Special Issue:
Issue I: Foundations of Hadronic Mathematics

Lead Guest Editor

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Introduction

20th century mathematics underlying mainstream physical and chemical theories is local-differential, thus solely permitting the representation of point-like masses. The Italian-American scientist R. M. Santilli accepted such a mathematics for the representation of particles when the masses are at large mutual distances, thus allowing point-like approximations, as it is the case for the atomic structure. Santilli then identified clear limitation of 20th century mathematics for the representation of extended charge distributions or wavepackets in conditions of partial or total mutual penetration, as it is the case for the synthesis of the neutron from a proton and an electron in the core of a star; for the structure of nuclei, stars and black holes; for the molecular bond of two identical valence electrons in singlet coupling; and other composite systems.

When at the Department of Mathematics of Harvard University in the late 1970s, Santilli developed a series of new mathematics for the representation of extended charge distributions or wavepackets when in condition of partial or total mutual penetration, resulting in:

1. The novel, single valued- isomathematics for the representation of composite matter-systems reversible over time of with extended constituents at short mutual distances;

2. The novel, single valued genomathematics for the representation of composite matter-systems or reactions irreversible over time with extended constituents at short mutual distance;

3. The novel multi-valued hypermathematics for the representation of biological matter-systems.

Additionally, Santilli constructed their anti-Hermitean isodual images for the representation of corresponding antimatter-systems in conditions of increasing complexity. These varieties of new mathematics are today collectively addressed by the name of hadronic mathematics, in view of their applications. The special issue of AJMP on the Foundations of Hadronic Mathematics shall review the above novel mathematics and present new advances for the use in subsequent special issues devoted to its applications.

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Outline of Hadronic Mathematics, Mechanics and Chemistry as Conceived by R. M. Santilli

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Abstract: In this paper, we outline the various branches of hadronic mathematics and their applications to corresponding branches of hadronic mechanics and chemistry as conceived by the Italian-American scientist Ruggero Maria Santilli. According to said conception, hadronic mathematics comprises the following branches for the treatment of matter in conditions of increasing complexity: 1) 20th century mathematics based on Lie's theory; 2) IsoMathematics based on Santilli's isotopies of Lie's theory; 3) GenoMathematics based on Santilli's formulation of Albert's Lie-admissibility; 4) HyperMathematics based on a multi-valued realization of genomathematics with classical operations; and 5) HyperMathematics based on Vougiouklis H_v hyperstructures expressed in terms of hyperoperations. Additionally, hadronic mathematics comprises the anti-Hermitean images (called isoduals) of the five preceding mathematics for the description of antimatter also in conditions of increasing complexity. The outline presented in this paper includes the identification of represented physical or chemical systems, the main mathematical structure, and the main dynamical equations per each branch. We also show the axiomatic consistency of various branches of hadronic mathematics as sequential coverings of 20th century mathematics; and indicate a number of open mathematical problems. Novel physical and chemical applications permitted by hadronic mathematics are presented in subsequent collections.

Keywords: Santilli Isomathematics, Genomathematics, Hypermathematics

1. 20th Century Mathematics, Mechanics and Chemistry

1.1. Represented Systems

Single-valued, closed-isolated, time-reversible systems of point-like particles moving in vacuum solely under action at a distance Hamiltonian interactions, such as the structure of atoms and molecules.

1.2. Main Mathematical Structure

Basic unit

$$I = +1 \tag{1}$$

Basic numeric fields n = real, complex, quaternionic numbers

$$F(n, \times, 1), n \tag{2}$$

Basic Associative product

$$nm = n \times m, 1 \times n = n \times 1 = n \; \forall n \in F \tag{3}$$

Measurement units of time, energy, etc. all positive
Ordinary functional analysis $f(r) \in F$,
Ordinary differential calculus
Conventional Lie theory

$$[X_i, X_j] = X_i \times X_j - X_j \times X_i == C_{ij}^k \times X_k, \tag{4}$$

$$A(w) = e^{X \times w \times i} \times A(0) \times e^{-i \times w \times X}. \tag{5}$$

Euclidean geometry and topology

$$E(r, \delta, 1), r = (r^k), k = 1,2,3, \delta = Diag.(1,1,1), \tag{6}$$

$$r^2 = r^i \times \delta_{ij} \times r^j = r_1^2 + r_2^2 + r_3^2 \in F, \tag{7}$$

Minkowskian geometry

$$M(x, \eta, I): x = (x^\mu), \mu = 1,2,3,4, x^4 = t, \tag{8}$$

$$\eta = Diag.(+1, +1, +1, -c^2), \tag{9}$$

$$x^2 = x^\mu \times \eta_{\mu\nu} \times x^\nu = x_1^2 + x_2^2 + x_3^2 - t^2c^2 \in F, \quad (10)$$

Riemannian geometry

$$R(x, g(x), I): x = (x^\mu), \mu = 1,2,3,4, x^4 = t, \quad (11)$$

$$x^2 = x^\mu \times g(x)_{\mu\nu} \times x^\nu \in F \quad (11)$$

$$x^2 = x^\mu \times g(x)_{\mu\nu} \times x^\nu \in F \quad (12)$$

Symplectic geometry.

$$\omega = dr^k \wedge dp_k \quad (13)$$

1.3. Dynamical equations

Newton equation

$$m \times \frac{dv}{dt} - F^{SA}(t, r, v,) = 0, \quad (14)$$

Variational principle

$$\delta A = \delta \int (p_k \times dr^k - H \times dt) = 0. \quad (15)$$

Hamilton's equations without external terms

$$\frac{dr^k}{dt} = \frac{\partial H(r,p)}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H(r,p)}{\partial r^k}, \quad (16)$$

Hilbert space H over C with states $|\psi>$ over (C)
Expectation value of a Hermitean operator A

$$< A > = < \psi| \times A \times |\psi> \in C, \quad (17)$$

Heisenberg equation

$$i \times \frac{dA}{dt} = [A, H] = A \times H - H \times A, \quad (18)$$

Schrödinger equations

$$H \times |\psi> = E \times |\psi> \quad (19)$$

$$p \times |\psi> = -i \times \partial_r |\psi> \quad (20)$$

Dirac equation

$$(\eta^{\mu\nu} \times \gamma_\mu \times p_\nu - i \times m \times c) \times |\psi> = 0. \quad (21)$$

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu \times \gamma_\nu + \gamma_n u \times \gamma_\mu = 2 \times \eta_{\mu\nu}, \quad (22)$$

Comments and References

The literature on 20th century mathematics, mechanics and chemistry is so vast and so easily identifiable to discourage discriminatory partial listings.

2. Isomathematcs, Isomechanics and Isochemistry

2.1. Represented Systems [1-5]

Single-value, closed-isolated, time-reversible system of extended-deformable particles with action at a distance Hamiltonian and contact non-Hamiltonian interactions, such as the structure of hadrons, nuclei and stars, in the valence electron bonds and other systems.

2.2. Main Mathematical Structure s [1-5]

Santilli IsoUnit $\hat{I}$ and isotopic element $\hat{T}$[1]

$$\hat{I} = \hat{I}(r, p, a, \psi, \ldots.) = 1/\hat{T}(r, p, a, \psi, \ldots.) > 0, \quad (23)$$

Santilli IsoFields

$$\hat{F}(\hat{n}, \hat{\times}, \hat{I}), \hat{n} = n \times \hat{I}_{,,} \quad (24)$$

Santilli isoproduct

$$\hat{n} \hat{\times} \hat{m} = \hat{n} \times \hat{T} \times \hat{m} \in \hat{F}, \quad (25)$$

$$\hat{I} \hat{\times} \hat{n} = \hat{n} \hat{\times} \hat{I} = \hat{n} \; \forall \hat{n} \in \hat{F}, \quad (26)$$

Representation via the isotopic element of extended-deformable particles under non-Hamiltonian interactions

$$\hat{T} = Diag.(\tfrac{1}{n_1^2}, \tfrac{1}{n_2^2}, \tfrac{1}{n_3^2}) \times e^{\Gamma(r,p,\psi,\partial\psi,\ldots)} \quad (27)$$

IsoCoordinates $\hat{r} = r \times \hat{I} \in \hat{F}$,
IsoFunctional analysis $\hat{f}(\hat{r}) = f(\hat{r}) \times \hat{I} \in \hat{F}$,
IsoDifferential Calculus

$$\hat{d}\hat{r} = dr + r \times \hat{T} \times d\hat{I}, \quad (28)$$

$$\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I} \times \frac{\partial \hat{f}(\hat{r})}{\partial \hat{r}}, \quad (29)$$

Santilli Lie-Isotopic Theory

$$[X_i \hat{,} X_j] = X_i \hat{\times} X_j - X_j \hat{\times} X_i == C_{ij}^k(r, p, \ldots) \times X_k, \quad (30)$$

$$A(w) = \hat{e}^{X \times w \times i} \hat{\times} A(0) \hat{\times} \hat{e}^{-i \times w \times X}. \quad (31)$$

Santilli Iso-Euclidean Geometry

$$\hat{E}(\hat{r}, \hat{\delta}, \hat{I}), \hat{\delta}(r, p, z, \psi, \ldots) = \hat{T}(r, p, z, \psi, \ldots) \times \delta, \quad (32)$$

$$\hat{T} = Diag.(1/n_1^2, 1/n_2^2, 1/n_3^2), \quad (33)$$

$$\hat{r}^{\hat{2}} = \hat{r}^i \hat{\times} \hat{\delta}_{ij} \hat{\times} \hat{r}^j = (\frac{r_1^2}{n_1^2} + \frac{r_2^2}{n_2^2} + \frac{r_3^2}{n_3^2}) \times \hat{I} \in \hat{F}, \quad (34)$$

Santilli Iso-Minkowskian Geometry

$$\hat{M}(\hat{x}, \hat{\eta}, \hat{I}): \hat{x} = (\hat{x}^\mu), \mu = 1,2,3,4, x_4 = t, \quad (35)$$

$$\hat{\eta}(x, \psi, \ldots) = \hat{T}(x, \psi, \ldots) \times \eta, \quad (36)$$

$$\hat{T} = Diag.(1/n_1^2, 1/n_2^2, 1/n_3^2, 1/n_4^2), \quad (37)$$

$$\hat{x}^{\hat{2}} = \hat{x}^\mu \hat{\times} \hat{\eta}_{\mu\nu} \hat{\times} \hat{x}^\nu = (\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} - t^2 \frac{c^2}{n_4^2}) \times \hat{I} \in \hat{F}, \quad (38)$$

[1]See Santilli's curriculum
http://www.world-lecture-series.org/santilli-cv
Prizes and Nominations
http://www.santilli-foundation.org/santilli-nobel-nominations.html
and scientific archive
http://www.santilli-foundation.org/news.html

Santilli Iso-Riemannian Geometry

$$\hat{R}(\hat{x},\hat{g},\hat{I}): \hat{g} = \hat{T}(x,v,\dots) \times g(x), \quad (39)$$

$$\hat{x}^{\hat{2}} = (\frac{g_{11}}{n_1^2} + \frac{g_{22}}{n_2^2} + \frac{g_{33}}{n_3^2} - \frac{g_{44}}{n_4^2}) \times \hat{I} \in \hat{F}, \quad (40)$$

Santilli Iso-Symplectic Geometry

$$\hat{\omega} = \hat{d}\hat{r}^k \hat{\wedge} \hat{d}\hat{p}_k \quad (41)$$

2.3. IsoDynamical IsoEquations s [1-5]

Newton-Santilli IsoEquation

$$\hat{m} \hat{\times} \frac{\hat{d}\hat{v}}{\hat{d}\hat{t}} - F^{SA}(t,r,p) = m \times \frac{dv}{dt} - F^{SA}(t,r,p) - F^{NSA}(t,r,p,\dots) = 0, \quad (42)$$

IsoVariational principle

$$\hat{\delta}\hat{A} = \hat{\delta}\hat{\int} (\hat{p}_k \hat{\times} \hat{d}\hat{r}^k - \hat{H} \hat{\times} \hat{d}\hat{t}) = 0. \quad (43)$$

Hamilton-Santilli IsoEquations

$$\frac{\hat{d}\hat{r}^k}{\hat{d}\hat{t}} = \frac{\hat{\partial}\hat{H}(\hat{r},\hat{p})}{\hat{\partial}\hat{p}_k}, \quad \frac{\hat{d}\hat{p}_k}{\hat{d}\hat{t}} = -\frac{\hat{\partial}\hat{H}(\hat{r},\hat{p})}{\hat{\partial}\hat{r}^k}, \quad (44)$$

Iso-Hilbert space $\hat{H}$ over C with states $|\hat{\psi}>$ over the isofield $\hat{C}$[2]

IsoExpectation value of a Hermitean operator $\hat{A}$ on $\hat{H}$

$$<\hat{A}> = <\hat{\psi}| \hat{\times} \hat{A} \hat{\times} |\hat{\psi}> \in \hat{C} \quad (45)$$

Heisenberg-Santilli IsoEquation

$$\hat{\imath} \hat{\times} \frac{\hat{d}\hat{A}}{\hat{d}\hat{t}} = [\hat{A}\hat{,}\hat{H}] = \hat{A} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{A} = \hat{A} \times \hat{T}(\hat{\psi},\dots) \times \hat{H}(\hat{r},\hat{p}) - \hat{H}(\hat{r},\hat{p}) \times \hat{T}(\hat{\psi},\dots) \times \hat{A} \quad (46)$$

Schrödinger-Santilli IsoEquation

$$\hat{H} \hat{\times} |\hat{\psi}> = \hat{H}(\hat{r},\hat{p}) \times \hat{T}(\hat{\psi},\hat{\partial}\hat{\psi},\dots) \times |\hat{\psi}> = \hat{E} \hat{\times} |\hat{\psi}> = E \times |\hat{\psi}>, \quad (47)$$

$$\hat{p} \hat{\times} |\hat{\psi}> = -\hat{\imath} \hat{\times} \hat{\partial}_{\hat{r}}|\hat{\psi}> = -i \times \hat{I} \times \partial_{\hat{r}}|\hat{\psi}>, \quad (48)$$

Dirac-Santilli IsoEquation

$$(\hat{\eta}^{\mu\nu} \hat{\times} \hat{\gamma}_\mu \hat{\times} \hat{p}_\nu - \hat{\imath} \hat{\times} \hat{m} \hat{\times} \hat{c}) \hat{\times} |\hat{p}si> = 0. \quad (49)$$

$$\{\hat{\gamma}_\mu\hat{,}\hat{\gamma}_\nu\} = \hat{\gamma}_\mu \hat{\times} \hat{\gamma}_\nu + \hat{\gamma}_\nu \hat{\times} \hat{\gamma}_\mu = \hat{2} \hat{\times} \hat{\eta}_{\mu\nu} = 2 \times \hat{\eta}_{\mu\nu}, \quad (50)$$

2.4. Comments and References

As it is well known, the local-differential calculus of 20th century mathematics can solely represent a finite set of isolated dimensionless points. In view of this structural feature, Newton formulated his celebrated equations (14) for *massive points,* resulted in a conception of nature that was adopted by Galileo and Einstein, became the dominant notion of 20th century sciences, and was proved to be valid for classical or quantum particles moving in vacuum at large mutual distances, such as for our planetary system or the atomic structure.

However, when bodies move within physical media, such as for a spaceship during re-entry in our atmosphere or for a proton in the core of a star, point-like abstractions of particles became excessive, e.g., because a macroscopic collection of point-particles cannot have entropy (since all known Hamiltonian interactions are invariant under time reversal), with consequential violation of thermodynamical laws and other insufficiencies.

Besides the clear identification of these insufficiencies, the first historical contribution by the Italian-American scientist Ruggero Maria Santilli (see Footnote 1) has been the *generalization of 20th century mathematics into such a form to admit a time invariant representation of extended, and therefore deformable particles under conventional Hamiltonian as well as contact non-Hamiltonian interactions, with implications for all quantitative sciences.*

The above central objective was achieved in monographs [1] originally written by Santilli during his stay at MIT from 1974 to 1977 (where they appeared as MIT preprints). Monographs [1] were then completed by Santilli during his stay at Harvard Universityfrom 1977 to 1982 under DOE support, and released for publication only following the delivery at Harvard of a post Ph. D. seminar Course in the field.

The representation of extended-deformable bodies moving within physical media was achieved via an axiom-preserving lifting, called *isotopy,* of the conventional associative product $AB = A \times B$ between generic quantities A, B (such as numbers,functions, matrices, operators, etc.) into the form $A \hat{\times} B = A \times \hat{T} \times B$, Eq. (25). Conventional interactions are represented via conventional Hamiltonian, while actual shape and non-Hamiltonian interactions are represented via realization of the quantity $\hat{T}$, called *isotopic element,* of the type (27).

Santilli then achieved in monographs [1] the axiom-preserving isotopies of the various branches of Lie's theory, e.g., Eqs. (30), (31,) including their elaboration via the initiation of the isotopies of functional analysis. In particular, Santilli showed that the isotopies of the rotational symmetry $SO(3)$ characterized by isotopic element (27) do represent extended, generally non-spherical and deformable bodies. Finally, Santilli proved in Vol. II of Ref. [1] the

[2]As shown in the seminal paper [6] of 1982, but vastly ignored for the past four decades, isomechanics formulated on iso-Hilbert spaces over isofields eliminates the divergencies of quantum mechanics and related scattering theories. This important feature is primarily due to the fact that, for all physical and chemical applications worked out to date, the isounit $\hat{I} = 1/\hat{T} > 0$ must have a large value of the exponential type (27) and, consequently, the isotopic element $\hat{T}$ must have a very small value. This occurrence eliminates the singularity of the Dirac delta "distribution" when lifted to the Dirac-Myung-Santilli delta "isofunction" as shown by the realization of the type

$$\hat{\delta}(r - r_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ik\hat{T}(r-r_0)}\, dk,$$

with $\hat{T} = \frac{N}{r-r_0}$, $N \ll 1$. Similarly, perturbative and other series with Hermitean operators that are divergent or slowly convergent in quantum mechanics can be lifted into isoseries of the type

$$A(w) = \hat{I} + \frac{w(A\hat{T}H - H\hat{T}A)}{1!} + \cdots$$

that are manifestly convergent for $w > 1$ but $\hat{T} \ll w$.As shown by A. O. E. Animalu and R. M. Santilli in five papers published proceedings [25], the above lack of divergences carries over to the covering of the scattering theory known as isoscattering theory, by therefore achieving numerical results without the use of infinities for the renormalization of divergent series.

significance of his Lie-isotopic theory by showing that it characterizes the Birkhoffian covering of classical Hamiltonian mechanics and its "direct universality" for the representation of all possible, non-singular, generally non-Hamiltonian Newtonian systems in the frame of the experimenter, which direct universality was subsequently proved to hold also for isotopic operator theories. The above advances were formulated on an ordinary numeric field.

Subsequently, Santilli discovered in 1993 [2] that the axioms of numeric fields with characteristic zero do not necessarily require that the basic multiplicative unit is the trivial number $+1$, since said axioms admit arbitrary generalized units, today called *Santilli isounits,* provided that they are positive-definite and are the inverse of the isotopic element, $\hat{I} = 1/\hat{T} > 0$. This second historical discovery identified new numbers today known as *Santilli isoreal, isocomplex and isoquaternionic numbers* of the First (Second) kind when the isounit is outside (an element of) the original field. This discovery prompted a flurry of reformulation over Santilli isofields of all preceding isotopies, including most importantly the reformulation of Santilli's Lie-isotopic theory.

Despite the above momentous advances, Santilli remained dissatisfied because the isotopic formulations of the early 1990s were not invariant under their time evolution, thus being unable to predict the same numerical values under the same conditions at different times. Since the entire 20th century mathematics had been isotonically lifted by the early 1990s, Santilli was left with no other choice than that of reinspecting the Newton-Leibnitz differential calculus by discovering that, contrary to a popular belief in mathematics and physics for some four centuries, the differential calculus is indeed dependent on the basic multiplicative unit. In this way, Santilli achieved in memoir [3] of 1996 the third historical discovery according to which the ordinary differential calculus needs generalizations of the type (28), (29) whenever the isounit depends on the local variable of differentiation. This discovery signaled the achievement of mathematical maturity of isomathematics that permitted numerous advances in physics and chemistry as well as novel industrial applications.

All in all, Santilli has written about 150 papers on the isotopies of all various aspects of 20th century mathematics. These contributions are reported in monographs [4] of 1995 that remain to this day the most comprehensive presentation on isotopies. In the subsequent series of monographs [5] of 2008, Santilli introduces the names of *Hadronic Mathematics, Mechanics and Chemistry* which have been adopted for this review due to their wide acceptance.

Numerous authors have made important contributions in Santilli isomathematics, among whom we quote: the mathematician H. C. Myung who initiated (with R. M. Santilli) [6] the isotopies of Hilbert Spaces, including the momentous elimination of the divergencies of quantum mechanics under sufficiently small values of the isotopic element $\hat{T}$; the mathematicians D. S. Sourlas and G. T. Tsagas [7] who conducted in 1993 the first comprehensive study of the Lie-Santilli isotheory; the theoretician J. V. Kadeisvili [8] who presented systematic studies of Santilli's isotopies of 20th century geometries and relativities; the mathematician Chun-Xuan Jiang [9] who conducted in 2001 systematic studies of Santilli IsoNumber Theory; the mathematicians R. M. Falcon Ganfornina and J. Nunez Valdes who wrote in 2001 the now historical, first mathematically rigorous treatment of Santilli isotopies [10], and the historical achieved isotopology [11] which provides the ultimate mathematical structure of the Newton-Santilli isoequations (42) for extended-deformable particles under Hamiltonian and non-hamiltonian interactions achieved in memoir [3]; the mathematician S. Georgiev who wrote one of the most monumental and important mathematical works in scientific history [12], by showing that Santilli's IsoDifferential Calculus implies a variety of fully consistent coverings of 20th century mathematics; the mathematician A. S. Muktibodh [13] who presented the first known generalization of Santilli isonumber theory for the case of characteristic $p \neq 0$; the physicists I. Gandzha and J. Kadeisvili who presented in 2011 [14] a comprehensive review of Santilli isomathematics and its applications in physics and chemistry; plus additional seminal advances presented in the subsequent papers of this collection.

3. Genomathematics, Genomechanics and Genochemistry

3.1. Represented Systems s [1-5]

Single-valued, time-irreversible system of extended-deformable particles under action at a distance Hamiltonian and contact non-Hamiltonian interactions, as occurring in nuclear reactions, biological structures and chemical reactions.

3.2. Main Mathematical Structure s [1-5]

Santilli Forward GenoUnit

$$\hat{I}^{>} = \hat{I}^{>}(t^{>} r^{>}, p^{>}, a^{>}, \psi^{>}, \partial^{>}\psi^{>}, \dots.) = 1/\hat{T}^{>} > 0, \quad (51)$$

Santilli Backward GenoUnit

$$^{<}\hat{I} = {^{<}\hat{I}}(^{<}r, ^{<}p, ^{<}a, ^{<}\psi, ^{<}\partial^{<}\psi, \dots.) = 1/^{<}\hat{T} > 0, \quad (52)$$

Condition for time-irreversibility

$$\hat{I}^{>} \neq \ ^{<}\hat{I} \quad (53)$$

Forward GenoFields

$$\hat{F}^{>}(\hat{n}^{>}, >, \hat{I}^{>}), \hat{n}^{>} = n \times \hat{I}^{>} \quad (54)$$

Backward GenoFields

$$^{<}\hat{F}(^{<}\hat{n}, <, ^{<}\hat{I}), \quad ^{<}\hat{n} = {^{<}\hat{I}} \times n, \quad (55)$$

Forward GenoProduct

$$\hat{n} > \hat{m} = \hat{n}^{>} \times \hat{T}^{>} \times \hat{m}^{>} \in \hat{F}^{>}, \quad (56)$$

$$\hat{I}^{>} > \hat{n}^{>} = \hat{n}^{>} > \hat{I}^{>} = \hat{n}^{>} \ \forall \, \hat{n}^{>} \in \hat{F}^{>} \quad (57)$$

Backward Genoproduct

$$^{<}\hat{n} <^{<} \hat{m} =^{<} \hat{n} \times^{<} \hat{T} \times^{<} \hat{m} \in^{<} \hat{F}, \quad (58)$$

$$^{<}\hat{I} <^{<} \hat{n} =^{<} \hat{n} <^{<} \hat{I} =^{<} \hat{n} \,\forall \quad ^{<}\hat{n} \in^{<} \hat{F}, \quad (59)$$

Representation of forward extended-deformable particles under non-Hamiltonian interactions

$$\hat{T}^{>} = Diag.(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2})^{>} \times e^{\Gamma(t,r,p,\psi,\partial\psi,\dots)^{>}} \quad (60)$$

Forward GenoCoordinates

$$\hat{r}^{>} = r \times \hat{I}^{>} \in \hat{F}^{>}, \quad (61)$$

Backward GenoCoordinates

$$^{<}\hat{r} =^{<} \hat{I} \times r \in^{<} \hat{F}, \quad (62)$$

Forward GenoFunctional analysis

$$\hat{f}^{>}(\hat{r}^{>}) = f(\hat{r}^{>}) \times \hat{I}^{>} \in \hat{F}^{>}, \quad (63)$$

Backward GenoFunctional analysis

$$^{<}\hat{f}(^{<}\hat{r}) = f(^{<}\hat{r}) \times^{<} \hat{I} \in^{<} \hat{F}, \quad (64)$$

Forward GenoDifferential Calculus

$$\hat{d}^{>}\hat{r}^{>} = dr + r \times \hat{T}^{>} \times d\hat{I}^{>}, \quad (65)$$

$$\frac{\hat{\partial}^{>}\hat{f}^{>}(\hat{r}^{>})}{\hat{\partial}^{>}\hat{r}^{>}} = \hat{I}^{>} \times \frac{\partial \hat{f}^{>}(\hat{r}^{>})}{\partial \hat{r}^{>}}, \quad (66)$$

Backward GenoDifferential Calculus

$$^{<}\hat{d}^{<}\hat{r} = dr + r \times^{<} \hat{T} \times d^{<}\hat{I}, \quad (67)$$

$$\frac{^{<}\hat{\partial}^{<}\hat{f}(^{<}\hat{r})}{^{<}\hat{\partial}^{<}\hat{r}} =^{<} \hat{I} \times \frac{\partial^{<}\hat{f}(^{<}\hat{r})}{\partial^{<}\hat{r}}, \quad (68)$$

Santilli Lie-Admissible Theory

$$(X_i \hat{,} X_j) = X_i < X_j - X_j > X_i = C_{ij}^k(t,r,p,\psi,\dots) \times X_k, \quad (69)$$

$$A(w) = \hat{e}_{>}^{X \times w \times i} > A(0) <_{<} \hat{e}^{-i \times w \times X}. \quad (70)$$

Santilli Forward Geno-Euclidean Geometry

$$\hat{E}^{>}(\hat{r}^{>}, \hat{\delta}^{>}, \hat{I}^{>}), \hat{\delta}^{>}(t,r,p,\psi,\dots) = \hat{T}^{>}(t,r,p,\psi,\dots) \times \delta, \quad (71)$$

$$\hat{r}^{>\hat{2}} = \hat{r}^{>i} > \hat{\delta}_{ij}^{>} > \hat{r}^{>j} \in F^{>}, \quad (72)$$

$$\hat{\delta}^{>} \neq \hat{\delta}^{>\, tranp} \quad (73)$$

Santilli Backward Geno-Euclidean Geometry

$$^{<}\hat{E}(^{<}\hat{r}, ^{<}\hat{\delta}, ^{<}\hat{I}), \quad ^{<}\hat{\delta}(t,r,p,\psi,\dots) =^{<} \hat{T}(t,r,p,\psi,\dots) \times \delta, \quad (74)$$

$$^{<\hat{2}}\hat{r} =^{<i} \hat{r} <^{<} \hat{\delta}_{ij} <^{<j} \hat{r} \in^{<} F, \quad (75)$$

$$^{<}\hat{\delta} \neq^{<\, transp} \hat{\delta} \quad (76)$$

Santilli Forward Geno-Minkowskian Geometry ($\mu =$ 1,2,3,4)

$$\hat{M}^{>}(\hat{x}^{>}, \hat{\eta}^{>}, \hat{I}^{>}): \hat{x}^{>} = (\hat{x}^{>\mu}), x_4^{>} = t^{>}, \quad (77)$$

$$\hat{\eta}^{>}(x,\psi,\dots) = \hat{T}^{>}(x,\psi,\dots) \times \eta, \quad (78)$$

$$\hat{x}^{>2} = \hat{x}^{>\mu} > \hat{\eta}_{\mu\nu}^{>} > \hat{x}^{>} \in \hat{F}^{>}, \quad (79)$$

$$\hat{\eta}^{>} \neq \hat{\eta}^{>\, transp} \quad (80)$$

Santilli Backward Geno-Minkowskian Geometry ($\mu =$ 1,2,3,4,)

$$^{<}\hat{M}(^{<}\hat{x}, ^{<}\hat{\eta}, ^{<}\hat{I}): \quad ^{<}\hat{x} = (\hat{x}^{\mu}), \quad ^{<}x_4 =^{<} t, \quad (81)$$

$$^{<}\hat{\eta}(x,v,\dots) =^{<} \hat{T}(x,v,\dots) \times \eta, \quad (82)$$

$$^{<}\hat{x}^{<2} =^{<\mu} \hat{x} <^{<} \hat{\eta}_{\mu\nu} <^{<\nu} \hat{x} \in^{<} \hat{F}, \quad (83)$$

$$^{<}\hat{\eta} \neq^{<\, transp} \hat{\eta} \quad (84)$$

Santilli Forward Geno-Riemannian Geometry

$$\hat{R}^{>}(\hat{x}^{>}, \hat{g}^{>}, \hat{I}^{>}): \hat{g}^{>} = \hat{T}^{>}(x,v,\dots) \times g(x), \quad (85)$$

$$\hat{x}^{>2} = \hat{x}^{>\mu} > \hat{g}_{\mu\nu}^{>} > \hat{x}^{>} \in \hat{F}^{>}, \quad (86)$$

$$\hat{g}^{>} \neq \hat{g}^{>\, transp} \quad (87)$$

Santilli Backward Geno-Riemannian Geometry

$$^{<}\hat{R}(^{<}\hat{x}, ^{<}\hat{g}, ^{<}\hat{I}): \quad ^{<}\hat{g} =^{<} \hat{T}(x,v,\dots) \times g(x), \quad (88)$$

$$^{<}\hat{x}^{<2} =^{<\mu} \hat{x} <^{<} \hat{g}_{\mu\nu} <^{<\nu} \hat{x} \in^{<} \hat{F}, \quad (89)$$

$$^{<}\hat{g} \neq^{<\, transp} \hat{g} \quad (90)$$

Santilli Forward Geno-Symplectic Geometry

$$\hat{\omega}^{>} = \hat{d}^{>}\hat{r}^{>k} \,\hat{\wedge}^{>}\, \hat{d}^{>}\hat{p}_k^{>} \quad (91)$$

Santilli Backward Geno-Symplectic Geometry

$$^{<}\hat{\omega} =^{<} \hat{d}^{<}\hat{r}^{<} \,\hat{\wedge}^{<}\, \hat{d}^{<}\hat{p}_k \quad (92)$$

3.3. GenoDynamical GenoEquations s [1-5]

Newton-Santilli Forward GenoEquation

$$\hat{m}^{>} > \frac{\hat{d}^{>}\hat{v}^{>}}{\hat{d}^{>}\hat{t}^{>}} - F^{>SA}(t,r,p) = [m \times \frac{dv}{dt}]^{>} - F^{SA>}(t,r,p) - F^{NSA>}(t,r,p,\dots) = 0, \quad (93)$$

Newton-Santilli Backward GenoEquation

$$^{<}\hat{m} < \frac{^{<}\hat{d}^{<}\hat{v}}{^{<}\hat{d}^{<}\hat{t}} - ^{<SA}F(t,r,p) = =^{<} [m \times \frac{dv}{dt}] - ^{<SA}F(t,r,p) - ^{<NSA}F(t,r,p,\dots) = 0, \quad (94)$$

Forward GenoVariational principle

$$\hat{\delta}^{>}\hat{A}^{>} = \hat{\delta}^{>} \hat{\int}^{>} (\hat{p}_k^{>} > \hat{d}^{>}\hat{r}^{>k} - \hat{H}^{>} > \hat{d}^{>}\hat{t}^{>}) = 0. \quad (95)$$

Backward GenoVariational principle

$$^{<}\hat{\delta}^{<}\hat{A} =^{<} \hat{\delta}^{<} \hat{\int} \, (^{<}\hat{p}_k <^{<} \hat{d}^{<}\hat{r}^{k} - ^{<}\hat{H} <^{<} \hat{d}^{<}\hat{t}) = 0. \quad (96)$$

Forward Hamilton-Santilli GenoEquations

$$[\frac{\hat{d}\hat{r}^k}{\hat{d}\hat{t}}]^> = [\frac{\hat{\partial}\hat{H}(\hat{r},\hat{p})}{\hat{\partial}\hat{p}_k}]^>, \quad [\frac{\hat{d}\hat{p}_k}{\hat{d}\hat{t}}]^> = -[\frac{\hat{\partial}\hat{H}(\hat{r},\hat{p})}{\hat{\partial}\hat{r}^k}]^>, \quad (97)$$

Backward Hamilton-Santilli GenoEquations

$$^<[\frac{\hat{d}\hat{r}^k}{\hat{d}\hat{t}} = \frac{\hat{\partial}\hat{H}(\hat{r},\hat{p})}{\hat{\partial}\hat{p}_k}], \quad ^<[\frac{\hat{d}\hat{p}_k}{\hat{d}\hat{t}}] = -^<[\frac{\hat{\partial}\hat{H}(\hat{r},\hat{p})}{\hat{\partial}\hat{r}^k}], \quad (98)$$

Forward Geno-Hilbert space $\hat{H}^>$ with states $|\hat{\psi}^> >$ over the isofield $\hat{C}^>$

GenoExpectation value of a Hermitean operator $\hat{A}$ on $\hat{H}^>$

$$< \hat{A}^> >=<^< \hat{\psi}| < \hat{A}^> > |\hat{\psi}^> >\in \hat{C} \quad (99)$$

Heisenberg-Santilli GenoEquation[3]

$$\hat{\imath} \hat{\times} \frac{\hat{d}\hat{A}}{\hat{d}\hat{t}} = (\hat{A},\hat{H}) = \hat{A} < \hat{H} - \hat{H} > \hat{A} = A \times^< T(\hat{\psi},\ldots) \times \hat{H}(\hat{r},\hat{p}) - \hat{H}(\hat{r},\hat{p}) \times \hat{T}^>(\hat{\psi},\ldots) \times \hat{A} \quad (100)$$

Forward Schrödinger-Santilli GenoEquation

$$\hat{H}^> > |\hat{\psi}^> >= \hat{H}^>(\hat{r},\hat{p}) \times \hat{T}^>(\hat{\psi},\hat{\partial}\hat{\psi},\ldots) \times |\hat{\psi}^> >= \hat{E}^> > |\hat{\psi}^> >= E^> \times |\hat{\psi}^> >, \quad (101)$$

$$\hat{p}^> > |\hat{\psi}^> >= -\hat{\imath}^> > \hat{\partial}_{\hat{r}}^> |\hat{\psi}^> >= -i \times \hat{I}^> \times \partial_{\hat{r}} |\hat{\psi}^> >, \quad (102)$$

Backward Schrödinger-Santilli GenoEquation

$$<^< \hat{\psi}| <^< \hat{H} =<^< \hat{\psi}| \times^< \hat{T}(\hat{\psi},\hat{\partial}\hat{\psi},\ldots) \times^< \hat{H}(\hat{r},\hat{p}) = <^< \hat{\psi}| <^< \hat{E} =<^< \hat{\psi}| \times^< E, \quad (103)$$

$$<^< \hat{\psi}| <^< \hat{p} = -<^< \hat{\psi}| <^< \hat{\imath} <_{\hat{r}}^< \hat{\partial} = -i \times <^< \hat{\psi}|_{\hat{r}}^< \partial \times^< \hat{I} \quad (104)$$

Forward Dirac-Santilli IsoEquation

$$(\hat{\eta}^{>\mu\nu} > \hat{\gamma}_\mu^> > \hat{p}_\nu^> - \hat{\imath}^> > \hat{m}^> > \hat{c}^>) > |\hat{p}si^> >= 0. \quad (105)$$

$$\{\hat{\gamma}_\mu \hat{,} \hat{\gamma}_\nu\}^> = [\hat{\gamma}_\mu \hat{\times} \hat{\gamma}_\nu + \hat{\gamma}_\nu \hat{\times} \hat{\gamma}_\mu]^> = \hat{2}^> > \hat{\eta}_{\mu\nu}^>, \quad (106)$$

Backward Dirac-Santilli GenoEquation

$$<^< \hat{\psi}| < (^<\hat{p}_\nu <^< \hat{\gamma}_\mu <^{<\mu\nu} \hat{\eta} -^< \hat{\imath} <^< \hat{m} <^< \hat{c}) = 0. \quad (107)$$

$$^<\{\hat{\gamma}_\mu \hat{,} \hat{\gamma}_\nu\} =^< [\hat{\gamma}_\mu \hat{\times} \hat{\gamma}_\nu + \hat{\gamma}_\nu \hat{\times} \hat{\gamma}_\mu] =^< \hat{2} <^< \hat{\eta}_{\mu\nu} = 2 \times^< \hat{\eta}_{\mu\nu}, \quad (108)$$

3.4. Comments and References

As it is also well known, all 20th century mathematical, physical or chemical formulations are reversible over time. Following research over half a century initiated during his Ph. D. studies at the University of Torino, Italy, in the mid 1960s [15, 17-23,4,5], R. M., Santilli has made the additional historical discovery of the first and only known, axiomatically consistent, generalization of 20th century mathematics as well as of its covering isomathematcs into a form embedding irreversibility over time in ordered forward and backward units, in corresponding ordered forward and backward products and, consequently, in all subsequent mathematical structures, resulting in the new mathematics nowadays known as *Santilli forward and backward genomathematics* with corresponding physical and chemical theories for the representation of irreversible processes.

Since the reversibility over time of 20th century theories can be reduced to the invariance under anti-Hermiticity of the Lie product between Hermitean operators, $[a,b] = ab - ba = -[a,b]^\dagger$, Santilli presented in 1967 [15] the first known (p, q)-deformation of the Lie product $(a,b) = pab - qba$, where p, q are scalars and the product ab is generally non-associative. Following an intense search in European mathematical libraries, Santilli discovered that the new product verifies the axiom of *Lie-admissibility* by the American mathematician A, A, Albert [16] in the sense that the attached anti-symmetric product $[a,b] = (a,b) - (b,a)$ verifies the axioms of a Lie algebra.

Since spaceship during re-entry are notoriously irreversible over time, Santilli was invited by the Center for Theoretical Physics of the University of Miami, Florida, under NASA support, where he moved with his wife Carla and newly born daughter Luisa inAugust 1967, and published a number of additional works in Lie-admissibility, including the first known Lie-admissible generalization of Hamilton and Heisenberg equations [17,18], nowadays considered at the foundation of hadronic mechanics and chemistry, as well as the first and only known Lie-admissible formulation of dissipative plasmas surrounding spaceships during reentry [19].

Santilli then spent seven years , from 1968 to 1974, at the Department of Physics of Boston University, and then three years, from 1974 to 1977, at MIT, during which tine he wrote, in his words, *Phys.. Rev of career-oriented papers nobody reads.* InSeptember 1977, Santilli joined Harvard University and was invited by the DOE to study irreversible processes because all energy releasing processes are irreversible over time. In April 1978, Santilli published under his DOE support his most important mathematical contribution [20] (see also monographs [21]) in which he achieved a Lie-admissible covering of the various branches of Lie's theory, Eqs. (69), (70), including the most general known time evolution whose brackets characterize an algebra, Eqs. (1000). It should be indicated that the isotopies of Lie's theory outlined in the preceding section were derived by Santilli as a particular case of the broader Lie-admissible theory of Ref. [20], and then published in monographs [1].

Subsequently, Santilli discovered in paper [2] of 1993 that the axiom of a numeric field, besides admitting a generalization of the multiplicative unit, also admit the restriction of the associative product to an ordered form to the right and, separately,to the left. In this way, Santilli discovered two additional classes of new numbers, today known as *Santilli forward and backward genoreal, genocomplex and genoquarternionic numbers.* In the seminal memoir [3] of

[3] By including the multi-valued (Section 4) and hyperstructural formulations (Section 5), Lie-admissible equations (100) are so broad that it will take centuries for their generalizations. For this reason,Santilli has requested in his will that his tombstone should have the engraving

$$i\dot{A} = A < H - H > A$$

below his name.

1996 Santilli discovered two additional coverings of the ordinary differential calculus and of its isotopic covering, today known as *Santilli forward and backward genodifferential calculi,* Eqs. (65) to (68). Santilli called a *genotopy* [20] the lifting of isomathematics into ordered formulations to the right and to the left in the Greek sense of inducing a covering of Lie's axioms, Eqs. (69), (70).

As it is well known, thousands of papers have been published beginning from the late 1980s on the so-called q-deformations of Lie algebras with product $(a, b) = ab - qba$ which are an evident particular case of Santilli Lie-admissible product [15]. Whatit is lesser known, or not admitted, all q-deformations did not achieve invariance over time, thus being afflicted by serious inconsistencies, since they consisted of non-unitary theories formulated via the mathematics of unitary theories. Santilli solved this problem in 1997 by achieving the first and only known invariant formulation of q- as well as of (p, q)-deformations [22].

We should indicate that Santilli's conception of a genotopic lifting of his preceding isomathematcs (indicated in Section 2 by "hat" on symbols plus the "arrow of time") is necessary to achieve a consistent representation of irreversibility because point-like particles can only experience action-at-a-distance interactions that are reversible over time. Therefore, a simple genotopy of 20th century mathematics based on the conventional associative product would be axiomatically inconsistent. Consequently, to represent irreversibility it is first necessary to lift 20th century mathematics into isomathematcs, with consequential representation of extended-deformable particles via realizations of type (27) so that extended particles can experience non-Hamiltonian interactions needed for irreversibility. It is then necessary to add irreversibility via the ordering of all products. It should also be indicated that, when formulated via time-dependent isounits, isomathematics can becomes genomathematics via the identifications $\hat{I}(t,\dots) = \hat{I}^{\dagger}(t,\dots) = \hat{I}^{>}, \hat{I}(-t,\dots) = \hat{I}^{\dagger}(-t,\dots) = {}^{<}\hat{I}, \hat{I}(t,\dots) \neq \hat{I}(-t)$, and the judicious addition of ordered products.

Systematic studies on the Lie-Admissible treatment of irreversible systems were presented in memoir [3] and monographs [4]. Santilli's subsequent memoir [23] of 2006 remains to this day the most comprehensive presentation of Lie-admissible treatments of irreversibility at the classical and operator levels. Monographs [5] of 2008 presented an update. Paper collection [24[presents all available independent contributions in Lie-admissibility up to [1984. The Proceedings of the *Third International Conference on Lie-admissible Treatment of Irreversible Systems* [25] present numerous additional independent contributions as well as references for the five *Workshops on Lie-0admissible Algebras* organized by Santilli at Harvard University, and for the preceding two international conference in Lie-admissibility, the first at the Université d'Orleans, France, in 1981 and the second at the Castle Prince Pignatelli, Italy, in 1995 (see also the general review [14] and large literature quoted therein).

As it is well known, there exists a large number of papers on Lie-admissible algebras within the context of non-associative algebras (see Tomber's Bibliography [26] listing all significant papers in the field up to 1986). It should be indicated that, regrettably, these studies have no connection with Santilli genomathematics since the latter deals with the irreversible generalizations of all aspects of 20th century mathematics.

4. Classical Hypermathematcs, Hypermechanics and Hyperchemistry

4.1. Represented Systems s [1-5]

Multi-valued, time-irreversible systems of extended -deformable particles or constituents under the most general known Hamiltonian and non-Hamiltonian interaction, as occurring for multi-valued universes or the structure of the DNA.

4.2. Main Mathematical Structure s [1-5]

Basic HyperUnits and HyperProducts

$$\hat{I}^{>} = \{\hat{I}_1^{>}, \hat{I}_2^{>}, \hat{I}_3^{>}, \dots\} = 1/\hat{S}, \tag{109}$$

$$^{<}\hat{I} = \{^{<}\hat{I}_1, ^{<}\hat{I}_2, ^{<}\hat{I}_3, \dots\} = \frac{1}{\hat{R}}, \tag{110}$$

Forward and Backward HyperProducts

$$A > B = \{A \times \hat{S}_1 \times B, A \times \hat{S}_2 \times B, A \times \hat{S}_3 \times B, \dots\}, \hat{I}^{>} > A = A > \hat{I}^{>} = A \times I, \tag{111}$$

$$A < B = \{A \times \hat{R}_1 \times B, A \times hatR_2 \times B, A \times \hat{R}_3 \times B, \dots\}^{<}\hat{I} < A = -A <^{<} \hat{I} = I \times A, \tag{112}$$

$$A = A^{\dagger}, B = B^{\dagger}, \hat{R} = \hat{S}^{\dagger}. \tag{113}$$

Classical hypermathematcs then follow as for genomathematcs with multi-valued units, quantities and operations.

4.3. Classical Hyper-Dynamical Equations s [1-5]

The same as those for genomathematics, but with multi-valued hyperunits, quantities and operations.

Comments and References

The multi-valued three-dimensional (rather than multi-dimensional) realization of genomathematics outlined in Section 4 emerged from specific biological needs. The Australian biologist C. Illert [27] confirmed that the *shape* of seashells can indeed be represented in a three-0dimensional Euclidean space as known since Fourier's time, but proved that the *growth in time* of a seashell cannot any longer be consistently represented in a conventional, three-dimensional Euclidean space, and achieved a consistent representation via the doubling of the three reference axis.

Santilli [27,28] confirmed Illert's findings because the conventional Euclidean geometry has no time arrow and, consequently, cannot consistently represent a strictly irreversible system, such as the growth of seashells. Additionally, Santilli proved thathis geno-Euclidean geometry, Eqs. (71) to (73), is equally unable to represent the growth in time of seashells despite its irreversible structure, however, an axiomatically consistent and exact representation of the growth of seashells was possible via the multi-valued realization of the forward geno-Euclidean geometry, thus beginning to illustrate the complexity of biological structures.

The multi-valued, rather than multi-dimensional character of classical hypermathematics is indicated by Santilli as follows [28] *We perceive the growth of a seashell specifically in three dimensions from our Eustachian lobes. Therefore, an irreversible mathematics suitable to represent the growth of sea shells must be perceived by us as being in three dimensions. However, Illert has shown the need to double the three Cartesian axis. Classical hypermathematics has been conceived and structured in such a way that the increase of the reference axes is complemented by a corresponding multi-valued hyperunit in such a way that a classical hyper-Euclidean geometry, when seen at the abstract level, remains indeed three-dimensional as necessary to achieve representation of biological structures compatible with our sensory perception.*

5. Hope Hypermathematics, Hypermechanics and Hyperchemistry

Represented Systems

The most complex known multi-valued, time-irreversible requiring extremely large number of data, such as the DNA code [31-35].

Comments and References

Despite the preceding structural generalization of 20th century mathematics, Santilli remained dissatisfied in view of the complexity of nature, particularly of biological entities because advances in the *structure* of the DNA are indeed possible via classical hypermathematics, as we shall see in the third collection of this series dedicated to chemistry (e.g., via Santilli hypermagnecules), but any attempt at representing the DNA *code* via any of the preceding mathematics can be proved to be excessively restrictive due to the volume, complexity, diversification and coordination of the information.

Therefore, Santilli approved one of the most important mathematicians in hyperstructures, T. Vougiouklis from Greece, and asked for his assistance in further generalizing the preceding mathematics via hyperstructures defined on hyperfields, as necessary for applications implying measurements, and formulated via hyperoperations (called "hope") permitting the needed broadening of the representational capability.

The above contact lead to the hypermathematics indicated in this section as presented in Refs. [29-33] which is based on Vougiouklis H_v hyperaxioms and which mathematics, in Santilli's words, *constitutes the most general mathematics that can be conceived nowadays by the human mind.*

6. Isodual Mathematics, Mechanics and Chemistry

6.1. Represented Systems

Single-valued, closed-isolated, time-reversible systems of classical and operatorpoint-like antiparticles moving in vacuum solely under action at a distance Hamiltonian interactions, such as the stricture of antimatter atoms and antimatter molecules [2,36-43].

6.2. Main Mathematical Structure [2,36-43]

Basic isodual unit

$$1^d = -1^\dagger = -1, \tag{114}$$

Isodual numeric fields

$$F^d(n^d, \times^d, 1^d), n^d = n \times 1^d, n^d \times^d m^d = n^d \times (1^d)_{-1} \times m^d \in F^d,$$

$$n^d = isodual!\, real, complex, quatern.!\, numbers, \tag{115}$$

Isodual functional analysis

$$f^d(r^d) = f(r^d) \times 1^d \in F^d \tag{116}$$

Isodual differential calculus

$$d^d r^d = (1)^{-1} \times dr^d = dr, \tag{117}$$

$$\frac{\partial^d f^d(r^d)}{\partial^d r^d} = 1^d \times \frac{\partial f^d(r^d)}{\partial r^d}, \tag{118}$$

Santilli Isodual Lie theory

$$[X_i, X_j]^d = (X_i \times X_j - X_j \times X_i)^d == -C_{ij}^k \times X_k, \tag{119}$$

$$A^d(w^d) = e_d^{X \times w \times i} \times^d A^d(0) \times^d e_d^{-i \times w \times X}. \tag{120}$$

Santilli isodual Euclidean geometry

$$E^d({}^d, \delta^d, 1^d), r^d = (r^{dk}), k = 1,2,3,$$

$$\delta^d = Diag.(-1,-1,-1), \tag{121}$$

$$r^{d2d} = r^{di} \times \delta_{ij} \times^d r^{dj} = (r_1^2 + r_2^2 + r_3^2) \times 1^d \in F^d, \tag{122}$$

Santilli Isodual Minkowskian geometry ($\mu = 1,2,3,4,$)

$$M^d(x^d, \eta^d, I^d): x^d = (x^{d\mu}), x^{d4} = t^d = t \times 1^d = -t, \tag{123}$$

$$\eta^d = Diag.(-1,-1,-1,+c^{d2d}), \tag{124}$$

$$x^{d2d} = (x^\mu \times \eta_{\mu\nu} \times x^\nu)^d = (x_1^2 + x_2^2 + x_3^2 - t^2c^2) \times 1^d \in F^d, \tag{125}$$

Isodual Riemannian geometry, Santilli Isodual Symplectic Geometry.

6.3. Isodual Dynamical Equations [2,36-43

Newton-Santilli Isodual Equation

$$m^d \times^d \frac{d^d v^d}{d^d t^d} - F^{dSA}(t^d, r^d, v^d) = 0, \quad (126)$$

Isodual Variational Principle

$$\delta^d A^d = \delta^d \int^d (p_k^d \times^d d^d r^{dk} - H^d \times^d d^d t^d) = 0. \quad (127)$$

Hamilton-Santilli Isodual Equations without external terms

$$\frac{d^d r^{dk}}{d^d t^d} = \frac{\partial^d H^d(r^d, p^d)}{\partial^d p_k^d}, \quad \frac{d^d p_k^d}{d^d t^d} = -\frac{\partial^d H^d(r^d, p^d)}{\partial^d r^{dk}}, \quad (128)$$

Isodual Hilbert space H^d over C with states $|\psi^d >= -< \psi|$ over C^d

Expectation value of a Hermitean operator A

$$< A^d >=< \psi| \times A^d \times |\psi >\in C^d m \quad (129)$$

Heisenberg-Santilli Isodual Equations

$$i^d \times^d \frac{d^d A^d}{d^d t^d} = [A, H]^d = (A \times H - H \times A)^d, \quad (130)$$

Schrödinger-Santilli Isodual Equations

$$H^d \times^d |\psi^d >= E^d \times^d |\psi^d >= -E \times |\psi > \quad (131)$$

$$p^d \times^d |\psi^d >= +i^d \times^d \partial_{rd}^d |\psi^d > \quad (132)$$

Dirac-Santilli Isodual Equation

$$(\eta^{d\mu\nu} \times^d \gamma_\mu^d \times^d p_\nu^d + i^d \times^d m^d \times^d c^d) \times |\psi >= 0. \quad (133)$$

$$\{\gamma_\mu, \gamma_\nu\}^d = (\gamma_\mu \times \gamma_\nu + \gamma_n u \times \gamma_\mu)^d = 2^d \times^d \eta_{\mu\nu}^d, \quad (134)$$

Comments and References

In addition to the the study of irreversible processes and the representation of extended-deformable particles, during his Ph. D. studies of the md 1960s Santilli was interested to ascertain whether a far away galaxy is made up of matter or of antimatter. He soon discovered that none of the mathematics and physics he had learned during his graduate studies was applicable for a quantitative study of the problem considered since, at that time, antimatter was solely represented in second quantization, while the study of far away antimatter galaxies requested their representation at the purely *classical and neutral* level. In this way, Santilli initiated a solitary scientific journey that lasted for half a century.

This occurrence created one of the biggest imbalances in scientific history because matter was treated at all possible levels, from Newtonian mechanics to second quantization, while antimatter was solely treated in second quantization. The imbalance originated from the fact that special and general relativities had been conceived decades before the discovery of antimatter and, therefore, they had no possibility of representing antimatter at the classical and neutral (as well as charged) level.

It should be stressed that the ongoing trend to extend the application of special and general relativities to the classical treatment of antimatter is afflicted by a number of serious inconsistencies, such as the impossibility to achieve a consistent representation of neutral antimatter, the impossibility to reach a consistent representation of matter-antimatter annihilation (evidently due to the lack of a suitable conjugation from matter to antimatter), violation of the PCT theorem and other inconsistencies that remain generally ignored.

Being an applied mathematician by instinct and training, Santilli knew that the imbalance was the result of a purely mathematical insufficiency because the transition from matter to antimatter is an anti-homomorphism. Consequently, the description of antimatter required a mathematics which is anti-homomorphic to conventional mathematics.

Santilli dedicated a decade to the search of the needed mathematics for antimatter. Following an additional extended search done at the Department of Mathematics of Harvard University under DOE support in the early 1980s, *Santilli concluded that a mathematics suitable for the joint classical and operator treatment of antimatter did not exist and had to be constructed.*

In the early 1980s, Since he had introduced the isoproduct $A \hat{\times} B = A \times \hat{T}\hat{B}, \hat{T} > 0$, Eq. (25). Consequently, it was natural to introduce its *negative-definite* counterpart which he called *isodual* and denoted with theupper index ^{d}, namely $A \hat{\times}^d B = A \times \hat{T}^d \hat{B}, \hat{T}^d = (\hat{T}^d)^\dagger < 0$. While constructing the isotopies of 20th century mathematics presented in Section 2, Santilli initiated the construction of their isodual image but published no paper in the new mathematics for over a decade.

This caution was due to the fact that, despite the lack of any visible mathematical inconsistency, Santilli remained skeptical on a mathematics based on a negative-definite product is afflicted by known physical inconsistencies, such as the violation of causality for negative time, energies and other physical quantities.

A breakthrough occurred in paper [2] of 1993. During the achievement of the broadest possible realizations of the abstract axioms of a numeric field (of characteristic zero), Santilli discovered that realizations with negative-definite units were simply unavoidable. This lead to the discovery of additional new numbers, today known as *Santilli isodual real, isodual complex and isodual quaternionic numbers* occurring for $I^d = -1$, Eq. (14), with isodual products (5), which are at the foundation of the isodual mathematics of this section and the additional numbers known as *Santilli isodual iso- and isodual geno-real, complex and quaternionic numbers* which are at the foundation of the isodual isomathematics and isodual genomathematics of Sections 7 and 8m respectively [2].

The discovery of isodual numbers is truly historical in our view due to its far reaching implications. In fact, the discovery established the existence of the desired *isodual mathematics* as an anti-isomorphic image of 20th century mathematics for the representation of antimatter. Additionally, the discovery permitted the resolution of the problems of causality for negative values of physical quantities.

To avoid insidious inconsistencies generally not seen by non-experts in the field, the isodual map must be applied for consistency to the *totality* of quantities and their operations. This lead to Santilli's conception of antimatter as possessing ıt negative-definite physical quantities for time, energy, momentum, frequency, etc, but such negative values are referred to *negative units* of measurements. Consequential a theory with negative time referred to negative units of time is as causal as our reality with a positive time referred to positive units, and the same holds for all other physical quantities.

Following the resolution of these basic issues, Santilli published in 1994 his first paper [36] specifically devoted to the isodual representation of antimatter. In mathematical memoir [3] of 1996, Santilli achieved the first isodual mathematical and physical representation of antimatter. In paper [37] of 1998, Santilli achieved his first goal of the early 1960s, namely, a consistent classical representation of neutral (as well as charged) antimatter.

By the early 1990s, Santilli had shown that *isodual mathematics represents all available experimental, data on antimatter at the classical and operator level.* Hence, he initiated the second phase of his studies, namely, the identification of new predictions for subsequent experimental verification.

A breakthrough occurred at the 1996 *First International Conference on Antimatter* help in Sepino, Italy [38]. By that time, Santilli had shown that the only conceivable representation of *neutral* antimatter required the conjugation of the sign of all physical quantities (jointly with the corresponding conjugation of their units of measurements). Since photons are neutral, the application of the same principle to light implies light emitted by antimatter, that Santilli called *isodual light,* is physically different than light emitted by matter in an experimentally verifiable way, e.g., because antimatter light is predicted to be *repelled* by a matter gravitational field.

Santilli then passed to a deeper geometric study of the gravitational field of antimatter. As indicated earlier, general relativity was formulated decades before the discovery of antimatter and, therefore, had no clue for the representation of the gravitational field of antimatter bodies. In Ref.[39] of 1998, Santilli conducted an in depth geometric study of antimatter, and in monograph [40) of 2006, Santilli completed the gravitational study of antimatter via the isodual Riemannian geometry.

All these studies concluded with the prediction of *gravitational repulsion* (antigravity) between matter and antimatter at all levels of analysis, from the isodual Newton-Santilli equations (26) to isodual second quantization. These aspects will be studied in the second collection of this series dedicated to hadronic mechanics.

Thanks to all the above advances, Santilli was finally in a position to address his original main aim of the 1960s, namely, ascertain whether a far away galaxy is made up of matter or of antimatter. The preceding studies had established that the light emitted by antimatter must have a *negative index of refraction* that, as such, require *concave* lenses for its focusing. Consequently, Santilli secured the construction of a revolutionary telescope with concave lenses. About fifty years following his original aim, Santilli finally published in 2013 [41[measurements of the night sky with his new telescope showing images that can be solely due to light with a negative index of refraction which light, in turn, can solely originate from far away antimatter stars or galaxies (see also the two independent confirmations [42,43]).

An intriguing aspect that should be of interest to pure mathematicians is the conclusion of these studies illustrating the power of new mathematics, to the effect that none of the large numbers of telescopes available nowadays can detect antimatter starsor galaxies since they all have *convex* lenses. Similarly, as humans evolved in a matter world, we will never be able to see antimatter with our eyes since our cornea is convex and, as such, it will disperse antimatter light all over the retina.

Needless to say, isodual mathematics and its application to antimatter have implications so intriguing that are stimulating the participation of a large number of scientists as we shall report in the second collection of this series

7. Isodual Isomathematics, Isodual Isomechanics and Isodual Isochemistry

7.1. Represented Systems [2,36-43

Single-value, closed-isolated, time-reversible system of classical or operator extended-deformable antiparticles with action at a distance Hamiltonian and contact non-Hamiltonian interactions, such as the structure of antimatter hadrons, nuclei and stars, in the antimatter valence electron bonds and other antimatter systems.

7.2. Main Mathematical Structure[2,36-43

Basic Isodual IsoUnit

$$\hat{I}^d = \hat{I}^d(r^d, p^d, a^d, \psi,^d \partial^d \psi^d, \ldots.) = 1^d /^d \hat{T}^d < 0, \quad (135)$$

Basic Isodual IsoFields

$$\hat{F}^d(\hat{n}^d, \hat{\times}^d, \hat{I}^d), \hat{n}^d = n \times \hat{I}^d, \hat{n}^d \hat{\times}^d \hat{m}^d = \hat{n}^d \times \hat{T}^d \times \hat{m}^d \in \hat{F}^d, \quad (136)$$

Isodual IsoCoordinates $\hat{r}^d = r \times \hat{I}^d \in \hat{F}^d$,

Isodual IsoFunctional analysis $\hat{f}^d(\hat{r}^d) == f(\hat{r}^d) \times \hat{I}^d \in \hat{F}^d$,

Isodual IsoDifferential Calculus

$$\hat{d}^d \hat{r}^d = dr - r^d \times \hat{T}^d \times d\hat{I}^d, \quad (137)$$

$$\frac{\hat{\partial}^d \hat{f}^d(\hat{r})}{\hat{\partial}^d \hat{r}^d} = \hat{I}^d \times \frac{\partial \hat{f}^d(\hat{r}^d)}{\partial \hat{r}^d}, \quad (138)$$

Santilli Isodual Lie-Isotopic Theory

$$[X_i \hat{,} X_j]^d = X_i \hat{\times} X_j - X_j \hat{\times} X_i)^d == -C_{ij}^k(r, p, \ldots) \times X_k, \quad (139)$$

$$A^d(w^d) = \hat{e}_d^{X^d \times w^d \times i^d} \hat{\times}^d A^d(0^d) \hat{\times}^d \hat{e}_d^{-i^d \times w^d \times X^d}. \quad (140)$$

Santilli Isodual Iso-Euclidean Geometry

$$\hat{E}^d(\hat{r}^d, \hat{\delta}^d, \hat{I}^d), \hat{\delta}^d(r^d, p^d, a^d, \psi, \ldots) =$$

$$\hat{T}^d(r^d, p,^d a^d, \psi^d, \dots) \times \delta, \quad (141)$$

$$\hat{T}^d = Diag.(1/n_1^2, 1/n_2^2, 1/n_3^2)^d, \quad (142)$$

$$\hat{r}^{d\hat{2}d} = (\hat{r}^i \hat{\times} \hat{\delta}_{ij} \hat{\times} \hat{r}^j)^d = (\frac{r_1^2}{n_1^2} + \frac{r_2^2}{n_2^2} + \frac{r_3^2}{n_3^2})^d \times \hat{I}^d \in \hat{F}^d, \quad (143)$$

Santilli Isodual Iso-Minkowskian Geometry ($\mu = 1,2,3,4$)

$$\hat{M}^d(\hat{x}^d, \hat{\eta}^d, \hat{I}^d): \hat{x}^d = (\hat{x}^{d\mu}), \hat{x}_4^d = \hat{t}^d = t \times \hat{I}^d, \quad (144)$$

$$\hat{\eta}^d(x^d, \psi^d, \dots) = \hat{T}^d(x^d, \psi^d, \dots) \times \eta, \quad (145)$$

$$\hat{T}^d = Diag.(1/n_1^2, 1/n_2^2, 1/n_3^2, 1/n_4^2)^d, \quad (146)$$

$$\hat{x}^{d\hat{2}d} = (\hat{x}^\mu \hat{\times} \hat{\eta}_{\mu\nu} \hat{\times} \hat{x}^\nu)^d == (\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} - t^2 \frac{c^2}{n_4^2})^d \times \hat{I}^d \in \hat{F}^d, \quad (147)$$

Santilli Isodual Iso-Riemannian Geometry

$$\hat{R}^d(\hat{x}^d, \hat{g}^d, \hat{I}^d): \hat{g}^d = \hat{T}^d(x^d, v^d, \dots) \times g(x), \quad (148)$$

$$\hat{x}^{d\hat{2}d} = (\frac{g_{11}}{n_1^2} + \frac{g_{22}}{n_2^2} + \frac{g_{33}}{n_3^2} - \frac{g_{44}}{n_4^2})^d \times \hat{I}^d \in \hat{F}^d, \quad (149)$$

Santilli Isodual Iso-Symplectic Geometry

$$\hat{\omega}^d = \hat{d}\hat{r}^{dk} \hat{\wedge}^d \hat{d}\hat{p}_k^d \quad (150)$$

7.3. Isodual IsoDynamical IsoEquation[2,36-43

Newton-Santilli Isodual IsoEquation

$$\hat{m}^d \hat{\times}^d \frac{\hat{d}^d \hat{v}^d}{\hat{d}^d \hat{t}^d} - F^{dSA}(r^d, p^d) == (m \times \frac{dv}{dt})^d - F^{dSA}(r^d, p^d) - F^{dNSA}(r^d, p^d, \dots) = 0^d = 0, \quad (151)$$

Isodual IsoVariational principle

$$\hat{\delta}^d \hat{A}^d = \hat{\delta}^d \hat{\int}^d (\hat{p}_k^d \hat{\times}^d \hat{d}^d hatr^{dk} - \hat{H}^d \hat{\times}^d \hat{d}^d \hat{t}^d) = 0^d = 0. \quad (152)$$

Hamilton-Santilli Isodual IsoEquations

$$\frac{\hat{d}^d \hat{r}^{dk}}{\hat{d}^d \hat{t}^d} = \frac{\hat{\partial}^d \hat{H}^d(\hat{r}^d, \hat{p}^d)}{\hat{\partial}^d \hat{p}_k^d}, \quad \frac{\hat{d}\hat{p}_k}{\hat{d}^d \hat{t}^d} = + \frac{\hat{\partial}^d \hat{H}^d(\hat{r}^d, \hat{p}^d)}{\hat{\partial}^d \hat{r}^{dk}}, \quad (153)$$

Isodual iso-Hilbert space $\hat{H}^d$ over C with states $|\hat{\psi}^d >= -< \hat{\psi}|$ over $\hat{C}^d$

Expectation value of a Hermitean operator A

$$< A^d >=< \hat{\psi}| \hat{\times} A^d \hat{\times} |\hat{\psi} >\in C^d \quad (154)$$

Heisenberg-Santilli Isodual IsoEquation

$$\hat{\imath}^d \hat{\times}^d \hat{d}^d hatA^d over \hat{d}^d \hat{t}^d = [\hat{A}\hat{,} \hat{H}]^d = (\hat{A} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{A})^d = (\hat{A} \times \hat{T}(\hat{\psi}, \hat{\partial}\hat{\psi}, \dots) \times \hat{H}(\hat{r}, \hat{p}) - \hat{H}(\hat{r}, \hat{p}) \times \hat{T}(\hat{\psi}, \hat{\partial}\hat{\psi}, \dots) \times \hat{A})^d, \quad (155)$$

Schrödinger-Santilli Isodual IsoEquation

$$(\hat{H} \hat{\times} |\hat{\psi} >)^d =< \hat{\psi}^d| \hat{\times}^d \hat{H}^d = (\hat{H}(\hat{r}, \hat{p}) \times \hat{T}(\hat{\psi}, \hat{\partial}\hat{\psi}, \dots) \times |\hat{\psi} >)^d = -< \hat{\psi}^d| \hat{\times}^d \hat{E}^d = -< \hat{\psi}^d| \times \hat{E}^d, \quad (156)$$

$$(\hat{p} \hat{\times} |\hat{\psi} >)^d =< \hat{\psi}^d| \hat{\times}^d \hat{\partial}_{\hat{r}^d} = -i \times < \hat{\psi}^d| \hat{\times}^d \hat{\partial}_{\hat{r}^d}^d, \quad (157)$$

Dirac-Santilli Isodual IsoEquation

$$[(\hat{\eta}^{\mu\nu} \hat{\times} \hat{\gamma}_\mu \hat{\times} \hat{p}_\nu - \hat{\imath} \hat{\times} \hat{m} \hat{\times} \hat{c}) \hat{\times} |\hat{p}si >]^d = 0. \quad (158)$$

$$\{\hat{\gamma}_\mu \hat{,} \hat{\gamma}_\nu\}^d = (\hat{\gamma}_\mu \hat{\times} \hat{\gamma}_\nu + \hat{\gamma}_\nu \hat{\times} \hat{\gamma}_\mu)^d = \hat{2}^d \hat{\times}^d \hat{\eta}_{\mu\nu}^d, \quad (159)$$

Comments and References

See monograph [40] with particular reference to the use of the isodual isomathematics for the achievement of a grand unification of electroweak and gravitational interactions inclusive of matter and antimatter.

8. Isodual Genomathematics, Isodual Genomechanics and Isodual Genochemistry

8.1. Represented Systems [2,36-43

Single-valued, time-irreversible system of extended-deformable antiparticles under action at a distance Hamiltonian and contact non-Hamiltonian interactions, as occurring in antimatter nuclear reactions, antimatter biological structures and antimatter chemical reactions.

8.2. Main Mathematical Structure [2,36-43

Backward Isodual GenoUnit

$$\hat{I}^{>c} = \hat{I}^{>d}(t^{>} r^{>}, p^{>d}, a^{>d}, \psi^{>d}, \partial^{>d}\psi^{>d}, \dots) = 1/\hat{T}^{>d} > 0, \quad (160)$$

Forward Isodual GenoUnit

$$^{<d}\hat{I} = ^{<d}\hat{I}(^{<d}r, ^{<d}p, ^{<d}a, ^{<d}\psi, ^{<d}\partial ^{<d}\psi, \dots) = 1/^{<d}\hat{T} > 0, \quad (161)$$

Condition for time-irreversibility

$$\hat{I}^{>d} \neq {}^{<d}\hat{I} \quad (162)$$

Backward Isodual GenoFields

$$\hat{F}^{>d}(\hat{n}^{>d}, >, \hat{I}^{>d}), \hat{n}^{>d} = n \times \hat{I}^{>d}, \hat{n}^{>d} >^d \hat{m}^{>d} = \hat{n}^{>d} \times \hat{T}^{>d} \times \hat{m}^{>d} \in \hat{F}^{>d}, \quad (163)$$

Forward Isodual GenoFields

$$^{<d}\hat{F}(^{<d}\hat{n}, <, ^{<d}\hat{I}), \quad ^{<d}\hat{n} = ^{<d}\hat{I} \times n, \quad ^{<d}\hat{n} <^d \quad ^{<d}\hat{m} = = ^{<d}\hat{n} \times ^{<d}\hat{T} \times ^{<d}\hat{m} \in ^{<d}\hat{F}, \quad (164)$$

Backward Isodual GenoCoordinates

$$\hat{r}^{>d} = r \times \hat{I}^{>d} \in \hat{F}^{>d}, \quad (165)$$

Forward Isodual GenoCoordinates

$$^{<d}\hat{r} = ^{<d}\hat{I} \times r \in ^{<d}\hat{F}, \quad (166)$$

Backward Isodual GenoFunctional analysis

$$\hat{f}^{>d}(\hat{r}^{>d}) = f(\hat{r}^{>d}) \times \hat{I}^{>d} \in \hat{F}^{>d}, \quad (167)$$

Forward Isodual GenoFunctional analysis

$$^{<d}\hat{f}(^{<d}\hat{r}) = f(^{<d}\hat{r}) \times^{<d} \hat{I} \in^{d} \hat{F} \quad (168)$$

Backward Isodual GenoDifferential Calculus

$$\hat{d}^{>d}\hat{r}^{>d} = dr + r \times \hat{T}^{>d} \times d\hat{I}^{>d}, \quad (169)$$

$$\frac{\hat{\partial}^{>d}\hat{f}^{>d}(\hat{r}^{>d})}{\hat{\partial}^{>d}\hat{r}^{>d}} = \hat{I}^{>d} \times \frac{\partial \hat{f}^{>d}(\hat{r}^{>d})}{\partial \hat{r}^{>d}}, \quad (170)$$

Forward Isodual GenoDifferential Calculus

$$^{<d}\hat{d}^{<d}\hat{r} = dr + r \times^{<d} \hat{T} \times d^{<d}\hat{I}, \quad (171)$$

$$\frac{^{<d}\hat{\partial}^{<d}\hat{f}(^{<d}\hat{r})}{^{<d}\hat{\partial}^{<d}\hat{r}} =^{<dd} \hat{I} \times \frac{\partial^{<d}\hat{f}(^{<d}\hat{r})}{\partial^{<d}\hat{r}}, \quad (172)$$

Santilli Isodual Lie-Admissible Theory

$$(X_i \hat{,} X_j)^d = (X_i < X_j - X_j > X_i)^d = -C_{ij}^{dk}(t^d, r^d, p^d, \psi^d, \dots) \times X_k, \quad (173)$$

$$A^d(w^d) = \hat{e}_{>d}^{X \times w \times i} >^d A(0) <_{<d}^{d} \hat{e}^{-i \times w \times X}. \quad (174)$$

Santilli Backward Geno-Euclidean Geometry

$$\hat{E}^{>d}(\hat{r}^{>d}, \hat{\delta}^{>d}, \hat{I}^{>d}), \hat{\delta}^{>d}(t, r, p, \psi, \dots) = \hat{T}^{>d}(t, r, p, \psi, \dots) \times \delta, \quad (175)$$

$$\hat{r}^{>d\hat{2}d} = (\hat{r}^{>di} >^d \hat{\delta}_{ij}^{>d} > \hat{r}^{>dj} = \in F^{>d}, \quad (176)$$

$$\hat{T}^{>d} \neq \hat{T}^{>d\, transp} \quad (177)$$

Santilli Forward Isodual Geno-Euclidean Geometry

$$^{<d}\hat{E}(^{<d}\hat{r}, ^{<d}\hat{\delta}, ^{<d}\hat{I}), \quad ^{<d}\hat{\delta}(t, r, p, \psi, \dots) =^{<d} \hat{T}(t, r, p, \psi, \dots) \times \delta, \quad (178)$$

$$^{<d\hat{2}d}\hat{r} =^{<di} \hat{r} <^{<d} \hat{\delta}_{ij} <^{d} \quad ^{<dj}\hat{r} \in^{<d} F, \quad (179)$$

$$^{<d}\hat{T} \neq^{<d\, transp} \hat{T} \quad (180)$$

Santilli Backward Isodual Geno-Minkowskian Geometry ($\mu = 1,2,3,4$)

$$\hat{M}^{>d}(\hat{x}^{>d}, \hat{\eta}^{>d}, \hat{I}^{>d}): \hat{x}^{>d} = (\hat{x}^{>d\mu}), x_4^{>d} = t^{>d}, \quad (181)$$

$$\hat{\eta}^{>d}(x, \psi, \dots) = \hat{T}^{>d}(x, \psi, \dots) \times \eta, \quad (182)$$

$$\hat{x}^{>d2d} = \hat{x}^{>d\mu} >^d \hat{\eta}_{\mu\nu}^{>d} >^d \hat{x}^{>dv} \in \hat{F}^{>d}, \quad (183)$$

$$\hat{\eta}^{>d} \neq \hat{\eta}^{>d\, transp} \quad (184)$$

Santilli Forward Isodual Geno-Minkowskian Geometry ($mu = 1,2,3,4$)

$$^{<d}\hat{M}(^{<d}\hat{x}, ^{<d}\hat{\eta}, ^{<d}\hat{I}): \quad ^{<d}\hat{x} = (\hat{x}^{\mu}), \quad ^{<d}x_4 =^{<d} t, \quad (185)$$

$$^{<d}\hat{\eta}(x, v, \dots) =^{<d} \hat{T}(x, v, \dots) \times \eta, \quad (186)$$

$$^{<d}\hat{x}^{<d2d} =^{<d\mu} \hat{x} <^{d} \quad ^{<d}\hat{\eta}_{\mu\nu} <^{d} \quad ^{<v}\hat{x} \in^{<d} \hat{F}, \quad (187)$$

$$^{<d}\hat{\eta} \neq^{<d\, transp} \hat{\eta} \quad (188)$$

Santilli Backward Isodual Geno-Riemannian Geometry

$$\hat{R}^{>d}(\hat{x}^{>d}, \hat{g}^{>d}, \hat{I}^{>d}): \hat{g}^{>d} = \hat{T}^{>d}(x, v, \dots) \times g(x), \quad (189)$$

$$\hat{x}^{>d2>} = x^{>d\mu}.^{d}\, \hat{g}_{\mu\nu}^{>d} >^d x^{>dv} \in \hat{F}^{>d} \quad (190)$$

$$\hat{T}^{>d} \neq \hat{T}^{>d\, transp} \quad (191)$$

Santilli Forward Isodual Geno-Riemannian Geometry

$$^{<d}\hat{R}(^{<d}\hat{x}, ^{<}\hat{g}, ^{<d}\hat{I}): \quad ^{<d}\hat{g} =^{<d} \hat{T}(x, v, \dots) \times g(x), \quad (192)$$

$$^{<d}\hat{x}^{<d2d} =^{<d\mu} \hat{x} <^{d} \quad ^{<d}\hat{g}_{\mu\nu} <^{d} \quad ^{<dv}\hat{x} \in^{<d} \hat{F}, \quad (193)$$

$$^{<d}\hat{g} \neq^{<d\, transp} \hat{g} \quad (194)$$

Santilli Backward Isodual Geno-Symplectic Geometry

$$\hat{\omega}^{>d} = \hat{d}^{>d}\hat{r}^{>dk} \hat{\wedge}^{>d} \hat{d}^{>d}\hat{p}_k^{>d} \quad (195)$$

Santilli Forward Isodual Geno-Symplectic Geometry

$$^{<d}\hat{\omega} =^{<d} \hat{d}^{<dl}\hat{r}^{<d} \hat{\wedge}^{<d} \hat{d}^{<d}\hat{p}_k \quad (196)$$

8.3. Isodual GenoDynamical GenoEquations [2,36-43

Newton-Santilli Backward Isodual GenoEquation

$$\hat{m}^{>d} > \frac{\hat{d}^{>d}\hat{v}^{>d}}{\hat{d}^{>d}\hat{t}^{>d}} - F^{>dSA}(t, r, p) = [m \times \frac{dv}{dt}]^{>d} - F^{SA>d}(t, r, p) - F^{NSA>d}(t, r, p, \dots) = 0, \quad (197)$$

Newton-Santilli Forward Isodual GenoEquation

$$^{<d}\hat{m} < \frac{^{<d}\hat{d}^{<d}\hat{v}}{^{<d}\hat{d}^{<d}\hat{t}} -^{<dSA}F(t, r, p) =^{<d} [m \times \frac{dv}{dt}] -^{<dSA}F(t, r, p) -^{<dNSA}F(t, r, p, \dots) = 0, \quad (198)$$

Backward Isodual GenoVariational principle

$$\hat{\delta}^{>d}\hat{A}^{>d} = \hat{\delta}^{>d} \hat{\int}^{>d} (\hat{p}_k^{>d} > \hat{d}^{>d}\hat{r}^{>dk} - \hat{H}^{>d} > \hat{d}^{>d}\hat{t}^{>d}) = 0. \quad (199)$$

Forward Isodual GenoVariational principle

$$^{<d}\hat{\delta}^{<d}\hat{A} =^{<d} \hat{\delta}^{<d} \hat{\int} \; (^{<d}\hat{p}_k <^{<d} \hat{d}^{<d}\hat{r}^{k} -^{<d}\hat{H} <^{<d} \hat{d}^{<d}\hat{t}) = 0. \quad (200)$$

Backward Isodual Hamilton-Santilli GenoEquations

$$\frac{\hat{d}^{>d}\hat{r}^{>dl}}{\hat{d}^{>d}\hat{t}^{>d}} = [\frac{\hat{\partial}\hat{H}(\hat{r}, \hat{p})}{\hat{\partial}\hat{p}_k}]^{>d}, \quad [\frac{\hat{d}^{>d}\hat{p}_k^{>d}}{\hat{d}^{>d}\hat{t}^{>d}}] = -[\frac{\hat{\partial}\hat{H}(\hat{r}, \hat{p})}{\hat{\partial}\hat{r}^k}]^{>d}, \quad (201)$$

Forward isodual Hamilton-Santilli GenoEquations

$$\frac{^{<d}\hat{d}^{<d}\hat{r}^{k}}{^{<d}\hat{d}^{<d}\hat{t}} =^{<d} \frac{\hat{\partial}\hat{H}(\hat{r}, \hat{p})}{\hat{\partial}\hat{p}_k}], \quad \frac{^{<d}\hat{d}^{<d}\hat{p}_k}{^{<d}hatd^{<d}\hat{t}}] = -^{<}[\frac{\hat{\partial}\hat{H}(\hat{r}, \hat{p})}{\hat{\partial}\hat{r}^k}], \quad (202)$$

Heisenberg-Santilli IsoDual GenoEqutions

$$\hat{\imath} \hat{\times} \frac{\hat{d}\hat{A}}{\hat{d}\hat{t}} = (\hat{A}, \hat{H}) = \hat{A} < \hat{H} - \hat{H} > \hat{A} = A \times^{<} T(\hat{\psi}, \hat{\partial}\hat{\psi}, \dots) \times \hat{H}(\hat{r}, \hat{p}) - \hat{H}(\hat{r}, \hat{p}) \times \hat{T}^{>}(\hat{\psi}, \hat{\partial}\hat{\psi}, \dots) \times \hat{A} \quad (203)$$

Schrödinger-Santilli Backward Isodual GenoEquations

$$\hat{H}^{>d} >^d |\hat{\psi}^{>d} >= \hat{H}^{>d}(\hat{r}, \hat{p}) \times \hat{T}^{>d}(\hat{\psi}, \hat{\partial}\hat{\psi}, \dots) \times |\hat{\psi}^{>} >=$$

$$\hat{E}^{>d} >^{d} |\hat{\psi}^{>d} >= E^{>d} \times |\hat{\psi}^{>d} >, \quad (204)$$

$$\hat{p}^{>d} > |\hat{\psi}^{>d} >= -\hat{\imath}^{>d} > \hat{\partial}_{\hat{r}}^{>d} |\hat{\psi}^{>} >= -i \times \hat{I}^{>} \times \partial_{\hat{r}} |\hat{\psi}^{>d} >, \quad (205)$$

Schrödinger-Santilli Forward Isodual GenoEquations

$$<^{<d} \hat{\psi}| <^{d} \quad {}^{<}\hat{H} =<^{<d} \hat{\psi}| \times^{<d} \hat{T}(\hat{\psi}, \partial\hat{\psi}, \dots) \times^{<} \hat{H}(\hat{r}, \hat{p}) = <^{<d} \hat{\psi}| <^{<} \hat{E} =<^{<d} \hat{\psi}| \times^{<d} E, \quad (206)$$

$$<^{<d} \hat{\psi}|<^{<d} \hat{p} = -<^{<d} \hat{\psi}| <^{d} \quad {}^{<d}\hat{\imath} <^{d} \quad {}_{\hat{r}}^{<d}\hat{\partial} = -i \times <^{<d} \hat{\psi}|_{\hat{r}}^{<d}\partial \times^{<d} \hat{I} \quad (207)$$

Dirac-Santilli Backward Isodual IsoEquation

$$\left(\hat{\eta}^{>d\mu\nu} >^{d} \hat{\gamma}_{\mu}^{>d} >^{d} \hat{p}_{\nu}^{>d} - \hat{\imath}^{>d} > \hat{m}^{>d} > \hat{c}^{>d}\right) > |\hat{p}si^{>d} >= 0. \quad (208)$$

$$\{\hat{\gamma}_{\mu} \hat{,} \hat{\gamma}_{\nu}\}^{>d} = [\hat{\gamma}_{\mu} \hat{\times} \hat{\gamma}_{\nu} + \hat{\gamma}_{\nu} \hat{\times} \hat{\gamma}_{\mu}]^{>} = \hat{2}^{>d} > \hat{\eta}_{\mu\nu}^{>d}, \quad (209)$$

Dirac-Santilli Forward Isodual GenoEquation

$$<^{<d} \hat{\psi}|< ({}^{<d}\hat{p}_{\nu} <^{<d} \hat{\gamma}_{\mu} <^{d} \quad {}^{<d\mu\nu}\hat{\eta} - {}^{<}\hat{\imath} <^{d} \quad {}^{<d}\hat{m} <^{d} \quad {}^{<}\hat{c}) = 0. \quad (210)$$

$${}^{<d}\{\hat{\gamma}_{\mu} \hat{,} \hat{\gamma}_{\nu}\} =^{<d} \left[\hat{\gamma}_{\mu} \hat{\times} \hat{\gamma}_{\nu} + \hat{\gamma}_{\nu} \hat{\times} \hat{\gamma}_{\mu}\right] =^{<d} \hat{2} <^{<} \hat{\eta}_{\mu\nu} = 2 \times^{<d} \hat{\eta}_{\mu\nu}, \quad (211)$$

Comments and References

See memoir [20] which constitutes the most comprehensive study of antimatter in irreducible conditions available at this writing.

9. Isodual Classical and Hope Isodual Hypermathematics

Isodual Hyper-Formulations are generally considered to be part of the Hyper-Formulations of Section 4 and 5 because the classification of ordered sets of hyperunits includes isodual realizations, as illustrated in the paper [44] and references quoted therein.

10. Simple Method for the Construction of Regular Hadronic Mathematics

10.1. Introduction [4.5]

Hadronic formulations are called *regular* when the *structure quantities* C_{ij}^{i} of Santilli's Lie-Isotopic algebras, Eqs. (3), Lie-admissible algebras, Eqs. (69) (zzz) and their isoduals, Eqs. (119-, (139), are *constant.* When the structure quantities are *functions of the local variables* $C_{ij}^{k}(t, r, p, \psi, \partial\psi, \dots)$, hadronic formulations are called *irregular*.

In this section, we shall review a very simple method for the construction of regular hadronic formulations via the mere use of non-unitary transformations of the corresponding conventional formulations. We shall then review the axiomatic consistency of hadronic formulations by showing that Santilli iso-, geno-, hyper-units and their isoduals are invariant under the transformations, thus implying the crucial invariance over time of extended-deformable shapes and their non-Hamiltonian interactions that are invariantly represented precisely nwith such generalized units.

No method exists to our knowledge at this writing (June 2015) for the construction of irregular hadronic formulations via maps of conventional formulations and, therefore, irregular hadronic formulations characterize a new axiomatic structure still mostlyunexplored.

10.2. Simple Construction of Regular Iso-Formulations [4.5]

A simple method has been identified in Refs. [4,5] for the construction of the Lie-Santilli isotheory, all its underlying isomathematics and all physical methods This method is important because it permits a simple implementation of conventional models into their isotopic covering without the need for advanced mathematics. The method consists in:

(i) Representing all conventional potential interactions with a Hamiltonian $H(r, p)$ and all extended-deformable shapes and their non-Hamiltonian interactions and effects with Santilli's isounit $\hat{I}(r, p, \psi, \partial\psi, ./..)$;

(ii) Identifying the latter interactions with a nonunitary transform

$$U \times U^{\dagger} = \hat{I} \neq I \quad (212)$$

and

(iii) Subjecting the *totality* of conventional mathematical and physical quantities and all their operations to the above nonunitary transform, resulting in expressions of the type

$$I \to \hat{I} = U \times I \times U^{\dagger} = 1/\hat{T}, \quad (213)$$

$$a \to \hat{a} = U \times a \times U^{\dagger} = a \times U \times U^{\dagger} = a \times \hat{I}, a \in F, \quad (214)$$

$$e^{A} \to U \times e^{A} \times U^{\dagger} = \hat{I} \times e^{\hat{T} \times \hat{A}} = \left(e^{\hat{A} \times \hat{T}}\right) \times \hat{I}, \quad (215)$$

$$A \times B \to U \times (A \times B) \times U^{\dagger} = (U \times A \times U^{\dagger}) \times (U \times U^{\dagger})^{-1} \times (U \times B \times U^{\dagger}) = \hat{A} \hat{\times} \hat{B}, \quad (216)$$

$$[X_i, X_j] \to U \times [X_i X_j] \times U^{\dagger} = [\hat{X}_i \hat{,} \hat{X}_j] = U \times (C_{ij}^{k} \times X_k) \times U^{\dagger} = \hat{C}_{ij}^{k} \hat{\times} \hat{X}_k = C_{ij}^{k} \times \hat{X}_k, \quad (217)$$

$$< \psi| \times |\psi > \to U \times < \psi| \times |\psi > \times U^{\dagger} =< \psi| \times U^{\dagger} \times (U \times U^{\dagger})^{-1} \times U \times |\psi > \times (U \times U^{\dagger}) = < \hat{\psi}| \hat{\times} |\hat{\psi} > \times \hat{I}, \quad (218)$$

$$H \times |\psi > \to U \times (H \times |\psi >) = (U \times H \times U^{\dagger}) \times (U \times U^{\dagger})^{-1} \times (U \times |\psi >) = \hat{H} \hat{\times} |\hat{\psi} >, etc. \quad (219)$$

Note that serious inconsistencies emerge in the event even 'one' single quantity or operation is not subjected to the above non-unitary map. In the absence of comprehensive liftings, we would have a situation equivalent to the elaboration of quantum spectral data of the hydrogen atom with isomathematics, resulting in large deviations from reality.

The construction of isodual iso-formulations is simply done via Santilli's isodual map, namely, via the simple anti-hermitean image of the above isotopic formulations.

10.3. Axiomatic consistency of Iso-Formulation [4.5]

Let us recall that Santilli's central assumption is the representation of extended-deformable shapes and their non-Hamiltonian interactions via the isounit. Therefore, any change of the numerical value of the isounit implies the inability to represent the same system over time, besides activating the *Theorem of Catastrophic Mathematical and Physical Inconsistencies of Non-Canonical and Non-Unitary Theories* when formulated via the mathematics of conventional canonical and unitary theories,respectively [23].

It is easy to see that the application of an additional nonunitary transform

$$W \times W^{\dagger} \neq I, \tag{220}$$

to the preceding expressions causes their *lack of invariance,* with consequential activation of the theorem of catastrophic inconsistencies. This is due to the *change of the value of the basic isounit* under additional non-unitary transformations

$$\hat{I} \rightarrow \hat{I}' = W \times \hat{I} \times W^{\dagger} \neq \hat{I}, \tag{221}$$

However, any given nonunitary transform can be identically rewritten in the isounitary form [3]

$$W \times W^{\dagger} = \hat{I}, \quad W = \widehat{W} \times \widehat{T}^{1/2}, \tag{222}$$

$$W \times W^{\dagger} = \widehat{W} \hat{\times} \widehat{W}^{\dagger} = \widehat{W}^{\dagger} \hat{\times} \widehat{W} = \hat{I}, \tag{223}$$

under which we have the invariance of the isounit and isoproduct [7]

$$\hat{I} \rightarrow \hat{I}' = \widehat{W} \hat{\times} \hat{I} \hat{\times} \widehat{W}^{\dagger} = \hat{I}, \tag{224}$$

$$\hat{A} \hat{\times} \hat{B} \rightarrow \widehat{W} \hat{\times} (\hat{A} \hat{\times} \hat{B}) \hat{\times} \widehat{W}^{\dagger} = (\widehat{W} \times \widehat{T} \times \hat{A} \times \widehat{T} \times \widehat{W}^{\dagger}) \times (\widehat{T} \times \widehat{W}^{\dagger})^{-1} \times \widehat{T} \times (\widehat{W} \times \widehat{T})^{-1} \times (\widehat{W} \times \widehat{T} \times \hat{B} \times \widehat{T} \times \widehat{W}^{\dagger}) = \hat{A}' \times (\widehat{W}^{\dagger} \times \widehat{T} \times \widehat{W})^{-1} \times \hat{B}' = \hat{A}' \times \widehat{T} \times \hat{B}' = \hat{A}' \hat{\times} \hat{B}', \; etc. \tag{225}$$

from which the invariance of the entire isotopic formalism follows.

Note that the invariance is ensured by the *numerically invariant values of the isounit and of the isotopic element under non-unitary-isounitary transformations,*

$$\hat{I} \rightarrow \hat{I}' \equiv \hat{I}, \tag{226}$$

$$A \hat{\times} B \rightarrow A' \hat{\times}' B' \equiv A' \hat{\times} B', \tag{227}$$

in a way fully equivalent to the invariance of Lie's theory and quantum mechanics, as expected to be necessarily the case due to the preservation of the abstract axioms under isotopies. The resolution of the inconsistencies for non-invariant theories is then consequential.

The proof of the invariance of Santilli isodual iso-formulations is an interesting exercise for non-initiated readers.

10.4. Simple Construction of Regular GenoMathematics and its IsoDual [4.5]

An important feature of the Lie-Santilli genotheory is its *form invariance* under the appropriate geno-transformations in a way fully similar to the invariance of the mathematical and physical structures of quantum mechanics under unitary transformations.

This feature can be shown via a *pair* of non-unitary transformations

$$V \times V^{\dagger} \neq I, W \times W^{\dagger} \neq I, V \times W^{\dagger} \neq I, W \times V^{\dagger} \neq I, \tag{228}$$

under which we have the characterization of the forward and backward genounits and related genoproduct

$$I \rightarrow V \times I \times W^{\dagger} = \hat{I}^{>}, eqno \tag{229}$$

$$A \times B \rightarrow V \times (A \times B) \times W^{\dagger} = A^{>} > B^{>} \tag{230}$$

$$I \rightarrow W \times I \times V = {}^{<}\hat{I}, \tag{231}$$

$$A \times B \rightarrow W \times (A \times B) \times V = {}^{<}A <^{<} B/ \tag{232}$$

10.5. Axiomatic Consistency of GenoMathematics and its Isodual [4.5]

It is easy to see that the above dual non-unitary transformations can always be identically rewritten as the *geno-unitary transforms* on geno-Hilbert spaces over complex genofields,

$$V \times V^{\dagger} \neq 1, V = {}^{<}\hat{V} \times \hat{R}^{1/2}, V \times V^{\dagger} = {}^{<}\hat{V} <^{<} \hat{V}^{\dagger} = {}^{<}\hat{V}^{\dagger} <^{<} \hat{V} = {}^{<}\hat{I}, \tag{233}$$

$$W \times W^{\dagger} \neq 1, W = \widehat{W}^{>} \times \hat{S}^{1/2}, W \times W^{\dagger} = \widehat{W}^{>} > \widehat{W}^{>\dagger} = \widehat{W}^{>\dagger} > \widehat{W}^{>} = \hat{I}^{>}, \tag{234}$$

under which we have indeed the following forward geno-invariance laws [3j]

$$\hat{I}^{>} \rightarrow \hat{I}'^{>} = \widehat{W}^{>} > \hat{I}^{>} > \widehat{W}^{>\dagger} = \hat{I}^{>}, \tag{235}$$

$$\hat{A} > \hat{B} \rightarrow \widehat{W}^{>} > (\hat{A} > \hat{B}) > \widehat{W}^{>\dagger} = \hat{A}' > \hat{B}', \tag{236}$$

$$\hat{H}^{>} > | >= \hat{E}^{>} > | >= E \times | > \rightarrow \widehat{W}^{>} > \hat{H}^{>} > | >= \hat{H}'^{>} > | >' = \widehat{W}^{>} > \hat{E}^{>} > | >= E \times | >', \tag{237}$$

with corresponding rules for the backward and classical counterparts.

The above rules confirm the achievement of the *invariance of the numerical values of genounits, geno-products and geno-eigenvalues*, thus permitting physically consistent applications.

The invariance of the isodual geno-formulations can then be proved via the isodual map applied to the above procedure.

11. Open Mathematical Problems

Among a predictable large number of basic open problems, we list for the interested readers the following ones:

Study methods to transform nonlinear models on

conventional spaces into isolinear models on isospaces over isofields;

See whether simple solutions of isolinear equations on isospaces over isofields provide at least ä" solution of their nonlinear projection on conventional spaces over conventional fields;

Study the removal of divergencies in quantum mechanics and scattering theories (Footnote 2) by isomechanics on an iso-Hilbert space over an isofield.

Study the regular and irregular isorepresentations of the Lie-Santilli isotheory;

Study Santilli isoMinkowskian geometry via the machinery of the Riemannian geometry, yet lack of curvature [39];

Study the Lie-admissible theory in Santilli's sense, that is, as a generalization of Lie's theory elaborated via genomathematics;

Study Santilli geno-Euclidean, geno-Minkowskian and geno-Riemannian geometries where irreversibility is embedded in the non symmetric character of the metric [23];

extend the Tsagas, Ganformina-Nunez isotopology to the genotopic form and their isoduals.

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Santilli's Isoprime Theory

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Abstract: We study Santilli's isomathematics for the generalization of modern mathematics via the isomultiplication $a \hat{\times} a = ab\hat{T}$ and isodivision $a \hat{\div} b = \frac{a}{b}\hat{I}$, where the new multiplicative unit $\hat{I} \neq 1$ is called Santilli isounit, $\hat{T}\hat{I} = 1$, and $\hat{T}$ is the inverse of the isounit, while keeping unchanged addition and subtraction, , In this paper, we introduce the isoaddition $a \hat{+} b = a + b + \hat{0}$ and the isosubtraction $a \hat{-} b = a - b - \hat{0}$ where the additive unit $\hat{0} \neq 0$ is called isozero, and we study Santilli isomathem,atics formulated with the four isooperations $(\hat{+}, \hat{-}, \hat{\times}, \hat{\div})$. We introduce, apparently for the first time, Santilli's isoprime theory of the first kind and Santilli's isoprime theory of the second kind. We also provide an example to illustrate the novel isoprime isonumbers.

Keywords: Isoprimes, Isomultiplication, Isodivision, Isoaddition, Isosubtraction

1. Introduction

Santilli [1] suggests the isomathematics based on the generalization of the multiplication × division ÷ and multiplicative unit 1 in modern mathematics. It is epoch-making discovery. From modern mathematics we establish the foundations of Santilli's isomathematics and Santilli's new isomathematics. We establish Santilli's isoprime theory of both first and second kind and isoprime theory in Santilli's new isomathematics.

1.1. Division and Multiplican in Modern Mathematics

Suppose that

$$a \div a = a^0 = 1, \tag{1}$$

where 1 is called multiplicative unit, 0 exponential zero.

From (1) we define division ÷ and multiplication ×

$$a \div b = \frac{a}{b}, b \neq 0, a \times b = ab, \tag{2}$$

$$a = a \times (a \div a) = a \times a^0 = a \tag{3}$$

We study multiplicative unit 1

$$a \times 1 = a, a \div 1 = a, 1 \div a = 1/a \tag{4}$$

$$(+1)^n = 1, (+1)^{a/b} = 1, (-1)^n = (-1)^n, (-1)^{a/b} = (-1)^{a/b} \tag{5}$$

The addition $+$, subtraction $-$, multiplication × and division ÷ are four arithmetic operations in modern mathematics which are foundations of modern mathematics. We generalize modern mathematics to establish the foundations of Santilli's isomathematics.

1.2. Isodivision and Isomultiplication in Santilli's Isomathematics

We define the isodivision $\hat{\div}$ and isomultiplication $\hat{\times}$ [1-5] which are generalization of division ÷ and multiplication × in modern mathematics.

$$a \hat{\div} a = a^{\bar{0}} = \hat{I} \neq 1, \quad \bar{0} \neq 0, \tag{6}$$

where $\hat{I}$ is called isounit which is generalization of multiplicative unit 1, $\bar{0}$ expeoenential isozero which is generalization of exponential zero.

We have

$$a \hat{\div} b = \hat{I}\frac{a}{b}, b \neq 0, a \hat{\times} b = a\hat{T}b, \tag{7}$$

Suppose that

$$a = a\hat{\times}(a\hat{\div}a) = a\hat{\times}a^{\bar{0}} = a\hat{T}\hat{I} = a. \quad (8)$$

From (8) we have

$$\hat{T}\hat{I} = 1 \quad (9)$$

where $\hat{T}$ is called inverse of isounit $\hat{I}$.

We conjectured [1-5] that (9) is true. Now we prove (9). We study isounit $\hat{I}$

$$a\hat{\times}\hat{I} = a,\ a\hat{\div}\hat{I} = a,\ \hat{I}\hat{\div}a = a^{-\hat{I}} = \hat{I}^2/a, \quad (10)$$

$$(+\hat{I})^{\hat{n}} = \hat{I}, (+\hat{I})^{\frac{\hat{a}}{b}} = \hat{I}, (-\hat{I})^{\hat{n}} = (-1)^n\hat{I}, (-\hat{I})^{\frac{\hat{a}}{b}} = (-1)^{\frac{a}{b}}\hat{I} \quad (11)$$

Keeping unchanged addition and subtraction, $(+,-,\hat{\times},\hat{\div})$ are four arithmetic operations in Santilli's isomathematics, which are foundations of isomathematics. When $\hat{I} = 1$, it is the operations of modern mathematics.

1.3. Addition and Subtraction in Modern Mathematics

We define addition and subtraction

$$x = a + b, \quad y = a - b \quad (12)$$

$$a + a - a = a \quad (13)$$

$$a - a = 0 \quad (14)$$

Using above results we establish isoaddition and isosubtraction

1.4. Isoaddition and Isosubtraction in Santilli's New Isomathematics

We define isoaddition $\hat{+}$ and isosubtraction $\hat{-}$.

$$a\hat{+}b = a + b + c_1,\ a\hat{-}b = a - b - c_2 \quad (15)$$

$$a = a\hat{+}a\hat{-}a = a + c_1 - c_2 = a \quad (16)$$

From (16) we have

$$c_1 = c_2 \quad (17)$$

Suppose that $c_1 = c_2 = \hat{0}$,

where $\hat{0}$ is called isozero which is generalization of addition and subtraction zero

We have

$$a\hat{+}b = a + b + \hat{0}, \quad a\hat{-}b = a - b - \hat{0} \quad (18)$$

When $\hat{0} = 0$, it is addition and subtraction in modern mathematics.

From above results we obtain foundations of santilli's new isomathematics

$$\hat{\times} = \times\hat{T}\times,\ \hat{+} = +\hat{0}+;\ \hat{\div} = \times\hat{I}\div,\ \hat{-} = -\hat{0}-;\ a\hat{\times}b = ab\hat{T},\ a\hat{+}b = a + b + \hat{0};$$

$$a\hat{\div}b = \frac{a}{b}\hat{I},\ a\hat{-}b = a - b - \hat{0};\ a = a\hat{\times}a\hat{\div}a = a,\ a = a\hat{+}a\hat{-}a = a;$$

$$a\hat{\times}a = a^2T, a\hat{+}a = 2a + \hat{0};\ a\hat{\div}a = \hat{I} \neq 1,\ a\hat{-}a = -\hat{0} \neq 0;\ \hat{T}\hat{I} = 1. \quad (19)$$

$(\hat{+},\hat{-},\hat{\times},\hat{\div})$ are four arithmetic operations in Santilli's new isomathematics.

Remark, $a\hat{\times}(b\hat{+}c) = a\hat{\times}(b + c + \hat{0})$, From left side we have

$a\hat{\times}(b\hat{+}c) = a\hat{\times}b + a\hat{\times}\hat{+} + a\hat{\times}c) = a\hat{\times}(b + \hat{+} + c)$

$= a\hat{\times}(b + \hat{0} + c)$, where $\hat{+} = \hat{0}$ also is a number.

$a\hat{\times}(b\hat{-}c) = a\hat{\times}(b - c - \hat{0})$. From left side we have

$a\hat{\times}(b\hat{-}c) = a\hat{\times}b - a\hat{\times}\hat{-} - a\hat{\times}c)$

$= a\hat{\times}(b - \hat{-} - c) = a\hat{\times}(b - \hat{0} - c)$, where $\hat{-} = \hat{0}$ also is a number.

It is satisfies the distributive laws. Therefore $\hat{+}, \hat{-}, \hat{\times}$ and $\hat{\div}$ also are numbers.

It is the mathematical problems in the 21st century and a new mathematical tool for studying and understanding the law of world.

2. Santilli's Isoprime Theory of the First Kind

Let $F(a,+,\times)$ be a conventional field with numbers a equipped with the conventional sum $a + b \in F$, multiplication $ab \in F$ and their multiplicative unit $1 \in F$. Santilli's isofields of the first kind $\hat{F} = \hat{F}(\hat{a},+,\hat{\times})$ are the rings with elements

$$\hat{a} = a\hat{I} \quad (20)$$

called isonumbers, where $a \in F$, the isosum

$$\hat{a} + \hat{b} = (a + b)\hat{I} \quad (21)$$

with conventional additive unit $0 = 0\hat{I} = 0, \hat{a} + 0 = 0 + \hat{a} = \hat{a}$, $\forall \hat{a} \in \hat{F}$ and the isomultiplications is

$$\hat{a}\hat{\times}\hat{b} = \hat{a}\hat{T}\hat{b} = a\hat{I}\hat{T}b\hat{I} = (ab)\hat{I}. \quad (22)$$

Isodivision is

$$\hat{a} \hat{\div} \hat{b} = \hat{I}\frac{a}{b} \tag{23}$$

We can partition the positive isointegers in three classes:
(1)The isouniti $\hat{I}$;
(2)The isonumbers: $\hat{1} = \hat{I}, \hat{2}, \hat{3}, \hat{4}, \hat{5}, \cdots$;
(3)The isoprime numbers: $\hat{2}, \hat{3}, \hat{5}, \hat{7}, \cdots$.
Theorem 1. Twin isoprime theorem

$$\hat{P}_1 = \hat{P} + \hat{2}. \tag{24}$$

Jiang function is

$$J_2(\omega) = \prod_{3 \le P}(P-2) \ne 0, \tag{25}$$

whre $\omega = \prod_{2 \le P} P$ is called primorial.

Since $J_2(\omega) \ne 0$, there exist infinitely many isoprimes $\hat{P}$ such that $\hat{P}_1$ is an isoprime.

We have the best asymptotic formula of the number of isoprimes less than $\hat{N}$

$$\pi_2(\hat{N}, 2) \sim \frac{J_2(\omega)\omega}{\phi^2(\omega)} \frac{N}{\log^2 N}, \tag{26}$$

where

$$\phi(\omega) = \prod_{2 \le P}(P-1).$$

Let $\hat{I} = 1$. From (24) we have twin prime theorem

$$P_1 = P + 2 \tag{27}$$

Theorem 2. Goldbach isoprime theorem

$$\hat{N} = \hat{P}_1 + \hat{P}_2 \tag{28}$$

Jiang function is

$$J_2(\omega) = \prod_{3 \le P}(P-2)\prod_{P|N}\frac{P-1}{P-2} \ne 0 \tag{29}$$

Since $J_2(\omega) \ne 0$ every isoeven number $\hat{N}$ greater than $\hat{4}$ is the sum of two isoprimes.

We have

$$\pi_2(\hat{N}, 2) \sim \frac{J_2(\omega)}{\phi^2(\omega)} \frac{N}{\log^2 N}. \tag{30}$$

Let $\hat{I} = 1$. From (28) we have Goldbach theorem

$$N = P_1 + P_2 \tag{31}$$

Theorem 3. The isoprimes contain arbitrarily long arithmetic progressions. We define arithmetic progressions of isoprimes:

$$\hat{P}_1, \hat{P}_2 = \hat{P}_1 + \hat{d}, \hat{P}_3 = \hat{P}_1 + \hat{2}\hat{\times}\hat{d}, \cdots, \hat{P}_k = \hat{P}_1 + (\hat{k} - \hat{I})\hat{\times}\hat{d}, (\hat{P}_1, \hat{d}) = \hat{I}. \tag{32}$$

Let $\hat{I} = 1$. From (32) we have arithmetic progressions of primes:

$$P_1, P_2 = P_1 + d, P_3 = P_1 + 2d, \cdots, P_k = P_1 + (k-1)d, (P_1, d) = 1. \tag{33}$$

We rewrite (33)

$$P_3 = 2P_2 - P_1, P_j = (j-1)P_2 - (j-2)P_1, 3 \le j \le k. \tag{34}$$

Jiang function is

$$J_3(\omega) = \prod_{3 \le P}[(P-1)^2 - \chi(P)], \tag{35}$$

$\chi(P)$ denotes the number of solutions for the following congruence

$$\prod_{j=3}^{k}[(j-1)q_2 - (j-2)q_1] \equiv 0 (\bmod P), \tag{36}$$

where $q_1 = 1, 2, \cdots, P-1; q_2 = 1, 2, \cdots, P-1$.

From (36) we have

$$J_3(\omega) = \prod_{3 \le P < k}(P-1)\prod_{k \le P}(P-1)(P-k+1) \ne 0. \tag{37}$$

We prove that there exist infinitely many primes P_1 and P_2 such that $P_3, \cdots, P_k$ are all primes for all $k \ge 3$.

We have the best asymptotic formula

$$\pi_{k-1}(N, 3) = \left|\{(j-1)P_2 - (j-2)P_1 = \text{prime}, 3 \le j \le k, P_1, P_2 \le N\}\right|$$

$$\sim \frac{J_3(\omega)\omega^{k-2}}{2\phi^k(\omega)} \frac{N^2}{\log^k N} = \frac{1}{2}\prod_{2 \le P < k}\frac{P^{k-2}}{(P-1)^{k-1}}\prod_{K \le P}\frac{P^{k-2}(P-k+1)}{(P-1)^{k-1}}\frac{N^2}{\log^k N}. \tag{38}$$

Theorem 4. From (33) we obtain

$$P_4 = P_3 + P_2 - P_1, \ P_j = P_3 + (j-3)P_2 - (j-3)P_1, 4 \le j \le k. \tag{39}$$

Jiang function is

$$J_4(\omega)=\prod_{3\le P}((P-1)^3-\chi(P)), \quad (40)$$

$\chi(P)$ denotes the number of solutions for the following congruence

$$\prod_{j=4}^{k}[q_3+(j-3)q_2-(j-3)q_1]\equiv 0(\bmod P), \quad (41)$$

where $q_i=1,2,\cdots,P-1, i=1,2,3.$

Frome (41) we have

$$J_4(\omega)=\prod_{3\le P<(k-1)}(P-1)^2\prod_{(k-1)\le P}(P-1)[(P-1)^2-(P-2)(k-3)]\ne 0\,. \quad (42)$$

We prove there exist infinitely many primes P_1, P_2 and P_3 such that $P_4,\cdots,P_k$ are all primes for all $k\ge 4$.

We have the best asymptotic formula

$$\pi_{k-2}(N,4)=\left|\{P_3+(j-3)P_2-(j-3)P_1=\text{prime},4\le j\le k,P_1,P_2,P_3\le N\}\right|\sim\frac{J_4(\omega)\omega^{k-3}}{6\phi^k(\omega)}\frac{N^3}{\log^k N}$$

$$=\frac{1}{6}\prod_{2\le P<(k-1)}\frac{P^{k-3}}{(P-1)^{k-2}}\prod_{(k-1)\le P}\frac{P^{k-3}[(P-1)^2-(P-2)(k-3)]}{(P-1)^{k-1}}\frac{N^3}{\log^k N} \quad (43)$$

The prime distribution is order rather than random. The arithmetic progressions in primes are not directly related to ergodic theory, harmonic analysis, discrete geometry and combinatorics. Using the ergodic theory Green and Tao prove there exist arbitrarily long arithmetic progressions of primes which is false [6,7,8,9,10].

Theorem 5. Isoprime equation

$$P_2=\hat{P}_1+2=P_1\hat{I}+2\,. \quad (44)$$

Let $\hat{I}$ be the odd number. Jiang function is

$$J_2(\omega)=\prod_{3\le P}(P-2)\prod_{P|\hat{I}}\frac{P-1}{P-2}\ne 0\,. \quad (45)$$

Since $J_2(\omega)\ne 0$, there exist infinitely primes P_1 such that P_2 is a prime.

We have

$$\pi_2(N,2)\sim\frac{J_2(\omega)\omega}{\phi^2(\omega)}\frac{N}{\log^2 N}\,. \quad (46)$$

Theorem 6. Isomprime equation

$$P_2=(\hat{P}_1)^{\hat{2}}+2=P_1^2\hat{I}+2\,. \quad (47)$$

Let $\hat{I}$ be the odd number. Jiang function is

$$J_2(\omega)=\prod_{3\le P}(P-2-X(P))\,, \quad (48)$$

where

$$X(P)=\begin{cases}(-\frac{-2I}{P})\\ -1 \quad \text{if} \quad P\big|\hat{I}\end{cases}$$

If $(\frac{-2I}{3})=-1$, there infinitely many primes P_1 such that P_2 is a prime. If $(\frac{-2I}{3})=1, J_2(3)=0$, there exist finite primes P_1 such that P_2 is a prime.

3. Santilli'S Isoprime Theory of the Second Kind

Santilli's isofields of the second kind $\hat{F}=\hat{F}(a,+,\hat{\times})$ (that is, $a\in F$ is not lifted to $\hat{a}=a\hat{I}$) also verify all the axioms of a field.

The isomultiplication is defined by

$$a\hat{\times}b=a\hat{T}b\,. \quad (49)$$

We then have the isoquotient, isopower, isosquare root, etc.,

$$a\hat{\div}b=(a/b)\hat{I},\ a^{\hat{n}}=a\hat{\times}\cdots\hat{\times}a\ (\text{ntimes})=a^n(\hat{T})^{n-1},a^{\hat{1}/2}=a^{1/2}(\hat{I})^{1/2}\,. \quad (50)$$

Theorem 7. Isoprime equations

$$P_2=P_1^{\hat{2}}+6,P_3=P_1^{\hat{2}}+12,P_4=P_1^{\hat{2}}+18 \quad (51)$$

Let $T=1$. From (51) we have

$$P_2=P^2+6,P_3=P_1^2+12,P_4=P_1^2+18\,, \quad (52)$$

Jiang function is

$$J_2(\omega)=2\prod_{5\le P}(P-4-(\frac{-6}{P})-(\frac{-3}{P})-(\frac{-2}{P}))\neq 0, \quad (53)$$

where $(\frac{-6}{P}), (\frac{-3}{P})$ and $(\frac{-2}{P})$ denote the Legendre symbols.

Since $J_2(\omega)\neq 0$, there exist infinitely many primes P_1 such that P_2, P_3 and P_4 are primes.

$$\pi_4(N,2)\sim\frac{J_2(\omega)\omega^3}{8\phi^4(\omega)}\frac{N}{\log^4 N} \quad (54)$$

Let $\hat{T}=5$. From (51) we have

$$P_2=5P_1^2+6, P_3=5P_1^2+12, P_4=5P_1^2+18. \quad (55)$$

Jiang function is

$$J_2(\omega)=8\prod_{7\le P}(P-4-(\frac{-30}{P})-(\frac{-15}{P})-(\frac{-10}{P}))\neq 0. \quad (56)$$

Since $J_2(\omega)\neq 0$, there exist infinitely many primes P_1 such that P_2, P_3 and P_4 are primes.

We have

$$\pi_4(N,2)\sim\frac{J_2(\omega)\omega^3}{8\phi^4(\omega)}\frac{N}{\log^4 N}. \quad (57)$$

Let $\hat{T}=7$. From (51) we have

$$P_2=7P_1^2+6, P_3=7P_1^2+12, P_4=7P_1^2+18. \quad (58)$$

We have Jiang function

$$J_2(5)=0. \quad (59)$$

There exist finite primes P_1 such that P_2, P_3 and P_4 are primes.

Theorem 8. Isoprime equations

$$P_2=P_1^{\hat{2}}+30, P_2=P_1^{\hat{2}}+60, P_4=P_1^{\hat{2}}+90, P_5=P_1^{\hat{2}}+120. \quad (60)$$

Let $\hat{T}=7$. From (60) we have

$$P_2=7P_1^2+30, P_3=7P_1^2+60, P_4=7P_1^2+90, P_5=7P_1^2+120 \quad (61)$$

Jiang function is

$$J_2(\omega)=48\prod_{11\le P}(P-5-\sum_{j=1}^{4}(\frac{-210j}{P}))\neq 0. \quad (62)$$

Since $J_2(\omega)\neq 0$, there exist infinitely many primes P_1 such that P_2, P_3, P_4 and P_5 are primes.

We have

$$\pi_5(N,2)\sim\frac{J_2(\omega)\omega^4}{16\phi^5(\omega)}\frac{N}{\log^5 N}. \quad (63)$$

Let $\hat{T}\ge 7$ be the odd prime. From (60) we have

$$P_k=P_1^2\hat{T}+30(k-1), k=2,3,4,5. \quad (64)$$

Jiang function is

$$J_2(\omega)=8\prod_{7\le P}(P-5-\chi(P))\neq 0. \quad (65)$$

If $P|\hat{T}$, $\chi(P)=4$; $\chi(P)=\sum_{j=1}^{4}(\frac{-30\hat{T}j}{P})$ otherwise.

Since $J_2(\omega)\neq 0$, there exist infinitely many primes P_1 such that P_2, P_3, P_4 and P_5 are primes.

We have

$$\pi_5(N,2)\sim\frac{J_2(\omega)\omega^4}{16\phi^5(\omega)}\frac{N}{\log^5 N}. \quad (66)$$

Theorem 9. Isoprime equation

$$P_3=P_2\hat{\times}(P_1+b)-b. \quad (67)$$

Let $\hat{T}=1$ Jiang function is

$$J_2(\omega)=\prod_{3\le P\le P_i}(P^2+3P+3-\chi(P))\neq 0, \quad (68)$$

where $\chi(P)=-P+2$ if $P|b$; $\chi(P)=0$ otherwise.

Since $J_3(\omega)\neq 0$, there exist infinitely many primes P_1 and P_2 such that P_3 is also a prime.

The best asymptotic formula is

$$\pi_2(N,3)=|\{P_1,P_2:P_1,P_2\le N; P_3=\text{prime}\}|\sim\frac{J_3(\omega)\omega}{4\phi^3(\omega)}\frac{N^2}{\log^3 N}. \quad (69)$$

Theorem 10. Isoprime equation

$$P_3=P_2\hat{\times}(P_1^{\hat{2}}+b)-b \quad (70)$$

Let $\hat{T}=1$ Jiang function is

$$J_3(\omega)=\prod_{3\le P\le P_i}(P^2-3P+3+\chi(P))\neq 0 \quad (71)$$

where $\chi(P)=P-2$ if $P|b$; $\chi(P)=(\frac{-b}{P})$ otherwise.

Since $J_3(\omega)\neq 0$, there exist infinitely many primes P_1 and P_2 such that P_3 is also a prime.

The best asymptotic formula is

$$\pi_2(N,3)=\left|\{P_1,P_2:P_1,P_2\le N;P_3=\text{prime}\}\right|\sim\frac{J_3(\omega)\omega}{6\phi^3(\omega)}\frac{N^2}{\log^3 N}. \quad (72)$$

Theorem 11. Isoprime equation

$$P_3=P_2^{\hat{2}}(P_1+1)-1. \quad (73)$$

Let $\hat{T}=1$. Jiang function is

$$J_2(\omega)=\prod_{3\le P\le P_i}(P^2-3P+4)\ne 0 \quad (74)$$

Since $J_3(\omega)\ne 0$, there exist infinitely many primes P_1 and P_2 such that P_3 is also a prime.

The best asymptotic formula is

$$\pi_2(N,3)=\left|\{P_1,P_2:P_1,P_2\le N;P_3=\text{prime}\}\right|$$

$$\sim\frac{J_3(\omega)\omega}{6\phi^3(\omega)}\frac{N^2}{\log^3 N}. \quad (75)$$

4. Isoprime Theory in Santilli's New Isomathematics

Theorem 12. Isoprime equation

$$P_3=P_1\hat{+}P_2=P_1+P_2+\hat{0}. \quad (76)$$

Suppose $\hat{0}=1$. From (76) we have

$$P_3=P_1+P_2+1. \quad (77)$$

Jiang function is

$$J_3(\omega)=\prod_{3\le P}(P^2-3P+3)\ne 0. \quad (78)$$

Since $J_3(\omega)\ne 0$, there exist infinitely many primes P_1 and P_2 such that P_3 is also a prime.

We have the best asymptotic formula is

$$\pi_2(N,3)\sim\frac{J_3(\omega)\omega}{2\phi^3(\omega)}\frac{N^2}{\log^3 N}. \quad (79)$$

Theorem 13. Isoprime equation

$$P_3=(P_1\hat{+}2)\hat{\times}(P_1\hat{-}2)\hat{+}P_2=\hat{T}[P_1^2-(2+\hat{0})^2]+P_2+\hat{0} \quad (80)$$

Suppose $\hat{T}=6$ and $\hat{0}=4$. From (80) we have

$$P_3=6(P_1^2-36)+P_2+4 \quad (81)$$

Jiang function is

$$J_3(\omega)=\prod_{3\le P}(P^2-3P+2)\ne 0. \quad (82)$$

Since $J_3(\omega)\ne 0$, there exist infinitely many primes P_1 and P_2 such that P_3 is also a prime.

We have the best asymptotic formula is

$$\pi_2(N,3)\sim\frac{J_3(\omega)\omega}{4\phi^3(\omega)}\frac{N^2}{\log^3 N}. \quad (83)$$

5. An Example

We give an example to illustrate the Santilli's isomathematics.

Suppose that algebraic equation

$$y=a_1\times(b_1+c_1)+a_2\div(b_2-c_2) \quad (84)$$

We consider that (84) may be represented the mathematical system, physical system, biological system, IT system and another system. (84) may be written as the isomathematical equation

$$\hat{y}=a_1\hat{\times}(b_1\hat{+}c_1)\hat{+}a_2\hat{\div}(b_2\hat{-}c_2)=a_1\hat{T}(b_1+c_1+\hat{0})+\hat{0}+a_2/\hat{T}(b_2-c_2-\hat{0}). \quad (85)$$

If $\hat{T}=1$ and $\hat{0}=0$, then $y=\hat{y}$.

Let $\hat{T}=2$ and $\hat{0}=3$. From (85) we have the isomathematical subequation

$$\hat{y}_1=2a_1(b_1+c_1+3)+3+a_2/2(b_2-c_2-3). \quad (86)$$

Let $\hat{T}=5$ and $\hat{0}=6$. From (85) we have the isomathematical subequation

$$\hat{y}_2=5a_1(b_1+c_1+6)+6+a_2/5(b_2-c_2-6). \quad (87)$$

Let $\hat{T}=8$ and $\hat{0}=10$. From (85) we have the isomathematical subequation

$$\hat{y}_3=8a_1(b_1+c_1+10)+10+a_2/8(b_2-c_2-10) \quad (88)$$

From (85) we have infinitely many isomathematical subequations. Using (85)-(88), $\hat{T}$ and $\hat{0}$ we study stability and optimum structures of algebraic equation (84).

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Measurable Iso-Functions

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Abstract: In this article are given definitions definition for measurable is-functions of the first, second, third, fourth and fifth kind. They are given examples when the original function is not measurable and the corresponding iso-function is measurable and the inverse. They are given conditions for the isotopic element under which the corresponding is-functions are measurable. It is introduced a definition for equivalent iso-functions. They are given examples when the iso-functions are equivalent and the corresponding real functions are not equivalent. They are deducted some criterions for measurability of the iso-functions of the first, second, third, fourth and fifth kind. They are investigated for measurability the addition, multiplication of two iso-functions, multiplication of iso-function with an iso-number and the powers of measurable iso-functions. They are given definitions for step iso-functions, iso-step iso-functions, characteristic iso-functions, iso-characteristic iso-functions. It is investigate for measurability the limit function of sequence of measurable iso-functions. As application they are formulated the iso-Lebesgue's theorems for iso-functions of the first, second, third, fourth and fifth kind. These iso-Lebesgue's theorems give some information for the structure of the iso-functions of the first, second, third, fourth and fifth kind.

Keywords: Measurable Iso-Sets, Measurable Is-Functions, Is-Lebesgue Theorems

1. Introduction

Genious idea is the Santilli's generalization of the basic unit of quantum mechanics into an integro-differential operator $\hat{I}$ which is as positive-definite as +1 and it depends of local variables and it is assumed to be the inverse of the isotopic element $\hat{T}$

$$+1 > 0 \to \hat{I}(t, r, p, a, E, \cdots) = \frac{1}{\hat{T}} > 0$$

and it is called Santilli isounit. Santilli introduced a generalization called lifting of the conventional associative product ab into the form

$$ab \to a\hat{x}b = a\hat{T}b$$

Called isoproduct for which

$$\hat{I}\hat{x}a = \frac{1}{\hat{T}}\hat{T}a = a\hat{x}\hat{I} = a\hat{T}\frac{1}{\hat{T}} = a$$

For every element a of the field of real numbers, complex numbers and quaternions. The Santilli isonumbers are defi ned as follows: for given real number or complex number or quaternion a,

$$\hat{a} = a\hat{I}$$

With isoproduct

$$\hat{a}\hat{x}\hat{b} = \hat{a}\hat{T}\hat{b} = a\frac{1}{\hat{T}}\hat{T}b\frac{1}{\hat{T}} = ab\frac{1}{\hat{T}} = \widehat{ab}.$$

If $a \neq 0$, the corresponding isoelement of $\frac{1}{a}$ will be denoted with $\hat{a}^{-1}$or $\hat{I} \searrow \hat{a}$.

With $\hat{F}_{\mathbb{R}}$ we will denote the field of the is-numbers $\hat{a}$ for which $a \in \mathbb{R}$ and basic isounit $\hat{I}_1$.

In [1], [3]-[12] are defined isocontinuous isofunctions and isoderivative of isofunction and in [1] are proved some of their properties. If $\hat{D}_1$ is an isoset in $\hat{F}_{\mathbb{R}}$, the class of isofunctions is denoted by $\widehat{FC}_{D_1}$ and the class of isodifferentiable isofunctions is denoted by $\widehat{FC}^1_{D_1}$, with the same basic isounit $\hat{I} = \frac{1}{\hat{T}}$, it is supposed

$$\hat{T} \in C^1(D_1), \hat{T} > 0 \; in \; D_1.$$

Here D_1 is the corresponding real set of $\hat{D}_1$. If x is an independent variable, then the corresponding lift is $\frac{x}{\hat{T}(x)}$, if f is real-valued function on D_1, then the corresponding lift of first kind is defined as follows

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f\left(\hat{t}(x)\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f(x)}{\hat{T}(x)}$$

and we will denote it by $\hat{f}^{\wedge\wedge}$.

In accordance with [1], the isodifferential is defined as follows

$$\hat{d}(\cdot) = \hat{T}(x)d(\cdot).$$

Then

$$\hat{d}(\hat{x}) = \hat{T}(x)d(\hat{x}) = \hat{T}(x)d\left(\frac{x}{\hat{T}(x)}\right) = \hat{T}(x)\left(\frac{1}{\hat{T}(X)} - X\frac{\hat{T}'(x)}{\hat{T}^2(x)}\right)dx = \left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx,$$

$$\hat{d}\left(\hat{f}^{\wedge}(\hat{x})\right) = \hat{T}(x)d\left(\hat{f}^{\wedge}(\hat{x})\right) = \widehat{T(x)d}\left(\frac{f(x)}{\hat{T}(x)}\right) = \left(f'(x) - f(x)\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx.$$

In accordance with [1], the first is-derivative of the is-function $\hat{f}^{\wedge\wedge}$ is defined as follows

$$\left(\hat{f}^{\wedge\wedge}\ (x)\right)_x^{\otimes} = \hat{d}\left(\hat{f}^{\wedge}(\hat{x})\right) \nearrow \hat{d}(\hat{x}) = \frac{1}{\hat{T}(x)}\frac{\hat{d}\left(\hat{f}^{\wedge}(\hat{x})\right)}{\hat{d}(\hat{x})} = \frac{f'(x)\hat{T}(x) - f(x)\widehat{T'(x)}}{\hat{T}^2(x) - x\hat{T}(x)\hat{T}'(x)}.$$

When $\hat{T}(x) \equiv 1$, then

$$\left(\hat{f}^{\wedge}(\hat{x})\right)_x^{\otimes} = f'(x).$$

Our aim in this article is to be investigated some aspects of theory of measurable iso-functions. The paper is organized as follows. In Section 2 are defined measurable iso-functions and they are deducted some of their properties. In Section 3 is investigated the structure of the measurable iso-functions.

2. The Definition and the Simplest Properties of Measurable Is-Functions

We suppose that A is a given point set, $\hat{T}: A \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$ for every $x \in A$, $\hat{T}_1 > 0$ be a given constant, $f: A \longrightarrow \mathbb{R}$ bea given real-valued function. With $\hat{f}$ we will denote the corresponding is-function of the first, second, third, fourth and fifth kind. More precisely,

1. $\hat{f}(x) \equiv \hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)}$, when $\hat{f}$ is an is-function of the first kind.
2. $\hat{f}(x) = \hat{f}^{\wedge}(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)}$, when $\frac{x}{\hat{T}(x)} \in A$ for $x \in A$, when $\hat{f}$ is an is-function of the second kind.
3. $\hat{f}(x) = \hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}, \frac{x}{\hat{T}(x)} \in A$ for $x \in A$, when $\hat{f}$ is an is-function of the third kind.
4. $\hat{f}(x) \equiv f^{\wedge}(x) = f(x\hat{T}(x))$, when $x\hat{T}(x) \in A$ for $x \in A$, when $\hat{f}$ is an is-function of the fourth kind.
5. $\hat{f}(x) = f^{\vee}(x) = f\left(\frac{x}{\hat{T}(x)}\right), \frac{x}{\hat{T}(x)} \in A$ for $x \in A$, when $\hat{f}$ is an is-function of the fifth kind.

For $a \in A$ with $A(\hat{f} > a)$ we will denote the set

$$A(\hat{f} > a) := \{x \in A: \hat{f}(x) > a\}.$$

We define the symbols $A(\hat{f} \geq a)$, $A(\hat{f} = a)$, $A(\hat{f} < a)$, $A(a < \hat{f} < b)$ and etc., in the same way.

If the set on which the is-function $\hat{f}$ is defined is designated by a letter C or D, we shall write $C(\hat{f}$>a) or $D(\hat{f}$>a).

Definition 2.1. The is-function $\hat{f}$ is said to be measurable if

1. The set A is measurable.
2. The set $A(\hat{f} > a)$ is measurable for all $a \in A$.

Theorem 2.3. Let $\hat{f}$ be a measurable is-function defined on the set A. If B is a measurable subset of A, then the is-function $\hat{f}(x)$, considered only for$x \in B$, is measurable.

Proof. Let $a \in \mathbb{R}$ be arbitrarily chosen and fixed. We will prove that

$$B(\hat{f} > a) = B \cap A(\hat{f} > a). \tag{1}$$

Really, let $x \in B(\hat{f} > a)$ be arbitrarily chosen. Then $x \in B$ and $\hat{f}$(x)>a. Since $B \subset A$, we have that $x \in A$. From $x \in A$ and $\hat{f}(x) > a$ it follows that $x \in A(\hat{f} > a)$. Because $x \in B(\hat{f} > a)$ was arbitrarily chosen and for it we get that it is an element of the set $B \cap A(\hat{f} > a)$, we conclude that

$$B \subset (\hat{f} > a) \subset B \cap A(\hat{f} > a). \tag{2}$$

Let now $x \in B \cap A(\hat{f} > a)$ be arbitrarily chosen. Then $x \in B$ and $x \in A(\hat{f} > a)$. Hence $x \in B$ and $\hat{f}(x) > a$. Therefore $x \in B(\hat{f} > a)$. Because $x \in B \cap A(\hat{f} > a)$ was arbitrarily chosen and we get that it is an element of $B(\hat{f} > a)$, we conclude that

$$B \cap A(\hat{f} > a) \subset B(\hat{f} > a).$$

From the last relation and from (2) we prove the relation (1).

Since the iso-function $\hat{f}$ is a measurable function on the set A, we have that $A(\hat{f} > a)$ is a measurable set. As the intersection of two measurable sets is a measurable set, we have that $B \cap A(\hat{f} > a)$ is a measurable set. Consequently, using (1), the set $B(\hat{f}$>a) is measurable set. In this way we have

1. B is a measurable set,
2. $B(\hat{f} > a)$ is a measurable set for all $a \in \mathbb{R}$.

Therefore the iso-function $\hat{f}$, considered only for $x \in B$, is a measurable is-function.

Theorem 2.4. Let $\hat{f}$ be defined on the set A, which is the union of a finite or denumerable number of measurable sets A_K, $A = \cup_K A_k$. If $\hat{f}$ is measurable on each of the sets A_k, then it is also measurable on A.

Proof. Let $a \in \mathbb{R}$ be arbitrarily chosen. We will prove that

$$A(\hat{f} > a) = \cup_k A_k(\hat{f} > a). \qquad (3)$$

Let $x \in A(\hat{f} > a)$ be arbitrarily chosen. Then $x \in A$ and $\hat{f}(x) > a$. Since $x \in A$ and $A = \cup_k A_k$, there exists k such that $x \in A_k$. Therefore $x \in A_k$ and $\hat{f}(x) > a$. Hence, $x \in A_k(\hat{f} > a)$ and $x \in \cup_k A_k(\hat{f} > a)$. Because $x \in A(\hat{f} > a)$ was arbitrarily chosen and for it we get that it is an element of $\cup_k A_k(\hat{f} > a)$, we conclude that

$$A(\hat{f} > a) \subset \cup_k A_k(\hat{f} > a). \qquad (4)$$

Let now $y \in \cup_k A_k(\hat{f} > a)$ be arbitrarily chosen. Then there exists l such that $y \in A_l(\hat{f}$>a). From here $x \in A_l$ and $\hat{f}(y) > a$. Hence, $y \in A = \cup_k A_k$ and $\hat{f}(y) > a$. Consequently $\in A(\hat{f} > a)$. Because $y \in \cup_k A_k(\hat{f} > a)$ was arbitrarily chosen and for it we get that it is an element of $A(\hat{f} > a)$ we conclude that

$$A(\hat{f} > a) \subset \bigcup_k A_k(\hat{f} > a).$$

From the last relation and from (4) we prove the relation (3).

Since the union of finite or denumerable number of measurable sets is a measurable set, using that the sets $A_k(\hat{f} > a)$ are measurable, we obtain that A and $A(\hat{f} > a)$ are measurable sets. Therefore $\hat{f}$ is a measurable is-function.

Definition 2.5. Two is-functions $\hat{f}$ and $\hat{g}$, defined on the same setr A, are said to be equivalent if

$$\mu\left(A(\hat{f} \neq \hat{g})\right) = 0.$$

We will write

$$\hat{f} \sim \hat{g}.$$

Remark 2.6. There is a possibility $f \nsim g$ and in the same time$\hat{f} \sim \hat{g}$.

Let

$$A = [1,2], f(x) = x, g(x) = x + 1,$$

$$\hat{T}(x) = \frac{-1 + \sqrt{1 + 4x^2}}{2x}, x \in A.$$

Then

$$f \nsim g.$$

On the oth-1+er hand,

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x}{\frac{-1 + \sqrt{1 + 4x^2}}{x}}$$

$$= \frac{2x^2}{-1 + \sqrt{1 + 4x^2}} = \frac{2x^2\left(1 + \sqrt{1 + 4x^2}\right)}{\left(\sqrt{1 + 4x^2} - 1\right)\left(\sqrt{1 + 4x^2} - 1\right)} = \frac{2x^2\left(1 + \sqrt{1 + 4x^2}\right)}{4x^2} = \frac{1 + \sqrt{1 + 4x^2}}{2},$$

$$g^{\wedge}(x) = g\left(x\hat{T}(x)\right) = x\hat{T}(x) + 1$$

$$= x\frac{-1 + \sqrt{1 + 4x^2} + 1}{2x} = \frac{-1 + \sqrt{1 + 4x^2}}{2} + 1 = \frac{1 + \sqrt{1 + 4x^2}}{2}.$$

We have that

$$\mu\left(A(\hat{f}^{\wedge\wedge} \neq g^{\wedge})\right) = 0$$

Or

$$\hat{f}^{\wedge\wedge} \sim g^{\wedge}.$$

Remark 2.7. There is a possibility $f \sim g$ and in the same time $\hat{f} \nsim \hat{g}$. Let

$$A = [1,2], f(x) = g(x) = x^2, \hat{T}(x) = x + 1, x \in A.$$

Then

$$f \sim g.$$

On the other hand,

$$f^{\wedge}(x) = f\left(x\hat{T}(x)\right) = x^2\hat{T}^2(x) = x^2(x+1)^2, g^{\vee}(x) = g\left(\frac{x}{\hat{T}(x)}\right) = \frac{x^2}{\hat{T}^2(x)} = \frac{x^2}{(x+1)^2}.$$

Then

$$f^{\wedge}(x) = g^{\vee}(x) \iff x^2(x+1)^2 = \frac{x^2}{(x+1)^2} \iff (x+1)^4 = 1 \iff x = 0 \notin A.$$

Therefore

$$\mu(A(f^{\wedge} = g^{\vee})) = 0,$$

Hence,

$$\mu(A(f^{\wedge} \neq g^{\vee})) = 1.$$

Consequently

$$f^{\wedge} \sim g^{\vee}.$$

Proposition 2.8. The fuAnctions f and g are equivalent if and only if the functions $\hat{f}^{\wedge\wedge}$ and $\hat{g}^{\wedge\wedge}$ are equivalent

Proof. We have

$$\mu(A(f \neq g)) = 0 \iff \mu\left(A\left(\frac{f}{\hat{T}} \neq \frac{g}{\hat{T}}\right)\right) = 0$$

$$\iff \mu\left(A(\hat{f}^{\wedge\wedge} \neq \hat{g}^{\wedge\wedge})\right) = 0.$$

Definition 2.9. Let some property P holds for all the points of the set A, except for the points of a subset B of the set A. If = 0 , we say that the property P holds almost everywhere on the set A, or for almost all points of A.

Definition 2.10. We say that two is-functions defined on the set A are equivalent if they are equal almost everywhere on the set A.

Theorem 2.11. If $\hat{f}(x)$ is a measurable is-function defined on the set A, and if $\hat{f} \sim \hat{g}$, then the is-function $\hat{g}(x)$ is also measurable.

Proof. Let

$$B := A(\hat{f} \neq \hat{g}), D := A \setminus B.$$

Because $\hat{f} \sim \hat{g}$ we have that

$$\mu\left(A(\hat{f} \neq \hat{g})\right) = 0$$

or $\mu B = 0$.

Since every function, definite on a set with measure zero is measurable on it, we have that the is-function $\hat{g}$ is measurable on the set B.

We note that the is-functions $\hat{f}(x)$ and $\hat{g}(x)$ are identical on D and since the is0-function $\hat{f}$ is measurable on D, we get that the is-function $\hat{g}$ is measurable on D.

Consequently the is-function $\hat{g}$ is measurable on

$$B \cup D = A.$$

Theorem 2.12. If the is-function $\hat{f}(x)$, defined on the set A, is measurable, then the sets

$$A(\hat{f} \geq a), A(\hat{f} = a), A(\hat{f} \leq a), A(\hat{f} < a)$$

Are measurable for all $a \in \mathbb{R}$.

Proof. We will prove that

$$A(\hat{f} \geq a) = \prod_{n=1}^{\infty} A\left(\hat{f} > a - \tfrac{1}{n}\right). \qquad (5)$$

Really, let $x \in A(\hat{f} \geq a)$ be arbitrarily chosen. Then $x \in A$ and $\hat{f}(x) \geq a$. Hence, for every $n \in \mathbb{N}$ we have $\hat{f}(x) > a - \frac{1}{n}$. Therefore

$$x \in \prod_{n=1}^{\infty} A\left(\hat{f} > a - \tfrac{1}{n}\right).$$

Because $x \in A(\hat{f} \geq a)$ was arbitrarily chosen and for it we obtain $x \in \prod_{n=1}^{\infty} A\left(\hat{f} > a - \frac{1}{n}\right)$,

We conclude that

$$A(\hat{f} \geq a) \subset \prod_{n=1}^{\infty} A\left(\hat{f} > a - \tfrac{1}{n}\right). \qquad (6)$$

Let now $x \in \prod_{n=1}^{\infty} A\left(\hat{f} > a - \frac{1}{n}\right)$ be arbitrarily chosen. Then $x \in A\left(\hat{f} > a - \frac{1}{n}\right)$ for every natural number n. From here $x \in A$ and

$$\hat{f}(x) > a - \frac{1}{n}$$

For all natural number n. Consequently

$$\lim_{n \to \infty} \hat{f}(x) \geq \lim_{n \to \infty} \left(a - \frac{1}{n}\right)$$

or

$$\hat{f}(x) \geq a$$

and $x \in A(\hat{f} \geq a)$. Since $x \in \prod_{n=1}^{\infty} A\left(\hat{f} > a - \frac{1}{n}\right)$ was arbitrarily chosen and we get that $x \in A(\hat{f} \geq a)$, we conclude

$$\prod_{n=1}^{\infty} A\left(\hat{f} > a - \frac{1}{n}\right) \subset A(\hat{f} \geq a).$$

From the last relation and from (6) we obtain the relation (5).

Because the intersection of denumerable measurable sets is a measurable set, using the relation (5) and the fact that all sets $A\left(\hat{f} > a - \frac{1}{n}\right)$ are measurable for all natural numbers n, we conclude that the set $A(\hat{f} \geq a)$ is a measurable set.

The set $A(\hat{f} = a)$ is a measurable set because

$$A(\hat{f} = a) = A(\hat{f} \geq a) \setminus A(\hat{f} > a).$$

The set $A(\hat{f} \leq a)$ is measurable set since

$$A(\hat{f} \leq a) = A \setminus A(\hat{f} > a).$$

The set $A(\hat{f} < a)$ is measurable since

$$A(\hat{f} < a) = A \setminus A(\hat{f} \geq a).$$

Remark 2.13. We note that if at least one of the sets

$$A(\hat{f} \geq a), A(\hat{f} = a), A(\hat{f} \leq a), A(\hat{f} < a)$$

Is measurable for all $a \in \mathbb{R}$, then the iso-function $\hat{f}$ is measurable on the set A.

Really, let $A(\hat{f} \geq a)$ is measurable for all $a \in \mathbb{R}$. Then, using the relation

$$A(\hat{f} > a) = \prod_{n=1}^{\infty} A\left(\hat{f} \geq a - \tfrac{1}{n}\right), \qquad (7)$$

we obtain that the set $A(\hat{f} > a)$ is measurable for all $a \in \mathbb{R}$.

If $A(\hat{f} \leq a))$ is measurable for all $a \in \mathbb{R}$, then using the relation

$$A(\hat{f} > a) = A \setminus A(\hat{f} \leq a),$$

we get that the set $A(\hat{f} > a)$ is measurable for all $a \in \mathbb{R}$.

If $A(\hat{f} < a)$ is measurable for all $a \in \mathbb{R}$, then using the relation

$$A(\hat{f} > a) = A \setminus A(\hat{f} \leq a),$$

We conclude that the set $A(\hat{f} > a)$ is measurable for all $a \in \mathbb{R}$.

Theorem 2.14. If $\hat{f}(x) = c = const$ for all points of a measurable set A, then the is-function $\hat{f}(x)$ is measurable.

Proof. For all $a \in \mathbb{R}$ we have that

$$A(\hat{f} > a) = A \text{ if } c > a \text{ and } A(\hat{f} > a) = \emptyset \text{ if } c \leq a.$$

Since the sets A and $\emptyset$ are measurable sets, then $A(\hat{f} > a)$

is measurable for all $a \in \mathbb{R}$. Therefore the is-function $\hat{f}(x)$ is measurable.

Definition 2.15. An is-function $\hat{f}(x)$ defined on the closed interval [a, b] is said to be a step is-function if there is a finite number of points

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$$

Such that $\hat{f}(x)$ is a constant on (a_i, a_{i+1}) , $i = 0, 1, 2, \cdots, n-1$.

Proposition 2.16. A step is-function is measurable.

Proof. Let $\hat{f}$(x) is a step is-function on the closed interval [a, b]. Let also,

$$a = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = b$$

be such that $\hat{f}(x)$ is a constant on (a_i, a_{i+1}) , $i = 0, 1, 2, \cdots, n-1$. From the previous theorem we have that $\hat{f}(x)$ is measurable on (a_i, a_{i+1}), $i = 0, 1, 2, \cdots, n$. We note that

the sets $\{a_i\}$, $i = 0, 1, 2, \cdots, n-1$, are sets with measure zero. Therefore the is-function

$\hat{f}(x)$ is measurable on $\{a_i\}$, $i = 0, 1, 2, \cdots, n$. From here, using that

$$[a, b] = \bigcup_{i=0}^{n} (a_i, a_{i+1}) \bigcup \bigcup_{i=0}^{n} \{a_i\},$$

We conclude that the is-function $\hat{f}(x)$ is measurable on [a, b].

Theorem 2.17. If the is-function $\hat{f}(x)$, defined on the set A is measurable and $c \in \mathbb{R}, c \neq 0$, then the is-functions

1. $\hat{f}(x) + c$,
2. $c\hat{f}(x)$,
3. $|\hat{f}(x),|$
4. $\hat{f}^2(x)$,
5. $\frac{1}{\hat{f}(x)}$,

are also measurable.

Proof. Let $a \in \mathbb{R}$ be arbitrarily chosen. The assertion follows from the following relations.

1. $A(\hat{f} + c > a) = A(\hat{f} > c - a)$.
2. $A(c\hat{f} > a) = A\left(\hat{f} > \frac{a}{c}\right)$ if c>0, $A(c\hat{f} > a) = A\left(\hat{f} < \frac{a}{c}\right)$ if c<0.
3. $A(|\hat{f}| > a) = A$ if a<0, $A(|\hat{f}| > a) = A(\hat{f} > a) \cup A(\hat{f} < -a)$ if $a \geq 0$.
4. $A(\hat{f}^2 > a) = A$ if a<0, [$A(\hat{f}^2 > a) = A(|\hat{f}| > \sqrt{a})$ if $a \geq 0$.
5. $A\left(\frac{1}{\hat{f}} > a\right) = A(\hat{f} > 0) \cap A\left(\hat{f} < \frac{1}{a}\right)$ if a>0, $A\left(\frac{1}{\hat{f}} > a\right) = A(\hat{f} > 0) \cup \left(A(\hat{f} < 0) \cap A\left(\hat{f} < \frac{1}{a}\right)\right)$ if a<0, $A\left(\frac{1}{\hat{f}} > a\right) = A(\hat{f} > 0)$ if a=0.

Definition 2.18. An is-function $\hat{f}$, defined on the closed interval [pa, b], is said to be is-step is-function, if there is a finite number of points

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b,$$

such that

$$\hat{f}(x) = \frac{c_i}{\hat{T}(x)}, x \in [a_i, a_{i+1}), c_i = const, i = 0, 1, \cdots, n-1.$$

Theorem 2.19. Let $\hat{T}$(x)>0 for every x ∈ [a, b] and $\hat{T}$(x) is measurable on [a, b]. Let also, $\hat{T}$(x) is an iso-step is-function on [a, b]. Then $\hat{f}$(x) is measurable on [a, b].

Proof. Let

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b,$$

be such that

$$\hat{f}(x) = \frac{c_i}{\hat{T}(x)}, x \in [a_i, a_{i+1}), c_i = const, i = 0, 1, \cdots, n-1.$$

From the last theorem it follows that $\frac{c_i}{\hat{T}(x)}$ is a measurable is-function on $[a_{i,}a_{i+1})$, $i = 0, 1, 2, \cdots, n-1$. Fromn-1 here and from

$$[a, b] = \bigcup_{i=0}^{n-1} [a_i, a_{i+1}) \cup \{b\}.$$

Since {b} is a set with measure zero, we conclude that the is-step is-function $\hat{f}$ is measurable on [a, b].

Definition 2.20. Let M be a subset of the closed interval [a, b]. The function $\varphi_M(x) = 0$ for $x \in [a, b] \setminus M$ and $\varphi_M = 1$ for $x \in M$, is called the characteristic function of the set M.

Theorem 2.21. If the set M is a measurable subset of the closed interval A=[a, b], then the characteristic function $\varphi_M(x)$ is measurable on [a, b].

Proof. The assertion follows from the following relations. $A(\varphi_M > a) = \emptyset$ if $a \geq 1$, $A(\varphi_M > a) = M$ if $1 > a \geq 0$, $A(\varphi_M > a) = A$ if a<0.

Definition 2.22. Let M be a subset of the set A=[a, b]. The iso- function $\hat{\varphi}_M(x) = 0$ if $x \in A \setminus M$ and $\hat{\varphi}_M = \frac{1}{\hat{T}(x)}$ if $x \in M$, will be called characteristic is-function of the set M.

Theorem 2.23. Let $\hat{T}(x)$ be a measurable function on A=[a, b], M be a measurable subset of A. Then the characteristic is-function $\hat{\varphi}_M(x)$ of the set M is measurable.

Proof. Let $a \in \mathbb{R}$ be arbitrarily chosen. Then

$$A(\hat{\varphi}_M > a) = (A \setminus M)(0 > a) \cup M\left(\frac{1}{\hat{T}(x)} > a\right),$$

From here, using that the sets $(A \setminus M)(0 > a)$ and $M\left(\frac{1}{\hat{T}(x)} > a\right)$ are measurable sets, we conclude that $A(\hat{\varphi}_M > a)$ is a measurable set. Because the constant a was arbitrarily chosen, we have that the characteristic function $\hat{\varphi}_M$ is a measurable is-function.

Theorem 2.24. Let f and $\hat{T}$ are continuous functions on the closed set A. Then the is-function $\hat{f}^\wedge(\hat{x})$ is measurable.

Proof. Let $a \in \mathbb{R}$ be arbitrarily chosen. Since every closed set is a measurable set, we conclude that the set A is a measurable set.

We will prove that the set $A(\hat{f}^{\wedge\wedge} \leq a)$ is a closed set.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements of the set $A(\hat{f}^{\wedge\wedge} \leq a)$ such that

$$\lim_{n\to\infty} x_n = x_0.$$

Since $A(\hat{f}^{\wedge\wedge} \leq a)$ is a subset of the set A we have that $\{x_n\}_{n=1}^{\infty} \subset A$. Because the set A is a closed set, we obtain that $x_0 \in A$. From the definition of the set $A(\hat{f}^{\wedge\wedge} \leq a)$ we have that

$$\hat{f}^{\wedge}(\hat{x}_n) = \frac{f(x_n)}{\hat{T}(x_n)} \leq a,$$

Hence, when $n \to \infty$, using that f and $\hat{T}$ are continuous functions on the set A, we get

$$\lim_{n\to\infty} \hat{f}^{\wedge}(\hat{x}_n) = \lim_{n\to\infty} \frac{f(x_n)}{\hat{T}(x_n)} = \frac{f(x_0)}{\hat{T}(x_0)} = \hat{f}^{\wedge}(\hat{x}_0) \leq a,$$

i.e., $x_0 \in A(\hat{f}^{\wedge\wedge} \leq a)$. Therefore the set $A(\hat{f}^{\wedge\wedge} \leq a)$ is a closed set. From here, the set $A(\hat{f}^{\wedge\wedge} \leq a)$ is a measurable set. Because the difference of two measurable sets is a measurable set, we have that the set

$$A(\hat{f}^{\wedge\wedge} > a) = A \setminus A(\hat{f}^{\wedge\wedge} \leq a)$$

Is a measurable set.

Since $a \in \mathbb{R}$ was arbitrarily chosen, we obtain that the is-function of the first kind $\hat{f}^{\wedge\wedge}$ is measurable.

Theorem 2.25. Let f and $\hat{T}$ are continuous functions on the closed set A. The the is-functions

$$\hat{f}^{\wedge}(x), \hat{f}(\hat{x}), f^{\wedge}(x), f^{\vee}(x)$$

are measurable on A.

Theorem 2.26. If two measurable is-functions $\hat{f}$ and $\hat{g}$ are defined on the set A, then the set $A(\hat{f} > \hat{g})$ is measurable.

Proof. We enumerate all rational numbers

$$r_1, r_2, r_3, \cdots.$$

We will prove that

$$A(\hat{f} > \hat{g}) = \cup_{k=1}^{\infty}\left(A(\hat{f} > r_k) \cap A(\hat{g} < r_k)\right). \quad (8)$$

Let

$$x \in A(\hat{f} > \hat{g})$$

Be arbitrarily chosen. Then

$$x \in A, \hat{f}(x) > \hat{g}(x).$$

There exists a rational number r_k such that

$$\hat{f}(x) > r_k > \hat{g}(x).$$

Therefore

$$x \in A \textit{ and } \hat{f}(x) > r_k; x \in A \textit{ and } r_k > \hat{g}(x),$$

i.e.,

$$x \in A(\hat{f} > r_k), x \in A(\hat{g} < r_k).$$

Consequently

$$x \in A(\hat{f} > r_k) \cap A(\hat{g} < r_k)$$

And

$$x \in \bigcup_{k=1}^{\infty}\left(A(\hat{f} > r_k) \cap A(\hat{g} < r_k)\right).$$

Because $x \in A(\hat{f} > \hat{g})$ was arbitrarily chosen and for it we get $x \in \cup_{k=1}^{\infty}\left(A(\hat{f} > r_k) \cap A(\hat{g} < r_k)\right)$, we conclude that

$$A(\hat{f} > \hat{g}) \subset \cup_{k=1}^{\infty}\left(A(\hat{f} > r_k) \cap A(\hat{g} < r_k)\right). \quad (9)$$

Let no

$$x \in \bigcup_{k=1}^{\infty}\left(A(\hat{f} > r_k) \cap A(\hat{g} < r_k)\right)$$

be arbitrarily chosen. Then there exists a natural k so that

$$x \in A(\hat{f} > r_k) \cap A(\hat{g} < r_k).$$

Hence,

$$x \in A(\hat{f} > r_k), x \in A(\hat{g} < r_k).$$

Then

$$x \in A, \hat{f}(x) > r_k, r_k < \hat{g}(x)$$

or

$$x \in A, \hat{f}(x) > r_k > \hat{g}(x).$$

Therefore

$$x \in A(\hat{f} > \hat{g}).$$

Because

$$x \in \bigcup_{k=1}^{\infty}\left(A(\hat{f} > r_k) \cap A(\hat{g} < r_k)\right)$$

Was arbitrarily chosen and for it we get that $x \in A(\hat{f} > \hat{g})$, we conclude that

$$\bigcup_{k=1}^{\infty}\left(A(\hat{f} > r_k) \cap A(\hat{g} < r_k)\right) \subset A(\hat{f} > \hat{g}).$$

From the last relation and from the relation (9) we get the relation (8).

Since $\hat{f}$ and $\hat{g}$ are measurable iso-functions on A, we have that the sets

$$A(\hat{f} > r_k), A(\hat{g} < r_k)$$

are measurable sets for every natural k, whereupon the sets

$$A(\hat{f} > r_k) \cap A(\hat{g} < r_k)$$

Are measurable sets for every natural k.

Therefore, using the relation (8), we obtain that the set $A(\hat{f} > \hat{g})$ is a a measurable set.

Theorem 2.27. Let $\hat{f}(x)$ and $\hat{g}(x)$ be finite measurable is-functions on the set A. Then each of the is-functions

1. $\hat{f}(x) - \hat{g}(x)$,
2. $\hat{f}(x) + \hat{g}(x)$,
3. $\hat{f}(x)\hat{g}(x)$,
4. $\frac{\hat{f}(x)}{\hat{g}(x)}$ if $\hat{g}(x) \neq 0$ on A,

Is measurable.

Proof.

1. Let $a \in \mathbb{R}$ be arbitrarily chosen. Since $\hat{g}(x)$ is measurable, then $a + \hat{g}(x)$ is measurable. From here and from the last theorem it follows that the set

$$A(\hat{f}(x) - \hat{g}(x) > a) = A(\hat{f}(x) > a + \hat{g}(x))$$

Is measurable. Because $a \in \mathbb{R}$ was arbitrarily chosen, we conclude that the function $\hat{f}(x) - \hat{g}(x)$ is measurable.

2. Since $\hat{g}$ is a measurable is-function, we have that the function $-\hat{g}$ is a measurable is-function. From here and from 1) we conclude that the is-function

$$\hat{f} + \hat{g} = \hat{f} - (-\hat{g})$$

Is measurable.

3. We note that

$$\hat{f}(x)\hat{g}(x) = \frac{1}{2}\left(\hat{f}(x) + \hat{g}(x)\right)^2 - \frac{1}{2}\left(\hat{f}(x) - \hat{g}(x)\right)^2. \quad (10)$$

Since $\hat{f}(x)$ and $\hat{g}(x)$ are measurable iso-functions, using 1) and 2) we have that

$$\hat{f}(x) + \hat{g}(x) \text{and } \hat{f}(x) - \hat{g}(x)$$

Are measurable is-functions. Hence the is-functions

$$\left(\hat{f}(x) + \hat{g}(x)\right)^2, \left(\hat{f}(x) - \hat{g}(x)\right)^2$$

Are measurable, whereupon

$$\frac{1}{2}\left(\hat{f}(x) + \hat{g}(x)\right)^2 \text{ and } \frac{1}{2}\left(\hat{f}(x) - \hat{g}(x)\right)^2$$

Are measurable. From here, using 1) and (10), we conclude that $\hat{f}(x)\hat{g}(x)$ is measurable.

4. Since $\hat{g}(x)$ is measurable and $\hat{g}(x) \neq 0$ on A, we have that the is-function $\frac{1}{\hat{g}(x)}$ is measurable. From here and from 3) the is-function

$$\frac{\hat{f}(x)}{\hat{g}(x)} = \hat{f}(x)\frac{1}{\hat{g}(x)}$$

Is measurable.

Theorem 2.28. Let $\{\hat{f}_n(x)\}_{n=1}^{\infty}$ be a sequence of measurable is-functions defined on the set A. If

$$\lim_{n\to\infty} \hat{f}_n(x) = \hat{f}(x) \quad (11)$$

Exists for every $x \in A$, then the is-function $\hat{f}(x)$ is measurable.

Proof. Let $a \in \mathbb{R}$ be arbitrarily chosen. For $n, k, m \in \mathbb{N}$ we define the sets

$$A_{m,k} := A\left(\hat{f}_k > a + \frac{1}{m}\right), B_{m,n} := \prod_{k=n}^{\infty} A_{m,k}.$$

We will prove that

$$A(\hat{f} > a) = \cup_{n,m} B_{m,n}. \quad (12)$$

Let

$$x \in A(\hat{f} > a)$$

Be arbitrarily chosen. Then

$$x \in A \text{ and } \hat{f}(x) > a.$$

Hence, there is enough large natural number m_1 such that

$$\hat{f}(x) > a + \frac{1}{m_1}.$$

Using (11), there are enough large natural numbers k and m such that

$$\hat{f}_k(x) > a + \frac{1}{m},$$

i.e., $x \in A_{m,k}$.

From here, it follows that there is enough large n so that $x \in A_{m,k}$ for every $k \geq n$, i.e., $x \in B_{m,n}$ and then $x \in \cup_{m,n} B_{m,n}$.

Since $x \in A(\hat{f} > a)$ was arbitrarily chosen and we get that it is an element of the set $\cup_{m,n} B_{m,n}$, we conclude that

$$A(\hat{f} > a) \subset \cup_{m,n} B_{m,n}. \quad (13)$$

Let now $x \in \cup_{m,n} B_{m,n}$ be arbitrarily chosen.

Then, there are $m_2, n \in \mathbb{N}$ so that

$$x \in B_{m_2,n_1} = \prod_{k=n_1}^{\infty} A_{m_2,k_1}$$

or

$$\hat{f}_{k_1}(x) > a + \frac{1}{m_2} \text{ for } \forall k \geq n_1.$$

Hence,

$$\lim_{k_1\to\infty} \hat{f}_{k_1}(x) \geq \lim_{k_1\to\infty}\left(a + \frac{1}{m_2}\right)$$

or

$$\hat{f}(x) \geq a + \frac{1}{m_2} > a.$$

Therefore

$$x \in A(\hat{f} > a).$$

Since $x \in \bigcup_{m,n} B_{m,n}$ was arbitrarily chosen and for it we obtain $x \in A(\hat{f} > a)$, we conclude that

$$\bigcup_{m,n} B_{m,n} \subset A(\hat{f} > a).$$

From the last relation and from (13) it follows the relation (12).

Since $\hat{f}_k(x)$ are measurable, we have that the sets $A_{m,k}$ are measurable for every $m, k \in \mathbb{N}$, hence $B_{m,n}$ are measurable for every $m, n \in \mathbb{N}$ and then, using (12), the set $A(\hat{f} > a)$ is measurable. Consequently the is-function $\hat{f}$ is measurable.

Theorem 2.29. be a sequence of measurable is-functions defined on the set A. If

$$\lim_{n\to\infty} \hat{f}_n(x) = \hat{f}(x) \quad (14)$$

Exists for almost everywhere $x \in A$, then the is-function $\hat{f}(x)$ is measurable.

Proof. Let B be the subset of A so that the relation (14) holds for every $x \in B$. From the previous theorem it follows that the is-function $\hat{f}(x)$ is measurable on the set B.

We note that

$$\mu(A \setminus B) = 0.$$

Therefore the is-function $\hat{f}(x)$ is measurable on $A \setminus B$. Hence, the is-function $\hat{f}(x)$ is measurable on A.

Let

$$\hat{T}_n, \hat{T}: A \to (0, \infty), f_n, f: A \to \mathbb{R},$$

$$0 < q_1 \leq \hat{T}_n(x), \hat{T}(x) \leq q_2 for\ x \in A, n \in \mathbb{N}.$$

Then

1. $\hat{f}_n^{\wedge}(\hat{x}) = \frac{f_n(x)}{\hat{T}_n(x)}, \hat{f}^{\wedge}(x) = \frac{f(x)}{\hat{T}(x)},$
2. $\hat{f}_n^{\wedge}(x) = \frac{f_n(x\hat{T}_n(x))}{\hat{T}_n(x)}, \hat{f}^{\wedge}(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)}$

   If

$$x\hat{T}_n(x), x\hat{T}(x), x \in A,$$

3. $\hat{f}_n(\hat{x}) = \frac{f_n\left(\frac{x}{\hat{T}_n(x)}\right)}{\hat{T}_n(x)}, \hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}$

   If

$$\frac{x}{\hat{T}_n(x)}, \frac{x}{\hat{T}(x)}, x \in A,$$

4. $f_n^{\wedge}(x) = f_n(x\hat{T}_n(x)), f^{\wedge}(x) = f(x\hat{T}(x)),$

   If

$$x\hat{T}_n(x), x\hat{T}(x), x \in A,$$

5. $f_n^{\vee}(x) = f_n\left(\frac{x}{\hat{T}_n(x)}\right), f^{\vee}(x) = f\left(\frac{x}{\hat{T}(x)}\right)$

   If

$$\frac{x}{\hat{T}_n(x)}, \frac{x}{\hat{T}(x)}, x \in A.$$

3. The Structure of the Measurable Is-Functions

Theorem 3.1. (is-Lebesgue theorem for is-functions of the first kind) Let there be given a sequence $\{f_n(x)\}_{n=1}^{\infty}$ of measurable functions on a set A, all of which are finite almost everywhere. Let also, $\{\hat{T}_n(x)\}_{n=1}^{\infty}$ be a sequence of measurable functions on the set A,

$$0 < q_1 \leq \hat{T}_n(x) \leq q_2$$

For all natural numbers n and for all $x \in A$, where q_1 and q_2 are positive constants. Suppose that

$$\lim_{n\to\infty} f_n(x) = f(x),$$

$$\lim_{n\to\infty} \hat{T}_n(x) = \hat{T}(x)$$

Almost everywhere on the set A, and $f(x)$ is finite almost everywhere on A,

$$q_1 \leq \hat{T}(x) \leq q_2$$

For all $x \in A$. Then

$$\lim_{n\to\infty} \mu A(|\hat{f}_n^{\wedge}(\hat{x}) - \hat{f}^{\wedge}(\hat{x})| \geq \sigma) = 0$$

For all $\sigma \geq 0$.

Proof. We will note that the limit functions f(x) and $\hat{T}(x)$ are measurable and the sets under considerations are measurable.

Let

$$A := A(|f| = \infty),$$

$$B_n := A(|f_n| = \infty),$$

$$C := A(f_n \nrightarrow f),$$

$$D := B \cup \left(\bigcup_{n=1}^{\infty} B_n\right) \cup C.$$

Since

$$\mu B = 0, \mu C = 0, \mu B_n = 0,$$

using the properties of the measurable sets, we have that

$$\mu \mathbb{Q} = 0.$$

Let

$$A_k(\sigma) = A\left(\left|\frac{f_k}{\hat{T}_k} - \frac{f}{\hat{T}}\right| \geq \sigma\right),$$

$$R_n(\sigma) = \bigcup_{k=n}^{\infty} A_k(\sigma),$$

$$M = \prod_{n=1}^{\infty} R_n(\sigma).$$

We have that

$$R_1(\sigma) \supset R_2(\sigma) \supset \cdots.$$

Hence,

$$\lim_{n\to\infty} \mu R_n(\sigma) = \mu M.$$

Let us assume that $x_0 \notin \mathbb{Q}$. Then, using the definition of the set $\mathbb{Q}$, we have

$$\lim_{n\to\infty} \frac{f_k(x_0)}{\hat{T}_k(x_0)} = \frac{f(x_{0)}.}{\hat{T}(x_0)}$$

Since

$$0 < q_1 \le \hat{T}_n(x), \widehat{T}(x) \le q_{2\,,k=1,2,\cdots,n,}$$

we have that

$$\frac{f_1(x_0)}{\hat{T}_1(x_0)}, \frac{f_2(x_0)}{\hat{T}_2(x_0)}, \dots, \frac{f_k(x_0)}{\hat{T}_k(x_0)}, \dots$$

and their limit

$$\frac{f(x_{0)}.}{\hat{T}(x_0)}$$

are finite.

Therefore there is an enough large natural n such that

$$\left|\frac{f_k(x_0)}{\hat{T}_k(x_0)} - \frac{f(x_{0)}.}{\hat{T}(x_0)}\right| < \sigma$$

for every $k \ge n$. Then $x_0 \notin A_k(\sigma), k \ge n$, where $x_0 \notin R_n(\sigma)$ and from here $x_0 \notin M$.

Consequently $M \subset \mathbb{Q}$.

Because $\mu\mathbb{Q} = 0$, from the last relation, we have that $\mu M = 0$, i.e.,

$$\lim_{n\to\infty} R_n(\sigma) = 0,$$

and since

$$A_n(\sigma) \subset R_n(\sigma),$$

$$\lim_{n\to\infty} R_n(\sigma) = 0$$

or

$$\lim_{n\to\infty} \mu A\left(\left|\hat{f}_n^{\wedge}(\hat{x}) - \hat{f}^{\wedge}(\hat{x})\right| \ge \sigma\right) = 0.$$

As in above one can prove the following results for the other kinds of is-functions.

Theorem 3.2. (is-Lebesgue theorem for is-functions of the second kind) Let there be given a sequence $\{f_n(x)\}_{n=1}^{\infty}$ of measurable functions on a set A, all of which are finite almost everywhere. Let also, $\{\hat{T}_n(x)\}_{n=1}^{\infty}$ be a sequence of measurable functions on the set A,

$$0 < q_1 \le \hat{T}_n(x) \le q_2$$

For all natural numbers n and for all $x \in A$, where q_1 and q_2 are positive constants. Suppose that

$$\lim_{n\to\infty} f_n(x) = f(x),$$

$$\lim_{n\to\infty} \hat{T}_n(x) = \hat{T}(x)$$

Almost everywhere on the set A, and $f(x)$ is finite almost everywhere on A,

$$q_1 \le \hat{T}(x) \le q_2$$

For all $x \in A$. Then

$$\lim_{n\to\infty} \mu A\left(\left|\hat{f}_n^{\wedge}(x) - \hat{f}^{\wedge}(x)\right| \ge \sigma\right) = 0$$

for all $\sigma \ge 0$.

Theorem 3.3. (is-Lebesgue theorem for is-functions of the third kind) Let there be given a sequence $\{f_n(x)\}_{n=1}^{\infty}$ of measurable functions on a set A, all of which are finite almost everywhere. Let also, $\{\hat{T}_n(x)\}_{n=1}^{\infty}$ be a sequence of measurable functions on the set A,

$$0 < q_1 \le \hat{T}_n(x) \le q_2$$

For all natural numbers n and for all $x \in A$, where q_1 and q_2 are positive constants. Suppose that

$$\lim_{n\to\infty} f_n(x) = f(x),$$

$$\lim_{n\to\infty} \hat{T}_n(x) = \hat{T}(x)$$

Almost everywhere on the set A, and $f(x)$ is finite almost everywhere on A,

$$q_1 \le \hat{T}(x) \le q_2$$

For all $x \in A$. Then

$$\lim_{n\to\infty} \mu A\left(\left|\hat{f}_n\ (\hat{x}) - \hat{f}\ \ (\hat{x})\right| \ge \sigma\right) = 0$$

for all $\sigma \ge 0$.

Theorem 3.4. (is-Lebesgue theorem for is-functions of the fourth kind) Let there be given a sequence $\{f_n(x)\}_{n=1}^{\infty}$ of measurable functions on a set A, all of which are finite almost everywhere. Let also, $\{\hat{T}_n(x)\}_{n=1}^{\infty}$ be a sequence of measurable functions on the set A,

$$0 < q_1 \le \hat{T}_n(x) \le q_2$$

For all natural numbers n and for all $x \in A$, where q_1 and q_2 are positive constants. Suppose that

$$\lim_{n\to\infty} f_n(x) = f(x),$$

$$\lim_{n\to\infty} \hat{T}_n(x) = \hat{T}(x)$$

Almost everywhere on the set A, and $f(x)$ is finite almost everywhere on A,

$$q_1 \leq \hat{T}(x) \leq q_2$$

For all $x \in A$. Then

$$\lim_{n\to\infty} \mu A(|f_n^{\wedge}(x) - f^{\wedge}(x)| \geq \sigma) = 0$$

For all $\sigma \geq 0$.

Theorem 3.5. (is-Lebesgue theorem for is-functions of the fifth kind) Let there be given a sequence $\{f_n(x)\}_{n=1}^{\infty}$ of measurable functions on a set A, all of which are finite almost everywhere. Let also, $\{\hat{T}_n(x)\}_{n=1}^{\infty}$ be a sequence of measurable functions on the set A,

$$0 < q_1 \leq \hat{T}_n(x) \leq q_2$$

For all natural numbers n and for all $x \in A$, where q_1 and q_2 are positive constants. Suppose that

$$\lim_{n\to\infty} f_n(x) = f(x),$$

$$\lim_{n\to\infty} \hat{T}_n(x) = \hat{T}(x)$$

Almost everywhere on the set A, and $f(x)$ is finite almost everywhere on A,

$$q_1 \leq \hat{T}(x) \leq q_2$$

For all $x \in A$. Then

$$\lim_{n\to\infty} \mu A(|f_n^{\vee}(x) - f^{\vee}(x)| \geq \sigma) = 0$$

for all $\sigma \geq 0$.

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Santilli Isomathematics for Generalizing Modern Mathematics

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Abstract: The establishment of isomathematics, as proposed by R. M. Santilli thirty years ago in the USA, and contributed to by Jiang Chun-Xuan in China during the past 12 years, is significant and has changed modern mathematics. At present, the primary teaching of mathematics is based on the simple operations of addition, subtraction, multiplication and division; a middle level teaching ofmathematics takes these four operations to a higher level, while the university teaching of mathematics extends them to an even higher level. These four arithmetic operations form the foundation of modern mathematics. Santilli isomathematics is a generalisation of these four fundamental operations and heralds a great revolution in mathematics. HIn this paper, we study the four generalized arithmetic operations of isoaddition, isosubtraction, isomultiplication and isodivision at the primary level of isomathematics. The material introduced here should be readily understandable by middle school pupils and university students.Santilli's isomathematics [1] ßßis based on a generalisation of modern mathematics. Isomultiplication is defined by $a \hat{\times} a = ab\hat{T}$, isodivision by $a \hat{\div} b = \frac{a}{b}\hat{I}$, where $\hat{I} \neq 1$ is called an isounit; $\hat{T}\hat{I} = 1$, where $\hat{T}$ is the inverse of the isounit. If addition and subtraction remain unchanged, $(\hat{+}, \hat{-}, \hat{\times}, \hat{\div})$ are the four arithmetic operations in Santilli's isomathematics[1-5]. Isoaddition $a \hat{+} b = a + b + \hat{0}$ and isosubtraction $a \hat{+} b = a + b + \hat{0}$, where $\hat{0} \neq 0$ is called the isozero, together with the operations of isomultiplication and isodivision introduced above, form the four arithmetic operations$(\hat{+}, \hat{-}, \hat{\times}, \hat{\div})$ in Santilli-Jiang isomathematics[6]. Santilli [1] suggests isomathematics based on a generalisation of multiplication ×, division ÷, and the multiplicative unit 1 of modern mathematics. It is an epoch making suggestion. From modern mathematics, the foundations of Santilli's isomathematics will be established.

Keywords: Santilli-Jiang Math, Isomultiplication, Isodivision, Isoaddition and Isosubtraction

1. Division and Multiplication in Modern Mathematics

Suppose that

$$a \div a = a^0 = 1 \tag{1}$$

where 1 is the multiplicative unit and 0 is exponent zero.

From (1), division ÷ and multiplication × are defined by

$$a \div b = \frac{a}{b}, b \neq 0, a \times b = ab \tag{2}$$

$$a = a \times (a \div a) = a \times a^0 = a \tag{3}$$

Now consider the multiplicative unit 1,

$$a \times 1 = a, a \div 1 = a, 1 \div a = 1/a \tag{4}$$

$$(+1)^n = 1, (+1)^{a/b} = 1, (-1)^n = (-1)^n, (-1)^{a/b} = (-1)^{a/b} \tag{5}$$

Addition +, subtraction -, multiplication ×, and division ÷ are the four operations forming the foundation of modern mathematics. The modern mathematics is generalised to establish the foundations of Santilli-Jiang isomathematics.

2. Isodivision and Isomultiplication in Santilli's Isomathematics

Isodivision $\hat{\div}$ and isomultiplication $\hat{\times}$ [1 – 5], which are generalisations of the division ÷ and multiplication × of modern mathematics, are now defined.

$$a\hat{\div}a = a^{\bar{0}} = \hat{I} \neq 1, \quad \bar{0} \neq 0 \tag{6}$$

where $\hat{I}$ is called the isounit and is a generalisation of the multiplicative unit 1 and $\bar{0}$ is the isoexponent zero which is a generalisation of the exponent zero 0. Then,

$$a\hat{\div}b = \hat{I}\frac{a}{b}, b \neq 0, a\hat{\times}b = a\hat{T}b \tag{7}$$

It is seen that

$$a = a\hat{\times}(a\hat{\div}a) = a\hat{\times}a^{\bar{0}} = a\hat{T}\hat{I} = a \tag{8}$$

from which it follows that

$$\hat{T}\hat{I} = 1 \tag{9}$$

where $\hat{T}$ is the inverse of the isounit $\hat{I}$.

The isounit $\hat{I}$ has the following properties[5,p93-95,isoexponents]:

$$a\hat{\times}\hat{I} = a, \ a\hat{\div}\hat{I} = a, \ \hat{I}\hat{\div}a = a^{-\hat{I}} = \hat{I}^2 / a \tag{10}$$

$$(+\hat{I})^{\hat{n}} = \hat{I}, (+\hat{I})^{\frac{\hat{a}}{\hat{b}}} = \hat{I}, (-\hat{I})^{\hat{n}} = (-1)^{n}\hat{I}, (-\hat{I})^{\frac{\hat{a}}{\hat{b}}} = (-1)^{\frac{a}{b}}\hat{I} \tag{11}$$

With addition and subtraction unchanged, $(+, -, \hat{\times}, \hat{\div})$ are the four arithmetic operations in Santilli's isomathematics and these form the foundations of Santilli isomathematics. When $\hat{I} = 1$, the operations revert to being those of the modern mathematics.

3. Addition and Subtraction in Modern Mathematics

These are defined by

$$\mathrm{x = a + b \text{ and } y = a - b} \tag{12}$$

$$\mathrm{a + a - a = a} \tag{13}$$

$$\mathrm{a - a = 0} \tag{14}$$

Isoaddition and isosubtraction may be established using these results.

4. Isoaddition and Isosubtraction in Santilli-Jiang Isomathematics

Isoaddition $\hat{+}$ and isosubtraction $\hat{-}$ are defined by

$$a\hat{+}b = a + b + c_1, \ a\hat{-}b = a - b - c_2 \tag{15}$$

$$\therefore a = a\hat{+}a\hat{-}a = a + c_1 - c_2 = a \tag{16}$$

Then, from (16), it follows that

$$c_1 = c_2 \tag{17}$$

Suppose that $c_1 = c_2 = \hat{0}$, where $\hat{0}$ is called the isozero which is a generalisation of the zero 0 of addition and subtraction[6]. Hence,

$$a\hat{+}b = a + b + \hat{0}, \quad a\hat{-}b = a - b - \hat{0} \tag{18}$$

When $\hat{0} = 0$, these equations are the usual laws of addition and subtraction of modern mathematics.

From the above results, the foundations of Santilli-Jiang isomathematics are readily established:

$$\begin{gathered}\hat{\times} = \times\hat{T}\times, \ \hat{+} = +\hat{0}+; \ \hat{\div} = \times\hat{I}\div, \ \hat{-} = -\hat{0}-; a\hat{\times}b = ab\hat{T}, a\hat{+}b = a + b + \hat{0}; \\ a\hat{\div}b = \frac{a}{b}\hat{I}, a\hat{-}b = a - b - \hat{0}; a = a\hat{\times}a\hat{\div}a = a, a = a\hat{+}a\hat{-}a = a; \\ a\hat{\times}a = a^2T, a\hat{+}a = 2a + \hat{0}; a\hat{\div}a = \hat{I} \neq 1, a\hat{-}a = -\hat{0} \neq 0; \hat{T}\hat{I} = 1.\end{gathered} \tag{19}$$

Here $(\hat{+}, \hat{-}, \hat{\times}, \hat{\div})$ are the four arithmetic operations in Santilli-Jiang isomathematics.

Remark:

$$a\hat{\times}(b\hat{+}c) = a\hat{\times}(b + c + \hat{0})$$

From the left-hand side, it follows

$$\begin{aligned} a\hat{\times}(b\hat{+}c) &= a\hat{\times}b + a\hat{\times}\hat{+} + a\hat{\times}c) = a\hat{\times}(b + \hat{+} + c) \\ &= a\hat{\times}(b + \hat{0} + c), \end{aligned}$$

where $\hat{+} = \hat{0}$ is a number also.

Again,

$$a\hat{\times}(b\hat{-}c) = a\hat{\times}(b - c - \hat{0})$$

From the left-hand side of this relation, it is seen that

$$\begin{aligned} a\hat{\times}(b\hat{-}c) &= a\hat{\times}b - a\hat{\times}\hat{-} - a\hat{\times}c) \\ &= a\hat{\times}(b - \hat{-} - c) = a\hat{\times}(b - \hat{0} - c), \end{aligned}$$

where $\hat{-} = \hat{0}$ is a number also.

The distributive laws are satisfied and $\hat{+}, \hat{-}, \hat{\times}$, $\hat{\div}$ are numbers.

This Santilli-Jiang isomathematics therefore, provides a new mathematical tool for studying the mathematical problems of the 21st century and helping in the understanding the mysteries of our universe.

5. An Illustrative Example

Consider the algebraic equation

$$y = a_1 \times (b_1 + c_1) + a_2 \div (b_2 - c_2) \quad (20)$$

(20) may represent a mathematical system, physical system, biological system,cryptogram system, IT system, or some other system. It may be written as the isomathematical equation

$$\hat{y} = a_1 \hat{\times} (b_1 \hat{+} c_1) \hat{+} a_2 \hat{\div} (b_2 \hat{-} c_2) = a_1 \hat{T}(b_1 + c_1 + \hat{0}) + \hat{0} + a_2 / \hat{T}(b_2 - c_2 - \hat{0}) \quad (21)$$

If $\hat{T} = 1$ and $\hat{0} = 0$ then $y = \hat{y}$.

Let $\hat{T} = 2$ and $\hat{0} = 3$. From (21) we have the isomathematical subequation

$$\hat{y}_1 = 2a_1(b_1 + c_1 + 3) + 3 + a_2 / 2(b_2 - c_2 - 3). \quad (22)$$

Let $\hat{T} = 5$ and $\hat{0} = 6$. From (21) we have the isomathematical subequation

$$\hat{y}_2 = 5a_1(b_1 + c_1 + 6) + 6 + a_2 / 5(b_2 - c_2 - 6) \quad (23)$$

Let $\hat{T} = 8$ and $\hat{0} = 10$. From (21) we have the isomathematical subequation

$$\hat{y}_3 = 8a_1(b_1 + c_1 + 10) + 10 + a_2 / 8(b_2 - c_2 - 10) \quad (24)$$

Therefore, (21) has infinitely many isomathematical subequations. Using (21) – (24), $\hat{T}$ and $\hat{0}$, the stability and optimum structure of the algebraic equation (20) may be studied.

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Hypermathematics, H_v-Structures, Hypernumbers, Hypermatrices and Lie-Santilli Addmissibility

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Abstract: We present the largest class of hyperstructures called H_v-structures. In H_v-groups and H_v-rings, the fundamental relations are defined and they connect the algebraic hyperstructure theory with the classical one. Using the fundamental relations, the H_v-fields are defined and their elements are called hypernumbers or H_v-numbers. H_v-matrices are defined to be matrices with entries from an H_v-field. We present the related theory and results on hypermatrices and on the Lie-Santilli admissibility.

Keywords: Representations, Hope, Hyperstructures, H_v-Structures

1. Introduction to Hypermathematics, the H_v-Structures

Hyperstructure is called an algebraic structure containing at least one hyperoperation. More precisely, a set H equipped with at least one multivalued map $\cdot: H\times H \rightarrow P(H)$, is called hyperstructure and the map hyperoperation, we abbreviate hyperoperation by hope. The first hyperstructure was the hypergroup, introduced by F. Marty in 1934 [25], [26], where the strong generalized axioms of a group wrere used. We deal with the largest class of hyperstructures called H_v-structures introduced in 1990 [40],[44],[45] which satisfy the weak axioms where the non-empty intersection replaces the equality.

Some basic definitions:

Definitions 1.1 In a set H with a hope $\cdot: H\times H\rightarrow P(H)$, we abbreviate by WASS the weak associativity: $(xy)z\cap x(yz)\neq\emptyset$, $\forall x,y,z\in H$ and by COW the weak commutativity: $xy\cap yx\neq\emptyset$, $\forall x,y\in H$.

The hyperstructure $(H,\cdot)$ is called H_v-semigroup if it is WASS and is called H_v-group if it is reproductive H_v-semigroup:

$$xH=Hx=H,\ \forall x\in H.$$

The hyperstructure $(R,+,\cdot)$ is called H_v-ring if (+) and $(\cdot)$ are WASS, the reproduction axiom is valid for (+) and $(\cdot)$ is weak distributive with respect to (+):

$$x(y+z)\cap(xy+xz)\neq\emptyset,\ (x+y)z\cap(xz+yz)\neq\emptyset,\ \forall x,y,z\in R.$$

For definitions, results and applications on H_v-structures, see books [44],[4],[10],[12] and papers [6],[7],[8],[9],[11], [17],[18],[19],[22],[24],[46]. An extreme class is defined as follows [41],[44]: An H_v-structure is very thin iff all hopes are operations except one, with all hyperproducts singletons except only one, which is a subset of cardinality more than one. Thus, a very thin H_v-structure is an H with a hope $(\cdot)$ and a pair $(a,b)\in H^2$ for which ab=A, with cardA>1, and all the other products, are singletons.

The main tools to study hyperstructures are the so called, fundamental relations. These are the relations β^* and γ^* which are defined, in H_v-groups and H_v-rings, respectively, as the smallest equivalences so that the quotient would be group and ring, respectively [38],[40],[44],[48],[49]. The way to find the fundamental classes is given as follows [44]:

Theorem 1.2 Let $(H,\cdot)$ be an H_v-group and let us denote by U the set of all finite products of elements of H. We define the relation β in H as follows: $x\beta y$ iff $\{x,y\}\subset u$ where $u\in U$. Then the fundamental relation β^* is the transitive closure of the relation β.

The main point of the proof is that β guaranties that the following is valid: Take elements x,y such that $\{x,y\}\subset u\in U$ and any hyperproduct where one of these elements is used. Then, if this element is replaced by the other, the new hyperproduct is inside the same fundamental class where the first hyperproduct is. Thus, if the 'hyperproducts'of the above β-classes are 'products', then, they are fundamental classes. Analogously for the γ in H_v-rings.

An element is called single if its fundamental class is a singleton.

Motivation for H_v-structures:

1. The quotient of a group with respect to an invariant subgroup is a group.

2. Marty states that, the quotient of a group with respect to any subgroup is a hypergroup.

3. The quotient of a group with respect to any partition is an H_v-group.

In H_v-structures a partial order can be defined [44].

Definition 1.3 Let $(H,\cdot)$, $(H,\otimes)$ be H_v-semigroups defined on the same H. $(\cdot)$ is smaller than $(\otimes)$, and $(\otimes)$ greater than $(\cdot)$, iff there exists automorphism $f\in Aut(H,\otimes)$ such that $xy\subset f(x\otimes y)$, $\forall x\in H$.

Then $(H,\otimes)$ contains $(H,\cdot)$ and write $\cdot\leq\otimes$. If $(H,\cdot)$ is structure, then it is called basic and $(H,\otimes)$ is an H_b-structure.

The Little Theorem [26]. Greater hopes of the ones which are WASS or COW, are also WASS and COW, respectively.

The fundamental relations are used for general definitions of hyperstructures. Thus, to define the general H_v-field one uses the fundamental relation γ*:

Definition 1.4 [40],[43],[44]. The H_v-ring $(R,+,\cdot)$ is an H_v-field if the quotient R/γ^* is a field.

The elements of an H_v-field are called hypernumbers. Let ω* be the kernel of the canonical map and from H_v-ring R to R/γ^*; then we call it reproductive H_v-field if:

$$x(R\text{-}\omega^*) = (R\text{-}\omega^*)x = R\text{-}\omega^*,\ \forall x\in R\text{-}\omega^*.$$

From this definition a new class is defined [51],[56]:

Definition 1.5 The H_v-semigroup $(H,\cdot)$ is called h/v-group if the H/β^* is a group.

An H_v-group is called cyclic [33],[44], if there is an element, called generator, which the powers have union the underline set, the minimal power with this property is the period of the generator. If there exists an element and a special power, the minimum one, is the underline set, then the H_v-group is called single-power cyclic.

To compare classes we can see the small sets. To enumerate and classify H_v-structures, is complicate because we have great numbers. The partial order [44],[47], restrict the problem in finding the minimal, up to isomorphisms, H_v-structures. We have results by Bayon & Lygeros as the following [2],[3]: In sets with three elements: Up to isomorphism, there are 6.494 minimal H_v-groups. The 137 are abelians; 6.152 are cyclic. The number of H_v-groups with three elements is 1.026.462. 7.926 are abelians; 1.013.598 are cyclic, 16 are very thin. Abelian H_v-groups with 4 elements are, 8.028.299.905 from which the 7.995.884.377 are cyclic.

Some more complicated hyperstructures can be defined, as well. In this paper we focus on H_v-vector spaces and there exist an analogous theory on H_v-modules.

Definition 1.6 [44],[50]. Let $(F,+,\cdot)$ be an H_v-field, $(M,+)$ be COW H_v-group and there exists an external hope

$$F\times M\rightarrow P(M): (a,x)\rightarrow ax,$$

such that, $\forall a,b\in F$ and $\forall x,y\in M$ we have

$$a(x+y)\cap(ax+ay)\neq\varnothing,\ (a+b)x\cap(ax+bx)\neq\varnothing,\ (ab)x\cap a(bx)\neq\varnothing,$$

then M is called an H_v-vector space over F.

The fundamental relation ε* is defined to be the smallest equivalence such that the quotient M/ε^* is a vector space over the fundamental field F/γ^*. For this fundamental relation there is an analogous to the Theorem 1.2.

Definitions 1.7 [51],[53],[55]. Let $(H,\cdot)$ be hypergroupoid. We remove $h\in H$, if we consider the restriction of $(\cdot)$ in the set H-{h}. We say that $h\in H$ absorbs $h\in H$ if we replace h by h and h does not appear in the structure. We say that $h\in H$ merges with $h\in H$, if we take as product of any $x\in H$ by h, the union of the results of x with both h, h, and consider h and h as one class, with representative h, therefore the element h does not appeared in the hyperstructure.

Let $(H,\cdot)$ be an H_v-group, then, if an element h absorbs all elements of its own fundamental class then this element becomes a single in the new H_v-group.

Theorem 1.8 In an H_v-group $(H,\cdot)$, if an element h absorbs all elements of its fundamental class then this element becomes a single in the new H_v-group.

Proof. Let $h\in\beta^*(h)$, then, by the definition of the 'absorb', h is replaced by h that means that $\beta^*(h)=\{h\}$. Moreover, for all $x\in H$, the fundamental property of the product of classes

$$\beta^*(x)\cdot\beta^*(h) = \beta^*(xh) \text{ becomes } \beta^*(x)\cdot h = \beta^*(xh),$$

and from the reproductivity ([44] p.19) we obtain $x\cdot h=\beta^*(xh)$, $\forall x\in\beta^*(x)$. This is the basic property that enjoys any single element [44].

Remark that in case we have a single element then we can compute all fundamental classes.

A well known and large class of hopes is given as follows [33],[37],[39],[44],[20]:

Definitions 1.9 Let $(G,\cdot)$ be a groupoid, then for every subset $P\subset G$, $P\neq\varnothing$, we define the following hopes, called P-hopes: $\forall x,y\in G$

$$P:\ xPy= (xP)y\cup x(Py),$$

$$P_r:\ xP_ry= (xy)P\cup x(yP),\quad P_l:\ xP_ly= (Px)y\cup P(xy).$$

The (G,P), (G,P_r) and (G,P_l) are called P-hyperstructures. In the case of semigroup $(G,\cdot)$: $xPy=(xP)y\cup x(Py)=xPy$ and (G,P) is a semihypergroup but we do not know about (G,P_r) and (G,P_l). In some cases, depending on the choice of P, the (G,P_r) and (G,P_l) can be associative or WASS.

A generalization of P-hopes is the following [13],[14]: Let $(G,\cdot)$ be abelian group and P a subset of G with more than one elements. We define the hope $\times_P$ as follows:

$$x\times_P y = x\cdot P\cdot y = \{x\cdot h\cdot y \mid h\in P\} \text{ if } x\neq e \text{ and } y\neq e$$

$$x\cdot y \text{ if } x=e \text{ or } y=e$$

we call this hope, P_e-hope. The hyperstructure $(G,\times_P)$ is an abelian H_v-group.

A general definition of hopes, is the following [57],[58]:

Definitions 1.10 Let H be a set with n operations (or hopes) $\otimes_1,\otimes_2,\ldots,\otimes_n$ and one map (or multivalued map) $f:H\to H$, then n hopes $\partial_1,\partial_2,\ldots,\partial_n$ on H are defined, called ∂-hopes by putting

$$x\partial_i y = \{f(x)\otimes_i y,\ x\otimes_i f(y)\},\ \forall x,y\in H,\ i\in\{1,2,\ldots,n\}$$

or in case where $\otimes_i$ is hope or f is multivalued map we have

$$x\partial_i y = (f(x)\otimes_i y)\cup(x\otimes_i f(y)),\ \forall x,y\in H,\ i\in\{1,2,\ldots,n\}$$

Let $(G,\cdot)$ groupoid and $f_i:G\to G$, $i\in I$, set of maps on G. Take the map $f_\cup:G\to P(G)$ such that $f_\cup(x)=\{f_i(x)\mid i\in I\}$, call it the union of the $f_i(x)$. We call the union ∂-hope (∂), on G if we consider the map $f_\cup(x)$. An important case for a map f, is to take the union of this with the identity id. Thus, we consider the map $f\equiv f\cup(id)$, so $f(x)=\{x,f(x)\}$, $\forall x\in G$, which is called b-∂-hope, we denote it by (∂), so we have

$$x\partial y = \{xy,\ f(x)\cdot y,\ x\cdot f(y)\},\ \forall x,y\in G.$$

Remark If $\otimes_i$ is associative then ∂_i is WASS. If ∂ contains the operation (·), then it is b-operation. Moreover, if $f:G\to P(G)$ is multivalued then the b-∂-hopes is defined by using the $f(x)=\{x\}\cup f(x)$, $\forall x\in G$.

Motivation for the definition of ∂-hope is the derivative where only multiplication of functions is used. Therefore, for functions s(x), t(x), we have $s\partial t=\{s't,st'\}$, (′) is the derivative.

Example. For all first degree polynomials $g_i(x)=a_ix+b_i$, we have

$$g_1\partial g_2 = \{a_1a_2x+a_1b_2,\ a_1a_2x+b_1a_2\},$$

so it is a hope in the set of first degree polynomials. Moreover all polynomials x+c, where c be a constant, are units.

There exists the uniting elements method introduced by Corsini–Vougiouklis [5] in 1989. With this method one puts in the same class, two or more elements. This leads, through hyperstructures, to structures satisfying additional properties.

Definition 1.11 The uniting elements method is the following: Let G be an algebraic structure and let d be a property, which is not valid. Suppose that d is described by a set of equations; then, consider the partition in G for which it is put together, in the same partition class, every pair of elements that causes the non-validity of the property d. The quotient by this partition G/d is an H_v-structure. Then, quotient out the H_v-structure G/d by the fundamental relation β^*, a stricter structure $(G/d)\beta^*$ for which the property d is valid, is obtained.

An interesting application of the uniting elements is when more than one property is desired, because some of the properties lead straight to the classes. The commutativity and the reproductivity property are easily applicable. The following is valid:

Theorem 1.12 [44] Let $(G,\cdot)$ be a groupoid, and

$$F = \{f_1,\ldots,f_m,f_{m+1},\ldots,f_{m+n}\}$$

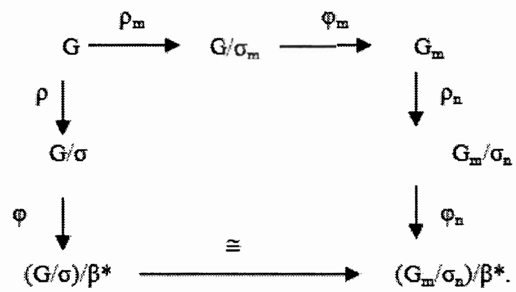
be a system of equations on G consisting of two subsystems

$$F_m = \{f_1,\ldots,f_m\} \text{ and } F_n = \{f_{m+1},\ldots,f_{m+n}\}.$$

Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using the systems F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the grupoid $G_m = (G/\sigma_m)/\beta^*$. Then

$$(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*.$$

i.e. the following diagram is commutative



From the above it is clear that the fundamental structure is very important, and even more so if this is known from the beginning. This is the problem to construct hyperstructures with desired fundamental structures [44].

Theorem 1.13 Let $(S,\cdot)$ be a commutative semigroup with one element $w\in S$ uch that the set wS is finite. Consider the transitive closure L^* of the relation L defined as follows: xLy iff there exists $z\in S$ such that $zx=zy$.

Then $<S/L^*,\circ>/\beta^*$ is finite commutative group, where (∘) is the induced operation on classes of S/L^*.

For the proof see [5],[44].

An application combining hyperstructures and fuzzy theory, is to replace the 'scale' of Likert in questionnaires by the bar of Vougiouklis & Vougiouklis [69],[70],[21],[27]:

Definition 1.14 In every question substitute the Likert scale with the 'bar' whose poles are defined with '0' on the left end, and '1' on the right end:

0 ———————————————— 1

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point they feel expresses their answer to the question.

The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm. The hyperstructure theory, offer innovating new suggestions to connect finite groups of objects. These suggestions are obtained from properties and special elements inside the hyperstructure.

2. Hyper-Representations

Representations (abbreviate by rep) of H_v-groups can be faced either by generalized permutations or by H_v-matrices [34],[36],[39],[43],[44],[52],[54],[66]. Reps by generalized permutations can be achieved by using translations [42]. We present an outline of the hypermatrix rep in H_v-structures and there exist the analogous theory for the h/v-structures.

Definitions 2.1 [44],[66] H_v-matrix is a matrix with entries elements of an H_v-field. The hyperproduct of two H_v-matrices $A=(a_{ij})$ and $B=(b_{ij})$, of type m×n and n×r respectively, is defined, in the usual manner,

$$A\cdot B = (a_{ij})\cdot(b_{ij}) = \{ C= (c_{ij}) \mid c_{ij}\in \oplus\Sigma a_{ik}\cdot b_{kj} \},$$

and it is a set of m×r H_v-matrices. The sum of products of elements of the H_v-field is the union of the sets obtained with all possible parentheses put on them, called n-ary circle hope on the hyperaddition.

The hyperproduct of H_v-matrices does not satisfy WASS.

The problem of the H_v-matrix reps is the following:

Definitions 2.2 For a given H_v-group $(H,\cdot)$, find an H_v-field $(F,+,\cdot)$, a set $M_R=\{(a_{ij}) \mid a_{ij}\in F\}$ and a map $T: H\to M_R:h\to T(h)$ such that

$$T(h_1h_2)\cap T(h_1)T(h_2) \neq \varnothing,\ \forall h_1,h_2\in H.$$

The map T is called H_v -matrix rep. If $T(h_1h_2)\subset T(h_1)T(h_2)$, $\forall h_1,h_2\in H$, then T is called inclusion rep. T is a good rep if $T(h_1h_2)=T(h_1)T(h_2)=\{T(h) \mid h\in h_1h_2\},\forall h_1,h_2\in H$. If T is one to one and good then it is a faithful rep.

The problem of reps is complicated since the hyperproduct is big. It can be simplified in cases such as: The H_v-matrices are over H_v-fields with scalars 0 and 1. The H_v-matrices are over very thin H_v-fields. On 2×2 H_v-matrices, since the circle hope coincides with the hyperaddition. On H_v-fields which contain singles, which act as absorbings.

The main theorem of reps is the following [44],[52]:

Theorem 2.3 A necessary condition in order to have an inclusion rep T of an H_v-group $(H,\cdot)$ by n×n H_v-matrices over the H_v-field $(F,+,\cdot)$ is the following:

For all classes $\beta^*(x)$, $x\in H$ there must exist elements $a_{ij}\in H$, $i,j\in\{1,\dots,n\}$ such that

$$T(\beta^*(a)) \subset \{A=(a'_{ij}) \mid a'_{ij}\in \gamma^*(a_{ij}),\ i,j\in\{1,\dots,n\}\}$$

Thus, every inclusion rep $T:H\to M_R:a\to T(a)=(a_{ij})$ induces a homomorphic rep T^* of the group H/β^* over the field F/γ^* by setting

$$T^*(\beta^*(a)) = [\gamma^*(a_{ij})],\ \forall\beta^*(a)\in H/\beta^*,$$

where $\gamma^*(a_{ij})\in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. T^* is called fundamental induced rep of T.

Denote $tr_\varphi(T(x)) = \gamma^*(T(x_{ii}))$ the fundamental trace, then the mapping

$$X_T : H \to F/\gamma^*: x\to X_T(x) = tr_\varphi(T(x)) = trT^*(x)$$

is called fundamental character.

Using special classes of H_v-structures one can have several reps [52],[66]:

Definition 2.4 Let $M=M_{m\times n}$ be vector space of m×n matrices over a field F and take sets

$$S=\{s_k:k\in K\} \subseteq F,\ Q=\{Q_j:j\in J\} \subseteq M,\ P=\{P_i:i\in I\} \subseteq M.$$

Define three hopes as follows

$$S: F\times M\to P(M):(r,A)\to rSA=\{(rs_k)A: k\in K\}\subseteq M$$

$$Q_+: M\times M\to P(M):(A,B)\to AQ_+B=\{A+Q_j+B: j\in J\}\subseteq M$$

$$P: M\times M\to P(M):(A,B)\to APB=\{AP^t_iB: i\in I\}\subseteq M$$

Then (M,S,Q_+,P) is a hyperalgebra over F called general matrix P-hyperalgebra.

The bilinear hope P, is strong associative and the inclusion distributivity with respect to addition of matrices

$$AP(B+C) \subseteq APB+APC,\ \forall A,B,C\in M$$

is valid. So (M,+,P) defines a multiplicative hyperring on non-square matrices.

In a similar way a generalization of this hyperalgebra can be defined considering an H_v-field instead of a field and using H_v-matrices instead of matrices.

In the representation theory several constructions are used, one can find some of them as follows [43],[44],[52], [54]:

Construction 2.5 Let $(H,\cdot)$ be H_v-group, then for all $(\oplus)$ such that $x\oplus y\supset\{x,y\}$, $\forall x,y\in H$, the $(H,\oplus,\cdot)$ is an H_v-ring. These H_v-rings are called associated to $(H,\cdot)$ H_v-rings.

In rep theory of hypergroups, in sense of Marty where the equality is valid, there are three associated hyperrings $(H,\oplus,\cdot)$ to $(H,\cdot)$. The $(\oplus)$ is defined respectively, $\forall x,y\in H$, by:

type a: $x\oplus y=\{x,y\}$, type b: $x\oplus y=\beta^*(x)\cup\beta^*(y)$, type c: $x\oplus y=H$

In the above types the strong associativity and strong or inclusion distributivity, is valid.

Construction 2.6 Let $(H,\cdot)$ be an H_V-semigroup and $\{v_1,\dots,v_n\}\cap H=\varnothing$, an ordered set, where if $v_i<v_j$, when $i<j$. Extend $(\cdot)$ in $H_n=H\cup\{v_1,v_2,\dots,v_n\}$ as follows:

$$x\cdot v_i = v_i\cdot x = v_i\ ,\ v_i\cdot v_j = v_j\cdot v_i = v_j\ ,\ \forall i<j \text{ and}$$

$$v_i\cdot v_i = H\cup\{v_1,\dots,v_{i-1}\},\ \forall x\in H,\ i\in\{1,2,\dots,n\}.$$

Then $(H_n,\cdot)$ is an H_V-group, called Attach Elements Construction, and $(H_n,\cdot)/\beta^*\cong Z_2$, where v_n is single [51],[55].

Some problems arising on the topic, are:

Open Problems.

a. Find standard H_v-fields to represent all H_v-groups.

b. Find reps by H_v-matrices over standard finite H_v-fields analogous to Z_n.

c. Using matrices find a generalization of the ordinary multiplication of matrices which it could be used in H_v-rep theory (see the helix-hope [68]).

d. Find the 'minimal' hypermatrices corresponding to the minimal hopes.

e. Find reps of special classes of hypergroups and reduce these to minimal dimensions.

Recall some definitions from [68],[16],[32]:

Definitions 2.7 Let $A=(a_{ij})\in M_{m\times n}$ be m×n matrix and $s,t\in N$ be natural numbers such that $1\le s\le m$, $1\le t\le n$. Then we define a characteristic-like map cst: $M_{m\times n}\to M_{s\times t}$ by corresponding to the matrix A, the matrix $Acst=(a_{ij})$ where $1\le i\le s$, $1\le j\le t$. We call it cut-projection of type st. We define the mod-like map st: $M_{m\times n}\rightarrow M_{s\times t}$ by corresponding to A the matrix $Ast=(a_{ij})$ which has as entries the sets

$$a_{ij}=\{a_{i+\kappa s,j+\lambda t} \mid 1\leq i\leq s, 1\leq j\leq t \text{ and } \kappa,\lambda\in N,\ i+\kappa s\leq m,\ j+\lambda t\leq n\}.$$

Thus we have the map

$$st: M_{m\times n}\rightarrow M_{s\times t}: A\rightarrow Ast=(a_{ij}).$$

We call this multivalued map helix-projection of type st. So Ast is a set of $s\times t$-matrices $X=(x_{ij})$ such that $x_{ij}\in a_{ij}, \forall i,j$.

Let $A=(a_{ij})\in M_{m\times n}$, $B=(b_{ij})\in M_{u\times v}$ matrices and $s=\min(m,u)$, $t=\min(n,u)$. We define a hope, called helix-addition or helix-sum, as follows:

$$\oplus: M_{m\times n}\times M_{u\times v}\rightarrow P(M_{s\times t}):$$

$$(A,B)\rightarrow A\oplus B=Ast+Bst=(a_{ij})+(b_{ij})\subset M_{s\times t},$$

where

$$(a_{ij})+(b_{ij})=\{(c_{ij})=(a_{ij}+b_{ij}) \mid a_{ij}\in a_{ij} \text{ and } b_{ij}\in b_{ij}\}.$$

And define a hope, called helix-multiplication or helix-product, as follows:

$$\otimes: M_{m\times n}\times M_{u\times v}\rightarrow P(M_{m\times v}): (A,B)\rightarrow A\otimes B=Ams\cdot Bsv=(a_{ij})\cdot(b_{ij})\subset M_{m\times v},$$

where

$$(a_{ij})\cdot(b_{ij})=\{(c_{ij})=(\sum a_{it}b_{tj}) \mid a_{ij}\in a_{ij} \text{ and } b_{ij}\in b_{ij}\}.$$

Remark. In $M_{m\times n}$ the addition of matrices is an ordinary operation, therefore we are interested only in the 'product'. From the fact that the helix-product on non square matrices is defined, the definition of the Lie-bracket is immediate, therefore the helix-Lie Algebra is defined [62], as well. This algebra is an H_v-Lie Algebra where the fundamental relation ε^* gives, by a quotient, a Lie algebra, from which a classification is obtained.

For more results on the topic see [16],[32],[61],[62].

In the following we denote E_{ij} any type of matrices which have the ij-entry 1 and in all the other entries we have 0.

Example 2.8 Consider the 2×3 matrices of the following form,

$$A_\kappa=E_{11}+\kappa E_{21}+E_{22}+E_{23},\ B_\kappa=\kappa E_{21}+E_{22}+E_{23},\ \forall\kappa\in\mathbb{N}.$$

Then we obtain $A_\kappa\otimes A_\lambda=\{A_{\kappa+\lambda},A_{\lambda+1},B_{\kappa+\lambda},B_{\lambda+1}\}$

Similarly, $B_\kappa\otimes A_\lambda=\{B_{\kappa+\lambda},B_{\lambda+1}\}$, $A_\kappa\otimes B_\lambda=B_\lambda=B_\kappa\otimes B_\lambda$.

Thus the set $\{A_\kappa,B_\lambda \mid \kappa,\lambda\in\mathbb{N}\}$ becomes an H_v-semigroup which is not COW because for $\kappa\neq\lambda$ we have

$$B_\kappa\otimes B_\lambda=B_\lambda\neq B_\kappa=B_\lambda\otimes B_\kappa,$$

however

$$(A_\kappa\otimes A_\lambda)\cap(A_\lambda\otimes A_\kappa)=\{A_{\kappa+\lambda},B_{\kappa+\lambda}\}\neq\varnothing,\ \forall\kappa,\lambda\in\mathbb{N}.$$

All elements B_λ are right absorbing and B_1 is a left scalar, because $B_1\otimes A_\lambda=B_{\lambda+1}$ and $B_1\otimes B_\lambda=B_\lambda$, A_0 is a unit.

3. Hyper-Lie-Algebras

Lie-Santilli admisibility

The general definition of an H_v-Lie algebra over an H_v-field is given as follows [61],[62]:

Definition 3.1 $(L,+)$ be H_v-vector space on H_v-field $(F,+,\cdot)$, $\varphi:F\rightarrow F/\gamma^*$ the canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the fundamental field F/γ^*. Moreover, let ω_L be the core of the canonical map φ': $L\rightarrow L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^*. Consider the bracket (commutator) hope:

$$[\ ,\ ]: L\times L\rightarrow P(L): (x,y)\rightarrow[x,y]$$

then L is called an H_v-Lie algebra over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\varnothing$$

$$[x,\lambda_1y_1+\lambda_2y]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\varnothing,$$

$$\forall x,x_1,x_2,y,y_1,y_2\in L \text{ and } \lambda_1,\lambda_2\in F$$

$$\text{(L2) } [x,x]\cap\omega_L\neq\varnothing,\ \forall x\in L$$

$$\text{(L3) } ([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L\neq\varnothing,\ \forall x,y\in L$$

Example 3.2 Consider all traceless matrices $A=(a_{ij})\in M_{2\times 3}$, in the sense that $a_{11}+a_{22}=0$. In this case, the cardinality of the helix-product of any two matrices is 1, or 2^3, or 2^6. These correspond to the cases: $a_{11}=a_{13}$ and $a_{21}=a_{23}$, or only $a_{11}=a_{13}$ either only $a_{21}=a_{23}$, or if there is no restriction, respectively. For the Lie-bracket of two traceless matrices the corresponding cardinalities are up to 1, or 2^6, or 2^{12}, resp. We remark that, from the definition of the helix-projection, the initial 2×2, block guaranties that in the result there exists at least one traceless matrix.

From this example it is obvious the following:

Theorem 3.3 Using the helix-product the Lie-bracket of any two traceless matrices $A=(a_{ij})$, $B=(b_{ij})\in M_{m\times n}$, $m<n$, contain at least one traceless matrix.

Last years, hyperstructures have a variety of applications in mathematics and other sciences. The hyperstructures theory can now be widely applicable in industry and production, too. In several books [4],[10],[12] and papers [1],[11],[17],[23],[31],[35],[50],[67],[70] one can find numerous applications.

The Lie-Santilli theory on isotopies was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed [28] a 'lifting' of the trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The isofields needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1996 and they are called e-hyperfields [29],[30],[59],[60],[64],[13],[14],[15] which are used in physics or biology. The H_v-fields can give e-hyperfields which can be used in the isotopy theory for applications.

The IsoMathematics Theory is very important subject in applied mathematics. It is a generalization by using a kind of the Rees analogous product on matrix semigroup with a sandwich matrix, like the P-hopes. It contains the classical theory but also can find easy solutions in different branches of mathematics. To compare this novelty we give two analogous examples: (1) The unsolved, from ancient times, problems in Geometry was solved in a different branch of mathematics, the Algebra with the genius Galois Theory. (2) With the Representation Theory one can solve problems in Lie Algebras and to transfer these in Lie Groups using the exponential map, and the opposite. One very important thing of the IsoMathematics Theory is that admits generalizations, as well. Two very important of them are the following: First, is the so called Admissible Lie-Santilli Algebras [28],[30], [62],[65] by using again a kind of Rees sandwich product. Second, is that one can extend this theory into the multivalued case, i.e. into H_v-structures.

Definitions 3.4 A hyperstructure $(H,\cdot)$ containing a unique scalar unit e, is called e-hyperstructure. We assume that $\forall x$, there is an inverse x^{-1}, i.e. $e \in x\cdot x^{-1} \cap x^{-1}\cdot x$. A hyperstructure $(F,+,\cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called e-hyperfield if the following are valid:

$(F,+)$ is abelian group with the additive unit 0, $(\cdot)$ is WASS,

$(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing: $0\cdot x = x\cdot 0 = 0$, $\forall x \in F$, there exist a multiplicative scalar unit 1, i.e. $1\cdot x = x\cdot 1 = x$, $\forall x \in F$, and $\forall x \in F$ there exists a unique inverse x^{-1}, such that $1 \in x\cdot x^{-1} \cap x^{-1}\cdot x$.

The elements of an e-hyperfield are called e-hypernumbers. In the case that the relation: $1 = x\cdot x^{-1} = x^{-1}\cdot x$, is valid, then we say that we have a strong e-hyperfield.

A general construction based on the partial ordering of the H_v-structures:

Construction 3.5 [13],[14],[15],[30] Main e-Construction. Given a group $(G,\cdot)$, where e is the unit, then we define in G, a large number of hopes $(\otimes)$ by extended $(\cdot)$, as follows:

$$x\otimes y = \{xy, g_1, g_2, \ldots\},\ \forall x,y \in G-\{e\},\ \text{and } g_1, g_2, \ldots \in G-\{e\}$$

Then $(G,\otimes)$ becomes an H_v-group, in fact is H_b-group which contains the $(G,\cdot)$. The H_v-group $(G,\otimes)$ is an e-hypergroup. Moreover, if $\forall x,y$ such that $xy=e$, so we have $x\otimes y = xy$, then $(G,\otimes)$ becomes a strong e-hypergroup.

Definition 3.6 Let $(H_o,+,\cdot)$ be the attached, by one element, H_v-field of the H_v-semigroup $(H,\cdot)$. Thus, for $(H,\cdot)$, take an element v outside of H, and extend $(\cdot)$ in $H_n = H \cup \{v\}$ by:

$$x\cdot v = v\cdot x = v,\ v\cdot v = H,\ \forall x \in H.$$

$(H_n,\cdot)$ is an H_V-group, called Attach Elements Construction, and $(H_n,\cdot)/\beta^* \cong Z_2$, where v, is single. If $(H,\cdot)$ has a left and right scalar unit e then $(H_o,+,\cdot)$ is an e-hyperfield, the attached H_v-field of $(H,\cdot)$.

Remark. The above main e-construction gives an extremely large class of e-hopes. These e-hopes can be used in the several more complicate hyperstructures to obtain appropriate e-hyperstructures. However, the most useful are the ones where only few products are enlarged.

Example 3.7 Take the finite-non-commutative quaternion group Q={1,-1, i,-i, j,-j, k,-k}. Using this operation one can obtain several hopes which define very interesting e-groups. For example, denoting i={i,-i}, j={j,-j}, k={k,-k} we may define the (*) hope by the Cayley table:

*	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	k	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

The hyperstructure (Q,*) is strong e-hypergroup because 1 is scalar unit and the elements -1,i,-i,j,-j,k and -k have unique inverses the elements -1,-i,i,-j,j,-k and k, resp., which are the inverses in the basic group. Thus, from this example one can have more strict hopes.

In [30],[62],[65] a kind of P-hopes was introduced which is appropriate to extent the Lie-Santilli admissible algebras in hyperstructures:

The general definition is the following:

Construction 3.8 Let $(L=M_{m\times n},+)$ be an H_v-vector space of $m\times n$ hyper-matrices over the H_v-field $(F,+,\cdot)$, $\varphi: F \to F/\gamma^*$, the canonical map and $\omega_F = \{x \in F: \varphi(x)=0\}$, where 0 is the zero of the fundamental field F/γ^*, ω_L be the core of the canonical map $\varphi': L \to L/\varepsilon^*$ and denote again by 0 the zero of L/ε^*. Take any two subsets $R,S \subseteq L$ then a Santilli's Lie-admissible hyperalgebra is obtained by taking the Lie bracket, which is a hope:

$$[,]_{RS}: L\times L \to P(L): [x,y]_{RS} = xR^t y - yS^t x.$$

Notice that $[x,y]_{RS} = xR^t y - yS^t x = \{ r^t y - ys^t x \mid r \in R \text{ and } s \in S\}$.

Special cases, but not degenerate, are the 'small' and 'strict':

$$\text{(a) } R=\{e\} \text{ then } [x,y]_{RS} = xy - yS^t x = \{xy - ys^t x \mid s \in S\}$$

$$\text{(b) } S=\{e\} \text{ then } [x,y]_{RS} = xR^t y - yx = \{xr^t y - yx \mid r \in R\}$$

$$\text{(c) } R=\{r_1,r_2\} \text{ and } S=\{s_1,s_2\} \text{ then}$$

$$[x,y]_{RS} = xR^t y - yS^t x =$$

$$\{xr_1{}^t y - ys_1{}^t x,\ xr_1{}^t y - ys_2{}^t x,\ xr_2{}^t y - ys_1{}^t x,\ xr_2{}^t y - ys_2{}^t x\}$$

4. Galois H_v-Fields and Low Dimensional H_v-Matrices

Recall some results from [63], which are referred to finite H_v-fields which we will call, according to the classical theory, Galois H_v-fields. Combining the uniting elements procedure with the enlarging theory we can obtain stricter structures or hyperstructures. So enlarging operations or hopes we can obtain more complicated structures.

Theorem 4.1 In the ring $(Z_n,+,\cdot)$, with n=ms we enlarge the multiplication only in the product of elements $0\cdot m$ by setting $0\otimes m=\{0,m\}$ and the rest results remain the same. Then

$$(Z_n,+,\otimes)/\gamma^* \cong (Z_m,+,\cdot).$$

Proof. First we remark that the only expressions of sums and products which contain more, than one, elements are the expressions which have at least one time the hyperproduct $0\otimes m$. Adding to this special hyperproduct the element 1, several times we have the equivalence classes modm. On the other side, since m is a zero divisor, adding or multiplying elements of the same class the results are remaining in one class, the class obtained by using only the representatives. Therefore, γ^*-classes form a ring isomorphic to $(Z_m,+,\cdot)$.

Remark. In the above theorem we can enlarge other products as well, for example $2\cdot m$ by setting $2\otimes m=\{2,m+2\}$, then the result remains the same. In this case the elements 0 and 1 remain scalars, so they are refered in e-hyperstructures.

From the above theorem it is immediate the following:

Corollary 4.2 In the ring $(Z_n,+,\cdot)$, with n=ps where p is a prime number, we enlarge the multiplication only in the product of the elements $0\cdot p$ by setting $0\otimes p=\{0,p\}$ and the rest results remain the same. Then the hyperstructure $(Z_n,+,\otimes)$ is a very thin H_v-field.

The above theorem provides the researchers with H_v-fields appropriate to the rep theory since they may be smaller or minimal hyperstructures.

Remarks 4.3 The above theorem in connection with Uniting Elements method leads to the fact that in H_v-structure theory it is able to equip algebraic structures or hyperstructures with properties as associativity, commutativity, reproductivity. This equipment can be applied independently of the order of the desired properties. This is crucial point since some properties are easy to be applied, so we can apply them first, and then the difficult ones. For example from an H_v-ring we first go to an H_v-integral domain, by uniting the zero divisors, and then to the H_v-field by reaching the reproductivity.

Construction 4.5 (Galois H_v-fields) In the ring $(Z_n,+,\cdot)$, with n=ps where p is prime, enlarge only the product of the elements 2 by p+2, i.e. $2\cdot(p+)$, by setting $2\otimes(p+2)=\{2,p+2\}$ and the rest remain the same. Then $(Z_n,+,\otimes)$ is a COW very thin H_v-field where 0 and 1 are scalars and we have:

$$(\mathbf{Z}_n,+,\otimes)/\gamma^* \cong (\mathbf{Z}_p,+,\cdot).$$

Proof. Straightforward.

Remark 4.6 Galois Hv-fields of the above type are the most appropriate in the representation theory since the cardinality of the products is low. Moreover, one can use more enlargements using elements of the same fundamental class, therefore, one can have several cardinalities. The low dimensional reps can be based on the above Galois Hv-fields, since they use infinite Hv-fields although the fundamental fields are finite.

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Santilli Autotopisms of Partial Groups

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Abstract: This paper deals with those partial groups that contain a given Santilli isotopism in their autotopism group. A classification of these autotopisms is explicitly determined for partial groups of order $n \leq 4$.

Keywords: Partial Group, Isotopism, Classification

1. Introduction

In 1942, Albert [1] introduced the concept of *isotopy* of algebras: Two algebras $(A_1, \cdot)$ and $(A_2, *)$ over a field K are said to be *isotopic* if there exist three regular linear transformations f, g and h from A_1 to A_2 such that

$$f(u) * g(v) = h(u \cdot v), \text{ for all } u, v \in A_1. \quad (1)$$

The algebra A_2 is then said to be *isotopic* to the algebra A_1, or, equivalently, A_2 *is* an *isotope* of A_1. The triple $\Theta = (f, g, h)$ is an *isotopy* or *isotopism* between both algebras A_1 and A_2. If $f = g = h$, then this is indeed an isomorphism. If the elements of A_1 and A_2 coincide, then the isotopism Θ is said to be *principal* if h is the trivial transformation Id, that is, if $h(u)=Id(u)=u$, for all $u \in A_1$. In this case, the algebra A_2 is said to be a *principal isotope* of A_1. In his original paper, Albert proposed the question as to whether a principal isotope of a Lie algebra is Lie. In this regard, he proved that a principal isotope A_2 of a Lie algebra A_1 with respect to a principal isotopism (f, g, Id) is a Lie algebra if and only if, for all $u, v, w \in A_1$, it is verified that

$$f(u) \cdot g(v) = -f(v) \cdot g(u). \quad (2)$$

$$f(f(u) \cdot g(v)) \cdot g(w) - f(f(u) \cdot g(w)) \cdot g(v) - f(u) \cdot g(f(v) \cdot g(w)) = 0. \quad (3)$$

In 1944, Bruck [2] introduced the concept of *isotopically simple algebra* as a simple algebra such that all their isotopic algebras are simple. He proved in particular that the Lie algebra of order $n \cdot (n-1)/2$, consisting of all skew-symmetric matrices over any subfield of the field of all reals, under the Lie product $[u, v] = u \cdot v - v \cdot u$, is isotopically simple. Further, the Lie algebra of order $n \cdot (n-1)$ consisting of all skew-Hermitian matrices in any field $R(i)$ (where R is a subfield of the reals and $i^2 = -1$), under the multiplication $[u, v]=u \cdot v - v \cdot u$, is an isotopically simple algebra over R.

More recently, in 1978, Santilli [3] generalized the associative product $u \cdot v$ between Hermitian generators of the universal enveloping associative algebra by considering the new product

$$u * v = u \cdot T \cdot v \quad (4)$$

where T is a positive-definite operator called the *isotopic element*, which can indeed depend on distinct factors

$$T = T(x, x', x', \ldots, \mu, \tau) \quad (5)$$

The product

$$[u, v] = u * v - v * u \quad (6)$$

preserves the Lie axioms and is called the *Lie-isotopic product*. The application to Lie's theory (enveloping algebras, Lie algebras and Lie groups) that emerges from this new product is the so-called *Lie-Santilli isotheory* (see [3, pp. 287-290 and 329-374] and also [4-9]).

In the development of the isotheory, Santilli extended the unit of the ground field to the generalized unit or *isounit*

$$I = I(x, x', x', \ldots, \mu, \tau) = T^{-1} \quad (7)$$

He defined then the isonumbers

$$u = u * I(x, x', x', \ldots, \mu, \tau), \text{ for all } u \in A. \quad (8)$$

and the *isoproduct*

$$[u, v] = u * v - v * u \quad (9)$$

This isoproduct constitutes the Lie product of an isomorphic Lie algebra of A whenever the isounit $\hat{I}$ is constant. In any other case, this determines a generalization of the classical notion (2) of isotopism. In order to analyze this fact, the authors [10] reinterpreted in 2006 the dependence on distinct factors of the isounit $\hat{I}$ as a family of classical Bruck's isotopisms. This reinterpretation became clearer shortly after [11] once the attention was focused not on isotopisms of algebras, but on isotopisms of partial quasigroups.

The term *quasigroup* was introduced in 1937 by Haussmann and Ore [12] to denote a nonempty set Q endowed with a product $\cdot$, such that if any two of the three symbols u, v and w in the equation $u \cdot v = w$ are given as elements of Q, then the third is uniquely determined as an element of Q. Its *order* is the cardinality of the underlying set, that is, the number of elements of the quasigroup Q. This is said to be a *loop* if it contains a unit element, that is, there exists an element $e \in Q$ such that $e \cdot u = u \cdot e = u$ for all $u \in Q$. Every associative loop is indeed a group. The multiplication table of a quasigroup of order n is a *Latin square of order n*, that is, an $n \times n$ array with elements chosen from a set of n distinct symbols such that each symbol occurs precisely once in each row and each column (see Figure 1).

2	3	4	1
3	4	1	2
4	1	2	3
1	2	3	4

Figure 1. *Latin square of order 4.*

A *partial Latin square of order n* is an $n \times n$ array with elements chosen from a set of n distinct symbols such that each symbol occurs at most once in each row and each column (see Figure 2). It constitutes the multiplication table of a *finite partial quasigroup* $(Q, \cdot)$ of order n. Let $u, v \in Q$. The product $u \cdot v$ is then an element of Q or it is undefined. This last case is denoted as $u \cdot v = \emptyset$. By abuse of notation, it is also considered that $u \cdot \emptyset = \emptyset \cdot u = \emptyset$, for all $u \in Q$ and hence, the partial quasigroup is *associative* if $(u \cdot v) \cdot w = u \cdot (v \cdot w)$, for all $u, v, w \in Q$. It is a *partial loop* if there exists an element $e \in Q$ such that $e \cdot u = u \cdot e \in \{u, \emptyset\}$ for all $u \in Q$ and there does not exist an element $e' \neq e$ such that $e' \cdot u = u$ or $u \cdot e' = u$. Every associative partial loop constitutes a *partial group*.

1			
	2		4
3			
4		3	

Figure 2. *Partial Latin square of order 4.*

In 1943-44, Albert [13, 14] together with Bruck [15] extended the definition of isotopy from algebras to quasigroups. Particularly, two quasigroups $(Q_1, \cdot)$ and $(Q_2, *)$ of the same order are said to be *isotopic* if there exist three bijections f, g and h between their sets of elements such that

$$f(u) * g(v) = h(u \cdot v), \text{ for all } u, v \in Q_1. \qquad (10)$$

The definition can be naturally extended to partial quasigroups once it is considered $h(\emptyset) = \emptyset$. The triple $\Theta = (f, g, h)$ is said to be an *isotopism* between Q_1 and Q_2 and it is denoted $Q_2 = Q_1^{\Theta}$. If $Q_2 = Q_1$, then the isotopism Θ is said to be an *autotopism* of Q_1 and f, g and h are permutations of the elements of Q_1. The set of autotopisms of a (partial) quasigroup constitutes, therefore, a group with the component-wise composition of permutations.

In 2007, the authors [11] introduced the concept of Santilli isotopism between partial quasigroups. Specifically, an isotopism $\Theta = (f, g, h)$ between two partial quasigroups $(Q_1, \cdot)$ and $(Q_2, *)$ is said to be a *Santilli isotopism* if there exist three elements i_f, i_g and i_h in Q_1 such that

$$f(u)=u\cdot i_f,\ g(u)=u\cdot i_g \text{ and } h(u)=u\cdot i_h, \text{ for all } u\in P_1. \quad (11)$$

The triple (i_f, i_g, i_h) is denoted by $S(\Theta,Q_1)$. If $Q_2 = Q_1$, then the isotopism Θ is said to be a *Santilli autotopism* of Q_1.

In [11], there were exposed several properties of the set of partial quasigroups having a Santilli autotopism that fixes at least one of the symbols. An exhaustive study of Santilli autotopisms is, however, currently required. This paper is established as a first contribution in this regard. In Section 2, some new general properties of the set of Santilli isotopisms of (associative) partial quasigroups, partial loops and partial groups are analyzed. In Section 3, a classification of the Santilli autotopisms of groups of order $n \leq 6$ is explicitly given. Remark that, even if the number of quasigroups is known for order up to 11 [16, 17], that of partial quasigroups is only known for order up to four [18, 19].

2. Santilli Autotopisms

From now on, every partial quasigroup of order n is considered to be formed by the set of elements $\{1,\ldots, n\}$. The set of isotopisms of partial quasigroups of order n is then denoted as $I_n = S_n \times S_n \times S_n$, where S_n is the symmetric group on $\{1,\ldots, n\}$. The set of fixed symbols in a permutation $\pi \in S_n$ is denoted as

$$Fix(\pi) = \{u \in \{1,\ldots n\} \text{ such that } \pi(u)=u\}. \qquad (12)$$

Let $\Theta \in I_n$ and let $SQ(\Theta)$, $SL(\Theta)$, $SAQ(\Theta)$ and $SG(\Theta)$ be, respectively, the sets of partial quasigroups, partial loops, associative partial quasigroups and partial groups that have Θ as a Santilli autotopism. The next results are satisfied.

Lemma 2.1. Let $\Theta = (f, g, h) \in I_n$ and $(Q, \cdot) \in SQ(\Theta)$ be such that $S(\Theta,Q)= (i_f, i_g, i_h)$. Then, $i_h = g(i_f)$. As a consequence,

$$(i \cdot i_f) \cdot (j \cdot i_g) = (i \cdot j) \cdot (i_f \cdot i_g), \text{ for all } i, j \in Q. \qquad (13)$$

Proof. Given $v\in Q$, let $u \in Q$ be such that $f(u)=v$. Then, $v \cdot i_h = h(v) = h(f(u)) = h(u\cdot i_f) = f(u) \cdot g(i_f) = v \cdot g(i_f)$ and the result holds from the fact that Q is a partial quasigroup and $h(v) \in Q$.

Proposition 2.2. Let $\Theta = (f, g, h) \in I_n$ and $(Q, \cdot) \in SQ(\Theta)$ be such that $S(\Theta,Q)= (i_f, i_g, i_h)$. If $h = f$, then $i_f \in Fix(g)$.

Proof. The result follows straightforward from Lemma 2.1 and the fact of being $h = f$.

Lemma 2.3. Let $\Theta = (f, g, h) \in I_n$. If there exist two permutations $\alpha, \beta \in \{f, g, h\}$ such that $\alpha(u_0) = \beta(u_0)$ for some $u_0 \in Q$, then $\alpha = \beta$.

Proof. Let $(Q, \cdot)$ be a partial quasigroup in $SQ(\Theta)$ and let $i_\alpha, i_\beta \in Q$ be such that $\alpha(u) = u \cdot i_\alpha$ and $\beta(u) = u \cdot i_\beta$ for all $u \in Q$. Particularly, $u_0 \cdot i_\alpha = \alpha(u_0) = \beta(u_0) = u_0 \cdot i_\beta$. This product is not undefined because $\alpha(u_0) \in Q$. Since Q is a partial quasigroup, it must be then $i_\alpha = i_\beta$ and hence, $\alpha = \beta$.

Proposition 2.4. Let $\Theta = (f, g, h) \in I_n$ be such that $Fix(g) = \emptyset$. Then, $f(u) \neq h(u)$ *for all* $u \in Q$.

Proof. Let $u \in Q$ be such that $f(u) = h(u)$. From Lemma 2.3 it must be $f = h$. Thus, from Lemma 2.1, it is $i_f = i_h = g(i_f)$ and hence, $i_f \in Fix(g)$, which is a contradiction.

Lemma 2.5. Let $\Theta = (f, g, h) \in I_n$ and $(Q, \cdot) \in SQ(\Theta)$ be such that $S(\Theta,Q) = (i_f, i_g, i_h)$. If there exists $u_0 \in Q$ such that $h^m(g(u_0)) = g(f^m(u_0))$ for some positive integer m, then $i_g \in Fix(g^m)$. As a consequence, if $Fix(g^m) = \emptyset$ for some positive integer m, then $h^m(g(u)) \neq g(f^m(u))$, for all $u \in Q$.

Proof. Let m be such that $h^m(g(u_0)) = g(f^m(u_0))$ for some $u_0 \in Q$. It is then $f^m(u_0) \cdot g^m(i_g) = h^m(u_0 \cdot i_g) = h^m(g(u_0)) = g(f^m(u_0)) = f^m(u_0) \cdot i_g$. This product is not undefined because $h^m(g(u_0)) \in Q$. Since Q is a partial quasigroup, it must be then $i_g \in Fix(g^m)$. The consequence is immediate.

Lemma 2.6. Let $\Theta = (f, g, h) \in I_n$ be such that $|Fix(f)| \cdot |Fix(g)| \cdot |Fix(h)| > 0$. Let $(Q, \cdot) \in SQ(\Theta)$ be such that $S(\Theta,Q) = (i_f, i_g, i_h)$. If there exist $u_0 \in Fix(f)$, $w_0 \in Fix(h)$ and $\alpha \in \{f, g, h\}$ such that $\alpha(u_0) = w_0$, then $i_\alpha \in Fix(g)$. Further, if $i_g \in Fix(g)$, then $g(u) \in Fix(h)$ for all $u \in Fix(f)$.

Proof. It is satisfied that $u_0 \cdot i_\alpha = \alpha(u_0) = w_0 = h(w_0) = h(u_0 \cdot i_\alpha) = f(u_0) \cdot g(i_\alpha) = u_0 \cdot g(i_\alpha)$. Since $w_0 \in Q$ and Q is a quasigroup, it must be $i_\alpha \in Fix(g)$. Let us suppose now that $i_g \in Fix(g)$ and let us consider an element $u \in Fix(f)$. Then $g(u) = u \cdot i_g = f(u) \cdot g(i_g) = h(u \cdot i_g) = h(g(u))$ and hence, $g(u) \in Fix(h)$.

The next three results deal with the set of partial loops $SL(\Theta)$ having a Santilli isotopism Θ in their autotopism group.

Proposition 2.7. Let $\Theta = (f, g, h) \in I_n$ and $(Q, \cdot) \in SL(\Theta)$ be a partial loop with unit element e. Then, $S(\Theta,Q) = (f(e), g(e), g(f(e)))$.

Proof. Let $S(\Theta,Q) = (i_f, i_g, i_h)$. The result follows straightforward from Lemma 2.1 and the fact that $\pi(e) \in Q$. Hence, $\pi(e) = e \cdot i_\pi = i_\pi$, for all $\pi \in \{f, g\}$.

Lemma 2.8. Let $\Theta = (f, g, h) \in I_n$. If there exists a permutation $\pi \in \{f, g, h\}$ such that $Fix(\pi) \neq \emptyset$, then $\pi = Id$.

Proof. Let $(Q, \cdot) \in SL(\Theta)$ and $S(\Theta,Q) = (i_f, i_g, i_h)$. Let $\pi \in \{f, g, h\}$ and $u_0 \in Q$ be such that $\pi(u_0) = u_0$. Since $u_0 = u_0 \cdot i_\pi$, the element i_π is the unit element of the partial loop. Let $u \in Q$. Since $\pi(u) \in Q$, it is $\pi(u) = u \cdot i_\pi = u$ and hence, $\pi = Id$.

Lemma 2.9. Let $\Theta = (f, g, h) \in I_n$ and $(Q, \cdot) \in SL(\Theta)$ be a partial loop with unit element e. If $e \in Fix(f^m)$ for some positive integer m, then $h^m = g^m$. Similarly, if $e \in Fix(g^m)$, then $h^m = f^m$.

Proof. Let us suppose that $e \in Fix(f^m)$ for some positive integer m. Let $u \in Q$. It is $g^m(u) = e \cdot g^m(u) = f^m(e) \cdot g^m(u) = h^m(e \cdot u)$. Since $g^m(u) \in Q$, it must be $e \cdot u = u$ and hence, $g^m(u) = h^m(u)$. The last assertion follows analogously.

We focus now on the set $SAQ(\Theta)$ of associative partial quasigroups having a Santilli autotopism in their autotopism group.

Proposition 2.10. Let $\Theta = (f, g, h) \in I_n$. If $SAQ(\Theta) \neq \emptyset$, then $h = g \circ f$.

Proof. Let $(Q, \cdot) \in SAQ(\Theta)$ and $S(\Theta,Q) = (i_f, i_g, i_h)$. From Lemma 2.1, we know that $i_h = g(i_f)$. Hence, for all $u \in Q$, it is verified that $h(u) = u \cdot i_h = u \cdot g(i_f) = u \cdot (i_f \cdot i_g) = (u \cdot i_f) \cdot i_g = g(f(u))$.

Lemma 2.11. Let $\Theta = (f, g, h) \in I_n$ be such that $SAQ(\Theta) \neq \emptyset$ and let $m \leq n$ be a positive integer. Then

a) $SAQ(\Theta) \subseteq SAQ(\Theta^m)$.

b) $SAQ(\Theta) = SAQ((f, g \circ f^m, h \circ f^m))$.

Proof. Let $(Q, \cdot) \in SAQ(\Theta)$ be such that $S(\Theta,Q) = (i_f, i_g, i_h)$ and let $m \leq n$ be a positive integer. Then

1. The isotopism Θ^m is an autotopism of $(Q, \cdot)$ because $h^m(u \cdot v) = h^{m-1}(f(u) \cdot g(v)) = \ldots = f^m(u) \cdot g^m(v)$, for all $u, v \in Q$. Since the quasigroup $(Q, \cdot)$ is associative, this is indeed a Santilli autotopism for which $S(\Theta^m,Q) = (i_f^m, i_g^m, i_h^m)$.
2. The isotopism $(f, g \circ f^m, h \circ f^m)$ is an autotopism of $(Q, \cdot)$ because $h(f^m(u \cdot v)) = h((u \cdot v) \cdot i_f^m) = h(u \cdot (v \cdot i_f^m)) = h(u \cdot f^m(v)) = f(u) \cdot g(f^m(v))$, for all $u, v \in Q$. Since the quasigroup $(Q, \cdot)$ is associative, this is indeed a Santilli autotopism for which $S((f, g \circ f^m, h \circ f^m),Q) = (i_f^m, i_f^m \cdot i_g, i_f^m \cdot i_h)$. Hence, $SAQ(\Theta) \subseteq SAQ((f, g \circ f^m, h \circ f^m))$.

Let us consider now an associative partial quasigroup $(Q', *) \in SAQ((f, g \circ f^m, h \circ f^m))$ such that $S((f, g \circ f^m, h \circ f^m),Q') = (i_1, i_2, i_3)$. It is then verified that Θ is a Santilli autotopism of $(Q', *)$ because, since $f^n = Id$, it is $h(u * v) = h(f^n(u * v)) = h(f^m(f^{n-m}(u * v)) = h(f^m(u * f^{n-m}(v)) = f(u) * g(f^m(f^{n-m}(v)) = f(u) * g(f^n(v)) = f(u) * g(v)$, for all $u, v \in Q'$. Further, $S(\Theta,Q') = (i_1, i_2 * i_1^{n-m}, i_3 * i_1^{n-m})$. Hence, $SAQ((f, g \circ f^m, h \circ f^m)) \subseteq SAQ(\Theta)$.

In general, given a positive integer $m \leq n$, it is not true that $SAQ(\Theta^m) \subseteq SAQ(\Theta)$. Thus, for instance, the isotopism $\Theta = ((1234), (1234), (13)(24))$ is a Santilli autotopism of the associative quasigroup whose multiplication table is the Latin square exposed in Figure 1. Nevertheless, even if the isotopism $\Theta^2 = ((13)(24),(13)(24),Id)$ is a Santilli autotopism of the associative partial quasigroup whose multiplication table is exposed in Figure 3, this is not contained in $SAQ(\Theta)$.

	3		1
	4		2
	1		3
	2		4

Figure 3. *Partial Latin square of order 4.*

Let us finish with a result about the set $SG(\Theta)$ of partial groups having a Santilli isotopism in their autotopism group.

Theorem 2.12. Let $\Theta = (f, g, h) \in I_n$. If $SG(\Theta) \neq \emptyset$ and $Fix(f) \neq \emptyset$, then $g = h$ and $f = Id$.

Proof. The result follows straightforward from Lemma 2.8 and Proposition 2.10.

3. Santilli Autotopisms of Partial Groups of Order $n \leq 4$

The results that have been exposed in Section 2 can be taken into account in order to determine explicitly the set of Santilli isotopisms that are autotopisms of partial groups of a given order. To this end, we say that two isotopisms $\Theta_1 = (f_1, g_1, h_1)$ and $\Theta_2 = (f_2, g_2, h_2)$ in I_n are *equivalent* if $f_2 = f_1$ and there exists a positive integer $m \leq n$ such that $g_2 = g_1 \circ f_1^m$ and $h_2 = h_1 \circ f_1^m$. From assertion (b) in Lemma 2.11, it is verified that $SAQ(\Theta_1) = SAQ(\Theta_2)$. To be equivalent is then an equivalence relation in the set I_n. Let $[\Theta]$ denote the equivalence class of $\Theta \in I_n$. We expose in Table 1 these equivalence classes for Santilli autotopisms of partial groups of order $n \leq 4$. We focus on the case of non-trivial isotopisms, that is, those that do not coincide with *(Id, Id, Id)*.

Table 1. *Santilli autotopisms of partial groups.*

n	[Θ]	SG(Θ)
2	[((12), (12), Id)]	A_2
	[(Id, (12),(12))]	
3	[(123), (123), (132)]	A_3
	[(132), (132), (123))]	
	[(Id, (123), (123))]	
	[(Id, (132), (132)]	
4	[(1234), (1234), (13)(24))]	A_4
	[(1432), (1432), (13)(24))]	
	[((13)(24), (1234), (1432)]	
	[(Id, (1234), (1234)]	
	[(Id, (1432), (1432)]	
	[(1243), (1243), (14)(23))]	B_4
	[(1342), (1342), (14)(23))]	
	[((14)(23), (1243), (1342)]	
	[(Id, (1243), (1243)]	
	[(Id, (1342), (1342)]	
	[(1324), (1324), (12)(34))]	C_4
	[(1423), (1423), (12)(34))]	
	[((12)(34), (1324), (1423)]	
	[(Id, (1324), (1324)]	
	[(Id, (1423), (1423)]	
	[((12)(34), (13)(24), (14)(23)]	D_4
	[((12)(34), (14)(23), (13)(24)]	
	[((13)(24), (12)(34), (14)(23)]	
	[((13)(24), (14)(23), (12)(34)]	
	[((14)(23), (12)(34), (13)(24)]	
	[((14)(23), (13)(24), (12)(34)]	
	[((12)(34), (12)(34), Id]	C_4, D_4, E_4
	[(Id, (12)(34), (12)(34)]	
	[((13)(24), (13)(24), Id]	A_4, D_4, F_4
	[(Id, (13)(24), (13)(24)]	
	[((14)(23), (14)(23), Id]	B_4, D_4, G_4
	[(Id, (14)(23), (14)(23)]	

We indicate for each class $[\Theta]$ in Table 1 the set $SG(\Theta)$ of partial groups that have all the isotopisms of the class in their corresponding autotopism group. The multiplication tables of the elements of these sets are described in Figures 4–12.

1	2	2	1
2	1	1	2

Figure 4. *Partial Latin squares related to A_2.*

1	2	3	3	1	2	2	3	1
2	3	1	1	2	3	3	1	2
3	1	2	2	3	1	1	2	3

Figure 5. *Partial Latin squares related to A_3.*

1	2	3	4	4	1	2	3	3	4	1	2	2	3	4	1
2	3	4	1	1	2	3	4	4	1	2	3	3	4	1	2
3	4	1	2	2	3	4	1	1	2	3	4	4	1	2	3
4	1	2	3	3	4	1	2	2	3	4	1	1	2	3	4

Figure 6. *Partial Latin squares related to A_4.*

1	2	3	4	3	1	4	2	2	4	1	3	4	3	2	1
2	4	1	3	1	2	3	4	4	3	2	1	3	1	4	2
3	1	4	2	4	3	2	1	1	2	3	4	2	4	1	3
4	3	2	1	2	4	1	3	3	1	4	2	1	2	3	4

Figure 7. *Partial Latin squares related to B_4.*

1	2	3	4	2	1	4	3	4	3	1	2	3	4	2	1
2	1	4	3	1	2	3	4	3	4	2	1	4	3	1	2
3	4	2	1	4	3	1	2	1	2	3	4	2	1	4	3
4	3	1	2	3	4	2	1	2	1	4	3	1	2	3	4

Figure 8. *Partial Latin squares related to C_4.*

1	2	3	4	2	1	4	3	3	4	1	2	4	3	2	1
2	1	4	3	1	2	3	4	4	3	2	1	3	4	1	2
3	4	1	2	4	3	2	1	1	2	3	4	2	1	4	3
4	3	2	1	3	4	1	2	2	1	4	3	1	2	3	4

Figure 9. *Partial Latin squares related to D_4.*

1	2			2	1					1	2			2	1
2	1			1	2					2	1			1	2
3	4			4	3					3	4			4	3
4	3			3	4					4	3			3	4

Figure 10. *Partial Latin squares related to E_4.*

1		3			1		3	3		1			3		1
2		4			2		4	4		2			4		2
3		1			3		1	1		3			1		3
4		2			4		2	2		4			2		4

Figure 11. *Partial Latin squares related to F_4.*

1			4		1	4			4	1		4			1
2			3		2	3			3	2		3			2
3			2		3	2			2	3		2			3
4			1		4	1			1	4		1			4

Figure 12. *Partial Latin squares related to G_4.*

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Hyper-Representations by Non Square Matrices Helix-Hopes

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Abstract: Hyperstructure theory can overcome restrictions which ordinary algebraic structures have. A hyperproduct on non-square ordinary matrices can be defined by using the so called helix-hyperoperations. We define and study the helix-hyperstructures on the representations and we extend our study up to Lie-Santilli theory by using ordinary fields. Therefore the related theory can be faced by defining the hyperproduct on the extended set of non square matrices. The obtained hyperstructure is an H_v-algebra or an H_v-Lie-alebra.

Keywords: Hyperstructures, H_v-Structures, H/V-Structures, Hope, Helix-Hope

1. Introduction

We deal with the largest class of hyperstructures called H_v-structures introduced in 1990 [23],[26], which satisfy the weak axioms where the non-empty intersection replaces the equality.

Basic definitions:

Definitions 1.1 In a set H equipped with a hyperoperation, which we abbreviate it by hope $\cdot$:$H\times H\to P(H)$, we abbreviate by WASS the weak associativity: $(xy)z\cap x(yz)\neq\varnothing$, $\forall x,y,z\in H$ and by COW the weak commutativity: $xy\cap yx\neq\varnothing$, $\forall x,y\in H$.

The hyperstructure $(H,\cdot)$ is called H_v-semigroup if it is WASS and is called H_v-group if it is reproductive H_v-semigroup: $xH=Hx=H$, $\forall x\in H$. $(R,+,\cdot)$ is called H_v-ring if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is weak distributive with respect to $(+)$:

$$x(y+z)\cap(xy+xz)\neq\varnothing,\ (x+y)z\cap(xz+yz)\neq\varnothing,\ \forall x,y,z\in R.$$

For more definitions and applications on H_v-structures, see books [26],[2],[8] and the survey papers [6],[25],[30]. An extreme class is the following [26]: An H_v-structure is very thin iff all hopes are operations except one, with all hyperproducts singletons except only one, which is a subset of cardinality more than one. Therefore, in a very thin H_v-structure in a set H there exists a hope $(\cdot)$ and a pair $(a,b)\in H^2$ for which ab=A, with cardA>1, and all the other products, with respect to any other hopes (so they are operations), are singletons.

The fundamental relations β^* and γ^* are defined, in H_v-groups and H_v-rings, respectively, as the smallest equivalences so that the quotient would be group and ring, respectively [22],[23],[26],[27],[28],[35]. The way to find the fundamental classes is given by analogous theorems to the following:

Theorem 1.2 Let $(H,\cdot)$ be an H_v-group and let us denote by U the set of all finite products of elements of H. We define the relation β in H as follows: $x\beta y$ iff $\{x,y\}\subset u$ where $u\in U$. Then the fundamental relation β^* is the transitive closure of the relation β.

The main point of the proof of this theorem is that β guaranties that the following is valid: Take two elements x,y such that $\{x,y\}\subset u\in U$ and any hyperproduct where one of these elements is used. Then, if this element is replaced by the other, the new hyperproduct is inside the same fundamental class where the first hyperproduct is. Therefore, if the 'hyperproducts'of the above β-classes are 'products', then, they are fundamental classes. Analogously for the γ in H_v-rings.

An element is single if its fundamental class is a singleton.

Motivation for H_v-structures:

We know that the quotient of a group with respect to an invariant subgroup is a group.

Marty states that, the quotient of a group with respect to any subgroup is a hypergroup.

Now, the quotient of a group with respect to any partition is an H_v-group.

Definition 1.3 Let $(H,\cdot)$, $(H,\otimes)$ be H_v-semigroups defined on the same set H. $(\cdot)$ is smaller than $(\otimes)$, and $(\otimes)$ greater than $(\cdot)$, iff there exists automorphism

$$f\in Aut(H,\otimes) \text{ such that } xy\subset f(x\otimes y),\ \forall x\in H.$$

Then $(H,\otimes)$ contains $(H,\cdot)$ and write $\cdot\leq\otimes$. If $(H,\cdot)$ is structure, then it is basic and $(H,\otimes)$ is an H_b-structure.

The Little Theorem [26]. Greater hopes of the ones which are WASS or COW, are also WASS and COW, respectively.

The fundamental relations are used for general definitions of hyperstructures. Thus, to define the general H_v-field one uses the fundamental relation γ^*:

Definition 1.4 [23],[26],[27]. The H_v-ring $(R,+,\cdot)$ is called H_v-field if the quotient R/γ^* is a field.

Let ω^* be the kernel of the canonical map from R to R/γ^*; then we call reproductive H_v-field any H_v-field $(R,+,\cdot)$ if the following axiom is valid:

$$x(R-\omega^*) = (R-\omega^*)x = R-\omega^*,\ \forall x\in R-\omega^*.$$

From the above a new class is introduced [31],[38]:

Definition 1.5 The H_v-semigroup $(H,\cdot)$ is called h/v-group if the H/β^* is a group.

Similarly the h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces etc, are defined. The h/v-group is a generalization of the H_v-group since the reproductivity is not necessarily valid. Sometimes a kind of reproductivity of classes is valid, i.e. if H is partitioned into equivalence classes $\sigma(x)$, then the quotient is reproductive $x\sigma(y)=\sigma(xy)=\sigma(x)y$, $\forall x\in H$ [31].

An H_v-group is cyclic [17],[26], if there is element, called generator, which the powers have union the underline set, the minimal power with this property is the period of the generator. If there exists an element and a special power, the minimum one, is the underline set, then the H_v-group is called single-power cyclic.

To compare classes we can see on small sets. The problem of enumeration and classification of H_v-structures, or of classes of them, is complicate in H_v-structures because we have great numbers. The partial order in H_v-structures, introduced in [26], restrict the problem in finding the minimal H_v-structures, up to isomorphism. We have results recently by Bayon & Lygeros as the following [1],[13]:

In sets with three elements: Up to isomorphism, there are 6.494 minimal H_v-groups. The 137 are abelians; the 6.152 are cyclic. The number of H_v-groups with three elements, up to isomorphism, is 1.026.462. The 7.926 are abelians; 1.013.598 are cyclic. 16 are very thin. Abelian H_v-groups with 4 elements are, 8.028.299.905, the 7.995.884.377.

Definitions 1.6 [25],[26],[38] Let $(R,+,\cdot)$ be H_v-ring, $(M,+)$ be COW H_v-group and there exists an external hope:

$$R\times M\to P(M): (a,x)\to ax,$$

such that, $\forall a,b\in R$ and $\forall x,y\in M$ we have

$$a(x+y)\cap(ax+ay)\neq\varnothing,\ (a+b)x\cap(ax+bx)\neq\varnothing,\ (ab)x\cap a(bx)\neq\varnothing$$

then M is called an H_v-module over R. In case of an H_v-field F instead of H_v-ring R, then the H_v-vector space is defined.

The fundamental relation ε^* is defined to be the smallest equivalence such that the quotient M/ε^* is a module (resp., a vector space) over the fundamental ring R/γ^* (resp. the fundamental field F/γ^*). The analogous to Theorem 1.2, is:

Theorem Let $(M,+)$ be H_v-module on the H_v-ring R. Denote by U the set of all expressions consisting of finite hopes either on R and M or the external hope applied on finite sets of elements of R and M. Define relation ε in M as follows: $x\varepsilon y$ iff $\{x,y\}\subset u$ where $u\in U$.

Then the relation ε^* is the transitive closure of the relation ε.

Definitions 1.7 [28],[29],[38]. Let $(H,\cdot)$ be hypergroupoid. We remove $h\in H$, if we consider the restriction of $(\cdot)$ in the $H-\{h\}$. We say that $h\in H$ absorbs $h\in H$ if we replace h by h and h does not appear in the structure. We say that $h\in H$ merges with $h\in H$, if we take as product of any $x\in H$ by h, the union of the results of x with both h, h, and consider h and h as one class, with representative h, therefore the element h does not appeared in the hyperstructure.

Let $(H,\cdot)$ be an H_v-group, then, if an element h absorbs all elements of its own fundamental class then this element becomes a single in the new H_v-group.

Definition 1.8 [35],[37] Let $(L,+)$ be H_v-vector space over the field $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$, the canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the fundamental field F/γ^*. Similarly, let ω_L be the core of the canonical map φ': $L\to L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^*. Consider the bracket (commutator) hope:

$$[\ ,\ ] : L\times L\to P(L): (x,y)\to[x,y]$$

then L is an H_v-Lie algebra over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\varnothing$$

$$[x,\lambda_1y_1+\lambda_2y]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\varnothing,$$

$$\forall x,x_1,x_2,y,y_1,y_2\in L \text{ and } \lambda_1,\lambda_2\in F$$

$$\text{(L2) } [x,x]\cap\omega_L\neq\varnothing,\ \forall x\in L$$

$$\text{(L3) } ([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L\neq\varnothing,\ \forall x,y\in L$$

A well known and large class of hopes is given as follows [17],[21]:

Definitions 1.9 Let $(G,\cdot)$ be a groupoid, then for every $P\subset G$, $P\neq\varnothing$, we define the following hopes, P-hopes: $\forall x,y\in G$

$$P:\ xPy=(xP)y\cup x(Py),$$

$$P_r:\ xP_ry=(xy)P\cup x(yP),\quad P_l:\ xP_ly=(Px)y\cup P(xy).$$

The (G,P), (G,P_r) and (G,P_l) are called P-hyperstructures. For semigroup $(G,\cdot)$, we have $xPy=(xP)y\cup x(Py)=xPy$ and (G,P) is a semihypergroup but we do not know about (G,P_r) and (G,P_l). In some cases, depending on the choice of P, the (G,P_r) and (G,P_l) can be associative or WASS.

A generalization of P-hopes is the following [9], [10]:

Let $(G,\cdot)$ be abelian group and P a subset of G with more than one elements. We define the hope $\times_P$ as follows:

$$x\times_P y = x\cdot P\cdot y = \{x\cdot h\cdot y \mid h\in P\} \text{ if } x\neq e \text{ and } y\neq e$$

$$x\cdot y \text{ if } x=e \text{ or } y=e$$

we call this, P_e-hope. The $(G,\times_P)$ is an abelian H_v-group.

A general definition of hopes, is the following [32],[35],[36],[37]:

Definitions 1.10 Let H be a set with n operations (or hopes) $\otimes_1,\otimes_2,\ldots,\otimes_n$ and one map (or multivalued map) f: H→H, then n hopes $\partial_1,\partial_2,\ldots,\partial_n$ on H are defined, called ∂-hopes, by putting

$$x\partial_i y = \{f(x)\otimes_i y,\ x\otimes_i f(y)\},\ \forall x,y\in H,\ i\in\{1,2,\ldots,n\}$$

or in case where $\otimes_i$ is hope or f is multivalued map we have

$$x\partial_i y = (f(x)\otimes_i y)\cup(x\otimes_i f(y)),\ \forall x,y\in H,\ i\in\{1,2,\ldots,n\}$$

Let $(G,\cdot)$ groupoid and f_i:G→G, i∈I, set of maps on G. Take the map $f_\cup$:G→P(G) such that $f_\cup(x)=\{f_i(x)\mid i\in I\}$, call it the union of the $f_i(x)$. We call the union ∂-hope (∂), on G if we consider the map $f_\cup(x)$. An important case for a map f, is to take the union of this with the identity id. Thus, we consider the map $f\equiv f\cup(id)$, so $f(x)=\{x,f(x)\},\forall x\in G$, which is called b-∂-hope, we denote it by (∂), so we have

$$x\partial y = \{xy,\ f(x)\cdot y,\ x\cdot f(y)\},\ \forall x,y\in G.$$

Remark. If $\otimes_i$ is associative then ∂_i is WASS. If ∂ contains the operation (·), then it is b-operation. Moreover, if f:G→P(G) is multivalued then the b-∂-hopes is defined by using the $f(x)=\{x\}\cup f(x)$, ∀x∈G.

Motivation for the definition of ∂-hope is the derivative where only multiplication of functions is used. Therefore, for functions s(x), t(x), we have $s\partial t=\{s't,st'\}$, (′) is the derivative.

Example. Take all polynomials of first degree $g_i(x)=a_ix+b_i$. We have

$$g_1\partial g_2 = \{a_1a_2x+a_1b_2,\ a_1a_2x+b_1a_2\},$$

so it is a hope in the set of first degree polynomials. Moreover all polynomials x+c, where c be a constant, are units.

In hyperstructures there is the uniting elements method. This is defined as follows [3],[26],[28]: Let G be a structure and d be a property, which is not valid, and d is described by a set of equations. Consider the partition in G for which it is put together, in the same class, every pair of elements that causes the non-validity of d. The quotient G/d is an H_v-structure. The quotient of G/d by β*, is a stricter structure (G/d)β* for which d is valid.

2. Matrix Representations

H_v-structures are used in Representation (abbr. by rep) Theory. Reps of H_v-groups can be considered either by generalized permutations or by H_v-matrices [18],[20],[24],[25],[26],[38]. The reps by generalized permutations can be achieved by using left or right translations. We present here the hypermatrix rep in H_v-structures and there exist the analogous theory for the h/v-structures.

Definitions 2.1 [20],[26] H_v-matrix is called a matrix with entries elements of an H_v-ring or H_v-field. The hyperproduct of two H_v-matrices $A=(a_{ij})$ and $B=(b_{ij})$, of type m×n and n×r respectively, is defined, in the usual manner,

$$A\cdot B = (a_{ij})\cdot(b_{ij}) = \{\ C=(c_{ij}) \mid c_{ij}\in \oplus\Sigma a_{ik}\cdot b_{kj}\ \},$$

and it is a set of m×r H_v-matrices. The sum of products of elements of the H_v-field is the union of the sets obtained with all possible parentheses put on them, called n-ary circle hope on the hyperaddition.

The hyperproduct of H_v-matrices does not necessarily satisfy WASS.

The problem of the H_v-matrix representations is the following:

Definitions 2.2 Let $(H,\cdot)$ be an H_v-group. Find an H_v-ring or an H_v-field $(F,+,\cdot)$, a set $M_R=\{(a_{ij})\mid a_{ij}\in R\}$ and a map

$$T:\ H\to M_R:\ h\to T(h)$$

such that

$$T(h_1h_2)\cap T(h_1)T(h_2)\neq\varnothing,\ \forall h_1,h_2\in H.$$

T is an H_v-matrix rep. If the $T(h_1h_2)\subset T(h_1)T(h_2)$, $\forall h_1,h_2\in H$ is valid, then T is an inclusion rep. If $T(h_1h_2)=T(h_1)T(h_2)=\{T(h)\mid h\in h_1h_2\}$, $\forall h_1,h_2\in H$, then T is a good rep and then an induced rep T* for the hypergroup algebra is obtained. If T is one to one and good then it is a faithful rep.

The problem of reps is complicated because the cardinality of the product of H_v-matrices is very big. It can be simplified in special cases such as the following: The H_v-matrices are over H_v-fields with scalars 0 and 1. The H_v-matrices are over very thin H_v-fields. On 2×2 H_v-matrices, since the circle hope coincides with the hyperaddition. On H_v-fields which contain singles, then these act as absorbing.

The main theorem of reps is the following [20],[25],[26]:

Theorem 2.3 A necessary condition in order to have an inclusion rep T of an H_v-group $(H,\cdot)$ by n×n H_v-matrices over the H_v-rind or H_v-field $(F,+,\cdot)$ is the following:

For all classes β*(x), x∈H there must exist elements $a_{ij}\in H$, $i,j\in\{1,\ldots,n\}$ such that

$$T(\beta^*(a))\subset\{\ A=(a'_{ij})\mid a'_{ij}\in\gamma^*(a_{ij}),\ i,j\in\{1,\ldots,n\}\ \}$$

So every inclusion rep $T:H\to M_R:a\to T(a)=(a_{ij})$ induces a homomorphic rep T* of the group H/β* over the field F/γ* by putting $T^*(\beta^*(a))=[\gamma^*(a_{ij})]$, $\forall\beta^*(a)\in H/\beta^*$, where the $\gamma^*(a_{ij})\in R/\gamma^*$ is the ij entry of the matrix T*(β*(a)). T* is called fundamental induced rep of T.

Denote $tr_\varphi(T(x))=\gamma^*(T(x_{ii}))$ the fundamental trace, then the mapping

$$X_T: H\to R/\gamma^*:\ x\to X_T(x) = tr_\varphi(T(x)) = trT^*(x)$$

is called fundamental character. There are several types of traces.

Using several classes of H_v-structures one can face several reps [26],[29],[30],[38]:

Definition 2.4 Let $M=M_{m\times n}$ be a module of m×n matrices over a ring R and take sets

$$S=\{s_k:k\in K\}\subseteq R,\ Q=\{Q_j:j\in J\}\subseteq M,\ P=\{P_i:i\in I\}\subseteq M.$$

Define three hopes as follows

$$S: R\times M\to P(M): (r,A)\to rSA=\{(rs_k)A: k\in K\}\subseteq M$$

$$Q_+: M\times M\to P(M): (A,B)\to AQ_+B=\{A+Q_j+B: j\in J\}\subseteq M$$

$$P: M\times M\to P(M): (A,B)\to APB=\{AP^t_iB: i\in I\}\subseteq M$$

Then (M,S,Q_+,P) is a hyperalgebra over R called general matrix P-hyperalgebra.

The hope P, which is a bilinear map, is a generalization of Rees' operation where, instead of one sandwich matrix, a set of sandwich matrices is used. The hope P is strong associative and the inclusion distributivity with respect to addition of matrices

$$AP(B+C)\subseteq APB+APC\ \forall A,B,C\in M$$

is valid. Thus, (M,+,P) defines a multiplicative hyperring on non-square matrices.

In a similar way a generalization of this hyperalgebra can be defined considering an H_v-ring or an H_v-field instead of a ring and using H_v-matrices instead of matrices.

Definition 2.5 Let $A=(a_{ij}),B=(b_{ij})\in M_{m\times n}$, we call (A,B) unitize pair of matrices if $A^tB=I_n$, where I_n denotes the n×n unit matrix.

The following theorem can be applied in the classical theory [37],[38].

Theorem 2.6 If m<n, then there is no unitize pair.

Proof. Suppose that n>m and that

$$A^tB=(c_{ij}),\ c_{ij}=\sum_{k=1}^{m} a_{ik}b_{kj}\,.$$

Denote by A_m the block of the matrix A such that $A_m=(a_{ij})\in M_{m\times m}$, i.e. we take the matrix of the first m columns. Then we suppose that we have $(A_m)^tB_m=I_m$, therefore we must have $\det(A_m)\neq 0$. Now, since n>m, we can consider the homogeneous system with respect to the 'unknowns' $b_{1n},b_{2n},\ldots,b_{mn}$:

$$c_{in}=\sum_{k=1}^{m} a_{ik}b_{kn}=0 \text{ for } i=1,2,\ldots,m.$$

From which, we obtain that $b_{1n}=b_{2n}=\ldots=b_{mn}=0$, since $\det(A_m)\neq 0$. Using this fact on the last equation, on the same unknowns,

$$c_{nn}=\sum_{k=1}^{m} a_{nk}b_{kn}=1$$

we have 0=1, absurd. ■

We recall some definitions from [18],[20],[25].

Definition 2.7 Let (G,·) hypergroupoid, is called set of fundamental maps on G, a set of onto maps

$$Q=\{\,q: G\times G\to G: (x,y)\xrightarrow{onto} z\,|\,z\in xy\,\}.$$

Any subset $Q_s\subset Q$ defines a hope $(\circ_s)$ on G as follows

$$x\circ_s y=\{\,z\,|\,z=q(x,y) \text{ for some } q\in Q\,\}$$

$\circ_s\leq\cdot$, and $Q_s\subset Q_{os}$, where Q_{os} is the set of fundamental maps with respect to $(\circ_s)$. A $Q_a\subset Q$ for which every $Q_s\subset Q_a$ has $(\circ_s)$ associative (resp. WASS) is called associative (resp. WASS). A hypergroupoid (G,·) is q-WASS if there exists an element $q_o\in Q$ which defines an associative operation (∘) in G. Remark that for H_v-groups we have $Q\neq\varnothing$.

If G is finite, cardG=$|G|$=n, it is q-WASS with associative $q_o\in Q$. In the set K[G] of all formal linear combinations of elements of G with coefficients from a field K, we define an operation (+):

$$(f_1+f_2)(g)=f_1(g)+f_2(g),\forall g\in G,f_1,f_2\in K[G]$$

and a hope (∗), the convolution,

$$f_1*f_2=\{\,f_q: f_q(g)=\sum_{q(x,y)=g} f_1(x)f_2(y),\ q\in Q\,\}.$$

(K[G],+,∗) is a multiplicative H_v-ring where the inclusion distributivity is valid, which is called hypergroupoid H_v-algebra.

For all q∈Q, g∈G, we have

$$|Q|\leq \prod_{(x,y) in GxG}(|xy|),\ 1\leq|q^{-1}(g)|\leq n^2-n+1$$

$$\text{and } \sum_{g in G}|q^{-1}(g)|=n^2.$$

The zero map f(x)=0 is a scalar element in K[G].

In the representation theory several constructions are used, some of them are the following [26],[28],[29],[30]:

Constructions 2.8 Let (H,·) be H_v-group, then for all (⊕) such that $x\oplus y\supset\{x,y\}$, ∀x,y∈H, the (H,⊕,·) is an H_v-ring. These H_v-rings are called associated to (H,·) H_v-rings.

In rep theory of hypergroups, in sense of Marty where the equality is valid, there are three associated hyperrings (H,⊕,·) to (H,·). The (⊕) is defined respectively, ∀x,y∈H, by: type a $x\oplus y=\{x,y\}$, type b $x\oplus y=\beta^*(x)\cup\beta^*(y)$, type c $x\oplus y=H$.

In the above types the strong associativity and strong or inclusion distributivity, is valid.

Let (H,·) be H_V-semigroup and $\{v_1,\ldots,v_n\}\cap H=\varnothing$, an ordered set, where if $v_i<v_j$, when i<j. Extend (·) in $H_n=H\cup\{v_1,v_2,\ldots,v_n\}$ as follows:

$$x\cdot v_i=v_i\cdot x=v_i,\ v_i\cdot v_j=v_j\cdot v_i=v_j,\ \forall i<j \text{ and}$$

$$v_i\cdot v_i=H\cup\{v_1,\ldots,v_{i-1}\},\ \forall x\in H,\ i\in\{1,2,\ldots,n\}.$$

Then $(H_n,\cdot)$ is an H_V-group (Attach Elements Construction). We have $(H_n,\cdot)/\beta^*\cong Z_2$ and v_n is single.

Some open problems arising on the topic of rep theory of hypergroups, are:

Open Problems.

a. Find standard H_v-rings or H_v-fields to represent all H_v-groups by H_v-matrices.

b. Find reps by H_v-matrices over standard finite H_v-rings analogous to Z_n.

c. Using matrices find a generalization of the ordinary multiplication of matrices which it could be used in H_v-rep theory (see the helix-hope [40]).

d. Find the 'minimal' hypermatrices corresponding to the minimal hopes.

e. Find reps of special classes of hypergroups and reduce these to minimal dimensions.

3. Helix-Hopes and Applications

Recall some definitions from [40],[16],[11]:

Definition 3.1Let $A=(a_{ij})\in M_{m\times n}$ be an m×n matrix and $s,t\in N$ be natural numbers such that $1\leq s\leq m$, $1\leq t\leq n$. Then we define a characteristic-like map cst: $M_{m\times n}\rightarrow M_{s\times t}$ by corresponding to the matrix A, the matrix $Acst=(a_{ij})$ where $1\leq i\leq s$, $1\leq j\leq t$. We call this map cut-projection of type st. In other words Acst is a matrix obtained from A by cutting the lines, with index greater than s, and columns, with index greater than t.

We can use cut-projections on several types of matrices to define sums and products, however, in this case we have ordinary operations, not multivalued.

In the same attitude we define hopes on any type of matrices:

Definition 3.2 Let $A=(a_{ij})\in M_{m\times n}$ be an m×n matrix and $s,t\in N$, such that $1\leq s\leq m$, $1\leq t\leq n$. We define the mod-like map st from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix Ast= $(\mathbf{a}_{ij})$ which has as entries the sets

$$\mathbf{a}_{ij} = \{a_{i+\kappa s, j+\lambda t} \mid 1\leq i\leq s,\ 1\leq j\leq t. \text{ and } \kappa,\lambda\in N,\ i+\kappa s\leq m,\ j+\lambda t\leq n\}.$$

Thus we have the map

$$st: M_{m\times n}\rightarrow M_{s\times t}: A\rightarrow Ast = (\mathbf{a}_{ij}).$$

We call this multivalued map helix-projection of type st. Thus Ast is a set of s×t-matrices $X=(x_{ij})$ such that $x_{ij}\in \mathbf{a}_{ij}, \forall i,j$. Obviously Amn=A. We may define helix-projections on 'matrices' of which their entries are sets.

Let $A=(a_{ij})\in M_{m\times n}$ be a matrix and $s,t\in N$ such that $1\leq s\leq m$, $1\leq t\leq n$. Then it is clear that we can apply the helix- projection first on the columns and then on the rows, the result is the same if we apply the helix-progection on both, rows and columns. Therefore we have

$$(Asn)st = (Amt)st = Ast.$$

Let $A=(a_{ij})\in M_{m\times n}$ be matrix and $s,t\in N$ such that $1\leq s\leq m$, $1\leq t\leq n$. Then if Ast is not a set of matrices but one single matrix then we call A cut-helix matrix of type s×t. Thus the matrix A is a helix matrix of type s×t, if Acst= Ast.

Definitions 3.3 (a) Let $A=(a_{ij})\in M_{m\times n}$ and $B=(b_{ij})\in M_{u\times v}$ be matrices and s=min(m,u), t=min(n,u). We define a hope, called helix-addition or helix-sum, as follows:

$$\oplus: M_{m\times n}\times M_{u\times v}\rightarrow P(M_{s\times t}):$$

$$(A,B)\rightarrow A\oplus B=Ast+Bst=(\mathbf{a}_{ij})+(\mathbf{b}_{ij})\subset M_{s\times t},$$

where

$$(\mathbf{a}_{ij})+(\mathbf{b}_{ij})= \{(c_{ij})= (a_{ij}+b_{ij}) \mid a_{ij}\in \mathbf{a}_{ij} \text{ and } b_{ij}\in \mathbf{b}_{ij}\}.$$

(b) Let $A=(a_{ij})\in M_{m\times n}$ and $B=(b_{ij})\in M_{u\times v}$ be matrices and s=min(n,u). We define a hope, called helix-multiplication or helix-product, as follows:

$$\otimes: M_{m\times n}\times M_{u\times v}\rightarrow P(M_{m\times v}):$$

$$(A,B)\rightarrow A\otimes B=Ams\cdot Bsv=(\mathbf{a}_{ij})\cdot(\mathbf{b}_{ij})\subset M_{m\times v},$$

where

$$(\mathbf{a}_{ij})\cdot(\mathbf{b}_{ij})= \{(c_{ij})=(\sum a_{it}b_{tj}) \mid a_{ij}\in \mathbf{a}_{ij} \text{ and } b_{ij}\in \mathbf{b}_{ij}\}.$$

The helix-addition is an external hope since it is defined on different sets and the result is also in different set. The commutativity is valid in the helix-addition. For the helix-multiplication we remark that we have A⊗B=Ams·Bsv so we have either Ams=A or Bsv=B, that means that the helix-projection was applied only in one matrix and only in the rows or in the columns. If the appropriate matrices in the helix-sum and in the helix-product are cut-helix, then the result is singleton.

Remark. In $M_{m\times n}$ the addition of matrices is an ordinary operation, therefore we are interested only in the 'product'. From the fact that the helix-product on non square matrices is defined, the definition of the Lie-bracket is immediate, therefore the helix-Lie Algebra is defined [36],[37], as well. This algebra is an H_v-Lie Algebra where the fundamental relation ε* gives, by a quotient, a Lie algebra, from which a classification is obtained.

In the following we restrict ourselves on the matrices $M_{m\times n}$ where m<n. We have analogous results in the case where m>n and for m=n we have the classical theory. In order to simplify the notation, since we have results on modm, we will use the following notation:

Notation. For given $\kappa\in \mathbb{N}-\{0\}$, we denote by $\underline{\kappa}$ the remainder resulting from its division by m if the remainder is non zero, and $\underline{\kappa}$=m if the remainder is zero. Thus a matrix

$A=(a_{\kappa\lambda})\in M_{m\times n}$, m<n is a cut-helix matrix if $a_{\kappa\lambda}=a_{\underline{\kappa}\underline{\lambda}}$, $\forall\kappa,\lambda\in \mathbb{N}-\{0\}$.

Moreover let us denote by $I_c=(c_{\kappa\lambda})$ the cut-helix unit matrix which the cut matrix is the unit matrix I_m. Therefore, since $I_m=(\delta_{\kappa\lambda})$, where $\delta_{\kappa\lambda}$ is the Kronecker's delta, we obtain that, $\forall\kappa,\lambda$, we have $c_{\kappa\lambda}=\delta_{\underline{\kappa}\underline{\lambda}}$.

Proposition 3.4 For m<n in $(M_{m\times n},\otimes)$ the cut-helix unit matrix $I_c=(c_{\kappa\lambda})$, where $c_{\kappa\lambda}=\delta_{\underline{\kappa}\underline{\lambda}}$, is a left scalar unit and a right unit. It is the only one left scalar unit.

Proof. Let $A,B\in M_{m\times n}$ then in the helix-multiplication, since m<n, we take helix projection of the matrix A, therefore, the result A⊗B is singleton if the matrix A is a cut-helix matrix of type m×m. Moreover, in order to have A⊗B=Amm·B=B, the matrix Amm must be the unit matrix. Consequently, $I_c=(c_{\kappa\lambda})$, where $c_{\kappa\lambda}=\delta_{\underline{\kappa}\underline{\lambda}}$, $\forall\kappa,\lambda\in \mathbb{N}-\{0\}$, is necessarily the left scalar unit

element.

Now we remark that it is not possible to have the same case for the right matrix B, therefore we have only to prove that cut-helix unit matrix I_c is a right unit but it is not a scalar, consequently it is not unique.

Let $A=(a_{uv})\in M_{m\times n}$ and consider the hyperproduct $A\otimes I_c$. In the entry $\kappa\lambda$ of this hyperproduct there are sets, for all $1\leq\kappa\leq m$, $1\leq\lambda\leq n$, of the form

$$\Sigma a_{\kappa s}c_{s\lambda} = \Sigma a_{\kappa s}\delta_{s\lambda}= a_{\kappa\lambda}\ni a_{\kappa\lambda}.$$

Therefore $A\otimes I_c\ni A$, $\forall A\in M_{m\times n}$. ■

In the following examples of the helix-hope, we denote E_{ij} any type of matrices which have the ij-entry 1 and in all the other entries we have 0.

Example 3.5 [38] Consider the 2×3 matrices of the following form,

$$A_\kappa=E_{11}+\kappa E_{21}+E_{22}+E_{23},\ B_\kappa=\kappa E_{21}+E_{22}+E_{23},\ \forall\kappa\in\mathbb{N}.$$

Then we obtain $A_\kappa\otimes A_\lambda=\{A_{\kappa+\lambda}, A_{\lambda+1},B_{\kappa+\lambda},B_{\lambda+1}\}$.

Similarly we have $B_\kappa\otimes A_\lambda=\{B_{\kappa+\lambda},B_{\lambda+1}\}$, $A_\kappa\otimes B_\lambda=B_\lambda=B_\kappa\otimes B_\lambda$.

Thus $\{A_\kappa,B_\lambda \mid \kappa,\lambda\in\mathbb{N}\}$ becomes an H_v-semigroup, not COW because for $\kappa\neq\lambda$ we have $B_\kappa\otimes B_\lambda=B_\lambda\neq B_\kappa=B_\lambda\otimes B_\kappa$, however

$$(A_\kappa\otimes A_\lambda)\cap(A_\lambda\otimes A_\kappa) = \{A_{\kappa+\lambda},B_{\kappa+\lambda}\}\neq\varnothing,\ \forall\kappa,\lambda\in\mathbb{N}.$$

All B_λ are right absorbing and B_1 is a left scalar, because $B_1\otimes A_\lambda=B_{\lambda+1}$ and $B_1\otimes B_\lambda=B_\lambda$. The A_0 is a unit.

Example 3.6 Consider the 2×3 matrices of the forms,

$$A_{\kappa\lambda}=E_{11}+E_{13}+\kappa E_{21}+E_{22}+\lambda E_{23},\ \forall\kappa,\lambda\in\mathbb{Z}.$$

Then we obtain $A_{\kappa\lambda}\otimes A_{st}=\{A_{\kappa+s,\kappa+t},A_{\kappa+s,\lambda+t},A_{\lambda+s,\kappa+t},A_{\lambda+s,\lambda+t}\}$.

Moreover $A_{st}\otimes A_{\kappa\lambda}=\{A_{\kappa+s,\lambda+s},A_{\kappa+s,\lambda+t},A_{\kappa+t,\lambda+s},A_{\kappa+t,\lambda+t}\}$, so $A_{\kappa\lambda}\otimes A_{st}\cap A_{st}\otimes A_{\kappa\lambda}=\{A_{\kappa+s,\lambda+t}\}$, thus (⊗) is COW.

The helix multiplication (⊗) is associative.

Example 3.7 Consider all traceless matrices $A=(a_{ij})\in M_{2\times3}$, in the sence that $a_{11}+ a_{22}=0$. In this case, the cardinality of the helix-product of any two matrices is 1, or 2^3, or 2^6. These correspond to the cases: $a_{11}=a_{13}$ and $a_{21}=a_{23}$, or only $a_{11}=a_{13}$ either only $a_{21}=a_{23}$, or if there is no restriction, respectively. For the Lie-bracket of two traceless matrices the corresponding cardinalities are up to 1, or 2^6, or 2^{12}, respectively. We remark that, from the definition of the helix-projection, the initial 2×2, block guaranties that in the result there exists at least one traceless matrix.

From this example it is obvious the following:

Theorem 3.8 The Lie-bracket of any two traceless matrices $A=(a_{ij})$, $B=(b_{ij})\in M_{m\times n}$, $m<n$, contain at least one traceless matrix.

Last years hyperstructures there is a variety of applications in mathematics and in other sciences. Hyperstructures theory can now be widely applicable in industry and production, too. In several books and papers [2],[4],[5],[7],[8],[10],[12],[19],[26],[33],[39] one can find numerous applications.

The Lie-Santilli theory on isotopies was born in 1970's to solve Hadronic Mechanics problems. The original theory is reconstructed such as to admit the new matrix as left and right unit. Isofields needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1996 and they are called e-hyperfields [9],[14],[15],[33], [36]. The H_v-fields can give e-hyperfields which can be used in the isotopy theory for applications.

Definitions 3.9 A hyperstructure $(H,\cdot)$ which contain a unique scalar unit e, is called e-hyperstructure, where we assume that $\forall x$, there exists an inverse x^{-1}, so $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$. A hyperstructure $(F,+,\cdot)$, where (+) is an operation and (·) is a hope, is called e-hyperfield if the following are valid:

(F,+) is abelian group with the additive unit 0, (·) is WASS,

(·) is weak distributive with respect to (+), 0 is absorbing: $0\cdot x=x\cdot 0=0$, $\forall x\in F$, exist a scalar unit 1, i.e. $1\cdot x=x\cdot 1=x$, $\forall x\in F$,

$\forall x\in F$ there exists unique inverse x^{-1}, s.t. $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

The elements of an e-hyperfield are called e-hypernumbers. In the case that the relation: $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then we say that we have a strong e-hyperfield.

A general construction based on the partial ordering of the H_v-structures:

Construction 3.10 [6],[36], Main e-Construction. Given a group $(G,\cdot)$, where e is the unit, then we define in G, a large number of hopes (⊗) by extended (·), as follows:

$x\otimes y = \{xy, g_1, g_2,\dots\}$, $\forall x,y\in G-\{e\}$, and $g_1, g_2,\dots\in G-\{e\}$

Then $(G,\otimes)$ becomes an H_v-group, in fact is H_b-group which contains the $(G,\cdot)$. The H_v-group $(G,\otimes)$ is an e-hypergroup. Moreover, if $\forall x,y$ such that $xy=e$, so we have $x\otimes y=xy$, then $(G,\otimes)$ becomes a strong e-hypergroup.

An application combining hyperstructures and fuzzy theory, is to replace the scale of Likert in questionnaires by the bar of Vougiouklis & Vougiouklis [41]:

Definition 3.11 In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:

0 ―――――――――――――――――――― 1

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point they feel expresses their answer to the question.

The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing [41]. The suggested length of the bar, according to the Golden Ratio, is 6.2cm.

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Rudiments of IsoGravitation for Matter and its IsoDual for AntiMatter

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Abstract: In this paper, we hope to initiate due scientific process on some of the historical criticisms of Einstein gravitation expressed by Einstein himself as well as by others. These criticisms have remained widely ignored for one century and deal with issues such as: the apparent lack of actual, physical curvature of space due to the refraction of star-light within the Sun chromosphere; the absence of a source in the field equations due to the electromagnetic origin (rather than the charge) of gravitational masses; the lack of clear compatibility of general relativity with special relativity, interior gravitational problems, electrodynamics, quantum mechanics and grand unifications; the lack of preservation over time of numerical predictions inherent in the notion of covariance; and other basic issues. We show that a resolution of these historical doubts can be apparently achieved via the use of the novel *isomathematics* and related *iso-Minkowskian geometry* based on the embedding of gravitation in *generalized isounits,* with isodual images for antimatter. Thanks to half a century of prior research, we then show that the resulting new theory of gravitation, known as *isogravitation,* preserves indeed Einstein's historical field equations although formulated on the iso-Minkowskian geometry over isofields *whose primary feature is to have null isocurvature.* We then show that isogravitation allows: Einstein field equations to achieve a unified treatment of generally inhomogeneous and anisotropic, exterior and interior gravitational problems; the achievement of a clear compatibility with 20th century sciences; the achievement of time invariant numerical predictions thanks to the strict invariance (rather than covariance) of gravitation under the Lorentz-Santilli isosymmetry; the apparent achievement of a consistent representation of the gravitational field of antimatter thanks ti the isodual iso-Minkowskian geometry; the apparent achievement of a grand unification inclusive of electroweak and gravitational interactions for matter and antimatter without known causality or structural inconsistencies; and other advances. We then present, apparently for the first time, the isogravitational isoaxioms characterized by the infinite family of isotopies of special relativity axioms as uniquely characterized by the Lorentz-Santilli isosymmetry which are applicable to both exterior and interior isogravitational problems of matter with their isodual for antimatter. We finally show, also for the first time, the apparent compatibility of isogravitation with current knowledge on the equivalence principle, matter black holes and other gravitational data.

Keywords: Gravitation, Isogravitation, Antimatter

1. Introduction

The author has stated several times in his writings that the theory developed by Lorentz [1], Poincaré [2], Einstein [3], Minkowski [4] and others, known as *special relativity,* has a majestic axiomatic structure and an impeccable body of experimental verifications under the conditions clearly stated by Einstein, namely, for: A) point-particles and electromagnetic waves; B) propagating in vacuum; and C) when referred to an inertial reference frame.

Whenever any of Einstein's conditions A), B), C) are violated, special relativity is at best approximately valid, and often it is completely inapplicable (rather than violated), in the sense that it produces no quantitative description at all, as it is the case for the synthesis of the neutron from the hydrogen in the core of a star for which any use of Dirac's equation has no scientific meaning [5].

By contrast, the author has stated various times that Einstein general relativity [6] is a scientific religion at this writing because of historical insufficiencies, some of which identified by Einstein himself, such as lack of clear compatibility of general relativity with special relativity, interior gravitational problems, electrodynamics, quantum mechanics and grand unifications, which insufficiencies have remained unaddressed by the "mainstream physics" for one full century, let alone resolved in peer reviewed journals [7] (see also the view by the late J. V. Kadeisvili [8] and papers quoted therein).

In this paper, the author reports half a century of research toward a resolution of the historical insufficiencies of general relativity via the use of a basically new mathematics and its ensuing new physical vistas in the origin of gravitation, besides its description, for the exterior and interior gravitational problems of matter and antimatter.

It should be noted that the literature accumulated in the field is very large. To avoid a prohibitive length, we only list the references of direct relevance to the problems addressed. A comprehensive presentation and list of references up to 2011 is available in the independent general review [41] with the suggestive title of *New Sciences for a New Era.*

2. Historical Insufficiency of General Relativity

2.1. First Historical Insufficiency of General Relativity: Ignoring the Refraction of Star-light Passing Through the Sun Chromosphere, with Consequential Lack of Evidence that Space is Actually, Physicallys Curved

As it is well known, the conjecture of an actual, physical, curvature of space was inferred from the 1.75 arc-second "bending" of star-light passing near the Sun. Half of this value, 0.87 arc-seconds, is known to be due to a purely Newtonian attraction of light.

To see it, we first recall that for Newton gravitation to be "universal" it must also attract light, and that the source of gravitation is the energy of a body since mass is a measure of our ignorance on inertia. Hence, the author always wrote Newton's equation in the identical form in terms of the energy rather than mass

$$F = g\frac{m_1 m_2}{r^2} = G\frac{E_1 E_2}{r^2}, \ G = \frac{g}{c^4}. \quad (1)$$

The calculation of the 0.87 arc-seconds deviation caused by Newton gravitation of star light passing near the Sun surface is then a good exercise for graduate students in physics by computing the energy equivalence $E_1 = mc^2$ of the Sun, and using the energy $E_2 = h\nu$ for a given frequency of visible light. `

The remaining 0.87 arc-seconds deviation have been known for a century, *not* to be due to the curvature of space, but to the refraction of sta-light when passing through the Sun chromosphere (see, e.g., Ref. [10] and references quoted therein). Additionally, the refraction of light passing through gaseous media is inherent in the experimental confirmations of Santilli IsoRedShift (IsoBlueShift) of light traveling through cold (hot) gases [11-15] (see Figures 1, 2, 3).

Irrespective of the above, the conjecture of curvature of space has been unable to represent without ambiguities truly basic gravitational events, such as the free fall of masses that has to be necessarily along a "straight" radial line, the weight of bodies in a gravitational field, and other basic events that are clearly represented by Newton gravitation.

Figure 1. *According to the first and perhaps most important unresolved historical criticism of Einstein gravitation, Sunset is a visual evidence of the lack of actual, physical, curvature of space because we still see the Sun at the horizon, while in reality it is already below the horizon due to the refraction of light passing through our atmosphere. Exactly the same refraction without curvature of space occurs for star-light passing through the Sun chromosphere, in which case the only "bending of light" is that due to Newton's gravitation in a flat space (see Section 2). Note that Einstein gravitation cannot represent light refraction because it requires a locally varying speed of light within a medium, first with increasing and then decreasing density. Hence, the representation of refraction via the curvature of space violates visual evidence, physical laws and experimental data [11I-15]. To achieve a credible proof that the bending of Star-light passing near the Sun is "evidence" of the curvature of space, Einstein supporters have to prove that star-light passing through the Sun chromosphere does not experience refraction. The impossible existence of such a proof is readily seen from the fact that Einstein gravitation was solely aimed at a description of "exterior gravitational problems in vacuum," while the propagation of star-light within the Sun chromosphere is strictly an "interior gravitational problem" treated later on in Section 5. Its description via the Riemannian geometry is beyond any realistic possibilities due to the need for a metric possessing a dependence on coordinates x, as well as density μ, temperature τ, frequency ω, etc. $g = g(x, \mu, \tau, \omega, \ldots)$ (see Sections 5-11 below).*

Despite one century of studies, the "actual" orbits of planets in our Solar system have not been represented in an accurate, unique and time invariant way via Einstein gravitation, while they are exactly and unambiguously represented by Newton's gravitation and Kepler's laws. In fact, calculations based on the Riemannian geometry of the actual orbits of planets, besides not being unique due to the non-linearity of the theory, are generally different than physical orbits, and are not the same over time (see below).

It should also be indicated that a concrete visualization of the curvature space require an increase of the number of space dimension. In fact, the curvature in a *two*-dimensional Riemannian space can only be seen in *three* dimensions, as well known. Consequently, a concrete visualization of the curvature of space in *three* dimensions requires the implausible assumption of a *fourth* space dimension.

Needless to say, gravitational waves [6] crucially depend on the curvature of space represented via the Riemannian geometry. Until we dismiss in peer reviewed journals the mathematical, theoretical, experimental and visual evidence against the curvature of space, studies on gravitational waves may well remain in suspended animation.

It goes without saying that a critical inspection of the conjecture of curvature of space creates great emotions in colleagues who have spent their research life on curved spaces. Yet, serious appraisals should be voiced only after identifying the huge limitations caused by curvature and only after inspecting the vast advances permitted by novel theories of gravitation on a flat space treated with the appropriate novel mathematics (Section 5).

2.2. Second Historical Insufficiency of General Relativity: Ignoring the Electromagnetic Origin of the Mass, with Consequential Invalidation of Einstein's Reduction of Gravitation to Pure Curvature Without Sources

As it is well known, the contribution to gravity of the total electric and magnetic field of a body is of the order of 10^{-30} or smaller. Consequently, following the assumption of the curvature of space, Einstein was forced to avoid any source in the r.h.s. of his field equations and reduce gravitation to pure geometry according to the the celebrated equations

$$G_{ij} = R_{ij} - g_{ij}R/2 = 0, i, j = 1,2,3,4 \quad (2)$$

In 1974, the author identified the electromagnetic origin of the mass via the full use of quantum electrodynamics, including advanced and retarded treatments, and showed that such an origin requires the necessary presence in the r.h.s. of the field equations of a source first order in magnitude, irrespective of whether the body is charged or neutral [16],

$$G_{ij} = R_{ij} - g_{ij}R/2 = kT_{ij,elm}, \quad (3)$$

where k is a unit-dependent constant, and the terms "first order in magnitude" are referred to the condition of *entirely representing the gravitational mass* of the body considered [16]

$$m_{grav} = \int T_{00} dv. \quad (4)$$

The most skeptical physicist should admit that the mass of the electron is of entire electromagnetic origin. Therefore, field equations (2) are insufficient to represent the gravitational field of the electron in favor of Eqs. (3)-(4).

But then, the same skeptical physicist should admit that exactly the same conclusion holds for the positronium, namely, the gravitational mass of the positronium is of entire electromagnetic origin despite the total charge and magnetic moment being null. Therefore, Einstein's field equations (2) are insufficient for the representation of the gravitational field of the positronium in favor of broader Eqs. (3)-(4).

Paper [16] essentially extended the above known reality to the π^0-meson under the assumption of being a bound state of a charged constituent and its anti-particle. Paper [16] then extended the results to all masses with null total charge and null total magnetic moments. The inclusion of gravitational contributions from total electromagnetic characteristics was trivial.

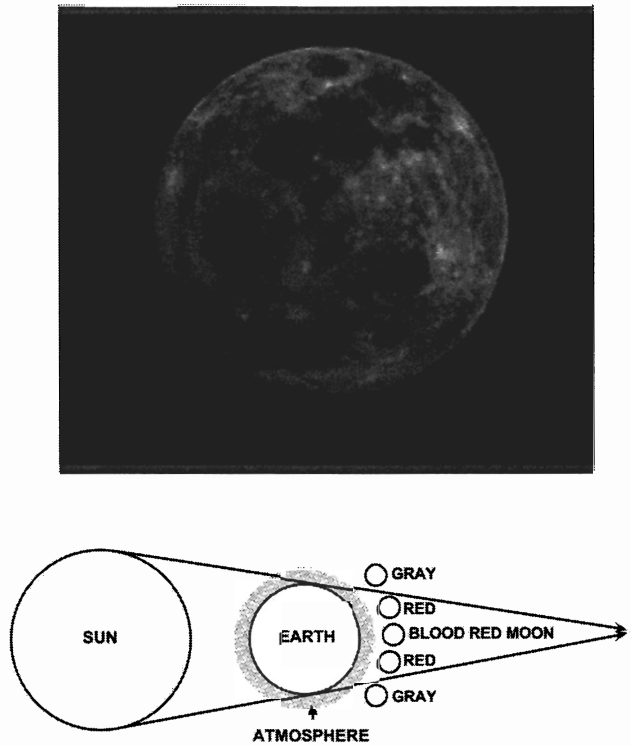

Figure 2. *The "blood red moon" (top view) during a Lunar eclipse is an additional visual evidence of the lack of curvature of space because Sunlight reaches the Moon even when it should be in total darkness (bottom view). Note that for both Sunsets and Lunar eclipses the entire spectrum of Sunlight is redshifted without relative motion, merely due to loss of energy by light to a cold medium (IsoRedShift). Note also that we are dealing with "direct Sunlight" traveling in empty space for which scattering and other interpretations have been dismissed in peer refereed journals [11-15]. Note finally that the "blood red moon" confirms the view by Einstein, Hubble, Fermi, Zwicky, Hoyle, de Broglie and others on the lack of expansion of the universe because, when our Sun is seen millions of light years away, we merely have the replacement of Earth's atmosphere with very cold intergalactic gases under which the entire spectrum of visible Sunlight will appear redshifted without any relative motion [11-15].*

In defense of Einstein, we have to recall that, contrary to his followers, Einstein always expressed serious doubts of field equations (2), for instance, by calling their r.h.s. *A house made of wood,* compared to the l.h.s. which he called *A house made of marble.* It is unfortunate for scientific knowledge that Einstein's own doubts have remained vastly ignored in the "mainstream literature" in gravitation.

We should also recall that, according to Ref. [16], the characterization of the inertial mass of a body requires the additional inclusion of all possible short range (e.g., weak and string) interactions, resulting in the need for an additional source in the r.h.s. of the equations whenever considering interior gravitational problems

$$G_{ij} = R_{ij} - g_{ij}R/2 = k_1 T_{ij,elm} + k_2 T_{ij,shortrange}, \quad (5)$$

such that (c = 1)

$$m_{inert} = \int (T_{00,elm} + T_{00,shortrange}) dv \quad (6)$$

Consequently, the inertial mass is predicted as bigger than the gravitational mass [16] (c = 1)

$$m_{inert} > m_{grav} \tag{7}$$

The expectation is that serious scientists will admit our lack of final experimental resolution on the relationship between the exterior gravitational and the interior inertial mass.

Besides the incontrovertible need for a source of first order in magnitude, the structure of Eqs. (5)-(6) is mandated by the fifth identity of the Riemannian geometry, the forgotten *Freud identity* [17] (see also the recent treatment by the late mathematician H. Rund [18]) which establishes the need on purely mathematical grounds of a source of first order in magnitude in the r.h.s of the field equations according precisely to Eqs. (5)-(6).

In fact, the source term of the Freud identity can be decomposed into a term with null trace, (evidently, the electromagnetic term), and a term with non-null trace (evidently, the source for short range interactions), thus providing a geometric confirmation of Eqs. (5)-(6).

We should indicate that the problem of a source in the gravitational field equations has been debated at length in the literature (see, e.g., Ref. [6]), although for its interpretation as a stress-energy tensor, or for other interpretations, while generally ignoring its electromagnetic origin.

Interested scholars should be aware of various claims in the literature that Einstein's gravitation verifies the Freud identity. These claims are based on the admission indeed of a source of electromagnetic nature, but restricted to the the total electromagnetic characteristics, thus violating condition (4) by a missing factor of 10^{30} or so.

Additionally, and perhaps more importantly, the Freud identity requires a source of first order in magnitude also for bodies with null total electromagnetic characteristics, thus confirming the lack of compliance of Einstein gravitation with the Freud identity.

Remember that gravitational waves are crucially dependent on Einstein's reduction of gravitation to pure geometry, Eqs, (2) [6]. However, physical and geometric needs mandate their extension to Eqs. (3), (4), for which gravitational waves cannot even be formulated, to our best knowledge at this writing.

Therefore, by noting the lack of independent addressing of the issues for the last four decades since the appearnce of paper [16], the theoretical prediction of gravitational waves will remain in suspended animation until the additional problem of theelectromagnetic origin of the gravitational mass is dismissed in refereed publications.

Again, the author has experienced over decades huge emotional reactions by colleagues at the instant of examining Einstein's reduction of gravitation to pure geometry, Eq. (2), without any in depth inspection of the advances permitted by a source term as in Eqs. (3)-(4). In a nutshell, the alternative between Eqs. (2) and (3), (4) bolls down to the belief of the existence of local infinities in the universe or not. Eqs. (2) do admit these local infinities, while covering Eqs. (3), (4) recover all main results of Eqs. (2) except replacing local infinities with large, yet finite values (Section 5 and Subsection 5.10 in particular).

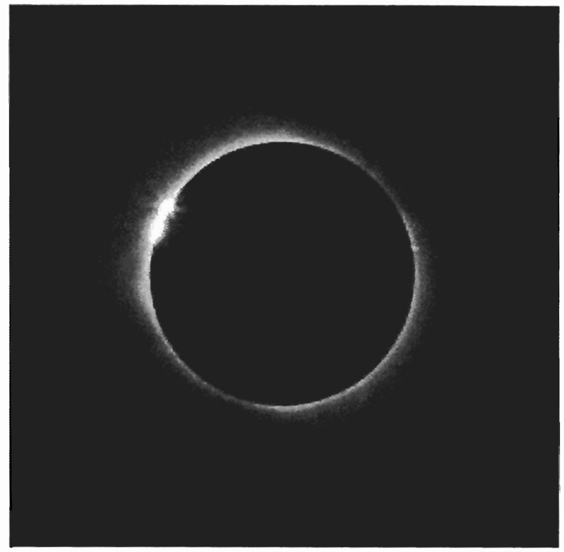

Figure 3. *A view of a Solar eclipse showing no "bending of light" because the Newtonian attraction of light by the moon is extremely small and there is no refraction due to the lack of lunar atmosphere. The faint luminescence at sea level is due to the diffraction of light in our atmosphere. In conclusion, final claims of "bending of light due to curvature of space" must be based on star light passing tangentially on a body without atmosphere or chromosphere and be proved to be greater than the Newtonian attraction.*

As a final note, the reader may have noted the lack of use of the *mathematical* terms "tensors" or "oseudotensors" and the use instead of the *physical* term "source." This is due to the fact that the clear physical content of the forgotten Freud identity is often dismissed on ground of purely mathematical differences in nomenclatures and personal mathematical interpretations without serious physical implications.

2.3. Third Historical Insufficiency of General Relativity: Abandoning the Majestic Lorentz and Poincaré "Invariance" of Special Relativity in Favor of the "Covariance" of General Relativity with Consequential Lack of Prediction of the Same Numerical Values under the Same Conditions at Different times

In our view, the above is perhaps the biggest insufficiency of Einstein gravitation because it implies the inability of gravitation to have *time invariance,* here referred to the prediction of the same numerical values under the same conditions at different times, while such a crucial requirement is verified by Galileo relativity and Einstein special relativity because of their Galilei and Poincare' symmetries, respectively.

In turn, the lack of time invariance establishes the lack of final character of all claims of "experimental verification of general relativity" [9] due to the absence of a physically consistent dynamical evolution.

In fact, "experimental verifications" of general relativity are done in ad hoc selected coordinate systems generally with no connection to the frame of the experimenter, thus prohibiting final experimental values, not only because said systems are different among themselves, but also because the needed experimental frame is generally not necessarily achievable via covariance.

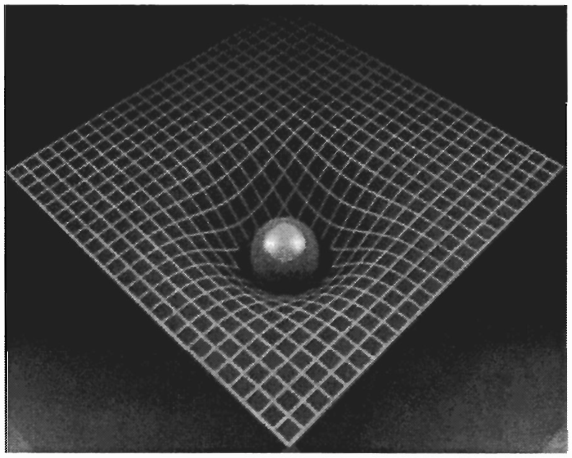

Figure 4. A typical representation of the claimed curvature of space caused by the gravitational field of a mass, which representation has been adopted for one full century. The historical, yet unresolved criticism is that the notion of physical curvature in one dimension requires a bigger dimension for its identification. In fact, the physical interpretation of the mathematical Riemannian curvature in two dimension can only be identified in three dimension as clearly illustrated by the above figure. Therefore, the additional historical criticism of Einstein gravitation that needs to be addressed is that the physical identification of the mathematical Riemannian curvature in three dimensions, as needed for realistic models of gravitations, requires four space dimensions that do not exist, thus confirming the lack of physical evidence for the actual physical curvature of space depicted in Figure 1, 2, 3. In any case, Einstein supporters are requested to illustrate with concrete geometric example the physical curvature needed for realistic models, not in two dimensions as done for one century, but in three dimensions.

Under the lack of invariance, general relativity could at best offer a kind of "polaroid picture" of gravitation [7,8]. However, such a static view of gravitation is dismissed by mathematical, physical, visual and experimental evidence on the lack of existence of the actual curvature of space.

Additional rather serious objections against published claims of "experimental verifications of Einstein gravitation" [9] stem from the fact that numerical predictions are, by far, not unique and/or unambiguous due to the non-linearity of the field equations. In fact, for any claim of "experimental verification" [9] we can assume a different PPN approximation with different expansions and show dramatic divergences with physical realities [7,8].

The lack of time invariance of Einstein's gravitation identifies an additional impossibility for gravitational waves to exist because any serious experimental verification should not only detect gravitational waves, which has been impossible for half a century despite the use of large public funds, but said gravitational waves should change in time without any change of the source, which is a blatant physical impossibility.

In defense of Einstein we should indicate that, once the Riemannian geometry is assumed for the representation of gravitation, no symmetry of the line element is possible for technical reasons similar to those of the *historical Lorentz problem.* We are here referring to Lorentz inability to achieve the invariance of the locally varying speeds of light of his time, that within physical media $C = c/n$, due to insurmountable technical difficulties in attempting to use Lie's theory for non-linear systems.

This is yet another case in which the author has experienced pre-judgments by colleagues mainly due to decades of research with covariance in gravitation without a serious inspection of qualified alternative views. In reality, serious judgments can only be expressed after a technical knowledge of the huge possibilities for further advances in gravitation permitted by alternative invariant theories (Section 5).

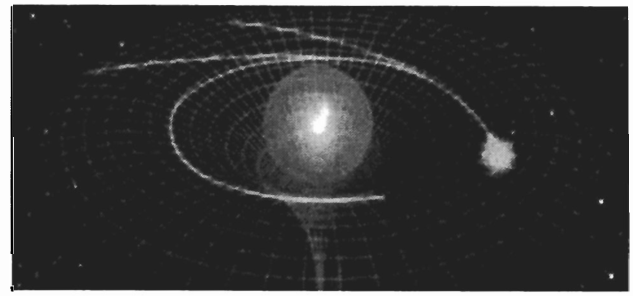

Figure 5. \ Another illustration of the insufficiencies of the one century old assumption that planets moving around the Sun in our Solar system actually move along a real, physical curvature of space. The historical criticism is that the above representation is purely mathematical because, to actually sense curvature in a three-dimensional space, the planet should move in a fort space dimension that does not exist.

2.4. Consequences of the Historical Insufficiencies of General Relativity: Incompatibility of Gravitation with Special Relativity, Interior Gravitational Problems, Electrodynamics, Quantum Mechanics, and Grand Unifications

There comes a point in the life of a scientist in which realities have to be admitted. The Riemannian geometry does indeed admit a unique and unambiguous reduction to the Minkowskian geometry via tangent, limit and other procedures.

However, it has been known for a century that general relativity does not admit a clear and unambiguous limit to special relativity of the type according to which special relativity uniquely and unambiguously admits a limit into the Galilei relativity. As one among many impossibilities, there exists no consistent limit of the covariance of general relativity into the fundamental Poincaré invariance of special relativity. The inconpatibilities that follow are then endless.

Another serious insufficiency is that the description by general relativity of "exterior gravitational problems" in vacuum is incompatible with "interior gravitational problems" that dominated the scientific scene in gravitation until the advent of Einstein's theory (e.g., Schwartzchild wrote *two* papers, one on the exterior and one on the interior gravitational problem [6], the second one being vastly ignored).

This is a serious incompatibility because its resolution prohibits the use of the Riemannian geometry due to the need of a geometry not only without curvature, but also (as indicated in Fig.1) with a metric having a dependence on coordinates x, as well as density μ, temperature τ, frequency ω, etc. $g = g(x, \mu, \tau, \omega, \dots)$ (see Section 5 for details).

Another aspect that should be admitted to prevent exiting from physical reality is the irreconcilable incompatibility between Einstein gravitation and electrodynamics to such an extent that [16]:

2A) Either one assumes Einstein's gravitation as being valid, in which case electrodynamics must be revised from its foundations so as to eliminate the electromagnetic origin of the mass, or

2B) One assumes electrodynamics and its inherent origination of the gravitational mass as being valid, in which case, Einstein gravitation must be revised from its foundations.

Yet another reality that has to be faced following one century of wide oblivion, is that Einstein's gravitation is incompatible with quantum mechanics as conventionally understood, (that is, a unitary theory on a Hilbert soace) for several reasons. The reason most important in our view is that a gravitational theory formulated on a Riemannian space is necessarily *non-canonical* at the classical level (variationally non-self-adjoint [20]).

Therefore, any consistent "quantization" of Einstein gravitation must be *non-unitary,* with the consequential activation of the *Theorems of Catastrophic Inconsistencies of Non-Canonical and Non-Unitary Theories* [19] and ensuing loss of physical value, e.g., due to the violation of causality laws.

The moment of truth also implies the admission that Einstein gravitation is incompatible with grand unified theories, if nothing else, because of failed attempts o [6]ver one full century, beginning with the failed attempt of unifying gravitation and electromagnetism by Einstein himself.

2.5. Problems to be Solved for an Axiomatically Consistent Grand Unification

Following studies on grand unifications for decades, the incompatibilities of a grand unification of Einstein gravitation with electroweak interactions are the following (see, later on monograph [40]):

2.I. The physical consistency of electroweak interactions on a flat Minkowski space cannot be salvaged when joined to a theory on the curved Riemannian space because the insufficiencies of the latter carry over to the former;

2.II. Within a grand unification, the covariance of Einstein's gravitation carries over to electroweak interactions, by therefore destroying their gauge invariance and,consequently, the very structure of electroweak interactions;

2.III. Electroweak interactions represent both particles and antiparticles, while Einstein gravitation solely represent matter, thus rendering any grand unification technically impossible and catastrophically inconsistent if attempted.

We should mention a recent trend of extending the applicability of special and general relativities to the *classical* representation of antimatter. Serious scholars should be alerted that this trend is afflicted by serious inconsistencies, such as the impossibility of admitting the annihilation of matter and antimatter precisely due to the lack of a conjugation in the transition from matter to antimatter, violation of the PCT theorem and other inconsistencies.

Another reality that should be faced by serious scholars in the field is that a consistent representation of the gravitational field of antimatter cannot be achieved by Einstein gravitation and a new theory must be constructed from its mathematical foundations.

3. Rudiments of IsoMathematics

The most important lesson the author has learned in fifty years of research is that *the protracted lack of resolution of physical problems is generally due to the use of insufficient mathematics, rather than to physical issues.*

We believe that this is precisely the case for gravitation, namely, all problems treated above are caused by the use of an excessively insufficient mathematics, that based on the differential calculus that dates back to the Newton-Leibniz times. Only after the achievement of a more adequate mathematics, open physical problems can be quantitatively and effectively addressed.

To see the case, note that for a theory of gravitation to resist the test of time, it is expected to possess an invariance similar to that of the Poincaré symmetry in special relativity so as to predict the same numerical values under the same conditions at different times.

The best known way to achieve an invariant theory of gravitation is via the use of Lie's theory. But the latter theory solely applies to linear systems. The necessary non-linearity of gravitation then precludes any realistic possibility of achieving an invariance via the use of 20th century mathematics.

The above occurrence forced the author to construct the *isotopies* (intended as axiom-preserving) of 20th century applied mathematics [20], today known as *isomathematics,* that was initiated by when author was at the Department of Mathematics of Harvard University in the late 1970s under DOE support.

Isomathematics is based on the isotopic lifting of the conventional associative product AB between generic quantities A, B (such as numbers, functions, matrices, etc.) into the isoproduct [19b]

$$A \hat{\times} B = A\hat{T}B \tag{8}$$

where the quantity $\hat{T}$, called the *isotopic element,* is positive definite but otherwise posses an arbitrary functional dependence on all needed local quantities, such as time t, coordinates r, velocities v, accelerations a, density μ, temperature τ, frequency ω, wavefunction ψ, etc. $\hat{T} = \hat{T}(t, r, v, a, \mu, \tau, \omega, \psi, \ldots) > 0$.

Product (8) was introduced for the primary intent of achieving an invariant representation of interior dynamical problems referred to extended, non-spherical and deformable particles moving within physical media, which is notoriously impossible via 20th century mathematics, but possible via isomathematics (see below for examples).

Therefore, isomathematics was suggested for the primary intent of achieving a generalization of Lie's theory into a form applicable for the first time to non-linear, non-local and non-Hamiltonian systems (that is, variationally non-self-adjoint systems not representable with a Hamiltonian [20a]).

A systematic isotopic lifting of the various branches of Lie's theory was presented in monograph [20b]. The resulting theory is today known as the *Lie-Santilli IsoTheory,* and it is based on the isoalgebra (the isotopues of Lie's second theorem)

$$[X_i \hat{,} X_j] = X_i \hat{\times} X_j - X_j \hat{\times} X_i = X_i \hat{T} X_j - X_j \hat{T} X_i = i C_{ij}^k X_k \quad (9)$$

and their integration into a finite isogroup here illustrated for simplicity via the one dimensional time evolution with the Hamiltonian $X = H$

$$A(t) = e^{X\hat{T}ti} A(0) e^{-it\hat{T}X} \quad (10)$$

with evident non-linear, non-local and non-Hamiltonian characters due to the presence of the isotopic element in the exponent.

In Vol. [20b], the author then presented a concrete realization of the Lie-Santilli isotheory given by the *Birkhoffian generalization of classical Hamiltonian mechanics* and its "direct universality," namely, the representation of all infinitely possible, well behaved, non-Hamiltonian systems directly in the frame of the experimenter.

However, the new mechanics activated the Theorems of Catastrophic Inconsistencies of Non-Canonical and Non-Unitary Theories when formulated via the mathematics of canonical and unitary theories, respectively [19].

Therefore, while visiting at the JINR in Dubna, Russia, the author was forced in 1993 [21] to reinspect the historical classification of numbers and discovered that the abstract axioms of a numeric field do not necessarily require that the basic multiplicative unit is the number +1, since they also admit realizations with arbitrary positive-definite units, provided that the associative product is lifted accordingly.

This lead to the discovery of new numbers, today known as *Santilli isonumbers*, with an arbitrary positive-=definite unit, called *Santilli isounit,* which is the inverse of the isotopic element of isoproduct (8)

$$\hat{I}(t, r, v, a, \mu, \tau, v, \psi, \ldots.) = 1/\hat{T}(t, r, v, a, \mu, \tau, v, \psi, \ldots..) \quad (11)$$

Applied mathematics was then reformulated on isofields. Yet, the fundamental invariance under the time evolution remained elusive. This forced the author to lift the Newton-Leibniz differential calculus into the form today known as *Santilli IsoDifferential Calculus* first presented in mathematical memoir [22] of 1996, with basic isodifferential

$$\hat{d}\hat{r} = dr + r\hat{T}d\hat{I} \quad (12)$$

and related isoderivative

$$\frac{\hat{\partial}\hat{F}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I}\frac{\partial\hat{F}(\hat{r})}{\partial\hat{r}}, \quad (13)$$

where the realizations $\hat{F} = F\hat{I}, \hat{r} = r\hat{I}$, etc. are necessary for the values to be isonumbers.

The isodifferential calculus permitted the achievement of maturity for mathematical, physical, and chemical developments, with ensuing numerous scientific as well as industrial applications. *Isomathematics* is today referred to the isotopies of the *totality* of 20th century mathematics formulated via isofunctional analysis, isodiufferential calculus, isoalgebras, isosymmetries, isogeometriesc, etrc., on Santgilli isofields.

A comprehensive presentation of isomathematics for physicists has been provided by the author in monographs [23]. A presentation of isomathematiccs for mathematicians is available in monograph [23] by R. M. Falcon Ganfornina and J. Nunez Valdes, while a monumental work on the isodifferential calculus and its bimplications for all of mathematics is available in the five monographs [25] by S. Georgiev.

4. Rudiments of IsoMechanics

The primary physical application of isomathematics is the isotopic lifting of Newton's equations, first presented in Ref. [22]

$$\hat{m} \hat{\times} \frac{\hat{d}\hat{v}}{\hat{d}\hat{t}} = F^{sa}(t, r, v).principle \quad (14)$$

today known as the *Newton-Santilli IsoEquations.*

Eqs. (14) allow the first known representation of the actual extended shape of bodies, for instance, via the isounit for the velocities

$$\hat{I}(t, r, v, a, \mu, \tau, \omega, \ldots.) = Diag.(n_1^2, n_2^2, n_3^2)e^{\Gamma},$$

$$n_k = n_k(t, r, v, a, \mu, \tau, v, \ldots.) > 0,$$

$$\Gamma = \Gamma(t, r, v, a, \mu, \tau, \omega, \ldots.) > 0, k = 1,2,3 \quad (15)$$

as well as the representation of non-Hamiltonian (variationally non-self-adjoint [20]) forces via the exponent of the isounit (15) and their embedded in the isodifferential $\hat{d}\hat{v} = \hat{d}(v\hat{I})$ in such a way that only Hamiltonian (variationally self-adjoint [20]) forces appear in the r.h.s. of the equations.

In view of these features, the Newton-Santilli isoequations for non-Hamiltonian systems admit the first known representation via *isoaction principle* [22]

$$\hat{\delta}\hat{A} = \int (\hat{p} \hat{\times} \hat{d}\hat{r} - \hat{H} \hat{\times} \hat{t}) = 0 \quad (16)$$

thus permitting the first known use of the optimal control theory for the shape, e.g., of a wing moving within a fluid.

In turn, the availability of the isoaction principle has allowed the isotopic lifting of classical Hamiltonian mechanics into its covering *Hamilton-Santilli isomechanics* with basic isotopies of the conventional Lagrange and Hamilton equations here ignored for brevity as well as of the *Hamilton-Jacobi-Santilli isoequations* [22,23]

$$\frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{t}} + \hat{H} = 0, \quad \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{r}} - \hat{p} = 0, \quad \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{p}} = 0. \quad (17)$$

Still in turn, the availability of the latter isoequations has permitted the first known, axiomatically consistent, unique and unambiguous, operator map of non-Hamiltonian systems into a covering of quantum mechanics introduced in 1978 under the name of hadronic mechanics [20], with Schrödinger-Santilli Isoequations [22]

$$\hat{\imath} \hat{\times} \frac{\hat{\partial} \hat{\psi}}{\hat{\partial} \hat{t}} = \hat{H} \hat{\times} \hat{\psi} =$$

$$= \hat{H}(\hat{r}, \hat{p}) \hat{T}(t, r, p, \mu, \tau, \nu, \psi, \ldots) \hat{\psi} =$$

$$= \hat{E} \hat{\times} \hat{\psi} = E \hat{\psi}, \tag{18}$$

related *isolinear momentum*

$$\hat{p} \hat{\times} \hat{\psi} = -\hat{\imath} \hat{\times} \frac{\hat{\partial} \hat{\psi}}{\hat{\partial} \hat{r}} = -i \hat{I} \frac{\partial \hat{\psi}}{\partial \hat{r}}, \tag{19}$$

and their isounitarily equivalent *Heisenberg-Santilli isoequations* [20,23] for the isotime evolution of an operator $\hat{A}$ in the infinitesimal form

$$\hat{\imath} \hat{\times} \frac{\hat{d} \hat{A}}{\hat{d} \hat{t}} = [\hat{A} \hat{,} \hat{H}] = \hat{H} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{A} =$$

$$= A \hat{T}(t, r, p, \mu, \tau, \nu, \psi, \ldots) \hat{H}(\hat{r}, \hat{p}) -$$

$$- \hat{H}(\hat{r}, \hat{p}) \hat{T}(t, r, p, \omega, \tau, \omega, \psi, \ldots) A, \tag{20}$$

and integrated finite form (10), where the "hat" denotes formulation on an iso-Hilbert space over the isofield of isocomplex numbers [23].

For readers not familiar with the field, we should recall that *hadronic mechanics is a non-unitary "completion" of quantum mechanics much along the celebrated argument by Einstein-Podolsky and Rosen* (see later on Ref. [36]). However, non-unitary theories formulated on a conventional Hilbert space over a conventional field violate causality [9,19]. Hence, the reformulation of non-unitary theories via isomathematics is crucial for the mathematical and physical consistency of hadronic mechanics at large and its isomechanical branch in particular (see monographs [23] for a comprehensive presentation).

We should also mention that *hadronic mechanics eliminates the divergencies of quantum mechanics* because the value of the isounit (15) is generally very big. Consequently, the value of the isotopic element $\hat{T}$ is very small, thus permitting the conversion of divergent or weakly convergent quantum series into strongly convergent isotopic forms via the systematic use of isoproduct (8). Additionally, the functional dependence of the isotopic element is unrestricted, thus allowing the removal of the singularity of the Dirac delta distributions under isotopy, which feature persists for the isotopies of the scattering theory. The absence of divergencies is particularly important for approximate solutions of exterior and interior dynamical problems, as well as of non-linear gravitational equations when reformulated in terms of isomathematics.

Finally, the non-initiated reader should be aware that *quantum mechanics and hadronic mechanics coincide at the abstract level by conception and construction* to such an extent that they can be expressed via the same symbols and equations, merely subjected to *different realizations.* Following decades of research in the field, we believe that the above features are iportant to assure consistency and causality of hadronic mechanics and its applications.

5. Rudiments of IsoGravitation for Matter

5.1. Elementary Formulation of IsoGravitation

The main result of the studies in gravitation herein reported is that *the conjecture of the actial curvature of space is the dominant origin of all problematic aspects of Einstein gravitation, including all its incompatibilities with 20th century sciences,* besides being disproved by visual, mathematical and experimental evidence (Figure 1-5).

Therefore, the main objectives of the studies herein reported are: A) the reformulation of Einstein field equations via a basically new geometry admitting the invariance of line elements without curvature; B) show the compatibility of said reformulation with 20th century sciences; and C) provide at least preliminary experimental verifications.

Following decades of preparatory research on the new isomathematics and isomechanics, isogravitation for matter was presented for the first time at the 1992 *Marcel Grossmann Meeting in Gravitation* [26] via the following elementary rules:

RULE 5-I: Decompose any non-singular Riemannian metric g(x) in (3+1)-dimensions into the product of the the Minkowski metric $\eta = Diag.(1,1,1,-1)$ and the $4 \times$ 4-dimensional *gravitational isotopic element* $\hat{T}_{gr}(x)$

$$g(x) = \hat{T}_{gr}(x) \eta \tag{21}$$

where the positive-definite character of $\hat{T}_{gr}(x)$ is assured by the topology of the Riemannian space;

RULE 5-II: Assume the inverse of the isotopic element as the *gravitational isounit*

$$\hat{I}_{gr}(x) = 1/\hat{T}_{gr}(x) > 0 \tag{22}$$

RULE 5-III: Reformulate the *totality* of Einstein gravitation into such a form admitting $\hat{I}_{gr}(x)$ as the correct left and right unit at all levels, including numbers, functional analysis, differential calculus, geometries, Christoffel symbols, etc.

As we shall see, the above simple rules will allow maintaining Einstein's field equations including its primary verifications, although formulated on a new geometry over new fields with null curvature.

5.2. Minkowski-Santilli IsoSpace

The spacetime of isogravitation verifying the above conditions is given by the infinite family of isotopies of the Minkowski space first introduced by the author in Ref. [26] of 1983 for the classical profile and Ref. [27] of the same year for the operator counterpart, and it is today known as the *Minkowski-Santilli IsoSpace.*

Consider the conventioinal Minkowski space $M(x, \eta, I)$ with spacetime coordinates $x = (\text{xi}), i = 1, 2, 3, 4,$ metric $\eta = Diag.(1,1,1,-c^2)$ and unit $I = Diag.(1,1,1,1)$. The Minkowski-Santilli isospace is denoted $\hat{M}(\hat{x}, \hat{\eta}, \hat{I})$, and it is characterized by the infinite family of isotopies for which coordinates are lifted into isocoordinates (a necessary condition for their value to be isonumbers) [26]

$$x \to \hat{x} = x\hat{I} \quad (23)$$

the Minkowski metric is lifted into the infinite family of isometrics

$$\eta \to \hat{\eta} = \hat{T}_{gr}\eta \quad (24)$$

the Minkowski unit is lifted into the isounits with related isotopic elements

$$\hat{I}_{gr}(t,r,p,\mu,\tau,\omega,\psi,\ldots) =$$

$$= Diag.(n_1^2, n_2^2, n_3^2, n_4^2) > 0 \ , n_i > 0, \quad (25)$$

$$\hat{T}_{gr}(t,r,p,\mu,\tau,\omega,\psi,\ldots) =$$

$$= Diag.(\frac{1}{n_1^2},\frac{1}{n_2^2},\frac{1}{n_3^2},\frac{1}{n_4^2}) > 0, \quad (26)$$

and line Minkowski element into the infinite family of *isoline elements*

$$\hat{x}^{\hat{2}} = \hat{x}^i \hat{\times} \hat{\Xi}_{ij} \hat{\times} \hat{x}^j = (x^i \hat{\eta}_{ij} x^j)\hat{I} =$$

$$= (\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} - t^2\frac{c^2}{n_4^2})\hat{I}, \quad (27)$$

where: $\hat{\Xi} = \hat{\eta}\hat{I}$ is a condition is a condition to have correct isomatrices, that is, matrices whose elements are isonumbers; one should note the multiplication of the isoline elements by the isounit which is also a necessary condition forf the line element to be isonumbers; and we have ignored for simplicity the exponential factor in the isounits and isotopic elements representing non-Hamiltonian interactions as in Eqs. (15) (see Refs. [23] for the full treatment).

The n-quantities are called the *characteristic quantities* of the gravitational field and they are illustrated in the verifications below. Readers are suggested to exercise caution for the popular interpretation of the n-quantities as being "free parameters" since this would literally imply that, for instance, the terms characterizing the Schwartzchild metric are "free parameters."

It is easy to see that the projection of the isoline element (27) in conventional spacetime is the most general possible symmetric (thus diagonalized) and non-singular line element in (3+1)-dimensions, thus including as particular cases all possible Minkowskian, Riemannian, Fynslerian and other line elements (it should be noted that non-symmetric line elements for the geometric representation of irreversible gravitational events require the broader Lie-admissible genomathematics [19,23])

5.3. Minkowski-Santilli IsoGeometry

The geometry of isospace $\hat{M}(\hat{c},\hat{\eta},\hat{I})$ was first studied in memoir [28] of 1998 and it is today known as the *Minkowski-Santilli isogeometry.* Its first important feature is the admission of the entire machinery of the Riemannian geometry, such as covariant derivative, Christoffel symbols, etc. merely reformulated in terms of the isodifferential calculus, Eqs. (12)-(13).

This is evidently due to the fact that, unlike the Minkowski metric η, its isotopic covering $\hat{\eta}$ admits the most general possible functional dependence, under the sole condition of positive-definiteness of the isotopic element, Eq. (26). Regrettably, an outline of the new geometry would be excessively advanced for the elementary character of this presentation.

The second important feature of the Minkowski-Santilli isogeometry is that of being *isoflat,* that is, its curvature is identically null when elaborated via isomathematics and defined over isofields.

An elementary way of seeing the second features is to note that, under isotopies, we have the mutation of the Minkowskian coordinates while the corresponding unit is mutated by the *inverse* amount,

$$x_k \to \hat{x}_k \hat{I}_k = \frac{x_k}{n_k^2} \quad (28)$$

$$I_k \to \hat{I}_k = n_k^2, \quad (29)$$

thus preserving the original flatness.

In any case, isotopies must preserve the original axioms by central condition and technical realization. This means that, when properly treated, the isotopies of the Minkowski space must preserve the original flatness despite the dependence of the isometric on local coordinates.

5.4. Lorentz-Santilli IsoSymmetry

Thanks to the prior construction of the Lie-Santilli isotheory [20], the universal isosymmetry of all possible isoline elements (27) was constructed for the first time in only one page of Ref. [26]; it is today called the *Lorentz-Santilli isosymmetry;* it is characterized by the original symmetry plus the isotopic element (26); and can be written for isotransformations in the (3, 4)-plane (see Refs. [23] for the general case)

$$x'^3 = \hat{\gamma}[x^3 - \hat{\beta}\frac{n_3}{n_4}x^4], \quad (30)$$

$$x'^4 = \hat{\gamma}[x^4 - \hat{\beta}\frac{n_4}{n_3}x^3]. \quad (31)$$

where

$$\hat{\gamma} = \frac{1}{\sqrt{1-\hat{\beta}^2}}, \quad \hat{\beta} = \frac{v/n_3}{c/n_4}; \quad (32)$$

As one can see, it is evident that the Lorentz-Santilli isosymmetry is locally isomorphic to the original symmetry by conception and realization. It is also evident that this local isomorphism is crucial for achieving compatibility of isogravitation with 20th century theories and for attempting a consistent grand unification of gravitation and electroweak interactions, as outlined below.

Following the original isotopies of the Lorentz symmetry

[26,27], systematic studies were done by the author on the isotopies of all most significant spacetime and internal symmetries. In fact, Ref. [29] was devoted to the isotopies $\hat{O}(3)$ of the rotational symmetry $O(3)$ to achieve the invariance of all topology preserving deformations of the sphere; Refs. [30,31] were devoted to the isotopies $\hat{S}U(2)$ of the $SU(2)$ spin symmetry; Ref. [32] presented for the first time the isotopies $\hat{P}(3.1)$ of the Poincaré symmetry $P(3.1)$ with the first proof of the universal invariance of all possible non-singular, Riemannian line elements; and Ref. [33] was devoted to the isotopies $P(3.1)$ of the spinorial covering of the Poincaré symmetry $P(3.1)$. Independent papers [34,35] confirmed the universal character of the Lorentz-Santilli isosymmetry for the invariance of all infinitely possible symmetric line elements in (3+1)-dimensions.

5.5. IsoGravitational IsoEquations

Another important feature of isogravitation is that of preserving Einstein's field equations (2), although necessarily extended to forms (3)-(6) and reformulated on the Minkowski-Santilli isogeometry without curvature.

Along these lines, we have the *isoequations for exterior gravitational problems*

$$\hat{G}_{ij} = \hat{R}_{ij} - \hat{\Xi}_{ij}(\hat{x},\hat{v}) \hat{\times} \hat{R}/\hat{2} =$$

$$= \hat{k} \hat{\times} \hat{T}_{ij,elm}(\hat{x},\hat{v}), \quad (33)$$

under the condition

$$m_{grav} = \int \hat{T}_{00,elm}(\hat{x}) \hat{\times} \hat{d}\hat{v} \quad (34)$$

where one should note the dependence of the source on isocoordinates and isovelocities, as typical for electromagnetic source.

Consequently, the isometric is equally dependent on isocoordinates and isovelocities, $\Xi(\hat{x},\hat{v}) = \hat{\eta}(\hat{x},\hat{v})\hat{I}$, a property forbidden by the Riemannian geometry but readily permitted by the Minkowski-Santilli isogeometry due to the unrestricted functional dependence of the isometric.

We also have the broader isoequations for the interior isogravitational problem

$$\hat{G}_{ij} = \hat{R}_{ij} - \hat{\Xi}_{ij}(\hat{t},\hat{r},\hat{v},\hat{a},\hat{\mu},\hat{\tau},\hat{\omega},\hat{\psi},\ldots) \hat{\times} \hat{R}/\hat{2} =$$

$$\hat{I}_{sch} = Diag.[1,1,(1-\frac{r_{sch}}{r}),(1-\frac{r_{sch}}{r})^{-1}]. \quad (41)$$

or with the isogravitational characteristic quantities

$$n_r^2 = 1-\frac{r_{sch}}{r},\ \ n_4^2 = (1-\frac{r_{sch}}{r})^{-1}, \quad (42)$$

where one should note the suggestive reformwhere one should note the suggestive reformulation of gravitational singularities in terms of the zeros of the space component of the isounit.

We now consider the isotopies of the Dirac equations introduced in Ref. [33], now called the *Dirac-Santilli IsoEquations,* and specialize then to the Schwartzchild metric

$$(\hat{\Xi}_{st}^{\mu\nu} \hat{\times} \hat{\Gamma}_\mu \hat{\times} \hat{\partial}_\nu + \hat{\imath} \hat{\times} \hat{m} \hat{\times} \hat{c}) \hat{\times} \hat{\psi} = (\hat{\eta}_{sch}^{\mu\nu}\hat{\gamma}_\mu\hat{\partial}_\nu + imc)\hat{\psi} = 0 \quad (43)$$

where $\Xi = \hat{\eta}\hat{I}, \hat{\Gamma} = \hat{\gamma}\hat{I}$ and the matrices $\hat{\gamma}$, known as the *Dirac-Santilli IsoGamma matrices,* are given by

$$\hat{\gamma}_k = \frac{1}{n_k}\begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix},$$

$$\hat{\gamma}_4 = i\frac{1}{n_4}\begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \quad (44)$$

with anti-isocommutation rules [25]

$$\{\hat{\gamma}_\mu \hat{,} \hat{\gamma}_\nu\} = \hat{\gamma}_\mu T_{st}\hat{\gamma}_\nu + \hat{\gamma}_\nu T_{st}\hat{\gamma}_\mu = 2\hat{\eta}_{\mu\nu,sch} \quad (45)$$

As one can see, Eqs. (43) did indeed succeed in embedding gravitation in the Dirac equation, for which reason Santilli proposed the name of the *Dirac-Schwartzchild IsoEquation* [25]. It's expected physical relevance is evident, e.g., as the first description on scientific records of an electron within an intense gravitational field in the surface of the Sun or near the event horizon of a black hole.

In closing, we would like to honor the memory of Einstein, Podolsky and Rosen [36] for their view on the "lack of completeness of quantum mechanics" which was instrumental for the birth of hadronic mechanics and its applications. In fact, operator isogravitation can be defined as an invariant non-unitary, axiom-preserving completion of relativistic quantum mechanics.

5.6. Compatibility of IsoGravitation with 20th Century Theories

The compatibility of isogravitation with 20th century sciences is direct and immediate. The compatibility of isogravitation with special relativity is immediately established by the fact that its universal isosymmetry is locally isomorphic to the conventional Poincaré symmetry. The compatibility of the physical laws of isogravitation with those of special relativity is then an immediate consequence.

The compatibility of isogravitation with the interior gravitational problem is established by the completely unrestricted functional dependence of the gravitational isometric. The compatibility of isogravitation with electromagnetism is established by the electromagnetic origin of the gravitational mass appearing in Eqs. (33).

The compatibility of isogravitation with quantum mechanics is inherent in the very notion of isotopies and it is used at the foundation of the very proposal of isogravitation [25]. The compatibility of isogravitation with grand unifications will be discussed in Section 7.

5.7. IsoGravitational IsoAxioms

The isotopies of the axioms of special relativity, today known as *IsoAxioms,* were initiated by Santilli in paper [26] of 1983; they received a first systematic formulation in monographs [37] of 1991; and they were finalized in monographs [23] of 1995 jointly with the discovery of the isodifferential calculus (see Ref. [41] for an independent review).

In works [23,26,37], the isoaxioms were specifically conceived and technically developed for quantitative treatments of relativistic interior dynamical problems, such as for the propagation of light within gaseous media (Figure 1), in which application they have received numerous experimental verifications (see, e.g., Refs. [11-15] and general review [41]).

The isoaxioms presented in Refs. [23,26,37] had no gravitational content. The application of the isoaxioms for a representation of gravity is presented for the first time in this paper under the proposed name of *IsoGravitational IsoAxioms.*

The presentation of this subsection is the most general possible for both the exterior and the interior gravitational problems characterized by a non-singular, symmetric isometric in (3+1)-dimension. This general formulation is merely achieved without any specification of the functional dependence of the isometric. In the verifications of the isogravitational isoaxioms of the next subsection, we will be forced to specify the isoaxioms to exterior or interior gravitational problems.

The first implication of the isotopies of special relativity is the abandonment of the speed of light in vacuum as the maximal causal speed in favor of a covering geometric notion. This is necessary for isogravitation because light is expected not to propagate within the hyperdense media inside planets or stars.

This occurrence is easily seen by specializing the isoline element (27) to the isolight isocone [23, 37]

$$\hat{x}^{\hat{2}} = \frac{x_k^2}{n_k^2} - t^2 \frac{c^2}{n_4^2}) = 0, \tag{46}$$

thus leading to the maximal Causal Speed V_{max} of IsoAxiom 5.1 below.

The remaining isoaxioms can be uniquely and unambiguously identified via a procedure parallel to the construction of the axioms of special relativity from the Poincaré symmetry [23,37].

The reader should be aware that isogravitation is generally *inhomogeneous* and *anisotropic* for both exterior and interior problems, as evidently intrinsic in the fact that the characteristic quantities n_k of isoelement (27) generally have different values for different space directions.

These features are necessary for a more realistic representation of exterior and interior gravitational fields of planets such as Earth. Inhomogeneity and anisotropy are then easily represented thanks to the arbitrary functional dependence of the characteristic quantities of the Minkowski-Santilli isogeometry.

A consequence of the inhomogeneity and anisotropy of isogravitation is that the isoaxioms are presented for one given direction in space, hereon denoted with the sub-index k, since the change of space direction generally implies a change in the explicit value of the characteristic quantities.

ISOAXIOM 5.I: The maximal causal speed in a given space direction k within an isogravitational field is given by

$$V_{max,k} = c\frac{n_k}{n_4}, \tag{47}$$

ISOAXIOM 5.II: The local isospeed of light within an isogravitational field is given by

$$\hat{c} = \frac{c}{n_4} \tag{48}$$

where c is the speed of light in intergalactic spaces w

where c is the speed of light in intergalactic spaces without any gravitational field.

ISOAXIOM 5.III: The addition of isospeeds in the k-direction within an isogravitational field follows the isotopic law

$$V_{tot,k} = \frac{v_{1,k}/n_k + v_{2,k/n_k}}{1 + \frac{v_{1,k}v_{2,k}}{c^2}\frac{n_4^2}{n_k^2}}. \tag{49}$$

ISOAXIOM 5.IV: The isodilatation of isotime, the isocontraction of isolengths, theiso variation of mass with isospeed, and the mass-energy isoequivalence principle follow the isotopic laws

$$\Delta t' = \widehat{\gamma_k}\,\Delta t, \tag{50}$$

$$\Delta \ell' = \widehat{\gamma_k}^{-1}\,\Delta \ell, \tag{51}$$

$$m' = \widehat{\gamma_k}\,m, \tag{52}$$

$$E = mV_{max}^2 = mc^3\frac{n_k^2}{n_4^2} \tag{53}$$

where $\hat{\gamma}$ and $\hat{\beta}$ have values (32).

where $\hat{\gamma}$ and $\hat{\beta}$ have values (32).

ISOAXIOM 5.V: The frequency isoshift of light propagating within an isogravitational field in the k-direction follows the Doppler-Santilli isotopic law

$$\omega_e = \omega_o\hat{\gamma}_k\left[1 \pm \frac{v/n_k}{c/n_4}\cos\,\alpha\right] \tag{54}$$

where ω_e is the experimentally measured value, ω_o is the value at the origin, and we have ignored for simplicity the isotopies of trigonometry (see Refs. [23] for brevity).

A technical understanding of the isoaxioms requires a technical knowledge of isomathematics. In fact, the isoaxioms presented below are given by their projection from the Minkowski-Santilli isospace over an isofield with isounit (25) into the conventional Minkowski space over a conventional field with isounit 1.

A main feature is that, when the isoaxioms are represented on isospace over isofields, they coincide with the conventional axioms of special relativity by conception and technical realization. In particular, the maximal causal speed $V_{max} \neq c$ solely occurs in the projection of the isoaxioms on Minkowski space because, at the isotopic level, the maximal causal speed is c for all possible isogravitational problems.

5.8. Verification of IsoGravitation for Exterior Problems without Source

It It is important for the self-consistency of this paper to initiate the appraisal of isogravitation via its application to the exterior gravitational problem without source in order to verify that Einctein field equations (2) can indeed be consistently formulated on a Minkowski-Santilli isospace.

In fact, all consistent experimental verifications of general relativity also apply to isogravitation without source because, for its own conception and technical realizations, isotopic liftings preserve all original numerical values (for brevity, see ref. [23b] with particular reference to the proof that the maximal causal speed on Minkowski-Santilli isospaces on isofields is the conventional speed of light in vacuum c).

In particular, it is easy to see that Einstein's Equivalence Principle [6,9] is maintained in its integrity for various independent reasons. First of all, the projection of isogravitation on the conventional Riemannian space over a conventional field coincides with Einstein gravitation with consequential trivial validity of Einstein';s Equivalence Principle. Additionally, the Equivalence Principle independently holds on the Minkowski-Santilli isospace over isofields by very conception of isotopies [23]. The verification of other serious experimental verifications of Einstein gravitation follows in the same way.

To verify the above general lines, let us assume the Schwartzchild metric (39) as a good *approximation* of the isometric for isoequations (33) for the case without source, and present the results for appraisal by interested readers.

Note that, under said assumption, we have the homogeneity and isotropy of the isogravitational field, thus eliminating the selected space direction k of the general isoaxioms.

Note that, under said assumption, we have the homogeneity and isotropy of the isogravitational field, thus eliminating the selected space direction k of the general isoaxioms.

Let us begin by recalling values (42) of the characteristic quantities for the Schwartzchild metric for which

$$\frac{\hat{v}}{\hat{c}} = \frac{v}{n_r}\ \frac{n_4}{c} =$$

$$= \frac{v}{1 - r_{sch}/r}\ \frac{1}{c(1 - r_{sch}/r)} =$$

$$= \frac{r/c}{(1 - r_{sch}/r)^2}, \tag{55}$$

and consequential expressions for the isogamma (32)

$$\hat{\gamma} = \frac{1}{\sqrt{1 - \hat{\beta}^2}} = \frac{1}{\sqrt{1 - \frac{\hat{v}^2}{\hat{c}^2}}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}\ \frac{n_4^2}{n_r^2}}} =$$

$$= \frac{1}{\sqrt{1 - (\frac{v}{1 - r_{sch}/r})^2\ (\frac{1}{c(1 - r_{sch}/r)})^2}} =$$

$$= \frac{1}{\sqrt{1 - \frac{v^2/c^2}{(1 - r_{sch}/r)^4}}} \approx$$

$$\approx \frac{1}{1 - \frac{v/c}{(1 - r_{sch}/r)^2}} \tag{56}$$

From the above values, we have the maximal causal speed in an isogravitational field

$$c{=}cn4{=}c(1{-}rschr) \tag{58}$$

which evidently tends to zero at the event horizon.

We believe that this occurrence is a significant confirmation of isogravitation because it provided a most effective, quantitative representation of the impossibility of matter to escape from a black hole.

Similarly, we have the expression for the isospeed of light

$$\hat{c} = \frac{c}{n_4} = c(1 - \frac{r_{sch}}{r}) \tag{58}$$

which also tends to zero at the event horizon and expectedly thereafter.

We believe that this is another supporting feature of isogravitation because the speed of light decreases for about $100{,}000 km/sec$ when propagating within water. It is then logical to assume that the speed of light is null when reaching the densest conceivable medium in the universe. The null value at the event horizon is also an effective way to represent the impossibility for light to escape from a black hole.

It should be noted that the conventional speed of light c is \ an invariant under the Lorentz-Santilli isosymmetry and related isogravitation because, e.g., the isosum of two light speeds c does not reproduce c as it is the case for special relativity.

However, isospeed (58) is indeed an isoinvariant because the isosum of two light isospeeds does indeed yield the light isospeed,

$$V_{tot,s} = \frac{c/n_4 + c/n_4}{1 + \frac{c^2}{c^2}} = \frac{c}{n_4} = c(1 - \frac{r_{sch}}{r}) \tag{59}$$

The reader should be aware of the fact that isogravitation predicts that the speed of light c in intergalactic spaces without any gravitational field is "bigger" then the speed of light $\hat{c}_{earth}$ measured on Earth, although for a very small amount,

$$\hat{c}_{earth} = \frac{c}{n_4} = c(1 - \frac{r_{sch}}{r}) < c, \tag{60}$$

By using isospeeds away from the observer, and values (42), we can write the first order approximation

$$\Delta t' = \Delta t\hat{\gamma} \approx \frac{\Delta t}{1 - \frac{v/c}{(1-r_{sch}/r)^2}} \quad (61)$$

which recovers the conventional time dilation of special relativity at a given distance r. However, the value of $\Delta t'$ within a gravitational field $(grav)$ is predicted to be smaller than that for special relativity (sr),

$$\Delta t'_{grav} < \Delta t'_{sr} \quad (62)$$

in such a way that time tends to zero at the event horizon, in full agreement with the behavior of the time component of the Schwartzchild metric (39),

$$Lim_{r \to sch} \Delta t = 0 \quad (63)$$

Similarly, we have the isolength isocontraction

$$\Delta \ell' = \frac{\Delta \ell}{\hat{\gamma}} \approx \Delta \ell 1 - \frac{v/c}{(1 - r_{sch}/r)^2} \quad (64)$$

which recovers the length contraction of special relativitywhich recovers the length contraction of special relativity for a given distance r. However, the value of $\Delta \ell'$ in the presence of a gravitational field is predicted to be bigger than that of special relativity

$$\Delta \ell'_{grav} > \Delta \ell'_{sr} \quad (65)$$

in such a manner that $\delta t'$ tends to infinity at the event horizon

$$Lim_{r \to sch} \Delta \ell = \infty \quad (66)$$

also in full agreement with the space component of the Schwartzchild metric (39).

We also have from isoaxiom (52) the isovariation of mass in an isogravitational field

$$m' = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}\frac{n_4^2}{n_r^2}}} \approx \frac{m}{\sqrt{1 - \frac{v^2/c^2}{(1-r_{sch}/r)^4}}} \quad (67)$$

illustrating the prediction based on the Schwartzchild metricthat the mass tends to zero at the event horizon.

Similarly, from the energy equivalence (53), we have in the vicinity of the event horizon

$$E' = m'V_{max}^2 = \frac{m}{\sqrt{1 - \frac{v^2/c^2}{(1-r_{sch}/r)^4}}} \text{ x}$$

$$x\, c^2 (1 - \frac{r_{sch}}{r})^4 \approx$$

$$\approx mc^2 \frac{(1 - r_{sch}/r)^6}{(1 - r_{sch}/r)^2 - v/c}, \quad (68)$$

illustrating the prediction that the energy isoequivalence of a particle tends to zero at the event horizon much faster than that for the mass.

We believe that the above features are an important verification of the isoaxioms for various reasons. Firstly, the expectation that Newton's inertia and other laws are valid within a black hole is nowadays rejected by the vast majority of scientists. Secondly, any expectation that particles may experience inertia when constrained within the densest medium in the universe without any possibility of motion, is manifestly illogical. Thirdly, and perhaps most importantly, the limitation for the divergent increase of mass and energy within a black hole appears to be an important mechanism set by nature to prevent the achievement of infinite mass under which one single black hole would swallow the entire universe.

It should be stressed to prevent misrepresentations that the null limit of the mass at the event horizon is similar to the singularity of the Schwartzchild metric and solely occur for the case of field equations (2) without source. As indicated in the next subsection, the presence of a source of first order in magnitude, Eq. (4), appears to avoid both the null value of the mass and the singularity at6 the event horizon by turning them into more realistic finite values.

For IsoAxiom 5.V, we have the Doppler-Santilli isoshift of the frequency of light within an exterior isogravitational field for the simple case of null aberration in the space k-direction

$$\omega_e \approx \omega_o [1 \pm \frac{v}{c}(1 - \frac{r_{sch}}{r})], \quad (69)$$

clearly showing Santilli isoRedShift [11,37], namely a redshift of the entire spectrum of visible light without any relative motion between the light source and the origin of the gravitational field.

The energy lost by light for the isoredshift when traversing a gravitational field is expected to be one of the continuous sources of energy needed for the Cosmic Background Radiation to exist in view of its weakness, in addition to the energy originating from the de-excitation of hydrogen atoms in intergalactic spaces when hit by light [11-15] which appears to be an additional source of the energy needed to maintain in time the Cosmic Backgrtound Radiation [11-15].

Note also that all frequencies of visible light become identically null at the event horizon. This feature is necessary for compatibility with the null value of the speed of light at the event horizon, thus confirming the plausible expectation that the conventional notion of electromagnetic waves becomes meaningless within the densest media existing in the universe. Needless to say, the energy lost by light to the event horizon is absorbed by the black hole.

In conclusion, to our best understanding at this writing, the predictions of isogravitational isoaxioms for matter appear to be supported by the behavior of the isotopic reformulation of the Schwartzchild metric, although more studies are evidently needed to achieve any conclusion due to the complexity of the problem and our rather limited final knowledge of black holes.

5.9. Verification of IsoGravitation for Exterior Problems with Source

As indicated earlier, the Schwartzchild metric (39) has a just place in the history of gravitation because it achieved for the first time a geometric understanding of gravitational singularities, besides other advances.

However, the Schwartzchild metric remains a first approximation of a rather complex physical reality because local infinities cannot exist in the universe as a condition for its continued existence.

Following decades of studies on the covering of the Schwartzchild metric suitable to avoid local infinities, the author has found no other consistent approach than that allowed by a first-order electromagnetic source in the r.h.s. of the field equations according to Eqs. (33).

This raises the question as to whether Einstein's Equivalence Principle also holds for exterior isogravitation with a source. Einstein supporters quickly voice their opinion that this is not the case for the intent of invalidating isogravitation. However, serious science is far from these unsubstantiated personal opinions because the problem is rather complex indeed and, to avoid a prohibitive length, it will be studied by the author in a subsequent paper.

At this moment, we limit ourselves to the indication that, apparently, the introduction of a source in the gravitational field equations implies numerical contributions in the verification of the Equivalence Principle well within experimental errors. Consequently, the introduction of a source *does not* invalidate the Equivalence Principle on serious scientific grounds until proved so with detailed calculations published in serious refereed journals.

The needed solution is scheduled for detailed studies in a subsequent paper. For the completeness of this paper, we limit ourselves to indicate that an approximate solutions of Eqs. (33) can be written

$$ds^2 = r^2(d\theta^2 + sin^2 d\theta^2 + d\phi^2) +$$

$$+(1 - \frac{r_{sch} + S(r,v)}{r})^{-1} dr^2 -$$

$$-(1 - \frac{r_{sch} + S(r,v)}{r}) dt^2 \equiv$$

$$\equiv \hat{T}_{sch} \times \eta \equiv \hat{\eta}_{sch}, \tag{70}$$

with characteristic quantities

$$= 1 - \frac{r_{sch} + S(r,v)}{r}, \quad n_4^2 =$$

$$= (1 - \frac{r_{sch} + S(r,v)}{r})^{-1}, \tag{71}$$

whose limit for $r \to 0$ (rather than for $r \to sch$) is such to avoid local singularities, e.g., of the type

$$Lim_{r \to 0}(1 - \frac{\hat{v}}{\hat{c}}) =$$

$$= Lim_{r \to 0}[1 - \frac{r/c}{1 - \frac{r_{sch} + S(r,v)}{r}}] =$$

$$= N \neq 0, \ N < \infty. \tag{72}$$

and the numerical value of N evidently requires the consideration of a specific black hole.

It then follows that isogravitational isoequations (33) with a first-order electromagnetic source recover all main historical results achieved by the Schwartzchild's metric, with the elimination of singularities that are not expected to exist in nature.

As a first illustration, the expected behavior of the isotime isodilation (61) acquires the form

$$Lim_{r \to 0} \Delta t' \approx Lim_{r \to 0} \frac{\Delta t}{(1 - \frac{\hat{v}}{\hat{c}})^2} =$$

$$= \frac{\Delta t}{N^2} > 0, \tag{73}$$

thus eliminating the singularity in time of the Schwartzchild metric (39)

Similarly, for the isolength isocontraction we have

$$Lim_{r \to 0} \Delta \ell \approx Lim_{r \to 0} \Delta \ell (1 - \frac{\hat{v}}{\hat{c}})^2 =$$

$$= N^2 \Delta \ell < \infty, \tag{74}$$

thus eliminating jointly the local singularity of the Schwartzchild metric for the space component.

Similar corrections occur for the remaining physical quantities studied in the preceding subsection, as the reader can verify.

Note the truly crucial role of the first-order nature of the electromagnetic source, that is, such to represent the entire gravitational mass, Eq. (34). In fact, the standard consideration of the total electromagnetic characteristics of a body leaves Schwartzchild's singularities completely unaffected since their contribution to the gravitational field is of the order of 10^{-30} or less.

In conclusion, we can state that the inclusion in the r.h.s. of the field equation of a first order source of electromagnetic character, essentially along Einstein's own intuition, besides achieving compatibility of gravitation with electrodynamics, does indeed offer realistic possibilities of avoiding local infinities in the universe, with ensuing significant advances in various gravitational problems.

5.10. Verification of IsoGravitation for the Interior Problem

Contrary to isogravitation, Einstein gravitation cannot even formulate interior gravitational problems in any realistic way, e.g., due to the inability to represent a locally varying speed of light. In this case there is the loss of credibility for Einstein supporters who even mention experimental verifications of Einstein gravitation, for the evident reason that we have no direct experimental tests in the interior of the Sun or planet.

The interior character of the Doppler-Santilli isolaw has been extensively studied in Refs. [11-15]. We hereby limit ourselves to consider the interior gravitational case of light passing through the Sun chromosphere.

In this case, the characteristic n -quantities have a functional dependence on the speed v, the distance d

covered within the physical medium, etc. thus admitting the expansion in the travwersed distance d by light within the medium

$$n_4 n_k \approx 1 \pm$$

$$\frac{n_4}{n_k} \approx 1 \pm \frac{c}{v} Hd, \tag{75}$$

where H is the Hubble constant with resulting Doppler-Santilli isoshift law [11]

$$\omega_e \approx \omega_0 [1 \pm vc(1 \pm Hd)] \tag{76}$$

Measurements [11-15] have established that, for a sufficiently dense chromosphere (or for a sufficiently long travel d in a thin atmosphere), the conventional Doppler term is ignorable, e.g., for the case of earth's rotation, and the Hubble term becomes dominant, resulting in the Hubble law for the cosmological shift

$$z = \pm Hd \tag{77}$$

which is uniquely and unambiguously characterized by the Poincaré-Santilli isosymmetry.

We hereby add, apparently for the first time, the extended version of the Doppler-Santilli isolaw within a transparent physical medium with the inclusion of a strong isogravitational, field from isoaxiom (64)

$$\omega_e \approx \omega_0 [1 \pm \frac{v}{c} (1 - \frac{r_{sch}}{r}) \pm Hd). \tag{78}$$

As one can see, the isoredshift caused by isogravitation in the vicinity of the event horizon is dominant over all others, as expected.

We believe that the above features provide a significant additional verification of isogravitation at large, and of its isoaxioms in particular.

In conclusion, we can safely state that isogravitation does indeed allow meaningful models of interior gravitational problems that are notoriously impossible for Einstein gravitation.

It should be noted that these interior gravitational models are reached via the same axioms of exterior problems under the uncompromisable condition that the metric has an unrestricted functional dependence on all possible interior local variable, which dependence is solely admitted at this writing by isogeometries.

6. Rudiments of IsoDual IsoGravitation for AntiMatter

Despite all the above advances, attempts at an axiomatically consistent grand unification of electroweak and gravitational interaction continued to be inconsistent and not worth their presentation in a scientific paper, because Einstein gravitation, as well as isogravitation, solely apply for *matter-bodies,* thus preventing any consistent unification with electroweak theories that are bona-fide theories of particles and antiparticles.

A solution of the latter problem required the construction of yet another new mathematics, specifically conceived for the *classical* representation of *neutral* (or charged) *antimatter-bodies.*

The transition from matter to antimatter required the new mathematics to be anti-isomorphic in general and anti-Hermitean in particular, to isomathematics, as a condition to be consistent with charge conjugation and experimental data, including matter-antimatter annihilation.

Following numerous failed attempts, when being at he Department of Mathematics of Harvard University in the early 1980s, the author finally succeeded in identifying the needed mathematics, called isodual mathematics and denoted with the upper symbol ^{d}.

In view of the above aspects, Santilli constructed the isodual image of 20th century mathematics and quantum mechanics under the condition of admitting the *isodual unit*

$$I^d = -1 \tag{79}$$

at all its mathematical and physical levels [40].

The above studies remained grossly insufficient to initiate studies on possible grand unifications due to the need of the anti-isomorphic image of isomathematics for antimatter whose need emerges even stronger from the model of isogravitation presented in this paper.

The latter mathematics was built via the systematic application of the *following isodual map*

$$\hat{I}(t, r, p, \mu, \tau, v, \psi, ...) \rightarrow \hat{I}^d =$$

$$-\hat{I}^\dagger(-t^\dagger, -r^\dagger, -v^\dagger, -a^\dagger, -\mu^\dagger, -\tau^\dagger, -v^\dagger, -\psi^\dagger, ...) \tag{80}$$

to the totality of quantities and the totality of their operations used for matter.

The resulting new mathematics is today known as *Santilli isodual isomathematics* and includes isodual isonumbers, isodual isofunctions, isodual isodifferential calculus, isodual isoalgebras, isodual isogeometries, etc. (see monograph [40] for a comprehensive study and Ref. [41] for an independent general review).

Following the construction of the isodual isomathematics it was necessary to construct the isodual image of classical and operator theories, with particular reference to the isodual Lorentz-Santilli isosymmetry and the axiomatically consistent classical representation of the gravitational field of neutral (or charged) antimatter-bodies. The compatibility of the emerging isodual theory of antimatter with experiental data was assured by the equivalence of the isodual map with charge conjugation (for brevity, one may inspect monograph [23]).

7. Rudiments of IsoGrandUnification

In our view, a most important implication of the search for axiomatically consistent grand unifications is the shift from the *description* of gravitation to a study of its *origin.* In fact, Ref. [16] is crucially dependent on the abandonment of the standard "unification" of gravitation and electromagnetic interactions in favor of their "identification" under appropriate field equations.

Ref. [16] also submitted experiments for the possible laboratory creation of a measurable gravitational field that appears feasible nowadays thanks to the availability of highly sensitive detectors, such as those based on neutron interferometry.

Only following the above scientific journey the author was finally in a position to present at the 1997 *Marcel Grossmann Meeting in Gravitatio,* a grand unification of electroweak and gravitational interactions with the inclusion of matter and antimatter at all classical and operator levels [38] (see also Ref. [39]).

The emerging grand unification essentially consistent in the embedding of gravitation in the gravitational isounit of electrostatic interactions under the universal isospinorial covering $P(3.1)$ of the Poincaré-Santilli isosymmetry $\hat{P}(3.1)$ the selected isotopic image of the selected gauge symmetry g for matter and their isodual for antimatter

$$\hat{S} = \{\hat{P}(3.1) \times \hat{G}\} \times \{\hat{P}^d(3.1) \times \hat{G}^d\} \quad (81)$$

which is the isosymmetry of the Dirac-Santilli isoequations (43) [33] and which, rather intriguingly, emerges as the isosymmetry of the universe at the limit of equal amounts of matter and antimatter (see monograph [40] for brevity).

Of course, we do not know whether the abovegrand unification is verified in nature, but we believe that the studies reported in this paper have provided at least much needed new vistas in gravitation [41] for further advances by interested colleagues.

To follow Albert Einstein teaching for powerful self-criticism, we note that the dynamics of test masses in a gravitational field is fully reversible in time. By contrast, the dynamics of a black holes is strictly irreversible over, since we are dealing with a *one way* absorption of matter and light.

By remembering that isomathematics and related isomechanics are reversible over time, \, a more accurate description of black holes may require a covering of isogravitation constructed via genomathematics with a Lie-admissible (rather than a Lie-isotopic) structure and related genogeometries with non-symmetric genometrics as a condition to embed irreversibility in the ultimate mathematical and physical structures [19,23].

All in all, the studies presented in this paper confirm that physics is a discipline that will never admit final theories.

Acknowledgments

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Comments on the Regular and Irregular IsoRepresentations of the Lie-Santilli IsoAlgebras

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Abstract: As it is well known, 20th century applied mathematics with related physical and chemical theories, are solely applicable to point-like particles moving in vacuum under Hamiltonian interactions (exterior dynamical problems). In this note, we study the covering of 20th century mathematics discovered by R. M. Santilli, today known as *Santilli isomathematics,* representing particles as being extended, non-spherical and deformable while moving within a physical medium under Hamiltonian and non-Hamiltonian interactions (interior dynamical problems). In particular, we focus the attention on a central part of isomathematics given by the isorepresentations of the Lie-Santilli isoalgebras that have been classified into *regular (irregular) isorepresentations* depending on whether the structure quantities of the isocommutation rules are constants (functions of local variables). The importance of the study of the isorepresentation theory for a number of physical and chemical applications is pointed out.

Keywords: Lie Theory, Lie-Santilli Isotheory, Isorepresentations

1. Introduction

As it is well known, 20th century applied mathematics at large, and the Lie theory in particular, can only represent *point-like particles moving in vacuum* (exterior dynamical problems), resulting in a body of methods that have proved to be effective whenever particles can be effectively abstracted as being point-like, such as for the structure of atoms, and crystals, particles moving in accelerators, and many other systems.

An important feature of exterior problems is that, being dimensionless, point-like particles can only experience *action-at-a-distance, potential and, therefore, Hamiltonian interactions,* which Hamiltonian character is a central condition for the very applicability of Lie's theory.

It is equally well known that point-like abstractions of particles are excessive for *extended, non-spherical and deformable particles moving within a physical medium* (interior dynamical problems), as it is the case for the structure of hadrons, nuclei and stars since their constituents are in a state of mutual penetration of their wavepackets and/or charge distribution.

An important feature of the finite size of particles in interior conditions is that they experience conventional action-a-distance, Hamiltonian interactions, as well as additional contact, non-potential and, therefore, non-Hamiltonian interactions, with the consequential inapplicability of 20th century applied mathematics at large, and of Lie's theory in particular.

In a series of pioneering works [1-11], R. M. Santilli has constructed a new mathematics, today known as *Santilli IsoMathematics,* for the representation of extended, non-spherical and deformable particles under Hamiltonian as well as non-Hamiltonian interactions, which new mathematics has seen contributions by numerous important mathematicians (see, e.g. Rfs., [12-21]).

In this note, the author would like to bring to the attention of the mathematical community the need for further studies on the central branch of isomathematics, namely, the *Lie-Santilli IsoTheory* [1], since the latter provides the only known time invariant methods for the lifting of the various applications of the conventional Lie theory from exterior to interior conditions.

In particular, we focus the attention on the

IsoRepresentations of the Lie-Santilli IsoAlgebras which have been classified into *regular* and *irregular,* depending on whether the structure quantities of the isocommutation rules are constant or functions of the local variables.

Besides Santilli's works, no study on the isorepresentation theory of the Lie-Santilli isoalgebras is on scientific record to our best knowledge, with consequential limitations on important applications, such as the search for much needed, new nuclear energies without the release of harmful radiations and other equally important applications outlined in Section 5.

It should be indicated that Santilli's pioneering works signal the historical transition from the notion of *massive point,* introduced by Newton, and adopted by Galileo and Einstein, to a new generation of physical and chemical theories representing particles as they are in the physical or chemical reality. This historical advance has so many implications for all of quantitative sciences that it has been referred to as characterizing *New Sciences for a New Era* in the title of Ref. [21].

2. The Lie-Santilli IsoTheory

Let L be an N-dimensional Lie algebra on a Hilbert space H over a field $F(n,\times,1)$ with elements n given by real, complex and quaternionic numbers, associative product hereon denoted $nm = n\times m \in F$, and multiplicative unit 1.

Let the generators of L be given by Hermitean operators $X_k, k=1,2,...,N$, on H over F. Let ξ be the universal enveloping associative algebra characterized by the infinite-dimensional set of ordered monomials according to the Poincaré-Birkhoff-Witt Theorem.

Let the Lie algebra L be isomorphic to the anti-symmetric algebra attached to the enveloping algebra $L \approx \xi^-$ with ensuing Lie's theorems and commutation rules. Let G be the Lie transformation group characterized by L.

In pioneering works done in 1978-1983 at the Department of Mathematics of Harvard University under DOE support, R. M. Santilli [1] proposed the axiom-preserving *isotopies* of 20th century applied mathematics at large, and of the Lie theory in particular, via the following isotropy of the associative product

$$X_i \hat{\times} X_j = X_i \times \hat{T} \times X_j \quad (1)$$

where $\hat{T}$, called the *isotopic element,* is solely restricted to be positive-definite, but otherwise possesses an arbitrary dependence on local variables such as time t, coordinates r, velocities v, density μ, temperature τ, index of refraction δ, frequency ω, wave functions ψ, etc.

The fundamental significance of Santilli's infinite class of isotopies (1) of the associative product is that they permit the representation of the actual extended, and deformable shape of the body considered under Hamiltonian interactions represented via the conventional Hamiltonian, and contact non-Hamiltonian interactions via realizations of the isotopic element of the type

$$\hat{T} = Diag.(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}) \times e^{-\Gamma(t,r,v,\mu,\tau,\delta,\omega,\psi,...)} \quad (2)$$

where $n_k^2 = n_k^2(t,r,v,\mu,\tau,\delta,\omega,\psi,...)$! $k = 1,2,3$, represents, in this case, the deformable semi-axes of the considered ellipsoid, and Γ is a positive-definite function representing all interactions not representable with the Hamiltonian.

Following the above basic assumptions, Santilli passed in monographs [1] to the construction of the isotopies of the various branches of Lie's theory over a conventional field F, and illustrated its significance via the *Birkhoffian generalization of Hamiltonian mechanics* which achieves "direct universality" for the representation of all possible (regular) non-Hamiltonian Newtonian systems directly in the frame of the experimenter. The resulting new theory is today known as the *Lie-Santilli IsoTheory.*

Following the above seminal advances, Santilli discovered that the original formulation [1] of the isotopies does not predict the same numerical values under the same conditions at different times (hereon referred to as *time invriance*), because the time evolution is *non-unitary* on H over F.

In summer 1993, while visiting the Joint Institute for Nuclear Research in Dubna, Russia, Santilli [2] discovered that the abstract axioms of a numeric field do not necessarily require that the basic unit be the number 1, since the multiplicative unit can be an arbitrary, positive definite quantity $\hat{I}$ irrespective of whether an element of the original field F or not, under the condition that it is the inverse of the isotopic element

$$\hat{I} = 1/\hat{T}, \quad (3)$$

and all possible associative products are lifted into form (1) under which $\hat{I}$ is the correct left and right multiplicative unit for all elements of the set considered

$$\hat{I} \hat{\times} X = X \hat{\times} \hat{I} = X \; \forall X \in L \quad (4)$$

This lead to the discovery of new fields, today known as *Santilli isofields* $\hat{F}(\hat{n},\hat{\times},\hat{I})$ with *isoreal, isocomplex and isoquaternionic isonumbers* $\hat{n} = n \times \hat{I}, n \in F$ equipped with the isoproduct $\hat{n}hat \times \hat{m} = n \times m \times \hat{I} \in \hat{F}$ [2].

Subsequently, Santilli discovered that, despite the reformulation over an isofield, the Lie-Santilli isotheory was still unable to achieve the crucial time invariance of the numerical prediction.

Following various trials and errors, while studying at the Institute for Basic research, Castle Prince Pignatelli in Italy, Santilli [3] discovered in 1995 that, contrary to a popular belief in mathematics and physics for centuries, the Newton-Leibnitz differential calculus depends indeed on the assumed basic multiplicative unit because, in the event such unit has a functional dependence on the differentiation variable, the conventional differential must be generalized into the *isodifferential*

$$\hat{d}\hat{r} = \hat{T} \times d[r \times \hat{I}(r,...)] = dr + r \times \hat{T} \times d\hat{I}(r,...), \quad (5)$$

with ensuing *isoderivative*

$$\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I} \times \frac{\partial \hat{f}(\hat{r})}{\partial \hat{r}}. \quad (6)$$

where, for consistency, $\hat{f}$ is an *isofunction* with the structure $\hat{f} = f \times \hat{I}$ and $\hat{r}$ is the *isovariable* with the structure $\hat{r} = r \times \hat{I}$ as an evident condition to have values in the isofield $\hat{F}$.

The discovery of isofields and of the isodifferential calculus signed the achievement in memoir [3] of mathematical maturity in the formulations of the isotopies of 20th century applied mathematics at large, and of the Lie-Santilli isotheory in particular, which maturity stimulated seminal, advances in mathematics as well as in physics and chemistry, including novel industrial applications indicated later on.

Nowadays, the *Lie-Santilli IsoTheory* is referred to the infinite family of isotopies of Lie's theory as defined in memoir [3], namely, formulated on an *iso-Hilbert space* $\hat{H}$ over an isofield $\hat{F}$ with iso-Hermitean generators $X_k, k = 1,2,\dots,N$, with all possible products lifted into the isoassociative form (1) and multiplicative isounit (3), the elaboration beng done via the *isofunctional analysis* and the *isodifferential calculus.*

A rudimentary outline of the Lie-Santilli isotheory comprises the following main branches [3,9]:

2.1) The *universal enveloping isoassociative isoalgebra* $\hat{\xi}$ with infinite-dimensional *isobasis* given by the *ordered isomonomials* of the *Poincaré-Birkhoff-Witt-Santilli isotheorem*

$$\hat{I}, X_k, X_i \times X_j, i \leq j, \dots . \quad (7)$$

with related *isoexponentiation*

$$\hat{e}^X = \hat{I} + \frac{X}{\hat{1}!} + \frac{X \hat{\times} X}{\hat{2}!} + \dots = \left(e^{X \times \hat{T}}\right) \times \hat{I} = \hat{I} \times \left(e^{\hat{T} \times X}\right) \quad (8)$$

and other isofunctions;

2.2) The *Lie-Santilli isoalgebras*

$$\hat{L} \approx \hat{\xi}^- \quad (9)$$

with *isocommutation rules*

$$[X_i \hat{,} X_j] = X_i \hat{\times} X_j - X_j \hat{\times} X_i = \hat{C}_{ij}^k \hat{\times} X_k \quad (10)$$

where $\hat{C}_{ij}^k = C_{ij}^k \times \hat{I}$ are the *isostructure quantities* of $\hat{L}$ with values in $\hat{F}$;

2.3) The *Lie-Santilli isogroups* $\hat{G}$ with structure for the one dimensional case ()

$$\hat{A}(\hat{w}) = \hat{e}^{\hat{H} \hat{\times} \hat{w} \hat{\times} \hat{i}} \hat{\times} \hat{A}(\hat{0}) \hat{\times} \hat{e}^{(-\hat{i} \hat{\times} \hat{w} \hat{\times} \hat{H})} =$$

$$= e^{H \times \hat{T} \times w \times i} \times A(0) \times e^{-i \times w \times] \hat{T} \times H} \quad (11)$$

where $\hat{H} = H \times \hat{I}$ is an *isomatrix,* namely, a matrix whose elements are isoscalars. The remaining aspect of the Lie-Santilli isotheory can be then constructed via axiom preserving isotopies of the *totality* of the conventional formulations with no exception known to the author.

Following the achievement in memoir [3] of a consistent formulation of the isotopies, Santilli applied the isotopies of Lie's isotheory them to a number of physical and chemical problems that cannot be even formulated with conventional Lie theory due to the need to represent of extended bodies under non-Hamiltonian interactions (see applications [6.7.8] with corresponding independent verifications and industrial applications [12,13,14], monograph [9] for a general treatment of the Lie-Santilli isotheory, and monographs [10,11] for applications in physics and chemistry, respectively).

In the author's view, Santilli's most salient achievement has been, not only the transition from the massive points of Newton, Galileo and Einstein theories to extended bodies, but also their representation under the most general (but non-singular) known non-linear, non-local and non-Hamiltonian interactions in a way as *invariant* as Hamiltonian formulations are.

This historical result was achieved via *the embedding of all non-Hamiltonian quantities in the generalized unit of the theory* because, whether conventional or generalized, the unit is indeed the basic invariant of any theory.

3. Classification of IsoRepresentations

The isorepresentations of Lie-Santilli isoalgebras are classified into [4,5,9]:

3.1) *Regular isorepresentations* occurring when the C's of rules (5) are constant; and

3.2) *Irregular isorepresentations* occurring when the C's of rules (5) are functions of local variables.

We should recall that "structure functions" are impossible for Lie's theory, and they are solely possible for the covering Lie-Santilli isotheory, by therefore establishing the non-trivial character of Santilli isotopies.

4. Regular IsoRepresentations

Let us recall that a given Lie algebra admits an infinite family of isotopies because a point-like particle in vacuum admits an infinite number of generalizations to extended particles moving within physical media.

Let us also recall that the extended shape of a particle and its non-Hamiltonian interactions are represented by the basic isounit or, equivalently, by the isotopic element [2].

Therefore, the transition from the conventional representations of a Lie algebra to the isorepresentation of the covering Lie-Santilli isoalgebras represents extended particles moving within physical media under conventional Hamiltonian interactions, as well as the most general known non-linear, non-local and non-Hamiltonian interactions.

Consider a given Lie algebra L and one of its representations. Santilli [4,5,9] has identified a simple method for the construction of the infinite family of regular isorepresentations of the Lie-Santilli covering $\hat{L}$ of L based on non-unitary transformations of the original Lie formulation. The method consists in:

4.1) Identifying the extended character of the particle considered and its non-Hamiltonian interactions represented via Santilli's isounit.

4.2) The identification of a non-unitary transform representing said isounit according to the rule

$$U \times U^{\dagger} = \hat{I} \quad (12)$$

where

$$U \times U^{\dagger} \neq I, \quad (13)$$

4.3) The application of the above nonunitary transform to the *totality* of the mathematics underlying the original representation of L , thus including numbers, spaces, algebras, geometries, symmetries, etc, with no known exception.

The bove method is illustrated by the transformations:

$$I \to \hat{I} = U \times I \times U^{\dagger} = 1/\hat{T}, \quad (14a)$$

$$n \to \hat{n} = U \times n \times U^{\dagger} = n \times U \times U^{\dagger} = n \times \hat{I} \in \hat{F},, n \in F, \quad (14b)$$

$$e^{A} \to U \times e^{A} \times U^{\dagger} = \hat{I} \times e^{\hat{T} \times \hat{A}} = (e^{\hat{A} \times \hat{T}}) \times \hat{I}, \quad (14c)$$

$$A \times B \to U \times (A \times B) \times U^{\dagger} = (U \times A \times U^{\dagger}) \times$$

$$(U \times U^{\dagger})^{-1} \times (U \times B \times U^{\dagger}) = \hat{A} \hat{\times} \hat{B}, \quad (14d)$$

$$[X_i, X_j] \to U \times [X_i X_j] \times U^{\dagger} = [\hat{X}_i \hat{,} \hat{X}_j] =$$

$$U \times (C_{ij}^{k} \times X_k) \times U^{\dagger} = \hat{C}_{ij}^{k} \hat{\times} \hat{X}_k = C_{ij}^{k} \times \hat{X}_k, \quad (14e)$$

$$< \psi | \times | \psi > \to U \times < \psi | \times | \psi > \times U^{\dagger} = <$$

$$\psi | \times U^{\dagger} \times (U \times U^{\dagger})^{-1} \times U \times | \psi > \times (U \times U^{\dagger}) = <$$

$$\hat{\psi} | \hat{\times} | \hat{\psi} > \times \hat{I}, \quad (14f)$$

$$H \times | \psi > \to U \times (H \times | \psi >) =$$

$$(U \times H \times U^{\dagger}) \times (U \times U^{\dagger})^{-1} \times (U \times | \psi >) =$$

$$\hat{H} \hat{\times} | \hat{\psi} >, etc. \quad (14g)$$

As an illustration, Santilli considered in Refs. [4,5] the two-dimensional irreducible representation of the SU(2) Lie algebra, which is given by the known Pauli matrices.

The regular isorepresentations of the Lie-Santilli isoalgebras $\widehat{SU}(2)$ can be constructed via the infinite family of non-unitary transformations with representative example

$$\hat{\sigma}_k = U \times \sigma_k \times U^{\dagger}, \quad (15a)$$

$$U = \begin{pmatrix} i \times g_1 & 0 \\ 0 & i \times g_2 \end{pmatrix},$$

$$U^{\dagger} = \begin{pmatrix} -i \times g_1 & 0 \\ 0 & -i \times g_2 \end{pmatrix}, \quad (15b)$$

$$g_1^2 = \frac{1}{g_2^2} = \lambda^2, \quad (15c)$$

where conditions (15c) is necessary for the isounitarity of the algebra and the g 's are well behaved nowhere null functions. The application of transformations (14) yields the *regular Pauli-Santilli isomatrices* [4,5,9]

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & g_1^2 \\ g_2^2 & 0 \end{pmatrix},$$

$$\hat{\sigma}_2 = \begin{pmatrix} 0 & -i \times g_1^2 \\ i \times g_2^2 & 0 \end{pmatrix},$$

$$\hat{\sigma}_3 = \begin{pmatrix} g_1^2 & 0 \\ 0 & g_2^2 \end{pmatrix}. \quad (16)$$

with isoalgebra isomorphic to the conventional $SU(2)$ algebra

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = \hat{\sigma}_i \times \hat{T} \times \hat{\sigma}_j - \hat{\sigma}_j \times \hat{T} \times \hat{\sigma}_i = 2 \times i \times \varepsilon_{ijk} \times \hat{\sigma}_k, \quad (17)$$

and consequential preservation of the conventional eigenvalues for spin $1/2$

$$\hat{\sigma}^{\hat{2}} \hat{\times} | \hat{\psi} >= (\hat{\sigma}_1 \times T \times \hat{\sigma}_1 + \hat{\sigma}_2 \times T \times \hat{\sigma}_2 +$$

$$+ \hat{\sigma}_3 \times T \times \hat{\sigma}_3) \times T \times | \hat{\psi} >= 3 \times | \hat{\psi} >, \quad (18a)$$

$$\hat{\sigma}_3 \hat{\times} | \hat{\psi} >= \hat{\sigma}_3 \times T \times | \hat{\psi} >= \pm 1 \times | \hat{\psi} >, \quad (18b)$$

Despite the apparent triviality, Santilli's isotopies of the SU(2)-spin algebra are not trivial because they introduce a new degree of freedom in the conventional spin $1/2$ given by the non-singular, but unrestricted parameter (or function) λ^2 of Eqs. (15c).

In turn, this new degree of freedom has permitted a number of novel applications, such as [4,5,9]: the reconstruction of the exact isospin symmetry in nuclear physics which was believed to be broken by weak interactions; the achievement of a concrete and explicit realization of hidden variables in quantum mechanics via the degrees of freedom λ^2 ; and rather seminal implications for local realism (see Ref. [5] for brevity).

5. Irregular IsoRepresentations

Santilli has additionally constructed in Refs. [4,5] the following example of irregular isorepresentation of the $\hat{\text{S}}$U(2) spin algebra

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & g_1^2 \\ g_2^2 & 0 \end{pmatrix},$$

$$\hat{\sigma}_2 = \begin{pmatrix} 0 & -i \times g_1^2 \\ i \times g_2^2 & 0 \end{pmatrix},$$

$$\hat{\sigma}_3 = \begin{pmatrix} w \times g_1^2 & 0 \\ 0 & w \times g_2^2 \end{pmatrix}. \quad (19)$$

which are known as the *irregular Pauli-Santilli isomatry,* and cannot any longer be constructed via non-unitary transformations of the Pauli matrices, and.

The irregular character of isomatrices (19) is established by the appearance of *structure functions* in the isocommutation rules

$$[\tilde{\sigma}_1 \hat{,} \tilde{\sigma}_2] = i \times w^{-1} \times \tilde{\sigma}_3, [\tilde{\sigma}_2 \hat{,} \tilde{\sigma}_3] = i \times w \times \tilde{\sigma}_1,$$

$$[\tilde{\sigma}_3 \hat{,} \tilde{\sigma}_2] = i \times w \times \tilde{\sigma}_1, \quad (20)$$

with the characterization of the following *mutation* (in Santilli's words) of the SU(2) -spin eigenvalues

$$\tilde{\sigma}^{\hat{2}} \hat{\times} |\hat{\psi}>=$$

$$(\tilde{\sigma}_1 \times T \times \tilde{\sigma}_1 + \tilde{\sigma}_2 \times T \times \tilde{\sigma}_2 + \tilde{\sigma}_3 \times T \times \tilde{\sigma}_3) \times T \times |\hat{\psi}>=$$

$$= (2 + w^2) \times |\hat{\psi}>, \quad (21a)$$

$$\tilde{\sigma}_3 \hat{\times} |\hat{\psi}>= \tilde{\sigma}_3 \times T \times |\hat{\psi}>= \pm w \times |\hat{\psi}>, w \neq 1, \quad (21b)$$

In essence, Santilli's irregular isorepresentation of $\hat{S}U(2)$ characterize a generalization of the conventional constant values of spin 1/2 into *locally variable spin isoeigenvalues.*

Rather than being a mathematical curiosity, the above spin mutation is expected to be important for a consistent representation of the spin of an electron, e.g., under the immense pressures, densities and temperature in the core of a star.

6. Independent Studies

Numerous mathematicians have made seminal contributions to the Lie-Santilli isotheory, among whom we quote: C-X, Jiang has conducted comprehensive studies [15] on the isonumber theory at the foundation of the Lie-Santilli isotheory; D. S. Sourlas and G. T. Tsagas have conducted the first comprehensive study of the Lie-Santilli theory [16], although prior to the discovery of isonumbers [2]; J. V. Kadeisvili has studied in detail the Lie-Santilli isotheory [17] following its formulation as in memoir [3]; R. M. Falc´ on and J. N. Valdés [18] have presented the most rigorous formulation to date of Santilli's isotopies; T. Vougiouklis [19] has developed the hyperstructural formulation of the Lie-Santilli isotheory which is the broadest possible formulation achievable with current mathematical knowledge; and S. Georgiev [20] has produced one of the most monumental works in mathematics showing the implications for all of mathematics of the isodifferential calculus which is nowadays called the *Santilli-Georgiev isodifferential calculus*. A comprehensive review with a large list of contributions has been produced by I. I. Gandzha and J. Kadeisvili, in monograph [21] with the suggestive title of *New Sciences for a New Era.*

7. Open Problems

The author has no words to recommend the study of regular and irregular isorepresentations of Lie-Santilli isoalgebras, with particular reference to the identification of a method for the construction of irregular isorepresentation parallel to that for the regular case of Section 4. The proposed study is important for a number of applications, such as:

7.1. Reconstruction of Exact Symmetries

Santilli has shown in Ref. [9] that the breaking of conventional spacetime and internal symmetries is the outcome of insufficient mathematics. because broken symmetries can be reconstructed as being exact at the covering isotopic level under the preservation of the conventional structure constants. This reconstruction has a number of important epistemological as well as technical implications. It is sufficient to note the reconstruction of parity under weak interactions or the maintaining of Einstein's abstract axioms of special relativity for interior conditions to illustrate the implications at hand. Their systematic study can be best done via the study of the isorepresentation of Lie-Santilli isoalgebras.

7.2. Invariant Representation of Hubble's Law

The regular Lorentz-Santilli isosymmetry has permitted an invariant derivation of the Hubble law on the cosmological redshift z = Hd via the mere admission that light loses energy to the cold intergalactic medium without any need for the hyperbolic conjecture of the expansion of the universe via the assumption z = Hd - v/c [6,12]. It is important to verify this occurrence via the study of the regular isorepresentations of the Lorentz-Santilli isosymmetry due to its implications for all of cosmology, since the elimination of the expansion of the universe will likely require the revision of all our cosmological knowledge.

7.3. Synthesis of the Neutron from the Hydrogen

In the author's view, the most important application and verification of isomathematics has been Santilli's exact and invariant representation at both the non-relativistic and relativistic levels of *all* characteristics of the neutron in its synthesis from the hydrogen (see review [21]). Such a synthesis is notoriously impossible for the conventional Hilbert space and related mathematics, e.g., because the rest energy of the neutron is *bigger* than the sum of the rest energies of the proton and the electron (a pure anathema for quantum mechanics); the Dirac equation, which is so effective for the representation of the electron orbiting around the proton in the hydrogen atom, becomes completely ineffective for the representation of the same electron when "compressed" (according to Rutherford) inside the proton; and for other reasons. The representation of the neutron synthesis was crucially dependent on the assumption of the proton and the electron as being *isoparticles,* that is, isounitary irreducible representations of the Galileo-Santilli or the Lorentz-Santilli isosymmetry whose study is evidently fundamental for true advances in particle physics, as well as in the structure of stars.

7.4. Nuclear Constituents as Extended Particles

One of the most important applications of isomathematics is the quantitative prediction of new nuclear energies without the release of harmful radiations (see review [21]). This prediction is based on the invariant representation of nuclear constituents as being extended and deformable charge distributions. Such a representation has been instrumental for the first achievement of the exact representation of nuclear magnetic moments and spin [7,10]. This new conception of the nuclear structure requires the representation of protons and neutrons as *isoparticles.* It is evident that important advances in nuclear physics and new clean energies will be curtailed until there are systematic studies on the isorepresentations of the Lie-Santilli isosymmetry.

7.5. Elimination of the Divergencies of Quantum Mechanics

Some of the biggest insufficiencies of quantum mechanics in particle physics are due to the singular character of Dirac's delta distribution at the origin, with ensuing divergencies of perturbative series that requiring the achievement of numerical results via the unreassuring subtraction of infinities. Santilli [9,10] has shown that isotopies of Dirac's delta *distribution* into a *function* without singularities at the origin. Additionally, in all known applications the absolute value of the isotopic element (2) is *very small,* with the consequential capability of turning divergent or slowly convergent quantum series into rapidly convergent ones (see the infinite series of isomonomials (8) for comparison). Due to the implications of these features for all quantitative sciences, it appears recommendable that they are confirmed and further developed via the study of the isorepresentation of the Galileo-Santilli or Lorentz-Santilli isosymmetries.

7.6. Electron Valence Bonds

According to the axioms of quantum mechanics and chemistry, two valence electrons, rather than forming any molecular bond, should *repel* each other due to the Coulomb repulsion of their equal charges $F = ke^2 / r^2$ which becomes extremely strong at the distances 10^{-13}cm of valence bonds. Santilli [11] has achieved a *strongly attractive* force between two electrons in singlet valence coupling via the admission that their wavepackets is in condition of total mutual penetration, resulting in non-Hamiltonian interactions represented with isotopic elements of type (2). In view of the predictable advances for all of chemistry, it is important to verify Santilli's strong valence bond via the study of the regular isorepresentations of the Lorentz-Santilli isosymmetry characterizing the valence electrons.

7.7. Nuclear and Chemical Reactions

The preceding applications can be sufficiently treated via regular isorepresentations since they deal with systems of extended particles assumed as being isolated from the rest of the universe. Santilli [9,10,11] has pointed out the insufficiency of the regular isorepresentations for nuclear and chemical reactions because they are *irreversible over time,* a feature that can only be represented via structure functions with an explicit time dependence of the type $C_{ij}^{k}(t,...) \neq C_{ij}^{k}(-t,...)$. Therefore, advances on much needed new energies without harmful radiation and on clean burning fuels will crucially depend on the availability of mathematical studies on irreducible isorepresentation of Lie-Santilli isoalgebras.

Due to their relevance, the R. M. Santilli Foundation has research funds for the writing of papers on the isorepresentations of the Lie-Santilli isotheory and their applications.

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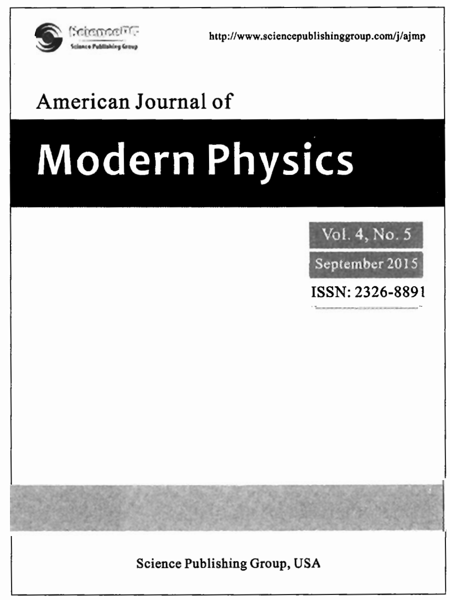

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