Categories of Topological Isospaces and Isogroups*

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Abstract

The objects of modern isogeometry, isotopology, like all objects of modern isomathematics, are sets of elements of arbitrary nature endowed with some mathematical isostructures, for example, isosymmetry etc. The typical way to think about isosymmetry is with the concept of a "isogroup". But to get a concept of isosymmetry that's really up to the demands put on it by modern isomathematics, we need — at the very least — to work with a "category" of isosymmetries, rather than a isogroup of isosymmetries. In this article we use Santilli functor for topological isostructures. There are constructed categories of topological vector isospaces, and topological isogroups, and some categorical structures on them.

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1 Introduction

Isomathematics was proposed by Santilli [1] in 1978, and subsequently studied by numerious pure and applied mathematicians as: S. Okubo, H. Myung, M. Tomber, Gr. Tsagas, D. Sourlas, C. Corda, J. Kadeisvili, A. Aringazin, A. Kirhukin, R. Ohemke, G. Wene, G. M. Benkart, J. Osborn, D. Britten, J. Lohmus, E. Paal, L. Sorgsepp, D. Lin, J. Voujouklis, P. Broadbridge, P. Chernooff, J. Sniatycku, S. Guiasu, E. Prugovecki, A. Sagle, C. Jiang, R. Falcon Ganfornina, J. Nunez Valdes, A. Davvaz, S. G. Georgiev and others.

As a result of these efforts, the new mathematics can be constructed via the systematic application of axiom - preservibg liftings, called isotopies, of the totality of all structures inthe mathematics: inluding all operators and their operations, icluding the isotopic lifting of numbers, functinal analysis, differential calculas, geometries, topologies, Lie theory and others [2–6].

The physical needs for isomathematics have been indicated in [3,4], and consists in the necessity for a representation of non-Hamiltonian scattering effects in a form that is invariant over time so as to admit the same numerical predictions under the same conditions at different times. Following the study of all possible alternatives, the latter condition required the representation of non-Hamiltonian scattering effects with an axiom-preserving generalization of the trivial (positive-deffinite) unit of quantum mechanics $\hat{h} = 1$ into the most general possible (positive-deffinite as a condition to characterize an isotopy), integro-differential operator $\hat{I}$ which is as positive - definite as $+1$, functional depending of local variables, that is assumed to be the inverse of the isotopic element $\hat{T}$

$$+1\hat{>}0 \longrightarrow \hat{I}(t, r, p, a, E, \ldots) = \frac{1}{\hat{T}}\hat{>}0$$

and it is called Santilli isounit. Santilli introduced a generalization called lifting of the conventional associative product ab into the form

$$ab \longrightarrow a\hat{\times}b = a\hat{T}b$$

called isoproduct for which:

$$\hat{I}\hat{\times}a = \frac{1}{\hat{T}}\hat{T}a = a\hat{\times}\hat{I} = a\hat{T}\frac{1}{\hat{T}} = a.$$

for every element a of the field of real numbers, complex numbers and quaternions.

The Santilli isonumbers are defined as follows: for given real number or complex number or quaternion a,

$$\hat{a} = a\hat{I},$$

with isoproduct

$$\hat{a}\hat{\times}\hat{b} = \hat{a}\hat{T}\hat{b} = a\frac{1}{\hat{T}}\hat{T}b\frac{1}{\hat{T}} = ab\frac{1}{\hat{T}} = \widehat{ab}.$$

If $a \neq 0$ the corresponding isoelement of $\frac{1}{a}$ will be denoted with $\hat{a}^{-1}$ or $\hat{I}\hat{/}\hat{a}$.

With $\hat{F}_{\mathbb{R}}$ or $\mathbb{R}$ we will denote the field of iso-reals. Below we will suppose that $\hat{T}_1 \in \hat{F}_{\mathbb{R}}$, $\hat{T}_1 > 0$ is an isotopic element which corresponds of $\hat{F}_{\mathbb{R}}$.

The text is organized and written in a "pedagogical style", rather than in a highly economical one. The paper is organized into five sections that represent natural "clusters" of topics. Following [13] the second section contains the basic category theory. Following [3–6] in the third section we present basic notions on Isotopies it also contains more recent research results obtained in [14, 15] in the realm of Santilli iso-functor and concrete categories of isogroups and vector isospaces. In order to make the flow of topics self-motivating, new categorical concepts of isotopology are introduced gradually, by moving from special cases of notions of the category **TOP** (with objects all topological spaces and morphisms all continuous functions between them), the category **TopVec** of topological vector spaces and continuous linear transformations and the category **TopGrp** of topological groups as objects and continuous homomorphisms as morphisms to the more general isotopological ones and categorical structures on them.

2 Basic Notions on Categories

Category theory groups together in categories the mathematical objects with some common structure (e.g., sets, partially ordered sets, groups, rings, and so forth) and the appropriate morphisms between such objects [7–13]. These morphisms are required to satisfy certain properties which make the set of all such relations coherent. Given a category, it is not the case that every two objects have a relation between them, some do and others dont. For the ones that do, the number of relations can vary depending on which category we are considering.

Definition 2.1. *A category is a quadruple* $(\mathrm{Ob}, \mathrm{Hom}, \mathrm{id}, \circ)$ *consisting of:*

(Cl) a class Ob *of objects;*

(C2) for each ordered pair (A, B) *of objects a set* $\mathrm{Hom}(A, B)$ *of morphisms;*

(C3) for each object A *a morphism* $\mathrm{id}_A \in \mathrm{Hom}(A, A)$*, the identity of* A*;*

(C4) a composition law associating to each pair of morphisms $f \in \mathrm{Hom}(A, B)$ *and* $g \in \mathrm{Hom}(B, C)$ *a morphism* $g \circ f \in \mathrm{Hom}(A, C)$*;*
which is such that:

(M1) $h \circ (g \circ f) = (h \circ g) \circ f$ *for all* $f \in \mathrm{Hom}(A, B)$*,* $g \in \mathrm{Hom}(B, C)$ *and* $h \in \mathrm{Hom}(C, D)$*;*

(M2) $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f$ *for all* $f \in Hom(A, B)$*;*

(M3) the sets $\mathrm{Hom}(A, B)$ *are pairwise disjoint.*

This last axiom is necessary so that given a morphism we can identify its domain A and codomain B, however it can always be satisfied by replacing $\mathrm{Hom}(A, B)$ by the set $\mathrm{Hom}(A, B) \times (\{A\}, \{B\})$.

Example 2.1. *The classic example is* **Set***, the category with sets as objects and functions as morphisms, and the usual composition of functions as composition.*

2.1 Functors and natural transformations

DEFINITION 2.2. *Let* **X** *and* **Y** *be two categories. A* **covariant functor** *from a category* **X** *to a category* **Y** *is a family of functions* $\mathcal{F}$ *which associates to each object* A *in* **X** *an object* $\mathcal{F}A$ *in* **Y** *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *a morphism* $\mathcal{F}f \in \mathrm{Hom}_{\mathbf{Y}}(\mathcal{F}A, \mathcal{F}B)$*, and which is such that:*

(FI) $\mathcal{F}(g \circ f) = \mathcal{F}g \circ \mathcal{F}f$ *for all* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *and* $g \in \mathrm{Hom}_{\mathbf{Y}}(B, C)$*;*
(F2) $\mathcal{F}\,\mathrm{id}_A = \mathrm{id}_{\mathcal{F}A}$ *for all* $A \in \mathrm{Ob}(\mathbf{X})$*.*

It is clear from the above that a covariant functor is a transformation that preserves both:

- The domains and the codomains identities.
- The composition of arrows, in particular it preserves the direction of the arrows.

DEFINITION 2.3. *Let* **X** *and* **Y** *be two categories. A* **contravariant functor** *from a category* **X** *to a category* **Y** *is a family of functions* $\mathcal{F}$ *which associates to each object* A *in* **X** *an object* $\mathcal{F}A$ *in* **Y** *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *a morphism* $\mathcal{F}f \in \mathrm{Hom}_{\mathbf{Y}}(\mathcal{F}A, \mathcal{F}B)$*, and which is such that:*

(FI) $\mathcal{F}(g \circ f) = \mathcal{F}f \circ \mathcal{F}g$ *for all* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *and* $g \in \mathrm{Hom}_{\mathbf{Y}}(B, C)$*;*
(F2) $\mathcal{F}\,\mathrm{id}_A = \mathrm{id}_{\mathcal{F}A}$ *for all* $A \in \mathrm{Ob}(\mathbf{X})$*.*

Thus, a contravariant functor in mapping arrows from one category to the next reverses the directions of the arrows, by mapping domains to codomains and vice versa. A contravariant functor is also called a presheaf.

So far we have defined categories and maps between them called functors. We will now abstract one step more and define maps between functors. These are called *natural transformations*.

DEFINITION 2.4. *Let* $\mathcal{F} : \mathbf{X} \to \mathbf{Y}$ *and* $\mathcal{G} : \mathbf{X} \to \mathbf{Y}$ *be two functors. A* **natural transformation** $\alpha : \mathcal{F} \to \mathcal{G}$ *is given by the following data:*

For every object A in $\mathbf{X}$ there is a morphism $\alpha_A : \mathcal{F}(A) \to \mathcal{G}(A)$ in $\mathbf{Y}$ such that for every morphism $f : A \to B$ in $\mathbf{X}$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\alpha_A} & \mathcal{G}(A) \\ \mathcal{F}(f)\downarrow & & \downarrow \mathcal{G}(f) \\ \mathcal{F}(B) & \xrightarrow{\alpha_B} & \mathcal{G}(B). \end{array}$$

Commutativity means (in terms of equations) that the following compositions of morphisms are equal: $\mathcal{G}(f) \circ \alpha_A = \alpha_B \circ \mathcal{F}(f)$.

The morphisms α_A, $A \in \mathrm{Ob}(\mathbf{X})$, are called the *components of the natural transformation* α.

2.2 Forgetful functor

In a category, two objects x and y can be equal or not equal, but they can be *isomorphic* or not, and if they are isomorphic, they can be isomorphic in many different ways. An isomorphism between x and y is simply a morphism $f : x \to y$ which has an inverse $g : y \to x$, such that $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$.

In the category **Sets** an isomorphism is just a one-to-one and onto function, i.e. a bijection. If we know two sets x and y are isomorphic we know that they are "the same in a way", even if they are not equal. But specifying an isomorphism $f : x \to y$ does more than say x and y are the same in a way; it specifies a *particular way* to regard x and y as the same.

In short, while equality is a yes-or-no matter, a mere *property*, an isomorphism is a *structure*. It is quite typical, as we climb the categorical ladder (here from elements of a set to objects of a category) for properties to be reinterpreted as structures [7–11].

DEFINITION 2.5. *We tell that a functor $\mathcal{F} : \mathbf{C} \to \mathbf{C}'$ define a additional $\mathscr{C}-$structure on objects of the category $\mathbf{C}'$ if*

1. *$\forall X, Y \in Ob(\mathbf{C})$ the map $F : \mathbf{C}(X, Y) \to \mathbf{C}'(\mathcal{F})X), \mathcal{F}(Y))$ is injective,*

2. $\forall X \in Ob(\mathbf{C}), Y \in Ob(\mathbf{C}')$ *and an isomorphism* $u : Y \to \mathcal{F}(X)$ *there is an object* $\tilde{Y} \in Ob$ *and an isomorphism* $\tilde{u} : \tilde{Y} \to X$ *such that* $\mathcal{F}(\tilde{Y}) = Y$ *and* $\mathcal{F}(\tilde{u}) = u$.

Such functor is called a **forgetful** *functor.*

3 Basic Notions on Isotopies

3.1 Isotopies of the unit and isospaces

The **isotheory** is based on the concept of fundamental **isotopy** which is the lifting $I \to \widehat{I}$ of the n–dimensional unit $I = diag\,(1, 1, ..., 1)$ of the Lie's theory into an $n \times n$–dimensional matrix

$$\widehat{I} = \left(I_j^i\right) = \widehat{I}\left(t, x, \dot{x}, \ddot{x}, \psi, \psi^+, \partial\psi, \partial\psi^+, \partial\partial\psi, \partial\partial\psi^+, ...\right)$$

called the isounit. For simplicity, we consider that maps $I \to \widehat{I}$ are of necessary Kadeisvili Class I (II), the Class III being considered as the union of the first two, i. e. they are sufficiently smooth, bounded, nowhere degenerate, Hermitian and positive (negative) definite, characterizing isotopies (isodualities).

One demands a compatible lifting of all associative products AB of some generic quantities A and B into the isoproduct $A * B$ satisfying the properties:

$$AB \Rightarrow A * B = A\widehat{T}B,\ IA = AI \equiv A \to \widehat{I} * A = A * \widehat{I} \equiv A,$$

$$A\,(BC) = (AB)\,C \to A * (B * C) = (A * B) * C,$$

where the fixed and invertible matrix $\widehat{T}$ is called the isotopic element.

To follow our outline, a conventional field $F\,(a, +, \times)\,,$ for instance of real, complex or quaternion numbers, with elements a, conventional sum $+$ and product $a \times b \doteq ab,$ must be lifted into the so–called isofield $\widehat{F}\,(\widehat{a}, +, *)\,,$ satisfying properties

$$F\,(a, +, *) \to \widehat{F}\,(\widehat{a}, +, *)\,,\ \widehat{a} = a\widehat{I}$$

$$\widehat{a} * \widehat{b} = \widehat{a}\widehat{T}\widehat{b} = (ab)\,\widehat{I},\ \ \widehat{I} = \widehat{T}^{-1}$$

with elements $\widehat{a}$ called isonumbers, $+$ and $*$ are conventional sum and isoproduct preserving the axioms of the former field $F\,(a,+,\times)$. All operations in F are generalized for $\widehat{F}$, for instance we have isosquares $\widehat{a}^{\widehat{2}} = \widehat{a} * \widehat{a} = \widehat{A}\widehat{T}\widehat{a} = a^2\widehat{I}$, isoquotient $\widehat{a}\widehat{/}\widehat{b} = (a/b)\,\widehat{I}$, isosquare roots $\widehat{a}^{1/2} = a^{1/2}\widehat{I}, ...;$ $\widehat{a} * A \equiv aA$. We note that in the literature one uses two types of denotation for isotopic product $*$ or $\widehat{\times}$ (in our work we shall consider $* \equiv \widehat{\times}$).

Let us consider, for example, the main lines of the isotopies of a n–dimensional Euclidean space $E^n\,(x,g,\mathbb{R})$, where $\mathbb{R}\,(n,+,\times)$ is the real number field, provided with a local coordinate chart $x = \{x^k\}, k = 1,2,...,n$, and n–dimensional metric $\rho = (\rho_{ij}) = diag\,(1,1,...,1)$. The scalar product of two vectors $x,y \in E^n$ is defined as

$$(x-y)^2 = \left(x^i - y^i\right)\rho_{ij}\left(x^j - y^j\right) \in \mathbb{R}\,(n,+,\times)$$

were the Einstein summation rule on repeated indices is assumed hereon.

The **Santilli's isoeuclidean** spaces $\widehat{E}\left(\widehat{x},\widehat{\rho},\widehat{\mathbb{R}}\right)$ of Class III are introduced as n–dimensional metric spaces defined over an isoreal isofield $\widehat{\mathbb{R}}\left(\widehat{n},+,\widehat{\times}\right)$ with an $n \times n$–dimensional real–valued and symmetrical isounit $\widehat{I} = \widehat{I}^t$ of the same class, equipped with the "isometric"

$$\widehat{\rho}\,(t,x,v,a,\mu,\tau,...) = (\widehat{\rho}_{ij}) = \widehat{T}\,(t,x,v,a,\mu,\tau,...) \times \rho = \widehat{\rho}^t,$$

where $\widehat{I} = \widehat{T}^{-1} = \widehat{I}^t$.

A local coordinate cart on $\widehat{E}\left(\widehat{x},\widehat{\rho},\widehat{\mathbb{R}}\right)$ can be defined in contravariant

$$\widehat{x} = \{\widehat{x}^k = x^{\widehat{k}}\} = \{x^k \times \widehat{I}^{\widehat{k}}_{\ k}\}$$

or covariant form

$$\widehat{x}_k = \widehat{\rho}_{kl}\widehat{x}^l = \widehat{T}^r_k \rho_{ri} x^i \times \widehat{I},$$

where $x^k, x_k \in \widehat{E}$. The square of "isoeuclidean distance" between two points $\widehat{x},\widehat{y} \in \widehat{E}$ is defined as

$$(\widehat{x} - \widehat{y})^{\widehat{2}} = \left[\left(\widehat{x}^i - \widehat{y}^i\right) \times \widehat{\rho}_{ij} \times \left(\widehat{x}^j - \widehat{y}^j\right)\right] \times \widehat{I} \in \widehat{R}$$

and the isomultiplication is given by

$$\widehat{x}^{\widehat{2}} = \widehat{x}^k \widehat{\times} \widehat{x}_k = \left(x^k \times \widehat{I} \right) \times \widehat{T} \times \left(x_k \times \widehat{I} \right) = \left(x^k \times x_k \right) \times \widehat{I} = n \times \widehat{I}.$$

Whenever confusion does not arise isospaces can be practically treated via the conventional coordinates x^k rather than the isotopic ones $\widehat{x}^k = x^k \times \widehat{I}$. The symbols $x, v, a, ...$ will be used for conventional spaces while symbols $\widehat{x}, \widehat{v}, \widehat{a}, ...$ will be used for isospaces; the letter $\widehat{\rho}\,(x, v, a, ...)$ refers to the projection of the isometric $\widehat{\rho}$ in the original space.

We note that an isofield of Class III, explicitly denoted as $\widehat{F}_{III}\left(\widehat{a}, +, \widehat{\times}\right)$ is a union of two disjoint isofields, one of Class I, $\widehat{F}_I\left(\widehat{a}, +, \widehat{\times}\right)$, in which the isounit is positive definite, and one of Class II, $\widehat{F}_{II}\left(\widehat{a}, +, \widehat{\times}\right)$, in which the isounit is negative–definite. The Class II of isofields is usually written as $\widehat{F}^d\left(\widehat{a}^d, +, \widehat{\times}^d\right)$ and called isodual fields with isodual unit $\widehat{I}^d = -\widehat{I} < 0$, isodual isonumbers $\widehat{a}^d = a \times \widehat{I}^d = -\widehat{a}$, isodual isoproduct $\widehat{\times}^d = \times \widehat{T}^d \times = -\widehat{\times}$, etc. For simplicity, in our further considerations we shall use the general terms isofields, isonumbers even for isodual fields, isodual numbers and so on if this will not give rise to ambiguities.

3.2 Isofunctions

An isofunction is a isorelation between a isoset of so - called isoinputs and a isoset of isopermissible so - called isooutputs with the property that each isoinput is isorelated of exactly one isooutput. An example is the isofunction that isorelates each isoreal isonumber $\hat{x}$ to its isosquare $\hat{x}\hat{\times}\hat{x}$. The isooutput of the isofunction $\hat{f}$ corresponding to a isoinput will be denoted with $\hat{f}^\wedge(\hat{x})$.

We will use the notation

$$\hat{f} : \hat{X} \hat{\longrightarrow} \hat{Y}$$

and in this context, the isoelements of $\hat{X}$ are called isoarguments of $\hat{f}$, $\hat{X}$ is called isodomain of $\hat{f}$ and $\hat{Y}$ is called isocodomain of $\hat{f}$

DEFINITION 3.1. *We will say that the isonumber $\hat{a}$ is isolimit of the isofunction $\hat{f} : \hat{X} \hat{\longrightarrow} \hat{Y}$ at the isopoint $\hat{x}_0$, $\hat{x}_0 \in \hat{X}$, if for every $\hat{\epsilon} \hat{>} \hat{0}$ there exists $\hat{\delta} = \hat{\delta}(\hat{\epsilon})$ such that from*

$$|\hat{x} - \hat{x}_0| \hat{<} \hat{\delta}$$

follows

$$|\hat{f}^\wedge(\hat{x}) - \hat{a}| \hat{<} \hat{\epsilon}.$$

We will write

$$\hat{f}^\wedge(\hat{x}) \hat{\longrightarrow}_{\hat{x} \hat{\longrightarrow} \hat{x}_0} \hat{a}$$

or

$$\hat{\lim}_{\hat{x} \hat{\longrightarrow} \hat{x}_0} \hat{f}^\wedge(\hat{x}) = \hat{a}.$$

Let $\hat{f}, \hat{g} : \hat{X} \hat{\longrightarrow} \hat{Y}$, $\hat{x}_0 \in \hat{X}$ and $\hat{\lim}_{\hat{x} \hat{\longrightarrow} \hat{x}_0} \hat{f}^\wedge(\hat{x}) = \hat{a}$, $\hat{\lim}_{\hat{x} \hat{\longrightarrow} \hat{x}_0} \hat{g}^\wedge(\hat{x}) = \hat{b}$. Then

1. $\hat{\lim}_{\hat{x} \hat{\longrightarrow} \hat{x}_0} (\hat{f}^\wedge(\hat{x}) \pm \hat{g}^\wedge(\hat{x})) = \hat{a} \pm \hat{b}$,
2. $\hat{\lim}_{\hat{x} \hat{\longrightarrow} \hat{x}_0} (\hat{\alpha} \hat{\times} \hat{f}^\wedge(\hat{x})) = \hat{\alpha} \hat{\times} \hat{a}$ for every $\hat{\alpha} \in \hat{F}_{\mathbb{K}}$,
3. $\hat{\lim}_{\hat{x} \hat{\longrightarrow} \hat{x}_0} (\hat{f}^\wedge(\hat{x}) \hat{\times} \hat{g}^\wedge(\hat{x})) = \hat{a} \hat{\times} \hat{b}$,
4. $\hat{\lim}_{\hat{x} \hat{\longrightarrow} \hat{x}_0} \hat{f}^\wedge(\hat{x}) \nearrow \hat{g}^\wedge(\hat{x}) = \hat{a} \nearrow \hat{b}$ if $\hat{b} \neq \hat{0}$.

DEFINITION 3.2. *An isofunction $\hat{f} : \hat{X} \hat{\longrightarrow} \hat{Y}$ will be called isocontinuous at the isopoint $\hat{x}_0 \in \hat{X}$ if for every $\hat{\epsilon} \hat{>} \hat{0}$ there exists $\hat{\delta}_1 = \hat{\delta}_1(\hat{\epsilon}) \hat{>} \hat{0}$ such that from*

$$|\hat{x} - \hat{x}_0| \hat{<} \hat{\delta}_1$$

follows

$$|\hat{f}^\wedge(\hat{x}) - \hat{f}^\wedge(\hat{x}_0)| \hat{<} \hat{\epsilon}.$$

Let $D \subset \mathbb{R}$. With $\hat{F}_D$ we will denote the isoset of the isonumbers $\hat{a}$ for which $a \in D$ and the corresponding basic unit in it is $\hat{I}_1 = \frac{1}{T_1}$, $T_1 \hat{>} 0$.

Let $T \in \mathcal{C}^1(D)$, $T \hat{>} 0$ in D. With $\hat{F\mathcal{C}}^1_D$ we will denote the isoset of all isofunctions $\hat{f}$ for which $f \in \mathcal{C}^1(D)$ and the corresponding basic unit in it is $\hat{I} = \frac{1}{T}$, the act of the isofunction $\hat{f} \in \hat{FC}^1_D$ on the isovariable $\hat{x}$ will be denoted as follows

$$\hat{f}\hat{}(\hat{x}) = \hat{f}(T_1 \hat{x}) = \frac{f}{T}(T_1 x \frac{1}{T_1}) = \frac{f}{T}(x)$$

and if $\hat{g} \in \hat{FC}^1_D$

$$\hat{f} \hat{\times} \hat{g} = \hat{f} T \hat{g}.$$

Definition 3.3. *Let $\hat{x} \in \hat{D}$. First isoderivative of the isofunctions $\hat{f} \in \hat{FC}^1_D$ at the isopoint $\hat{x}$ will be called*

$$(\hat{f})^{\circledast} := \hat{\lim}_{\hat{h} \hat{\longrightarrow} \hat{0}} (\hat{f}\hat{}(\hat{x} + \hat{h}) - \hat{f}\hat{}(\hat{x})) \curlywedge \hat{h}.$$

In this case we will say that $\hat{f}$ is isodifferentiable at the isopoint $\hat{x}$.

$$
\begin{aligned}
&(\hat{f}\hat{}(\hat{x} + \hat{h}) - \hat{f}\hat{}(\hat{x})) \curlywedge \hat{h} \\
&= \hat{f}(T_1(\hat{x} + \hat{h})) - \hat{f}(T_1 \hat{x})) T_1 \tfrac{1}{h} \tfrac{1}{T_1} \\
&= \Big(\frac{f}{T}(x+h) - \frac{f}{T}(x) \Big) \frac{1}{h} \longrightarrow_{h \longrightarrow 0} \frac{f'(x)T(x) - f(x)T'(x)}{T^2(x)} \\
&= f'(x) \frac{1}{T(x)} - f(x) \frac{1}{T(x)} \frac{T'(x)}{T(x)} \\
&= \hat{f}' - \hat{f} \hat{\times} \hat{T}' \hat{\times} (\hat{\hat{I}}),
\end{aligned}
$$

consequently the representation of the isoderivative in iso-language is

$$(\hat{f})^{\circledast} = \hat{f}' - \hat{f} \hat{\times} \hat{T}' \hat{\times} (\hat{\hat{I}}). \tag{1}$$

3.3 Santilli iso-functor

As we have seen the Santilli's central idea [3–5] is the generalization of the fundamental unit of the number theory, from its trivial n-dimensional form $I = \mathrm{diag}(1, 1, \ldots)$ to an n-dimensional matrix $\hat{I}$ with the general dependence of all essential variables

$$I = \mathrm{diag}(1, 1, \ldots) \Rightarrow \hat{I} = \hat{I}(s, x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, \ldots) \tag{2}$$

under the condition of preserving the original axioms of the unit (nondegeneracy, hermiticity, and positive-definiteness).

The "lifting" $I \Rightarrow \hat{I}$ requires, naturally, for necessary compatibility, a generalization of the conventional associative multiplication $x \circ y$ into the so-called isomultiplication

$$x \circ y \Rightarrow x \,\hat{\circ}\, y := xTy\,, \quad T = \text{fixed}\,, \tag{3}$$

where the quantity T is called the *isotopic* element. Then $\hat{I} = T^{-1}$ is a correct left and right unit element of the theory with respect the new multiplication $\hat{\circ}$ and it is called the *isounit.*

Definition 3.4. *Let* **X** *and* **Y** *be two categories. A covariant* **Santilli isofunctor** *from* **X** *to* **Y** *is a family of isofunctions* $\hat{I}$ *which associates to each object* A *in* **X** *an object* $\hat{I}A$ *in* **Y** *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *a morphism* $\hat{I}f \in \mathrm{Hom}_{\mathbf{Y}}(\hat{I}A, \hat{I}B)$, *and which is such that:*

(FI) $\hat{I}(g \circ f) = \hat{I}g \circ \hat{I}f$ *for all* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *and* $g \in \mathrm{Hom}_{\mathbf{Y}}(B, C)$*;*

(F2) $\hat{I}\,\mathrm{id}_A = \mathrm{id}_{\hat{I}A}$ *for all* $A \in \mathrm{Ob}(\mathbf{X})$.

It is clear from the above that a covariant functor is a transformation that preserves both:

- The domains and the codomains identities.
- The composition of arrows, in particular it preserves the direction of the arrows.

DEFINITION 3.5. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two categories. A contravariant* **Santilli iso-functor** *from* $\mathbf{X}$ *to* $\mathbf{Y}$ *is a family of isofunctions* $\hat{F}$ *which associates to each object* A *in* $\mathbf{X}$ *an object* $\hat{F}A$ *in* $\mathbf{Y}$ *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *a morphism* $\hat{F}f \in \mathrm{Hom}_{\mathbf{Y}}(\hat{F}A, \hat{F}B)$*, and which is such that:*

(FI) $\hat{F}(g \circ f) = \hat{F}f \circ \hat{F}g$ *for all* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *and* $g \in \mathrm{Hom}_{\mathbf{Y}}(B, C)$*;*
(F2) $\hat{F}\,\mathrm{id}_A = \mathrm{id}_{\hat{F}}A$ *for all* $A \in \mathrm{Ob}(\mathbf{X})$*.*

Thus, a contravariant Santilli functor in mapping arrows from one category to the next reverses the directions of the arrows, by mapping domains to codomains and vice versa. A contravariant Santilli functor is also called a Santilli presheaf. These types of Santilli functors will be the principal objects which we will study when discussing Santilli quantum isotheory in the language of topos theory.

DEFINITION 3.6. *Given two categories* C *and* D*, the collection of all covariant (or contravariant) Santilli functors* $F : C \to D$ *is actually a category which will be denoted as* D^C*. This is called the category of Santilli functors and has as objects covariant (or contravariant) Santilli functors and as map natural transformations between Santilli functors.*

3.4 Category of Isogroups

One of the simplest algebraic structures is the structure of a group [13]. A set G of elements of any kind is said to be a *group* if a group operation $a \circ b$ is defined in it satisfying the following axioms:

$G.1^\circ$ For any two elements a and b there exists an element

$$c = a \circ b \tag{4}$$

$G.2^\circ$ This operation is associative, that is, for any three elements a, b, c,

$$(a \circ b) \circ c = a \circ (b \circ c). \tag{5}$$

$G.3^\circ$ There exists a *neutral element* e, i. e. an element such that for every element a,

$$a \circ e = e \circ a = a. \tag{6}$$

$G.4^\circ$ For each element a there exists a *symmetric element* $\bar{a}$ such that

$$a \circ \bar{a} = \bar{a} \circ a = e. \tag{7}$$

If the group operation $a \circ b$ is called *addition*, we write $c = a + b$ and the element c is called the *sum*, the neutral element is called *zero* and is written as 0, the symmetric element is called the *opposite* and is written as $-a$, and the group is called *additive*.

If the group operation $a \circ b$ is called *multiplication*, we write $c = a \cdot b$, or $c = ab$, the element c is called the *product*, the neutral element is called *unit* and is written as 1, the symmetric element is called the *inverse* and is written as a^{-1}, and the group is called *multiplicative*.

If the group satisfies in addition the axiom

$G.5^\circ$. For any two elements a and b

$$a \circ b = b \circ a, \tag{8}$$

then the group is called *commutative* or *Abelian*.

A set of elements endowed with an operation $a \circ b$ without the properties $G.2^\circ$, $G.3^\circ$ and $G.4^\circ$ is called a *magma*. A magma with the property $G.3^\circ$ is called a *unital magma*, a magma with the property $G.2^\circ$ is called a *semigroup*. A magma in which the equations $a \circ x = b$ and $x \circ a = b$ are solvable for all a and b is called a *quasigroup*. A unital semigroup is called a *monoid*, a unital quasigroup is called a *loop*. All these structures (as also the ones to be introduced yet) are termed infinite, respectively finite, if the underlying set is infinite, respectively finite.

Groups are algebraic systems with one internal composition law. More complicated (and hence, richer) systems are obtained if we introduce a second internal composition law, which is related to the first.

If, in a set of elements of any kind two operations $a+b$ and ab are defined such that

$R.1^\circ$ The set is a commutative group with respect to the operation $a+b$;
$R.2^\circ$ The set is a semigroup with respect to the operation ab;
$R.3^\circ$ The operation ab is distributive with respect to the operation $a+b$:

$$a(b+c) = ab + ac, \qquad (a+b)c = ac + bc, \tag{9}$$

the set is called a *ring*.

A ring in which the set of elements without 0 is a commutative group with respect to the operation ab is called a *field*.

Definition 3.7. *A map $f : G \longrightarrow G'$ between two groups $(G, \circ)$ and $(G', \Box)$ is called* **homomorphism**, *if the following property holds:*

$$(\forall a, b \in G)[f(a \circ b) = f(a) \Box f(b)] \tag{10}$$

Thus a homomorphism "carries" the composition law $\circ$ on G to the composition law $\Box$ on G'. Homomorphisms of groups are well visualized in some important aspects with the help of two concepts, the image Im(f) and the kernel Ker(f) of the homeomorphism.

Definition 3.8. *If $f : G \longrightarrow G'$ is a group homomorphism, then we define:*

$$a)\mathrm{Im(f)} = f(a)/a \in G \tag{11}$$

$$b)\mathrm{Ker(f)} = a \in G/f(a) = e' \in G'. \tag{12}$$

It is well known that Im(f) is a subgroup of G' and Ker(f) is a subgroup of G.

Definition 3.9. *A homomorphism f between two groups G and G' is called* **isomorphism** *if f is bijective. In the case where $G = G'$ an homomorphism f is called* **endomorphism** *and an isomorphism is called* **automorphism**.

Definition 3.10. **Grp** *is the category with groups as objects and homomorphisms as morphisms.*

Definition 3.11. *Let* **Grp** *and* $\widehat{\mathbf{Grp}}$ *be two categories.*
A Santilli functor $\mathcal{I}$ *from* **Grp** *associates to each object* G *in* **Grp** *category an object* $\hat{G}$ *in* $\widehat{\mathbf{Grp}}$ *i.e., we reconstruct the elements for each object* $\hat{G}$ *of the category* $\widehat{\mathbf{Grp}}$ *as*

$$\mathbf{Grp} \ni a \longrightarrow \hat{a} \equiv a\hat{I} \in \widehat{\mathbf{Grp}}, \tag{13}$$

where the isounit $\hat{I}$ *is defined with the help of an invertible element*

$$T : \hat{I} = T^{-1}, \tag{14}$$

called isotopic element, and the new composition law is defined by

$$(\forall\, \hat{a}, \hat{b} \in \widehat{\mathbf{Grp}})[\hat{a} \,\hat{\circ}\, \hat{b} \equiv \hat{a}\, T\, \hat{b}]. \tag{15}$$

satisfying the following axioms:
$\widehat{G}.1^\circ$ *For any two elements* $\hat{a}$ *and* $\hat{b}$ *there exists an element*

$$\hat{c} = \hat{a} \,\hat{\circ}\, \hat{b} \tag{16}$$

$\widehat{G}.2^\circ$ *This operation is associative, that is, for any three elements* $\hat{a}, \hat{b}, \hat{c}$

$$(\hat{a} \,\hat{\circ}\, \hat{b}) \hat{\circ}\, \hat{c} = \hat{a} \,\hat{\circ}\, (\hat{b} \,\hat{\circ}\, \hat{c}). \tag{17}$$

$\widehat{G}.3^\circ$ *There exists a isounit* $\hat{I}$*, i. e. an element such that for every element* $\hat{a}$

$$\hat{a} \,\hat{\circ}\, \hat{I} \equiv \hat{a}\, T\, \hat{I} = \hat{a}\, T\, T^{-1} = \hat{a}. \tag{18}$$

$\widehat{G}.4^\circ$ *For each element* $\hat{a}$ *there exists a symmetric element* $\hat{a}^{-1}$ *such that*

$$\hat{a} \,\hat{\circ}\, \hat{a}^{-1} = \hat{a}^{-1} \,\hat{\circ}\, \hat{a} = \hat{I}. \tag{19}$$

If the group satisfies in addition the axiom
$\widehat{G}.5^\circ$*. For any two elements* $\hat{a}$ *and* $\hat{b}$

$$\hat{a} \,\hat{\circ}\, \hat{b} = \hat{b} \,\hat{\circ}\, \hat{a}, \tag{20}$$

then the isogroup $\hat{G}$ is called commutative or Abelian.

And the Santilli functor $\mathcal{I}$ from **Grp** *also associates to each morphism $f \in \mathrm{Hom}_{\mathbf{Grp}}(G, G')$ a morphism $\hat{I}f \in \mathrm{Hom}_{\widehat{\mathbf{Grp}}}(\hat{G}, \hat{G}')$, if the following property holds:*

$$(\forall \hat{a}, \hat{b} \in \hat{G}) \ \ [\hat{f}(\hat{a} \,\hat{\circ}\, \hat{b}) = \hat{f}(\hat{a}) \,\widehat{\Box}\, \hat{f}(\hat{b})]. \tag{21}$$

A map $\hat{f} : \hat{G} \longrightarrow \hat{G}'$ between two isogroups $(\hat{G}, \hat{\circ})$ and $(\hat{G}', \widehat{\Box})$ is called **isohomomorphism**. Thus an isohomomorphism "carries" the composition law $\hat{\circ}$ on $\hat{G}$ to the composition law $\widehat{\Box}$ on $\hat{G}'$. It can be proved easily, that if $\widehat{\mathbf{Grp}}$ is a monoid and also a groupoid for the fixed isotopic element T, with the above internal composition, it can become a isogroup $\hat{G}$ with unit $\hat{I}$.

DEFINITION 3.12. *Let $\mathcal{I} : \mathbf{Grp} \to \widehat{\mathbf{Grp}}$ and $\mathcal{I}' : \mathbf{Grp} \to \widehat{\mathbf{Grp}}$ be two Santilli functors. A natural transformation $\hat{\alpha} : \mathcal{I} \to \mathcal{I}'$ is given by the following data: For every object A in* **Grp** *there is a morphism $\hat{\alpha}_A : \mathcal{I}(A) \to \mathcal{I}'(A)$ in $\widehat{\mathbf{Grp}}$ such that for every morphism $f : A \to B$ in* **Grp** *the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{I}(A) & \stackrel{\hat{\alpha}_A}{\longrightarrow} & \mathcal{I}'(A) \\ \mathcal{I}(f)\downarrow & & \downarrow \mathcal{I}'(f) \\ \mathcal{I}(B) & \stackrel{\hat{\alpha}_B}{\longrightarrow} & \mathcal{I}'(B). \end{array}$$

Commutativity means (in terms of equations) that the following compositions of morphisms are equal: $\mathcal{I}(f) \star \hat{\alpha}_A = \hat{\alpha}_B \star \mathcal{I}'(f)$.

The morphisms $\hat{\alpha}_A$, $A \in \mathrm{Ob}(\mathbf{Grp})$, are called the *components of the natural transformation $\hat{\alpha}$.*

3.5 Category of Vector Isospaces

A set L^n of elements of any kind, called *vectors*, is said to be an *n-dimensional vector space* if in this set the operations of *addition* and of *multiplication by scalars*, that is real numbers, are defined, satisfying:

$VI.1^\circ$ - 5° Addition of vectors satisfies axioms $G.1 - 5^\circ$ for a commutative group;

$VII.1^\circ$ For any vector a and any scalar λ there exists a vector

$$\mathbf{b} = \mathbf{a} \cdot \lambda = \mathbf{a}\lambda \tag{22}$$

called the product of $\mathbf{a}$ by λ;

$VII.2^\circ$ Multiplication by 1 does not change a vector:

$$\mathbf{a} \cdot 1 = \mathbf{a}; \tag{23}$$

$VII.3^\circ$ Multiplication of vectors by scalars is distributive with respect to addition of scalars:

$$\mathbf{a}(\lambda + \mu) = \mathbf{a}\lambda + \mathbf{a}\mu; \tag{24}$$

$VII.4^\circ$ Multiplication of vectors by scalars is distributive with respect to addition of vectors:

$$(\mathbf{a} + \mathbf{b})\lambda = \mathbf{a}\lambda + \mathbf{b}\lambda; \tag{25}$$

$VII.5^\circ$ Multiplication of vectors by scalars is associative:

$$(\mathbf{a}\lambda)\mu = \mathbf{a}(\lambda\mu); \tag{26}$$

and *axioms* $VIII.1^\circ$ - 2° *of dimension*, which are based on the notions of linear independence and dependence of vectors. Vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are said to be *linearly independent* if a linear combination $\mathbf{a}_1\lambda_1 + \mathbf{a}_2\lambda_2 + \ldots + \mathbf{a}_m\lambda_m$ is equal to zero only if all coefficients $\lambda_i = 0$, and *linearly dependent* if there are nonzero coefficients λ_i such that this linear combination is equal to zero.

$VIII.1^\circ$ There exist n linearly independent vectors;

$VIII.2^\circ$ Any $n+1$ vectors are linearly dependent.

If we have chosen n linearly independent vectors $\mathbf{e}_1\, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in L_n, then each vector can be written as

$$\mathbf{x} = \sum_i \mathbf{e}_i x^i = \mathbf{e}_i x^i. \tag{27}$$

The numbers x^i are called the *coordinates* of the vector $\mathbf{x}$, the vectors $\mathbf{e}_i$ are called *basis vectors*. Later we will write the sums (27) only in the last form and when in our formulas the same upper and lower indices appear we will always mean summation with respect to these indices.

DEFINITION 3.13. *A subset U of a vector space V is called* **vector subspace** *if it is a subsystem which obeys the axioms of vector space in itself, that is U is closed under vector addition and scalar multiplication.*

DEFINITION 3.14. *The notions of* Ker(f) *and* Im(f) *are defined by the relations*

$$a)\ \mathrm{Ker(f)} = [x \in V\ /\ f(x) = 0 \in U], \tag{28}$$

$$b)\ \mathrm{Im(f)} = [f(x) \in U\ /\ x \in V]. \tag{29}$$

It easy proved that Ker(f) and Im(f) are subspaces of V and U respectively.

DEFINITION 3.15. *Let V and U two vector spaces over the same field $\mathbb{F}$ (not necessarily of the same dimension). A map $f : V \longrightarrow U$ is called* **linear map** *or* **linear transformation** *if the following property is holds:*

$$(\forall \alpha, \beta \in \mathbb{F})\,(\forall x, y \in V)\,[f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)]. \tag{30}$$

In case $V = U$, the map f is called **linear operator**.

Two vector spaces are called *isomorphic* if there is a bijection between them that preserves addition of vectors and multiplication of vectors by scalars, and *homomorphic* if there is a surjection or an injection between them that preserves these operations. Isomorphisms of a vector space onto itself and homomorphisms of a vector space into itself are called *automorphisms* and *endomorphisms* of this vector space respectively. Automorphisms and endomorphisms of a vector space are called *linear transformations* in it.

DEFINITION 3.16. $\mathbf{Vect}_k$ *category consists of vector spaces over a field $\mathbb{F}$ as objects and k-linear maps as morphisms.*

From the definition of vector space one can see that we cannot construct an isotopy of a vector space without first introducing an isotopy of the field, because the multiplicative unit I of the space is that of the underlying field. Note that we are lifting of the field, but the elements of the linear space remain unchanged.

DEFINITION 3.17. *Let* **Vect** *and* $\widehat{\mathbf{Vect}}$ *be two categories,* V *be a vector space over the field* $\mathbb{F}$ *and* $\hat{\mathbb{F}}$ *be an isofield of* $\mathbb{F}$*. A Santilli functor* $\mathcal{I}$ *from* **Vect** *associates to each object* A *in* **Vect** *category an object* $\hat{I}A$ *in* $\widehat{\mathbf{Vect}}$ *category by "isovector space" as the vector isospace* $\widehat{V}$*, (which has the same set of the axioms as for a vector space* V*), over the isofield* $\hat{\mathbb{F}}$ *equipped with a new external operation* $\diamond$ *which is such to verify all the axioms for a vector isospace, i.e,*

$\widehat{V}I.1^\circ$ - 5° *Addition of vectors satisfies axioms* $\widehat{G}.1{-}5^\circ$ *for a commutative isogroup;*

$\widehat{V}II.1^\circ$ *For any vector* $\mathbf{x}$ *and any isoscalar* $\hat{\alpha}$ *there exists a vector*

$$(\forall\ \hat{\alpha}, \in \hat{\mathbb{F}})\ \ (\forall\ \mathbf{x}, \mathbf{y} \in \mathbf{V})\ \ [\mathbf{y} = \hat{\alpha} \diamond \mathbf{x}]; \tag{31}$$

called the product of $\mathbf{x}$ *by* $\hat{\alpha}$ *;*

$\widehat{V}II.2^\circ$ *Multiplication by* $\hat{I}$ *does not change a vector:*

$$(\forall\ \mathbf{x} \in V)\ [\hat{I} \diamond \mathbf{x} = \mathbf{x} \diamond \hat{I} = \mathbf{x}]; \tag{32}$$

$\widehat{V}II.3^\circ$ *Multiplication of vectors by isoscalars is distributive with respect to addition of isoscalars:*

$$(\forall\ \hat{\alpha}, \hat{\beta} \in \hat{\mathbb{F}})\ \ (\forall\ \mathbf{x} \in \mathbf{V})\ \ [(\hat{\alpha} + \hat{\beta}) \diamond \mathbf{x}) = \hat{\alpha} \diamond \mathbf{x} + \hat{\beta} \diamond \mathbf{x}]; \tag{33}$$

$\widehat{V}II.4^\circ$ *Multiplication of vectors by isoscalars is distributive with respect to addition of isovectors:*

$$(\forall\ \hat{\alpha}, \in \hat{\mathbb{F}})\ \ (\forall\ \mathbf{x}, \mathbf{y} \in \mathbf{V})\ \ [\hat{\alpha} \diamond (\mathbf{x} + \mathbf{y}) = \hat{\alpha} \diamond \mathbf{x} + \hat{\beta} \diamond \mathbf{y}]; \tag{34}$$

$\widehat{V}II.5^\circ$ *Multiplication of vectors by isoscalars is associative:*

$$(\forall\ \hat{\alpha}, \hat{\beta} \in \hat{\mathbb{F}})\ \ (\forall\ \mathbf{x} \in \mathbf{V})\ \ [\hat{\alpha} \diamond (\hat{\beta} \diamond \mathbf{x}) = (\hat{\alpha} \star \hat{\beta}) \diamond \mathbf{x}]; \tag{35}$$

$\widehat{V}III.1^\circ$ *There exist* n *linearly independent isovectors;*

$\widehat{V}III.2^\circ$ *Any* $n+1$ *isovectors are linearly dependent.*

and the Santilli functor $\mathcal{I}$ from **Vect** *associates to each morphism* $f \in \mathrm{Hom}_{\mathbf{Vect}}(A,B)$ *an continuous linear isotransformation as a morphism* $\hat{I}f \in \mathrm{Hom}_{\widehat{\mathbf{Vect}}}(\hat{I}A, \hat{I}B)$*:*

$$\hat{f} : \widehat{V} \longrightarrow \widehat{V'}, \tag{36}$$

between two vector isospaces $\widehat{V}$ *and* $\widehat{V'}$ *over the same isofield* $\hat{\mathbb{F}}$ *which preserves the sum and isomultiplication, i.e., which is such that*

$$(\forall\ \hat{\alpha},, \hat{\beta} \in \hat{\mathbb{F}})\ \ (\forall\ \mathbf{x}, \mathbf{y} \in V)\ \ [\hat{f}(\hat{\alpha} \diamond \mathbf{x} + \hat{\beta} \diamond \mathbf{y}) = \hat{\alpha} \diamond \hat{f}(\mathbf{x}) + \hat{\beta} \diamond \hat{f}(\mathbf{y})]. \tag{37}$$

4 Category of topological vector isospaces

4.1 Topological vector spaces

A set T of elements is said to be a *topological vector space* if in it subsets called *closed subsets* are singled out and the following axioms are fulfilled:

$T.1°$ The union of a finite number of closed subsets is closed;

$T.2°$ The intersection of arbitrary many closed subsets is closed;

$T.3°$ The whole vector space T is a closed set;

$T.4°$ The empty set $\emptyset$ is a closed set.

A topological structure also can be defined by means of *open sets*, which are complements of closed sets, by means of *closures of sets* (the closure $\overline{M}$ of a set M is the intersection of all closed sets containing M), by means of *interiors of sets* (the interior of a set M is the union of all open sets contained in M), and by means of *neighborhoods* (i.e. open sets such that any open set can be represented as a union of these sets).

The elements in topological spaces are called *points*, and a neighborhood U containing a point x: is called *neighborhood* $U(x)$ *of this point.* A point x all neighborhoods $U(x)$ of which contain points of a set M different from x is called a *limit point* of M.

If the set of neighborhoods in a space T is countable, the space is called a *topological vector space with countable base.*

A topological vector space in which the only closed sets are the whole space and the empty set is called a *trivial vector space*. A topological space in which all subsets are closed is called a *discrete vector space*.

The most important topological vector spaces are the Hausdorff vector spaces and regular vector spaces. *Hausdorff vector spaces* satisfy the axioms:

$T.5^\circ$ All points are closed subsets;

$T.6^\circ$ Any two points in the space have disjoint neighborhoods.

Regular vector spaces satisfy the axiom $T.5^\circ$ and

$T.6'$ Any point in the vector space and any closed set in this vector space which does not contain this point have disjoint open sets containing this point and this closed set, respectively.

The natural topology in the field $\mathbb{R}$, with closed and open sets defined as in usual real Calculus, can be specified by the countable system of neighborhoods consisting of the intervals with rational ends.

4.2 Categories of topological vector spaces and isospaces and their subcategories

DEFINITION 4.1. *The* **TopVec** *is the category with topological vector spaces as objects and continuous linear transformations as morphisms.*

The most important subcategories of the category of topological vector spaces **TopVec** are the category of Hausdorff vector spaces **HausVec** and the category of regular vector spaces **RegVec**.

DEFINITION 4.2. *The* **HausVec** *is the category with Hausdorff vector spaces as objects and continuous linear transformations as morphisms.*

DEFINITION 4.3. *The* **RegVec** *is the category with regular vector spaces as objects and continuous linear transformations as morphisms.*

The notion of n–dimensional isomanifold was studied by Tsagas and Sourlas (we refer the reader for details in [5]). Their constructions are based on idea that every isounit of Class III can always be diagonalized into the form

$$\widehat{I} = diag\left(B_1, B_2, ..., B_n\right), B_k\left(x, ...\right) \neq 0, k = 1, 2, ..., n.$$

In result of this one defines a Santilli functor for isotopology $\widehat{\tau}$ on $\widehat{\mathbb{R}}^n$ which coincides everywhere with the conventional topology τ on $\mathbb{R}^n$ except at the isounit $\widehat{I}$.

Definition 4.4. *Let* **TopVec** *and* $\widehat{\textbf{TopVec}}$ *be two categories, T be a topological vector space over the field $\mathbb{F}$ and $\hat{\mathbb{F}}$ be an isofield of $\mathbb{F}$. A Santilli functor $\mathcal{I}$ from* **TopVec** *associates to each object T in* **TopVec** *category an object $\hat{I}T$ in* $\widehat{\textbf{TopVec}}$ *category by "isotopological vector space" as the topological vector isospace $\widehat{T}$, which satisfy the following axioms:*

$\widehat{T}.1^\circ$ *The union of a finite number of closed subsets is closed;*

$\widehat{T}.2^\circ$ *The intersection of arbitrary many closed subsets is closed;*

$\widehat{T}.3^\circ$ *The whole vector isospace $\hat{T}$ is a closed set;*

$\widehat{T}.4^\circ$ *The empty set $\emptyset$ is a closed set.*

and the Santilli functor $\mathcal{I}$ from **TopVec** *also associates to each morphism $f \in \mathrm{Hom}_{\textbf{TopVec}}(A, B)$ an continuous linear isotransformation as a morphism $\hat{I}f \in \mathrm{Hom}_{\widehat{\textbf{TopVec}}}(\hat{I}A, \hat{I}B)$:*

$$\hat{f} : \widehat{T} \longrightarrow \widehat{T}', \tag{38}$$

between two topological vector isospaces $\widehat{T}$ and $\widehat{T}'$ over the same isofield $\hat{\mathbb{F}}$ which preserves the sum and isomultiplication, i.e., which is such that

$$(\forall\ \hat{\alpha}, , \hat{\beta} \in \hat{\mathbb{F}})\ \ (\forall\ \mathbf{x}, \mathbf{y} \in T)\ \ [\hat{f}(\hat{\alpha} \diamond \mathbf{x} + \hat{\beta} \diamond \mathbf{y}) = \hat{\alpha} \diamond \hat{f}(\mathbf{x}) + \hat{\beta} \diamond \hat{f}(\mathbf{y})]. \tag{39}$$

Definition 4.5. *Let* **HausVec** *and* $\widehat{\textbf{HausVec}}$ *be two categories, T be a Hausdorff vector space over the field $\mathbb{R}$ and $\hat{\mathbb{R}}$ be an isofield of $\mathbb{R}$. A Santilli functor $\mathcal{I}$ from* **HausVec** *associates to each object T in* **HausVec** *category an object $\hat{I}T$ in* $\widehat{\textbf{HausVec}}$ *category by "iso-Hausdorff vector space" as the Hausdorff vector isospace $\widehat{T}$, which satisfy the following axioms:*

$\widehat{TH}.1^\circ$ *The union of a finite number of closed subsets is closed;*

$\widehat{TH}.2^\circ$ *The intersection of arbitrary many closed subsets is closed;*

$\widehat{TH}.3^\circ$ *The whole vector isospace $\hat{T}$ is a closed set;*

$\widehat{TH}.4^\circ$ *The empty set $\emptyset$ is a closed set;*

$\widehat{TH}.5^\circ$ *All points are closed subsets;*

$\widehat{TH}.6^\circ$ *Any two points in the vector isospace have disjoint neighborhoods. and the Santilli functor $\mathcal{I}$ from* **HausVec** *also associates to each morphism $f \in \mathrm{Hom}_{\mathbf{HausVec}}(A, B)$ an continuous linear isotransformation as a morphism $\hat{I}f \in \mathrm{Hom}_{\widehat{\mathbf{HausVec}}}(\hat{I}A, \hat{I}B)$:*

$$\hat{f} : \widehat{T} \longrightarrow \widehat{T}', \tag{40}$$

between two Hausdorff vector isospaces $\widehat{T}$ and $\widehat{T}'$ over the same isofield $\hat{\mathbb{R}}$ which preserves the sum and isomultiplication, i.e., which is such that

$$(\forall\ \hat{\alpha}, , \hat{\beta} \in \hat{\mathbb{R}})\ \ (\forall\ \mathbf{x}, \mathbf{y} \in T)\ \ [\hat{f}(\hat{\alpha} \diamond \mathbf{x} + \hat{\beta} \diamond \mathbf{y}) = \hat{\alpha} \diamond \hat{f}(\mathbf{x}) + \hat{\beta} \diamond \hat{f}(\mathbf{y})]. \tag{41}$$

regular vector spaces

Definition 4.6. *Let* **RegVec** *and* $\widehat{\mathbf{RegVec}}$ *be two categories, T be a regular vector space over the field $\mathbb{R}$ and $\hat{\mathbb{R}}$ be an isofield of $\mathbb{R}$. A Santilli functor $\mathcal{I}$ from* **RegVec** *associates to each object T in* **RegVec** *category an object $\hat{I}T$ in* $\widehat{\mathbf{RegVec}}$ *category as the regular vector isospace $\widehat{T}$, which satisfy the following axioms:*

$\widehat{TR}.1^\circ$ *The union of a finite number of closed subsets is closed;*

$\widehat{TR}.2^\circ$ *The intersection of arbitrary many closed subsets is closed;*

$\widehat{TR}.3^\circ$ *The whole vector isospace $\hat{T}$ is a closed set;*

$\widehat{TR}.4^\circ$ *The empty set $\emptyset$ is a closed set;*

$\widehat{TR}.5^\circ$ *All points are closed subsets;*

$\widehat{TR}.6$ *Any point in the vector isospace and any closed set in this vector isospace which does not contain this point have disjoint open sets containing this point and this closed set, respectively.*

and the Santilli functor $\mathcal{I}$ from **RegVec** *also associates to each morphism $f \in \mathrm{Hom}_{\mathbf{RegVec}}(A, B)$ an continuous linear isotransformation as a morphism $\hat{I}f \in \mathrm{Hom}_{\widehat{\mathbf{RegVec}}}(\hat{I}A, \hat{I}B)$:*

$$\hat{f} : \widehat{T} \longrightarrow \widehat{T}', \tag{42}$$

between two regular vector isospaces $\widehat{T}$ and $\widehat{T}'$ over the same isofield $\hat{\mathbb{R}}$ which preserves the sum and isomultiplication, i.e., which is such that

$$(\forall\ \hat{\alpha}, , \hat{\beta} \in \hat{\mathbb{R}})\ \ (\forall\ \mathbf{x}, \mathbf{y} \in T)\ \ [\hat{f}(\hat{\alpha} \diamond \mathbf{x} + \hat{\beta} \diamond \mathbf{y}) = \hat{\alpha} \diamond \hat{f}(\mathbf{x}) + \hat{\beta} \diamond \hat{f}(\mathbf{y})]. \tag{43}$$

The natural isotopology in the field $\hat{\mathbb{R}}$, with closed and open sets defined as in usual real iso-Calculus [6], can be specified by the countable system of neighborhoods consisting of the intervals with rational ends.

4.3 Subspaces of topological vector isospaces

A subset of a topological vector isospace where for closed subsets we take the intersections with closed subsets of this space is called a *subisospace*. A topological vector isospace which cannot be divided into two closed non-empty subsets with empty intersection is called *isocontinuous* or *isoconnected*.

A topological vector isospace admitting such a division is called *isononconnected* and consists of isoconnected components.

A topological vector isospace, or a subset of it, is called *isocompact* if each infinite subset of it has a limit point. In every covering of a compact topological vector isospace we can choose a finite covering.

If $\widehat{T}_1, \widehat{T}_2, \ldots, \widehat{T}_n$ are topological vector isospaces, the n-tuples $\{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n\}$ consisting of points $\hat{x}_i$ in $\widehat{T}_i$ form a new topological vector isospace, whose closed sets are sets of such n-tuples for which every point $\hat{x}_i$ runs through a closed subset in $\widehat{T}_i$ and arbitrary intersections of these sets. This new space is called the *topological isoproduct* of the topological vector isospaces $\widehat{T}_i$.

4.4 Isocontinuous mappings and isohomeomorphisms

A mapping from a topological vector isospace $\widehat{T}$ onto a topological vector isospace $\widehat{T}'$ is called *isocontinuous* if for each neighborhood $V(\hat{x}')$ of a point $\hat{x}'$ in $\widehat{T}'$ there is a neighborhood $U(\hat{x})$ of the corresponding point $\hat{x}$ in $\widehat{T}$ such that the images of all points in $U(\hat{x})$ belong to $V(\hat{x}')$.

A bijection from $\widehat{T}$ onto $\widehat{T}'$ which is isocontinuous together with its inverse bijection is called a *isohomeomorphism*; in this case the topological vector isospaces $\widehat{T}$ and $\widehat{T}'$ are called *isohomeomorphic*. The sets of closed subsets of two isohomeomorphic topological vector isospaces are mapped onto each other.

If $\hat{f}$ is a isocontinuous mapping from $\widehat{T}$ to $\widehat{T}'$ and all preimages of points of $\widehat{T}'$ in $\widehat{T}$ are subisospaces isohomeomorphic to a topological vector isospace $\widehat{S}$, the space $\widehat{T}'$ is called the *quotient isospace* of $\widehat{T}$ by $\widehat{S}$ and is written as $\widehat{T}/\widehat{S}$.

4.5 Categorical structures on topological isospaces

Among the structures on topological spaces we can select that one, which is compatible with the topology. Let **Top** be a category of some topological spaces with a forgetful functor $\mathcal{F} : \mathbf{Top} \to \mathbf{Set}$.

The categories associated with a topological space $T \in Ob(\mathbf{Top})$ are as follows:

– The category $\mathbf{T}(T)$, where $Ob(\mathbf{T})$ is the set of all open subsets of T, and $Mor(T', T''))$ are all their inclusions.

– The category (pseudogroup) **DE**, where $Ob(\mathbf{DE})$ is the set of all open subsets of T, and $Mor(T', T'')$ are all their homeomorphisms.

Functors $\mathcal{PRESH} : \mathbf{T} \to \mathbf{Set}$ are called *presheaves* of sets on T. Some of them are called sheaves. Thus we have the inclusions

$$\mathcal{SH}(T) \subset \mathcal{PRESH}(T) \subset \mathcal{FUNCT}(\mathbf{T}, \mathbf{Set}).$$

A *Grothendieck topology* on a category is defined by saying which families of maps into an object constitute a *covering* of the object when certain axioms are fulfilled. A category together with a Grothendieck topology on it is called a *site*. For a site $\mathfrak{C}$ one define the full subcategories $\mathcal{SH}(\mathfrak{C}) \subset \mathcal{PRESH}(\mathfrak{C}) \subset \mathcal{FUNCT}(\mathfrak{C}^\circ, \mathbf{Set})$. The objects of $\mathcal{FUNCT}(\mathfrak{C}^\circ, \mathbf{Set})$ are called *presheaves* on the site $\mathfrak{C}$, and the objects of $\mathcal{SH}(\mathfrak{C})$ are called *sheaves* on $\mathfrak{C}$.

For any category there exists the finest topology such that all representable presheaves are sheaves. It is called the *canonical* Grothendieck topology. *Topos* is a category which is equivalent to the category of sheaves for the canonical topology on them.

Hence, the topology is already transfered on a category. So now it is natural to consider on language of toposes and sheaves in all questions connected to local properties.

Here we shall not consider local structures on toposes in general, and we shall restrict ourselves to the consideration of the elementary case of the category **Top**.

DEFINITION 4.7. *A structure defined by a forgetful functor $\mathcal{F} : \mathbf{C} \to \mathbf{Top}$ is called a* **local structure** *if*

$\forall C \in Ob(\mathbf{C})$ and any inclusion map $i : U \to \mathcal{F}(C)$ of the open subset U an object $\tilde{U} \in Ob(\mathbf{C})$ and a morphism $\tilde{i} \in \mathbf{Mor}(\tilde{U}, C)$ exist such that $\mathcal{F}(\tilde{U}) = U$ $\mathcal{F}(\tilde{i}) = i$. This $C-$structure $\tilde{U}$ is denoted by $C|U$ and called a **restriction** *of C on U.*

In other words we can restrict ourselves to local structures on open subsets.

For a local structure $\mathcal{F} : \mathbf{C} \to \mathbf{Top}$ and each object $X \in Ob(\mathbf{Top})$ there is the presheaf of categories

$$\mathbf{T}(X)^\circ \to \mathbf{Cat} : U \mapsto \mathcal{F}^{-1}(U, id_U).$$

Often this presheaf is a sheaf.

5 Category of Topological Isogroups

A set of elements is said to be a *topological group* if

$TG.1°$ This set is a group (see (4)–(7));

$TG.2°$ This set is a topological vector space (see $T.1°$–$T.4°$);

$TG.3°$ The group operations $x \to ax$, $x \to xa$ and $x \to x^{-1}$, where a is a constant element, are continuous mappings from this topological space onto itself.

The group and the topological space participating in the definition of a topological group are called the *basis group* and the *basis space* of this topological group. A topological group is called *commutative* or *noncommutative*, *additive* or *multiplicative* together with its basis group, and *connected* or *nonconnected*, *compact* or *noncompact* together with its basis space. A subset of a topological group is called a *topological subgroup* if it is a subgroup of the basis group and a closed subset of the basis space of the topological group.

Definition 5.1. *The* **TopGrp** *is the category with topological groups as objects and continuous homomorphisms as morphisms.*

Definition 5.2. *Let* **TopGrp** *and* $\widehat{\textbf{TopGrp}}$ *be two categories.*
A Santilli functor $\mathcal{I}$ *from* **TopGrp** *associates to each object* G *in* **TopGrp** *category an object* $\hat{I}G$ *in* $\widehat{\textbf{TopGrp}}$ *category as the topological isogroup* $\widehat{G}$*, i.e., we reconstruct a set of elements for each object* $\hat{G}$ *of the category* $\widehat{\textbf{TopGrp}}$ *as A set of elements is said to be a* topological group *if*

$\widehat{TG}.1°$ *This set have to be a isogroup (see (16)–(19));*

$\widehat{TG}.2°$ *This set have to be a topological vector isospace (see* $\widehat{T}.1°$*–*$\widehat{T}.4°$*);*

$\widehat{TG}.3°$ *The isogroup operations* $\hat{x} \to \hat{\alpha}\,\hat{\circ}\,\hat{x}$*,* $\hat{x} \to \hat{x}\,\hat{\circ}\,\hat{\alpha}$ *and* $\hat{x} \to \hat{x}^{-1}$*, where* α *is a isoscalar, are isocontinuous mappings from this topological vector isospace onto itself. And the Santilli functor* $\mathcal{I}$ *from* **TopGrp** *also associates to each morphism in* **TopGrp** *a isocontinuous homomorphism in* $\widehat{\textbf{TopGrp}}$.

6 Conclusions

In 1996, Santilli generalized in [16] (pages 24-25) the work of Tsagas and Sourlas [5] for the case of isofields of second kind. Later, in 2003, R. M. Falcón and J. Núñez have shown in [17] a possible generalization of Tsagas-Sourlas-Santilli isotopology, by studying the possibility of working with fields that cannot be arranged. In that last work the notion of isoorder is defined and general notions of (iso)topological isospace are also introduced, to get the definition of isotopology proposed by Tsagas and Sourlas to be a peculiar case of the ones there proposed. We are thinking in a future to make a generalization of Santilli fuctor with isodifferentiable isomanifolds which used in the isodifferential calculus introduced by Santilli in 1996 (see [16]). That is to say, starting from the generalization of Tsagas-Sourlas-Santilli isotopology give us a tool to build the theory of isocobordism and then the isoopological quantum field theory too.

References

[1] R. M. Santilli, "On a possible Lie-admissible covering of Galilei's relativity in Newtonian mechanics for nonconservative and Galilei form-noninvariant systems," Hadronic J. 1, 223-423 (1978), available in free pdf download from http://www.santilli-foundation.org/docs/Santilli-58.pdf

[2] R. M. Santilli, Rendiconti Circolo Matematico Palermo, Suppl. 42, 7-82 (1996), available as free download from http://www.santilli-foundation.org/docs/Santilli-37.pdf

[3] R. M. Santilli, *Elements of Hadronic Mechanics*, Vol. I (), Vol. II. Kiev: Naukova Dumka Publishers, 1995. Available in free pdf downloads from http://www.santilli-foundation.org/docs/Santilli-300.pdf http://www.santilli-foundation.org/docs/Santilli-301.pdf

[4] R. M. Santilli, *Hadronic Mathematics, Mechanics and Chemistry*, Vol. I, II, III, IV and V, International academic press, 2008. Available as free downlaods from http://www.i-b-r.org/Hadronic-Mechanics.htm

[5] D. S. Sourlas, G. T. Tsagas. *From Mathematical Foundations of the Lie-Santilli theory.* Kyiv: Naukova Dumka Publishers, 1993.

[6] S.G. Georgiev, *Foundation of Iso-Differential Calculus*, Vol. 1. New York: Nova Science Publishers, 2014.

[7] S.S. Moskaliuk and A.T. Vlassov, *On some categorical constructions in mathematical physics*, Proc. of the 5th Wigner Symposium, Singapore, World Scientific (1998) 162-164.

[8] S.S. Moskaliuk and A.T. Vlassov, "Double categories in mathematical physics", *Ukr. J. Phys.* **43** (1998) 162-164.

[9] S.S. Moskaliuk, "The method of additional structures on the objects of a category as a background for category analysis in physics", *Ukr. J. Phys.* **46** (2002) 51-58.

[10] P.V. Golubtsov and S.S. Moskaliuk, "Method of Additional Structures on the Objects of a Monoidal Kleisli Category as a Background for Information Transformers Theory", *Hadronic Journal* **25(2)** (2002) 179–238 [arXiv: math-ph/0211067].

[11] P.V. Golubtsov and S.S. Moskaliuk, "Fuzzy logic, informativeness and Bayesian decision-making problems", *Hadronic Journal* **26** (2003) 589.

[12] S.S. Moskaliuk, "Weil representations of the Cayley-Klein hermitian symplectic category", *Ukrainian J. Phys.* **48** (2003) 350-384.

[13] S.S. Moskaliuk. *From Cayley-Klein Groups to Categories*, volume 11 of Series "Methods of Mathematical Modelling" . Kyiv: TIMPANI Publishers, 2006, 352 p.

[14] S.S. Moskaliuk, "On Santilli Isofunktor for Mathematical Isostructures", *Algebras, Groups and Geometries*, **31**, No. 1 (2014) 1-22.

[15] S.S. Moskaliuk, "Categories of the Basic Santilli's Isostructures", *Algebras, Groups and Geometries*, **31**, No. 1 (2014) 101-116.

[16] R. M. Santilli, "Nonlocal-integral isotopies of differential calculus, mechanics and geometries", *Rendiconti del Circolo Matematico di Palermo Serie II, Supl.* **42** (1996), 7-82.

[17] R. M. Falcón and J. Núñez, "Studies on the Tsagas-Sourlas-Santilli isotopology", *Algebras, Groups and Geometries* **20**, No. 1 (2003), 1-100.