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Lie-isotopic liftings of Lie symmetries. I. General considerations

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In a number of cases, specific physical conditions may cause the deformation of a given metric into a functionally more complex form, with the consequent loss of the original Lie symmetry. We present a general method for the construction of the symmetry of the new metric which: (1) is based on a Lie-isotopic, step-by-step generalization of the conventional Lie theory (enveloping associative algebra, Lie algebra, and Lie group); (2) is applicable to an infinite family of different deformations of the original metric; and (3) admits the original symmetry as a particular case. Moreover, the method has been conceived to permit the explicit computation of the new symmetry via the knowledge only of the original symmetry and of the new metric. Applications to the rotations and to the Lorentz group are presented in subsequent papers.

I. STATEMENT OF THE PROBLEM

As is well known, when absolute rigidity is relaxed to admit the deformations occurring in the physical reality, perfectly spherical objects in Euclidean spaces

$$r'\delta r = xx + yy + zz = 1, \qquad 2 \in E(3), \qquad (1)$$

can be deformed into ellipsoids

$$r'gr = xb_1^2x + yb_2^2y + zb_3^2z = 1 \qquad (2)$$

with the consequent manifest loss of the rotational symmetry.

In this paper we shall attempt a general method for the construction of a generalization of the original symmetry which: (a) provides the form invariance of all possible new metrics; (b) is achieved via a generalization of the structure of the original symmetry (unit, enveloping algebra, Lie algebra, Lie group, composition law, etc.), while being isomorphic to the original symmetry; and (c) assures that the original symmetry is recovered identically whenever the local physical conditions are such as to recover the original metric.

The specific application to the generalization of the group of rotations for the invariance of all possible ellipsoidical deformations of the sphere will be studied in the accompanying paper. The method presented in this paper is, however, of sufficient generality to permit its application to broader settings, such as the possible generalization of the Minkowski invariant due to the local functional dependence of the speed of light when propagating within material media. In subsequent papers we shall therefore attempt a generalization of the Lorentz group under conditions (a), (b), (c) above. Specific applications to elementary-particle physics are also contemplated.

Our main tool is the notion of *isotopy* of the theory of abstract algebras, but applied specifically to the isotopic generalization of Lie's theory. The notion of *algebraic isotopy* is rather old and dates back to the early stages of set theory. According to R. H. Bruck[1] (p. 56), "the notion is so natural to creep in unnoticed." In the more recent specialized literature on (nonassociative) algebras, the notion of isotopy was applied to the case of the Jordan algebras by K. McCrimmon[2] and others. Despite that, the notion remained largely ignored in general treatises in abstract algebras, as it is still today.

What is apparently the first application of the isotopy to Lie algebras was presented in a memoir,[3] jointly with the identification of the main lines of the *Lie-isotopic theory*, that is, the Lie-isotopic generalization of the Poincaré-Birkhoff-Witt theorem; of Lie's first, second, and third theorems; of the Baker–Campbell–Hausdorff theorem; etc.

The proposal of the Lie-isotopic theory was submitted as a particular case of the broader *Lie-admissible generalization of Lie's theory* (which for brevity will not be considered in this paper).

Since that time (1978), the general lines of the theory have been subjected to a systematic study by mathematicians at the yearly Workshops[4,5] and at the Orléans International Conference.[6] For mathematical studies, we refer the interested reader to the contributions by G. M. Benkart, D. J. Britten,

Y. Ilamed, M. Kôiv, J. Lôhmus, H. C. Myung, R. H. Oehmke, S. Okubo, J. M. Osborn, A. A. Sagle, L. Sorgsepp, M. L. Tomber, G. P. Wene, *et al.* (see the bibliography by M. L. Tomber *et al.*[7] and the subsequent Proceedings[5,6]).

The contributions here referred to are devoted to the Lie-admissible algebras, of which the Lie-isotopic ones are a particular case. For recent mathematical contributions more specifically related to the Lie isotopy, we refer the reader to G. Benkart, J. M. Osborn, and D. Britten,[8] H. C. Myung,[9] J. M. Osborn,[10] and A. A. Sagle.[11] A review for physicists of the state of the art of our knowledge in the Lie-isotopic theory is provided in a monograph[12] (Chapter 6, in particular).

The first physical application of the Lie-isotopic theory was submitted jointly with the original proposal[3] within the context of the problem of symmetries and conserved quantities. In fact, there exist equivalence transformations of a Lagrangian (sometime referred to as *isotopic transformations*[12]) in which the original symmetries are generally lost without evidently affecting the conserved quantities. The symmetries of different but equivalent Lagrangians (or Hamiltonians) which lead, via Noether's theorem, to the same conserved quantities were then called *isotopically related symmetries*.

This is the case, for instance, for the O(3) symmetry of the Lagrangian

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \tfrac{1}{2}(x^2 + y^2 + z^2), \quad (3)$$

and the O(2.1) symmetry of the equivalent Lagrangian

$$L^* = \tfrac{1}{2}(\dot{x}^2 - \dot{y}^2 + \dot{z}^2) - \tfrac{1}{2}(x^2 - y^2 + z^2), \quad (4)$$

which both lead to the conservation of the angular momentum, as pointed out by G. Marmo and E. J. Saletan.[13] It was shown in Ref. 3 (pp. 287–290) that the Lie-isotopic generalization of O(3) characterized by the isotopic element $g = \mathrm{diag}(+1, -1, +1)$ leads exactly to O(2.1). In this sense O(2.1) can be interpreted as a Lie isotope of O(3). The notion of isotopically related Lie symmetries was subsequently studied in detail in Ref. 14 and in other papers.

It should be also recalled here that the understanding of the mechanism of isotopy in symmetries and conservation laws stimulated the conception of the Lie-isotopic generalization of Lie's theory as considered in this paper. In fact, the problem for the Lagrangian (4) was to reach a realization of the *Lorentz* group in (2.1) dimensions whose generators are those of the *rotation* group (the latters being precisely the conserved quantities which cannot be altered by assumption). The achievement of a realization of the Lorentz group O(2.1) with the angular momentum components as generators evidently called for the abandonment of the simplest possible Lie product $AB - BA$ of current use and its generalization into the form[3]

$$[A, B]^* = A * B - B * A = AgB - BgA, \quad (5)$$

where g is precisely the new metric of structure (4).

The isotopic relationship between O(2.1) and O(3) will be inspected in more detail in paper II of this series, and it will be extended to O(3.1) and O(4) in Paper III.

The reader should be aware that we shall not look for the most general possible Lie isotopy. In fact, our objective is to illustrate the implications of Lie products more general than the simplest possible one, $AB - BA$, of current use in the mathematical and physical literature. For this purpose, the Lie-isotopic product (5) is sufficient.

The reader might also be interested in knowing that, besides the product (5), only the following additional Lie-isotopic product is known at this writing[16]:

$$\begin{aligned}[A, B]^* &= A * B - B * A \\ &= WAWBW - WBWAW, \end{aligned} \quad (6)$$

where W is idempotent ($W^2 = W$). For brevity, the formulation of Lie-isotopic symmetries in terms of the alternative product above will not be considered in these papers and will be left to the interested reader.

Studies on the classification of all possible Lie-isotopic formulations of the Lie product have been conducted in Refs. 17, 18. These studies were based on the generalization of enveloping associative algebras into nonassociative, Lie-admissible forms called *genotopes*, with evident considerable increase of the degrees of freedom. Nevertheless, when the attached Lie algebras are reformulated in terms of an equivalent associative-isotopic form, only the forms (5) and (6) emerge as possible or otherwise known until now. Throughout our studies we shall only consider associative-isotopic enveloping algebras, and leave to the reader their reformulation in terms of nonassociative genotopes.

In closing these introductory words, we should stress that, despite the efforts conducted until now, the Lie-isotopic generalization of the conventional realization of Lie's theory is at its very beginning, and much remains to be done.

II. ELEMENTS OF THE LIE-ISOTOPIC THEORY ON METRIC SPACES

For the purposes of this paper, we shall use the term *metric spaces* for the n-dimensional topological spaces M over the field $\mathbb{F}$ of real numbers $\mathbb{R}$, or complex numbers $\mathbb{C}$ or quaternions $\mathbb{Q}$, equipped with a nonsingular, sesquilinear, and Hermitian composition (x, y), $x, y \in M$, characterizing the mapping

$$(x, y): M \times M \to \mathbb{F}. \tag{7}$$

Let $e = (e_1, \ldots, e_n)$ be a basis of M, and define the *metric tensor* via the familiar rules

$$(e_i, e_j) = g_{ij}. \tag{8}$$

Then, the condition of nonsingularity is intended to ensure the existence of the inverse

$$\hat{I} = g^{-1}, \qquad g = (g_{ij}), \tag{9}$$

with the consequent characterization of covariant and contravariant quantities

$$x_i = g_{ij}x^j, \qquad x^i = I^{ij}x_j. \tag{10}$$

The condition of sesquilinearity

$$(x, \alpha y + \beta z) = \alpha(x, y) + \beta(x, z), \tag{11a}$$

or

$$(\alpha x + \beta y, z) = \bar{\alpha}(x, z) + \bar{\beta}(y, z), \tag{11b}$$

where the overbar represents complex conjugation in $\mathbb{F}$, permit the realization of the composition

$$(x, y) = x^\dagger g y = x^{\dagger i} g_{ij} x^j, \tag{12}$$

where the dagger represents Hermitian conjugation in M.

Finally, the condition of Hermiticity can be formulated via the rules

$$(x, gy) = (g^\dagger x, y) = (gx, y), \tag{13}$$

and it is introduced for reasons to be identified below.

Additional conditions, such as the positive-definite character of the metric, are not recommendable for a general view of the Lie-isotopic theory, and they will not be considered at this time.

Whenever appropriate, metric spaces will be indicated with the notation

$$M = M(n, g, \mathbb{F}), \qquad \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{Q}. \tag{14}$$

Some of the metric spaces we are admitting for $\mathbb{F} = \mathbb{R}$ are: the Euclidean space $E(3, \delta, \mathbb{R})$, $\delta = \mathrm{diag}(+1, +1, +1)$; the Minkowski space $M(3 + 1, g, \mathbb{R})$, $g = \mathrm{diag}(+1, +1, +1, -1)$; the Riemannian space $R(n, g(x), \mathbb{R})$, with $g(x)$, $x \in M$, being symmetric and positive definite; the Finsler space $F(n, g(x, \dot{x}), \mathbb{R})$, where $g(x, \dot{x}) = \frac{1}{2}(\partial^2 f(x, \dot{x})/\partial x^i \partial x^j)$ is positive definite (for non-null $\dot{x}$) and of rank n; and others with corresponding spaces for the fields $\mathbb{F}$ of complex numbers and quaternions. Thus, we shall assume that the metric g is nonsingular, is Hermitian, and verifies the needed continuity conditions (e.g., analyticity) in all variables, and we write

$$\det g \neq 0, \qquad g^\dagger = g, \qquad g = g(t, x, \dot{x}, \ldots). \tag{15}$$

As one can see, our definition of a metric is as general as possible, and *does not* coincide with the more restrictive definition conventionally used in specific geometries, such as the symplectic or the Riemannian ones. This situation is permitted by the Lie-isotopic theory because it does not require restrictions on g beyond those considered here. The formalization of the metric and its restriction to specific cases would then imply particularizations (such as the removal of the dependence on the velocities) which are not warranted or recommendable for a general study in Lie isotopy.

We consider now an m-parameter, continuous Lie transformation group $G(m)$ on $M(n, g, \mathbb{F})$, i.e., a topological space $G(m)$ equipped with a binary mapping

$$\phi: G(m) \times G(m) \to G(m) \tag{16}$$

verifying the conditions for $G(m)$ to be a topological group, and an additional mapping

$$f: G(m) \times M \to M \tag{17}$$

characterized by n analytic functions $f(w; x)$ depending on m parameters w and the local coordinates $x \in M$, which verify the conditions for $G(m)$

to be a Lie transformation group (closure, associativity, identity, and inverse).

We shall furthermore assume that the group $G(m)$ acts linearly on M, i.e.,

$$x' = f(w; x) = A(w)x, \tag{18}$$

under which the group conditions can be realized in the form

$$A(0) = I, \tag{19a}$$

$$A(w)A(w') = A(w''),$$
$$w'' = w + w' \tag{19b}$$

$$A(w)A(w^{-1}) = A(w^{-1})A(w) = I,$$
$$w^{-1} = -w, \tag{19c}$$

where I is the unit matrix in n dimensions.

Among the rather large number of aspects of the theory of linear, continuous, m-parameter Lie transformation groups, we now focus our attention on the following ones.

(1) The *universal enveloping associative algebra*[19] $\mathscr{E}$ of $G(m)$, which we shall indicate with the symbolic expression of the basis (see Ref. 19 for technical definitions)

$$\mathscr{E}: I, X_r, \underset{r\leq s}{X_r X_s}, \underset{r\leq s\leq t}{X_r X_s X_t}, \dots,$$
$$r, s, t, \dots = 1, 2, \dots, n, \tag{20}$$

where I is now the $n \times n$ identity of $\mathscr{E}$,

$$IX_r = X_r I = X_r. \tag{21}$$

The X's are the generators of $G(m)$ in their fundamental $(n \times n)$ representation verifying the skew-Hermiticity property

$$X_r^\dagger = -X_r; \tag{22}$$

the product $X_r X_s$ is the conventional associative product of matrices; and the attached Lie algebra is given by the familiar rule

$$\mathscr{E}^- : [P_r, P_s]_{\mathscr{E}} = P_r P_s - P_s P_r, \tag{23}$$

where the P's are polynomials in the X's.

(2) The *Lie group*[20] $G(m)$ of transformations on M for the case of the action to the right as in Eq. (19), which we shall write in the symbolic exponentiated form for continuous transformations (see Ref. 20 for technical definitions)

$$G(m): A(w) = e^{X_1 w_1} e^{X_2 w_2} \cdots e^{X_m w_m}$$
$$= \prod_{k=1}^{m} e^{X_k w_k} \tag{24}$$

and which will be reduced to the appropriate exponential form whenever we consider specific cases. The corresponding action to the left,

$$X^{\dagger\prime} = X^\dagger A^\dagger(w), \tag{25}$$

can be characterized by the operation of Hermitian conjugation, which we shall write in the symbolic form

$$G(m): \hat{A}^\dagger(w) = \left(\prod_{k=1}^{m} e^{X_k w_k}\right)^\dagger \tag{26}$$

and whose explicit form will be computed whenever the reduced form of Eq. (24) is known (see, again, the case of rotations in Paper II).

(3) The *Lie algebra*[20] $\mathbf{G}(m)$ of $G(m)$, characterized by the closure rules

$$\mathbf{G}(m): [X_r, X_s]_{\mathscr{E}} = X_r X_s - X_s X_r = C_{rs}^t X_t. \tag{27}$$

The underlying methodology we shall tacitly imply is the familiar one consisting of the Poincaré-Birkhoff-Witt theorem for the characterization of the basis (20); the Baker-Campbell-Housdorff theorem for the composition of the exponentials (24) and (26); Lie's first, second, and third theorems for the characterization of the closure rules (27); the representation theory; etc.

The idea of the *Lie-isotopic theory*[3] is that of generalizing the structure of the enveloping algebra $\mathscr{E}$, of the Lie group $G(m)$, and of the Lie algebra $\mathbf{G}(m)$ in such a way to preserve the Lie character of the theory (in order to qualify for isotopy). The generalization is done via the replacement of the simplest possible associative Lie-admissible product $X_r X_s$ of the conventional theory into a form denoted by $X_r * X_s$ which is still associative and Lie admissible (i.e., its attached product $X_r * X_s - X_s * X_r$ is Lie); nevertheless, it is given by the structurally more general form

$$X_r * X_s = X_r g X_s. \tag{28}$$

It is evident that the generalization of the product of $\mathscr{E}$ implies a step-by-step generalization of the entire formulation of Lie's theory, from the basis (20) to the groups (24) and (26) to the algebra (27), etc.

In this paper, we are specifically interested, not in the Lie-isotopic theory per se, but in its formulation for the action on a metric space. We therefore need the generalization of the structure of the metric space permitting a consistent action of the Lie-isotopic theory.

To outline these ideas, we shall first introduce a notion of *metric isotopy*, that is, a generalization of a given metric space which preserves its metric character. We shall then review the corresponding Lie-isotopic theory. Finally, we shall apply the results to the case when the considered Lie and Lie-isotopic groups constitute symmetries of the metric and its isotope, respectively. This latter result will be presented via a theorem on the symmetry properties of isotopy which is at the foundation of the next paper on rotations, and of the subsequent one on Lorentz transformations.

Consider the simplest possible metric spaces, the Euclidean space $E(n, \delta, \mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{Q}$, with composition law

$$(x, y) = x^{\dagger i}\delta_{ij}x^j. \tag{29}$$

Suppose that the metric δ has to be modified into a form of the generic type (15). The emerging generalized space can be expressed via the notion of metric isotopy as follows.

Let $\hat{I} = g^{-1}$ be the inverse of the new metric tensor according to (9). Following Refs. 15, we introduce the isotopic lifting of the field

$$\hat{\mathbb{F}} = \{\hat{N} \mid \hat{N} = N\hat{I},\ N \in \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{Q}\}. \tag{30}$$

As one can see, $\hat{\mathbb{F}}$ is still a field, which essentially generalizes the conventional unit element $1 \in \mathbb{F}$ into a matrix form $\hat{I} \in \hat{\mathbb{F}}$.

The composition of elements of the field with elements of the metric space is now done according to the redefinition of the product

$$\hat{N} * x = \hat{N}gx = N\hat{I}gx = Nx. \tag{31}$$

Thus, the lifting $\hat{\mathbb{F}}$ of $\mathbb{F}$ essentially permits the use of a generalized composition $\hat{N} * x$ which, while being characterized by the new metric g, preserves the old values Nx.

Next, we generalize the metric space $E(n, \delta, \mathbb{F})$ into a form $\hat{E}$ that accommodates the new metric g under a mapping of the type

$$\hat{m}: \hat{E} \times \hat{E} \to \hat{\mathbb{F}}. \tag{32}$$

This implies that the generalized composition law must have value in $\hat{\mathbb{F}}$. A realization is given by the form patterned along the isotopic lifting of the Hilbert spaces of Ref. 15:

$$(x \hat{;} y) = \hat{I}(x, gy) = \hat{I}x^{\dagger i}g_{ij}x^j = (x, gy)\hat{I} = (gx, y)\hat{I} \tag{33}$$

We shall define as *isotopic liftings of the Euclidean space* all possible spaces $\hat{E}(n, g, \hat{\mathbb{F}})$ over the field $\hat{\mathbb{F}} = \hat{\mathbb{R}}, \hat{\mathbb{C}}, \hat{\mathbb{Q}}$, equipped with mappings (32) realized via composition (33), where g is the new metric tensor.

It is evident that, by construction, *all possible metrics are isotopes of the Euclidean metric.* This includes the Minkowskian, Riemannian, Finslerian, and other metrics.

Note that, strictly speaking, the metric spaces $M(n, g, \mathbb{F})$ cannot be considered as isotopes of $E(n, \delta, \mathbb{F})$, owing to the lack of lifting of the field. Nevertheless, this technical point can be ignored in practical applications owing to the identity $\hat{N} * x = Nx$. While conceding the insufficiency of the technical rigor (see also below), we can then assume that all possible metric spaces of n dimensions over the field $\mathbb{F}$ are isotopes of the Euclidean space $E(n, \delta, \mathbb{F})$.

Note that, since $\hat{\mathbb{F}}$ is still a field, $\hat{E}(n, g, \hat{\mathbb{F}})$ is also a metric space in the sense indicated earlier.

It is evident that the original Lie group $G(m)$ cannot act consistently on the new space. In fact, to begin, the action of the group on the space cannot be formulated according to the old composition (18), and must be modified into the form

$$x' = \hat{A}(w) * x = \hat{A}(w)gx \tag{34}$$

[where the quantities $\hat{A}(w)$ will be identified shortly]. In turn, this implies that the old composition laws (19) cannot be consistently preserved, and must be generalized into the form

$$\hat{A}(0) = \hat{I}, \tag{35a}$$

$$\hat{A}(w) * \hat{A}(w') = \hat{A}(w + w'), \tag{35b}$$

$$\hat{A}(w) * \hat{A}(-w) = \hat{A}(-w) * \hat{A}(w) = \hat{I}, \tag{35c}$$

which are precisely the defining conditions of a *Lie-isotopic transformation group*[3,12] $\hat{G}(m)$.

The most important property of generalized laws (35) is the replacement of the old unit I with the new unit $\hat{I} = g^{-1}$. Thus, the dominant feature of the isotopy under consideration is the assumption of the inverse $\hat{I}$ of the new metric g as the generalized identity of the group. Since the original identity I can be interpreted as the inverse of the metric δ of the Euclidean space, when the original group $G(m)$ is a symmetry of δ, we expect its isotopic image $\hat{G}(m)$ to constitute a symmetry of g, as we shall see.

To achieve this result, we need the following main lines of the Lie-isotopic theory:

(1) *Isotopic lifting of the universal enveloping associative algebra.*[3] The Poincaré-Birkhoff-Witt theorem admits a consistent isotopic generalization (Ref. 3, p. 353, and Ref. 12, p. 156), resulting in a new basis, which we shall again write in the symbolic form (see Ref. 12 for a more rigorous treatment)

$$\hat{\mathscr{E}}: \hat{I},\ X_r,\ \underset{r<s}{X_r * X_s},\ \underset{r\leq s\leq t}{X_r * X_s * X_t},\ \ldots,$$

$$r, s, t, \ldots = 1, 2, \ldots, n, \qquad (36)$$

where the identity $\hat{I}$ is the same as that of the group composition laws (35a), and the generators X_r are the same as those of $\mathscr{E}$ (in their fundamental representation); the attached Lie algebra is now given by the isotope

$$\hat{\mathscr{E}}^- : [P_r, P_s]_{\hat{\mathscr{E}}} = P_r * P_s - P_s * P_r$$

$$= P_r g P_s - P_s g P_r \stackrel{\text{def}}{=} [P_r \hat{,} P_s], \qquad (37)$$

and the algebra $\hat{\mathscr{E}}$ is still "universal" and "enveloping"—not, of course, with respect to the algebra $\mathscr{E}^-$, but with respect to $\hat{\mathscr{E}}^-$. We see in this way that the generalized metric g enters into the very structure of the Lie product, Eq. (37), as expected, because *g^{-1} is the identity of the group composition law.*

(2) *Isotopic lifting of the Lie group.* The new basis (36) permits the construction of the new group elements $\hat{A}(w)$ via the so-called *isotopic exponentiation*. For one-parameter actions to the right, this exponentiation is characterized by the old generator X of $G(m)$ but now expanded in the new envelope according to the rule (Ref. 3, p. 334; Ref. 12, p. 171; and Ref. 15, p. 1301)

$$\hat{G}(1): \hat{A}(w) = \hat{I} + \frac{1}{1!}(Xw) + \frac{1}{2!}(Xw)^{\hat{2}} + \frac{1}{3!}(Xw)^{\hat{3}} + \cdots$$

$$= \hat{I} + \frac{1}{1!}(Xw) + \frac{1}{2!}(Xw)g(Xw) + \frac{1}{3!}(Xw)g(Xw)g(Xw) + \cdots$$

$$= e^{Xw}|_{\hat{\mathscr{E}}} \stackrel{\text{def}}{=} \hat{e}^{Xw}, \qquad (38)$$

which, for clarity of computation, can be reexpressed via the following expansion in the old envelope (Ref. 15, Theorem 2.14, p. 1303):

$$\hat{G}(1): \hat{A}(w) = \Big[1 + \frac{1}{1!}(Xgw) + \frac{1}{2!}(Xgw)(Xgw) + \frac{1}{3!}(Xgw)(Xgw)(Xgw) + \cdots\Big]\hat{I}$$

$$= \left(e^{Xgw}|_{\mathscr{E}}\right)\hat{I} = e^{X*w}\hat{I}$$

$$= \hat{I}\left(e^{wgX}|_{\mathscr{E}}\right) = \hat{I}e^{w*X}. \qquad (39)$$

It is evident that the elements $\hat{A}(w)$ so constructed verify all the rules (35), and thus they constitute the desired Lie-isotopic lifting of $G(1)$. The generalization to more than one dimension is permitted by the *Lie-isotopic generalization of the Campbell-Baker-Hausdorff theorem* (Ref. 3, p. 335; Ref. 12, p. 172; and Ref. 15, p. 1303),

$$\hat{e}^{\alpha} * \hat{e}^{\beta} = \hat{e}^{\gamma},$$

$$\gamma = \alpha + \beta + \tfrac{1}{2}[\alpha \hat{,} \beta] + \tfrac{1}{12}[(\alpha - \beta)\hat{,}[\alpha \hat{,} \beta]] + \cdots, \qquad (40)$$

under which we have the desired Lie-isotopic lifting of the Lie transformation group (24), here written, again, in symbolic form

$$\hat{G}(m): \hat{A}(w) = \hat{e}^{X_1 w_1} * \hat{e}^{X_2 w_2} * \cdots * \hat{e}^{X_m w_m}$$

$$= \prod_{k=1}^{m}{}^{*}\, \hat{e}^{X_k w_k}$$

$$= \left(e^{X_1 * w_1} e^{X_2 * w_2} \cdots e^{X_m * w_m}\right)\hat{I}$$

$$= \left(\prod_{k=1}^{m} e^{X_k * w_k}\right)\hat{I}, \qquad (41)$$

with the intent of computing the explicit, reduced form in specific cases. The action of the Lie-isotopic group to the left,

$$X^{\dagger\prime} = x^{\dagger} * \hat{A}^{\dagger}(w), \tag{42}$$

is given, for the one-parameter case, by the expansion of the old generator $X^{\dagger}$ in the new envelope $\hat{\mathscr{E}}$, according to the rule

$$\begin{aligned} \hat{G}(1): \hat{A}^{\dagger}(w) &= \hat{I} + \frac{1}{1!}(wX^{\dagger}) + \frac{1}{2!}(wX^{\dagger})^{\hat{2}} \\ &\quad + \frac{1}{3!}(wX^{\dagger})^{\hat{3}} \\ &= \hat{I} + \frac{1}{1!}(wX^{\dagger}) \\ &\quad + \frac{1}{2!}(wX^{\dagger})g(wX^{\dagger}) + \cdots \\ &= e^{wX^{\dagger}}|_{\hat{\mathscr{E}}} = \hat{e}^{wX^{\dagger}} = \hat{e}^{-wX}, \end{aligned} \tag{43}$$

with reformulation in $\mathscr{E}$ given by

$$\begin{aligned} \hat{G}(1): \hat{A}^{\dagger}(w) &= \hat{I}\Big[1 + \frac{1}{1!}(wgX^{\dagger}) \\ &\quad + \frac{1}{2!}(wgX^{\dagger})(wgX^{\dagger}) + \cdots\Big] \\ &= \hat{I}\left(e^{wgX^{\dagger}}|_{\mathscr{E}}\right) = \hat{I}e^{wgX^{\dagger}}, \\ &= e^{X^{\dagger}gw}\hat{I} = \hat{I}e^{-w*X}, \end{aligned} \tag{44}$$

and m-parameter expression here symbolically written

$$\hat{G}(m): \hat{A}^{\dagger}(w) = \hat{I}\left(\prod_{k=1}^{m} e^{X_k * w_k}\right)^{\dagger}, \tag{45}$$

whose explicit form will be computed in specific cases (see, again, the case of the isotopic rotations in Paper II). It remains to prove that the operation of Hermitian conjugation, as conventionally defined, also acts consistently under isotopy in $\hat{E}(n, g, \hat{\mathbb{F}})$. The fact that this is not the case in general is now known.[15,5] Nevertheless, the operation of Hermiticity persists for the particular case under consideration here, that for which the isotopic element of the envelope coincides with that of the composition, as is readily seen by using the property (13) and definition (33):

$$\begin{aligned} (X\hat{,}\, \hat{A} * y) &= \hat{I}(X, g\hat{A}gy) \\ &= \hat{I}\left((g\hat{A})^{\dagger}x, gy\right) = \hat{I}(\hat{A}^{\dagger}gx, gy) \\ &= \hat{I}(\hat{A}^{\dagger} * x, y), \end{aligned} \tag{46}$$

for which

$$(e^{Xgw})^{\dagger} = e^{wg^{\dagger}X^{\dagger}} = e^{-wgX}. \tag{47}$$

(3) *Isotopic lifting of the Lie algebra.* This is characterized by the isotopic generalization of Lie's first, second, and third theorems (Ref. 3, pp. 331–334; Ref. 18, pp. 163–172) according to the rules

$$\begin{aligned} \hat{G}(m): [X_r\hat{,}\, X_s] &= X_r * X_s - X_s * X_r \\ &= X_r g X_s - X_s g X_r \\ &= \hat{D}^t_{rs}(x) * X_t, \\ \hat{D}^t_{rs} &= D^t_{rs}\hat{I}, \end{aligned} \tag{48}$$

where the D's are called *structure functions*. As was the case for the expansion (38), the rules (48) can also be reformulated in $\mathscr{E}$ according to either of the following two expressions:

$$\begin{aligned} [X_r\hat{,}\, X_s] &= X_r g X_s - X_s g X_r \\ &= [X_r g, X_s g]\hat{I} \\ &= [X_r, X_s]g + X_r[g, X_s] + X_s[X_r, g] \\ &= \hat{I}[gX_r, gX_s] \\ &= g[X_r, X_s] + [X_r, g]X_s + [g, X_s]X_r, \end{aligned} \tag{49}$$

each one derivable from the other via the Jacobi law.

The primary lines of the Lie-isotopic theory as outlined above are sufficient for our objectives. We shall therefore pass to the main task of this section, that dealing with the problem of the symmetries of arbitrary metrics (15).

Suppose that the original (conventional) Lie transformation group $G(m)$ is a symmetry group of the composition (x, y) in $E(n, \delta, \mathbb{F})$, or, equivalently, of the metric δ, according to the familiar conditions

$$x^{\dagger\prime}x' \equiv x^{\dagger\prime}\delta x' = x^\dagger A^\dagger \delta A x = x^\dagger \delta x \equiv x^\dagger x, \quad (50)$$

which can hold identically iff

$$A^\dagger \delta A \equiv A^\dagger A = A A^\dagger \equiv A\delta A^\dagger = I = \delta^{-1}, \quad (51)$$

i.e.,

$$A^\dagger = A^{-1}, \quad (52a)$$

$$(\det A)^2 = (\det I)^2 = 1. \quad (52b)$$

As is well known,[20] when the conditions (52) are verified, we have the *orthogonal groups* $\mathrm{O}(n,\mathbb{R})$, the *unitary groups* $\mathrm{U}(n,\mathbb{C})$, and others. When realizations of the continuous type (24) are considered, we have the *special orthogonal groups* $\mathrm{SO}(n,\mathbb{R})$ or the *special unitary groups* $\mathrm{SU}(n,\mathbb{C})$. In this latter case, the determinant of the transformations is 1, and the discrete transformations (e.g., inversions) are excluded.

We are interested in investigating the behavior of the symmetry (50) under an isotopic lifting of the Euclidean space $E(n,\delta,\mathbb{F})$ and of the group $G(m)$ to a form characterized by an arbitrary metric (15). For this purpose, we recall that the composition law of $\hat{E}(n,g,\hat{\mathbb{F}})$ is based on the term

$$x^\dagger * x = x^\dagger g x. \quad (53)$$

We therefore have a symmetry when the following conditions are identically verified:

$$x^{\dagger\prime} * x' = x^\dagger * \hat{A}^\dagger * \hat{A} * x = x^\dagger * x, \quad (54)$$

which can hold iff

$$\hat{A}^\dagger g \hat{A} = \hat{A} g \hat{A}^\dagger = \hat{I}, \quad (55a)$$

i.e., iff

$$\hat{A}^\dagger = \hat{A}^{-\hat{1}}, \quad (56a)$$

$$(\det \hat{A})^2 = (\det \hat{I})^2, \quad (56b)$$

where the inverse is computed, of course, with respect to $\hat{I}$.

It is easy to see that, when the original transformations verify the conditions (50), their images under lifting necessarily verify the new conditions (54). In fact, for the case of continuous transformations, we have, from Eqs. (41) and (44),

$$\hat{A}^\dagger(w) = \hat{A}(-w), \quad (57)$$

Therefore, the conditions (55) are reduced to one of the conditions for the very existence of a Lie-isotopic group, Eq. (35c).

The rules (55) can be expressed in a form particularly suitable for practical applications. Redefine the elements of $\hat{G}(m)$ according to the forms

$$\hat{A}(w) = B(w)\hat{I}, \quad B(w) = \prod_{k=1}^{m} e^{X_k * w_k}, \quad (58a)$$

$$\hat{A}^\dagger(w) = \hat{I}B^\dagger(w), \quad B^\dagger(w) = \left(\prod_{k=1}^{m} e^{X_k * w_k}\right)^\dagger. \quad (58b)$$

Then, conditions (56) can be equivalently expressed as

$$B^\dagger g B = g, \quad (59a)$$

$$(\det B)^2 = 1, \quad (59b)$$

which hold identically under the Lie-isotopic liftings of continuous transformations owing to the identity

$$e^{-wgX} g e^{Xgw} = g - w(gXg - gXg) + \tfrac{1}{2}w^2(gXgXg - gXgXg) + \cdots = g. \quad (60)$$

For the case of discrete transformations, we introduce the following *Lie-isotopic lifting of inversions*

$$\hat{\mathscr{P}} * x = (\mathscr{P}\hat{I})gx = \mathscr{P}x = -x, \quad (61)$$

where $\mathscr{P}$ is the conventional total inversion. The preservation of the symmetry then results from known expressions of the type

$$\mathscr{P}g\mathscr{P} = g, \quad (62)$$

whose validity is trivial.

We reach in this way our man result, which can be formulated as follows.

Theorem 2.1. Let $G(m)$ be an m-parameter Lie symmetry group of the composition $z^\dagger \delta z$ of an

n-dimensional Euclidean space $E(n, \delta, \mathbb{F})$ over the field $\mathbb{F}$ of real numbers $\mathbb{R}$, of complex numbers $\mathbb{C}$, or of quaternions $\mathbb{Q}$. Then the isotopic lifting $\hat{G}(m)$ of $G(m)$ characterized by a nonsingular, Hermitian, and sufficiently smooth metric g in the local variables leaves invariant the generalized composition $z^{\dagger}gz$ of the isotopic space $\hat{E}(n, g, \hat{\mathbb{F}})$, $\hat{\mathbb{F}} = \mathbb{F}\hat{I}$, $\hat{I} = g^{-1}$.

The remaining papers of this series can be considered as dealing with applications of the above theorem to specific cases of physical relevance.

Note that the explicit construction of the Lie-isotopic transformations (as well as of the entire theory) can be done following the knowledge only of the original symmetry and of the new metric.

We close this section with the indication that all Lie algebras (27) admit the following *trivial Lie isotopy*

$$\begin{aligned}\hat{\mathbb{G}}(m) : [\hat{X}_r \hat{,} \hat{X}_s] &= \hat{X}_r * \hat{X}_s - \hat{X}_s * \hat{X}_r \\ &= (X_r X_s - X_s X_r)\hat{I} = C^k_{rs}\hat{X}_k, \\ \hat{X} &= X\hat{I}, \quad X \in \mathbb{G}(m), \qquad (63)\end{aligned}$$

with a self-evident isomorphism $\hat{\mathbb{G}}(m) \approx \mathbb{G}(m)$. The above trivial isotopy should be excluded from the content of Theorem 2.1 because it does not provide the invariance of the generalized composition law. This can be readily seen from the fact that the exponentials (41) and (45), when realized for the generators $\hat{X}_k$, coincide with the original exponentials (except for the factorization of the new unit), and no genuine lifting has actually occurred.

III. CONCLUDING REMARKS

A few comments are in order, not only to identify better certain aspects that are relevant for the subsequent analysis, but also to indicate a number of intriguing, open problems we cannot possibly investigate here.

A rather frequent misrepresentation of the Lie-isotopic theory is the expectation that the theory will produce new Lie algebras. This evidently cannot be the case, because all Lie algebras (over a field of characteristic zero, the only ones considered in these papers) are known. The objective of the Lie-isotopic theory is merely that of expressing known Lie algebras in a structurally more general way (see the comments in Sec. I in regard to our current lack of knowledge of the most general possible Lie isotopy).

A further aspect deserving a comment is that a given Lie algebra $\mathbb{G}(m)$ and its isotope $\hat{\mathbb{G}}(m)$ are not necessarily isomorphic. This property was studied in Ref. 14, where it was also shown that isotopic liftings do not preserve the compact or noncompact character of the original algebra. Along similar lines, one can easily see that the isotopic liftings do not necessarily preserve the Abelian or non-Abelian character of the original algebra. Even the preservation of the semisimplicity or nonsemisimplicity and the notion of radical deserve specific studies (which will not be conducted at this time).

The reader may have noticed our reference to the m-parameter character of $\hat{\mathbb{G}}(m)$ (and our silence on the dimensionality of the Lie-isotopic algebra). In fact, under the assumed conditions on the metric, the numbers of identities (51) and (55) coincide, therefore implying the preservation of the number of independent parameters under lifting.

The corresponding situation for the dimensionality of the underlying Lie-isotopic algebra is in need of additional study. In fact, as one can see from rule (49), the isotopic commutation rules (48) imply the commutations of the old generators X_r with the new metric g which generally produces elements outside the original envelope $\mathscr{E}$. However, closure under isotopy is characterized by structure functions, and this opens up the possibility of the preservation of (at least) the finite-dimensional character of the isotopic algebra.

A detailed study of this aspect is much needed for the Lie-isotopic theory, although it is not essential for the objectives of these papers, where the identity of the number of parameters and that of generators will be verified in cases of physical relevance.

Another aspect deserving a comment is the intrinsically nonlinear character of the Lie-isotopic transformations, even though expressed via a formally linear theory. In fact, the transformation laws are formally linear, although in the isotopic sense

$$x' = \hat{A} * x = Bx. \qquad (64)$$

However, when the transformations are explicitly written down, their intrinsic nonlinearity emerges transparently, and we shall write

$$x' = \hat{A}(w; x, \dot{x}, \dots) * x = B(w; x, \dot{x}, \dots)x, \qquad (65)$$

because the elements B are constructed via powerseries expansions (38) in terms of the metric tensor g, which is explicitly dependent on local coordinates, their derivatives with respect to independent parameters, and any other needed quantities:

$$B = \exp[Xg(x, \dot{x}, \ldots)w]. \tag{66}$$

As a matter of fact, one of the intriguing features of the Lie-isotopic lifting of Lie's theory is the possibility of turning a conventionally nonlinear transformation theory into an isotopically linear one, with evident computational advantages.

As a further comment, the isotopic liftings of Euclidean spaces considered in this paper are expected to be extendable in such a way to accommodate antisymmetric metrics and their symplectic symmetry groups. In fact, liftings (39) and (45) are possible also for antisymmetric metrics. The restriction to Hermitian metrics has been introduced in this paper for the compatibility condition (47), having in mind quantum-mechanical applications[5] based on the completion of the Euclidean spaces into Hilbert spaces.

As a final comment, we would like to indicate that, under certain topological restrictions on the new metric (to be identified in the subsequent papers), all distinctions between the original symmetry and its isotopic image cease to exist at the level of abstract, realization-free formulations of Lie algebras. In fact, the envelope can be treated in terms of an abstract associative product, say, ab, and, as such, it admits different realizations in terms of matrices, such as AB, AgB, or $WAWBW$ ($W^2 = W$). A corresponding situation occurs at the level of Lie algebras, Lie groups, etc.

The Lie-isotopic theory is therefore essentially concerned with generalized realizations of known, abstract Lie structures. The understanding of this property is important for the proper setting of the isotopic liftings of space-time symmetries.

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[1]R. H. Bruck, *A Survey of Binary Systems* (Springer, Berlin, 1958).

[2]K. McCrimmon, Pacific J. Math. **15**, 925 (1965).

[3]R. M. Santilli, Hadronic J. **1**, 223 (1978).

[4]"Proceedings of the Second Workshop on Lie—admissible Formulations, Vol. A: Review papers," Hadronic J. **2**, 1252–2019 (1979); "Vol. B: Research Papers," *ibid*. **3**, 1–725 (1979); "Proceedings of the Third Workshop on Lie-admissible Formulations, Vol. A: Mathematics," Hadronic J. **4**, 183–607 (1981); "Vol. B: Theoretical physics," *ibid*. **4**, 608–1165 (1981); "Vol. C: Experimental physics and bibliography," **4**, 1166–1625 (1981).

[5]*Proceedings of the First Workshop on Hadronic Mechanics, edited by J. Fronteau, R. Mignani, H. C. Myung, and R. M. Santilli (Hadronic Press, Nonantum, Mass.* 02195, 1983).

[6]"Proceedings of the First International Conference on Nonpotential Interactions and Their Lie-admissible Treatment, Vol. A: Invited papers," Hadronic J. **5**, 245–678 (1982); "Vol. B: Invited papers," *ibid*. **5**, 679–1193 (1982); "Vol. C: Contributed papers," *ibid*. **5**, 1194–1626 (1982); "Vol. D: Contributed papers," *ibid*. **5**, 1627–1947 (1982).

[7]M. L. Tomber *et al*., "A nonassociative algebra bibliography," Hadronic J. **3**, 507 (1979); "Addenda to 'A Nonassociative Algebra Bibliography,'" *ibid*. **4**, 1318 (1981); "A subject index of works related to nonassociative algebras," *ibid*. **4**, 1444 (1981).

[8]G. Benkart, J. M. Osborn, and D. J. Britten, Hadronic J. **4**, 497 (1981).

[9]H. C. Myung, Hadronic J. **5**, 771 (1982).

[10]J. M. Osborn, Hadronic J. **5**, 904 (1982).

[11]A. A. Sagle, Hadronic J. **5**, 1546 (1982).

[12]R. M. Santilli, *Foundations of Theoretical Mechanics, II: Birkhoffian Generalization of Hamiltonian Mechanics* (Springer, New York/Heidelberg/Berlin, 1982).

[13]G. Marmo and E. J. Saletan, Hadronic J. **1**, 955 (1978), *and cited papers.*

[14]*R. M. Santilli, Phys. Rev. D* ***20****, 555 (1979).*

[15]H. C. Myung and R. M. Santilli, Hadronic J. **5**, 1277 (1982); **5**, 1367 (1982).

[16]R. M. Santilli, Hadronic J. **4**, 1166 (1981).

[17]Y. Ilamed, Hadronic J. **5**, 1734 (1982).

[18]A. J. Kalnay, Hadronic J. (in press).

[19]N. Jacobson, *Lie Algebras* (Wiley, New York, 1962).

[20]A. A. Sagle and R. E. Walde, *Introduction to Lie Groups and Lie Algebras*, (Academic, New York, 1973).