

## THE SANTILLI'S THEORY 'INVASION' IN HYPERSTRUCTURES

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**Abstract.** In the quiver of hyperstructures Professor R. M. Santilli in early 90'es, tried to find algebraic structures in order to express his pioneer Lie-Santilli Theory. Santilli's theory on 'isotopies' and 'genotopies', born in 1960's, desperately needs 'units  $e$ ' on left or right, which are nowhere singular, symmetric, real-valued, positive-defined for  $n$ -dimensional matrices based on the so called isofields. These elements can be found in hyperstructure theory, especially in  $H_V$ -structure theory introduced in 1990. This connection appeared first in 1996 and actually several  $H_V$ -fields, the  $e$ -hyperfields, can be used as isofields or genofields so as, in such way they should cover additional properties and satisfy more restrictions. Meanwhile, the hyperstructure theory obtained a lot of results and applications in mathematics as well as in other applied sciences.

This presentation aims to review applicable hyperstructures in Lie Santilli theory especially when multivalued problems appeared, either in finite or in infinite case.

**Key words:** Lie-Santilli theory, hyperstructures, hope,  $H_V$ -structures.

**AMS Subject Classification:** 20N20, 16Y99

## 1 INTRODUCTION

The hyperstructures were introduced by F.Marty in 1934 [18] when he first defined the hypergroup as a set equipped with an associative and reproductive hyperoperation. The motivating example was the quotient of a group by any, not necessarily normal, subgroup. M.Koskas in 1970 [17] was introduced the fundamental relation  $\beta^*$ , which it turns to be the main tool in the study of hyperstructures. T.Vougiouklis in 1990 [27] was introduced the  $H_V$ -structures, by defining the weak axioms. The motivating example of those hyperstructures is the

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quotient of any group by any partition. Therefore the class of  $H_v$ -structures is the largest class of hyperstructures. Therefore we have:

**Motivation for  $H_v$ -structures:**

*The quotient of a group with respect to an invariant subgroup is a group.  
Marty states that, the quotient of a group with respect to any subgroup is a hypergroup.*

*Now, the quotient of a group with respect to any partition is an  $H_v$ -group.*

In [47] we introduced the abbreviation: **hyperoperation=hope**. Thus there is a definition: In algebraic hyperstructures there is a hope, in the classical structures there is not any hope!

In 1996 R.M.Santilli and T.Vougiouklis [21], point out that in physics the most interesting hyperstructures are the one called e-hyperstructures. These hyperstructures contain a unique left and right scalar unit, which is the most important tool in Lie-Santilli theory. In what follows we present the related hyperstructure theory mainly from the paper [21], enriched with some new results on the related hyperstructures. However one can see the books by P.Corsini [5], T.Vougiouklis [30], P.Corsini-V.Leoreanu [6] and B.Davvaz- V.Leoreanu-Fotea [12], for more definitions as well as the site: [aha.eled.duth.gr](http://aha.eled.duth.gr), for an extensive bibliography on the concept. Moreover, in this site one can see the Vougiouklis's point of view on the birth and the history of  $H_v$ -structures in the above site: *An  $H_v$ -interview, i.e. weak, with Th. Vougiouklis*, Interviewer N.Lygeros.

## 2 BASIC DEFINITIONS ON HOPES

In a set  $H$  is called **hyperoperation** (abr. **hope**) or **multivalued operation**, any map from  $H \times H$  to the power set of  $H$ . Therefore, in a hope

$$\cdot : H \times H \rightarrow \wp(H) : (x,y) \rightarrow x \cdot y \subset H$$

the result is a subset of  $H$ , instead of an element as we have in usually operations.

In a set  $H$  equipped with a hope  $\cdot : H \times H \rightarrow \mathcal{P}(H) - \{\emptyset\}$ , we abbreviate by

*WASS the weak associativity:*  $(xy)z \cap x(yz) \neq \emptyset, \forall x,y,z \in H$  and by

*COW the weak commutativity:*  $xy \cap yx \neq \emptyset, \forall x,y \in H$ .

The hyperstructure  $(H, \cdot)$  is called  **$H_v$ -semigroup** if it is *WASS* and it is called  **$H_v$ -group** if it is reproductive  $H_v$ -semigroup, i.e.  $xH = Hx = H, \forall x \in H$ . The hyperstructure  $(R, +, \cdot)$  is called  **$H_v$ -ring** if  $(+)$  and  $(\cdot)$  are *WASS*, the reproduction axiom is valid for  $(+)$  and  $(\cdot)$  is *weak distributive* with respect to  $(+)$ :

$$x(y+z) \cap (xy+xz) \neq \emptyset, \quad (x+y)z \cap (xz+yz) \neq \emptyset, \quad \forall x,y,z \in R.$$

An extreme class of hyperstructures is the following [26]: An  $H_V$ -structure is called *very thin* iff all hopes are operations except one, which has all hyperproducts singletons except one, which is a subset of cardinality more than one.

An  $H_V$ -group is called *cyclic* [22], if there is an element, called *generator*, which the powers have union the underline set. The minimal power with the above property is called *period* of the generator. Moreover if there exist an element and a special power, the minimum one, is the underline set, then the  $H_V$ -group is called *single-power cyclic*.

The main tool to study all hyperstructures are the fundamental relations  $\beta^*$ ,  $\gamma^*$  and  $\varepsilon^*$ , which are defined, in  $H_V$ -groups,  $H_V$ -rings and  $H_V$ -vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively [23, 27, 30]. A way to find the fundamental classes is given by analogous theorems to the following:

**Theorem 2.1** Let  $(H, \cdot)$  be an  $H_V$ -group and  $U$  be all finite products of elements of  $H$ . We define the relation  $\beta$  by setting  $x\beta y$  iff  $\{x,y\} \subset u$ ,  $u \in U$ . Then  $\beta^*$  is the transitive closure of  $\beta$ .

The main point of the proof of this theorem is that the relation  $\beta$  guaranties the validity of the following: Take two elements  $x,y$  such that  $\{x,y\} \subset u \in U$  and any hyperproduct where one of these elements is used. Then, if this element is replaced by the other, the new hyperproduct is inside the same fundamental class where the first hyperproduct is. Therefore, if the 'hyperproducts' of the above  $\beta$ -classes are 'products', then, they are fundamental classes.

Analogous theorems for the relations  $\gamma^*$  in  $H_V$ -rings and  $\varepsilon^*$  in  $H_V$ -modules and  $H_V$ -vector spaces, are also proved.

An element is called *single* if its fundamental class is singleton.

The fundamental relations are used for general definitions. Thus, to define the  $H_V$ -field the  $\gamma^*$  is used: A  $H_V$ -ring  $(R, +, \cdot)$  is called  *$H_V$ -field* if  $R/\gamma^*$  is a field. In the sequence the  *$H_V$ -vector space* is defined [30].

Let  $(H, \cdot)$ ,  $(H, *)$  be  $H_V$ -semigroups defined on the same set  $H$ .  $(\cdot)$  is called *smaller* than  $(*)$ , and  $(*)$  *greater* than  $(\cdot)$ , iff there exists an

$$f \in \text{Aut}(H, *) \text{ such that } xy \subset f(x^*y), \quad \forall x,y \in H.$$

Then we write  $\cdot \leq^*$  and we say that  $(H, *)$  *contains*  $(H, \cdot)$ . If  $(H, \cdot)$  is a structure then it is called *basic structure* and  $(H, *)$  is called  *$H_b$ -structure*.

**The little Theorem 2.2** Greater hopes than the ones which are *WASS* or *COW*, are also *WASS* or *COW*, respectively.

The definition of the  $H_v$ -field introduced a new class of hyperstructures[40,43]:

**Definition 2.3** The  $H_v$ -semigroup  $(H, \cdot)$  is called *h/v-group* if the quotient  $H/\beta^*$  is a group.

The h/v-groups are a generalization of the  $H_v$ -groups because in h/v-groups the reproductivity is not necessarily valid. However, 'sometimes a kind of *reproductivity of classes* is valid. This leads the quotient to be reproductivity. In a similar way the *h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces* etc, are defined.

The motivation to this concept is the following: Let  $\omega^*$  be the kernel of the canonical map from  $R$  to  $R/\gamma^*$ ; then we call *reproductive  $H_v$ -field* any  $H_v$ -field  $(R, +, \cdot)$  if the following axiom is valid:

$$x(R-\omega^*) = (R-\omega^*)x = R-\omega^*, \quad \forall x \in R-\omega^*.$$

So the h/v-group is a generalization of the  $H_v$ -group since the reproductivity is not necessarily valid. Sometimes a kind of *reproductivity of classes* is valid, i.e. if  $H$  is partitioned into equivalence classes  $\sigma(x)$ , then the quotient is reproductivity  $x\sigma(y) = \sigma(xy) = \sigma(x)y, \forall x \in H$ .

Hopes on any type of matrices can be defined:

**Definition 2.4** [48] Let  $A = (a_{ij}) \in \mathbf{M}_{m \times n}$  be matrix and  $s, t \in \mathbf{N}$ , with  $1 \leq s \leq m, 1 \leq t \leq n$ .

Then *helix-projection* is a map  $\underline{st}: \mathbf{M}_{m \times n} \rightarrow \mathbf{M}_{s \times t}: A \rightarrow A \underline{st} = (\underline{a}_{ij})$ , where  $A \underline{st}$  has entries

$$\underline{a}_{ij} = \{ a_{i+\kappa s, j+\lambda t} \mid 1 \leq i \leq s, 1 \leq j \leq t \text{ and } \kappa, \lambda \in \mathbf{N}, i+\kappa s \leq m, j+\lambda t \leq n \}$$

Let  $A = (a_{ij}) \in \mathbf{M}_{m \times n}, B = (b_{ij}) \in \mathbf{M}_{u \times v}$  be matrices and  $s = \min(m, u), t = \min(n, v)$ . We define a hyper-addition, called *helix-addition*, by

$$\oplus : \mathbf{M}_{m \times n} \times \mathbf{M}_{u \times v} \rightarrow \mathcal{P}(\mathbf{M}_{s \times t}): (A, B) \rightarrow A \oplus B = A \underline{st} + B \underline{st} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset \mathbf{M}_{s \times t}$$

where  $(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij}) \mid a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}$ .

Let  $A = (a_{ij}) \in \mathbf{M}_{m \times n}, B = (b_{ij}) \in \mathbf{M}_{u \times v}$  and  $s = \min(n, u)$ . We define the *helix-multiplication*, by

$$\otimes : \mathbf{M}_{m \times n} \times \mathbf{M}_{u \times v} \rightarrow \mathcal{P}(\mathbf{M}_{m \times v}): (A, B) \rightarrow A \otimes B = A \underline{ms} \cdot B \underline{sv} = (\underline{a}_{ij}) \cdot (\underline{b}_{ij}) \subset \mathbf{M}_{m \times v}$$

where  $(\underline{a}_{ij}) \cdot (\underline{b}_{ij}) = \{(c_{ij}) = (\sum a_{it} b_{tj}) \mid a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}$ .

The helix-addition is commutative, WASS but not associative. The helix-multiplication is WASS, not associative and it is not distributive, not even weak, to the helix-addition. For all matrices of the same type, the inclusion distributivity, is valid.

The problem of enumeration and classification of  $H_v$ -structures, was started from the beginning [18]. However, the problem becomes more complicate in  $H_v$ -structures because we have very great numbers in this case. The partial order in  $H_v$ -structures [27] and the Little Theorem, transfers and restrict the problem in finding the minimal, *up to isomorphisms*,  $H_v$ -structures. In this direction we have results by Bayon & Lygeros [3]:

Let  $H=\{a,b\}$  a set of two elements. There are 20  $H_v$ -groups, up to isomorphism,

Suppose in  $H=\{e,a,b\}$ , a hope is defined and there exists a scalar unit, then, there are 13 *minimal*  $H_v$ -groups. The number of all  $H_v$ -groups with three elements, up to isomorphism, which have a scalar unit, is 292.

In a set with three elements there are, exactly 6.494 minimal  $H_v$ -groups. 137 are abelians and the 6.357 are non-abelians; the 6.152 are cyclic and the 342 are not cyclic.

The number of  $H_v$ -groups with three elements, up to isomorphism, is 1.026.462. More precisely, there are 7.926 abelians and 1.018.536 non-abelians; the 1.013.598 are cyclic and the 12.864 are not cyclic, the 16 are very thin.

The number of all  $H_v$ -groups with four elements, up to isomorphism, which have a scalar unit, is 631.609. There are 10.614.362 abelian hopes from which the 10.607.666 are cyclic and the 6.696 are not. There are 8.028.299.905 abelian  $H_v$ -groups from which the 7.995.884.377 are cyclic and the 32.415.528 are not.

### 3 ENLARGING AND REDUCING HYPERSTRUCTURES

In [37] the 'enlarged' hyperstructures were examined in the sense that an extra element, outside the underlying set, appears in one result. In both directions, enlargement or reduction, most useful in representation theory, are those  $H_v$ -structures with the same fundamental structure: Suppose we have a structure and one element, outside of the structure, then we can attach this element in order to have a hyperstructure which becomes h/v-structure. Moreover we have the opposite problem: How one can remove at least one element of an  $H_v$ -structure or a classical structure?

**The Attach Construction 3.1** [37,41]. Let  $(H, \cdot)$  be an  $H_v$ -semigroup and  $v \notin H$ . We extend the  $(\cdot)$  into  $\underline{H} = H \cup \{v\}$  as follows:  $x \cdot v = v \cdot x = v, \forall x \in H$ , and  $v \cdot v = H$ . The  $(\underline{H}, \cdot)$  is a h/v-group where  $(\underline{H}, \cdot) / \beta^* \cong Z_2$  and  $v$  is a single element. We call the hyperstructure  $(\underline{H}, \cdot)$  the attach h/v-group of  $(H, \cdot)$ .

**Remarks.** The core of  $(\underline{H}, \cdot)$  is the set  $H$ . All scalar elements of  $(H, \cdot)$  are also scalars in  $(\underline{H}, \cdot)$  and any unit element of  $(H, \cdot)$  is also a unit of  $(\underline{H}, \cdot)$ . Finally, if  $(H, \cdot)$  is *COW* (resp. commutative) then  $(\underline{H}, \cdot)$  is also *COW* (resp. commutative). The motivation of the attach construction is the first kind very thin  $H_v$ -groups [26].

In the representation theory of  $H_v$ -groups by  $H_v$ -matrices one needs  $H_v$ -rings or  $H_v$ -fields which have non-degenerate fundamental structures in addition with only few of hypersums and hyperproducts to have cardinals greater than one.

**Theorem 3.2** Let  $(G, \cdot)$  be semigroup and  $v \notin G$  be an element appearing in a product  $ab$ , where  $a, b \in G$ , thus the result becomes a hyperproduct  $a \otimes b = \{ab, v\}$ . Then the minimal hyperoperation  $(\otimes)$  extended in  $G' = G \cup \{v\}$  such that  $(\otimes)$  contains  $(\cdot)$  in the restriction on  $G$ , and such that  $(G', \otimes)$  is a minimal  $H_v$ -semigroup which has fundamental structure isomorphic to  $(G, \cdot)$ , is defined as follows:

$$a \otimes b = \{ab, v\}, \quad x \otimes y = xy, \quad \forall (x, y) \in G^2 - \{(a, b)\}$$

$$v \otimes v = abab, \quad x \otimes v = xab \quad \text{and} \quad v \otimes x = abx, \quad \forall x \in G.$$

Therefore  $(G', \otimes)$  is a very thin  $H_v$ -semigroup.

If  $(G, \cdot)$  is commutative then the  $(G', \otimes)$  becomes strongly commutative.

Generally we have:

**Definitions 3.3** [37,41]. Let  $(H, \cdot)$  be a hypergroupoid.

We say that we **remove**  $h \in H$ , if we simply consider the restriction of  $(\cdot)$  on  $H - \{h\}$ .

We say that  $\underline{h} \in H$  **absorbs**  $h \in H$  if we replace  $h$ , whenever it appears, by  $\underline{h}$ .

We say that  $\underline{h} \in H$  **merges with**  $h \in H$ , if we take as product of  $x \in H$  by  $\underline{h}$ , the union of the results of  $x$  with both  $h$  and  $\underline{h}$ , and consider  $h$  and  $\underline{h}$  as one class, with representative  $\underline{h}$ .

The *uniting elements* method was introduced by Corsini–Vougiouklis [7] in 1989. With this method one puts in the same class, two or more elements. This leads, through hyperstructures, to structures satisfying additional properties.

The *uniting elements* method is the following: Let  $G$  be algebraic structure and let  $d$  be a property, which is not valid and it is described by a set of equations; then, consider the partition in  $G$  for which it is put together, in the same partition class, every pair of elements that causes the non-validity of the property  $d$ . The quotient by this partition  $G/d$  is an  $H_v$ -structure. Then, quotient out the  $H_v$ -structure  $G/d$  by the fundamental relation  $\beta^*$ , a stricter structure  $(G/d)/\beta^*$  for which the property  $d$  is valid, is obtained.

An interesting application of the uniting elements is when more than one properties are desired. The reason for this is that some of the properties lead straighter to the classes than others. So, it is better to apply the straightforward classes followed by the more complicated ones. The commutativity and reproductivity are easy applicable properties. One can do this because the following is valid.

**Theorem 3.4** [30] Let  $(G, \cdot)$  be a groupoid, and  $F = \{f_1, \dots, f_m, f_{m+1}, \dots, f_{m+n}\}$  be a system of equations on  $G$  consisting of two subsystems  $F_m = \{f_1, \dots, f_m\}$  and  $F_n = \{f_{m+1}, \dots, f_{m+n}\}$ . Let  $\sigma, \sigma_m$  be the equivalence relations defined by the uniting elements procedure using the systems  $F$  and  $F_m$  resp., and let  $\sigma_n$  be the equivalence relation defined using the induced equations of  $F_n$  on the groupoid  $G_m = (G/\sigma_m)/\beta^*$ . Then

$$(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*.$$

## 4 THE $\partial$ -HOPES

In [44] a hope, in a groupoid with a map on it, called *theta*  $\partial$ , is introduced.

**Definitions 4.1** Let  $(G, \cdot)$  be groupoid (resp., hypergroupoid) and  $f: G \rightarrow G$  be a map. We define a hope  $(\partial)$ , called *theta-hope* and we write  *$\partial$ -hope*, on  $G$  as follows

$$x\partial y = \{ f(x) \cdot y, x \cdot f(y) \}, \quad \forall x, y \in G. \quad (\text{resp. } x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall x, y \in G)$$

If  $(\cdot)$  is commutative then  $(\partial)$  is commutative. If  $(\cdot)$  is *COW*, then  $(\partial)$  is *COW*.

Let  $(G, \cdot)$  be a groupoid (resp. hypergroupoid) and  $f: G \rightarrow P(G) - \{\emptyset\}$  be any multivalued map. We define the  $(\partial)$ , on  $G$  as follows

$$x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall x, y \in G.$$

Let  $(G, \cdot)$  be a groupoid,  $f_i: G \rightarrow G, i \in I$ , be a set of maps on  $G$ . The

$$f_{\cup}: G \rightarrow P(G): f_{\cup}(x) = \{f_i(x) \mid i \in I\},$$

is the union of  $f_i(x)$ . We have the *union theta-hope*  $(\partial)$ , on  $G$  if we take  $f_{\cup}(x)$ .

If we take  $f \equiv f \cup (\text{id})$ , then we have the *b-theta-hope*:  $\underline{\partial}$ .

This definition can be generalized as follows:

**Definition 4.2** Let  $H$  be a set equipped with  $n$  operations (or hopes)  $\otimes_1, \otimes_2, \dots, \otimes_n$  and a map (or multivalued map)

$f: H \rightarrow H$  (or  $f: H \rightarrow P(H)$ , resp.), then  $n$  hopes  $\partial_1, \partial_2, \dots, \partial_n$  on  $H$  can be defined, called *theta-hopes* by putting

$$x\partial_i y = \{f(x)\otimes_i y, x\otimes_i f(y)\}, \quad \forall x, y \in H \text{ and } i \in \{1, 2, \dots, n\}$$

or, in case where  $\otimes_i$  is hope or  $f$  is multivalued map, we have

$$x\partial_i y = (f(x)\otimes_i y) \cup (x\otimes_i f(y)), \quad \forall x, y \in H \text{ and } i \in \{1, 2, \dots, n\}$$

If  $\otimes_i$  is associative then  $\partial_i$  is WASS.

**Motivation** for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. Therefore, in these terms, for two functions  $s(x), t(x)$ , we have  $s\partial t = \{s't, st'\}$  where  $(')$  denotes the derivative.

**Example 4.3** Taking the application on the derivative, consider all polynomials of first degree  $g_i(x) = a_i x + b_i$ . We have

$$g_1\partial g_2 = \{a_1 a_2 x + a_1 b_2, a_1 a_2 x + b_1 a_2\},$$

so this is a hope in the set of the first degree polynomials. Moreover all polynomials  $x+c$ , where  $c$  be a constant, are units.

**Properties 4.4** [45,46]. If  $(G, \cdot)$  is a semigroup then:

For every  $f$ , the hyperoperation  $(\partial)$  is WASS, and the  $b$ -theta-operation  $(\partial)$  is WASS.

If  $f$  is homomorphism and projection, then  $(\partial)$  is associative.

*Reproductivity.* If  $(\cdot)$  is reproductivity then  $(\partial)$  is also reproductivity.

*Commutativity.* If  $(\cdot)$  is commutative then  $(\partial)$  is commutative. If  $f$  is into the centre of  $G$ , then  $(\partial)$  is commutative. If  $(\cdot)$  is COW then,  $(\partial)$  is COW.

*Unit elements.*  $u$  is a unit element if  $f(u)=e$ , where  $e$  be a unit in  $(G, \cdot)$ . The elements of the kernel of  $f$ , are the units of  $(G, \partial)$ .

*Inverse elements.* Let  $(G, \cdot)$  be a monoid with unit  $e$  and  $u$  be a unit in  $(G, \partial)$ , then  $f(u)=e$ . The elements  $x' = (f(x))^{-1}u$  and  $x' = u(f(x))^{-1}$ , are the right and left inverses, respectively. We have two-sided inverses iff  $f(x)u = uf(x)$ .

**Proposition 4.5** Let  $(G, \cdot)$  be a group then, for all  $f: G \rightarrow G$ , the  $(G, \partial)$  is an  $H_v$ -group.

In order to see a connection of the merge with the  $\partial$ -hope, consider the map  $f$  such that  $f(h)=h$  and  $f(x)=x$  in the rest cases.

**Example 4.6 P-hopes.** Let  $(G, \cdot)$  be commutative semigroup and  $P \subset G$ . Consider the multivalued map  $f$  such that

$f(x) = P \cdot x, \quad \forall x \in G.$  Then we have

$$x\partial y = x \cdot y \cdot P, \quad \forall x, y \in G.$$

So the  $\partial$ -hope coincides with the well known class of P-hopes [22].

One can define theta-hopes on rings and other more complicate structures, where more than one theta-hopes can be defined.

**Definition 4.7** Let  $(R, +, \cdot)$  be a ring and  $f: R \rightarrow R, g: R \rightarrow R$  be two maps. We define two hopes  $(\partial_+)$  and  $(\partial_-)$ , called both *theta-hopes*, on  $R$  as follows

$$x\partial_+y = \{f(x)+y, x+f(y)\} \quad \text{and} \quad x\partial_-y = \{g(x)\cdot y, x\cdot g(y)\}, \quad \forall x, y \in R.$$

A hyperstructure  $(R, +, \cdot)$ , where  $(+), (\cdot)$  be hopes which satisfy all  $H_V$ -ring axioms, except the weak distributivity, will be called  *$H_V$ -near-ring*.

**Proposition 4.8** Let  $(R, +, \cdot)$  ring and  $f: R \rightarrow R, g: R \rightarrow R$  maps. The hyperstructure  $(R, \partial_+, \partial_-)$ , called *theta*, is an  $H_V$ -near-ring. Moreover  $(+)$  is commutative.

**Proposition 4.9** Let  $(R, +, \cdot)$  ring and  $f: R \rightarrow R, g: R \rightarrow R$  maps, then  $(R, \partial_+, \partial_-)$ , is an  $H_V$ -ring.

**Properties 4.10** The theta hyperstructure  $(R, \partial_+, \partial_-)$  takes new form in special cases:

(a) If  $f(x) \equiv g(x), \forall x \in R$ , i.e. the two maps coincide, then we have

$$x\partial_-(y\partial_+z) \cap (x\partial_+y)\partial_-(x\partial_+z) = \emptyset.$$

If  $f$  is homomorphism and projection, then  $(R, \partial_+, \partial_-)$  is an  $H_V$ -ring.

(b) If  $f(x) = x, \forall x \in R$ , then  $(R, +, \partial_-)$  becomes a multiplicative  $H_V$ -ring:

$$x\partial_-(y+z) \cap (x\partial_+y) + (x\partial_+z) = \{g(x)y + g(x)z\} \neq \emptyset.$$

## 5 THE $H_V$ -LIE ALGEBRAS

**Definition 5.1** [40]. Let  $(F, +, \cdot)$  be an  $H_V$ -field,  $(V, +)$  be a COW  $H_V$ -group and there exists an external hope

$$\cdot : F \times V \rightarrow P(V) : (a, x) \rightarrow ax$$

such that, for all  $a, b$  in  $F$  and  $x, y$  in  $V$  we have

$$a(x+y) \cap (ax+ay) \neq \emptyset, \quad (a+b)x \cap (ax+bx) \neq \emptyset, \quad (ab)x \cap a(bx) \neq \emptyset,$$

then  $V$  is called an  *$H_V$ -vector space* over  $F$ .

In the case of an  $H_V$ -ring instead of  $H_V$ -field then the  *$H_V$ -modulo* is defined.

In the above cases the fundamental relation  $\varepsilon^*$  is the smallest equivalence relation such that the quotient  $V/\varepsilon^*$  is a vector space over the fundamental field  $F/\gamma^*$ .

The general definition of an  $H_v$ -Lie algebra over a field  $F$  is the following [40]:

**Definition 5.2** Let  $(L,+)$  be an  $H_v$ -vector space over the field  $(F,+,\cdot)$ ,  $\varphi:F \rightarrow F/\gamma^*$ , the canonical map and  $\omega_F = \{x \in F : \varphi(x) = 0\}$ , where 0 is the zero of the fundamental field  $F/\gamma^*$ . Similarly, let  $\omega_L$  be the core of the canonical map  $\varphi': L \rightarrow L/\varepsilon^*$  and denote by the same symbol 0 the zero of  $L/\varepsilon^*$ . Consider the bracket (commutator) hope:

$$[ , ] : L \times L \rightarrow P(L): (x,y) \rightarrow [x,y]$$

then  $L$  is an  $H_v$ -Lie algebra over  $F$  if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \emptyset, [x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \emptyset$$

for all  $x, x_1, x_2, y, y_1, y_2 \in L$  and  $\lambda_1, \lambda_2$  in  $F$

(L2)  $[x, x] \cap \omega_L \neq \emptyset$  for all  $x$  in  $L$

(L3)  $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset$  for all  $x, y$  in  $L$

This is a general definition thus one can use special cases in order to face problems in applied sciences. Moreover, we see how the weak properties can be defined as the above weak linearity (L1), anti-commutativity (L2) and the Jacobi identity (L3). Similarly the  $h/v$ -rings,  $h/v$ -fields,  $h/v$ -modulus,  $h/v$ -vector spaces etc, are defined.

We present here a direction to obtain results from special cases by applying  $\partial$ -hopes on more complicated structures, in the sense that they have more than one operation.

**Theorem 5.3** Consider the ring of integers  $(Z,+,\cdot)$  and let  $n \neq 0$ . Consider the map  $f$  such that  $f(0) = n$  and  $f(x) = x, \forall x \in Z - \{0\}$ . Then  $(Z, \partial_+, \partial \cdot)$  is an  $H_v$ -near-ring, with  $(Z, \partial_+, \partial \cdot) / \gamma^* \cong Z_n$ .

**Proposition 5.4** Let  $(V,+,\cdot)$  be an algebra over the field  $(F,+,\cdot)$  and  $f:V \rightarrow V$  be a map. Consider the  $\partial$ -hope defined only on the multiplication of the vectors  $(\cdot)$ , then  $(V,+,\partial)$  is an  $H_v$ -algebra over  $F$ , where the related properties are weak. If, moreover  $f$  is linear then we have more strong properties.

**Definition 5.5** Let  $L$  be a Lie algebra, defined on an algebra  $(V,+,\cdot)$  over the field  $(F,+,\cdot)$  where the Lie bracket  $[x,y] = xy - yx$ . Consider any map  $f:L \rightarrow L$ , then the  $\partial$ -hope is defined as follows

$$x \partial y = \{f(x)y - f(y)x, f(x)y - yf(x), xf(y) - f(y)x, xf(y) - yf(x)\}$$

**Proposition 5.6** Let  $(V, +, \cdot)$  be an algebra over the field  $(F, +, \cdot)$  and  $f: V \rightarrow V$  be a linear map. Consider the  $\partial$ -hope defined only on the multiplication of the vectors  $(\cdot)$ , then  $(V, +, \partial)$  is an  $H_V$ -algebra over  $F$ , with respect to Lie bracket, where the weak anti-commutativity and the inclusion linearity is valid.

We can see that the weak linearity is valid, more precisely, the inclusion linearity is valid:

$$[\lambda_1 x_1 + \lambda_2 x_2, y] \subset \lambda_1 [x_1, y] + \lambda_2 [x_2, y].$$

Remark that one can face the weak Jacobi identity in analogous to the above propositions as well. One can use well known maps as constants or linear.

We conclude this paragraph with a definition of a hypergroupoid algebra [23,30].

**Definition 5.7** Let  $(G, \cdot)$  hypergroupoid, is called *set of fundamental maps* on  $G$ , the set of onto maps

$$Q = \{q: G \times G \rightarrow G: (x, y) \xrightarrow{\text{onto}} z \mid z \in xy\}.$$

Any subset  $Q_s \subset Q$  defines a hope  $(\circ_s)$  on  $G$  as follows  $x \circ_s y = \{z \mid z = q(x, y) \text{ for some } q \in Q_s\}$

$\circ_s \leq \cdot$  and  $Q_s \subset Q_{os}$ , where  $Q_{os}$  is the set of fundamental maps with respect to  $(\circ_s)$ . A  $Q_a \subset Q$  for which every  $Q_s \subset Q_a$  has  $(\circ_s)$  associative (resp. *WASS*) is called *associative* (resp. *WASS*). A hypergroupoid  $(G, \cdot)$  is  $q$ -*WASS* if there exists an element  $q_0 \in Q$  which defines an associative operation  $(\circ)$  in  $G$ . Remark that for  $H_V$ -groups we have  $Q \neq \emptyset$ .

Suppose  $G$  is finite,  $\text{card}G = |G| = n$ , it is  $q$ -*WASS* with associative  $q_0 \in Q$ . In the set  $K[G]$  of all formal linear combinations of elements of  $G$  with coefficients from a field  $K$ , we define an operation  $(+)$ :

$$(f_1 + f_2)(g) = f_1(g) + f_2(g), \quad \forall g \in G, f_1, f_2 \in K[G]$$

and a hope  $(*)$ , the *convolution*,

$$f_1 * f_2 = \{f_q: f_q(g) = \sum_{q(x,y)=g} f_1(x) f_2(y), q \in Q\}.$$

**Definition 5.8** The hyperstructure  $(K[G], +, *)$  is a multiplicative  $H_V$ -ring where the inclusion distributivity is valid. This  $H_V$ -ring is called *hypergroupoid  $H_V$ -algebra*.

$\forall q \in Q, g \in G$ , we have  $|Q_s| \leq \prod_{(x,y) \in G \times G} (|xy|)$ ,  $1 \leq |q^{-1}(g)| \leq n^2 - n + 1$  and  $\sum_{g \in G} |q^{-1}(g)| = n^2$ .

The zero map  $f(x) = 0$  is a scalar element in  $K[G]$ .

## 6 REPRESENTATIONS

Representations (we abbreviate by *rep*) of  $H_v$ -groups, can be considered either by generalized permutations [28] or by  $H_v$ -matrices [23,25,29]. First we present the matrix reps.

In the classical books on representations we find the following definitions: Let  $G$  be a group and  $V$  be a finite dimensional vector space over the field  $F$ . A *representation* of  $G$  is a homomorphism  $\rho: G \rightarrow \text{Aut}(V)$  of  $G$  into the set of automorphisms of  $V$ .

Analogous definitions are given for complicate structures: Let  $L$  be a Lie algebra then a *rep* of  $L$  is a homomorphism  $\rho: L \rightarrow \text{gl}(V)$ , from  $L$  into linear transformations on  $V$  over  $F$ .

Since there exists 1-1 correspondence on the sets of all endomorphisms with  $n \times n$  matrices, where  $n = \dim V$ , any rep corresponds to each element, of a finite group, a matrix, and this set of matrices acts exactly as the group. Notice that the addition of matrices and the external multiplication of scalars by matrices are 'natural operations', by contrast the multiplication of matrices is a 'strange' operation because it is dictated by the composition of linear maps or automorphisms or by the composition of maps.

With this theory, mathematicians try to transfer the study of the several structures into the study of matrices which is clear and easy. Ado's theorem -and the related ones- states that every finite-dimensional Lie algebra has a faithful finite-dimensional rep. The two steps in rep theory: first, by the Cayley's theorem every group has a faithful rep by permutations. Second, every permutation group of order  $n$  can be represented by  $n \times n$  *monomial matrices*, i.e. matrices with only one entry 1 in every row and column and the rest entries are 0. The above steps are clear but the obtained reps are not useful since the matrices are of type  $n \times n$ . Thus, the main attempt is to reduce the dimension of reps. Most important is to find the irreducible reps over the field of real or complex numbers.

The rep theory is very important because it represents all groups in one form so that they can be compared and studied in the same way. Thus the low dimensional reps are most useful, i.e. the reps by  $2 \times 2$  matrices is the simplest non degenerate case.

$H_v$ -*matrix* (or *h/v-matrix*) is called a matrix with entries elements of an  $H_v$ -ring or  $H_v$ -field (or *h/v-ring* or *h/v-field*). The hyperproduct of  $H_v$ -matrices  $A=(a_{ij})$  and  $B=(b_{ij})$ , of type  $m \times n$  and  $n \times r$ , respectively, is a set of  $m \times r$   $H_v$ -matrices, defined in a usual manner:

$$\mathbf{A} \cdot \mathbf{B} = (a_{ij}) \cdot (b_{ij}) = \{ \mathbf{C} = (c_{ij}) \mid c_{ij} \in \bigoplus \Sigma a_{ik} \cdot b_{kj} \},$$

where  $(\bigoplus)$  denotes the *n-ary circle hope* on the hyperaddition [30]: that is the sum of products of elements of the  $H_v$ -ring is considered to be the union of the sets obtained with all possible parentheses. However, in the case of  $2 \times 2$   $H_v$ -matrices the 2-ary circle hope which coincides with the hyperaddition in the  $H_v$ -ring. Notice that the hyperproduct of  $H_v$ -matrices does not necessarily satisfy *WASS*.

*The rep problem by  $H_v$ -matrices is the following:*

**Definition 6.1** Let  $(H, \cdot)$  be  $H_v$ -group,  $(R, +, \cdot)$  be  $H_v$ -ring and  $\mathbf{M}_R = \{(a_{ij}) \mid a_{ij} \in R\}$ , then any

$$T: H \rightarrow \mathbf{M}_R: h \mapsto T(h) \text{ with } T(h_1 h_2) \cap T(h_1)T(h_2) \neq \emptyset, \quad \forall h_1, h_2 \in H,$$

is called  *$H_v$ -matrix rep*. If  $T(h_1 h_2) \subset T(h_1)T(h_2)$ , then  $T$  is an *inclusion rep*, if  $T(h_1 h_2) = T(h_1)T(h_2)$ , then  $T$  is a *good rep* an induced rep  $T^*$  for the hypergroup algebra is obtained. If  $T$  is one to one and good then it is a *faithful rep*.

In reps of  $H_v$ -groups by  $H_v$ -matrices, there are two difficulties: To find an  $H_v$ -ring and an appropriate set of  $H_v$ -matrices.

The problem of reps is very complicated mainly because the cardinality of the product of two  $H_v$ -matrices is normally very big. The problem can be simplified in several special cases such as the following:

- (a) The  $H_v$ -matrices are over  $H_v$ -rings with 0 and 1 and if these are scalars. Thus the e-hyperstructures are interesting in the rep theory.
- (b) The  $H_v$ -matrices are over *very thin*  $H_v$ -rings.
- (c) The case of  $2 \times 2$   $H_v$ -matrices, since the 2-ary circle hope coincides with the hyperaddition in  $H_v$ -rings. This is the lowest dimensional, non degenerate, rep.
- (d) The case of  $H_v$ -rings in which the strong associativity in hyperaddition is valid.
- (e) The case of  $H_v$ -rings which contains singles, then these act as absorbings.

The main theorem of reps on  $H_v$ -structures [30], is the following:

**Theorem 6.2** A necessary condition in order to have an inclusion rep  $T$  of an  $H_v$ -group  $(H, \cdot)$  by  $n \times n$   $H_v$ -matrices over the  $H_v$ -ring  $(R, +, \cdot)$  is the following:

For all classes  $\beta^*(x)$ ,  $x \in H$  there must exist elements  $a_{ij} \in H$ ,  $i, j \in \{1, \dots, n\}$  such that

$$T(\beta^*(a)) \subset \{ A = (a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}), \quad i, j \in \{1, \dots, n\} \}$$

Therefore, every inclusion rep  $T: H \rightarrow \mathbf{M}_R: a \mapsto T(a) = (a_{ij})$  induces a homomorphic rep  $T^*$  of the group  $H/\beta^*$  over the ring  $R/\gamma^*$  by setting  $T^*(\beta^*(a)) = [\gamma^*(a_{ij})]$ ,  $\forall \beta^*(a) \in H/\beta^*$ , where the element  $\gamma^*(a_{ij}) \in R/\gamma^*$  is the  $ij$  entry of the matrix  $T^*(\beta^*(a))$ . Then  $T^*$  is called *fundamental induced rep* of  $T$ .

Denote  $\text{tr}_\phi(T(x)) = \gamma^*(T(x_{ii}))$  the fundamental trace, then the mapping

$$X_T : H \rightarrow R/\gamma^* : x \mapsto X_T(x) = \text{tr}_\phi(T(x)) = \text{tr}T^*(x)$$

is called *fundamental character*. There are several types of traces.

Second, we present the rep problem is by *Generalized Permutations* (we write *gp* ).

**Definitions 6.3** [28,30] Let  $X$  be a set, then a map  $f: X \rightarrow P(X) - \{\emptyset\}$ , is a *gp of  $X$*  if the reproduction axiom is valid  $\cup_{x \in X} f(x) = f(X) = X$ . Denote by  $M_X$  the set of all gps on  $X$ . For an  $H_V$ -group  $(X, \cdot)$  and  $a \in X$ , the gp  $f_a$  defined by  $f_a(x) = ax$  is an *inner gp*. Arrow of  $f$  is any  $(x, y) \in X^2$  with  $y \in f(x)$ .  $f_2 \in M_X$  contains  $f_1 \in M_X$  or  $f_1$  is a *sub-gp* of  $f_2$ , if  $f_1(x) \subset f_2(x)$ ,  $\forall x \in X$ , then we write  $f_1 \subset f_2$ . If, moreover,  $f_1 \neq f_2$ , then  $f_1$  is a proper sub-gp of  $f_2$ . A  $f \in M_X$  is called *minimal* if it has no proper sub-gp. Denote  $\underline{M}_X$  the set of all minimal gps of  $M_X$ . The gp  $t$  with  $t(x) = X$ ,  $\forall x \in X$ , is called *universal* and contains all elements of  $M_X$ . The *converse* of a gp  $f$  is the gp  $\underline{f}$  defined by  $\underline{f}(x) = \{z \in X: f(z) \ni x\}$ , thus  $\underline{f}$  is obtained by reversing arrows. We call *associated* to  $f \in M_X$  the gp  $f \circ \underline{f}$ , where  $(\circ)$  is the map composition. The *union*  $f = \cup_{i \in I} f_i$  of a family of gps  $\{f_i: i \in I\}$ , is defined by  $f(x) = \cup_{i \in I} f_i(x)$ ,  $\forall x \in X$ .

For finite  $X$ , we reach a minimal gp, by the *deleting arrows* method.

**Theorem 6.4** Let  $f \in M_X$ , then  $f \in \underline{M}_X$  if and only if, the following condition is valid: if  $a \neq b$  and  $f(a) \cap f(b) \neq \emptyset$ , then  $f(a) = f(b)$  and  $f(a)$  is a singleton.

**Corollary 6.5** If  $f \in \underline{M}_X$  then  $\underline{f} \in \underline{M}_X$ .

An explicit description of  $\underline{M}_X$ :  $(f \circ \underline{f})(x) = f\{u: f(u) \ni x\} = \cup_{f(u) \ni x} f(u)$ ,  $\forall x \in X$ . So  $(f \circ \underline{f})(x) = \{y: \exists u \in X, \{x, y\} \subset f(u)\}$ . So, if  $I$  is the identity permutation, then  $I \subset f \circ \underline{f}$ ,  $\forall f \in M_X$ .

There is a direct relation of  $\beta^*$  and the associated gp  $f \circ \underline{f}$ . We see this relation, for finite  $X$ , in the following theorem:

**Theorem 6.6** If  $f \in \underline{M}_X$  then  $(f \circ \underline{f})(x) = \{y \in X: f(y) = f(x)\}$ .

In order to face the rep problem by the gp one can see that the usual map composition can not be used because it is not multivalued. Therefore the main *open problem* is to find standard hope on  $M_X$ .

Several constructions can be used to obtain appropriate  $H_V$ -rings and  $H_V$ -fields:

(i) Let  $(H, \cdot)$  be  $H_V$ -group, then for every  $(\oplus)$  such that  $x \oplus y \supset \{x, y\}$ ,  $\forall x, y \in H$ , the  $(H, \oplus, \cdot)$  is an  $H_V$ -ring. These  $H_V$ -rings are called *associated to  $(H, \cdot)$   $H_V$ -rings*.

In reps theory of hypergroups, in sense of Marty, there are three associated hyperrings  $(H, \oplus, \cdot)$  to  $(H, \cdot)$ . The  $(\oplus)$  is defined respectively,  $\forall x, y \in H$ , as follows:

type a:  $x \oplus y = \{x, y\}$ , type b:  $x \oplus y = \beta^*(x) \cup \beta^*(y)$ , type c:  $x \oplus y = H$ .

In the above types the strong associativity and strong or inclusion distributivity, is valid.

(ii) Let  $(H, +)$  be  $H_v$ -group, then for all hopes  $(\otimes)$  such that  $x \otimes y \supset \{x, y\}$ ,  $\forall x, y \in H$ , the  $(H, +, \otimes)$  is an  $H_v$ -ring.

A variation of this is the following: Let  $(H, +)$  be  $H_v$ -group with a scalar zero 0. Then  $\forall (\otimes)$ :  $x \otimes y \supset \{x, y\}$ ,  $\forall x, y \in H - \{0\}$ ,  $x \otimes 0 = 0 \otimes x = 0$ ,  $\forall x \in H$ , the  $(H, +, \otimes)$  is an  $H_v$ -ring.

(iii) Let  $(H, \cdot)$  be  $H_v$ -group. Take a  $0 \notin H$  and set  $H' = H \cup \{0\}$ . Define a hope  $(+)$  by:  $0 + 0 = 0$ ,  $0 + x = H = x + 0$ ,  $x + y = 0$ ,  $\forall x, y \in H$ ,

and extend  $(\cdot)$  in  $H'$  by  $0 \cdot 0 = 0$ ,  $0 \cdot x = x \cdot 0 = 0$ ,  $\forall x, y \in H$ .

Then  $(H', +, \cdot)$  is reproductive  $H_v$ -field with  $H'/\gamma^* \cong Z_2$  where 0 is absorbing and single.

## 7 THE SANTILLI'S e-CONSTRUCTIONS

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli [20], proposed a 'lifting' of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1999 [21] and they are called *e-hyperfields*. The  $H_v$ -fields or  $h/v$ -fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology. We present in the following the main definitions and results restricted in the  $H_v$ -structures.

**Definition 7.1** A hyperstructure  $(H, \cdot)$  which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we normally assume that for every element x, there exists an inverse element  $x^{-1}$ , i.e.  $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$ . Remark that the inverses are not necessarily unique.

**Definition 7.2** A hyperstructure  $(F, +, \cdot)$ , where  $(+)$  is an operation and  $(\cdot)$  is a hope, is called *e-hyperfield* if the following axioms are valid:

1.  $(F, +)$  is an abelian group with the additive unit 0,
2.  $(\cdot)$  is WASS,
3.  $(\cdot)$  is weak distributive with respect to  $(+)$ ,
4. 0 is absorbing element:  $0 \cdot x = x \cdot 0 = 0$ ,  $\forall x \in F$ ,

5. there exists a multiplicative scalar unit, i.e.  $1 \cdot x = x \cdot 1 = x, \forall x \in F,$
6. for every element  $x \in F$  there exists a unique inverse  $x^{-1}$ , such that  $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x.$

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation  $1 = x \cdot x^{-1} = x^{-1} \cdot x.$  is valid, then we say that we have a *strong e-hyperfield*.

Now we introduce a general construction which is based on the partial ordering of the  $H_v$ -structures and on the Little Theorem.

**The Main e-Construction 7.3** Given a group  $(G, \cdot)$ , where e is the unit, then we can define in G, a large number of hopes  $(\otimes)$  as follows:  $x \otimes y = \{xy, g_1, g_2, \dots\}, \forall x, y \in G - \{e\},$  and  $g_1, g_2, \dots$  are elements from  $G - \{e\}$  which are not necessarily the same for each pair  $(x, y)$ . Then  $(G, \otimes)$  becomes an  $H_v$ -group which in fact is an  $H_b$ -group which contains the basic group  $G - \{e\}$ . The  $H_v$ -group  $(G, \otimes)$  is an e-hypergroup. Moreover, if for each  $x, y$  such that  $xy = e$  then we have  $x \otimes y = xy,$  then  $(G, \otimes)$  becomes a strong e-hypergroup.

*Proof.* The proof is immediate since for both cases we enlarge the results of the group by putting elements from the set G and applying the Little Theorem. Moreover one can easily see that the unit e is a unique scalar element and for each x in G, there exists a unique inverse  $x^{-1}$ , such that  $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x.$  Finally if the last condition is valid then we have  $1 = x \cdot x^{-1} = x^{-1} \cdot x,$  so the hyperstructure  $(G, \otimes)$  is a strong e-hypergroup.

*Remark.* The above main e-construction gives an extremely large class of e-hopes. These e-hopes can be used in the several more complicate hyperstructures to obtain appropriate e-hyperstructures. However, we remark that the most useful are the ones where only few products are enlarged and, even more, the extra elements are one or two. This means that we have analogous situation to the rep theory.

*Example.* Consider the non-commutative quaternion group  $Q = \{1, -1, i, -i, j, -j, k, -k\}$  whose multiplication is given by the table

.	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j

j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

Using this operation one can obtain several hopes which are e-groups.  
 For example, denoting  $\underline{i}=\{i,-i\}$ ,  $\underline{j}=\{j,-j\}$ ,  $\underline{k}=\{k,-k\}$  we define  $(\star)$  hope by the table:

$\star$	1	-1	1	-i	j	-j	k	-k
1	1	-1	1	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	<u>k</u>	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	<u>i</u>	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
K	k	<u>k</u>	i	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

The hyperstructure  $(Q, \star)$  is strong e-hypergroup because 1 is scalar unit and the elements -1, i, -i, j, -j, k and -k have unique inverses the elements -1, -i, i, -j, j, -k and k, respectively, which are the inverses in the basic group.

The important thing of this example is that one can have more strict hopes, so for example, one can have a strong commutative hope,

$\circ$	1	-1	1	-i	j	-j	k	-k
1	1	-1	1	-i	j	-j	k	-k

-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	<u>k</u>	<u>k</u>	i	i
-i	-i	i	1	-1	<u>k</u>	<u>k</u>	i	i
j	j	-j	<u>k</u>	<u>k</u>	-1	1	i	i
-j	-j	j	<u>k</u>	<u>k</u>	1	-1	i	i
K	k	-k	i	i	i	i	-1	1
-k	-k	k	i	i	i	i	1	-1

therefore this strong e-hypersrtructure  $(Q, \circ)$  is a commutative e-hypergroup.

The reason we gave the above example is to see that there is a large variety of e-hyperstructures we can construct from given classical structures. One can see in the papers [21,39] and in the book [12] some classes of e-hyperstructures and their properties and results connected them with the classical theory. The representation theory and the Lie algebras as well as in hypermatrix theory large classes of e-hyperstructures appear and can offer to Lie-Santilli algebraic theory models to represent their theory. In the Lie admissible algebras the P-hopes can be used as one can see the replacement of the unit matrix in the Lie-bracket (in isoproducts): by any set of matrices P:

$$[X, Y]_P = (XP)Y - (YP)X, \text{ for all matrices or hypermatrices } X \text{ and } Y.$$

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