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Lie-Santilli admissibility with helix-hyper-product

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Abstract

Helix-hyper-operations, are defined on any type of matrices via multi-valued operations. Thus, a helix-product is defined on a set of non-square matrices and the results are sets of matrices, of the same type, instead of a single matrix. The characteristic point is that the helix-operations take into account all the entries of the factors. Einstein's legacy imply that quantum mechanics and chemistry are 'incomplete' theories for the description of complex time-irreversible systems of extended constituents with internal non-potential interactions. Due to the extremely large number of constituents and the extreme complexity of the multi-valued internal communications, it is convenient to use a representation via two hyper-operations, left and right, Lie-admissible H_v-hyperstructures. We achieve this by using the helix-product of non-square matrices on ordinary real, imaginary infinite or finite fields. The type of used matrices must be as simple as possible in order to reduce the cardinality of the results.

Keywords: Lie-Santilli iso-theory, H_v-hyperstructures, helix-hyper-operations.

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1. Santilli’s new conception of living organisms

Einstein accepted the validity of quantum mechanics for the representation of the atomic structure and other systems, but never accepted quantum mechanics as being a final theory. For this reason, Einstein expressed the view in 1935, jointly with B. Podolsky & N. Rosen, that *‘Quantum mechanics is not a complete theory’* (EPR argument) [7],[8], in the sense that quantum mechanics could admit enlargements for the representations of more complex systems. Moreover, Einstein did not accepted the uncertainties of quantum mechanics being final, but there could exist conditions recovering classical determinism. For this, Einstein made his famous quote: *‘God does not play dice with the universe.’*

The most evident illustration if the validity of the lack of ‘completeness’ of quantum mechanics (and of quantum chemistry) is given by the fact that *quantum mechanics and chemistry can only represent systems of point-like particles that are invariant under time-reversal*. This is due to the invariance under anti-Hermiticity of the quantum mechanical Lie product between Hermitean operators

$$[A,B] = AB - BA = -[A,B]^{\dagger},$$

where AB is the classical associative product. In fact, the Lie product characterizes Heisenberg’s time evolution of an observable A in terms of the Hamiltonian H,

$$i dA/dt = [A,H] = AH - HA.$$

Physical, chemical and biological processes such as nuclear fusion and living organisms, are irreversible over time. The verification of Einstein’s legacy via irreversible processes was first identified by R. M. Santilli in the mid 1960s. In fact, Santilli’s Ph.D. thesis, published in the 1967, provided the first known confirmation of the EPR argument via the following Lie-admissible ‘completion’ of quantum mechanical Lie algebras for the representation of irreversible processes

$$(A,B) = ARB-BSA = (ATB-BTA)+(AJB+BJA), \quad R=T-J,\ S=T+J \neq 0,$$

where the new product (A,B) is Lie-admissible when the attached antisymmetric product

$$[A,B]^{*} = (A,B) - (B,A) = ATB - BTA$$

verifies the Lie axioms whenever T is nowhere singular. Also, the product (A,B) is called *Jordan-admissible* when the attached symmetric product

$$\{A,B\}^{*} = (A,B) + (B,A) = AJB + BJA$$

verifies the axioms of Jordan algebras.

Santilli called *hadronic mechanics* and *hadronic chemistry* the ‘completion’ of quantum mechanics and chemistry, respectively, with a Lie-admissible structure for the representation of irreversible structures and processes.

The mathematics Lie-admissible formulations, known as *geno-mathematics* [4], [5], [6], [7], [8], [10], [18] can be summarized as follows. A general assumption of classical, numeric fields underlying Lie’s theory is that *the multiplication of two numbers to the* *right* $n{-}{-}{>}3$ *is equal to the multiplication to the left,* $2{<}{-}{-}3 = 2{-}{-}{>}3$. Consequently, the indicated order of the multiplication is ignored in classical number theory, and we write $2\times 3 = 6$. In the transition from Lie theory to the covering Lie-admissible, the above ordering of the multiplication is not ignorable since the one to the right $2>3 = 2S3$ is not equal to the multiplication to the left $2<3 = 2R3 \neq 2>3$. This has permitted the identification of two numeric fields for Lie-admissible formulations:

1) The *forward genofields* $F^{>}(n^{>},>,I^{>})$ with *forward genounit* $I^{>}=1/S$, *forward geno-numbers* $n^{>}= nI^{>}$, and *forward genoproduct* $n^{>}>m^{>} = n^{>}Sm^{>}$, where n, m represent ordinary numbers; and

2) The *backward genofields* $F^{<}(n^{<},<,I^{<})$ with *backward genounit* $I^{<}=1/R$, *backward geno-numbers* $n^{<} = I^{<}n$, and *backward genoproduct* $n^{<}< m^{<} = n^{<}Rm^{<}$.

Recall that Lie algebras can be constructed via the universal enveloping associative algebras ξ with classical, associative, modular product AB. The indicated inequivalence of the multiplications to the right and to the left implies the existence for Lie-admissible theories of *two universal, enveloping, geno-associative geno-algebars,* that to the right $\xi^{>}$ (left $\xi^{<}$) with geno-associative geno-product to the right $A>B$ (left $A<B$). The indicated bimodular formulations characterize the *time-irreversible, Lie-admissible, Heisenberg-Santilli geno-equation* [7], [8], [9], [17].

$$i dA/dt = (A, H) = ARH - HSA = A < H - H > A.$$

The modular associative multiplication to the right of an operator H to a Hilbert stat, $H\psi(t,r) = E\psi(t,r)$ yields the same eigenvalues E for the one to the left $\psi(t,r)H = \psi(t,r)E$. The Lie-admissible ‘completion’ of the above bimodular structure:

1) The geno-associative action to the right via the Schrödinger-Santilli geno-equation

$$H(r,p) > \psi^{>}(t^{>},r^{>}) = H(r,p)S(\psi^{>},\ldots)\psi(t,r) = E^{>}\psi^{>}(t^{>},r^{>})^{>}, \qquad \text{and}$$

2) The geno-associative action to the left via Schrödinger-Santilli geno-equation

$$\psi^{<}(t^{<},r^{<}) < H(r,p) = \psi^{<}R(\psi^{<},\ldots)H(r,p) = \psi^{<}(t^{<},r^{<})E^{<}$$

where $E^{>} \neq E^{<}$.

Following the above mathematics, with experimental and industrial verifications of hadronic mechanics and chemistry, Santilli proved Einstein’s legacy that ‘quantum mechanics is not a complete theory’. The results were achieved via the representation of the *extended, overlapping character of the constituents of irreversible systems in terms of the forward genotopic element* with realizations of the type

$$\hat{T} = \Pi_{k=1,\ldots,N}\, \mathrm{Diag.}(1/n_{1k}^{2}, 1/n_{2k}^{2}, 1/n_{3k}^{2}, 1/n_{4k}^{2})e^{-\Gamma(\psi,\partial\psi,\ldots)},$$

where n_{1k}^{2}, n_{2k}^{2}, n_{3k}^{2}, are the semi-axes of the k-particle normalized to $n_{\mu k}^{2}=1$, $\mu=1,2,3$; n_{4k}^{2} is the density of the k-particle normalized to $n_{4k}^{2}=1$; and $\Gamma(\psi,\partial\psi)$ is the non-linear, non-local, non-potential interactions by mutual entanglement of the particles.

Particles originally in conditions of mutual entanglement of their wave packets and then separated, have been experimentally proved to instantly influence each other at a distance. Santilli has achieved a quantitative representation via the extended character of the wavepacket of particles resulting in their entanglement at a distance of their center of mass. All studies [5], [7], [8], [9], [10], [17], [18], have established that the instantaneous communication of entangle particles at a distance occurs without any use of energy.

2. Hyper-structues

We deal with the class of hyper-structures called *H_v-structures* and introduced in 1990 [14]. These satisfy the *weak axioms* where the non-empty intersection replaces equality. Basic definitions:

A set H equipped with at least one *hyper-operation*: $\cdot: H\times H \to P(H)-\{\varnothing\}$, is called *hyper-structure* and we write $(H,\cdot)$. The hyper-operation $(\cdot)$ is called *weak associative,* if $(xy)z\cap x(yz) \neq \varnothing$, $\forall x,y,z\in H$ and it is called the *weak commutative* if $xy\cap yx \neq \varnothing$, $\forall x,y\in H$. The hyper-structure $(H,\cdot)$ is called *H_v-semigroup* if it is weak associative, and it is called *H_v-group* if it is *reproductive* H_v-semigroup: $xH=Hx=H$, $\forall x\in H$.

In a similar way more complicated hyperstructures can be defined:

$(R,+,\cdot)$ is *H_v-ring* if $(+)$ and $(\cdot)$ are weak associative, the reproduction axiom is valid for $(+)$ and, finally, $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y+z)\cap(xy+xz)\neq\varnothing, \quad (x+y)z\cap(xz+yz)\neq\varnothing, \quad \forall x,y,z\in R.$$

Let $(R,+,\cdot)$ be H_v-ring, $(M,+)$ is a weak commutative H_v-group and there exists an external hyper-operation $\cdot : R\times M\to P(M): (a,x)\to ax$ such that, $\forall a,b\in R$ and $\forall x,y\in M$, we have

$$a(x+y)\cap(ax+ay) \neq \varnothing, \quad (a+b)x\cap(ax+bx) \neq \varnothing, \quad (ab)x\cap a(bx) \neq \varnothing,$$

then M is an *H_v-module* over F. In the case that we have of an H_v-field F instead of the H_v-ring R, then the *H_v-vector space* is defined.

For more definitions and applications on H_v-structures one can see books and papers as [1], [2], [4], [5], [14], [15], [16], [20], [21].

Definition 2.1 The *fundamental relations β^*, γ^** and ε^*, are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively [5], [14], [15].

The way to find the fundamental classes is given by Theorems as the following:

(1) Let $(H,\cdot)$ be an H_v-group and denote by U the set of finite products of elements of H. We define the relation β in H by setting $x\beta y$ if $\{x,y\}\subset u$, where $u\in U$. Then β^* is the transitive closure of β.

(2) Let $(R,+,\cdot)$ be an H_v-ring. Denote by U the set of finite polynomials of elements from R. We define the relation γ in R as follows: $x\gamma y$ if $\{x,y\}\subset u$ where $u\in U$. Then the relation γ^* is the transitive closure of γ.

An element is called *single* if its fundamental class is singleton.

Fundamental relations are used for general definitions. Thus, an H_v-ring $(R,+,\cdot)$ is called *H_v-field* if R/γ^* is a field. *Hyper-numbers* or *H_v-numbers* are called the elements of H_v-fields.

Let $(H,\cdot)$, $(H,*)$ be H_v-semigroups on H. $(\cdot)$ is called *smaller* than $(*)$, if there is an

$$f\in Aut(H,*) \text{ such that } xy\subset f(x*y),\ \forall x,y\in H,$$

then we say that $(H,*)$ *contains* $(H,\cdot)$.

Theorem 2.2 (The Little Theorem). Greater hyper-operations than the ones which are weak associative or weak commutative, are also weak associative or weak commutative, respectively.

This Theorem leads to a partial order on H_v-structures. The problem of enumeration of hyper-structures, started from the beginning, it is complicate in H_v-structures because we have great numbers. The Little Theorem, transfers and restrict the problem in finding the minimal, up to isomorphisms, H_v-structures. The number of H_v-groups with three elements, up to isomorphism, is 1.026.462. There are 7.926 abelian.

Some interesting large classes of H_v-structures, are the following [5], [14], [16]:

An H_v-structure is *very-thin* if all hyper-operations are operations except one, with all hyper-products singletons except one, which is a set with more than one element.

Let $(G,\cdot)$ be groupoid, then for all $P\subset G$, $P\neq\varnothing$, we define the *P-hyper-operations*:

$$\underline{P}:\ x\underline{P}y=(xP)y\cup x(Py),\ \forall x,y\in G,$$

and $(G,\underline{P})$ are called *P-hyper-structures*. If $(G,\cdot)$ is semigroup, then $x\underline{P}y=xPy$ and $(G,\underline{P})$ is a semi-hypergroup.

A generalization of P-hyperoperation, need in Lie Santilli's theory, is the following:

Let $(G,\cdot)$ be abelian group and P subset of G. We define the *P_e-hyper-operation $\times_P$* by

$$\begin{cases} x\times_P y = x\cdot P\cdot y=\{x\cdot h\cdot y \mid h\in P\} & \text{if } x\neq e \text{ and } c\neq e \\ x\cdot y & \text{if } x=e \text{ or } y=e \end{cases}$$

The hyper-structure $(G,\times_P)$ is an abelian H_v-group.

The general definition of an H_v-Lie algebra was given as follows [3], [5], [19], [22]:

Definition 2.3 Let $(L,+)$ be H_v-vector space on $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$ the canonical map, $\omega_F=\{x\in F:\varphi(x)=0\}$, 0 is zero of F/γ^*. Let ω_L the core of $\varphi':L\to L/\varepsilon^*$, 0 the zero of L/ε^*. Consider the bracket hyper-operation:

$$[\ ,\ ]: L\times L\to P(L): (x,y)\to[x,y]$$

then L is an *H_v-Lie algebra* over F if the following axioms are satisfied:

(L1) The bracket hyper-operation is bilinear:

$$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\varnothing$$

$$[x,\lambda_1y_1+\lambda_2y_2]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\varnothing,\ \ \forall x,x_1,x_2,y,y_1,y_2\in L,\ \forall\lambda_1,\lambda_2\in F$$

(L2) $[x,x]\cap\omega_L\neq\varnothing,\ \ \forall x\in L$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L\neq\varnothing,\ \ \forall x,y\in L.$

The Representations of H_v-groups can be, mainly, achieved by H_v-matrices and has been created and studied in [5], [14], [15], [16], [20].

Definitions 2.4 *H_v-matrix* is a matrix with entries from an H_v-ring or H_v-field . The hyper-product of two H_v-matrices $A=(a_{ij})$ and $B=(b_{ij})$, of type m×n and n×r respectively, is defined, in the usual manner, and it is a set of m×r H_v-matrices:

$$A\cdot B=(a_{ij})\cdot(b_{ij})=\{C=(c_{ij})\mid c_{ij}\in\oplus\Sigma a_{ik}\cdot b_{kj}\}.$$

Let $(H,\cdot)$ be H_v-group, F be H_v-field, then it is called *H_v-matrix representation* on a set $M_F=\{(a_{ij})\mid a_{ij}\in F\}$ any map

$$T:H\to M_F:h\mapsto T(h)\ \text{ such that }\ T(h_1h_2)\cap T(h_1)T(h_2)\neq\varnothing,\ \forall h_1,h_2\in H.$$

If $T(h_1h_2)\subset T(h_1)T(h_2),\ \forall h_1,h_2\in H$, then T is called *inclusion representation.*

Theorem 2.5 A necessary condition to have an inclusion representation T of an H_v-group $(H,\cdot)$ by n×n H_v-matrices over $(F,+,\cdot)$ is the following:

For all $\beta^*(x)$, $x\in H$ there must exist $a_{ij}\in H$, $i,j\in\{1,\dots,n\}$ such that

$$T(\beta^*(a))\subset\{A=(a'_{ij})\mid a'_{ij}\in\gamma^*(a_{ij}),\ i,j\in\{1,\dots,n\}\}$$

We can obtain a large number of very-thin hyper-structures from given structures, or hyper-structures, by enlarging only one of the results. Therefore, we have very interesting results, as the following [5], [20], [23]:

Theorem 2.6 In the ring $(\boldsymbol{Z}_n,+,\cdot)$, with n=ms we enlarge the product only $0\cdot m$ by setting $0\otimes m=\{0,m\}$ and the rest results are the same. Then $(\boldsymbol{Z}_n,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_m,+,\cdot)$.

We can enlarge other products as well, as $2\cdot m$ setting $2\otimes m=\{2,m+2\}$, then the result remains the same. In this case 0 and 1 remain scalars.

Corollary. In $(\boldsymbol{Z}_n,+,\cdot)$, with n=ps, p prime, we enlarge only $0\cdot p$ by $0\otimes p=\{0,p\}$ and the rest remain the same. Then $(\boldsymbol{Z}_n,+,\otimes)$ is very-thin H_v-field.

The *isofields* needed in *isotopies* correspond to hyper-structures introduced by Santilli & Vougiouklis in 1999 and they are called *e-hyper-fields*. The H_v-fields can give e-hyper-fields which can be used in applications as in physics or biology [9], [17], [18] [19], [22].

Definitions 2.7 A hyper-structure $(H,\cdot)$ containing a unique scalar unit e, is called *e-hyper-structure*. In an e-hyper-structure, we assume that for every element x, there exists an inverse x^{-1}: $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$. Remark that the inverses are not necessarily unique.

A hyperstructure $(F,+,\cdot)$, where $(+)$ is operation and $(\cdot)$ is hyper-operation, is called *e-hyper-field* if the following axioms are valid:

1. $(F,+)$ is abelian group,
2. $(\cdot)$ is weak associative,
3. $(\cdot)$ is weak distributive to $(+)$,
4. 0, the additive unit, is absorbing: $0\cdot x = x\cdot 0 = 0$, $\forall x\in F$,
5. there is a multiplicative scalar unit 1: $1\cdot x=x\cdot 1=x$, $\forall x\in F$,
6. for every $x\in F$ there is unique inverse x^{-1}, that is $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

The elements of an e-hyper-field are called *e-hyper-numbers*. If $1=x\cdot x^{-1}=x^{-1}\cdot x$, then we say that the e-hyper-field is *strong*.

Definition 2.8 *The Main e-Construction.* In a given group $(G,\cdot)$, e the unit, we define, a large number of hyper-operations $(\otimes)$ as follows:

$$x\otimes y = \{xy,g_1,g_2,\ldots\},\ \forall x,y\in G-\{e\},\ g_1,g_2,\ldots\in G-\{e\}$$

$g_1,g_2,\ldots$ are not the same for each pair (x,y). Then, $(G,\otimes)$ becomes an H_v-group, because it contains the $(G,\cdot)$, so it is an e-hyper-group. Moreover, if for each x,y such that $xy=e$, so, $x\otimes y=xy$, then $(G,\otimes)$ becomes a strong e-hypergroup.

3. Helix-hyper-operations

Hyper-operations on any type of ordinary matrices can be defined [4], [11], [21], [23].

Definition 3.1 Let $A=(a_{ij})\in M_{m\times n}$ be $m\times n$ matrix and $s,t\in N$, $1\le s\le m$, $1\le t\le n$. We define the *helix-$\underline{st}$* map, by

$$\underline{st}: M_{m\times n}\to M_{s\times t}: A\to A\underline{st}=(\underline{a}_{ij}),$$

where $\underline{a}_{ij}=\{a_{i+\kappa s,j+\lambda t} \mid 1\le i\le s,\ 1\le j\le t,\ \kappa,\lambda\in N,\ i+\kappa s\le m,\ j+\lambda t\le n\}$.

Definitions 3.2 (a) Let $A=(a_{ij})\in M_{m\times n}$, $B=(b_{ij})\in M_{u\times v}$, $s=\min(m,u)$, $t=\min(n,u)$. The hyper-operation called *helix-sum*, is defined by

$$\oplus: M_{m\times n}\times M_{u\times v}\to P(M_{s\times t}): (A,B)\to A\oplus B=A\underline{st}+B\underline{st}=(\underline{a}_{ij})+(\underline{b}_{ij})\subset M_{s\times t},$$

where $(\underline{a}_{ij})+(\underline{b}_{ij})= \{(c_{ij})= (a_{ij}+b_{ij}) \mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij}\}$.

(b) Let $A=(a_{ij})\in M_{m\times n}$, $B=(b_{ij})\in M_{u\times v}$, $s=\min(n,u)$. The *helix-product*, is defied by

$$\otimes: M_{m\times n}\times M_{u\times v}\to P(M_{m\times v}): (A,B)\to A\otimes B=A\underline{ms}\cdot B\underline{sv}=(\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset M_{m\times v},$$

where $(\underline{a}_{ij})\cdot(\underline{b}_{ij})= \{(c_{ij})=(\sum a_{it}b_{tj}) \mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij}\}$.

The helix-sum is commutative and the helix-product is weak associative.

The *helix-Lie Algebra* is defined as well [11], [12], [13], [21].

We study the matrices $M_{m\times n}$ where m<n. since we have analogous cases if m>n and for m=n we have the classical theory.

Several classes appropriate in helix-operations are defined [12], [13], [23].

Definition 3.3 Let $A=(a_{ij})\in M_{m\times n}$ and $s,t\in N$, $1\leq s\leq m$, $1\leq t\leq n$. We call *S-helix matrix of type s×t*, a matrix A if the $A\underline{st}$ has on the diagonal entries which are but elements. An S-helix matrix A, is called *S_o-helix matrix* if, moreover, the condition $a_{11}\dots a_{mm}\neq 0$, is valid.

Properties. The set of S-helix matrices $A=(a_{ij})\in M_{m\times n}$ with m<n, is closed under the helix-product. It has a helix-$\underline{st}$ unit matrix I_c, which is left scalar. The S_0-helix matrices X have inverses X^{-1}. Thus, if I_c is the unit matrix, we have $I_c\in X\otimes X^{-1}\cap X^{-1}\otimes X$.

In order to explain the way that helix-hyper-operation acts we claim that this replaces some elements and shifts them together along with the corresponding elements, treating them in the same way. This modulo-like procedure looks like the 'repetition' in teaching.

Example 3.4 Consider the set of 4×7 matrices on a field of real or complex numbers or on any finite field. Take

$$X=\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{11} & x_{16} & x_{17}\\ 0 & x_{22} & x_{23} & x_{24} & 0 & x_{22} & x_{27}\\ 0 & 0 & x_{33} & x_{34} & 0 & 0 & x_{33}\\ 0 & 0 & 0 & x_{44} & 0 & 0 & 0 \end{pmatrix},\quad Y=\begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} & y_{11} & y_{16} & y_{17}\\ 0 & y_{22} & y_{23} & y_{24} & 0 & y_{22} & y_{27}\\ 0 & 0 & y_{33} & y_{34} & 0 & 0 & y_{33}\\ 0 & 0 & 0 & y_{44} & 0 & 0 & 0 \end{pmatrix}$$

we have

$$X\otimes Y=\begin{pmatrix} x_{11} & \{x_{12},x_{16}\} & \{x_{13},x_{17}\} & x_{14}\\ 0 & x_{22} & \{x_{23},x_{27}\} & x_{24}\\ 0 & 0 & x_{33} & x_{34}\\ 0 & 0 & 0 & x_{44} \end{pmatrix}\cdot\begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} & y_{11} & y_{16} & y_{17}\\ 0 & y_{22} & y_{23} & y_{24} & 0 & y_{22} & y_{27}\\ 0 & 0 & y_{33} & y_{34} & 0 & 0 & y_{33}\\ 0 & 0 & 0 & y_{44} & 0 & 0 & 0 \end{pmatrix}$$

Then, denoting C_{ij} the ij entry of the result, we have

$C_{11}=\{x_{11}y_{11}\}$, $C_{12}=\{x_{11}y_{12}+\{x_{12},x_{16}\}y_{22}\}$, $C_{13}=\{x_{11}y_{13}+\{x_{12},x_{16}\}y_{23}+\{x_{13},x_{17}\}y_{33}\}$,

$C_{14}=\{x_{11}y_{14}+\{x_{12},x_{16}\}y_{24}+\{x_{13},x_{17}\}y_{34}+x_{14}y_{44}\}$, $C_{15}=\{x_{11}y_{11}\}$, $C_{16}=\{x_{11}y_{16}+\{x_{12},x_{16}\}y_{22}\}$,

$C_{17}=\{x_{11}y_{17}+\{x_{12},x_{16}\}y_{27}+\{x_{13},x_{17}\}y_{33}\}$,

$C_{21}=\{0\}$, $C_{22}=\{x_{22}y_{22}\}$, $C_{23}=\{x_{22}y_{23}+\{x_{23},x_{27}\}y_{33}\}$, $C_{24}=\{x_{22}y_{24}+\{x_{23},x_{27}\}y_{34}+x_{24}y_{44}\}$,

$C_{25}=\{0\}$, $C_{26}=\{x_{22}y_{22}\}$, $C_{27}=\{x_{22}y_{27}+\{x_{23},x_{27}\}y_{33}\}$, $C_{31}=\{0\}$, $C_{32}=\{0\}$, $C_{33}=\{x_{33}y_{33}\}$,

$C_{34}=\{x_{33}y_{34}+x_{34}y_{44}\}$, $C_{35}=\{0\}$, $E_{36}=\{0\}$, $C_{37}=\{x_{33}y_{33}\}$, $C_{41}=\{0\}$, $C_{42}=\{0\}$, $C_{43}=\{0\}$,

$C_{44}=\{x_{44}y_{44}\}$, $C_{45}=\{0\}$, $C_{46}=\{0\}$, $C_{47}=\{0\}$.

Therefore, this helix product is a set with cardinality up to 2^{11}.

In Santilli's isotheory, we focus on *small non-degenerate H_v-fields* on $(Z_n,+,\cdot)$, which satisfy the conditions:

(1) very-thin minimal, (2) weak commutative, (3) 0,1 scalars, (4) unique inverses

Thus, we enlarge the ring product by putting one more element. We cannot enlarge the result if it is 1 and we cannot put 1 in the enlargement, [10], [12], [13].

Example 3.5 Take above, n=10 and only hyper-result $2\otimes 4=\{3,8\}$. Fundamental classes: $[0]=\{0,5\}$, $[1]=\{1,6\}$, $[2]=\{2,7\}$, $[3]=\{3,8\}$, $[4]=\{4,9\}$ and $(\boldsymbol{Z}_{10},+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_5,+,\cdot)$.

Consider the 4×7 S-helix matrices on the above H_v-field and take the

$$A=\begin{pmatrix}7&2&2&6&7&7&4\\0&1&3&8&0&1&4\\0&0&3&2&0&0&3\\0&0&0&2&0&0&0\end{pmatrix},\qquad B=\begin{pmatrix}4&9&3&5&4&6&7\\0&1&0&4&0&1&2\\0&0&8&4&0&0&8\\0&0&0&3&0&0&0\end{pmatrix}$$

then, we have

$$A\otimes B=\begin{pmatrix}7&\{2,7\}&\{2,4\}&6\\0&1&\{3,4\}&8\\0&0&3&2\\0&0&0&2\end{pmatrix}\cdot\begin{pmatrix}4&9&3&5&4&6&7\\0&1&0&4&0&1&2\\0&0&8&4&0&0&8\\0&0&0&3&0&0&0\end{pmatrix}=$$

$$\begin{pmatrix}8&3+\{2,7\}&1+8\{2,4\}&5+\{2,7\}4+\{2,4\}4+8&8&2+\{2,7\}&9+\{2,7\}2+\{2,4\}8\\0&1&8\{3,4\}&4+\{3,4\}4+4&0&1&2+\{3,4\}8\\0&0&4&2+6&0&0&4\\0&0&0&6&0&0&0\end{pmatrix}=$$

$$\begin{pmatrix}8&\{0,5\}&\{3,7\}&\{2,4,7,9\}&8&\{4,9\}&\{5,9\}\\0&1&\{2,4\}&\{0,4\}&0&1&\{4,6\}\\0&0&4&8&0&0&4\\0&0&0&6&0&0&0\end{pmatrix}$$

Therefore, this helix product is a set with cardinality up to 2^9.

4. Aplications of helix-hyper-operations

Hyperstructures have applications in mathematics and other sciences. They range from bio-mathematics -conchology, inheritance- and hadronic physics or leptons, Santilli's iso-theory, to mention but a few. The hyper-structure theory is related to fuzzy theory; thus, can be applicable in linguistic, sociology, industry and production, too. The fundamental relations connect, by quotients, the H_v-structures with the corresponding classical ones, and they are used to define hyperstructures as H_v-fields, H_v-Lie algebras [3], [18], [19].

A new field in hyper-mathematics comes from Santilli's Admissibility. We transfer Santilli's theory in admissibility for representations either by using ordinary matrices and a hyper-operation on them, or using hypermatrices and ordinary operations on them. The Lie-Santilli admissibility in H_v-structures is defined as follows [3], [6], [10], [12], [22]:

Definition 4.1 Let L be H_v-vector space on F, $\varphi:F\to F/\gamma^*$, canonical, $\omega_F=\{x\in F:\varphi(x)=0\}$, 0 is zero of F/γ^*, ω_L core of $\varphi':L\to L/\varepsilon^*$, 0 zero of L/ε^*. Take $R,S\subset L$ then a *Lie-Santilli admissible hyperalgebra* is obtained by taking the hyper-operation *Lie-bracket*

$$[\ ,\ ]_{RS}: L\times L\to P(L): [x,y]_{RS}=(xR)y-(yS)x=\{xry-ysx \mid r\in R,\ s\in S\}$$

Special cases are:

(a) When only *S* is considered, then $[x,y]_S=xy-ySx$

(b) When only *R* is considered, then $[x,y]_R=xRy-yx$

According to Santilli's iso-theory, on a field $F=(F,+,\cdot)$, an isofield $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ is defined to be a field with elements $\hat{a}=a\times\hat{1}$, *isonumbers*, where $a\in F$, and $\hat{1}$ is a positive-defined element outside F, equipped with operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with the additive unit 0, and $\hat{\times}$ is a new product

$$\hat{a}\,\hat{\times}\,\hat{b}:=\hat{a}\times\hat{T}\times\hat{b},\ \text{ with }\ \hat{1}=\hat{T}^{-1},\ \forall\hat{a},\hat{b}\in\hat{F} \qquad \text{(i)}$$

called *iso-product*, for which $\hat{1}$ is the left and right unit of F,

$$\hat{1}\,\hat{\times}\,\hat{a}=\hat{a}\times\hat{1}=\hat{a},\ \forall\hat{a}\in\hat{F} \qquad \text{(ii)}$$

called *iso-unit*. The rest properties of a field are reformulated analogously.

In order to transfer this theory into the hyper-structure case we generalize only the new product $\hat{\times}$ from (i), by replacing with a hyper-operation including the old one. There is a general construction on this direction:

Construction 4.2 *General enlargement.* On a field F and its isofield $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ replace in the results of the iso-product

$$\hat{a}\,\hat{\times}\,\hat{b}=\hat{a}\times\hat{T}\times\hat{b},\ \text{ with }\ \hat{1}=\hat{T}^{-1}$$

of the element $\hat{T}$ by a set of elements $\hat{H}_{ab}=\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}$ where $\hat{x}_1,\hat{x}_2,\ldots\in\hat{F}$, containing $\hat{T}$, for all $\hat{a}\,\hat{\times}\,\hat{b}$ for which $\hat{a},\hat{b}\notin\{\hat{0},\hat{1}\}$ and $\hat{x}_1,\hat{x}_2,\ldots\in\hat{F}-\{\hat{0},\hat{1}\}$. If one of $\hat{a}$, $\hat{b}$, or both, is equal to $\hat{0}$ or $\hat{1}$, then $\hat{H}_{ab}=\{\hat{T}\}$. Thus, the new iso-hope is

$$\hat{a}\,\hat{\times}\,\hat{b}=\hat{a}\times\hat{H}_{ab}\times\hat{b}=\hat{a}\times\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}\times\hat{b},\ \forall\hat{a},\hat{b}\in\hat{F} \qquad \text{(iii)}$$

$\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ becomes *very-thin isoH_v-field.* The elements of $\hat{F}$ are called *isoH_v-numbers* or *isonumbers*.

Remarks. More important hyper-operations, are the ones where only for few pairs $(\hat{a},\hat{b})$ the result is enlarged, even more, the extra elements $\hat{x}_i$, are only few. Thus, if there exists only one pair $(\hat{a},\hat{b})$ for which

$$\hat{a}\,\hat{\times}\,\hat{b}=\hat{a}\times\{\hat{T},\hat{x}\}\times\hat{b},\ \forall\hat{a},\hat{b}\in\hat{F}$$

and the rest are ordinary, then we have a hyper-structure called *very-thin isoH_v-field.*

The assumption $\hat{H}_{ab}=\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}$, $\hat{a}$ or $\hat{b}$, is equal to $\hat{0}$ or $\hat{1}$, and $\hat{x}_i$, are not $\hat{0}$ or $\hat{1}$, give that the isoH_v-field has a scalar absorbing $\hat{0}$, a scalar $\hat{1}$, and $\forall\hat{a}\in\hat{F}$, has one inverse.

Construction 4.3 *The Living Organism Construction.* In a set G equipped with several operations we take a product (·), where (G,·) is group. Suppose that e is the unit, then we define in G, a large number of hopes (⊗) as follows:

$$e\otimes x = x\otimes e = x,\ \forall x\in G,$$

$$x\otimes y = \{xy, g_{xy1}, g_{xy2}, \ldots\},\ \forall x,y\in G\text{-}\{e\},\ \text{where}\ \ g_{xy1}, g_{xy1}, \ldots\in G\text{-}\{e\}$$

$g_{xy1}, g_{xy2}, \ldots$ depend on (x,y). Then (G,⊗) becomes an H_v-group, because it contains (G,·), which is an e-hypergroup. Moreover, if for each x,y such that xy=e, so we have x⊗y=xy, then (G,⊗) is a strong e-hypergroup.

Example 4.4 We fix the real 3×8 S_0-helix matrices

$$R=\begin{pmatrix}1&0&2&1&1&0&1&2\\0&5&1&0&5&5&0&5\\0&0&1&0&0&1&0&0\end{pmatrix},\quad S=\begin{pmatrix}1&2&0&1&1&2&1&0\\0&1&0&0&1&1&0&1\\0&0&5&0&0&5&0&0\end{pmatrix}$$

then, for the S_0-helix matrices

$$A=\begin{pmatrix}1&1&2&1&1&1&1&2\\0&1&1&0&1&3&0&1\\0&0&5&0&0&5&0&0\end{pmatrix},\quad B=\begin{pmatrix}5&3&3&5&5&3&5&0\\0&1&3&0&1&1&0&1\\0&0&1&0&0&1&0&0\end{pmatrix}$$

we have

$$[A,B]_{RS} = (A\otimes R)\otimes B - (B\otimes S)\otimes A =$$

$$\left(\begin{pmatrix}1&\{1,2\}&\{1,2\}\\0&1&\{1,3\}\\0&0&5\end{pmatrix}\begin{pmatrix}1&0&2&1&1&0&1&2\\0&5&1&0&5&5&0&5\\0&0&1&0&0&1&0&0\end{pmatrix}\right)\otimes B\ -$$

$$-\left(\begin{pmatrix}5&\{0,3,5\}&3\\0&1&\{1,3\}\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&0&1&1&2&1&0\\0&1&0&0&1&1&0&1\\0&0&5&0&0&5&0&0\end{pmatrix}\right)\otimes A\ =$$

$$\begin{pmatrix}1&\{5,6,7,10,11,12\}&\{4,5,6,7,11,12\}\\0&5&\{2,4,6.8\}\\0&0&5\end{pmatrix}\begin{pmatrix}5&3&3&5&5&3&5&0\\0&1&3&0&1&1&0&1\\0&0&1&0&0&1&0&0\end{pmatrix}\ -$$

$$-\begin{pmatrix}5&\{0,3,5,8,10,13,15\}&\{15,25,28,30\}\\0&1&\{5,6,15,16\}\\0&0&5\end{pmatrix}\begin{pmatrix}1&1&2&1&1&1&1&2\\0&1&1&0&1&3&0&1\\0&0&5&0&0&5&0&0\end{pmatrix}\ =$$

From which we obtain the final result.

We remark that we can simplify the results in the finite cases. Thus, let us see the above example on the finite ring $\boldsymbol{Z}_6$. Then, we have

$$[A,B]_{RS} = (A\otimes \mathit{R})\otimes B - (B\otimes \mathit{S})\otimes A =$$

$$\begin{pmatrix} 1 & \{0,1,4,5\} & \{0,1,4,5\} \\ 0 & 5 & \{0,2,4\} \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 5 & 3 & 3 & 5 & 5 & 3 & 5 & 0 \\ 0 & 1 & 3 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} -$$

$$-\begin{pmatrix} 5 & \boldsymbol{Z}_6 & \{0,1,3,4\} \\ 0 & 1 & \{0,3,4,5\} \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 5 & 0 & 0 & 5 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 5 & \{1,2,3,4\} & \boldsymbol{Z}_6 & 5 & \{0,3,4,5\} & \boldsymbol{Z}_6 & 5 & \{0,1,4,5\} \\ 0 & 5 & \{1,3,5\} & 0 & 5 & \{1,3,5\} & 0 & 5 \\ 0 & 0 & 5 & 0 & 0 & 5 & 0 & 0 \end{pmatrix} -$$

$$-\begin{pmatrix} 5 & \boldsymbol{Z}_6 & \boldsymbol{Z}_6 & 5 & \boldsymbol{Z}_6 & \{1,2,4,5\} & 5 & \boldsymbol{Z}_6 \\ 0 & 1 & \{0,3,4,5\} & 0 & 1 & \{0,1,2,3\} & 0 & 1 \\ 0 & 0 & 5 & 0 & 0 & 5 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \boldsymbol{Z}_6 & \boldsymbol{Z}_6 & 0 & \boldsymbol{Z}_6 & \boldsymbol{Z}_6 & 0 & \boldsymbol{Z}_6 \\ 0 & 4 & \boldsymbol{Z}_6 & 0 & 4 & \boldsymbol{Z}_6 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We remark that in helix-hyper-operations have results with big cardinality if for m×n matrices the difference n-m is big.

5 Conclusions

The mathematical model called ‘helix’, which defines a hyper-product on non-square matrices, as well, might give solutions in cases where the ordinary product is not defined. In the helix-hyper-product, every element is present and every element maintains its independence. The helix-hyper-operations on minimal H_v-fields are used in order to shorten the results. The Lie-Santilli’s theory can be represented in a new way, to give solutions on problems on living organisms, which are irreversible over time.

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