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ON A POSSIBLE LIE-ADMISSIBLE COVERING OF THE GALILEI RELATIVITY IN NEWTONIAN MECHANICS FOR NONCONSERVATIVE AND GALILEI FORM-NONINVARIANT SYSTEMS

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Abstract

In order to study the problem of the relativity laws of nonconservative and Galilei form-noninvariant systems, two complementary methodological frameworks are presented. The first belongs to the so-called Inverse Problem of Classical Mechanics and consists of the conventional analytic, algebraic and geometrical formulations which underlie the integrability conditions for the existence of a Lagrangian or, independently, of a Hamiltonian. These methods emerge as possessing considerable effectiveness in the identification of the mechanism of Galilei relativity breaking in Newtonian Mechanics by forces not derivable from a potential. Nevertheless, they do not exhibit a clear constructive capability for a possible covering relativity. For this reason, the second methodological framework is presented. It belongs to the so-called Lie-Admissible Problem in Classical Mechanics and consists of the covering analytic, algebraic and geometrical formulations which are needed for the equations originally conceived by Lagrange and Hamilton, those with external terms. These formulations are characterized by the Lie-admissible algebras which are known to be genuine algebraic covering of Lie algebras, and which in this paper are identified as possessing (a) a direct applicability in Newtonian Mechanics for the case of forces not derivable from a potential, (b) an analytic origin fully parallel to that of Lie algebras, i.e., via the brackets of the time evolution law, (c) a covering of the conventional canonical formulations as classical realizations, (d) an implementation at a number of levels of Lie's theory, including a fundamental realization as enveloping nonassociative algebras, (e) a generalization of symplectic and contact geometry as geometrical backing and (f) the capability of recovering conventional formulations identically at the limit of null external forces, here interpreted as relativity breaking forces. A covering of the Galilei relativity, called Galilei-admissible relativity, is then conjectured for independent scrutiny by interested researchers. A number of potential implications, particularly for hadron physics, are then briefly considered for future detailed treatment.

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"Let us not forget, the problem of interactions is still a mystery."

E.P. WIGNER

Symmetry and Reflections

Indiana University Press, p. 18
Bloomington, Indiana (1967)

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Ruggero Maria Santilli

CONTENTS

1: STATEMENT OF THE PROBLEM

The objective of this paper is to attempt the construction of a covering of the Galilei relativity which is applicable to nonconservative and Galilei form-noninvariant systems and which is capable of recovering the Galilei relativity identically at the limit of null relativity breaking forces. The paper then presents a few conjectural arguments for the possible relevance of such covering relativity beyond the framework of Newtonian Mechanics, for subsequent more detailed treatment.

Clearly, such a task is of rather delicate nature. In particular, it implies the study of a possible generalization of Galilei's relativity ideas which, within a Newtonian context, have remained unchanged for centuries.

Almost needless to say, a problem of this nature goes beyond my capabilities as an isolated researcher. As a result, the analysis of this paper must be considered as conjectural, tentative and yet inconclusive on both mathematical and physical grounds.

In essence, I will have achieved my objective if I succeed in stimulating the awareness of our community of basic studies on the need to reexamine the problem of the relativity laws of Newtonian Mechanics. Equivalently, this paper is an expression of my personal belief that Theoretical Physics is a Science which will never admit terminal disciplines. To state it explicitly, I do not believe that the Galilei relativity is the terminal relativity of Newtonian mechanics, particularly for the case of the systems of our everyday experience, that is, genuinely nonconservative.

Permit me to begin with the following introductory remarks.

(1) The need of a generalization of the Galilei relativity. Predictably, this need is not immune to controversial aspects. Pending the identification of more technical tools, the argument can be summarized as follows. An "arena of unequivocal applicability" of the Galilei relativity in Newtonian Mechanics is that of the systems whose forces are not only conservative, but also form invariant under the Galilei transformations, and I shall write

$$m_k \ddot{r}_{ka} - f_{ka}(\underset{\sim}{r}) = 0, \quad f_{ka} = -\frac{\partial V(\underset{\sim}{r})}{\partial r^{ka}}, \quad k = 1, 2, \ldots N; \; a = x, y, z, \qquad (1.1)$$

where the r's are the Cartesian coordinates of the Euclidean space of the experimental detection of the system, customarily assumed to be representative of an inertial frame of reference.
My problem consists in attempting the identification of a covering relativity for local, class C^{∞}, Newtonian systems whose forces are generally not derivable from a potential (nonconservative) as well as generally form-noninvariant under the Galilei transformations, and I shall write

$$\begin{aligned} & m_k \ddot{r}_{ka} - f_{ka}(\underset{\sim}{r}) - F_{ka}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) = 0, \\ & f_{ka} = \frac{\partial V}{\partial r^{ka}}, \quad F_{ka} \neq -\frac{\partial U}{\partial r^{ka}} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}^{ka}}, \end{aligned} \qquad (1.2)$$

where the r's represent, again, the system of Cartesian coordinates of the experimental detection of the system considered.

When confronting a system of type (1.2) the customary attitude is that of transforming it into an equivalent system in new coordinates, say r'^{ka} which is consistent with the Galilei relativity. This demands the transformation of system (1.2) into an equivalent system of the type

$$m_k \ddot{r}'^{k}_{ka} - f'_{ka}(\underset{\sim}{r}') = 0, \quad f'_{ka} = -\frac{\partial V'(\underset{\sim}{r}')}{\partial r^{ka}}, \qquad (1.3)$$

where the forces f'_{ka} are now derivable from a potential and Galilei form-invariant, or into equivalent system of free particles in the r'-space.

$$m_k \ddot{r}'_{ka} = 0. \qquad (1.4)$$

As we shall see during the course of our analysis, a transformation of this type is indeed generally possible. Neverthless, this relativity approach to systems of type (1.2) will be left to the interested reader for a number of reasons.

First of all, the conclusion that system (1.2) is consistent with the Galilei relativity (as currently known), because there exists an equivalent system in new coordinates which is consistent with such relativity, is equivalent to the following opposite conclusion. Consider a Newtonian system in the representation space of the experimental verification which is strictly consistent with the Galilei relativity. By using the inverse transition from Eqs. (1.3) to an equivalent form (1.2), such system can be transformed into an equivalent system in new coordinates which is incompatible with the Galilei relativity. The formal equivalence of the direct argument indicated above with its inverse would then imply that the original system is inconsistent with the Galilei relativity, contrary to the experimental evidence.

Secondly, the transition from system (1.2) to the equivalent form (1.3) has a number of physical implications which will be indicated during the course of the analysis. At this point it is sufficient to indicate that the transition considered implies a profound modification of the structure of the acting forces, that is, from a genuinely nonconservative a Galilei form-non-invariant form, as experimentally detected, to "new" forces in a new space which are derivable from a potential and are form-invariant under the Galilei transformations. Clearly, care must be exercised before extracting physical conclusions within such an equivalent mathematical approach. In the final analysis, the dominant physical character of the original system is that of being nonconservative, and any physically effective relativity characterization must represent this physical profile in its entirety.

Thirdly, the transition from system (1.2) to an equivalent form of type (1.3) or (1.4) is rather complex in practical realization. In particular, as we shall see better later on, it often

demands, as a necessary condition, that the new variables r'^{ka} depend on the old variables r^{ka} as well as their derivatives in a generally nonlinear way. This implies that, if the original system of coordinates is inertial, the new system is generally non-inertial, as well as generally non-realizable in an experimental set up.

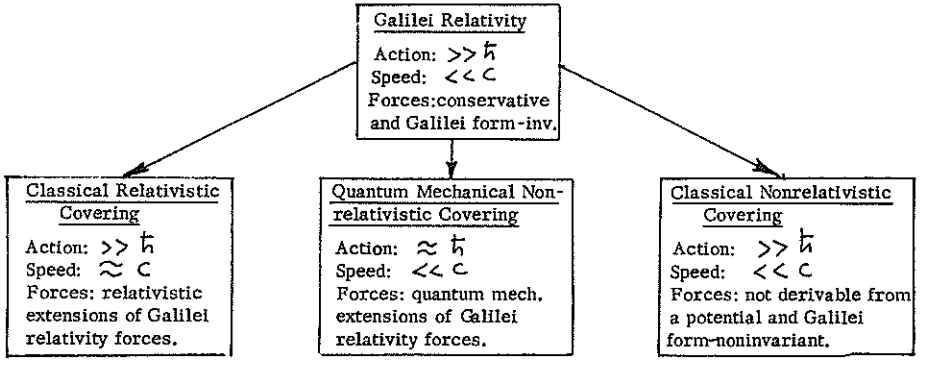
I hope that these introductory remarks indicate the need of confronting the problem of the applicable relativity laws to system (1.2) in the system of coordinates of its experimental detection. Once this problem has been resolved, then the study of the relativity aspect within the context of mathematical spaces of new coordinates can acquire its proper methodological role.

(2) The covering nature of the intended generalization. As is well known, new insights in Theoretical Physics never "destroy" previous accomplishments of proved physical relevance. They only implement them in a broader conceptual, physical and methodological context. The problem of the intended generalization of the Galilei relativity would be inconsistent in its very formulation unless such generalization is a covering (e.g., in the sense of ref.[1]) of the conventional Galilei relativity. In particular, the generalized and conventional relativities must be compatible in the sense, e.g., that there must exist clear limiting procedures of clear physical meaning which reduce the new relativity to the old and viceversa. Also, the new relativity must constitute a nontrivial generalization of the old for a nontrivially different physical context.

As we all know, the Galilei relativity has already been subjected to a number of coverings. The fundamental ones are those offered by Einstein special relativity and quantum mechanics. In the former case we have a classical covering of the Galilei relativity for speed of the order of that of light. In the latter case we have a covering of quantum mechanical nature for values of the action of the order of magnitude of the Planck constant, while the speed remains much smaller than the speed of light. These two coverings of the Galilei relativity can then be considered at the basis of two corresponding series of coverings. The methodological context of the former is that of classical field theory or the general theory of gravitation, while that of the second series is relativistic quantum mechanics, or quantum field theory.

The covering of the Galilei relativity which is attempted in this paper is according to none of these lines. The intended covering is purely classical in nature and, thus, quantum mechanical considerations are excluded at this time. Also, the intended covering is purely nonrelativistic and, thus, relativistic generalizations are excluded too, at this time. As a matter of fact the possible novelty of my efforts relies precisely in the intent of identifying a covering of the Galilei relativity which is independent from all existing coverings.

This objective appears to be rendered identifyiable, pending independent verifications, by the central topic of the study, the nonconservative nature of the acting forces, rather than the value of the speed or of the action. It is hoped that the following diagram is of some assistance in the identification of the objective of the study and its relationship with other relativity profiles.



The reader should be aware that the above characterization is mainly qualitative, pending the identification of methodological tools, to be outlined later on, which are capable of providing a technical characterization of the nature of the acting forces in the transition from one relativity to the other.

To summarize, the covering Galilei relativity which is attempted in this paper is purely classical and nonrelativistic and it is centered on the transition from conservative to nonconservative systems. This is intended to provide the nontrivially different physical context indicated earlier. Also, the fundamental requirement of compatibility of the classical relativistic and quantum mechanical nonrelativistic coverings is provided by the clear limiting procedures of clear physical meaning: $v/c \to 0$ (or Inönü-Wigner contraction) and ($\hbar$/action) $\to 0$ (or the Correspondence Principle), respectively. The corresponding, but different, limit for the classical nonrelativistic covering is: (Galilei relativity breaking forces) $\to 0$.

(3) The methodological tools of the intended generalization. Although not on a full time basis owing to my involvement in other research topics, I have been interested in the problem of the applicable relativity laws to nonconservative Newtonian systems since the time of my graduate studies in theoretical physics (at the University of Turin, Italy, from 1963 to 1966). However, it was not without surprise that an initial library search (conducted in 1963-1964) revealed that the methodology for the treatment of forces not derivable from a potential had remained virtually ignored in the physical and mathematical literatures, to the best of my knowledge.

This accounts for the rather considerable period of time which has passed from the identification of the problem to this tentative presentation. And indeed, in order to be able to even partially confront the problem, I had first to identify the rudiments of the methodology for the treatment of these forces.

As we all know, the virtual totality of the methodological context of Analytic Mechanics enters, either in a direct way or in a subtle indirect way, into the characterization of the Galilei relativity. I am here referring to the conventional analytic formulations (e.g., Lagrange's and Hamilton's equations, canonical transformation theory, etc), algebraic formulations (e.g., Lie algebras, universal enveloping associative algebras, Lie groups, etc.) and geometrical formulations (e.g., symplectic geometry, Lie derivatives, etc.). The problem of the construction of a possible covering of the Galilei relativity for nonconservative forces cannot be studied without the prior identification of at least the rudiments of the methodology which is applicable to the forces considered.

The primary objective of this paper is to outline my efforts in this respect and then to indicate a possible covering relativity which can be conjectured on the basis of the emerging methodology.

In essence, Newtonian systems with forces derivable from a potential can be fully and consistently treated with the indicated analytic, algebraic and geometrical formulations. The situation for systems with forces not derivable from a potential appears to be different. And indeed, at least in principle, these systems can be studied within the context of the following dual methodological profile.

(I) Formulations based on Lagrange's and Hamilton's equations without external terms. Within the context of conventional treatments of Analytic Mechanics, the Lagrangian and Hamiltonian are often assumed as possessing the conventional trivial structure $L = T - V$ and $H = T + V$, respectively. However, within the context of the broader discipline known as the Calculus of Variations, these functions can have an arbitrary functional structure (provided that certain continuity and regularity conditions are satisfied). The transition from the conventional to an arbitrary structure of a Lagrangian or a Hamiltonian essentially implies, at the Newtonian level, the transition from systems with forces derivable from a potential to systems with arbitrary Newtonian forces. The net effect is that the conventional Lagrange's and Hamilton's equations can indeed effectively represent nonconservative Newtonian systems. The knowledge of these functions then immediately implies the applicability of all the established analytic, algebraic and geometrical formulations to the systems considered. Explicitly stated, the knowledge of a Hamiltonian for a system of type (1.2) implies the applicability, say, of the canonical transformation theory, Noether's theorem, Lie's theory,

symplectic geometry, etc., by therefore bringing nonconservative systems up to the methodological context of systems with forces derivable from a potential.

However, in order for such an approach to have any practical effectiveness, the fundamental problem consists of the integrability conditions for the existence of a Lagrangian or a Hamiltonian for systems (1.2), that is, the necessary and sufficient conditions for systems with forces not derivable from a potential to admit an analytic representation in terms of the conventional Lagrange's and Hamilton's equations (without external terms).

I have been involved in the study of this problem, although also not on a full time basis, from 1973 until recently. My efforts for the Newtonian profile of the problem are presented in the forthcoming monographs of refs.[2a,2b] and their extension to classical field theories are presented in refs.[3] Understandably, no relativity aspect is treated in these preliminary studies, apart from few incidental remarks.

This first methodological profile for the treatment of Newtonian systems with forces not derivable from a potential can be identified as belonging to the so-called Inverse Problem of Classical Mechanics, where these terms can be referred not only to the integrability conditions for the existence of a Lagrangian or a Hamiltonian, but also to the methods for their construction as well as the consequential enclosure of all the available analytic, algebraic and geometrical techniques.

(II) Formulations based on Lagrange's and Hamilton's equations with external terms. The most natural way of representing Newtonian forces not derivable from a potential is that originally conceived by Lagrange and Hamilton, that is, with external terms. In essence, Lagrange and Hamilton appeared to be fully aware that the Newtonian forces are generally non-derivable from a potential. The presence of external terms in their equations was thus essential to avoid an excessive approximation of physical reality. Oddly, it has been only since the beginning of this century that Lagrange's and Hamilton's equations have been "truncated" with the removal of the external terms by acquiring the form which is almost universally used in current physical literature.

This is not an occurrence of marginal relevance. Instead, it could indicate that the virtual totality of our current theoretical knowledge based on analytic techniques at all presently known levels, such as classical, quantum mechanical and quantum field theoretical, can be considered as solidly established, provided that the underlying systems possess forces derivable from a potential, that is, structures of the type $L = T - V = L_{free} + L_{int}$, and $H = T + V = H_{free} + H_{int}$, represent the systems in their entirety (I shall elaborate this aspect both in this paper as well as in subsequent papers more specifically devoted to this issue).

My first research interest has been devoted to the study of these equations.[4] The initial library search conducted in 1963-1964 also revealed a rather sizable methodological gap existing between the analytic equations without and with external terms which, to the best of my knowledge, still persists as of today. In essence, while the study of analytic equations without external terms has developed into the beautiful and articulated body of interrelated methodological tools, known as analytic, algebraic and geometrical tools (to ignore other profiles), no comparable development has occurred, to the best of my knowledge, for the case of analytic equations with external terms.

For instance, questions for me of fundamental relevance, such as the algebraic structure which underlies these latter equations, or their transformation theory, resulted to be untreated in the available literature despite my laborious search.

The study of the methodology related to Lagrange's and Hamilton's equations with external terms was clearly mandatory for my objective to attempt the construction of a covering of the Galilei relativity. On analytic grounds these equations can be interpreted as constituting a covering of the conventional equations in the sense of being directly applicable to broader systems (that is, applicable without changes of the local variables) while capable of recovering the conventional equations identically at the limit of null forces not derivable from a potential. Also, and most importantly, within these broader equations the Lagrangian and the Hamiltonian can represent not only the free motion, but also all Galilei form-invariant forces derivable from a potential, i.e., Eqs., (1.1), while the external terms can represent precisely the Galilei breaking forces, i.e., the F-forces of Eqs. (1.2).

It was however easy to see that the presence of external forces is, by far, nontrivial on methodological grounds. It is sufficient, in this respect, to indicate the nonapplicability of the conventional canonical transformation theory; the fact that the brackets of the emerging generalized time evolution law violate the Lie algebra identities; and the inevitable, consequential, nonapplicability of the symplectic geometry.

Rather than considering these occurrences as drawbacks, I interpreted them as promising on methodological grounds. In essence, the fact that the brackets of the time evolution laws violate the Lie algebra laws, by no means, implies that these brackets are unable to characterize a well defined (nonassociative) algebra. And indeed, as we shall see later on, when properly written, these brackets characterize a (nonassociative) algebra called Lie-admissible algebra[4] which results to be an algebraic covering of the Lie algebras, that is, (a) directly applicable to a broader physical context, (b) admitting a consistent analytic origin fully parallel to that of the Lie algebras, and, last but not least, (c) admitting a realization of the product which

recovers the conventional Poisson brackets identically at the limit of null external forces.

These features essentially indicate that the Lie algebras are not "lost" in the broader Lie-admissible algebras. Instead, they are fully present, although in an embedded form.

The identification of this algebraic character of Hamilton's equations with external terms was my first step, as embrionically presented in refs.[4a,4b,4c]. This step was clearly crucial for any subsequent study. And indeed, the existence of a consistent algebraic covering of the Lie algebras gave hope for the existence of covering analytic, algebraic and geometrical formulations which (1) are applicable to the broader class of systems with forces not derivable from a potential via analytic equations with external terms, (2) possess the same interrelations and analytic origin of the conventional formulations, and (3) are capable of recovering the conventional formulations identically at the limit of null external forces.

Most of my subsequent efforts have been devoted to the study of the possible existence of these covering formulations. These efforts are presented in the forthcoming monographs of refs.[5a,5b,5c]. They can be identified as belonging to what I have tentatively called the Lie-admissible Problem of Classical Mechanics, where these terms can be referred to the analytic, algebraic and geometrical formulations based on analytic equations with external terms.

As we shall see, my conjectural arguments related to the possible existence of a covering of the Galilei relativity are based on these broader formulations. To be explicit on this crucial point, I do not believe that a genuinely new covering of the Galilei relativity along lines different than those of the existing coverings can be effectively attempted without first identifying at least the rudiments of the coverings of the central methodological tools of current relativity ideas: the analytic, algebraic and geometrical tools.

(III) Joint use of Lagrange's and Hamilton's equations without and with external terms for the representation of the same nonconservative systems. As we shall see, one of the most insidious aspects of the problem of the relativity laws for nonconservative systems is of conceptual, rather than technical nature. This is due to the fact that, owing to extended use, the primary contemporary emphasis is in the study of "symmetries and conservation laws". The proper study of nonconservative systems appears to demand a profound conceptual departure from this context. On symmetry grounds, the emphasis is shifted to that of broken symmetries. These are familiar terms in contemporary theoretical physics, but there is a central difference between their conventional meaning in current literature and their meaning in this paper, which is advisable to identify already at this introductory stage.

In essence, the terms "broken symmetries" are customarily referred to broken internal symmetries (e.g., the SU(3) breaking due to strong interactions) or to broken discrete symmetries

(e.g., the parity violation in weak interactions). The terms "broken symmetries" in this paper are specifically referred to broken continuous, connected, space-time symmetries, of course, at the Newtonian level.

An example is here crucial to understand the meaning of these terms, as well as the nature of the breaking of the Galilei relativity provided by systems of type (I.2). Consider the spinning top under gravity. The conventional treatment of this system is often restricted to its conservative abstraction with consequential dominant role of the exact symmetry under the group of rotations, SO(3). However, if this exact symmetry were actually realized in our environment, it would literally imply the existence of the perpetual motion, trivially, from the conserved nature of the angular momentum. The physical reality appears to be different. Experimental evidence indicates that the angular momentum of the spinning top is not conserved. In turn, this implies that the symmetry under rotation is broken for the system considered, as we shall identify later, on more technical grounds. This inevitably implies the loss of the group of rotation as a methodological tool of any effectiveness. As a matter of fact, in order to properly represent the system considered as it occurs in the physical reality, all my efforts will be centered in producing the highest possible breaking of the symmetry under rotations. And indeed, this implies the existence of drag torques which are responsible for the decaying in time of the angular momentum.

In conclusion, in the study of Newtonian systems with forces not derivable from a potential and Galilei form-noninvariant the conceptual attitude is shifted from that of the conventional "exact space-time symmetries" to that of "broken space-time symmetries", with particular emphasis on the fundamental part of these symmetries, the group of rotations SO(3) and related Lie algebra $\underline{SO}(3)$.

This conceptual profile becomes even more insidious when passing to the complementary part of the physical conservation laws. And indeed, to comply with the experimental evidence that the physical quantities of the systems considered are nonconserved, the emphasis is now shifted to the physical non conservation laws. The following remark may be of assistance in identifying the insidious nature of this profile. As we shall see, the use of the techniques of the Inverse Problem sometimes yields a Hamiltonian for the representation of nonconservative systems which does not depend explicitly on time. This is the case, for instance, for the damped oscillator. The use of the techniques of symmetries and conservation laws trivially yields that such a Hamiltonian is indeed conserved. The issue which is however relevant is the physical meaning of the mathematical occurrence $\dot{H} = 0$ when the represented system is nonconservative by assumption, that is, when the experimental evidence indicates that the physical energy of the system decays in time, as trivial for the damped oscillator.

We reach in this way a crucial aspect of the problem of the relativity laws for nonconservative Newtonian systems: the applicable methodology must be capable of characterizing broken space-time symmetries and physical nonconservation laws. This is exactly the opposite in conceptual attitude of the corresponding setting for Newtonian systems which obey the Galilei relativity.

As we shall see, the Lie-admissible formulations appear to satisfy this crucial requirement. And indeed, they break the space-time symmetries to the point of rendering all Lie algebras inapplicable "ab initio", whenever the external terms are nonnull. The intriguing aspect is that the broken symmetries do not remain algebraically undefined, as in conventional (classical) treatment. Instead, they acquire a broader algebraic structure which appears to be parallel in physical effectiveness to that of the Lie treatement of exact symmetries, although the conceptual and methodological context is now profoundly altered. For, instance, with reference to the case of the spinning top under gravity, the Lie algebra of the group of rotation has the precise physical meaning of representing conserved quantities via its generators, the angular momentum components. In the transition to the case of the nonconservative spinning top represented with Lie-admissible formulations, this Lie algebra $\underline{SO}(3)$ becames undefinable in a consistent way because the basic analytic equations are non-Lie in algebraic character. However, the $\underline{SO}(3)$ Lie algebra results to be replaced by an $\underline{SO}(3)$ Lie-admissible algebra which is not only fully defined on algebraic grounds, but also such to directly express the nonconservation of the angular momentum components. As we shall indicate in details, this $\underline{SO}(3)$-admissible algebra results to be an algebraic covering of the conventional $\underline{SO}(3)$ algebra in the sense of (a) possessing an analytic origin fully parallel to that of the latter, (b) being different as algebraic structure, that is, being a non-Lie algebra, and (c) capable of recovering the latter identically at the limit of null nonconservative forces. Most importantly, while the equations are form-noninvariant under the conventional rotations by central requirement, the Lie-admissible context appears to produce generalized transformations which leave form-invariant the nonconservative (nonlinear) equations of motion.

In conclusion, the covering of the Galilei relativity which will be conjectured in this paper is based on the attempt of embedding the Galilei algebra into a covering Galilei-admissible structure. The embedding will be technically realized via the embedding of the universal enveloping associative algebra into a nonassociative but Lie-admissible covering which preserves the base manifold, the parameters and the generators of the original structure. In turn, this Lie-admissible envelope will open the possibility of having, on one side, a Lie-admissible behaviour in the neighborhood of the identity while, on the other side, producing generalized, connected transformations under suitable integrability conditions. In turn, these latter transformations will open the possibility of leaving form-invariant the nonlinear, nonconservative systems.

Predictably, in a program of this nature, the technical difficulties which I shall identify (without any claim of solving any of them) are expected to be conspicuous. It is in this respect where the dual methodological approach to the same systems acquires its full light. I am here referring to the joint use of the analytic equations without and with external terms for the representation of the same system, and of the related methodologies (the Inverse Problem and the Lie-admissible Problem, respectively).

It is relevant here to indicate that my initial efforts at the construction of, for instance, a covering of the canonical transformation theory for Hamilton's equations with external terms have encountered such severe consistency problems, to force me into the laborious study of the Inverse Problem. And indeed, since the analytic equations without and with external terms represent the same system by assumption, the knowledge of a Hamiltonian for the former via the Inverse Problem finally allowed me to construct the transformation theory of the latter as an "image" of the conventional canonical transformation theory. The consistency of the approach was now guaranteed. But then for the approach to be of any practical usefulness, the knowledge of a Hamiltonian for nonconservative systems (1.2) was mandatory. This is, in essence the spirit of the methodology of the Inverse Problem.

In conclusion, conservative systems can be effectively treated with only the conventional analytic equations (i.e., those without external terms). When nonlinear, nonconservative systems are considered, the situation is different. In this case, owing to the complexity of the problems to be confronted and, in due time, solved, the most recommendable attitude is that of using the totality of the available techniques, whenever possible. These techniques can be classified into two groups, here called those of the Inverse Problem (for analytic equations without external terms) and of the Lie-admissible Problem (for analytic equations with external terms). It is hoped that a judicious interplay of these two complementary methodological profiles will result to be of assistance in the study of the problems to be confronted. For instance, each insight reached within the context of one approach can be subject to consistency verification within the context of the other. Similarly, aspects which are of difficult treatement within the context of one approach may result to be more treatable within the con text of the other.

My use of this dual methodological profile will be the following. That of the Lie-admissible Problem will be used as the fundamental constructive tool of the intended covering relativity, while that of the Inverse Problem will be used as a methodological backing only. The use of the same methodologies but with different roles, however, is not excluded, but actually encouraged.

The organization of this paper is the following. In Section 2, I shall present the rudiments of the methodology of the Inverse Problem as a review of refs.[2a, 2b] while, in Section 3, I shall presents the rudiments of the methodology of the Lie-admissible problem

as a review of refs.[4, 5]. The reader should be alterted that, to reduce this paper to a minimal length, the proof of all the theorems presented in these two parts is either left to the interested reader or to the inspection of the detailed presentation of the quoted references. In Section 4, I shall then present the conjecture of the Lie-admissible covering of the Galilei relativity, called Galilei-admissible relativity, and work out few simple examples. Finally, in Section 5, I shall present few highly conjectural remarks related to the possible physical relevance of the analysis for non-Newtonian frameworks. I am here referring to possible classical relativistic and quantum mechanical extensions.

It is rather tempting in this latter respect to recall the fact that, irrespective of whether actually constructed or only identified as plausible, any new relativity idea has always proved to have a deep impact in our representation of physical reality. Most notably, this was the case of the physical role of the Einstein special relativity for our representation of the electromagnetic interactions in general, and of the atomic structure in particular. The intended Lie-admissible covering of the Galilei relativity will be presented for its arena of clear poten tial significance, the Newtonian systems of our everyday experience. However, let me confess that the intended arena of applicability, upon a number of technical implementations, is that of the old idea that strong interactions in general, and the strong hadronic forces in particular, are not derivable from a potential, that is, they are precisely of type (1.2) at the primitive Newtonian level.

In essence, the moment I was taught the profound physical differences which exist between the electromagnetic and the strong interactions, I had difficulties in accepting for the latter interactions basic concepts, laws and principles which are essentially the same as those of the former interactions. The reason was due to the fascinating physical effectiveness of established disciplines for the electromagnetic interactions versus the lack of any comparable physical effectiveness of the same tools, when applied to the strong interactions. If the strong interactions are assumed as analytically equivalent to the electromagnetic interactions (i.e., both derivable from a potential), I saw no way of escaping from the inflexible laws of established disciplines. The representation of the strong interactions (and the strong hadronic forces in particular) as still local, but analytically nonequivalent to the electromagnetic interactions (i.e., nonderivable from a potential), appeared to me as sufficiently interesting to deserve a study prior to the confrontation of more complex models, e.g., in terms of nonlocal forces not derivable from a potential. A part from a new methodological horizon which appears to be stimulated by this line of study, a most intriguing aspect is that the approach appears to produce a profound differentiation of the electromagnetic and the strong interactions in the physical space of their experimental verification (Euclidean or Minkowski).

In conclusion, the hope which stimulated this work is that of being able to study, in due time, a possible differentiation of the atomic and the hadronic structure via the relativity laws.[5]

2. RUDIMENTS OF THE METHODOLOGY OF THE INVERSE PROBLEM

The Direct Problem of Newtonian Mechanics is the conventional approach according to which one assigns a Lagrangian $L(t, q, \dot{q})$ and then computes the equations of motion with Lagrange's equations

$$L_k(q) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} = \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^i}\ddot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k \partial q^i}\dot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k \partial t} - \frac{\partial L}{\partial q^k} = 0, \quad (2.1)$$

$$k = 1, 2, \ldots, n$$

(throughout this section we shall use for conciseness the terms "Lagrange's equations" to denote those without external terms and the presentation will mainly deal with generalized coordinates).

The Inverse Problem of Newtonian Mechanics can be empirically defined as consisting of the inverse approach according to which an arbitrary (quasilinear) system of second-order ordinary differential equations is assigned

$$F_k(q) = A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0, \quad (2.2)$$

and the knowledge of a Lagrangian for the representation of these equations with Eqs. (2.1) is requested.

At a closer inspection the problem essentially consists of the following aspects:

(a) the necessary and sufficient conditions (integrability conditions) for the existence of a Lagrangian (or, independently, of a Hamiltonian) for the analytic representation of generally nonconservative Newtonian systems (i.e., second-order ordinary differential equations which are linear in the second order derivative $\ddot{q}^k$, but generally nonlinear in the first-order derivatives $\dot{q}^k$ and in the generalized coordinates q^k as well as generally depending explicitly on time);

(b) the methods for the computation of a Lagrangian (or, independently, of a Hamiltonian) from the given equations of motion when their existence is ensured by the integrability conditions; and

(c) the significance of the underlying methodology (inclusive of the established analytic, algebraic and geometrical formulations) for the study of nonconservative systems, e.g., transformation theory, symmetries and first integrals, etc..

On rigorous terms, the problem is known under the name of Inverse Problem of the Calculus of Variations in which grounds it can be technically identified. However, we are not interested in this paper in the extremal aspect of the problem and, as such, this latter profile will be ignored. The reader, however, should keep in mind that, even tough such extremal aspect can be effectively ignored in the study of problems (a), (b) and (c), the underlying techniques

I shall use were conceived within the context of the Calculus of Variations and remain strictly variational in nature. I shall make a genuine effort in being as simple as possible, Neverthless, nowadays, the Inverse Problem can be studied with modern, effective and rigorous mathematical tools such as within the context of

(1) Differential Geometry. In essence, the conditions for a vector field on a (Hausdorff, second countable, ∞-differentiable, 2n-dimentional symplectic, or (2n+1)-dimensional contact) manifold to be globally Hamiltonian[6] can be reformulated to provide the integrability conditions for the existence of a Hamiltonian. A corresponding approach holds for the Lagrangian case.

(2) Functional Analysis. In this case the computation of the Gateau differential of Eqs. (1.2) reinterpreted as nonlinear operators and the condition of potentiality yield the integrability conditions for the existence of a Lagrangian;[7]

(3) Cohomology Theory. In this case differential operators are used to construct cochain complexes on star-shaped sets of field functions. The use of concomitants then yields the integrability conditions for the existence of a Lagrangian.[8]

In this section I shall outline an approach based on what appears to be a simple but most effective tool, known under the name of variational approach to selfadjointness, with an economical use of its prerequisites, e.g., the existence theory of ordinary differential equations and the Calculus of Variations, and its complementary aspects, e.g., the calculus of differential forms in general and the Converse of the Poincaré Lemma in particular.

As a result, the differential, functional and cohomology approach will be largely ignored. In any case, a study of the issue has indicated that the ultimate explicit form of the integrability conditions for the existence of a Lagrangian or, independently, of a Hamiltonian constructed with mathematically different approaches either coincide or are trivially equivalent.[2a] It is this property which allows the restriction of the treatment to only the variational approach to selfadjointness. (which appears to be preferable for explicit computations, e.g., the explicit construction of a Lagrangian). In any case, the reader with a serious interest in the relativity problem of nonconservative Newtonian systems is urged to study also the geometrical, functional and cohomology treatment with an understanding that the rudimentary review of the variational approach to selfadjointness outlined in this section is largely insufficient.

For conciseness, the main arguments will be presented in sequential tables. The detailed proof of all statements and theorems is presented in refs.[2,3]. The assumptions which will be tacitly used troughout this section are that (1) all differential equations are local (nonlocal forces are excluded), (2) all equations of motion (and, thus, including the acting forces) are of class C^∞ in their region of definition, and (3) the functional matrices of all equations of motion (Hessian matrices for all Lagrangians) are regular (for forces without acceleration couplings this essentially means the regularity of the mass tensor). All systems are finite-dimensional.

TABLE 2.1: THE CONTROVERSY ON THE REPRESENTATION OF NONCONSERVATIVE NEWTONIAN SYSTEMS WITH THE CONVENTIONAL HAMILTON'S PRINCIPLE. A problem which has been controversial for over one century in the physical literature is whether nonconservative Newtonian systems can be represented with the conventional Hamilton's principle

$$\delta \int_{t_1}^{t_2} dt\, L(t,q,\dot{q}) = -\int_{t_1}^{t_2} dt\, L_k(q)\,\delta q^k = 0\,, \qquad \delta q^k(t_s) = 0,\ s = 1,2\,. \qquad (2.1.1)$$

To the best of my knowledge, this controversy, somewhat inherited from contrasting statements dating back from the past century, reached a climactic stage in the early 30's as a result of the following corollary of a theorem by P. S. BAUER (1931)[9a]

> "The equations of motion of a dissipative linear dynamical system with constant coefficients are not given by a variational principle."

This statement prompted the publication of a disproof by H. BATEMAN[9b] (BAUER's paper was submitted as a Harvard note on March 21, 1931 and BATEMAN's rebuff was submitted as a CALTEC note on June 17, 1931). Nevertheless, BATEMAN's paper was based on the use of a method, today's known as BATEMAN's prolongation theory, which implies the doubling of the number of equations (which is outside the context of the Inverse Problem as commonly understood). As a result, the controversy did not ended, but was taken up again by a number of authors, such as L. J. SINGE[9c] (BATEMAN had properly published his paper in The Physical Review).

In the final stage, this controversy resulted in negative positions in more recent textbooks on mechanics. For instance, C. LANCZOS, in his textbook on variational principles[9d] states on p. xxi (1949 edition and subsequent reprints)

> "Forces of frictional nature, which have no work function, are outside the realm of variational principles."

Similarly, on p. 19-7 of Vol. II of the FEYNMAN Lectures (R. P. FEYNMAN, R. B. LEIGHTON and M. S. SANDS[9e], 1966 edition and subsequent reprints) one can read

> "The principle of least action only works for conservative systems-where all forces can be gotten from a potential function."

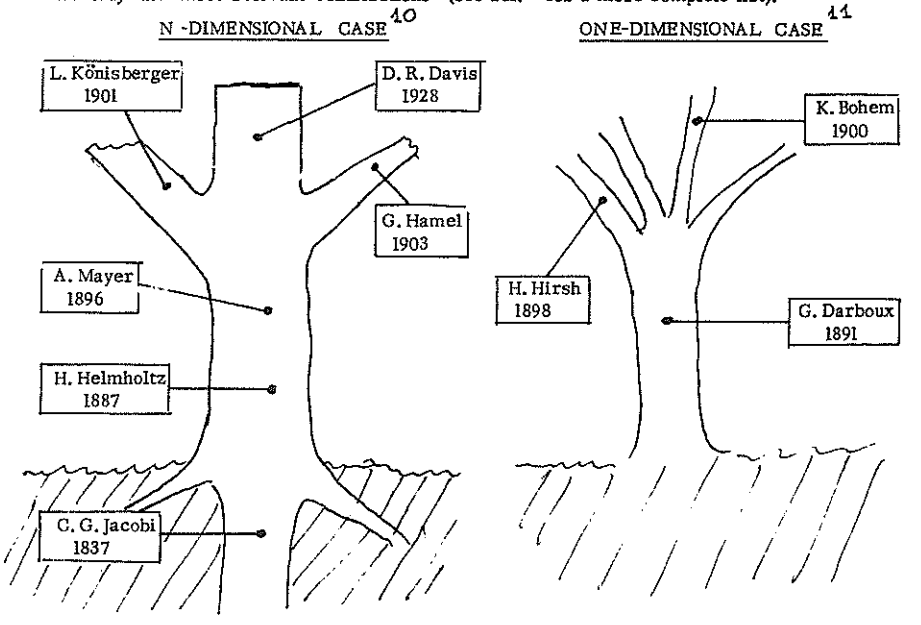
As we shall outline in the following tables, the Inverse Problem allows the resolution of this controversy.[2] The net result will be that the arena of representational capabilities of the conventional Hamilton's principle in Newtonian Mechanics is rather vast indeed. Of course, this will crucially depend on the notion of "analytic representation" which, perhaps, was at the basis of the controversy.

In any case, it was unfortunate that none of the authors quoted in this table was aware of the fact that, by the late 20's, the methodology of the Inverse Problem was sufficiently well established in the specialized literature of the Calculus of Variation, as outlined in the next table.

TABLE 2.2: THE TENTATIVE GENEALOGICAL TREE OF THE INVERSE PROBLEM.

One of the most time consuming aspects of the research project presented in this paper has been the identification of the prior state of the art on the Inverse Problem. An initial library search conducted in 1973 soon revealed that problems (a), (b) and (c) (of the introduction of this section) were not identified, let alone treated, in all textbooks in Newtonian Mechanics, Calculus of Variations and other disciplines I was able to inspect. However, these problems are at the very heart of Lagrange's and Hamilton's equations and, as such, they "had" to be treated in the existing literature. It was not after a laborious search which I conducted in the libraries of the Boston area by moving backward in time, that my determination was finally rewarded. And indeed I finally succeeded in identifying a number of contributions which established, to the best of my knowledge, the foundations of the methodology of the Inverse Problem, the first and perhaps most important contributions dating back from the last part of the past century.[10,11]

The results of my search are presented below with a strict understanding that they should not be interpreted as historical notes. They are simply the results of my personal findings and, as such, at a more detailed scrutiny, they may result to be grossly deficient. Notice that I quote below only the most relevant contributions (see ref.[2a] for a more complete list).



The one-dimensional case was treated in details and solved, apparently for the first time, by G. DARBOUX in 1891 by using conventional techniques (for that time) of partial differential equations. Subsequently, the problem was extended to the case of higher order derivatives by a number of authors (we are here solely interested to the second-order case). This problem is trivial by today's standard because it implies the solution of one partial differential equation in one unknown, the Lagrangian L, i.e.,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = A(t,q,\dot{q})\ddot{q} + B(t,q,\dot{q}). \qquad (2.2.1)$$

As such, a solution is guaranteed by the existence theorems for partial differential equations under certain technical conditions (Table 2.6).

The n-dimensional case, on the contrary, is nontrivial because it consists of n partial differential equations in only one unknown, again the Lagrangian, i.e.,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} = A_{ki}(t,q,\dot{q})\ddot{q}^i + B_k(t,q,\dot{q}). \qquad (2.2.2)$$

The system is now overdetermined and a solution does not necessarily exists (in the form presented above, pending the generalizations outlined in the subsequent tables).

The integrability conditions for Eqs. (2.2.2) were apparently identified for the first time by H. HELMHOLTZ (1887) on beautiful intuitional grounds. In essence, HELMHOLTZ's starting point was the property that Lagrange's equations are always selfadjoint (Table 2.5), a property which goes back to a contribution by C. G. JACOBI of 1837 . He then argued that the conditions of (variational) selfadjointness were both necessary and sufficient for the existence of L. This approach was reinspected by a number of authors, most notably in my opinion, A. MAYER in a first contribution of 1896. The most comprehensive treatement of which I am aware on the study of the joint necessity and sufficiency of the conditions of selfadjointness is the thesis by D. R. DAVIS at the Department of Mathematics of the University of Chicago under the supervision of G. A. BLISS, subsequently expanded and published in three articles of 1928, 1929 and 1931. Perhaps, equally notable is the study by L. KÖNISBERGER of 1901 (oddly there is no direct quotation in DAVIS's papers of HELMOLTZ's and KÖNISBERGER's contributions).

In conclusion, it appears that by the late 20's the problem of the integrability conditions for the existence of a Lagrangian was well established. Most importantly, the studies were extended to the inclusion of integrating factors (which are ignored in the initial formulation (2.2.2). As we shall outline in Table 2.9, a proper use of these techniques allow the disproof of Bauer's statement (Table 2.1) without an increase of the number of equations. For a specific study of the case of linear equations with constant coefficients see reference [2b].

However, the techniques used in the joint proof of the necessity and sufficiency of the conditions of selfadjointness were those available at that time. At a closer inspection, a number of aspects remained still open. I confronted the problem along the same conceptual lines, that is, the variational approach to selfadjointness, but I used what is nowdays considered a more effective tool for the study of the integrability conditions, the calculus of differential forms in general and the converse of the Poincaré lemma in particular. My proof of the joint necessity and sufficiency was first published in ref. [3b] for the field theoretical case. Ref. [2a] contains the reduction of the proof to the Newtonian case. The problem along different methodological lines had already been solved by M. M. VAINBERG [7a] in 1964 by using the functional approach to nonlinear operators for the case of first-order Lagrangians, that is, Lagrangians $L(t,q,\dot{q})$. The case of the representation of the same systems with second-order Lagrangians $L(t,q,\dot{q},\ddot{q})$ had also been solved by G. W. HORNDESKI [8] in 1974 within the context of the cohomology theory and cochain complexes, but this approach implies the use of third order analytic equations which are uncommon in Newtonian Mechanics. As R. W. ATHERTON and G. M. HOMSY [7c] put it, VAINBERG's approach was so abstract to remain "inaccessible to many applied mathematicians and engineers". It is here tempting to say that a similar comment perhaps applies also to HORNDESKY's approach. My efforts were therefore motivated by the intent of achieving a proof which was accessible to the physics and engineering community at large.

Notice that the genealogical reference tree for the n-dimensional case has been truncated. This is due to the fact that the Inverse Problem remained largely ignored after the early 30's, to the best of my knowledge and with very few exceptions known to me. [12] For a complete list of all relevant contributions on the Inverse Problem of which I am aware see ref. [2a]

TABLE 2.3: THE CONCEPT OF ANALYTIC REPRESENTATION

The most direct way to define an analytic representation is that of imposing that the totality of solutions of the equations of motion coincides with that of Lagrange's equations. Our systems, however, are generally nonlinear and such an approach is in practice faced with severe difficulties. There exists a number of ways to overcome these difficulties. The assumed continuity and regularity conditions ensure the applicability of the theorem on implicit functions to Eqs. (2.2) and, most importantly, the uniqueness of the system of implicit functions. As a result, a first definition of analytic representation can be introduced by requiring that the systems of implicit functions of the equations of motion and of Lagrange's equations coincide. In the following we shall say that a (local, class C^∞ and regular) Newtonian system admits an ordered indirect

analytic representation in terms of Lagrange's equations (without external terms and in first-order Lagrangians) when there exists a class C^∞ and regular matrix of factor functions such that the following identifications[2a,3a]

$$\frac{d}{dt}\frac{\partial L(t,q,\dot q)}{\partial \dot q^k} - \frac{\partial L(t,q,\dot q)}{\partial q^k} \equiv h_k{}^i(t,q,\dot q)\left[A_{ij}(t,q,\dot q)\ddot q^j + B_i(t,q,\dot q)\right], \quad (2.3.1)$$
$$k = 1,2,\ldots,n,$$

hold in a given ordering of the index k=1,2,3,...,n. The regularity of the matrix of factor functions is intended to yield the identity of the systems of implicit functions of the equations of motion (as originally given) and of Lagrange's equations, i.e., the uniqueness of the system

$$\ddot q^k - f^k(t,q,\dot q) = 0, \quad (f) = -(A)^{-1}(B), \quad (2.3.2)$$

for both members of identifications (2.3.1). This, in turn, implies the identity of the totality of solutions. When representation (2.3.1) exists[2a,3a] with $(h_k{}^i) = (\delta_k{}^i)$ we shall say that we have an ordered direct analytic representation. As we shall see, the notion of ordering plays a crucial role, particularly for the necessity of the conditions of selfadjointness. Notice that the maximal admissible functional dependence of the integrating factors is $h_k{}^i = h_k{}^i(t,q,\dot q)$.

TABLE 2.4: VARIATIONAL APPROACH TO SELFADJOINTNESS. The finite part of the first-order variation of the equations of motion is called the system of variational forms, can be written

$$M_k(\eta) = \frac{1}{w}\delta^{(1)}F_k(q) = \frac{\partial F_k}{\partial q^i}\eta^i + \frac{\partial F_k}{\partial \dot q^i}\dot\eta^i + \frac{\partial F_k}{\partial \ddot q^i}\ddot\eta^i, \quad (2.4.1)$$
$$\delta q = \eta w, \quad w \approx 0,$$

and can be computed along any admissible variation, that is, along any function $\eta(t)$ possessing the same continuity properties of the solutions. The adjoint system of variational forms can be written

$$\tilde M_k(\tilde\eta) = \tilde\eta^i\frac{\partial F_i}{\partial q^k} - \frac{d}{dt}\left(\tilde\eta^i\frac{\partial F_i}{\partial \dot q^k}\right) + \frac{d^2}{dt^2}\left(\tilde\eta^i\frac{\partial F_i}{\partial \ddot q^k}\right), \quad (2.4.2)$$

and (under the assumed conditions) is uniquely defined by the so-called Lagrange identity

$$\tilde\eta^i M_i(\eta) - \eta^i\tilde M_i(\tilde\eta) = \frac{d}{dt}\varrho(\eta,\tilde\eta), \quad (2.4.3a)$$

$$\varrho(\eta,\tilde\eta) = \tilde\eta^i\frac{\partial F_i}{\partial \dot q^j}\eta^j + \tilde\eta^i\frac{\partial F_i}{\partial \ddot q^j}\dot\eta^j + \frac{d}{dt}\left(\tilde\eta^i\frac{\partial F_i}{\partial \ddot q^j}\right)\eta^j, \quad (2.4.3b)$$

where η and $\tilde\eta$ are generally different admissible variations. A system of second-order ordinary differential equations is called (variationally) selfadjoint when its variational forms coincide with the adjoint systems for all admissible variations, i.e.,

$$M_k(\eta) \equiv \tilde M_k(\eta), \quad k = 1,2,\ldots,n. \quad (2.4.4)$$

Simple calculations then yield the following[2a,3a]

THEOREM 2.4.1: A necessary and sufficient condition for a Newtonian system in the form (here referred to as the kinematical form)

$$\ddot q - f(t,q,\dot q) = 0, \quad k = 1,2,\ldots,n, \quad f^k \in C^2(R), \quad (2.4.5)$$

to be selfadjoint in a region R of points $(t,q,\dot q)$ is that the acting forces are linear in the velocities, i.e., the system is of the form

$$\ddot q_k - \rho_{ki}(t,q)\dot q^i - \sigma_k(t,q) = 0, \quad (2.4.6)$$

and all the following conditions of selfadjointness

$$\rho_{ij} + \rho_{ji} = 0, \quad (2.4.7a)$$

$$\frac{\partial \rho_{ij}}{\partial q^k} + \frac{\partial \rho_{jk}}{\partial q^i} + \frac{\partial \rho_{ki}}{\partial q^j} = 0, \quad (2.4.7b)$$

$$\frac{\partial \rho_{ij}}{\partial t} = \frac{\partial \sigma_i}{\partial q^j} - \frac{\partial \sigma_j}{\partial q^i}, \quad (2.4.7c)$$

are identically verified in the subregion $R' \in R$ of points (t,q).

The notion of region used hereon is that of an open and connected set. In practice, it can be restricted to a (regular) point of the variables and its neighborhood. In conclusion, the physically relevant aspect of the above theorem is that in order for Newtonian systems as originating from Newton's second law, e.g., for the unconstrained case

$$m_k\ddot x_{ka} - F_{ka}(t,\underline{x},\underline{\dot x}) = 0, \quad k = 1,2,\ldots,N, \quad a = x,y,z, \quad n = 3N, \quad (2.4.8)$$

to be variationally selfadjoint, the acting forces must be utmost linear in the velocities and then satisfy the conditions of selfadjointness. Predictably, all forces derivable from a potential and, most notably, the Lorentz force, satisfy Theorem 2.4.1. As a matter of fact, after trivial implementation to the Minkoski space (see in this respect ref.[5a]) the relativistic generalization of Newtonian forces derivable from a potential and, again most importantly, the Lorentz force, satisfy a "formally equivalent" theorem, that is, the conditions of variational selfadjointness.

We shall therefore say that the acting Newtonian forces are self-adjoint when they satisfy Theorem 2.4.1. Similarly, we shall say that the

Newtonian forces are nonselfadjoint when either nonlinear in the velocities or violate some of conditions (2.4.7).

Clearly, the condition of linearity in the velocities is highly restrictive for the objectives of this paper. This restriction can be lifted by passing from the kinematical form (2.4.5) to an equivalent general form induced by a class C^∞ and regular matrix of factor functions. And indeed, a simple reformulation of the procedure to derive Theorem 2.4.1 yields the following [10f]

THEOREM 2.4.2: A necessary and sufficient condition for a Newtonian system in the form (here referred to as the general form in configuration space)

$$A_{ki}(t,q,\dot q)\ddot q^i + B_k(t,q,\dot q) = 0, \quad k=1,2,\dots,n, \quad A_{ki}, B \in C^2(R), \tag{2.4.9}$$

to be selfadjoint in a region R of points $(t, q, \dot q)$ is that all the following conditions

$$A_{ij} = A_{ji}, \quad \frac{\partial A_{ik}}{\partial \dot q^j} = \frac{\partial A_{jk}}{\partial \dot q^i}, \tag{2.4.10a}$$

$$\frac{\partial B_i}{\partial \dot q^j} + \frac{\partial B_j}{\partial \dot q^i} = 2\left\{\frac{\partial}{\partial t} + \dot q^k \frac{\partial}{\partial q^k}\right\} A_{ij}, \tag{2.4.10b}$$

$$\frac{\partial B_i}{\partial q^j} - \frac{\partial B_j}{\partial q^i} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot q^k \frac{\partial}{\partial q^k}\right\}\left(\frac{\partial B_i}{\partial \dot q^j} - \frac{\partial B_j}{\partial \dot q^i}\right), \tag{2.4.10c}$$

are identically verified in R.

Conditions of selfadjointness (2.4.10) now clearly admit a nonlinear dependence in the velocities (as well as the coordinates). Notice that the regularity condition on the equations of motion implies that the functional determinant is nonnull in R (except a finite number of isolated zeros), i.e., $|A_k^{\,l}|(R) \neq 0$.

In conclusion, the variational approach to selfadjointness results in a set of conditions on the $A_k^{\,l}$ and B_k terms of the equations motion which must be identically verified along any admissible path (i.e., trajectory in q-space which possesses the same continuity properties of the solution) or, more empirically, as functions. The important point is that the identification whether a system is selfadjoint or not does not demand the knowledge of a solution. This is crucial for our program. And indeed, if the methods for, say, the computation of a Lagrangian demand the prior knowledge of a solution, they would be of little practical significance.

Simple examples of selfadjoint systems are given by

$$m\ddot r + k r = 0, \tag{2.4.11a}$$

$$m_k \ddot r_k + \frac{\partial V}{\partial r^k} = 0, \tag{2.4.12b}$$

$$\begin{cases} (\dot q_1 + 2\dot q_2)\ddot q_1 + 2(\dot q_1 + \dot q_2)\ddot q_2 + q_2\dot q_2 - q_1\dot q_2 + q_1 q_2 + \frac{1}{2} q_2^2 = 0, \\ 2(\dot q_1 + \dot q_2)\ddot q_1 + (\dot q_2 + 2\dot q_1)\ddot q_2 + q_1\dot q_1 - q_2\dot q_1 + q_1 q_2 + \frac{1}{2} q_1^2 = 0, \end{cases} \tag{2.4.12c}$$

Equally simple examples of nonselfadjoint systems are given by

$$\frac{1}{r}(m\ddot r + k) = 0, \quad r \neq 0, \tag{2.4.13a}$$

$$\ddot r_k + \frac{1}{m_k}\frac{\partial V}{\partial r^k} = 0 \quad (\text{No Summ.}), \quad m_1 \neq m_2 \neq m_3 \dots, \quad \frac{\partial^2 V}{\partial r^{ka}\partial r^{jb}} \neq 0, \tag{2.4.13b}$$

$$\begin{cases} 2(\dot q_1 + \dot q_2)\ddot q_1 + (\dot q_2 + 2\dot q_1)\ddot q_2 + q_1\dot q_1 - q_2\dot q_1 + q_1 q_2 + \frac{1}{2} q_1^2 = 0, \\ (\dot q_1 + 2\dot q_2)\ddot q_1 + 2(\dot q_1 + \dot q_2)\ddot q_2 + q_2\dot q_2 - q_2\dot q_1 + q_1 q_2 + \frac{1}{2} q_2^2 = 0. \end{cases} \tag{2.4.13c}$$

The reader is here urged to verify that the mere division of Eqs. (2.4.12b) by the mass is sufficient to break the selfadjointness. As we shall see, this means that a Lagrangian for the ordered direct representation of the system exist in its "natural" form (2.4.12b) as derived from Newton's second law, and not in the equivalent form (2.4.13b). Equally intriguing is the fact that the simple permutation of the ordering in the transition from Eqs. (2.4.12c) to their equivalent form (2.4.13c) is sufficient to break the selfadjointness of the system. This is the reason, as we shall see better later on, for the necessity of the use of the concept of ordering in the notion of analytic representation. The interested reader is here urged to work out other cases. Notice that Theorem 2.4.1 trivially extends to the equations of motion in the "natural form" (2.4.8), i.e., the multiplication of the acceleration by the mass terms leaves the conditions of selfadjointness unaffected. This is the reason why we have used Theorem 2.4.1 for the definition of the notion of selfadjointness (or nonselfadjointness) for Newtonian forces (rather than systems). Equivalently, the reader can reach the same results by using the context of Theorem 2.4.2 with the A-terms substituted with the mass tensor.

TABLE 2.5: THE FUNDAMENTAL ANALYTIC THEOREMS OF THE INVERSE PROBLEM

The significance of the variational approach to selfadjointness for our program is expressed by the following property which will have a significant impact at virtually all levels of our analysis.

THEOREM 2.5.1: Lagrange's equations in class C^4 and regular Lagrangians are variationally selfadjoint.

The proof of the property can be conducted in a number of ways. First, one can compute the variational forms of Lagrange's equations

$$J_k(\eta) = \frac{1}{w}\delta^{(1)} L_k(\eta) = \frac{d}{dt}\frac{\partial J}{\partial \dot{\eta}^k} - \frac{\partial J}{\partial \eta^k}, \quad \delta q = \eta w, \; w \approx 0, \tag{2.5.1a}$$

$$J = \frac{1}{2}\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\dot{\eta}^i \dot{\eta}^j + 2\frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{\eta}^i \eta^j + \frac{\partial^2 L}{\partial q^i \partial q^j}\eta^i \eta^j\right), \tag{2.5.1b}$$

and then see that they coincide with their adjoint system computed via Eqs. (2.4.2). Equivalently, one can see that Eqs. (2.1) satisfy all the conditions of Theorem 2.4.2. The conditions that the Lagrangians be (at least) of class C^4 is introduced to ensure the continuity of the fourth-order derivatives appearing in the adjoint system and it is a customary condition of the Calculus of Variations. The case when a Lagrangian satisfies weaker continuity conditions will be ignored because inessential for the objective of this paper. Theorem 2.5.1 extends to Lagrangians which are degenerate (also sometimes called nonstandard or singular), that is, when the Hessian

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \equiv 0 \tag{2.5.2}$$

is identically null as a function, although the methodological context is now considerably more involved because it demands the reformulation of the variational approach to selfadjointness on the hypersurface of the subsidiary constraints which are implicit in the degeneracy property. This aspect too is inessential for our objectives and it will be ignored.

The reader should keep in mind that the property expressed by Theorem 2.5.1 goes back to C. G. JACOBI[10a], as indicated in Table 2.2. And indeed, the equations of variation of Lagrange's equations, $J_k(\eta) = 0$, (rather than the variational forms $J_k(\eta)$) are customarily called Jacobi's equations in the literature of the Calculus of Variations.[13] Notice that while the former equations are generally nonlinear in q^k and $\dot{q}^k$, the associated Jacobi's equations are always linear in η^k and $\dot{\eta}^k$ (both equations are always linear in the second-order derivatives). As a result, while the former equations are generally of quite difficult solution (as typical of nonlinear equations), the latter can always be solved with conventional techniques. Conceivably, the joint use of Lagrange's equations and their associated Jacobi's equations could be useful for the study of nonlinear systems, although I am not aware of studies along this line.

The first important consequence of the selfadjointness property of Lagrange's equations for "all" Lagrangians (of the considered class) is expressed by the following theorem which has been called in ref.[2a] the Fundamental Analytic Theorem for configuration space formulations.

THEOREM 2.5.2: A necessary and sufficient condition for a quasilinear system of second-order ordinary differential equations which is well-defined, of (at least) class C^2 and regular in a star-shaped region R* of its variables to admit an ordered direct analytic representation in terms of Lagrange's equations (without external terms) in first-order Lagrangians, is that the system is selfadjoint in R*.

The content of this theorem was, in the final analysis, HELMHOLTZ's intuition. The conditions of selfadjointness of the equations of motion result to be not only necessary, from the selfadjointness of Lagrange's equations, but also sufficient. Under the conditions of the theorem, we shall therefore write

$$\left[L_k(q)\right]^{C^2,R}_{SA} \equiv \left[A_{ki}\ddot{q}^i + B_k\right]^{C^2,R}_{SA}, \quad R = \text{Regular}, \tag{2.5.3}$$

$$k = 1, 2, \ldots, n,$$

where SA stands for selfadjointness here interpreted as a property of the left-hand-side and as a condition for the right-hand-side.

Theorem 2.5.2 is presented in the Newtonian limit of the field theoretical proof I worked out in ref.[3b]. It contains a number of restrictions which are customarily ignored in previous treatments. As such, they deserve a brief comment. The minimal continuity conditions (the system is of at least class C^2) corresponds to the continuity property $L \in C^4$. It can be reduced by using canonical formulations, but in any case it is inessential for this paper (for relativity considerations all systems will be assumed to be of class C^∞ in order to be able to incorporate geometrical methods). The condition of regularity is nontrivial. And indeed, the extension of Theorem 2.5.2 to degenerate systems is expected to exist, but its explicit proof is expected to be considerably more involved, as typical of all systems with (generally nonintegrable) subsidiary constraints. In any case the condition of regularity is introduced on precautional grounds, in the sense that the removal of the condition of regularity should be performed after an explicit proof has appeared in the literature. The condition of ordering has a more subtle meaning. If it is not introduced in the notion of analytic representation, one would arrive at the conclusion that, say, the nonselfadjoint system (2.4.13c) admits a direct analytic representation (that is, the two members of Eqs. (2.5.3) would be identified as systems, rather than equation per equation). The most important restriction which I have introduced in Theorem 2.5.2 is that the systems are well behaved in a star-shaped (rather than an ordinary) region R*. This means that they must be well behaved for all values $t' = t$, $q' = \tau q$ and $\dot{q}' = \tau \dot{q}$, with $0 \leq \tau \leq 1$,

and for $(t, q, \dot{q}) \in R$. This restriction is typical of the central methodological tool used in the proof, the converse of the Poincaré lemma, as formulated within the context of the calculus of differential form (see, for instance, ref.[14a]). The practical meaning of this condition will be commented below in this table. Its removal is a rather delicate problem which is left to the interested researcher.

A first significance of the use of the converse of the Poincaré lemma in the proof of Theorem 2.5.2 is that it actually allows the computation of a solution, that is a Lagrangian, under the given integrability conditions. We reach in this way the third fundamental analytic theorem of the Inverse Problem, which can be formulated as follows.

THEOREM 2.5.3:[22] A Lagrangian for the ordered direct analytic representation of Newtonian systems

$$\left[A_{ki}(t,q,\dot{q})\ddot{q}^i + B_k(t,q,\dot{q}) \right]_{SA}^{C^2,R} = 0, \qquad (2.5.4)$$

which are well-defined, of (at least) class C^2, regular and selfadjoint in a star-shaped region of the points $(t, q, \dot{q})$ is given by

$$L(t,q,\dot{q}) = K(t,q,\dot{q}) + D_k(t,q)\dot{q} + C(t,q), \qquad (2.5.5)$$

where the (n+2) functions K, D_k and C are a solution of the quasilinear, overdetermined, system of partial differential equations

$$\frac{\partial^2 K}{\partial \dot{q}^i \partial \dot{q}^j} = A_{ij}, \qquad (2.5.6a)$$

$$\frac{\partial D_i}{\partial q^j} - \frac{\partial D_j}{\partial q^i} = \frac{1}{2}\left(\frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i}\right) + \left(\frac{\partial^2}{\partial q^i \partial \dot{q}^j} - \frac{\partial^2 K}{\partial \dot{q}^i \partial q^j}\right) = Z_{ij}(t,q), \qquad (2.5.6b)$$

$$\frac{\partial C}{\partial q^i} = \frac{\partial D_i}{\partial t} - B_i - \frac{\partial K}{\partial q^i} + \frac{\partial^2 K}{\partial \dot{q}^i \partial t} + \left[\frac{\partial^2 K}{\partial q^i \partial \dot{q}^k} + \frac{1}{2}\left(\frac{\partial B_i}{\partial \dot{q}^k} - \frac{\partial B_k}{\partial \dot{q}^i}\right)\right]\dot{q}^k = W_i(t,q), \qquad (2.5.6c)$$

given by

$$K = \dot{q}^i \int_0^1 d\tau' \left\{ \left[\int_0^1 d\tau\, A_{ij}(t,q,\tau\dot{q}) \right] \dot{q}^j \right\}(t,q,\tau'\dot{q}), \qquad (2.5.7a)$$

$$D_i = \left[\int_0^1 d\tau\, \tau\, Z_{ij}(t,\tau q) \right] q^j, \qquad (2.5.7b)$$

$$C = q^i \int_0^1 d\tau\, W_i(t,\tau q). \qquad (2.5.7c)$$

The practical meaning of the restriction to a star-shaped region can now be identified. In essence, it ensures the existence of the integrals (2.5.7), and, thus, a Lagrangian. Notice that there is no need to verify the consistency of the overdetermined system (2.5.6) under the conditions of selfadjointness in R*. As a matter of fact, the necessity and sufficiency of the conditions of selfadjointness are precisely centered on the proof that such a system is consistent. In turn, this illustrates the nontriviality of the proof of Theorem 2.5.2. Notice that Eqs. (2.5.7) must be computed in sequential order. Notice also that the method of Theorem 2.5.3 appears to be computerizable. This method was introduced in ref.[3b] for the field theoretical case and then its Newtonian reduction was presented in ref.[2a].

As an example, the reader is urged to verify that the method of Theorem 2.5.3 applies for system (2.4.12c), yielding as a Lagrangian

$$L = \frac{1}{6}(\dot{q}_1^3 + \dot{q}_2^3) + \dot{q}_1^2\dot{q}_2 + \dot{q}_1\dot{q}_2^2 + \frac{1}{3}(q_2^2\dot{q}_1 + q_1^2\dot{q}_2) - \frac{1}{3}(\dot{q}_1 + \dot{q}_2)q_1q_2 - \frac{1}{2}(q_1^2q_2 + q_1q_2^2). \qquad (2.5.8)$$

TABLE 2.6: THE INDIRECT LAGRANGIAN REPRESENTATIONS

The fundamental analytic theorems of the Inverse Problem have little pratical significance in the given form, particularly for the objective of this paper, because the Newtonian systems with forces not derivable from a potential are always nonselfadjoint as derived from Newton's second law (that is, ma – F = 0). As a result, a Lagrangian for their analytic representation according to Theorems 2.5.2 and 2.5.3 does not exist. More generally, the nonselfadjointness of a quasilinear system of ordinary differential equations is the rule and its selfadjointness is the exception. Perhaps, this is a reason why active studies on the Inverse Problem were virtually abandoned since the early 30's, as indicated in Table 2.2.

Clearly, to reach a methodology of practical usefulness for the problem of the relativity laws of Newtonian systems with forces not derivable from a potential, I had to confront this issue.[3c,3d] The results of my studies were first presented for the field theoretical profile in ref. and then the Newtonian reduction was worked out in ref.[2b]. Here is a summary.

Theorems 2.5.2 and 2.5.3 are formulated for "direct" analytic representations. Clearly, a first broadening of the representational capability of these theorems can be achieved by removing this restriction and considering instead the broader case of "indirect" representations.

This immediately yields the following [3b,2b]

THEOREM 2.6.1: A necessary and sufficient condition for Newtonian systems

$$\left[A_{ki}(t,q,\dot q)\ddot q^{i} + B_k(t,q,\dot q)\right]^{C^2,R}_{NSA} = 0\,,\quad k=1,2,\dots,n, \qquad (2.6.1)$$

which are of at least class C^2, regular and nonselfadjoint in a region R of the variables, to admit the ordered indirect analytic representation

$$\left[L_k(q)\right]^{C^2,R}_{SA} \equiv \left[h_k{}^{i}\left(A_{ij}\ddot q^{j} + B_i\right)^{C^2,R}_{NSA}\right]^{C^2,R}_{SA}, \qquad (2.6.2)$$

is that all the following conditions of selfadjointness in the equivalent system

$$A^*_{ij} = A^*_{ji}\,,\quad \frac{\partial A^*_{ik}}{\partial \dot q^{j}} = \frac{\partial A^*_{jk}}{\partial \dot q^{i}}\,,\quad (A^*) = (h)(A)\,, \qquad (2.6.3a)$$

$$\frac{\partial B^*_i}{\partial \dot q^{j}} + \frac{\partial B^*_j}{\partial \dot q^{i}} = 2\left\{\frac{\partial}{\partial t} + \dot q^{k}\frac{\partial}{\partial q^{k}}\right\}A^*_{ij}\,,\quad (B^*) = (h)(B)\,, \qquad (2.6.3b)$$

$$\frac{\partial B^*_i}{\partial q^{j}} - \frac{\partial B^*_j}{\partial q^{i}} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot q^{k}\frac{\partial}{\partial q^{k}}\right\}\left(\frac{\partial B^*_i}{\partial \dot q^{j}} - \frac{\partial B^*_j}{\partial \dot q^{i}}\right), \qquad (2.6.3c)$$

are verified and such system is well behaved in a star-shaped extension R* of R. In this case a Lagrangian is given by

$$L^*(t,q,\dot q) = K^*(t,q,\dot q) + D^*_k(t,q)\dot q^{k} + C^*(t,q), \qquad (2.6.4a)$$

$$K^* = \dot q^{i}\int_0^1 d\tau'\left\{\left[\int_0^1 d\tau\, A^*_{ij}(t,q,\tau\dot q)\right]\dot q^{j}\right\}(t,q,\tau'\dot q), \qquad (2.6.4b)$$

$$D^*_k = \left[\int_0^1 d\tau\, \tau Z^*_{ij}(t,\tau q)\right]q^{j}, \qquad (2.6.4c)$$

$$C^* = q^{i}\int_0^1 d\tau\, W^*_i(t,\tau q)\,, \qquad (2.6.4d)$$

$$Z^*_{ij} = \frac{1}{2}\left(\frac{\partial B^*_i}{\partial \dot q^{j}} - \frac{\partial B^*_j}{\partial \dot q^{i}}\right) + \left(\frac{\partial^2 K^*}{\partial q^{i}\partial \dot q^{j}} - \frac{\partial^2 K^*}{\partial \dot q^{i}\partial q^{j}}\right), \qquad (2.6.4e)$$

$$C^*_i = \frac{\partial D^*_i}{\partial t} - B^*_i - \frac{\partial K^*}{\partial q^{i}} + \frac{\partial^2 K^*}{\partial \dot q^{i}\partial t} + \left[\frac{\partial^2 K^*}{\partial q^{i}\partial \dot q^{k}} + \frac{1}{2}\left(\frac{\partial B^*_i}{\partial \dot q^{k}} - \frac{\partial B^*_k}{\partial \dot q^{i}}\right)\right]\dot q^{k}. \qquad (2.6.4f)$$

The above theorem clarly produces a nontrivial broadening of the arena of applicability of the Inverse Problem with the inclusion of genuine nonconservative systems. For instance, the use of Theorem 2.6.1 for the damped oscillator yields the indirect analytic representation

$$\left[\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q}\right]^{C^\infty,R}_{SA} \equiv \left[e^{\gamma t}\left(\ddot q + \gamma\dot q + \omega_0^2 q\right)^{C^\infty,R}_{NSA}\right]^{C^\infty,R}_{SA}, \qquad (2.6.5a)$$

$$L = e^{\gamma t}\frac{1}{2}\left(\dot q^2 - \omega_0^2 q\right), \qquad (2.6.5b)$$

and for the nonlinear, nonconservative system

$$\begin{pmatrix} \ddot q_1 + \frac{g}{2}q_1\dot q_1 + g\, q_2\dot q_1\dot q_2 - \frac{g}{2}q_1\dot q_2^{\,2} \\ \ddot q_2 + \frac{g}{2}q_2\dot q_2 + g\, q_1\dot q_1\dot q_2 - \frac{g}{2}q_2\dot q_1^{\,2} \end{pmatrix}^{C^\infty,R}_{NSA} = 0\,, \qquad (2.6.6)$$

we have

$$L = e^{\frac{g}{2}(q_1^2+q_2^2)}\,\frac{1}{2}\left(\dot q_1^{\,2} + \dot q_2^{\,2}\right). \qquad (2.6.7)$$

However, Theorem 2.6.1 is still restrictive in the sense that it applies only when a solution of the conditions of selfadjointness (2.6.3) in the unknown factor functions $h_k{}^{i}$ (with fixed $A_k{}^{i}$ and B_k terms) exists. Clearly, such a system is generally overdetermined and, as such, a solution does not necessarily exist.

We reach in this way a point which will apparently be crucial for relativity considerations. Theorem 2.6.1 essentially characterizes a class of nonconservative systems (to be better identified later on in Table 2.9) which admits an analytic representation within the coordinate system of its experimental detection. And indeed, any further extension of the representational capability of the Inverse Problem demands the use of coordinate transformations. In turn, this necessarily demands the abandonement of the (inertial) system of the experimental set up and the construction of a new system of coordinates.

At this point a second crucial aspect emerges. Suppose that the forces not derivable from a potential are such to render inconsistent the integrability conditions for the existence of a Lagrangian in the Cartesian coordinates of its experimental verification, here assumed as caraterizing an inertial system. Can (class C^2, invertible) transformations $t \to t'$ and $q^k \to q'^k$ be identified in such a way that (I) a Lagrangian exists in such new coordinate system, (II) this new system of coordinate is equally realizable with experimental setups and (III) it is inertial. As we shall see, the asnwer to problem (I) is affermative, but that to problems (II) and (III) is generally negative.

These problems can be more effectively treated within the context of canonical formulations and, as such, we shall consider them in more details later on. At this point it is sufficient to note that an ordinary (generally nonlinear) point transformation $t \to t' = t$ and $q^k \to q'^k = q'^k(q)$ is insufficient to produce a Lagrangian under the assumed condition (inconsistency of system (2.6.3) for given A and B terms, i.e., for given implicit functions). And indeed, the existence of a Lagrangian in such new system implies the existence of also a Lagrangian in the old system via an inverse transform, a part the multiplication of the (regular) Jacobian matrix. But then this matrix would represent the matrix of the solution of system (2.6.3), contrary to assumption.

In conclusion, the transformations capable of inducing a Lagrangian when the conditions of Theorem 2.6.1 are violated must produce a change in the structure of the system such that its image in the original variables leads to an inconsistent system (2.6.3). As we shall see these transformations exists and are of the type

$$t \to t'(t,q,\dot{q}), \quad q^k \to q'^k(t,q,\dot{q}), \; k=1,2,\dots,n, \qquad (2.6.8)$$

as familiar in the Calculus of variations (see, for instance, ref.[13b]), although rarely used in Analytic Mechanics. In particular, a generally nonlinear dependence in the velocities will result to be essential to produce the desired result. In turn, this implies, in general, the practical impossibility of realizing the new systems of coordinates with an experimental set up and, most importantly, they are generally noninertial.

The net result is that the class of systems whose relativity laws we are interested in, is such that they admit a Lagrangian representation in a new coordinate system which is generally noninertial and nonrealizable in experiments. This is the reason why, as indicated in Section I, we are primarily interested in the study of the relativity laws of the systems considered in the representation space of their experimental verification, and we shall leave the study of the same relativity profile in equivalent systems to the interested researcher.

The significance of the Inverse Problem as a methodological backing, however persists, as will be more transparent when considering the complementary Lie-admissible Problem of Section 3. We shall therefore continue to outline the former methodology.

TABLE 2.7: THE INDEPENDENT INVERSE PROBLEM FOR HAMILTONIAN FORMULATIONS.

One of the intriguing aspects of the Inverse Problem is that it can be equivalently formulated for Hamiltonian formulations without any prior knowledge of a Lagrangian. That is, given a Newtonian system, one can construct an equivalent system of first-order differential equations

and identify within this context the integrability conditions for the existence of a Hamiltonian.

The procedure can be summarized as follows. Consider a Newtonian system in the general form (2.6.1) and introduce arbitrary prescriptions for the characterization of new (independent) variables, say, y_k, in the form[2a]

$$G_k(t,q,\dot{q},y) = \alpha_{ki}(t,q,y)\dot{q}^i + \beta^k(t,q,y) = 0, \qquad (2.7.1)$$

which are such to admit a unique system of (single-valued) implicit functions in the velocities,

$$\dot{q}^k = g^k(t,q,y). \qquad (2.7.2)$$

The subsitution of this latter system in Eqs. (2.4.9) then yields the equivalent first-order system

$$\left[\begin{pmatrix} (h) & (h') \\ (h'') & (h''') \end{pmatrix}\begin{pmatrix} \alpha_{ki}(t,q,y)\dot{q}^i + \beta_k(t,q,y) \\ \tilde{\alpha}_k{}^i(t,q,y)\dot{y}_i + \tilde{\beta}_k(t,q,y) \end{pmatrix}\right]^{C^1_1 R} = 0, \quad \det\begin{pmatrix} h & h' \\ h'' & h''' \end{pmatrix} \neq 0, \qquad (2.7.3a)$$

$$\tilde{\alpha}_k{}^i = A_{ki}\frac{\partial g^i}{\partial y_i}, \quad \tilde{\beta}_k = A_{ki}\frac{\partial g^i}{\partial q^r}g^r + A_{ki}\frac{\partial g^i}{\partial t} + B_k(t,q,g). \qquad (2.7.4a)$$

The variational approach to selfadjointness then yields the following[2a]

THEOREM 2.7.1: A necessary and sufficient condition for the system of 2n first order ordinary differential equations

$$C_{\mu\nu}(t,a)\dot{a}^\nu + D_\mu(t,a) = 0, \quad \mu = 1,2,\dots,2n, \qquad (2.7.5a)$$

$$(C_{\mu\nu}) = \begin{pmatrix} (h) & (h') \\ (h'') & (h''') \end{pmatrix}\begin{pmatrix} (\alpha) & 0 \\ 0 & (\tilde{\alpha}) \end{pmatrix}, \quad (D_\mu) = \begin{pmatrix} (h) & (h') \\ (h'') & (h''') \end{pmatrix}\begin{pmatrix} (\beta) \\ (\tilde{\beta}) \end{pmatrix}, \quad (a^\mu) = \begin{pmatrix} q^k \\ y_k \end{pmatrix}, \qquad (2.7.5b)$$

which is of (at least) class C^1 and regular in a region R of the variables (t, a) to be selfadjoint in R, is that all the following conditions

$$C_{\mu\nu} + C_{\nu\mu} = 0, \qquad (2.7.6a)$$

$$\frac{\partial C_{\mu\nu}}{\partial a^\tau} + \frac{\partial C_{\nu\tau}}{\partial a^\mu} + \frac{\partial C_{\tau\mu}}{\partial a^\nu} = 0, \qquad (2.7.6b)$$

$$\frac{\partial C_{\mu\nu}}{\partial t} = \frac{\partial D_\mu}{\partial a^\nu} - \frac{\partial D_\nu}{\partial a^\mu}, \quad \mu,\nu = 1,2,\dots,2n \qquad (2.7.6c)$$

are identically satisfied in R.

Notice that the identification of the conditions of selfadjointness is quite simple in the space of the a-variables (via a simple application of the approach of Table 2.4). However, the formulation of the same conditions in the space of the q^k, and y_k variables is considerably more involved. The reader should also be aware that the y-variables are not necessarily canonical, i.e., of the type $y_k = \partial L/\partial \dot{q}^k$, trivially, because a Lagrangian for the representation of the original system (2.6.1) is unknown.

COROLLARY 2.7.1.A: When system (2.7.5a) is of the form

$$\omega_{\mu\nu}\dot{a}^\nu - \Xi_\mu(t,a) = 0, \qquad (2.7.7a)$$

$$(\omega_{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ 1_{n\times n} & 0_{n\times n} \end{pmatrix}, \qquad (2.7.7b)$$

the conditions of selfadjointness reduce to

$$\frac{\partial \Xi_\mu}{\partial a^\nu} - \frac{\partial \Xi_\nu}{\partial a^\mu} = 0, \quad \mu,\nu = 1,2,\dots,2n. \qquad (2.7.8)$$

And indeed, matrix (2.7.7b) is a trivial solution with constant coefficient of Eqs. (2.7.6a) and (2.7.6b). This is the first contact with the symplectic geometry. And indeed, structure (2.7.7b) is the familiar fundamental symplectic form. For more comments in this respect see Table 2.8.

The canonical equivalent of the selfadjointness property of Lagrange's equations, Theorem 2.5.1, can now be formulated as follows.[2a]

THEOREM 2.7.2: Hamilton's equations (without external terms)

$$\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu} = 0, \quad \{a^\mu\} = \{q,p\}, \quad \mu = 1,2,\dots,2n, \qquad (2.7.9)$$

in Hamiltonians of at least class C^2 are always variationally selfadjoint.

Again, the proof of this fundamental property by using the conventional way of writing Hamilton's equations

$$\dot{p}_k + \frac{\partial H}{\partial q^k} = 0, \quad \dot{q}^k - \frac{\partial H}{\partial p_k} = 0, \qquad (2.7.10)$$

is rather involved. For the form of the unified notation (2.7.9) the proof of the same property is trivial. And indeed, by using Corollary 2.7.1.A, the conditions of selfadjointness reduce to the commutativity of the second order derivatives in the a-variables which is implicit in the assumed continuity conditions.

For geometrical reasons we call equations of type (2.7.7a) covariant normal form of Newton's equations. It can be quite simply constructed as follows. Consider the original system in the (unique) kinematical form (2.4.5) and introduce the prescriptions (2.7.2). Then the notation

$$(a^\mu) = \begin{pmatrix} q^k \\ y_k \end{pmatrix}, \quad \begin{pmatrix} \Xi^\mu \end{pmatrix} = \begin{pmatrix} \tilde{g}_k \\ g^k \end{pmatrix}, \quad \dot{q} = g^k(t,q,y), \quad \tilde{g}_k = -\frac{\partial y_k}{\partial q^i}\left[f_i(t,q,g) - \frac{\partial g^i}{\partial q^r} g^r\right], \qquad (2.7.11)$$

yields a covariant normal form. (2.7.7a)

The integrability conditions for the existence of a Hamiltonian are then easily identified.[2a]

THEOREM 2.7.3: A necessary and sufficient condition for a Newtonian system in the covariant normal form (2.7.7a) which is of at least class C^1 and well-defined in a star-shaped region R* of its variables to admit the ordered direct analytic representation in terms of Hamilton's equations (2.7.9), i.e.,

$$\omega_{\mu\nu}\dot{a}^\nu - \Xi_\mu \equiv \omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}, \quad \mu = 1,2,\dots,2n, \qquad (2.7.12)$$

is that each and all the following conditions of selfadjointness

$$\frac{\partial \Xi_\mu}{\partial a^\nu} - \frac{\partial \Xi_\nu}{\partial a^\mu} = 0, \qquad (2.7.13)$$

are identically verified in R*, in which case a Hamiltonian is given by

$$H(t,a) = a^\mu \int_0^1 d\tau\, \Xi_\mu(t,\tau a). \qquad (2.7.14)$$

The proof is trivial. Conditions (2.7.13) are necessary and sufficient for the one-form $\Xi_1 = \Xi_\mu da^\mu$ to be exact, i.e.,

$$\Xi_1 = \Xi_\mu da^\mu = d\,\Xi_0 = dH = \frac{\partial H}{\partial a^\mu} da^\mu. \qquad (2.7.15)$$

The simplicity of the proof of the Theorem 2.7.3 (called in ref.[2a] the Fundamental Analytic Theorem for canonical formulations) should be compared with the rather involved nature of the proof for the configuration space case, i.e., Theorem 2.5.2 (refs.[3b]).

In practice, the g^k fuctions of presciptions (2.7.2) can be interpreted as the unknowns, and the conditions of selfadjointness (2.7.8) can be used to attempt a solution in these functions. If a solution exists, the y_k variables are canonical (i.e., the a-variables span a phase space). As a simple example, the particle with damping force

$$(\ddot{q} + \gamma \dot{q})^{C^\infty, R}_{NSA} = 0. \qquad (2.7.16)$$

is characterized by a nonselfadjoint one-dimensional, second-order equation. As such, Theorem 2.5.2 does not apply. Instead of using Theorem 2.6.1, one can use the independent approach to the computation of a Hamiltonian indicated in this section. Eqs. (2.7.8) in the unknown function g(t, q, y) yield, as a solution, the form

$$g = e^{y} \quad , \tilde{g} = \gamma \quad , y = p = \text{canon.} \qquad (2.7.17)$$

A Hamiltonian, via Eq. (2.7.14) is then given by

$$H(t,a) = a^1 \int_0^1 d\tau\, \Xi_1(\tau a) + a^2 \int_0^1 d\tau\, \Xi_2(\tau a) = e^{p} + \gamma q - 1. \qquad (2.7.18)$$

A Lagrangian, if needed, can then be computed via an (inverse) Legendre transform by reaching the expression

$$L = \dot{q} \ln \dot{q} - \gamma q. \qquad (2.7.19)$$

The aspect which is relevant for relativity considerations is that the system of partial differential equations (2.7.8) in the unknown functions g^k of prescriptions (2.7.2) is generally overdetermined and, as such, a solution does not necessarily exists. This was, after all expected from the content of Table 2.6. And indeed, this property is the canonical counterpart of the "lack of universality" of Theorem 2.6.1 on indirect Lagrangian representations. The emphasis is however different. The "lack of universality" of Theorem 2.7.3 for Newtonian systems with forces not derivable from a potential implies the inability, for the systems considered, of introducing a central methodological tool of the Galilei relativity, the canonical formalism, in the inertial system of Cartesian coordinates of the experimental verification and their canonically conjugate momenta.

TABLE 2.8: ANALYTIC, ALGEBRAIC AND GEOMETRICAL SIGNIFICANCE OF THE CONDITIONS OF VARIATIONAL SELFADJOINTNESS. To make further progress, we must reinspect the conditions of selfadjointness for general covariant forms, Eqs. (2.7.6) and identify their methodological significance. It is advisable to consider first autonomous systems, that is, systems without an explicit dependence on time, and inspect conditions (2.7.7) within the context of the following three methodological aspects.

I. Analytic significance of the conditions of variational selfadjointness. For the case of autonomous systems the equations at hand are given by

$$[\Omega_{\mu\nu}(a)\dot{a}^\nu - \Gamma_\mu(a)]^{C^\infty, R} = 0, \quad \mu = 1, 2, \ldots, 2n, \qquad (2.8.1)$$

and their conditions of selfadjointness reduce to

$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0, \qquad (2.8.2a)$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial a^\tau} + \frac{\partial \Omega_{\nu\tau}}{\partial a^\mu} + \frac{\partial \Omega_{\tau\mu}}{\partial a^\nu} = 0, \qquad (2.8.2b)$$

$$\frac{\partial \Gamma_\mu}{\partial a^\nu} - \frac{\partial \Gamma_\nu}{\partial a^\mu} = 0. \qquad (2.8.2c)$$

It is easy to see that a solution of Eqs. (2.8.2a) and (2.8.2b) can always be written

$$\Omega_{\mu\nu} = \frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial R_\nu}{\partial a^\mu}. \qquad (2.8.3)$$

As a result, the conditions of variational selfadjointness are the integrability conditions for the existence of an ordered direct analytic representation in terms of Birkhoff's equations (rather than Hamilton's equations), i.e.,

$$\left[\left(\frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial R_\nu}{\partial a^\mu}\right)\dot{a}^\nu - \frac{\partial H^B}{\partial a^\mu}\right]^{C^\infty, R}_{SA} \equiv [\Omega_{\mu\nu}\dot{a}^\nu - \Gamma_\mu]^{C^\infty, R}_{SA}, \qquad (2.8.4a)$$

$$H^B = a^\mu \int_0^1 d\tau\, \Gamma_\mu(\tau a). \qquad (2.8.4b)$$

Only as a particular case under the limit

$$R_\mu(a) \to \tfrac{1}{2}\omega_{\mu\nu} a^\nu, \qquad (2.8.5)$$

the conditions of selfadjointness ensure the existence of an analytic representation in terms of Hamilton's equations.

Regrettably, Birkhoff's equations have remained largely ignored in the literature of Analytic Mechanics since their identification in ref.[15d], with only few exceptions known to me such as refs.[15b-15f]. One reason might be due to the fact that they are actually inessential in the sense that, by using geometrical arguments (see below in this table) they can always be reduced to a Hamiltonian form. As a result, they do not play a fundamental role within the context of the methodology of the Inverse Problem. However, the study of nonconservative systems brings into focus a number of aspects which are ignorable for conservative systems. In particular, Birkhoff's equations will emerge as possessing a precise methodological function within the context of the Lie-Admissible Problem (see Table 3.4). As a result, they emerge as possessing a significant role for our relativity treatment of nonconservative systems.

The aspect which must be here stressed in that Birkhoff's equations are essentially equivalent, on methodological grounds, to Hamilton's equations, even though there exists a

number of predictable technical differences.

First of all Birkhoff's equations are derivable via transforms of second-order equations in full analogy with the derivation of Hamilton's equations via the Legendre transform. And indeed, the general method of transforming second-order into first-order systems outlined in Table 2.7 yields precisely Birkhoff's equations under the conditions of selfadjointness. In particular, this generalized transform can be performed without any necessary knowledge of a Lagrangian (if a Lagrangian is known, the same method can be applied to Lagrange's equations by turning them into Birkhoff's equations). The main difference between Hamilton's and Birkhoff's equations is due to the fact that the variables of the former span a phase space while this is not necessarily the case for the variables of the latter. This is the same as saying that the generalized transform of Table 2.7 is generally noncanonical (although there exists a class of Birkhoff's equations with R-quantities other than those of Eqs. (2.8.5) which characterizes a sphace space, see ref.[5b]). This might be considered as a drawback by some. My personal attitude is that Birkhoff's equations are potentially significant precisely because they do not span a phase space (see the problem of computation of first integrals of Table 2.12).

Secondly, Birkhoff's equations possess a dynamical meaning fully parallel to that of Hamilton's equations. This can be seen as follows. Both equations can be written in the contravariant forms

$$\dot{a}^\mu - \omega^{\mu\nu}\frac{\partial H(a)}{\partial a^\nu} = 0 \ , \quad (\omega^{\mu\nu}) = (\omega_{\mu\nu})^{-1}, \tag{2.8.6a}$$

$$\dot{a}^\nu - \Omega^{\mu\nu}(a)\frac{\partial H^B(a)}{\partial a^\nu} = 0 \ , \quad (\Omega^{\mu\nu}) = (\Omega_{\mu\nu})^{-1}, \tag{2.8.6b}$$

and, thus, both equations yield a fully defined time evolution law, i.e.,

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu}\dot{a}^\mu = \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} = [A, H]_{(a)} \tag{2.8.7a}$$
$$= \frac{\partial A}{\partial q^k}\frac{\partial H}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial H}{\partial q^k} \ ,$$

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu}\dot{a}^\mu = \frac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}(a)\frac{\partial H}{\partial a^\nu} = [A, H]^*_{(a)} \ . \tag{2.8.7b}$$

The main difference is that the brackets of the time evolution law of Hamilton's equations are the conventional Poisson brackets, while those of Birkhoff's equations are the so-called generalized Poisson brackets. The important point is that both brackets satisfy the Lie algebra identities, i.e., verify the laws

$$A \circ B + B \circ A = 0, \tag{2.8.8a}$$

$$(A \circ B) \circ C + (B \circ C) \circ A + (C \circ A) \circ B = 0, \tag{2.8.8b}$$

where $A \circ B = [A, B]_{(a)}$ or $[A, B]^*_{(a)}$ (for details see below in this table). As a matter of fact, this property is so relevant that Birkhoff's equations can be interpreted as a Lie covering of Hamilton's equations, that is, a generalization of the latter which preserves the underlying Lie algebra structure via the brackets of the time evolution law. On similar grounds, the generalized transform of second-order into selfadjoint first-order systems (Table 2.7) can be interpreted as a Lie covering of the Legendre transform, that is, a generalization of the latter which preserves the underlying Lie algebra structure. To restate these findings in the language of the Inverse Problem, Birkhoff's equations are the most general form of selfadjoint, first-order, regular, analytic equations, where the regularity property is expressed by the nondegeneracy of the matrix $(\Omega^{\mu\nu})$.

Thirdly, Birkhoff's equations are derivable from a variational principle in a way fully parallel to that of Hamilton's equations, although, in a predictable generalized way. This is a typical "casework" for the Inverse Problem. The solution is straightforward and can be written[5b]

$$\delta A^g = \delta\int_{t_1}^{t_2} dt\, F(a,\dot{a}) = -\int_{t_1}^{t_2} dt \left(\frac{d}{dt}\frac{\partial F}{\partial \dot{a}^\mu} - \frac{\partial F}{\partial a^\mu}\right)\delta a^\mu \tag{2.8.9a}$$
$$\equiv -\int_{t_1}^{t_2} dt \left[\left(\frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial R_\nu}{\partial a^\mu}\right)\dot{a}^\nu - \frac{\partial H^B}{\partial a^\mu}\right]\delta a^\mu = 0,$$

$$F(a,\dot{a}) = -a^\mu\left[\int_0^1 d\tau\, \tau\, \Omega_{\mu\nu}(\tau a)\right]\dot{a}^\nu + H^B. \tag{2.8.9b}$$

The Hamiltonian limit is significant because different than the conventional Hamilton's principle for phase space formulations, i.e.,

$$\delta A = \delta\int_{t_1}^{t_2} dt\,[p_k\dot{q}^k - H] = \int_{t_1}^{t_2} dt\left\{\left[\dot{q}^k - \frac{\partial H}{\partial p_k}\right]\delta p_k - \left[\dot{p}_k + \frac{\partial H}{\partial q^k}\right]\delta q^k\right\} = 0. \tag{2.8.10}$$

And indeed, under limit (2.8.5) we have[2a, 2b, 5a, 5b]

$$A^g = \int_{t_1}^{t_2} dt\, F(a,\dot{a}) = \int_{t_1}^{t_2} dt\left[-\frac{1}{2}a^\mu\omega_{\mu\nu}\dot{a}^\nu + H\right], \tag{2.8.11a}$$

$$\delta A^g = -\int_{t_1}^{t_2} dt\left(\frac{d}{dt}\frac{\partial F}{\partial \dot{a}^\mu} - \frac{\partial F}{\partial a^\mu}\right)\delta a^\mu = -\int_{t_1}^{t_2} dt\left(\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right)\delta a^\mu = 0. \tag{2.8.11b}$$

Notice that variational principles (2.8.10) and (2.8.11b) are equivalent because they yield the same equations, Hamilton's equations, only written in different notations. However, algorithms (2.8.10) and (2.8.11b) are generally nonequivalent as variational problems, because the former belongs to the class of variational problems with fixed end points, while the latter is a (subcase) of the so-called Problem of Bolza of the Calculus of Variations[13a]. The point is that principle (2.8.11b) appears to be preferable over principle (2.8.10), particularly for relativity considerations, because it is directly expressed in terms of the fundamental symplectic structure, while this structure is undefinable for principle (2.8.10). In turn, this has a number of consequences (e.g., for the transformation theory) of crucial relevance for relativity problems.

To summarize, the analytic significance of the conditions of variational selfadjointness for first-order sistems is that of identifying a Lie algebra preserving covering of Hamilton's equations. As a particular case for the covariant normal forms (2.7.a) they constitute the integrability conditions for the existence of a Hamiltonian.

II. Algebraic significance of the conditions of variational selfadjointness. For the brackets

$$[A,B]^*_{(a)} = \frac{\partial A}{\partial a^\mu}\,\Omega^{\mu\nu}(a)\,\frac{\partial B}{\partial a^\nu}\,, \qquad (2.8.12)$$

the Lie algebra identities, Eqs. (2.8.8), are equivalent to the conditions

$$\Omega^{\mu\nu} + \Omega^{\nu\mu} = 0, \qquad (2.8.13a)$$

$$\Omega^{\mu\rho}\frac{\partial\Omega^{\nu\tau}}{\partial a^\rho} + \Omega^{\nu\rho}\frac{\partial\Omega^{\tau\mu}}{\partial a^\rho} + \Omega^{\tau\rho}\frac{\partial\Omega^{\mu\nu}}{\partial a^\rho} = 0, \qquad (2.8.13b)$$

for all tensors $\Omega^{\mu\nu}$ with a nontrivial dependence in the a-variables (i.e., other than constants).

It is a simple exercise to prove that Eqs. (2.8.13) are equivalent to conditions of selfadjointness (2.8.2a) and (2.8.2b). This property was apparently identified for the first time (independently from the context of the Inverse Problem) by W. PAULI[16a,16b]. Of course, this equivalence crucially depends on the regularity of the matrix $(\Omega_{\mu\nu})$ and, more specifically, on the existence (and regularity) of the inverse matrix

$$(\Omega^{\mu\nu}):\ \Omega^{\mu\rho}\Omega_{\rho\nu} = \Omega_{\nu\rho}\Omega^{\rho\mu} = \delta^\nu_\mu\,. \qquad (2.8.14)$$

On equivalent grounds, this property can be seen as follows. It is known that the covariant form $(\omega_{\mu\nu}) = (\omega^{\mu\nu})^{-1}$ characterizes the "inverse" of the conventional Poisson brackets, i.e., the conventional Lagrange brackets

$$\{A,B\}_{(a)} = \frac{\partial a^\mu}{\partial A}\,\omega_{\mu\nu}\,\frac{\partial a^\nu}{\partial B} = \frac{\partial p_k}{\partial A}\frac{\partial q^k}{\partial B} - \frac{\partial q^k}{\partial A}\frac{\partial p_k}{\partial B}\,, \qquad (2.8.15)$$

which satisfy the identities

$$\{A,B\} + \{B,A\} = 0, \qquad (2.8.16a)$$

$$\frac{\partial}{\partial A}\{B,C\} + \frac{\partial}{\partial B}\{C,A\} + \frac{\partial}{\partial C}\{A,B\} = 0. \qquad (2.8.16b)$$

On similar grounds, Birkhoff's tensor $\Omega_{\mu\nu}$, interpreted as the covariant form of a general Lie-tensor $\Omega^{\mu\nu}$, i.e., according to Eq. (2.8.6b), characterizes the "inverse" of the generalized Poisson brackets, i.e., the generalized Lagrange brackets

$$\{A,B\}^*_{(a)} = \frac{\partial a^\mu}{\partial A}\,\Omega_{\mu\nu}(a)\,\frac{\partial a^\nu}{\partial B}\,. \qquad (2.8.17)$$

The point is that these generalized brackets preserve identities (2.8.16). The analogy is then completed by the properties

$$\sum_{k=1}^{m} [f_i, f_k]\{f_k, f_j\} = \delta_{ij}\,, \qquad (2.8.18a)$$

$$\sum_{k=1}^{m} [f_i, f_k]^*\{f_k, f_j\}^* = \delta_{ij}\,, \qquad (2.8.18b)$$

which are identically verified by the conventional and the generalized brackets.

In conclusion, the algebraic significance of the conditions of variational selfadjointness for first-order systems is that their subset (2.8.2a) and (2.8.2b) is equivalent to the Lie algebra identities (under the tacit regularity condition). In particular, they recover the conventional Poisson brackets for the particular case $\Omega^{\mu\nu} = \omega^{\mu\nu}$. This confirms the Lie covering character of Birkhoff's equations over Hamilton's equations.

III. Geometrical significance of the conditions of variational selfadjointness. Eqs. (2.8.1) can be interpreted as characterizing a vector field on a (Hausdorff, second countable, ∞-differentiable, 2n-dimensional) Manifold $M(a,\Omega_2)$ with local coordinates a^μ and structure

$$\Omega_2 = \Omega_{\mu\nu}(a)\,da^\mu \wedge da^\nu. \qquad (2.8.19)$$

Under the conditions of regularity, it is easy to see that the conditions of selfadjointness (2.8.2a) and (2.8.2b) guarantee that this two-form is closed and, thus, $M(a,\Omega_2)$ is a symplectic manifold[2a]. And indeed, the closure conditions can be written

$$\delta^{\nu_1\nu_2\nu_3}_{\mu_1\mu_2\mu_3}\frac{\partial\Omega_{\nu_1\nu_2}}{\partial a^{\nu_3}} = 0, \qquad \delta^{\nu_1\nu_2\nu_3}_{\mu_1\mu_2\mu_3} = \begin{vmatrix} \delta^{\nu_1}_{\mu_1} & \delta^{\nu_1}_{\mu_2} & \delta^{\nu_1}_{\mu_3} \\ \delta^{\nu_2}_{\mu_1} & \delta^{\nu_2}_{\mu_2} & \delta^{\nu_2}_{\mu_3} \\ \delta^{\nu_3}_{\mu_1} & \delta^{\nu_3}_{\mu_2} & \delta^{\nu_3}_{\mu_3} \end{vmatrix}. \qquad (2.8.20)$$

and coincide with Eqs. (2.8.2 b) under antisymmetry properties (2.8.2a). This result was, after all, expected from the Lie character of the contravariant version $\mathcal{R}^{\mu\nu}$ of $\mathcal{R}_{\mu\nu}$. Of course, this geometrical meaning is strictly in local coordinates and, as such, does not realize the coordinate-free treatment of symplectic manifolds. Neverthless, the potential significance for relativity considerations persists.

We can now also say that Birkhoff's equations consitute a symplectic covering of Hamilton's equations as the geometrical counterpart of the algebraic property of Birkhoff's equations of being a Lie covering of Hamilton's equations. This aspect will play a crucial role for Lie-admissible formulations. Thus, it deserve few comments.

First of all, our unified notation $\{a^\mu\} = \{q^k, p_k\}$ (which, as indicated earlier, is crucial for the speedy identification of the conditions of selfadjointness) is not customarily used in the available literature of symplectic geometry.[6] It is, therefore, of some usefulness to indicate its equivalence with the conventional notation for phase space variables. Secondly, it is of some significance, for later needs, to reformulate the conditions for a vector field to be either globally or locally Hamiltonian within the context of such notation. Finally, the reformulation of the Lie derivative within such notational setting will also be useful. The reader should again be aware that we are primarily interested in these notions expressed in local coordinates. For the rigorous coordinate-free treatment we therefore refer the interested reader to the existing literature.[6]

The fundamental symplectic form is customarily written as the (exterior) two-form

$$\theta = d p_k \wedge d q^k \quad (= - d q^k \wedge d p_k). \qquad (2.8.21)$$

It is trivially nowhere degenerate and closed. Thus, it is symplectic. In addition, it is exact because derivable via the (exterior) derivative of a one-form, the canonical form

$$\theta_1 = p_k d q^k. \qquad (2.8.22)$$

In our notation we shall write for the canonical form

$$\omega_1 = -a_\mu d a^\mu = -a^\nu \omega_{\nu\mu} d a^\mu = p_k d q^k - q^k d p_k. \qquad (2.8.23)$$

The fundamental symplectic form then becomes

$$\omega_2 = -\omega_{\mu\nu} d a^\mu \wedge d a^\nu = 2 (d p_k \wedge d q^k), \qquad (2.8.24)$$

(the relationship between the forms ω_2 and θ_2 or ω_1 and θ_1 will be investigated later on in Table 2.13).

The reason for our selection of these forms is that they allow the easy identification of the parallelism with a general symplectic form characterized by Birkhoff's tensor $\mathcal{R}_{\mu\nu}$, Eqs. (2.8.3). And indeed, by generalizing Eq. (2.8.23) into the form

$$\mathcal{R}_1 = -2 R_\mu(a) d a^\mu, \qquad (2.8.25)$$

we have

$$\mathcal{R}_2 = d \mathcal{R}_1 = -2 \frac{\partial R_\mu}{\partial a^\nu} d a^\nu \wedge d a^\mu = \left(\frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial R_\nu}{\partial a^\mu} \right) d a^\mu \wedge d a^\nu = \mathcal{R}_{\mu\nu}(a) d a^\mu \wedge d a^\nu. \qquad (2.8.26)$$

Thus, the symplectic form (2.8.19) is exact.

Let $\xi^\mu(a)$ be a (contravariant) vector field in a symplectic manifold $M(a, \omega_2)$. The inner product of ξ^μ with ω_2 will be written

$$\xi_1^{\omega_2} = -\frac{1}{2} \xi \rfloor \omega_2 = -\frac{1}{2} \delta^{\nu_1 \nu_2}_{\mu_1 \mu_2} \omega_{\nu_1 \nu_2} \xi^{\mu_1} d a^{\mu_2} = \omega_{\mu\nu} \xi^\nu d a^\mu = \xi_\mu d a^\mu. \qquad (2.8.27)$$

We shall say that the vector field ξ^μ is globally Hamiltonian (or Hamiltonian for short) when the one-form $\xi_1^{\omega_2}$ is exact, that is, at a point $m \subset M(a, \omega_2)$ there exists a neighborhood $N(m)$ and a function $H(a)$, the Hamiltonian, on $N(m)$ such that

$$\xi_1^{\omega_2} = \omega_{\mu\nu} \xi^\nu d a^\mu = \xi_\mu d a^\mu = d H = \frac{\partial H}{\partial a^\mu} d a^\mu. \qquad (2.8.28)$$

The notation has therefore the following advantages for our program. First of all it clearly indicates that when the vector field ξ^μ is Hamiltonian, the tensor $\omega_{\mu\nu}$ acquires the geometrical meaning of lovering the contravariant index μ. An equivalent meaning then holds for the tensor $\omega^{\mu\nu}$, but, this time, for raising covariant indeces. Secondly, the notation allows a geometrical formulation of what we have called the Fundamental Analytic Theorem of the Inverse Problem for canonical formulations, Theorem 2.7.3.
And indeed, the integrability conditions for the one-form $\xi_1^{\omega_2}$ to be exact are precisely the conditions of selfadjointness (2.7.13). Thus, the Inverse Problem for Hamiltonian formulations is, in essence, a formulation in local variables of the geometrical notion of a vector field of being (globally) Hamiltonian.

However, as is well known, a vector field ξ^μ is not necessarily (globally) Hamiltonian. We thefefore consider the inner product

$$\xi_1^{\mathcal{R}_2} = -\frac{1}{2} \xi \rfloor \mathcal{R}_2 = \mathcal{R}_{\mu\nu} \xi^\nu d a^\mu = \xi_\mu d a^\mu. \qquad (2.8.29)$$

If the one-form $\Xi^{\Omega_2}_{1}$ is exact, we shall say that the vector field Ξ^{μ} is locally Hamiltonian, according to the conventional terminology, and Birkhoffian in our terminology. The aspect which is relevant for subsequent steps of our analysis is the geometrical analogy between Hamiltonian and Birkhoffian vector fields. And indeed, in both cases the crucial geometrical role of lowering the indeces is played by the fundamental tensor of the analytic equations, the tensor $\omega_{\mu\nu}$ for the former case and the tensor $\Omega_{\mu\nu}$ for the latter case. The point is that both tensors characterize a symplectic form. In conclusion, if a vector field is not Hamiltonian it can be Birkhoffian, in which case the tensor $\Omega_{\mu\nu}$ (and not $\omega_{\mu\nu}$) is the proper tensor for lowering the indeces.

The next step is the transition from Birkhoffian forms (or vector fields) to a Hamiltonian form. This is provided by Darboux's Theorem here presented in the version known as the Darboux-Weinstein Theorem [6c]

> THEOREM 2.8.1: Let M_1 be a submanifold of a manifold M and let Ω_2 and Ω'_2 be two nowhere degenerate, closed two-forms on M such that $\Omega_2|M_1 = \Omega'_2|M_1$. Then there exists a neighborhood $N(M_1)$ and a diffeomorphism $f:N(M_1) \to M$ such that
> (a) $f(m) = m$ for all $m \in M_1$ and
> (b) $f^* \Omega_2 = \Omega'_2$

The transformations of this theorem, within the context of our analysis, essentially guarantee that Birkhoff's equations can always be reduced to Hamilton's form. It is in this sense that a Birkhoffian field is locally Hamiltonian. For a reformulation of this geometrical treatment in local coordinates see the paper by W. SARLET and F. CANTRIJN [15b] in this issue.

The reformulation of the Lie derivative [6b]

$$\mathcal{L}_X F = \lim_{t\to 0} \frac{F\circ G_a(t) - F\circ G_a(0)}{t} = XF, \qquad (2.8.30)$$

in our notation is then trivial. Suppose that the vector field Ξ^{μ} is Hamiltonian. Then the realization of the generator X of the one-parameter Lie group $G_a(t)$ is given by

$$X = \omega^{\mu\nu}\,\Xi_\nu \frac{\partial}{\partial a^\mu} = \omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}\frac{\partial}{\partial a^\mu} = \Xi^\mu \frac{\partial}{\partial a^\mu}. \qquad (2.8.31)$$

As a result, we can recover the time evolution law of Hamilton's equations via the Lie derivative, i.e.,

$$\mathcal{L}_{\Xi} F = X_{\Xi} F = \omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}\frac{\partial F}{\partial a^\mu} = [F, H], \qquad (2.8.32)$$

by, at the same time, reaching a geometrical interpretation of conservation laws via the Lie derivative (when $\mathcal{L}_{\Xi} F = 0$).

A point which will be important for our conjectured Galilei-admissible covering of the Galilei relativity is that realization (2.8.32) is not unique within the con text of the symplectic geometry. And indeed, as we shall see in more details later on (Table 3.6), the vector field Ξ can be Birkhoffian and still generate a one-parameter group of translations in time. This yields the "symplectic covering" of realization (2.8.32)

$$X^* = \Omega^{\mu\nu}(a)\,\Xi_\nu \frac{\partial}{\partial a^\mu} = \Omega^{\mu\nu}\frac{\partial H^B}{\partial a^\nu}\frac{\partial}{\partial a^\mu}, \qquad (2.8.33a)$$

$$\mathcal{L}_{X^*} F = X^* F = \Omega^{\mu\nu}\frac{\partial H^B}{\partial a^\nu}\frac{\partial F}{\partial a^\mu} = [F, H^B]^*, \qquad (2.8.33b)$$

that is, the geometrical interpretation of the time evolution law of Birkhoff's equations, now expressed in terms of the generalized Poisson brackets. This concludes our rudimentary remarks for the autonomous case.

To summarize, the geometrical significance of the conditions of variational selfadjointness of general first-order covariant systems (2.8.1) is that they guarantee that the underlying geometry is the symplectic geometry for locally Hamiltonian (Birkhoffian in our terminology) vector fields. As a particular case when $\Omega_{\mu\nu} = \omega_{\mu\nu}$, the vector fields are Hamiltonian.

The extension of the above findings to the case of nonautonomous systems, that is, systems with an explicit dependence on time

$$\left[\Omega_{\mu\nu}(t,a)\,\dot a^\nu - \Gamma'_\mu(t,a)\right]^{C^\infty, R} = 0, \qquad (2.8.34)$$

will be essentially left to the interested reader. Let us only indicate that (a) Birkhoff's equations are now extended to the form

$$\left[\left(\frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial R_\nu}{\partial a^\mu}\right)\dot a^\mu - \frac{\partial H^B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right]^{C^\infty, R}_{SA} = 0, \qquad (2.8.35)$$

which we shall call SARLET-CANTRIJN form and which constitutes the most general first-order form of selfadjoint systems of ordinary differential equations,[15b] (b) the underlying algebra is still a Lie algebra, and (c) the applicable geometry is now that of contact manifolds (rather than symplectic manifolds). The analytic, algebraic and geometrical meanings of the conditions of selfadjointness, now given by the full system (2.7.6), also admit an extension to this broader system. To see it, let us only indicate that conditions (2.7.6) can be written in the unified notation

$$\bar\Omega_{ij} + \bar\Omega_{ji} = 0, \qquad (2.8.36a)$$

$$\frac{\partial \bar\Omega_{ij}}{\partial \bar a^k} + \frac{\partial \bar\Omega_{jk}}{\partial \bar a^i} + \frac{\partial \bar\Omega_{ki}}{\partial \bar a^j} = 0, \qquad (2.8.36b)$$

$$\{\bar{a}^i\} = \{t, a^\mu\}, \quad \bar{\Omega}_{\mu\nu} = \Omega_{\mu\nu}, \quad \bar{\Omega}_{\mu 0} = -\bar{\Omega}_{0\mu} = \Gamma_\mu, \quad \bar{\Omega}_{00} = 0,$$
$$i = 0, 1, 2, \ldots, 2n, \quad \bar{a}^0 = t, \quad \bar{a}^\mu = a^\mu. \tag{2.8.36c}$$

Under these conditions we can introduce a (2n+1)-dimensional manifold $M(\bar{a}, \bar{\Omega}_2)$ with local coordinates $\bar{a}^i$ equipped with the two form

$$\bar{\Omega}_2 = \bar{\Omega}_{ij}(\bar{a})\, d\bar{a}^i \wedge d\bar{a}^j. \tag{2.8.37}$$

This form is of maximal rank (i.e., its restriction to $M(a, \Omega_2)$ is nowhere degenerate) and it is closed. Thus, the full set of conditions of selfadjointness are necessary and sufficient for the two form $\bar{\Omega}_2$ to be a contact form (or structure).[15b] It is an instructive exercise for the interested reader to work out the generalization of Eqs. (2.8.28) and (2.8.29) and see that the underlying analytic equations are indeed of the form (2.8.35). A generalization of Darboux's theorem for contact manifolds exists[6] and it is applicable for the reduction of Eqs (2.8.35) to the Hamiltonian form (2.7.12). The Hamiltonian, however, now acquires an explicit dependence on time. For the same reduction expressed in terms of the transformation theory, see W. SARLET and F. CANTRIJN.[15b]

In conclusion, the conditions of variational selfadjointness provide a symbiotic characterization of certain elemental aspects of Analytic Mechanics, Lie Algebras and Differential Geometry. As such, they constitute a valuable arena for the study, in general, of the deep interrelations among these disciplines and for the study, in particular, of relativity aspects.

TABLE 2.9: THE THEOREM OF INDIRECT UNIVERSALITY OF THE INVERSE PROBLEM.

We are now sufficiently equipped to outline the following crucial property identified by the methodology of the Inverse Problem.[2b]

THEOREM 2.9.1: Local, class C^∞ and regular Newtonian systems always admit an indirect analytic representation in the neighborhood of a regular point of their local variables.

The proof of this theorem can be outlined according to the following steps.

Step 1: It consists of the reduction of Newton's equations as derived from the second law, i.e.,

$$\left[\mu_k \ddot{r}_{ka} - F_{ka}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})\right]^{C^\infty, R}_{SA} = 0, \quad k = 1, 2, \ldots, N, \quad a = x, y, z, \tag{2.9.1}$$

into an equivalent, general, covariant, first-order, form

$$\left[C_{\mu\nu}(t, a)\, \dot{a}^\nu + D_\mu(t, a)\right]^{C^\infty, R}_{SA} = 0, \quad \mu = 1, 2, \ldots 2n, \quad n = 3N, \tag{2.9.2}$$

via the use of the method of Section 2.7. This implies, in particular, that the emerging variables $\{a^\mu\} = \{r^{ka}, y_{ka}\}$ are not, in general, canonically conj gate and that system (2.9.2) is not, in general, selfadjoint.

Step 2: Construction of an equivalent, general, first-order, covariant and selfadjoint form. This construction can be done as follows. The "degrees of freedom" at hand are constituted by the functions $g^k(t, q, y)$ of prescriptions (2.7.2) and an additional set of $4n^2$ functions $h_\mu^{\ \nu}(t, a)$ which are of class C^∞ and whose matrix $(h_\mu^{\ \nu})$ is regular in the neighborhood of the considered (regural) point (t, a) and we write

$$\left\{h_\mu^{\ \nu}(t,a)\left[C_{\nu\rho}(t,a)\, \dot{a}^\rho + D_\nu(t,a)\right]\right\}^{C^\infty, R}_{SA} = \left[\Omega_{\mu\rho}\, \dot{a}^\rho - \Gamma_\mu\right]^{C^\infty, R}_{SA}, \tag{2.9.3}$$

where now SA stands for a condition on the unknown g^k and $h_\mu^{\ \nu}$. A study of this system indicates that, in the neighborhood of a regular point, it is consistent, namely, it always admit a solution in g^k and $h_\mu^{\ \nu}$. See in this respect W. SARLET and F. CANTRIJN[15b] and, also, P. HAVAS[17a]. As a result, Newtonian systems (2.9.1) always admit (in the neighborhood of a regular point) an indirect analytic representation in terms of Birkhoff's equations, i.e.,

$$\left[\Omega_{\mu\rho}\, \dot{a}^\rho - \Gamma_\mu\right]^{C^\infty, R}_{SA} \equiv \left[\left(\frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial R_\nu}{\partial a^\mu}\right)\dot{a}^\nu - \frac{\partial H^B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right]^{C^\infty, R}_{SA}. \tag{2.9.4}$$

Step 3: Reduction of the Birkhoffian representation to a Hamiltonian form, i.e.,

$$\bar{a}^i \longrightarrow \bar{b}^i = \bar{b}^i(\bar{a}), \quad \{\bar{a}^i\} = \{t, a^\mu\}, \quad \{\bar{b}^i\} = \{t, b^\mu\}, \tag{2.9.5a}$$

$$\left[\Omega_{\mu\nu}\, \dot{a}^\nu - \frac{\partial H^B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t}\right]^{C^\infty, R}_{SA} = 0 \longrightarrow \tag{2.9.5b}$$

$$\longrightarrow \left[\omega_{\mu\nu}\, \dot{b}^\nu - \frac{\partial H}{\partial b^\nu}\right]^{C^\infty, R}_{SA} = 0, \quad H(\bar{b}) = H^B(\bar{a}(\bar{b})) + \frac{\partial a^\mu}{\partial t} R_\mu(\bar{a}(\bar{b})).$$

The existence of this reduction is guaranteed by the generalization of Darboux's Theorem to contact manifolds (the so-called contact charts). For the use of the transformation theory to prove the reduction, see ref.[15b] This concludes the outline of the three major steps for the proof of the theorem. For details, see ref.[2b]

Theorem 2.9.1, in essence, expresses a known result, the property that a vector field, under the indicated consitions, can always be transformed to an equivalent form which is Hamiltonian. Equivalently, the theorem expresses the property that Lie's theory is always applicable, up to invertible changes of the (local) coordinates, to an (even) dimensional system of first-order ordinary differential equations, as implicit in the Lie-Koening theorem.[17a, 17e] As a result, Theorem 2.9.1 and the outlined three steps of its proof are essentially intended to provide a working grounds for the explicit construction of an indirect analytic representation of

Newtonian systems (2.9.1) with forces not derivable from a potential in terms of the conventional Hamilton's equations. A Lagrangian, if needed, can then be computed via the Legendre transform.

A number of remarks are here in order. Theorem 2.9.1 allows the following classification of Newtonian systems which will result to be crucial for our relativity considerations.

CLASS I : ESSENTIALLY SELFADJOINT NEWTONIAN SYSTEMS.[5a] These are (local, class C^∞, regular, unconstrained) Newtonian systems in the (inertial) reference frame of their experimental detection which are selfadjoint as derived from Newton's second law, and we shall write

$$\left[m_k \ddot{r}_{ka} - f_{ka}(t, r_m, \dot{r}_m) \right]_{ESA}^{C^\infty, R} = 0, \qquad (2.9.6)$$

where ESA stands for essential selfadjointness in the above sense. It is hoped that the terms "essentially selfadjoint" here referred to a variational property of systems of ordinary differential equations does not create confusion with the corresponding terms used in the theory of linear operators on vector spaces. In actuality, these terms have been selected because of a close parallelism between the variational approach to selfadjointness and the corresponding approach within the context of the Functional Analysis[7]. For details, see ref.[2a]

In relations to the proof of Theorem 2.9.1, step 3 is redundant in the sense that the vector fields are (globally) Hamiltonian. In essence, systems (2.9.6) can be reduced to the normal form (2.7.7a) and the conditions of selfadjointness (2.7.8), interpreted as equations on the unknown prescriptions g^k, are always consistent. This yields a representation of the system in terms of Hamilton's equations in the variables $\{a^\mu\} = \{r^{ka}, p_{ka}\}$ which now span a phase space. Notice tha lack of use of a Lagrangian representation in this approach, as typical of the independent formulation of the Inverse Problem for canonical formulations.

The use of the Lagrangian representation yields the same result. For system (2.9.6) to be essentially selfadjoint, all the acting forces must be derivable from a potential, i.e., must satisfy the conditions of Theorem 2.4.1. As a result, the computation of a Lagrangian L is in this case trivial. The use of the prescriptions $p_{ka} = \partial L / \partial \dot{r}^{ka}$ and of the Legendre transform then yields a Hamiltonian representation. This is equivalent to the approach indicated above in the sense that the conditions of selfadjointness (2.7.8) essentially yield a solution in the function g^k which characterize the implicit form (2.7.2) of the prescriptions $p_k = \partial L / \partial \dot{r}^{ka}$, by therefore yielding the same Hamiltonian (up to all admissible equivalence transformations, such as those characterizable by the "Newtonian gauge"

$$L(t, r_m, \dot{r}_m) \rightarrow L'(t, r_m, \dot{r}_m) = L(t, r_m, \dot{r}_m) + \frac{d}{dt} G(t, r_m), \quad G \in C^\infty. \qquad (2.9.7)$$

CLASS II : NONESSENTIALLY NONSELFADJOINT NEWTONIAN SYSTEMS.[5a] These are (local, class C^∞, regular, unconstrained) Newtonian systems in the reference frame of their experimental detection which, as derived from Newton's second law, are nonselfadjoint but such to satisfy Theorem 2.6.1. This essentially means that there exist a class C^∞ and regular matrix of integrating factors capable of producing an equivalent selfadjoint form without changing the local variables, and we shall write

$$\left[m_k \ddot{r}_{ka} - F_{ka}(t, r_m, \dot{r}_m) \right]_{NENSA}^{C^\infty, R} = 0. \qquad (2.9.8)$$

With respect to the steps of the proof of Theorem 2.9.1, step 3 is still absent. However, step 2 now acquires an essential role in the sense that, in addition to the freedom in the g^k functions, the multiplicative functions h_μ^ν must be used to induce a Hamiltonian form, with the h_μ^ν clearly playing the canonical role of the h_k^i functions of Eqs. (2.6.2). Again, no Lagrangian representation is used in this approach. Its use would yield the same result. The computation of a Lagrangian L^* via Theorem 2.6.1, the use of the prescriptions $p^*_k = \partial L^* / \partial \dot{r}^{ka}$ and the Legendre transform, do indeed give rise to a Hamiltonian representation without necessarily going through the intermediate Birkhoffian representation. Notice that, unlike Case I, the canonical momentum p^*_k is now generalized in the sense that it cannot represent the physical linear momentum (see below for comments).

CLASS III : ESSENTIALLY NONSELFADJOINT NEWTONIAN SYSTEMS.[5a] These are (local, class C^∞, regular, unconstrained) Newtonian systems in the (inertial) reference system of their experimental detection which, as derived from Newton's second law, are nonselfadjoint and such to violate the conditions of Theorem 2.6.1. This essentially means that the systems do not admit an equivalent selfadjoint form within the same coordinate system, and we shall write

$$\left[m_k \ddot{r}_{ka} - F_{ka}(t, r_m, \dot{r}_m) \right]_{ENSA}^{C^\infty, R} = 0. \qquad (2.9.9)$$

In this case all three steps of the proof are used to construct a Hamiltonian representation. In particular, the intermediate Birkhoffian representation plays a crucial role to identify a symplectic (for autonomous systems) or contact (for nonautonomous systems) characterization of the systems. The symplectic (or contact) charts, respectively, then ensure the reduction to a Hamiltonian form.

The reader can now see the reason for our efforts in rendering a Lagrangian representation inessential, And indeed, since Theorem 2.6.1 is violated by assumption, the use of the transformation theory is necessary to induce an equivalent selfadjoint form. But

point transformations $r^{ka} \to r'^{ka}(r)$ are insufficient in this case. This necessarily implies the use of more general transformations of type (2.6.8). The proof of the existence of a Lagrangian within such a setting appears to be more involved that that of a Hamiltonian. Notice that transformations (2.6.8) are the configuration space image of the most general transformations in a contact manifold, i.e.,

$$\{\bar{a}^i\} \quad \{t, a^\mu\} \longrightarrow \{\bar{b}^i\} = \{\bar{t}, b^\mu\} = \{\bar{b}^i(\bar{a})\} = \{t'(\bar{a}), b^\mu(\bar{a})\}$$
$$= \{t'(t, \underset{\sim}{r}, \underset{\sim}{y}), \underset{\sim}{r}'(t, \underset{\sim}{r}, \underset{\sim}{y}), \underset{\sim}{y}'(t, \underset{\sim}{r}, \underset{\sim}{y})\}, \qquad (2.9.10)$$

the point is that a dependence of the new coordinates r'^{ka} in both the old coordinates and the "momenta" $y_{ka} = f_{ka}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})$ is now necessary to achieve the reduction. The configuration space image is then of type (2.6.8).

We are now in a position to identify the class of Newtonian systems in which we are interested for relativity consideration. It is that of essentially nonselfadjoint systems. And indeed, this class is such to possess truly nontrivial forces not derivable from a potential and, as such, it is the class that will predictably produce the highest possible breaking of the Galilei relativity (Table 2.14). From now on, unless explicitly stated, nonselfadjointness stands for essential nonselfadjointness. Notice that, on formal grounds, the class of essentially nonselfadjoint systems can be considered as inclusive of that of nonessentially nonselfadjoint and of selfadjoint systems.

The reader should be aware that the Hamiltonian representations of essentially nonselfadjoint systems and, more properly, their Lagrangian image constructed via a Legendre transform, according to the remarks of Table 2.6, occurs within a system of coordinates r'^{ka} which is generally noninertial and nonrealizable in experiments.

The reader should also recall, from Section 2, that all considered systems are tacitly assumed to be finite-dimensional.

TABLE 2.10: THE STRUCTURE OF A LAGRANGIAN OR A HAMILTONIAN AND THEIR DEGREES OF FREEDOM. After having outlined the existence theorems for analytic representations, it is of some significance to indicate the structure of the emerging Lagrangians and Hamiltonians. This is one of the topics of the theory of nonconservative systems which demands a departure from the costumary conceptual attitude of conservative mechanics. In essence, as a result of extended use, "the" Lagrangian or "the" Hamiltonian in Analytic Mechanics are often associated with the structures L = T-V and H = T + V. The Inverse Problem, however, essentially brings Analytic Mechanics up to the level of the Calculus of Variations as far the structure of these functions is concerned, that is, they can exhibit an arbitrary structure, provided

that the assumed continuity and regularity conditions are satisfied. As a result, in the study of the problem considered it is recommendable to use the conceptual context of the Calculus of Variations, rather than that of Analytic Mechanics for conservative systems.

It is advisable to identify the structure of "a" Lagrangian or "a" Hamiltonian for each of the three classes of Newtonian systems introduced in the preceding table.

I. Case of essentially selfadjoint systems. In this case the structure of the direct Lagrangian representation reads

$$\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^{ka}} - \frac{\partial L}{\partial r^{ka}}\right]^{C^\infty, R}_{SA} \equiv \left[m_k \ddot{r}_{ka} - f_{ka}\right]^{C^\infty, R}_{ESA}, \qquad (2.10.1)$$

the forces $f_{\underset{\sim}{m}k}$ are selfadjoint (i.e., verify Theorem 2.4.1) and, thus, derivable from a potential, and the emerging Lagrangian has the conventional structure

$$f_{ka} = -\frac{\partial U}{\partial r^{ka}} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}^{ka}}, \qquad (2.10.2a)$$

$$L = T - U = \frac{1}{2} m_k \dot{\underset{\sim}{r}}_k \dot{\underset{\sim}{r}}^k - U(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}). \qquad (2.10.2b)$$

An intriguing aspect, however, is that such conventional structure does not exhaust the possibilities which are rendered identifyable by the Inverse Problem. And indeed, Theorem 2.6.1 on indirect representations, even though presented for nonselfadjoint systems, is equally applicable to the essentially selfadjoint systems. Under the assumption that the conditions of the theorem are satisfied, these systems therefore admit the indirect Lagrangian representation[3c,2b]

$$\left[\frac{d}{dt}\frac{\partial L^*}{\partial \dot{r}^{ka}} - \frac{\partial L^*}{\partial r^{ka}}\right]^{C^\infty, R}_{SA} \equiv \left[h_{ka}{}^{jb}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})\left(m_j \ddot{r}_{jb} - f_{jb}\right)^{C^\infty, R}_{SA}\right]^{C^\infty, R}_{SA}. \qquad (2.10.3)$$

The net effect is that the conventional structure (2.10.2b) is generally lost already at the level of essentially selfadjoint systems, e.g., conservative systems. And indeed, the integrating factors $h_k{}^i$ now enter into the structure of an admissible Lagrangian via Eqs. (2.6.4), yielding a generalized structure which can be written in any of the following equivalent forms[2,3]

$$L^*(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) = K(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) + D_{ka}(t, \underset{\sim}{r})\dot{r}^{ka} + C(t, \underset{\sim}{r})$$
$$= \sum_{k=1}^{N} \sum_{a}^{x,y,z} L^{(ka)}_{\text{int}, I}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) L^{(ka)}_{\text{free}}(\dot{r}^{ka}) + L_{\text{int}, II}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})$$
$$= \frac{1}{2}\left[\dot{r}^{ia} G_{iajb} \dot{r}^{jb} + 2\dot{r}^{ia} F_{iajb} r^{jb} + r^{ia} E_{iajb} r^{jb}\right]. \qquad (2.10.4)$$

These generalized Lagrangians are fully acceptable on analytic grounds because the system of the implicit functions of Lagrange's equations in these functions and that of the equations of motion coincide. As a result, they provide an analytic representation in the sense of Table 2.3 .

But system (2.6.3) on the integrating factors h_k^i is a system of partial differential equations, that is, a type of system which, when consistent, admits solution with a functional degree of freedom. This implies that there may exist a family of equivalent Lagrangians within the same system of Cartesian coordinates, with a corresponding family of equivalent Hamiltonians.

For instance, all the following functions

$$L^*_1 = \frac{1}{6}\dot{q}^3\cos t + \frac{1}{2}q^2\dot{q}^2\sin t - q^2\dot{q}\cos t, \qquad (2.10.5a)$$

$$L^*_2 = 2\frac{\dot{q}}{q}\operatorname{arctg}\frac{\dot{q}}{q} - \ln(\dot{q}^2+q^2), \quad q\neq 0, \qquad (2.10.5b)$$

$$H^*_2 = 2\ln\left|q\sec\frac{1}{2}q\,p^*\right|, \quad p^*_k = \partial L^*_2/\partial\dot{q}^k, \qquad (2.10.5c)$$

$$H^*_3 = 2\ln\left|q\sec\left(\frac{1}{2}q\,p' - \operatorname{arctg}\frac{c}{q}\right)\right|, \quad q=0,\ c=\text{const}, \qquad (2.10.5d)$$

$$H^*_4 = q(\tan t)\sec t\left|\left(e^{p''}\operatorname{cotg}\frac{1}{2}t\right)^{\cos t} - \ln\left(e^{p''}\operatorname{cotg}\frac{1}{2}t\right)^{\cos t} - 1\right|, \qquad (2.10.5e)$$

represent the same system, the one-dimensional harmonic oscillator with equation

$$\ddot{q} + q = 0, \quad m = k = 1, \quad L = \frac{1}{2}(\dot{q}^2 - q^2). \qquad (2.10.6)$$

We are here clearly facing a degree of freedom of the Lagrangian structure which is not derivable with conventional means, e.g., the "Newtonian gauges" (2.9.7). And indeed, it is a new degree of freedom directly produced by the integrability conditions for the existence of a Lagrangian representation. In essence, these integrability conditions yield the representation not only of one selfadjoint form of the equations of motion, but more properly of all the equivalent selfadjoint forms. For this reason we shall use the term "a" Lagrangian, rather than "the" Lagrangian.

The degrees of freedom we are here referring to were apparently identified for the first time by using conventional techniques by D.G. CURRIE and E.G. SALETAN[18a] for the case of one dimensional systems and called "fouling transformations". The Inverse Problem essentially produce the necessary and sufficient conditions for their existence (Theorem 2.6.1). We shall call these transformation isotopic from the meaning[2b,3c,5a]

"ἰσότοπος" "τοπος" = "preserve" "configuration",

here interpreted as class C^∞, invertible, selfadjointness preserving transformation. Two Lagrangians $L(t,\underline{x},\dot{\underline{x}})$ and $L^*(t,\underline{x},\dot{\underline{x}})$ will be said to be isotopically related when they satisfy the rule[3c,2b]

$$\left[\frac{d}{dt}\frac{\partial L^*}{\partial \dot{x}^{ka}} - \frac{\partial L^*}{\partial x^{ka}}\right]^{C^\infty,R}_{SA} \equiv \left[h_{ka}{}^{jb}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^{jb}} - \frac{\partial L}{\partial x^{jb}}\right)^{C^\infty,R}_{SA}\right]^{C^\infty,R}_{SA}, \qquad (2.10.7)$$

where, as by now familiar, the repetion of the symbol C^∞ and R, stands for the condition that the matrix (h) is of class C^∞ and regular.

The reason for the selection of the terms "isotopic" will be indicated in Table 2.13 . The reader should be aware that this concept of isotopy will play a crucial role for our conjecture of a Galilei-admissible covering relativity in the sense that it constitutes a first step toward a more general concept (that of genotopy) which will be actually used in the construction of our conjectured relativity. Even then, the notion of isotopy will persist in an associated form.

Notice that structure (2.10.4) is the Newtonian limit of structures which are called chiral Lagrangians in field theory[19a]. For use in subsequent papers we shall say that a Lagrangian structure is nonessentially chiral[19c] when there exist an equivalent conventional structure within the same system of variables, i.e., when there exists a Lagrangian L = T-V related to the generalized Lagrangian via rule (2.10.7). Thus, nonessentially chiral Lagrangians can represent conservative systems, despite their generalized structure.

II. Case of nonessentially nonselfadjoint systems. This the class of systems for which the indirect Lagrangian representations according to Theorem 2.6.1

$$\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^{ka}} - \frac{\partial L}{\partial x^{ka}}\right]^{C^\infty,R}_{SA} \equiv \left[h_{ka}{}^{jb}\left(m_j\ddot{x}_{jb} - F_{jb}\right)^{C^\infty,R}_{NENSA}\right]^{C^\infty,R}_{SA}, \qquad (2.10.8)$$

exist. In this case the structure of a Lagrangian is, of course, generalized, i.e., of type (2.10.4). However, this generalized structure is now necessary to represent the system. And indeed, if there exists a Lagrangian L = T - V within the same coordinates for the representation of the systems considered, this implies that all acting forces are derivable

from a potential, contrary to a central property of the systems considered. We shall then say that the Lagrangians for representations (2.10. 8) are essentially chiral.[19c]

The isotopic degrees of freedom, however, persist . And indeed, system (2.10. 8) may admit a family of functionally different solutions. For instance, the damped harmonic oscillator admit the following isotopically mapped Lagrangians

$$L = e^{\gamma t}\frac{1}{2}\left(\dot{q}^2 - \omega_0^2 q^2\right) \rightarrow L^* = \frac{2\dot{q}+\gamma q}{2q\omega}\, tg^{-1}\left(\frac{2\dot{q}+\gamma q}{2q\omega}\right) - \frac{1}{2}\ln\left(\dot{q}^2 + \gamma q\dot{q} + \omega_0^2 q^2\right), \qquad \omega^2 = \omega_0^2 - \gamma/4 . \tag{2.10.9a, 2.10.9b}$$

In essence with the terms nonessentially and essentially chiral Lagrangians we intend to express the fact that (both at a Newtonian and a field theoretical level) a generalized structure of a Lagrangian does not necessarily guarantee the existence of a generalized system. It is hoped that examples (2.10. 5) can be of assistance in this respect. The attitude which is recommended in relation to this issue is as naive as possible. When a generalized Lagrangian structure is studied, the best way to ascertain that the system is actually generalized is to compute the equations of motion and, in particular, the implicit functions of the system. This is the most direct and unequivocal way to reach conclusions of physical nature from the mathematical algorithm represented by a Lagrangian.

The extension of these remarks to Hamiltonian structures is trivial, by reaching structures of the type

$$\begin{aligned} H(t, \underset{\sim}{r}, \underset{\sim}{p}) &= \tilde{T}(t, \underset{\sim}{r}, \underset{\sim}{p}) + \tilde{D}^{ka}_{ka}(t, \underset{\sim}{r})\, p_{ka} + \tilde{C}(t, \underset{\sim}{r}) \\ &= \sum_{k=1}^{N}\sum_{a=x,y,z} H^{(ka)}_{int,\,I}(t, \underset{\sim}{r}, \underset{\sim}{p})\, H^{(ka)}_{free}(p_{ka}) + H_{int,\,II}(t, \underset{\sim}{r}, \underset{\sim}{p}) \\ &= \frac{1}{2}\left[p_{ia}\tilde{G}^{iajb}(t, \underset{\sim}{r}, \underset{\sim}{p})\, p_{jb} + 2\, p_{ia}\tilde{F}^{ia}{}_{jb}(t, \underset{\sim}{r})\, r^{jb} + r^{ia}\tilde{E}_{iajb}(t, r)\, r^{jb}\right], \end{aligned} \tag{2.10.10}$$

which, again, can be either essentially or nonessentially generalized. Notice that these structures are not treatable with Riemannian manifolds (as currently known) even at the limit of null forces derivable from a potential (but non null forces not derivable from a potential). This property will be significant for relativistic generalizations to be considered in subsequent papers (see Section 5 for introductory comments).

III. Case of essentially nonselfadjoint systems. In this case the use of the transformation theory is essential. Thus, the admissible Lagrangian and Hamiltonian structures are defined in a new system of variables. It is easy to see that this structure can be either of the conventional type (2.10.2 b) or of the generalized type (2.10. 4), but now in a new system of

variables. Most intriguingly, the use of symplectic (or contact) charts may lead to straight trajectories.[6] In this case the Lagrangian and Hamiltonian structures are not only conventional, but actually those for a free particle, i. e.,

$$L'(\underset{\sim}{r}') = \frac{1}{2}\mu_k\, \dot{\underset{\sim}{r}}'_k \cdot \dot{\underset{\sim}{r}}'^k, \qquad H'(\underset{\sim}{p}') = \frac{1}{2\mu_k}\, \underset{\sim}{p}'_k \cdot \underset{\sim}{p}'^k . \tag{2.10.11}$$

This is, in essence, our argument (indicated in Section I) according to which systems which break the Galilei relativity in the reference frame of their experimental detection can be transformed into an equivalent system in new variables which is fully compatible with the Galilei relativity. This is the line of study of the relativity problem of Newtonian Mechanics which we shall leave to the interested reader for the reasons indicated earlier.

TABLE 2. II: THE REPRESENTATIONAL CAPABILITIES OF VARIATIONAL PRINCIPLES.

Theorem 2. 9 . 1 essentially indicates that variational principles can represent "all" Newtonian systems of class C^∞, regular, unconstrained and local.[2b] It is of some significance for our problem to point out the mechanics of the "representation" and the nature of the "variational principles". Again, the reader is here discouraged to use the mental attitude of Analytic Mechanics for conservative systems and use instead that of the Calculus of Variations. More specifically, the terms "variational principles" should be referred to the algorithms of the established "variational problems" in which the estremal part is ignored. It is again useful to outline the arguments for the three separate classes of Newtonian systems of Table 2. 9 .

I. Case of essentially selfadjoint systems. This is the typical variational setting of conservative mechanics and we shall write

$$\delta\int_{t_1}^{t_2} dt\, L = -\int_{t_1}^{t_2} dt\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^{ka}} - \frac{\partial L}{\partial r^{ka}}\right)^{C^\infty, R}_{SA} \delta r^{ka} = -\int_{t_1}^{t_2} dt\left(\mu_k \ddot{r}_{ka} - f_{ka}\right)^{C^\infty, R}_{ESA} \delta r^{ka} = 0 . \tag{2.11.1}$$

We shall then say that the conventional Hamilton's principle is a selfadjoint variational principle[2a] because it induces selfadjoint analytic equations.

The existence of the isotopic degrees of freedom of a Lagrangian for the systems considered provides a first departure from conventional patterns. And indeed, for these Lagrangians we now have

$$\delta\int_{t_1}^{t_2} dt\, L^* = -\int_{t_1}^{t_2} dt\left(\frac{d}{dt}\frac{\partial L^*}{\partial \dot{r}^{ka}} - \frac{\partial L^*}{\partial r^{ka}}\right)^{C^\infty, R}_{SA} \delta r^{ka} = -\int_{t_1}^{t_2} dt\left[h_{ka}{}^{jb}\left(\mu_j \ddot{r}_{jb} - f_{jb}\right)^{C^\infty, R}_{SA}\right]^{C^\infty, R}_{SA} \delta r^{ka} = 0 \tag{2.11.2}$$

Predictably, these variational algorithms may be discarted by some reader because unconventional.

The point which we would like to bring to the reader's attention is that the variations used in principle (2.11.1) are the simplest possible variations of the Calculus of Variations, i.e., of the type

$$\delta r^{ka} = \eta^{ka}(t)\, w\,, \quad w \approx 0, \tag{2.11.3}$$

called weak variations. In actuality, variations can have considerably more involved structures. In principle any implicit or explicit functional dependence of the variations in the independent variable, time in our case, which satisfies the desired continuity properties is acceptable, and we shall arite $\delta r^{ka} = (\delta r^{ka})(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})$. And indeed, since the first order variation of the action must be computed along the actual path to yield identity (2.11.1), this has the effect of reducing the functional dependence of the variations to only that of time, i.e., $(\delta r^{ka})(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})\big|_{\underset{\sim}{r}^0} = (\delta r^{ka})(t)$, where $\underset{\sim}{r}^0$ is the actual path (that is, the path which renders null variation (2.11.1)).

As a result of this occurrence we can assume the following form of the variations

$$\delta^{*-1} r^{ka} = h^{-1}{}_{jb}{}^{ka}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})\, \delta r^{jb}\,, \quad (h^{-1}) = (h)^{-1}, \tag{2.11.4}$$

under which principle (2.11.2) becomes

$$\delta^{*-1} \int_{t_1}^{t_2} dt\, L^* = \delta \int_{t_1}^{t_2} dt\, L\,. \tag{2.11.5}$$

Equivalently, we can write

$$\delta \int_{t_1}^{t_2} dt\, L^* = \delta^* \int_{t_1}^{t_2} dt\, L\,. \tag{2.11.6}$$

We shall call principle (2.11.5) an isotopically mapped variational principle[2b,3c]. Its net effect is that of producing the equations of motion in the "natural form" as derived from Newton's second law. In conclusion, if structure (2.11.2) of the variational algorithm is undesired because unconventional, the isotopic degrees of freedom of a Lagrangian can be eliminated by the corresponding (inverse) degrees of freedom of the variational algorithm.

II. Case of nonessentially nonselfadjoint systems. In this case the use of Lagrange's equations and a Lagrangian produced by Theorem 2.6.1 yield the indirect analytic representation of nonconservative systems

$$\delta \int_{t_1}^{t_2} dt\, L = -\int_{t_1}^{t_2} dt \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^{ka}} - \frac{\partial L}{\partial r^{ka}} \right)_{SA} \delta r^{ka} \equiv -\int_{t_1}^{t_2} dt \left[h_{ka}{}^{jb} \left(m_j \ddot{r}_{jb} - F_{jb} \right)_{NENSA}^{C^\infty, R} \right]_{SA}^{C^\infty, R} \delta r^{ka}. \tag{2.11.7}$$

The principle, however, is still selfadjoint, contrary to the nonselfadjoint nature of the equations of motion in their "natural form".

This discrepancy can be removed with the use of variations of type (2.11.4) which now yield the principle

$$\delta^* \int_{t_1}^{t_2} dt\, L = -\int_{t_1}^{t_2} dt \left[g_{ka}{}^{jb} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^{jb}} - \frac{\partial L}{\partial r^{jb}} \right)_{SA}^{C^\infty, R} \right]_{NSA}^{C^\infty, R} \delta r^{ka} \equiv -\int_{t_1}^{t_2} dt \left(m_k \ddot{r}_{ka} - F_{ka} \right)_{NENSA}^{C^\infty, R} \delta r^{ka} = 0. \tag{2.11.8}$$

Here an occurrence of particular relevance for our analysis emerges. Principle (2.11.8) is a nonselfadjoint variational principle[2b] because it induces analytic equations of nonselfadjoint type, i.e.,[3a,3b,3c]

$$\left\{ g_{ka}{}^{jb}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) \left[\frac{d}{dt}\frac{\partial L(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})}{\partial \dot{r}^{jb}} - \frac{\partial L(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})}{\partial r^{jb}} \right]_{SA}^{C^\infty, R} \right\}_{NSA}^{C^\infty, R} = 0. \tag{2.11.9}$$

By construction, this is a trivial equivalence transformation of Lagrange's equations induced by the multiplication of a class C^∞ and regular matrix of factor functions. However, it will have nontrivial implications for the Lie-admissible formulations because these formulations are nonselfadjoint by central requirement. As a result, the analytic equations in the nonselfadjoint form (2.11.9) rather than the conventional selfadjoint form (2.1) will be useful for the direct transition from the analytic equations of the Inverse Problem to those of the Lie-admissible problem (see Table 3.4).

At this point let us content ourselves with the remark that nonselfadjoint variational principles allows the direct analytic representations of nonconservative (nonessentially nonselfadjoint) Newtonian systems in their "natural form". This possibility is, in essence, implicit in the same definition of "analytic representation". And indeed, identities (2.3.1) are trivially equivalent to their nonselfadjoint version (2.11.8)

$$(g_{ka}{}^{jb}) = (h_{ka}{}^{jb})^{-1}. \tag{2.11.10}$$

What is again important on representational grounds is that, irrespective of the form of the variational principle (selected on grounds of personal preference) the system of implicit functions of the (selfadjoint or nonselfadjoint) analytic equations and those of the equations of motion coincide.

III. Case of essentially nonselfadjoint systems. In this case the conventional structure of the principles and the conventional structure of the integrands are fully admissible, although now acting in a new system of local coordinates. The drawback is that such a system is generally noninertial and generally nonrealizable with experimental set ups.

In this respect the use of Birkhoff's equations might play a significant role. Let us recall from the proof of Theorem 2.9.1 that in the construction of the analytic representation in terms

Birkhoff's equation (step 2 of the proof), the space components of the variables are those of the experimental set up by construction, although the y_k components are generally noncanonical. The net effect is that the representation of essentially nonselfadjoint systems with principle (2.8.9), i.e.,

$$\delta\int_{t_1}^{t_2} dt\, F(a,\dot a) = -\int_{t_1}^{t_2} dt \left(\Omega_{\mu\nu}\,\dot a^{\nu} - \frac{\partial H^B}{\partial a^{\mu}}\right)^{C^{\infty}R}_{SA} \delta a^{\mu} = 0, \qquad (2.11.11)$$

$$\{a^{\mu}\} = \{\underset{\sim}{x}, \underset{\sim}{y}\},$$

might play a significant role for relativity considerations. This aspect will be left to the interested reader. Notice, however, that principle (2.11.11) is selfadjoint and admits nonselfadjoin t generalizatios of type (2.11.8). Notice also that the integrand of this principle is totally degenerate, in the sense that not only the Hessian determinant is identically null, but actually each element of the Hessian matrix is identically null, trivially, because the F-function is linear in the first-order derivatives.

It is of some significance also to indicate that the methodology of the Inverse Problem allows the identification of a series of generalizations of Hamilton's principle, e.g., to include the integrability conditions for the existence of a Lagrangian directly in the variational algorithm[22a]

$$\delta A = -\int_{t_1}^{t_2} dt \left(\frac{d}{dt}\frac{\partial L}{\partial \dot z^{k_a}} - \frac{\partial L}{\partial z^{k_a}}\right)\delta z^{k_a} = 0, \qquad (2.11.12a)$$

$$\lim_{\tilde\delta\to\delta} \frac{1}{2}\left[\delta(\tilde\delta A) - \tilde\delta(\delta A)\right] = \lim_{\tilde\delta\to\delta}\int_{t_1}^{t_2} dt \left[\tilde\delta z^{k_a} J_{k_a}(\delta z) - \delta z^{k_a}\tilde J_{k_a}(\tilde\delta z)\right] = 0, \qquad (2.11.12b)$$

or to include Lagrange's equations and their associated Jacobi's equations, or to include the additional presence of end points contributions (in addition to the symplectic generalization (2.8.11) and the nonselfadjoint generalization (2.11.8) indicated here). For these generalizations we refer the reader to refs.[22a,22b].

As a final remark, our interest in variational principles for nonconservative systems is not that of constituting an alternative to the analytic equations. Instead, it is mainly of methodological nature, with particular reference to the problem of quantization of nonconservative forces (to be considered in a subsequent paper) via the Hamilton-Jacobi equations derived, as customary from variational principles with end points contributions.

TABLE 2.12: THE NOTIONS OF SYMMETRIES AND FIRST INTEGRALS FOR NONCONSERVATIVE SYSTEMS. One of our central objectives is that of identifying the mechanism of Galilei symmetry breaking produced by nonconservative forces. It is therefore recommendable to first identify the notion of exact symmetry for a nonconservative system. In Table 2.10 we indicated the

potentially misleading nature of the conventional attitude of conservative mechanics in the study of the Lagrangian structure for nonconservative systems. The remark was repeated in relation to the variational approach of Table 2.11. There is no doubt that the potentially misleading nature of the conventional attitude of conservative mechanics reaches its climax in relation to the problem of symmetries and first integrals of nonconservative systems. Let me indicate from the outset that the potential difficulties are solely conceptual in nature, because the available techniques for the study of symmetries and first integral are fully established on unequivocal technical grounds.

The first area of potential misrepresentation is constituted by the used terminology. And indeed, the use of the conventional terms "conservation laws" is clearly misleading because, as a selfevident condition, nonconservative systems violate the conservation laws (e.g., a necessary condition for a system to be nonconservative is that its total physical energy is non conserved).

In the following we shall use the following terminology. With the term symmetry (or exact symmetry) we refer to the rather universally accepted definition, that is, a (class C^{∞}, invertible) transformation of the independent and dependent variables under which a Lagrangian preserves its functional structure up to terms with null Lagrange's derivatives

$$L(t,\underset{\sim}{z},\dot{\underset{\sim}{z}}) = L'(t',\underset{\sim}{z}',\dot{\underset{\sim}{z}}')\frac{\partial t'}{\partial t}, \quad L'(t',\underset{\sim}{z}',\dot{\underset{\sim}{z}}') = L(t',\underset{\sim}{z}',\dot{\underset{\sim}{z}}') + \frac{d}{dt}G'(t',\underset{\sim}{z}'). \qquad (2.12.1)$$

This is equivalent to the definition that the transformations leave form-invariant the underlying equations of motion. The terms manifest symmetries will be referred to symmetries of a Lagrangian or of the equations of motion which are identifyable with simple means, often a visual inspection (e.g., the symmetry of conservative systems under translations in time). The terms nonmanifest symmetries will be referred to symmetries which are of complex identification, usually, via indirect techniques. Discrete symmetries (i.e., symmetries under space-time inversions) will be ignored for simplicity and we shall restrict the outline to the case of connected Lie symmetries in the conventional sense. These symmetries will be classified into: (a) contemporaneous, when they occur at a fixed value of time (e.g., rotations), (b) noncontemporaneous, when they include time transformations, (c) first-order, when they are infinitesimal of the first-order, (d) order p, when they are infinitesimal of order p, and (e) finite, when they are characterized by finite, connected, Lie transformations.

A set of functions $I_s(t,q,\dot q)$, $r=1,2,\ldots,m$, are called first integrals when they are conserved along the actual path, i.e.,

$$\dot{I}_s = \frac{\partial I_s}{\partial t} + \frac{\partial I_s}{\partial q^k}\dot{q}^k + \frac{\partial I_s}{\partial \dot{q}^k} f^k \equiv 0, \qquad (2.12.2)$$

where the f's are the implicit functions of the system. The functions I_s will be said to represent physical conservation laws if and only if they represent physical quantities as commonly understood, i.e., the physical total energy, linear momentum and angular momentum. The distintion between conservation laws and first integrals is truly crucial for the study of the problem of the relativity laws of Newtonian Mechanics. In essence, there is a basic distinction between a mathematical and a physical content. On mathematical grounds the occurrence that a quantity is conserved along the actual path is basically insensitive as to whether the represented system is conservative or not. On physical grounds the situation is different. And indeed, a quantity which is conserved is not necessarily representative of a physical "law". An example is here useful to illustrate this crucial distinction. For the case of the one-dimensional conservative harmonic oscillator the quantity

$$\dot{E} = \frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = \dot{H}(q,p) = 0, \quad L = \frac{1}{2}(\dot{q}^2 - q^2), \qquad (2.12.3)$$

represents not only a first integral but also a conservation law because the mathematical algorithm "H" represents a physical quantity. In the transition to the nonconservative damped oscillator the situation is different. And indeed, this system admits the quantity[12c]

$$H(q,p) = \ln q - \ln\cos(\omega p q) - \frac{1}{2}\gamma q p, \qquad (2.12.4)$$

which is conserved along the actual path. As such, it does constitute a first integral, but not a conservation law. And indeed, the assumption that the mathematical algorith "H" in this case also represents a physical quantity would be in contradiction with the experimental evidence that the physical energy decays in time. In conclusion, the attitude which is recommended for non-conservative systems is that their "conserved quantities" are, in general, only first integrals and they are not representative of physical conservation laws.

A second possible area of misconceptions is related to the methodology which associates symmetries to first integrals. As is weell known, this topic, within the context of conservative mechanics, is dominated by Noether's theorem.[20] A few remarks are here in order. The first is that this theorem essentially guarantees that, whenever a Lagrangian possesses a symmetry under an n-parameter connected Lie group, there exist n first integrals. The point is that, by no means, this theorem ensures that the first integrals are representative of physical conservation laws, nor I am aware of any intent by Emmy NOETHER to this effect. Another point which is not often emphasized in the existing literature (see, however, ref.[17a]) is that Noether's theorem does not guarantee that the n first integrals generated by an n-dimensional connected

Lie symmetry are actually independent. For instance, the 10 conserved quantities generated by the Galilei symmetry of the Lagrangian for the free motion, $L = \frac{1}{2} m \dot{\underline{r}}^2$, are (necessarily) nonindependent among themselves.

But a more controversial issue may be that related to the effectiveness of Noether's approach for symmetries and first integrals. It is known that for the conservative two-body problem the identification of all manifest symmetries (the Galilei group) leads, via Noether's theorem, to the identification of all first integrals needed for the solution of the system by quadrature. However, in the transition to the three-body problem the use of this approach does not produce the identification of all needed (18) first integrals. Clearly, if additional (independent) first integrals exist, they are associated to nonmanifest symmetries, in which case Noether's approach, as conventionally known, is ineffective.

In the transition to nonconservative systems these occurrences acquire a more definite light. And indeed, nonconservative systems are such that they seldom admit manifest symmetries. This brings into focus in a natural way the problem of the methodology for the identification of first integrals.

In this respect the following possibilities are conceivable. First of all, one might attempt to complement Noether's approach with additional insights. For instance, the use of isotopically mapped Lagrangians may be of assistance in this respect, because they may turn nonmanifest symmetries of the original Lagrangian into manifest symmetries of the new Lagrangian.[3c,2b] As a simple example, consider the particle with drag force

$$\ddot{q} + \gamma\dot{q} = 0. \qquad (2.12.5)$$

Two independent first integrals are needed for its solution by quadrature. A first Lagrangian

$$L^* = \dot{q}\ln\dot{q} - \gamma q \qquad (2.12.6)$$

exhibits the manifest symmetry under translations in time yielding the first integral

$$I_1 = \dot{q} + \gamma q. \qquad (2.12.7)$$

But, if one insists in considering only Lagrangian (2.11.6), the identification of the second first integral becomes rather involved. The use of the isotopically mapped Lagrangian

$$L = e^{\gamma t}\frac{1}{2}\dot{q}^2, \qquad (2.12.8)$$

instead, produces a trivial solution. And indeed, this equivalent Lagrangian now exhibit a manifest symmetry under translations in space, by therefore yielding the second (independent) first integral

$$I_2 = e^{\gamma t}\dot{q}. \qquad (2.12.9)$$

The reader should be aware that first integrals are conserved in virtue of the equations of motion (or Lagrange's equations). Thus, both quantities (2.12.7) and (2.12.9) are conserved for Lagrange's equations in each Lagrangian (2.12.6) or (2.12.8). This implies that, say, Lagrangian (2.12.6) possesses a nonmanifest symmetry which leads to quantity (2.12.9) and a similar situation occurs for Lagrangian (2.12.8) and quantity (2.12.7). For ean explicit calculation see ref.[2b] It is in this sense that the mechanism of isotopical mapping of a Lagrangian can turn a nonmanifest symmetry of the original Lagrangian (i.e., a symmetry of difficult identification as such) into a manifest symmetry of the isotopic image.

Similarly, the representation of the same system with both, Hamilton's and Birkhoff's equations can be of assistance. And indeed, the manifest symmetries of the Hamiltonian are not expected to be generally preserved by the "Birkhoffian". The potential relevance of the latter equations relies precisely in this general loss of the original symmetries. And indeed, this may imply that the "Birkhoffian" has new manifest symmetries which, as such, can be useful for the identification of first integrals.[2b]

Notice that, even when the original manifest symmetries are preserved in the transition from the original Hamiltonian to either an isotopic image or to a "Birkhoffian", this generally implies the identification of different first integrals (trivially, from the nontrivially different functional differences of these functions).

Despite these auxiliary implementations of Noether's approach, the need of a more effective methodological approach persists. This problem can be classified into the following two aspects.

A. Identification of the first integrals associated to the manifest symmetries of the equations of motion (rather than a Lagrangian). The insufficiency of Noether's approach, as currently known, for the resolution of this problem can be indicated by the property that the class of manifest symmetries of the equations of motion is generally larger than that of each individual Lagrangian for its analytic representation.[2b,3d] For instance, Eq. (2.12.5) possesses two manifest symmetries, translations in space and in time, while each individual Lagrangian (2.12.6) or (2.12.8) possesses only one symmetry (I have been unable to identify one Lagrangian for the system considered via the techniques of the Inverse Problem which exhibits two manifest symmetries). In conclusion, it appears that for an effective solution of the problem considered, the methods should be independent from Lagrangian representations. The most remarkable approach along these lines of which I am aware is the geometrical treatement by S. STERNBERG[21] which the interested reader is here urged to inspect.

B. Identification of the first integrals associated to the nonmanifest symmetries of the equations of motion. This problem goes at the very foundation of the problem of the relativity laws of nonconservative Newtonian systems. And indeed, one of the central aspects of this

problem is the identification of transformations which leave form-invariant nonconservative, (generally nonlinear and explicitly time dependent) equations of motion. Clearly these symmetries are expected to be of highly nonmanifest nature.

The study of these nonmanifest symmetries is one of the central objectives of the Lie-admissible formulations and, as such, it will be trated in Sections 3 and 4.

I cannot close this section without touching on another area of potential misrepresentations. I am here referring to the fact sometimes explicitly stated or implied by available treatments according to which conventional symmetries (e.g., translations in time, translations in space, rotations, etc.)lead to conventional physical conservation laws (total physical energy, linear momentum, angular momentum, respectively). Equivalently, I am here referring to an often implied unique association of the physical conservation laws and the symmetries for their derivation. The techniques of the Inverse Problem allow a disprof of these beliefs in the sense that, when a conventional physical conservation law occurs, a Lagrangian for its analytic representation deos not necessary exhibits the conventional symmetry. Viceversa, when a Lagrangian exhibits one of the indicated conventional symmetries, the induced first integral is not necessarily the conventionally associated quantity.

It is best to illustrate this point with the following occurrences.

OCCURRENCE 1: When the total physical energy of a system is conserved, a Lagrangian for its analytic representation is not necessarily invariant under translations in time. This occurrence is illustrated by Lagrangian (2.10.5a) which is explicitly dependent in time, neverthless, the represented system is conservative (the one-dimensional harmonic oscillator).

OCCURRENCE 2: When the total physical energy of a system is nonconserved, a Lagrangian for its analytic representation can be invariant under translations in time. This occurrence is illustrated by Lagrangian (2.10.9b) which is manifestly invariant under translations in time, neverthless the system is nonconservative (damped harmonic oscillator).

OCCURRENCE 3: When the total physical linear momentum of a system is conserved, a Lagrangian for its analytic representation can violate the symmetry under translations in space. This occurrence is illustrated by the system

$$\ddot{r}_x + \ddot{r}_y = 0, \qquad m = 1, \qquad (2.12.10a)$$

$$\ddot{r}_x - \ddot{r}_y + 2r_x = 0, \qquad (2.12.10b)$$

whose first equation expresses the conservation of the total linear momentum. Neverthless, the following Lagrangian for its analytic representation

$$L = \frac{3}{2}\dot{r}_x^2 + \dot{r}_x\dot{r}_y + \frac{1}{2}r_y^2 - r_x^2, \qquad (2.12.11)$$

is not invariant under translations in space.

OCCURRENCE 4: When the total physical linear momentum of a system is nonconserved, a Lagrangian for its analytic representation can be invariant under translations in space. This occurrence is illustrated by Eq. (2.12.5) for which the linear momentum decays (exponentially) in time. Neverthless, Lagrangian (2.12.8) for the representation of the system is manifestly invariant under translations in space.

OCCURRENCE 5: When the total physical angular momentum of a system is conserved, a Lagrangian for its analytic representation can violate the symmetry under rotation. This occurrence has been studied by G. MARMO and E. J. SALETAN [18b]. Consider the three-dimensional harmonic oscillator

$$\ddot{\underset{\sim}{r}} + \underset{\sim}{r} = 0 \quad , \quad m = k = 1 \, , \; \underset{\sim}{r} = (r_x, r_y, r_z), \qquad (2.12.12)$$

with trivial conservation laws of the angular momentum. A fully admissible Lagrangian for this system is given by

$$L^* = \frac{1}{2}\left(\dot{r}_x^2 - \dot{r}_y^2 + \dot{r}_z^2\right) - \frac{1}{2}\left(r_x^2 - r_y^2 + r_z^2\right), \qquad (2.12.13)$$

and, sas such, it violates the symmetries under rotations (see also Table 2.13).

OCCURRENCE 6: When the total physical angular momentum of a system is nonconserved, a Lagrangian for its representation can be invariant under rotations. This occurrence is illustrated by Lagrangian (2.6.7) which is manifestly invariant under rotations. Neverthless, the represented system is highly nonconservative (and nonlinear in the velocity terms).

These occurrences inevitably lead to the following aspect of particular significance for relativity considerations.

OCCURRENCE 7: The symmetry of a Lagrangian under the Galilei group does not necessarily imply the validity of the physical Galilei conservation laws (total physical energy, linear momentum, and angular momentum and uniform motion of the center of mass).

A few comments are here in order. Notice the emphasis on the word "physical" when used in the context of the conservation laws. This is suggested by a possible trap of nonconservative systems (generally absent for conservative settings) according to which, say, the mathematical algorithm represented by the canonical momentum "$\underset{\sim}{P}$" or the canonical angular momentum "$\underset{\sim}{M}$" are mathematical quantities of the type

$$\underset{\sim}{P} = \underset{\sim}{\gamma} \sinh^{-1} \beta \dot{\underset{\sim}{r}}^2 \quad , \; \gamma, \beta = \text{const},$$

$$\underset{\sim}{M} = \sinh^{-1} \beta \dot{\underset{\sim}{r}}^2 (\underset{\sim}{r} \times \underset{\sim}{\gamma}). \qquad (2.12.14)$$

To be explicit, in conservative mechanics the symbol "p" generally represents a physical quantity, the linear momentum $m\dot{\underset{\sim}{r}}$. As such the use of the term "physical" when referring to "p" is inessen tial. In the transition to nonconservative systems the situation is different. Here the Lagrangians must possess a generalized structure. In turn, this means that the algorith $\underset{\sim}{P} = \partial L / \partial \dot{\underset{\sim}{r}}$ has, in general, no direct physical significance. The term "physical" is then used for the intend of differentiating between the canonical quantity $\underset{\sim}{P}$ and the physical quantity $m\dot{\underset{\sim}{r}}$. Similarly, the "physical an gular momentum" of system (2.6.6) is the conventional quantity $\underset{\sim}{M} = \underset{\sim}{r} \times \underset{\sim}{p} = \underset{\sim}{r} \times m\dot{\underset{\sim}{r}}$, Occurrence 6 refers to the nonconservation of this quantity. If canonical quantities are considered, the situation is different. And indeed, the manifest symmetry of Lagrangian (2.6.7) under rotations certainly leads to a conserved quantity (the canonical angular momentum). The point is that this quantity does not coincide with the physical angular momentum.

For a detailed discussion of the occurrences indicated above, as well as for alternative examples, see refs. [2b, 3d, 19c]

[Note added in proof: in relation to the physical implications in the selection of the phase space variables the reader should also consult A. P. BALACHANDRAN, T. R. GOVINDARAJAN and B. VIJAYALAKSHMI, Syracuse University preprint SU-4211-110, January 1978].

TABLE 2.13: THE NOTIONS OF ALGEBRAIC, GEOMETRIC AND ANALYTIC ISOTOPY.

Let U be an (associative or nonassociative) algebra with elements a, b, c, ... over a field F with elements $\alpha, \beta, \gamma, \ldots$ equipped with the (abstract) product ab satisfying given laws (associativity, commutativity, Lie, etc.). An isotopic mapping $U \longrightarrow U^*$ of U is the mapping from U to an algebra U* which coincides with U as vector space (that is, the elements of U and U* coincides) and which is equipped with a new product a*b such to preserve the algebraic laws of U (that is, if ab is associative or Lie, a*b is equally associative or Lie, respectively). The algebra U* is then called an isotope of U.

An isotopic mapping of the product ab can be realized in a variety of ways. Suppose that ab is associative. Then the mapping

$$ab \longrightarrow a*b = \alpha ab \quad , \quad \alpha \in F, \qquad (2.13.1)$$

is isotopic because trivially preserves the associativity laws. However, isotopic mappings can be realized also in terms of elements of U. Let c be an invertible element of U.

If mapping of the types

$$ab \to a^* b = (ac)b \,, \quad \text{or,} \quad a^* b = (ac)b + (a(1-c))b \,, \qquad (2.13.2)$$
$$c = \text{fixed}, \quad a, b \in U,$$

preserve the algebraic laws of U, they are isotopic.

In essence, the notion of algebraic isotopy is intended to express the "degrees of freedom" of the product to satisfy given algebraic laws without changing the algebra as vector space. This notion is rather old, and actually dates back to the early stages of set theory[22]. And indeed, the notion was apparently identified for the first time within the context of the Latin squares. Two latin squares were called isotopically related if they could be made to coincide by using permutations. But Latin squares can be interpreted as the multiplication table of quasigroups. The extension to quasigroups, groups and, then, algebras, is then direct. As R. H. BRUCK put it, the concept of algebraic isotopy is "so natural to creap in unnoticed".[22a] And indeed, this notion has received rather little attention in recent times, to the best of my knowledge.

During the course of our analysis we shall attempt the identification of the possible existence of the notion of isotopy at several levels of study, e.g., Lie algebra, symplectic geometry, Lie's transformation theory. The reason for our interest is that this notion appears to be relevant for the study of Lie-admissible algebras.

In this table we shall outline the rudiments of the algebraic, geometrical and analytic realizations of the notion of isotopy which are relevant in Newtonian Mechanics.

Consider the Lagrangians

$$L = \frac{1}{2}\left[(\dot{r}_x^2 + \dot{r}_y^2 + \dot{r}_z^2) - (r_x^2 + r_y^2 + r_z^2)\right] \leftrightarrow L^* = \frac{1}{2}\left[(\dot{r}_x^2 - \dot{r}_y^2 + \dot{r}_z^2) - (r_x^2 - r_y^2 + r_z^2)\right], \qquad (2.13.3)$$

which we have called isotopically related (Table 2.10). They represent the same system, a part the trivial multiplication by (-1) of the second equation. The angular momentum is conserved for both Lagrangians. The symmetry of L which leads to this conservation law is the group of rotations SO(3). It is possible to prove that the symmetry of L* which leads to the same conservation law is the Lorentw group SO(2.1). What we would like to indicate is that the Lie algebras SO(3) and SO(2.1) are isotopically related.

Here a departure from conventional classical realizations of Lie algebras is essential. Typically, the algebras SO(3) and SO(2.1) are realized in terms of different generators because nonisomorphic. This conventional realization would now be inconsistent. The mathematical and physical meaning of this SO(2.1) symmetry is that of leading to the conservation law of the angular momentum, that is, to the generators of SO(3). Thus, to be consistent with mapping

(2.13.3), SO(2.1) must be realized in terms of the angular momentum components as generators. To reach with these generators an algebra which is nonisomorphic to SO(3), there is then only one possibility left: perform an invertible Lie algebra preserving mapping of the product, that is, an isotopic mapping. And indeed, simple calculations yields the solution[19c]

$$SO(2.1): [M_x, M_y]^* = M_z, \quad [M_y, M_z]^* = -M_x, \quad [M_z, M_x]^* = M_y, \qquad (2.13.4a)$$
$$SO(3): [M_x, M_y] = M_z, \quad [M_y, M_z] = M_x, \quad [M_z, M_x] = M_y, \qquad (2.13.4b)$$
$$[A,B] = \frac{\partial A}{\partial \alpha^\mu}\,\omega^{\mu\nu}\,\frac{\partial B}{\partial \alpha^\nu}, \quad (\mathcal{R}^{\mu\nu}) = \begin{pmatrix} 0_{3N\times 3N} & \begin{pmatrix} +1 & 0 \\ 0 & -1 \\ 0 & +1 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ +1 & \\ 0 & -1 \end{pmatrix} & 0_{3N\times 3N} \end{pmatrix}, \quad (\omega^{\mu\nu}) = \begin{pmatrix} 0_{3N\times 3N} & 1_{3N\times 3N} \\ -1_{3N\times 3N} & 0_{3N\times 3N} \end{pmatrix}.$$
$$[A,B]^* = \frac{\partial A}{\partial \alpha^\mu}\,\mathcal{R}^{\mu\nu}\,\frac{\partial B}{\partial \alpha^\nu}, \qquad (2.13.4c)$$

In conclusion, the algebras SO(3) and SO(2.1) in the above realizations are isotopically related (that is, one is the isotope of the other), because (a) they coincide as vector spaces (that is, they are realized in terms of the same generators), (b) they preserve the parameters(but not necessarily their range, one algebra being of compact and the other of noncompact type), and base manifold (that is, they are both defined in terms of the same phase space coordinates) and (c) they are defined in terms of different Lie products.

Notice that the transition from the conventional Poisson brackets $[A,B]$ of SO(3) to the generalized brackets $[A,B]^*$ is defined in terms of elements of the field. This is clearly a particular case, the most general case being that defined in terms of elements of the base manifold. We reach in this way the conclusion that the transition from the conventional to the generalized Poisson brackets without changing the base manifold

$$[A,B]_{(a)} \longrightarrow [A,B]^*_{(a)}, \qquad (2.13.5)$$

is a Newtonian realization of the notion of Lie algebra isotopy. Intriguingly, the roots of this notion rest on the property that the symmetry of a Newtonian system capable of characterizing first integrals (or conservation laws) is not necessarily unique (Table 2.12).

For a detailed treatement of this notion, see refs.[2b,5b]. Here let us only recall that the Lie isotopies do not necessarily preserve the compact or noncompact, semisimple or nonsemisiple and Abelian or nonAbelian character of the original algebra. For instance, another isotope of SO(3) can be, at least in principle, an Abelian three-dimensional algebra. Of course, the notion here considered necessarily preserves the dimensionality of the original algebra. This implies in particular that the isotopically mapped product must be such to yield a closed algebra with the generators of the original algebra.

As we shall see in Section 3, the notion of Lie algebra isotopy admits a consistent group

image. Here let us indicate that the algebraic isotopy (2.13.4) admits a direct geometrical counterpart, the symplectic isotopy (for autonomous cases)

$$\omega_2 = \omega_{\mu\nu}\, da^\mu \wedge da^\nu \longrightarrow \Omega_2 = \Omega_{\mu\nu}\, da^\mu \wedge da^\nu, \qquad (2.13.6)$$

or the corresponding contact isotopies for the nonautonomous case. And indeed, the realizations of SO(3) and SO(2.1) indicate earlier are such that the covariant versions $\omega_{\mu\nu}$ and $\Omega_{\mu\nu}$ of the respective Lie tensors $\omega^{\mu\nu}$ and $\Omega^{\mu\nu}$ characterize symplectic structures. But the mapping occurs within the same base manifold. Thus, it is a case of symplectic isotopy in our terminology.

Another example is provided by the transition from the conventional canonical form (2.8.2.) to our realization (2.8.2), i.e.,

$$\theta_1 = p_k\, dq^k \longrightarrow \omega_1 = -a_\mu da^\mu = p_k dq^k - q^k dp_k. \qquad (2.13.7)$$

Again, the base manifold is not changed. Neverthless, both forms lead (via exterior derivation) to a symplectic structure. Thus, mapping (2.13.7) is an example of symplectic isotopy.

In general we can say that a Lie algebra isotope always admit a corresponding symplectic image via the covariant version of the Lie tensors.

To conclude, let us briefly indicate the analytic origin of these algebraic and geometric isotopies. Consider the following equivalence transformation of Hamilton's equations within the same base manifold

$$\left\{ h_\mu{}^\nu(a) \left[\omega_{\nu\rho}\, \dot{a}^\rho - \frac{\partial H(a)}{\partial a^\nu} \right]^{C^\infty, R}_{SA} \right\}^{C^\infty R}_{SA} = 0. \qquad (2.13.8)$$

Under the assumption that, for a given Hamiltonian, the equivalent system is selfadjoint, we reach Birkhoff's equations, i.e.,

$$\Omega_{\mu\nu}\, \dot{a}^\nu - \frac{\partial H^B}{\partial a^\mu} = 0, \quad \Omega_{\mu\nu} = h_\mu{}^\nu \omega_{\nu\rho} = \frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial R_\nu}{\partial a^\mu}, \qquad (2.13.9)$$

$$\frac{\partial H^B}{\partial a^\mu} = h_\mu{}^\nu \frac{\partial H}{\partial a^\nu}.$$

Thus, the analytic origin of the notions here considered is that of a selfadjointness preserving equivalence transformation of the analytic equations within the same system of variables. This illustrates the reason for calling Lagrangians (2.13.3) isotopically related.

It is an instructive exercise for the interested reader to work out the SO(2.1) invariant representation of the harmonic oscillator in terms of Birkhoff's (rather than Hamilton's) equations.

TABLE 2.14: CLASSIFICATION OF THE BREAKINGS OF THE GALILEI SYMMETRY IN NEWTONIAN MECHANICS. We are now sufficiently equipped to study the mechanism of Galilei symmetry breaking due to nonconservative forces. This problem is studied in details in ref.[5a]. In essence, the use of the Inverse Problem allows the identification of the following five classes of Galilei symmetry breakings in Newtonian Mechanics.

ISOTOPIC BREAKING[5a]. This is a selfadjointness preserving Galilei symmetry breaking induced by the multiplication of a class C^∞ and regular matrix of factor terms to a conservative and Galilei form-invariant system, and I shall write

$$\left\{ h_{ka}{}^{jb}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) \left[m_j \ddot{r}_{jb} - f_{jb}(\underset{\sim}{r}) \right]^{GFI}_{SA} \right\}^{GFNI}_{SA} = 0, \qquad (2.14.1)$$

where SA stands for selfadjointness and GFI (GFNI) stands for Galilei form-invariance (Galilei form-noninvariance). At the Lagrangian level the breaking is characterized by an isotopic mapping of the type (Table 2.10)

$$\left[\frac{d}{dt} \frac{\partial L^*}{\partial \dot{r}^{ka}} - \frac{\partial L^*}{\partial r^{ka}} \right]^{GFNI}_{SA} = \left\{ \dot{h}_{ka}{}^{jb} \left[\frac{d}{dt} \frac{\partial L}{\partial \dot{r}^{jb}} - \frac{\partial L}{\partial r^{jb}} \right]^{GFI}_{SA} \right\}^{GFNI}_{SA}, \qquad (2.14.2)$$

where now the isotopically mapped Lagrangian is Galilei noninvariant owing to the integrating factors $h_{ka}{}^{jb}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})$ which enter into its structure via Eqs. (2.6.4).

On relativity grounds this breaking is the "weakest possible" to the point of being purely formal. This is due to the property recalled earlier in Table 2.12, according to which conserved quantities are conserved in virtue of the equations of motion. This property is left unaffected by equivalence transformations of type (2.14.2) (e.g., in the conservation law, according to Eq. (2.12.2), only the (unique) system of implicit functions enters). The net effect is that the physical conservation laws of the original system (total physical energy, linear momentum, etc.) persist for the isotopically mapped Lagrangian. This implies the existence of nonmanifest, first-order, noncontemporaneous, symmetries of L* which lead to the conserved quantities of L. We thus have a case of isotopically mapped Galilei algebra,[19c] that is, the generators, base manifold and parameters are unchanged, but the Lie brackets are now generalized. This isotope G*(3.1) of the Galilei algebra G(3.1) can be practically computed by using the techniques of the known inverse Noether approach (the conserved quantities and related Lagrangian are known and the symmetry leading to such conserved quantities is computed).[20r-20*f] Under the assumption that the approach extends to higher orders and that the integrability conditions for the exponential mapping are verified, we have the isotopically mapped Galilei group G*(3.1) (see Table 3.7 for more details). The interested

reader is here urged to work out the case, say, for the free particle.

In conclusion, the isotopic breakings of the Galilei symmetry are purely formal on relativity grounds because the original system verifies the Galilei relativity and this physical occurrence is not altered by equivalence transformations of type (2.14.1). Neverthless, this class of breakings is methodologically significant because it indicates the possibility of characterizing the conventional physical conservation laws of the Galilei relativity via a symmetry algebra $\underline{G}^*(3.1)$ which is generally nonisomorphic to the Galilei algebra, as typical of all isotopic mappings.

As we shall see in Section 4, the conjectured covering of the Galilei relativity for nonconservative systems is based on a generalization of the above notion of isotopy of the relativity algebra.

5a

SELFADJOINT BREAKINGS. This is a selfadjointness preserving Galilei symmetry breaking induced by the addition of Galilei form-noninvariant forces to a conservative and Galilei form-invariant system, and I shall write

$$\left\{ \left[m_k \ddot{r}_{ka} - f_{ka}(\underset{\sim}{r}) \right]^{GFI}_{ESA} - \hat{f}_{ka}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) \right\}^{GFNI}_{ESA} = 0. \qquad (2.14.3)$$

This is, in essence, the reformulation in the language of the Inverse Problem of the conventional Lagrangian approach to the (classical) breaking of any symmetry, i.e.,

$$L^{GI} \longrightarrow L^{GNI} = L^{GI} + L_0^{GNI}. \qquad (2.14.4)$$

And indeed, the condition of preservation of the selfadjointness of the equations of motion by the additive force $\hat{f}_{ka}$ is rendered equivalent to the addition of the term L_0^{GNI} at the Lagrangian level by the existence theorems of the Inverse Problem.

This class of breakings is not trivial on relativity grounds because it implies the general loss of the physical conservation laws (e.g., one can add a time dependent applied force derivable from a potential which induces the nonconservation of the physical energy) as well as the general loss of the form-invariance of the equations of motion, i.e.,

$$G(3.1): \left\{ \left[m_k \ddot{r}_{ka} - f_{ka}(\underset{\sim}{r}) \right]^{GFI}_{ESA} - \hat{f}_{ka}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) \right\}^{GFNI}_{ESA} = 0 \qquad (2.14.5)$$

$$\longrightarrow \left\{ \left[m_k \ddot{r}'_{ka} - f_{ka}(\underset{\sim}{r}') \right]^{GFI}_{ESA} - \hat{f}^{*}_{ka}(t', \underset{\sim}{r}', \dot{\underset{\sim}{r}}') \right\}^{GFNI}_{ESA} = 0.$$

Neverthless, this class of breakings is rare in the physical reality because the Newtonian forces are generally nonderivable from a potential.

5a

SEMICANONICAL BREAKING. This is a nonessentially nonselfadjoint breaking induced by the addition of Galilei form-invariant forces not derivable from a potential to a conservative and Galilei form-invariant system, and I shall write

$$\left\{ \left[m_k \ddot{r}_{ka} - f_{ka}(\underset{\sim}{r}) \right]^{GFI}_{ESA} - F_{ka}(\underset{\sim}{r}, \dot{\underset{\sim}{r}}) \right\}^{GFI}_{NENSA} = 0. \qquad (2.14.6)$$

In essence, the additive forces F_{ka} are such to (a) to be genuinely nonconservative (nonseldadjoint or, also equivalently, not derivable from a potential), (b) be capable of admitting an indirect representation without changing the coordinates r^{ka} according to Theorem 2.6.1 (nonessentially nonselfadjoint systems) and (c) be Galilei form-invariant. The Lagrangian representations are then of the type

$$\left[\frac{d}{dt} \frac{\partial L}{\partial \dot{r}^{ka}} - \frac{\partial L}{\partial r^{ka}} \right]_{SA} \equiv \left\{ h_{ka}^{ib} \left[\left(m_{ib} \ddot{r}_{ib} - f_{ib} \right)^{GFI}_{ESA} - F_{ib} \right]^{GFI}_{NENSA} \right\}_{SA}, \qquad (2.14.7)$$

with an essentially chiral Lagrangian structure (Table 2.10).

This class of breaking is also a rare occurrence in the Newtonian systems of our everyday experience and it is here quoted mainly for completeness. In essence, the aspect which is relevant in this class of breakings is that the physical conservation laws of the Galilei relativity can be lost due to forces which are Galilei form-invariant, but not derivable from a potential. The breakings are called "semicanonical" because (under the assumption that the integrating factors of Eqs. (2.14.7) are Galilei form-invariant), the canonical formalism of the Galilei relativity is fully definable, neverthless, it does not lead to the conventional physical conservation laws (for instance, the algorithm "$\underset{\sim}{P}$" $= \partial L / \partial \underset{\sim}{r}$ is a mathematical quantity which does not directly represent the physical linear momentum, etc.).

As a result, this class of breakings has its own methodological function. In particular, it focuses the attention on a dichotomy of canonical generators of physical transformations versus physical quantities, which is absent in the conventional conservative mechanics.

Permit me to elaborate on this point by reviewing first the conventional conservative and Galilei form-invariant case. Here, the physical quantities (total physical energy, linear momentum, etc.) coincide with the canonical generators of the corresponding physical transformations (translations in time, translations in space, etc.). This symbiotic meaning of the generators of the Galilei algebra is lost when nonconservative forces are included.

An example is here useful to illustrate this occurrence. Consider the breaking of the symmetry under translations in time produced by the addition of a linear velocity dependent drag force to, say, the harmonic oscillator, i.e., the conventional damped oscillator

$$T_1(t): \quad \left\{ \left[\ddot{r} + \omega_0^2 r \right]^{FI}_{SA} + \gamma \dot{r} \right\}^{FI}_{NENSA} = 0. \qquad (2.14.8)$$

The breaking is semicanonical in our terminology because the force not derivable from a potential is fully invariant under translations in time. Neverthless, the conservation law of the physical energy is lost. And indeed, experimental evidence indicates that the energy is dissipated and the motion tends to rest in a finite period of time. These occurrences are well known (see, for instance, ref.[23]). We are here interested in the mechanism of this breaking. First of all, a Hamiltonian for the representation of system (2.14.8) without an explicit time dependence exists and it is given by Eq. (2.12.4), i.e,

$$H = \ln q - \ln \cos(\omega p q) - \frac{1}{2}\gamma p q, \qquad (2.14.9)$$
$$p = p(t, q, \dot{q}), \quad \omega^2 = \omega_0^2 - \gamma^2/4 > 0.$$

This confirms the semicanonical nature of the breaking: the Hamiltonian is invariant under translations in time (physical transformation), neverthless the energy is not conserved (physical nonconservation law). But the canonical realization of the translations in time is fully defined and its generator is given by Hamiltonian (2.14.9). This, then illustrates the dichotomy indicated earlier: in nonconservative mechanics the generators of physical transformations do not coincide with physical quantities.

Two additional comments are here in order. The reader might be surprised at the terms "physical transformations" which are definitively absent in conservative mechanics. The intent of these terms is the following. Within the context of the canonical formalism, any class C^∞ function of the phase space variables induces perfectly acceptable transformations. Thus, rather than using Hamiltonian (2.14.9) one can use the physical energy

$$E = \frac{1}{2}\left(\dot{q}^2 + \omega_0^2 q^2\right), \qquad (2.14.10)$$

reexpressed in the (q, p) variables as a generator of a transformation of system (2.14.8). The aspect in which we are concerned is the physical meaning of such a transformation. It is easy to see that it <u>is not</u> a translation in time. The interested reader is here urged to work out the details to see that the transformation induced by quantity (2.14.10) is a highly involved transformation which carries no resemblance or connection with physically relevant transformations.

As a result, for nonconservative (nonessentially nonselfadjoint) systems, the generators of "physical transformations" (that is, translations in time, translations in space, etc.) do not coincide with "physical quantities" (the total physical energy, linear momentum, etc.). Viceversa, the use of these total physical quantities as generators does not lead to physical transformations as commonly understood.

Finally, we remain with the question "what is a physical quantity for a nonconservative system?" This concept is trivial for conservative mechanics, but the extension of the same notion to nonconservative systems does not appear to be trivial. Oddly, it is not immune of controversy and, thus, of personal viewpoints. The answer we shall use in the following is as naive as possible (actually, from undergraduate textbooks,[23] because, at a graduate level, forces nonderivable from a potential have remained largely ignored in recent times). The energy of system (2.14.6) will be assumed to be given by the sum of the kinetic energy and the potential energy of all forces derivable from a potential. The linear momentum is the conventional quantity $\underset{\sim}{p} = m \dot{\underset{\sim}{r}}$ and the angular momentum is also the familiar form $\underset{\sim}{M} = \underset{\sim}{r} \times \underset{\sim}{p} = \underset{\sim}{r} \times m\dot{\underset{\sim}{r}}$.

In the language of the inverse problem we can see that <u>the physical quantities of nonselfadjoint system (2.14.8) are given by the canonical quantity of the maximal associated selfadjoint system</u>, that is, the selfadjoint subsystem within the inner brackets of Eqs. (2.14.8). And indeed, since all F-forces are nonconservative by assumption, what we refer to as the "total physical energy" is the Hamiltonian (or energy integral) of the maximal associated selfadjoint system. A similar situation occurs for the other quantities.

The study of other viewpoints on the notion of physical quantities for nonconservative systems, and their bearing on the problem of the applicable relativity laws, will be left to the interested reader.

5a

<u>CANONICAL BREAKING</u>. This is a nonessentially nonselfadjoint breaking of the Galilei symmetry induced by the addition of Galilei form-noninvariant forces not derivable from a potential to a conservative and Galilei form-invariant system, and shall write

$$\left\{ \left[m_k \ddot{r}_{ka} - \hat{F}_{ka}(\underset{\sim}{r}) \right]^{GFI}_{ESA} - F_{ka}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) \right\}^{GFNI}_{NENSA} \qquad (2.14.11)$$

In this case the symmetry breaking forces F_{ka} are such to (a) be nonconservative, (b) be capable of inducing a nonessentially nonselfadjoint system and (c) be form-noninvariant under the Galilei transformations. The underlying Lagrangian representations are of the type

(2.4.7), i.e.

$$\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^{ka}} - \frac{\partial L}{\partial r^{ka}}\right]_{SA} \equiv \left\{ h_{ka}{}^{jb}\left[\left(m_j \ddot{r}_{jb} - f_{jb}\right)^{GFI}_{ESA} - F_{jb}\right]^{GFNI}_{NENSA} \right\}. \tag{2.14.12}$$

The breaking is called canonical because, even though the canonical formalism is fully definable via a Legendre transform, the breaking occurs at the level of the canonical formalism of the Galilei relativity, as necessary from the lack of invariance of the Hamiltonian.

This is a type of breaking which is more realized in the physical reality than the preceding breakings. It constitutes a class of particular methodological significance. To state it explicitly, these breakings should not be interpred as occurrences of marginal relevance. Instead, they should be interpred at the utmost of their conceptual, technical and physical implications. The best way to emphasize this profile is by focusing the attention on the breaking of a central methodological tool: the group of rotations (seethe remarks related to the nonconservative spinning top of Section 1). The issue which is then raised is whether this broken context should remain as currently is, methodologically undefined, or broader methods capable of characterizing this broken SO(3) symmetry should be attempt ed. This is an objective of Sections 3 and 4.

The reader should be aware that, despite the Galilei form-noninvariance of the F-forces, the canonical breakingsare still restrictive because they assume the validity of Theorem 2.6.1. We reach in this we the last class ofGalilei symmetry breakings characterizable by the Inverse Problem.

5a

ESSENTIALLY NONSELFADJOINT BREAKINGS. This is a breaking induced by the addition of Galilei form-noninvariant forces not derivable from a potential to a conservative and Galilei form-invariant system in such a way to violate the integrability conditions for the existence of an indirect Lagrangian representation within the reference frame of the experimental detection, and we shall write

$$\left\{ \left[m_k \ddot{r}_{ka} - f_{ka}(\underset{\sim}{r}) \right]^{GFI}_{ESA} - F_{ka}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) \right\}^{GFNI}_{ENSA} = 0. \tag{2.14.13}$$

This is the most general class of Galilei symmetry breaking(via local forces)which is rendered identifyable by the methodology of the Inverse Problem and, as such, it can be considered as inclusive of weaker forms of breakings.

The methodological implications of these breakings are rather deep. On a comparative basis between the canonical and the essentially nonselfadjoint breakings, in the former the canonical formalism is fully definable and the breaking occurs only at the level of the formalism of the Galilei relativity, while in the latter the entire canonical formalism is not definable under the condition that the space component of the underlying manifold is constituted by the Cartesian coordinates of the systems of the experimental verification. With the terms "canonical formalism" we here refer to that based on Hamilton's equations. In particular, this implies the inability of introducing all Lie algebras via the con ventional Poisson brackets

$$[A,B] = \frac{\partial A}{\partial a^{\mu}}\,\omega^{\mu\nu}\,\frac{\partial B}{\partial a^{\nu}} = \frac{\partial A}{\partial r^{ka}}\frac{\partial B}{\partial p_{ka}} - \frac{\partial A}{\partial p_{ka}}\frac{\partial B}{\partial r^{ka}}, \tag{2.14.14}$$

under the indicated restriction on the physical meaning of the r^{ka} variables.

This is the class of breakings in which we are primarily interested from here on. As an incidental note, the reader should be aware that these breakings cannot occur for one-dimensional systems because these systems can ut most be nonessentially nonselfadjoint. Explicitly, it is possible to prove that all one-dimensional (class C^{∞} , regular)and nonselfadjoint systems satisfy Theorem 2.6.1 for an indirect analytic representation. As a result, the conditions of this theorem can be broken only for a sufficiently high dimensionality.

By looking in retrospective, it has been for me rewarding to see that the methodology of the Inverse Problem has indeed fulfilled all my expectations, particularly on relativity grounds. And indeed, it provides a valuable method for the characterization of the acting forces in the transition from one relativity to another (in the sense of Section 5), for the identification of the mechanics of the Galilei symmetry breakings in Newtonian Mechanics, for the study of formulations of Lie-admissible type (see Section 3), etc. . Intriguingly, the methodology is of some significance also for nonrelativity related problems, such as nonlinear nonconservative plasma equations, electric circuits inclusive of internal losses, trajectory problems in atmosphere, etc. The reader interested in an outline of these possibilities, may consult refs.[22,2b].

As not unusual for theoretical formulations, the methodology of the Inverse Problem also exhibits rather precise limitations, particularly from a relativity profile. And indeed, despite my best efforts, I have been unable to confront the problem of the relativity laws of nonconservative mechanics within the context of only this methodological framework. To be more specific in this rather crucial point, besides effective possibilities for studying the Galilei relativity breakings, the methodology exhibit no constructive capacity for a generalized relativity, to the best of my understanding at this moment.

3: RUDIMENTS OF THE METHODOLOGY OF THE LIE-ADMISSIBLE PROBLEM.

The Inverse Problem of Newtonian Mechanics outlined in Section 2 could also be called the Lie Problem in the sense that the efforts are devoted to the representation of nonconservative systems in terms of a methodology whose underlying algebraic structure is a Lie algebra.

The Lie-admissible Problem of Newtonian Mechanics can be conceived as a body of methodological tools for the study of nonconservative systems whose underlying algebraic structure is not, this time, a Lie algebra by central requirement, but it is instead a Lie-admissible algebra (in the sense of Table 3.3).

It should be indicated that the terms "Lie-admissible problem" are here tentatively introduced mainly for reference to the content of this section and that a number of other terms could equivalently refer to the same topic. Notice that the only terms known in mathematical literature are "Lie-admissible algebras".

As by now familiar, the Inverse Problem or, more appropriately in this context, the Lie problem does admit a solution for the considered class of nonconservative systems. However, it appears that clear limitations of physical effectiveness emerge. I am here referring to the lack of constructive role of the methods for a generalized relativity, the loss of direct physical significance of the algorithms at hand ($\underset{\sim}{P}$, H, $\underset{\sim}{r} \times \underset{\sim}{P}$, etc.), the inability to produce Hamiltonian characterizations in a base manifold whose space coordinates are those of the reference frame of the experimental detection of the system considered, the generally noninertial nature of the coordinate systems of the indirect Lagrangian representations and their general nonrealizability with experimental set ups, etc.

The hope of the Lie-admissible Problem is that of identifying methods which avoid these difficulties. The fundamental starting point is the representation of essentially nonselfadjoint systems (2.14.13) in the reference frame of their experimental identification. This is clearly crucial for relativity considerations. Since the conventional Lagrange's equations are unable to satisfy this requirement, they will be modified in a suitable form capable of producing the desired "direct universality", that is, applicability to all systems (2.14.13) as given. On equivalent grounds, Hamilton's equations will be modified into a form capable of representing the equations of motion considered such that: (A) all algorithms at hand have a direct physical significance, that is, the symbol "r^{ka}" has the indicated inertial meaning, the symbol "p_{ka}" represents the physical linear momentum ($m_k \dot{r}_{ka}$), the symbol "H" represents the physical energy (sum of the kinetic energy and potential energy of all forces derivable from a potential or, more specifically, the Hamiltonian of the maximal selfadjoint associated system), the

symbol "$\underset{\sim}{M}$" represents the physical angular momentum ($\underset{\sim}{r} \times m\dot{\underset{\sim}{r}}$), etc., where the term "physical" (while obviously inessential in conservative mechanics) is here introduced to stress the difference with canonical quantities of type (2.12.14). In conclusion, the generalization of Hamilton's equations I shall be looking for is based on the preservation of all the algorithms of the Galilei form-invariant subsystem of Eqs. (2.14.13). The Galilei breaking forces will then be represented with a modification of the structure of the conventional Hamilton's equations. As we shall see in Section 4, this appears to be at the very foundation of the possibility of identifying a group of nonmanifest symmetries for the form-invariance of the Galilei-breaking system (2.14.13), provided that the brackets of the generalized time evolution law characterize a Lie-admissible algebra.

The reader should also keep in mind, from the content of Table 2.14. that the objective of this paper is the study of broken Lie, space-time, symmetries. As a result, the fundamental constructive problem is that of attempting the identification of algebraic-group theoretic methods for the treatment of broken symmetries which are fully parallel, although generalized, to the established methods for the treatment of exact symmetries.

To restate this situation in different terms, the mere identification of the breakings of the Galilei relativity in Newtonian Mechanics is, "per sé", sterile. To achieve a physically productive context, the central problem is that of the identification of effective methods for the treatment of such broken context. It is precisely in this respect that Lie-admissible algebras appear to be particularly intriguing. And indeed, on one side they guarantee the breaking of the Lie symmetry algebra while, on the other side, constitute a covering algebraic framework for the treatment of the broken Lie symmetry.

For conciseness, I shall again present the essential aspects of the analysis in sequential tables. To avoid a prohibitive length of the manuscript, the proofs of all theorems and major steps will be omitted. This section, however, is a summary of Volume II of ref.[5] In this reference, therefore, the interested reader can inspect all proofs of the theorems of this section. The assumptions which are tacitly implemented are the same as those of Section 2,

As a personal note, permit me to indicate that, without any doubt, the study of Lie-admissible algebras along the three profiles outlined in this section (analytic, algebraic and geometrical profiles) has been the most interesting, stimulating and rewarding research topic of my academic life. I hope that this paper will suceed in communicating some of my enthusiasm to receptive readers because this line of study is at the very beginning and so much remains to be done. I would like also to take this opportunity to express my appreciation to C. N. KTORIDES for calling in his papers[24] Santilli algebras the Lie-admissible algebras.

TABLE 3.1: THE NON-LIE ALGEBRA CHARACTER OF HAMILTON'S EQUATIONS WITH EXTERNAL TERMS. The customary form of Hamilton's equations with external terms

$$\dot{r}^{ka} = \frac{\partial H}{\partial p_{ka}}, \quad \dot{p}_{ka} = -\frac{\partial H}{\partial r^{ka}} + F_{ka}(t, \underset{\sim}{r}, \underset{\sim}{p}), \quad \begin{matrix} k=1,2,\cdots,N \\ a=x,y,z \end{matrix}, \qquad (3.1.1)$$

implies the following generalization of the time evolution law (2.8.7a)

$$\dot{A}(\underset{\sim}{r}, \underset{\sim}{p}) = \frac{\partial A}{\partial r^{ka}}\frac{\partial H}{\partial p_{ka}} - \frac{\partial A}{\partial p_{ka}}\frac{\partial H}{\partial r^{ka}} + \frac{\partial A}{\partial p_{ka}} F_{ka} = A \times H. \qquad (3.1.2)$$

Assume that, for fixed values of the external forces, this broader law characterizes generalized brackets here denoted with the symbol " AxB ". A simple inspection soon reveals that these brackets violate the Lie algebra identities, i.e.,

$$A \times B - B \times A \neq 0, \qquad (3.1.3a)$$

$$(A \times B) \times C + (B \times C) \times A + (C \times A) \times B \neq 0. \qquad (3.1.3b)$$

Thus, Hamilton's equations with external terms are non-Lie in algebraic character.

As indicated in Section 1, this occurrence is not negative "per sé". As a matter of fact, it can be considered methodologically intriguing because of the possible existence of a broader algebra underlying Eqs. (3.1.1).

However, for consistency, the brackets AxB must satisfy certain properties to characterize an algebra as commonly understood. In particular, the brackets must satisfy the right and left distributive laws and the scalar laws.[25] A simple inspection also reveals that the brackets AxB satisfy the left distributive law, but violate the right version of the same law, i.e.

$$(A + B) \times C = A \times C + B \times C, \qquad (3.1.4a)$$

$$A \times (B + C) \neq A \times B + A \times C, \qquad (3.1.4b)$$

Also, brackets AxB satisfy a right version of the scalar law but violate the left version of the same, i.e.,

$$\alpha \times (A \times B) = A \times (\alpha \times B) = (\alpha \times A) \times B, \quad \alpha = \text{const.} \qquad (3.1.5a)$$

$$(A \times B) \times \alpha \neq A \times (B \times \alpha) \neq (A \times \alpha) \times B. \qquad (3.1.5b)$$

As a result, the brackets AxB of the time evolution law of Hamilton's equations with external terms do not characterize an algebra, that is, Eqs. (3.1.1) are not only non-Lie but actually non-algebraic in nature. This situation indicates that, despite their preservation for over one century, Hamilton's equations with additive external terms must be modified to yield an acceptable algebraic structure.

TABLE 3.2: THE TENTATIVE GENEALOGICAL TREE OF THE LIE-ADMISSIBLE PROBLEM

By inspecting the occurrence of Table 3.1, one can see that the violation of the right distributive and scalar rules by the brackets AxB of law (3.1.2) is due to the additive nature of the external forces. This indicates that, if the forces not derivable from a potential are represented with multiplicative (rather than additive) terms to the derivatives of the Hamiltonian with respect to the local variables, the brackets of the emerging time evolution law are expected to characterize a fully acceptable (nonassociative) algebra.

For simplicity, let me consider the case of one space dimension. The modification of Eqs. (3.1.1) in which I have been initially interested can be written[4a, 4c, 4d]

$$\dot{r} = \frac{\partial H}{\partial p}, \quad \dot{p} = s(t, \underset{\sim}{r}, \underset{\sim}{p})\frac{\partial H}{\partial r}, \qquad (3.2.1a)$$

$$s = -1 + F/(\partial H/\partial r), \quad \frac{\partial H}{\partial r} \neq 0. \qquad (3.2.1b)$$

Where the last condition is assumed to be always satisfied for the argument of this table. If it is not, one can add and substract ficticious forces derivable from a potential.

Eqs. (3.2.1) characterize the following generalized time evolution law

$$\dot{A}(\underset{\sim}{r}, \underset{\sim}{p}) = \frac{\partial A}{\partial r}\frac{\partial H}{\partial p} + \frac{\partial A}{\partial p} s(t, \underset{\sim}{r}, \underset{\sim}{p})\frac{\partial H}{\partial r} = (A, H). \qquad (3.2.2)$$

It is easy to see that the transition from Eqs. (3.1.1) to their equivalent form (3.2.1) permits the characterization of a fully acceptable algebraic structure. And indeed, at a fixed value of the s-term, law (3.2.2) can be interpreted as characterizing generalized brackets here denoted with the symbol (A,B). Again, these brackets are non-Lie, i.e., they are such that for all nonnull values of the external forces (and, thus, for all values $s \neq -1$) the brackets (A,B) violate the Lie algebra identities

$$(A,B) - (B,A) \neq 0, \qquad (3.2.3)$$

$$((A,B),C) + ((B,C),A) + ((C,A),B) \neq 0. \qquad (3.2.4)$$

However, this time they do characterize a (nonassocoative) algebra because they satisfy the left and right distributive laws

$$(A + B, C) = (A,C) + (B,C), \qquad (3.2.5a)$$

$$(A, B + C) = (A,B) + (A,C), \qquad (3.2.5b)$$

the scalar rule

$$(A,\alpha) = (\alpha, A) = 0, \quad \alpha = \text{const.} \tag{3.2.6}$$

as well as the left and right differential rules

$$(AB, C) = (A,C)B + A(B,C), \tag{3.2.7a}$$

$$(A, BC) = (A,B)C + B(A,C). \tag{3.2.7b}$$

This was the status of my first understanding of noncanonical equations. After having ensured the existence of an acceptable algebraic structure, my next problem was the identification of the type of algebra characterized by brackets (A, B). On grounds similar to those of Table 2.2, the identification of the prior state of the art in the algebra characterized by brackets (A, B) turned out to be another very time consuming part of the project of this paper.

An initial search (at the libraries of the University of Torino, Italy, in 1965) revealed that the algebra of brackets (A, B) was simply not treated in available treateses in Abstract Algebra (it is still the case as of today, to the best of my knowledge, as the interested reader is encouraged to verify). I therefore initiated a second search in the specialized mathematical and physical literatures. This search turned out to be fruitless because the brackets (A, B) essentially violate most of the identities of the algebraic structures of general interest among mathematicians. For instance, the brackets are neither symmetric nor antisymmetric (for an arbitrary F-force)

$$(A,B) \neq \pm (B,A), \tag{3.2.8}$$

and, thus, this excludes both, the Lie algebras and the commutative Jordan algebras. Next, the brackets violate the alternative laws

$$((A,A),B) \neq (A(A,B)), \quad (B,(A,A)) \neq ((B,A),A), \tag{3.2.9}$$

and, thus, alternative algebras are excluded. Next, they violate the flexibility and the Jordan laws,

$$((A,B),A) \neq (A,(B,A)), \tag{3.2.10a}$$

$$(((A,A),B),A) \neq ((A,A),(B,A)), \tag{3.2.10b}$$

and thus, noncommutative Jordan algebras are excluded. Next, they generally violate the power associative law

$$((A,A),A) \neq (A,(A,A)), \quad ((A,A),(A,A)) \neq (((A,A),A),A), \tag{3.2.11}$$

and, thus, power associative algebras are excluded too. And so on.

However, it is well known that the types of algebras identified by mathematicians are quite numerous indeed. I therefore decided to enter into a detailed library search (which I conducted at the University of Torino and at the International Centre for Theoretical Physics of Trieste, Italy, in 1966-1967). My determination to pay tribute to previous contributions was finally rewarded. And indeed, I finally identified a paper by A. A. ALBERT of 1948[26a] in which he introduced the definition of a Lie-admissible algebra, although without any detailed treatement. The brackets (A, B) are indeed Lie-admissible because the algebra characterized by the attached brackets $[A, B]^* = (A,B) - (B,A)$ is Lie (see next table for more details). The only additional papers specifically devoted to the study of Lie-admissible algebras which I succeeded in identifying were a paper by L. M. WEINER of 1957[26b] and a paper by P. J. LAUFER and M. L. TOMBER of 1962[26c]. My rudimentary first papers on this subject[4a-4d] were primarily devoted to the understanding that Hamilton's equations with external terms, when properly written, are Lie-admissible in algebraic character.

Since that time a number of contributions have appeared in both the mathematical and physical literature. Within the former context, most notable is a series of studies on Lie-admissible algebras (of flexible type, see Table 3.3) by H. C. MYUNG conducted from 1971 until recently.[26d-l] See also the contributions by A. A. SAGLE (1971)[26m], D. R. SCRIBNER (1971)[26n] and H. STRADE (1972).[26o] Within the latter context, most notable is a paper by C. N. KTORIDES of 1975[24a] in which the generalization of the Poincaré-Birkhoff-Witt Theorem to Lie-admissible algebra is apparently studied for the first time. See also the studies by M. KŎIV and J. LŎHMUS (1972)[26q] on the covering nature of Lie-admissibility over the deformation theory and that by P. P. SRIVASTAVA (1976)[26r] which is sufficient to indicate the covering nature of Lie-admissible algebras over that of graded (supersymmetric) algebras.

These studies were sufficient to establish the following properties.

(A) The Lie-admissible algebras have a direct physical significance for systems with forces not derivable from a potential, where the term "direct" is here referred to the property of being applicable in the space of the coordinates of the reference frame of the experimental detection of the system and the physical momentum (the reader should keep in mind from Section 2 that this direct applicability is precluded to Lie algebras).

(B) The Lie-admissible algebras constitute an algebraic covering of the Lie algebras in a sense to be outlined in this section, which, in particular, has an analytic origin (the time evolution law) fully parallel to that of Lie algebras, although of generalized nature.

(C) The Lie-admissible algebras constitute an algebraic covering also of other astructures of current interest in theoretical physics, such as the deformation theory and the graded algebras of supersymmetric models.

$$A \circ B = \frac{\partial A}{\partial b^\mu} S^{\mu\nu}(t,b) \frac{\partial B}{\partial b^\nu}, \qquad (3.3.11)$$

where the $S^{\mu\nu}$ tensor satisfies the following continuity and regularity conditions

$$S^{\mu\nu} \in C^\infty(R) \quad , \quad \det\left(S^{\mu\nu}\right)(R) \neq 0, \qquad (3.3.12)$$

which will be tacitly assumed throughout this section. The space $\mathcal{U}$ equipped with product (3.3.11) is an algebra because Eqs. (3.3.1) and (3.3.2) are satisfied. Product (3.3.11) also verify the differential rules (3.2.7).

The tensor $S^{\mu\nu}$ is, in general, neither totally symmetric nor totally antisymmetric in the μ,ν indices. When exhibiting an explicit dependence in the b-variables, the tensor $S^{\mu\nu}$ and related brackets (3.3.11) will be called nontrivial.

The first central step of our program is turning the algebraic laws of Lie-admissibility into a system of quasilinear partial differential equations. The analytic brackets of the Lie-admissible formulations can then be characterized by the solutions of such a system. This objective is realized by the following [5b]

THEOREM 3.3.1: A necessary and sufficient condition for nontrivial brackets (3.3.11) to satisfy the general Lie-admissibility condition (3.3.8) is that all the following partial differential equations in the $S^{\mu\nu}$ tensor

$$\begin{aligned}&\left(S^{\mu\rho} - S^{\rho\mu}\right)\frac{\partial}{\partial b^\rho}\left(S^{\nu\tau} - S^{\tau\nu}\right)\\ &+ \left(S^{\nu\rho} - S^{\rho\nu}\right)\frac{\partial}{\partial b^\rho}\left(S^{\tau\mu} - S^{\mu\tau}\right)\\ &+ \left(S^{\tau\rho} - S^{\rho\tau}\right)\frac{\partial}{\partial b^\rho}\left(S^{\mu\nu} - S^{\nu\mu}\right) = 0,\end{aligned} \qquad (3.3.13)$$

are identically verified in the considered region of the local variables.

The existence of a nontrivial physical relevance of Lie-admissible algebras in Newtonian Mechanics is constituted by the fact that system (3.3.13) is consistent, that is, it admits solutions (other than constants) with a nontrivial degree of functional arbitraryness, as we shall see in the next Table.

Theorem 3.3.1 relates to general Lie-admissible algebras. For the second layer of flexible Lie-admissible algebra we have the following [5b]

THEOREM 3.3.2: A necessary and sufficient condition for nontrivial brackets (3.3.11) to satisfy the flexible Lie-admissibility conditions (3.3.9) is that all the following partial differential equations in the $S^{\mu\nu}$ tensor

$$S^{\mu\rho}\frac{\partial S^{\tau\nu}}{\partial b^\rho} + S^{\tau\rho}\frac{\partial S^{\nu\mu}}{\partial b^\rho} = \frac{\partial S^{\mu\nu}}{\partial b^\rho} S^{\rho\tau} + \frac{\partial S^{\tau\nu}}{\partial b^\rho} S^{\rho\mu}, \qquad (3.3.14a)$$

$$\left(S^{\mu\rho} - S^{\rho\nu}\right)\frac{\partial S^{\nu\tau}}{\partial b^\rho} + \left(S^{\nu\rho} - S^{\rho\nu}\right)\frac{\partial S^{\tau\mu}}{\partial b^\rho} + \left(S^{\tau\rho} - S^{\rho\tau}\right)\frac{\partial S^{\mu\nu}}{\partial b^\rho} = 0, \qquad (3.3.14b)$$

are identically verified in the considered region of the local variables.

It is an instructive exercise for the interested reader to prove that all solutions $S^{\mu\nu}$ of Eqs. (3.3.14) are also solutions of Eqs. (3.3.13). This is the equivalent notion in terms of partial differential equations of the algebraic notion that all flexible Lie-admissible algebras are general Lie-admissible algebras.

The third and last layer of Lie-admissibility recovers the conventional Lie properties and is expressible with the [16]

THEOREM 3.3.3: A necessary and sufficient condition for nontrivial brackets (3.3.11) to satisfy the Lie algebra identities (3.3.10) is that all the following conditions on the $S^{\mu\nu}$ tensor

$$S^{\mu\nu} + S^{\nu\mu} = 0, \qquad (3.3.15a)$$

$$S^{\mu\rho}\frac{\partial S^{\nu\tau}}{\partial b^\rho} + S^{\nu\rho}\frac{\partial S^{\tau\mu}}{\partial b^\rho} + S^{\tau\rho}\frac{\partial S^{\mu\nu}}{\partial b^\rho} = 0, \qquad (3.3.15b)$$

are identically verified in the considered region of the local variables.

As is well known, Eqs. (3.3.15) ensure that brackets (3.3.11) are the generalized Poisson brackets (2.8.12) and, thus, they are Lie. The conventional Poisson brackets are then recovered as a particular case.

Physicists interested in commutative Jordan algebras [25h, 25i] might be intrigued to know that, at the abstract level, Lie-admissible algebras are often jointly Jordan-admissible, that is, they possess a well defined content of both Lie and commutative Jordan algebras (this is the case e.g., of the Lie-admissible algebras constructed via the fundamental representations of SU(n)). However, , no classical realization of Lie-admissible algebras we shall be involved with is also Jordan-admissible. This opens intriguing perspectives (on commutative Jordan algebras) for quantization via Lie-admissible techniques whose classical limit is of the so-called bonded type.

A classical (quantum mechanical) study of this issue is conducted in ref.[5b] (ref.[5c]).

Before entering into the presentation of the application of Lie-admissible algebras in Newtonian Mechanics, it might be of some interest to indicate the current status of the abstract study of these structures.

The first point of some relevance is that the studies of ref.[26a-26p] conducted by mathematicians have been (properly so) devoted to the first fundamental step, the flexible Lie-admissible algebras. This is not a deficiency, but simply an indication of the novelty of these studies. In particular, I am aware of no study conducted by mathematicians on what I have called the general Lie-admissible algebra (this term or, equivalently, any other term differentiating algebras (3.3.8) from algebras (3.3.9), does not appear to exist in mathematical literature to the best of my knowledge).

As we shall see, the flexible Lie-admissible algebras do have a physical significance, such as for the construction of the Gell-Mann-Okubo mass formula[4d,24a], for a Lie-admissible quantization of forces not derivable from a potential[5c] or of couplings not derivable from a potential[24b]. Thus, studies[26a-26p] have a direct physical significance.

Neverthless, the algebras which appear to have the major physical role are the general, rather than the flexible, Lie-admissible algebras. As we shall see, this is the case for possible covering relativities. The net effect is that while the study of flexible Lie-admissible algebras should be continued, studies on general Lie-admissible algebra are urged.

The study of general Lie-admissible algebras of ref.[5b] is essentially based by the use of as many methodological tools offered by the theory of Abstract Algebras as possible, such as the use of the associative multiplicative algebra, the Lie multiplicative algebra, the Pierce decomposition, the Cartan decomposition for Lie algebras, Jordan algebras and nonassociative algebras in general, the (solvable, nilpotent, associator nilpotent, f-solvable, f-nilpotent, Jacobson, Levitzki, McCoy, Brown, Amitsur, Nagata, etc.) radical approach, etc. But this is only a truly rudimentary first step and the number of open aspects is too large to suggest an outline. Besides, this algebraic approach will not be outlined in this paper to avoid an excessive length.

For the reader interested in these algebraic aspects I suggest, as a first reading, textbooks[25a-25f], as second reading, monographs[25g-25v] and, as third reading, research monographs[25z-25'g]. Papers[28] appear to be particularly valuable for Lie-admissible algebras. The reader, however, should be aware that all references[25] and [28] are devoted to the study of algebras other than Lie algebras and that none of them treats or even defines a Lie-admissible algebra. Neverthless, as indicated earlier, they provide methods which, under a number of technical implementations, are often applicable to the Lie-admissible algebras.

In conclusion, we can state that there exists a hierarchy of three classes of Lie-admissible algebras satisfying the following enclosure properties (and which can be interpreted as a corresponding hierarchy of Newtonian forces according to the analysis of Table 3.4):

$$\left\{\begin{matrix}\text{Lie}\\ \text{algebras}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{Flexible Lie-admissible}\\ \text{algebras}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{General Lie-admissible}\\ \text{algebras}\end{matrix}\right\}. \quad (3.3.16)$$

Within such a context, the Lie algebras emerge as being the simplect possible Lie-admissible algebras. In the transition to the two subsequent layers of generalization, the Lie-admissible character persists. Neverthless, the totally antisymmetric nature of the product is lost. In such a transition, however, the Lie algebras are not "lost". Instead, they are preserved in a doubled embedded form: (a) the attached form $U \approx L$ and (b) the limiting form (under anticommutativity of the Lie-admissible product) $U_{lim} \approx L'$, where, in general, $L \not\approx L'$.

As we shall see later on, these features are such to offer some genuine hope of constructing a Lie-admissible covering of Lie's theory.

TABLE 3.4: HAMILTON-ADMISSIBLE COVERING OF HAMILTON'S EQUATIONS AS THE ANALYTIC ORIGIN OF THE LIE-ADMISSIBLE ALGEBRAS.

Clearly, generalization (3.2.1) of Hamilton's equations was purely indicative. The proper generalization within the context of the Lie-admissible formulations must be constructed according to Theorem 3.3.1. A study of this problem (which is reported in details in ref.[5b]) leads to the following property which we shall refer to as the Theorem of Direct Universality of the Lie-admissible Formulations.

THEOREM 3.4.1: Local, class C^∞, regular, nonconservative, (essentially nonselfadjoint) Newtonian systems (2.14.13), i.e.,

$$\left\{\left[\mu_k \ddot{r}_{ka} - f_{ka}(r)\right]^{C^\infty, R}_{SA} - F_{ka}(t, \underline{r}, \dot{\underline{r}})\right\}^{C^\infty, R}_{ENSA} = 0, \quad \begin{matrix} k=1,2,\dots,N\ (<\infty) \\ a=x,y,z \end{matrix} \quad (3.4.1)$$

can always be directly represented in the (neighborhood of a regular point of the) variables $\{b^\mu\} = \{r^{ka}, p_{ka}\}$ (where r^{ka} represents the Cartesian coordinates of the reference frame of the experimental detection of the system and p_{ka} represents the linear momenta $m_k \dot{r}_{ka}$) in terms of the contravariant equations, here called Hamilton-admissible equations,

$$\dot{b}^\mu - S^{\mu\nu}(t,b)\frac{\partial H(b)}{\partial b^\nu} \equiv \dot{b}^\mu - \frac{\partial b^\mu}{\partial R_\nu}\frac{\partial H}{\partial b^\nu} = 0, \quad \mu = 1,2,\dots,6N, \quad (3.4.2a)$$

$$\det(S^{\mu\nu}) = |S^{\mu\nu}| \equiv \left|\frac{\partial b^\mu}{\partial R_\nu}\right| \neq 0, \quad (3.4.2b)$$

or in terms of the equivalent covariant forms

$$S_{\mu\nu}(t,b)\,\dot{b}^{\nu} - \frac{\partial H(b)}{\partial b^{\mu}} \equiv \frac{\partial R_{\mu}}{\partial b^{\nu}}\,\dot{b}^{\nu} - \frac{\partial H}{\partial b^{\mu}} = 0, \qquad (3.4.3a)$$

$$(S_{\mu\nu}) = \left(\frac{\partial R_{\mu}}{\partial b^{\nu}}\right) = (S^{\mu\nu})^{-1} = \left(\frac{\partial b^{\mu}}{\partial R_{\nu}}\right)^{-1}, \qquad (3.4.3b)$$

where the functions R_{μ} are the solution of the system

$$\frac{\partial R_{\mu}(t,b)}{\partial b^{\rho}}\left[f^{\rho}(b) - F^{\rho}(t,b)\right] = f_{\mu}(b), \qquad (3.4.4a)$$

$$\{f^{\rho}\} = \left\{\frac{\partial H}{\partial p_{ka}}, -\frac{\partial H}{\partial r^{ka}}\right\}, \{F^{\rho}\} = \{0, F_{ka}\}, \{f_{\mu}\} = \left\{\frac{\partial H}{\partial b^{\mu}}\right\}. \qquad (3.4.4b)$$

The brackets of the time evolution law

$$\dot{A}(b) = \frac{\partial A}{\partial b^{\mu}}\, S^{\mu\nu}\, \frac{\partial H}{\partial b^{\mu}} = (A, H), \qquad (3.4.5)$$

then characterize a general Lie-admissible algebra, i.e., violate the Lie algebra identities, but satisfy the covering law of Lie-admissibility

$$((A,B),C) + ((B,C),A) + ((C,A),B) + (C,(B,A)) + (B,(A,C)) + (A,(C,B))$$
$$= (A,(B,C)) + (B,(C,A)) + (C,(A,B)) + ((C,B),A) + ((B,A),C) + ((A,C),B). \qquad (3.4.6)$$

A few comments are here in order. The "universality" of the approach originates from the fact that Eqs. (3.4.4) constitute a system of 6N linear, inhomogeneous, first-order, partial differential equations in 6N unknowns, the functions $R_{\mu}(t,b)$. A solution, under the assumed conditions, is then ensured by the existence theory of partial differential equations.

The crucial property of Lie-admissibility of brackets (3.4.5) can be proved in a number of ways.

(A) Direct proof of Lie-admissibility. The brackets

$$(A,B) = \frac{\partial A}{\partial b^{\mu}}\, S^{\mu\nu}\, \frac{\partial B}{\partial b^{\nu}} = \frac{\partial A}{\partial b^{\mu}}\,\frac{\partial b^{\mu}}{\partial R_{\nu}}\,\frac{\partial B}{\partial b^{\nu}} = \frac{\partial A}{\partial R_{\nu}}\,\frac{\partial B}{\partial b^{\nu}}, \qquad (3.4.7)$$

characterize a general Lie-admissible algebra because the attached brackets

$$(A,B) - (B,A) = [A,B]^{*} = \frac{\partial A}{\partial R_{\nu}}\,\frac{\partial B}{\partial b^{\nu}} - \frac{\partial B}{\partial R_{\nu}}\,\frac{\partial A}{\partial b^{\nu}}, \qquad (3.4.8)$$

satisfy the Lie algebra identities, while the original brackets (3.4.7) are (nonassociative and) non-Lie.

(B) Algebraic proof of Lie-admissibility. The tensor $S^{\mu\nu} = \partial b^{\mu}/\partial R_{\nu}$ can be always written in the form $S^{\mu\nu} = \omega^{\mu\alpha}\,\partial b^{\nu}/\partial T^{\alpha}$, where the functions T^{α} are uniquely characterized

by the functions R_{μ}. Brackets (3.4.5) characterize a general Lie-admissible algebra because the tensor $\omega^{\mu\alpha}\partial b^{\nu}/\partial T^{\alpha}$ satisfy Theorem 3.3.1, as the reader can verify by inspection.

(C) Geometrical proof of Lie-admissibility. The tensor $S_{\mu\nu}$ is nondegenerate, from condition (3.4.2b), but it is not totally antisymmetric. As such, it does not, strictly speaking, characterize a symplectic geometry. Nevertheless, the attached tensor

$$\Omega_{\mu\nu} = S_{\mu\nu} - S_{\nu\mu} = \frac{\partial R_{\mu}}{\partial b^{\nu}} - \frac{\partial R_{\nu}}{\partial b^{\mu}}, \qquad (3.4.9)$$

is precisely the nondegenerate, antisymmetric Birkhoff's tensor (2.8.3) which, as such, characterizes a symplectic form (2.8.26). This is sufficient to ensure the Lie-admissibility of brackets (3.4.5) because it ensures that the attached tensor $\Omega^{\mu\nu} = S^{\mu\nu} - S^{\nu\mu}$ is Lie.

Notice that, in conventional notations, Eqs. (3.4.5) can be written

$$\{R_{\mu}\} = \{\tilde{R}_{ka}(t, \underline{r}, \underline{p}), r^{ka}\}, \qquad (3.4.10a)$$

$$\begin{cases} \dfrac{\partial \tilde{R}_{ka}}{\partial r^{jb}}\,\dot{r}^{jb} + \dfrac{\partial \tilde{R}_{ka}}{\partial p_{jb}}\,\dot{p}_{jb} - \dfrac{\partial H}{\partial r^{ka}} = 0, & (3.4.10b) \\ \dot{r}^{ka} - \dfrac{\partial H}{\partial p_{ka}} = 0. & (3.4.10c) \end{cases}$$

Thus, the (crucial) regularity condition (3.4.2b) can be equivalently written

$$\det(S_{\mu\nu}) = |S_{\mu\nu}| = \begin{vmatrix} \left(\dfrac{\partial \tilde{R}_{ia}}{\partial r^{jb}}\right) & \left(\dfrac{\partial \tilde{R}_{ia}}{\partial p_{jb}}\right) \\ 1_{3N\times 3N} & 0_{3N\times 3N} \end{vmatrix} = \left|\frac{\partial \tilde{R}_{ia}}{\partial p_{jb}}\right| \neq 0. \qquad (3.4.11)$$

Notice that this property can be satisfied even when all forces derivable from a potential are null.

The analytic counterpart of the concept of Lie-admissibility of Table 3.3 is also two-fold. First, the "analytic content" of Eqs. (3.4.3) is expressed by Birkhoff's (rather than Hamilton's) equations in the sense of the attached form (3.4.9). Secondly, Hamilton's equations are recovered identically at the limit of null forces not derivable from a potential according to

$$\lim_{F^{\mu}\to 0}\left(S_{\mu\nu}\,\dot{b}^{\nu} - \frac{\partial H}{\partial b^{\mu}}\right) = \lim_{R_{\mu}\to\omega_{\mu\alpha}b^{\alpha}}\left(\frac{\partial R_{\mu}}{\partial b^{\nu}}\,\dot{b}^{\nu} - \frac{\partial H}{\partial b^{\mu}}\right) \qquad (3.4.12)$$
$$= \omega_{\mu\nu}\,\dot{b}^{\nu} - \frac{\partial H}{\partial b^{\mu}}.$$

This also illustrates the reason for the selected terms "Hamilton-admissible equations".

In the following we give few indicative examples of representation of nonselfadjoint Newtonian systems in terms of our Hamilton-admissible equations.

$$\begin{cases} [(\ddot{r})_{SA} + \gamma \dot{r}]_{NSA} = 0, \quad m = 1, \\ \{R_\mu\} = \{(-p - \gamma r), r\}, \quad H = \frac{1}{2} p^2, \quad p = \dot{r}, \end{cases} \tag{3.4.13a}$$

$$\begin{cases} [(\ddot{r} + k r)_{SA} + \gamma \dot{r}]_{NSA}, \quad m = 1, \\ \{R_\mu\} = \{(-p - \gamma r), r\}, \quad H = \frac{1}{2}(p^2 + k r^2), \quad p = \dot{r}, \end{cases} \tag{3.4.13b}$$

$$\begin{cases} [(\ddot{r} + k r)_{SA} + \gamma \dot{r} - f(t)]_{NSA} = 0, \\ \{R_\mu\} = \{(-p - \gamma r), r\}, \quad H = \frac{1}{2}[p^2 + k r^2 - f(t) r], \quad p = \dot{r}, \end{cases} \tag{3.4.13c}$$

$$\begin{cases} \begin{pmatrix} m \ddot{r}_x + c t \underset{\sim}{r} \cdot \dot{\underset{\sim}{r}} - \alpha \\ m \ddot{r}_y + \alpha \end{pmatrix}_{NSA} = 0, \quad H = \frac{1}{2m}(\underset{\sim}{p}^2 + \alpha r_y), \quad \underset{\sim}{p} = m \dot{\underset{\sim}{r}}, \\ \{R_\mu\} = \{(-p_x - p_y - \frac{c}{2} t \underset{\sim}{r}^2), (-p_y), (r_x), (r_y)\}, \end{cases} \tag{3.4.13d}$$

$$\begin{cases} \begin{pmatrix} m \ddot{r}_x + a r_y^2 \dot{r}_x^2 \dot{r}_y \\ m \ddot{r}_y + b \dot{r}_y^3 (r_x \dot{r}_y + r_y \dot{r}_x) \end{pmatrix}_{NSA} = 0, \quad H = \frac{1}{2m} \underset{\sim}{p}^2, \quad \underset{\sim}{p} = m \dot{\underset{\sim}{r}}, \\ \{R_\mu\} = \left\{ \left(\frac{1}{p_x} - \frac{c}{3} r_y^3\right), \left(\frac{1}{2 p_y^2} - d\, r_x r_y\right), (r_x), (r_y) \right\}, \\ c = m a, \quad d = m b. \end{cases} \tag{3.4.13e}$$

The use of Eqs. (3.4.3) is here tacitly assumed. The reformulation of the above representations in terms of the equivalent form (3.4.2) is left to the interested reader. Notice how the Hamiltonian is representative of the kinetic energy as well as of all the forces derivable from a potential.

The reader should be aware that the forms (3.4.2) and (3.4.3) of writing our Hamilton-admissible equations are not unique and several additional alternatives can be formulated. These additional forms are significant on methodological grounds, depending on the aspect under consideration. We therefore give below some of the most representative contravariant and covariant ways of writing the Hamilton-admissible equations.

$$\begin{aligned} & \dot{b}^\mu - S^{\mu\nu}(t,b) \frac{\partial H(b)}{\partial b^\mu} \\ & \equiv \dot{b}^\mu - \frac{\partial b^\mu}{\partial R_\nu} \frac{\partial H}{\partial b^\nu} \equiv \dot{b}^\mu - \frac{\partial b^\mu}{\partial U^\alpha} \omega^{\alpha\nu} \frac{\partial H}{\partial b^\nu} \\ & \equiv \dot{b}^\mu - \frac{\partial b^\nu}{\partial Z_\mu} \frac{\partial H}{\partial b^\nu} \equiv \dot{b}^\mu - \frac{\partial H}{\partial Z_\mu} \\ & \equiv \dot{b}^\mu - \omega^{\mu\alpha} \frac{\partial b^\nu}{\partial T^\alpha} \frac{\partial H}{\partial b^\nu} \equiv \dot{b}^\mu - \omega^{\mu\alpha} \frac{\partial H}{\partial T^\alpha} = 0, \end{aligned} \tag{3.4.14a}$$

$$\begin{aligned} & S_{\mu\nu}(t,b)\, \dot{b}^\mu - \frac{\partial H(b)}{\partial b^\mu} \\ & \equiv \frac{\partial R_\mu}{\partial b^\nu} \dot{b}^\nu - \frac{\partial H}{\partial b^\mu} \equiv \dot{R}_\mu - \frac{\partial H}{\partial b^\mu} - \frac{\partial R_\mu}{\partial t} \\ & \equiv \frac{\partial Z_\nu}{\partial b^\mu} \dot{b}^\nu - \frac{\partial H}{\partial b^\mu} = \frac{\partial}{\partial b^\mu}\left(Z_\nu \dot{b}^\nu - H\right) \\ & \equiv \frac{\partial T^\alpha}{\partial b^\mu} \omega_{\alpha\nu} \dot{b}^\nu - \frac{\partial H}{\partial b^\mu} = \omega_{\mu\alpha} \frac{\partial U^\alpha}{\partial b^\nu} \dot{b}^\nu - \frac{\partial H}{\partial b^\mu} = 0, \end{aligned} \tag{3.4.14b}$$

where the functions Z_μ, T^μ and U^μ are (uniquely) characterized by the functions R_μ of Eqs. (3.4.4).

For instance, by using the contravariant form in the Z_μ functions, one can easily identify the generalization of the conventional Lagrange's equations which is induced by the Lie-admissible formulations. It is given by the equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}^{ka}} - \frac{\partial E}{\partial Z^{ka}} = 0, \qquad \begin{aligned} & L(\underset{\sim}{r}, \dot{\underset{\sim}{r}}) = T(\dot{\underset{\sim}{r}}) - V(\underset{\sim}{r}), \\ & E = \dot{r}^{ka} \frac{\partial L}{\partial \dot{r}^{ka}} - L = E(\underset{\sim}{r}, \dot{\underset{\sim}{r}}), \\ & Z^{ka} = Z^{ka}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}), \end{aligned} \tag{3.4.15}$$

which we shall call <u>Lagrange-admissible equations</u> because they recover the conventional Lagrange's equations identically at the limit $\dot{r}^{ka} \to -r^{ka}$, i.e.,

$$\lim_{Z^{ka} \to -\dot{r}^{ka}} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{r}^{ka}} - \frac{\partial E}{\partial Z^{ka}} \right) = \frac{d}{dt} \frac{\partial L}{\partial \dot{r}^{ka}} - \frac{\partial L}{\partial r^{ka}}, \tag{3.4.16}$$

as it can be seen by writing the equations in the equivalent form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^{ka}} + \frac{\partial \dot{r}^{jb}}{\partial Z^{ka}}\frac{\partial L}{\partial \dot{r}^{jb}} - \left(\frac{\partial}{\partial Z^{ka}}\frac{\partial L}{\partial \dot{r}^{jb}}\right)\dot{r}^{jb} = 0. \quad (3.4.17)$$

Viceversa, starting from Eqs. (3.4.17), the conventional Legendre transform induces the Hamilton-admissible equations in the contravariant form in the Z-functions, from which the other equivalent forms can be constructed. Thus, the Lagrange-admissible equations constitute the generalization of Lagrange's equations (with and without external terms) which is consistent with the Lie-admissible formulations.

A fundamental property of both the Lagrange-admissible and the Hamilton-admissible equations is that they are, in general, essentially nonselfadjoint, and we shall write

$$\left[S_{\mu\nu}\dot{b}^{\nu} - \frac{\partial H}{\partial b^{\mu}}\right]^{C^{\infty},R}_{NSA} = 0, \quad \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^{ka}} - \frac{\partial E}{\partial Z^{ka}}\right]^{C^{\infty},R}_{NSA} = 0. \quad (3.4.18)$$

As a matter of fact, this is precisely the reason why these equations are capable of producing a direct analytic representation of essentially nonselfadjoint systems which is prohibited for the conventional Lagrange's and Hamilton's equations.

Eqs. (3.4.14) will also be called canonical-admissible equations because they offer a genuine hope of constructing, in due time, a covering of the conventional canonical theory. In particular,

(a) the canonical-admissible equations are directly applicable to a broader physical context, that is, applicable without reformulations of the variables;

(b) they are non-trivially different than Hamilton's equations in the sense that they are non-Lie in algebraic character;

(c) they embed the forces not derivable from a potential (Galilei relativity breaking forces) into the Lie-admissible tensor $S^{\mu\nu}$ (rather than into the Hamiltonian);

(d) they recover Hamilton's equations identically at the limit of null forces not derivable from a potential; and, last but not least,

(e) their departure from Hamilton's equations is a measure of the latter forces, i.e.,

$$(S^{\mu\nu} - \omega^{\mu\nu})\frac{\partial H}{\partial b^{\nu}} = F^{\mu}, \quad \{F^{\mu}\} = \{0, F_{ka}\}. \quad (3.4.19)$$

Eqs. (3.4.14) are at the foundations of the Lie-admissible formulations. All my efforts (reported in ref.[5]) are essentially devoted to attempt an initial understanding of the relativity and quantum mechanical implications of the direct applicability to physical systems of general Lie-admissible algebras, as established by Theorem 3.4.1. It should be stressed in this respect that the emerging Lie-admissible algebras are of general, rather than flexible type. For a study of the (rather restrictive) conditions under which the flexible Lie-admissible algebras occur, see ref.[5b].

The reader should be aware that the variables r^{ka} and p_{ka} of Eqs. (3.4.14) do not span a phase space as commonly understood. Notice that this is the case even though $p_{ka} = \partial L/\partial \dot{r}^{ka}$. The point is that the functions L and H represent only partially the system within the context of the formulations under consideration. The space of the b-variables of Eqs. (3.4.14) has been called dynamical space in ref.[5b]. It will be geometrically identified in Table 3.8 .

The transition from the analytic equations of the Inverse Problem to those of the Lie-admissible Problem will play a crucial function for the subsequent steps of this paper. We here restrict ourselves for brevity only to the case of nonessentially nonselfadjoint systems. Then, the use of Eqs. (2.11.9) allows the direct identities

$$\left[g_{ka}{}^{jb}\left(\frac{d}{dt}\frac{\partial L^{g}}{\partial \dot{r}^{jb}} - \frac{\partial L^{g}}{\partial r^{jb}}\right)^{C^{\infty},R}_{SA}\right]^{C^{\infty},R}_{NENSA} \quad (3.4.20)$$

$$\equiv \left[\frac{d}{dt}\frac{\partial L^{c}}{\partial \dot{r}^{ka}} + \frac{\partial \dot{r}^{jb}}{\partial Z^{ka}}\frac{\partial L^{c}}{\partial r^{jb}} - \left(\frac{\partial}{\partial Z^{ka}}\frac{\partial L^{c}}{\partial \dot{r}^{jb}}\right)\dot{r}^{jb}\right]^{C^{\infty},R}_{NENSA}$$

$$\equiv \left[(m_{k}\ddot{r}_{ka} - f_{ka})^{C^{\infty},R}_{ESA} - F_{ka}\right]^{C^{\infty},R}_{NENSA}.$$

This indicates the reason for our insistence in the generalized form (2.11.9) of Lagrange's equations rather than the conventional form (2.1). And indeed, it is the nonselfadjoint nature of these equations which allows a direct link between Lagrange's and Lagrange-admissible equations. Notice that in the former the forces not derivable from a potential are represented with the generalized structure L^{g} of the admissible Lagrangians (essentially chiral Lagrangians), while in the latter the forces not derivable from a potential are represented with the $\dot{Z}^{ka}$ functions by allowing in this way the Lagrangians L^{c} of rule (3.4.20) to have the conventional structure $L^{c} = T - V$.

The transition from Hamilton's to Hamilton-admissible equations (again for nonessentially nonselfadjoint systems) is, in essence, a particular type of mapping under which the Hamiltonian does not transform as a scalar. We can write in this case

$$\{a^{\mu}\} = \{r^{ka}, P_{ka}\} \longrightarrow \{b^{\mu}\} = \{r^{ka}, p_{ka}\}, \quad (3.4.21a)$$

$$P_{ka} = \partial L^{g}/\partial \dot{r}^{ka} \longrightarrow p_{ka} = \partial L^{c}/\partial \dot{r}^{ka}, \quad (3.4.21b)$$

$$H^{g} \longrightarrow H^{c} : \quad H^{g}(t, b(t,a)) = \tilde{\rho} H^{c} + \tilde{\sigma}. \quad (3.4.21c)$$

For the transition of the equations in r^{ka} we trivially have (by construction)

$$\dot{r}^{ka} = \frac{\partial H^{g}}{\partial P_{ka}} \equiv \frac{\partial H^{c}}{\partial p_{ka}}. \quad (3.4.22)$$

For the transition of the equations in p_{ka} notice that

$$P_{ka} = \frac{\partial L^g}{\partial \dot{r}^{ka}} = \frac{\partial}{\partial \dot{r}^{ka}}\left(\rho L^c + \sigma\right) = P_{ka}(t, \underline{r}, \underline{p}), \qquad (3.4.23a)$$

$$p_{ka} = \frac{\partial L^c}{\partial \dot{r}^{ka}} = \frac{\partial}{\partial \dot{r}^{ka}}\left(\gamma L^g + \delta\right) = p_{ka}(t, \underline{r}, \underline{P}), \qquad (3.4.23b)$$

Then, under the assumed conditions, there always exists a (class C^∞ and regular) matrix $(h_{ka}{}^{jb})$ such that

$$\dot{P}_{ka} + \frac{\partial H^g}{\partial r^{ka}} = h_{ka}{}^{jb}\left(\dot{p}_{jb} - \frac{\partial H^c}{\partial Z^{jb}}\right). \qquad (3.4.24)$$

We can thus write in unified notation

$$\dot{a}^\mu - \omega^{\mu\nu}\frac{\partial H^g}{\partial a^\nu} \equiv h^\mu{}_\nu\left(\dot{b}^\nu - S^{\mu\nu}\frac{\partial H^c}{\partial b^\nu}\right), \qquad (3.4.25a)$$

$$(h^\mu{}_\nu) = \begin{pmatrix} 1 & 0 \\ 0 & (h) \end{pmatrix}, \quad S^{\mu\nu} = \frac{\partial b^\nu}{\partial Z_\mu}, \quad \{z_\mu\} = \{p_{ka}, z^{ka}\}. \qquad (3.4.25b)$$

Finally, let us recall that the Lie tensor $\omega^{\mu\nu}$ of the conventional Poisson brackets has a rather special physical significance, it represents the <u>fundamental Poisson brackets of the Inverse Problem</u> in the unified notation

$$(\omega^{\mu\nu}) = \begin{pmatrix} ([r^{ia}, r^{jb}]) & ([r^{ia}, P_{jb}]) \\ ([P_{ia}, r^{jb}]) & ([P_{ia}, P_{jb}]) \end{pmatrix} = \begin{pmatrix} 0_{3N\times 3N} & 1_{3N\times 3N} \\ -1_{3N\times 3N} & 0_{3N\times 3N} \end{pmatrix}. \qquad (3.4.26)$$

For Lie-admissible formulations these brackets are lost. We have instead the expressions [4c,5b]

$$(S^{\mu\nu}) = \begin{pmatrix} ((r^{ia}, r^{jb})) & ((r^{ia}, p_{jb})) \\ ((p_{ia}, r^{jb})) & ((p_{ia}, p_{jb})) \end{pmatrix} = \begin{pmatrix} 0_{3N\times 3N} & 1_{3N\times 3N} \\ \left(\frac{\partial r^{jb}}{\partial Z^{ia}}\right) & \left(\frac{\partial p_{jb}}{\partial Z^{ia}}\right) \end{pmatrix}, \qquad (3.4.27)$$

which represent the <u>fundamental dynamical brackets of the Lie-admissible Problem</u>.

A comparative analysis of this dual methodological context for the representation of the same (nonessentially nonselfadjoint) system is instructive. In essence, within the context of the Inverse Problem all conventional formulations (e.g., Hamilton's equations, Lie algebras, etc.) are preserved, but the mathematical algorithms at hand (the symbols "P", "H^g", etc.) lose their direct physical significance. Within the context of the Lie-admissible problem exactly the opposite situation occurs in the sense that, by construction, all algorithms of the approach (e.g., the symbols "p", "H^c", etc.) have a direct physical significance, but the conventional formulations are lost. It is hoped that a judicious interplay between these two

complementary formulations, rather than the individual use of any of them, can be effective for the study of nonconservative mechanics, with particular reference to a most insidious aspect, the physical meaning of the mathematical algorithms at hand. For a study on the implications for, say, the physical electric current and conductivity tensor of a nonlinear, nonconservative plasma see ref. [4e]. Notice again the insistence on the term "physical" which is customarily absent in current literature. The reason is that the quantities which are computed in conventional studies are the "canonical" electric current and conductivity tensor. Our contention is that, under the assumption of nonconservative forces (a not so rare occurrence in in plasma physics), the canonical quantities do not represent physical quantities. These occurrences can be best expressed by using the dynamical space of physical variables r^{ka} and p_{ka} (rather than the phase space of mathematical quantities); by carefully formulating physical quantities in such a space; by representing the nonconservative forces with our canonical-admissible equations and consequential generalization of, say, the Liouville's equations; [4e,5b] and finally by comparing the physical predictions of such Lie-admissible approach with the canonical predictions of the conventional phase space approach.

Regrettably, we are forced to ignore a number of aspects for conciseness. For instance, the conventional Legendre transform, as indicated earlier, does not induce canonical quantities in the transition from Eqs. (3.4.15) to (3.4.14). The net effect is that the Legendre transform, while crucial for the Inverse Problem, is inessential for the Lie-admissible problem. In ref. [5b] I present a simple <u>Lie-admissible covering of the Legendre transform</u>, that is, a <u>non</u>canonical generalization of this transform of the type

$$H = p_{ka}\dot{r}^{ka} + p_{ka}G^{ka}(\underline{r}, \underline{p}) - L, \qquad \underline{p}_{ka} = \frac{\partial L}{\partial \dot{r}^{ka}}, \qquad (3.4.28)$$

which, when applied to Eqs. (3.4.15), yields a generalized version of Eqs. (3.4.14) of the type

$$\dot{b}^\mu - S^{*\mu\nu}\frac{\partial H}{\partial b^\nu} = 0, \quad S^{*\mu\nu}\frac{\partial H}{\partial b^\nu} = \frac{\partial b^\mu}{\partial R^*_\nu}\frac{\partial H}{\partial b^\nu} = \omega^{\mu\nu}\frac{\partial H}{\partial b^\nu} + F^\mu, \qquad (3.4.29)$$

$$\{R^*_\nu\} = \{S^*_{ka}, T^{*ka}\}, \qquad \{F^\mu\} = \{F^{ka}, F_{ka}\},$$

which, however, is Lie-admissible in algebraic character. In essence, this is the Lie-admissible covering of the Lie covering of the Legendre transform, that is, the noncanonical generalization of the transform of Table 2.7 which induces Birkhoff's equations.

It is here appropriate to recall that one of the central hopes of this analysis is to be able to study, in due time, strong interactions as not derivable from a potential. The reader is then encouraged to consider transform (3.4.28), say, within the context of the problem of a possible Lie-admissible covering of the canonical perturbation theory. The quantities G^{ka} can also be, as

a particular case, constants and infinitesimal, by therefore offering new possibilities of expansions which are nonexistent in conventional canonical perturbation theory. In turn, this is another indication of the conceptual departure from conventional conservative settings which is needed to study forces not derivable from a potential: the generalized transform (3.4.28) is potentially significant precisely because noncanonical.

On algebraic grounds, transform (3.4.28) essentially provides the analytic origin of the isotopic degrees of freedom of the, this time, Lie-admissible brackets, i.e., the class C^∞ invertible transformations of the Lie-admissible brackets within the same b-variables which preserve the Lie-admissibility law, and we shall write

$$(A,B)_{(b)} = \frac{\partial A}{\partial b^\mu} S^{\mu\nu} \frac{\partial B}{\partial b^\nu} \longrightarrow (A,B)^*_{(b)} = \frac{\partial A}{\partial b^\mu} S^{*\mu\nu} \frac{\partial B}{\partial b^\nu}, \quad (3.4.30a)$$

$$(S^{\mu\nu}) = \left(\frac{\partial b^\mu}{\partial R_\nu}\right), \quad (S^{*\mu\nu}) = \left(\frac{\partial b^\mu}{\partial R^*_\nu}\right) = \begin{pmatrix} \left(\frac{\partial z}{\partial S^*}\right) & \left(\frac{\partial p}{\partial S^*}\right) \\ \left(\frac{\partial z}{\partial T^*}\right) & \left(\frac{\partial p}{\partial T^*}\right) \end{pmatrix}. \quad (3.4.30b)$$

This yields the notion of Lie-admissible isotopy as an algebraic covering of that of Lie isotopy, that is, the transition from the conventional to the generalized Poisson brackets

$$[A,B]_{(a)} = \frac{\partial A}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial B}{\partial a^\nu} \rightarrow [A,B]^*_{(a)} = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu} \frac{\partial B}{\partial a^\nu}. \quad (3.4.31)$$

In other words, Eqs. (3.4.29) are a Lie-admissible covering of Eqs. (3.4.14) in a way similar to that according to which Birkhoff's equations are a Lie covering of Hamilton's equations.

Finally, it might be of some significance to indicate that the Lie-admissible brackets of this table are not Jordan-admissible, that is, the attached brackets $A \bullet B = (A,B) + (B,A)$ violate the Jordan law (3.2.10b) and, thus, no "Jordan content" occurs (at the given classical level[5b, 5c]).

TABLE 3.5: CANONICAL-ADMISSIBLE COVERING OF CANONICAL FORMULATIONS

One of the fundamental properties of Galilei's transformations is that they constitute canonical transformations, that is, transformations which preserve the time evolution of the systems considered. Clearly, in order to attempt the construction of a Lie-admissible covering of the Galilei relativity one of the necessary prerequisites is the identification of the rudiments of the expected covering of the transformation theory characterizable by our Hamilton-admissible equations. The objective of this table is to outline my studies on this problem for the finite and infinitesimal case, as a preliminary step for the reinspection of the problem as a generalization of Lie's theory (see Tables 3.6 and 3.7). The reader should be aware that the study of the transformation theory within the broader Lie-admissible context brings into focus a number of aspects of the conventional canonical transformation theory which are nonessential for its customary presentation. It is therefore advisable to reinspect the known transformation theory first,

and then enter into the problem of its Lie-admissible generalization. Again, I am here interested in nonconservative systems. This means that the conventional canonical transformation theory I shall review below is referred specifically to these systems. In the final analysis this is one of the methodological insights which is rendered applicable to nonconservative systems by the Inverse Problem.

Permit me to begin with a redefinition of the conventional canonical transformations which appears useful for their generalization for Hamilton-admissible equations. Under an arbitrary (but of class C^∞ and invertible) transformations $a^\mu \longrightarrow a'^\mu = a'^\mu(a)$ the conventional Poisson brackets are transformed into new brackets

$$[A,B]_{(a)} = \frac{\partial A}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial B}{\partial a^\nu} = \frac{\partial A'}{\partial a'^\rho} \frac{\partial a'^\rho}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial a'^\sigma}{\partial a^\nu} \frac{\partial B'}{\partial a'^\sigma} = \frac{\partial A'}{\partial a'^\rho} T^{\rho\sigma}_{(a')} \frac{\partial B'}{\partial a'^\sigma}$$
$$= \langle A', B' \rangle_{(a')}, \quad (3.5.1)$$

which are generally non-Lie, that is, violate laws (3.3.10). The first subset of the transformations considered which is relevant for the canonical transformation theory is that of the Lie isotopic transformations, that is, the transformations which preserve the Lie algebra identities, i.e., perform the transition from the conventional Poisson brackets in the a-variables to the generalized Poisson brackets in the a'-variables

$$[A,B]_{(a)} \longrightarrow [A',B']^*_{(a')}. \quad (3.5.2)$$

This means that the analytic equations of the former are Haminton's equations, while those of the latter are Birkhoff's equations.

Of course, the Lie isotopic transformations are not, in general, canonical transformations, e.g., because they do not necessarily preserve the value of the fundamental Poisson brackets (3.4.24) . However, we can define the conventional canonical transformations as the Lie identity isotopic transformations of the fundamental tensor $\omega^{\mu\nu}$, that is, the transformations which not only preserve the Lie algebra, but actually preserve the value of the brackets,

$$\omega^{\mu\nu} \longrightarrow \Omega^{\mu\nu} = \frac{\partial a'^\mu}{\partial a^\rho} \omega^{\rho\sigma} \frac{\partial a'^\nu}{\partial a^\sigma} = \omega^{\mu\nu}. \quad (3.5.3)$$

This, however, is only a first layer of the transformation theory, i.e., that for Hamilton's equations. The existence of a Lie covering of Hamilton's equations, Birkhoff's equations, suggests the existence of a Lie covering of the canonical transformation theory. Such covering is known in the existing literature (see, for instance, ref.[16]) under the name of generalized canonical transformations, although they are rarely interpreted in the way essential for this paper, that is,

as the transformation theory of Birkhoff's equations. In a way fully equivalent to Eqs. (3.5.3), we can define the generalized canonical transformations as the Lie identity isotopic transformations of the generalized tensor $\mathcal{R}^{\mu\nu}$, i.e., the transformations which not only preserve the Lie algebra, but also are such that

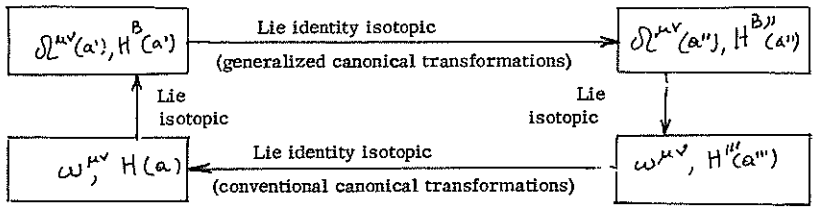
$$\mathcal{R}^{\mu\nu}(a) \longrightarrow \mathcal{R}'^{\mu\nu}(a') = \frac{\partial a'^{\mu}}{\partial a^{\rho}} \mathcal{R}^{\rho\sigma}(a(a')) \frac{\partial a'^{\nu}}{\partial a^{\sigma}} = \mathcal{R}^{\mu\nu}(a'). \tag{3.5.4}$$

Clearly, the generalized canonical transformations contain the conventional canonical transformations as a particular case. We can therefore focus our primary attention in the former. The following property proved by W. SARLET and F. CANTRIJN[15b] is relevant for our analysis.

LEMMA 3.5.1: A necessary and sufficient condition for a class C^{∞}, invertible transformation $a \rightarrow a'(a)$ to be a Lie identity isotopic transformation of Birkhoff's tensor is that there exist a class C^{∞} function $G(a')$, called the generator of the transformation, such that

$$R'_{\mu}(a') = R_{\mu}(a') + \frac{\partial G(a')}{\partial a'^{\mu}}. \tag{3.5.5}$$

But, under the necessary conditions, Birkhoff's equations are reducible to Hamilton's equations. This implies the following diagram[15b]



which is closed and invertible. This confirms the methodological equivalence of the transformation theory of Birkhoff's and Hamilton's equations. The following property is well known.

LEMMA 3.5.2: The set of all possible generalized canonical transformations forms a group, called generalized canonical group.

Notice that transformations $a \rightarrow a'(a)$ which are canonical, are so with respect to all Hamiltonians(of the admitted class) and, thus, all systems. This is no longer the case for the

generalized canonical transformations in the sense that if a transformation $a \rightarrow a'(a)$ is of this type for one tensor $\mathcal{R}^{\mu\nu}(a)$, the same transformation is not necessarily a generalized canonical for another tensor $\mathcal{R}'^{\mu\nu}(a')$. But Birkhoff's tensor has a dynamical role in the sense that it is representative of the acting forces jointly with the Birkhoffian. We therefore reach the conclusion that in the transition from the transformation theory of Hamilton's equations to that of Birkhoff's equations the methodology apply to all tensors $\mathcal{R}^{\mu\nu}$ of the admitted class, but the theory "per sé" becomes dependent on the tensor $\mathcal{R}^{\mu\nu}$ considered.

We are now equipped to consider the case of the transformation theory of our Hamilton-admissible equations. At this point a rather profound conceptual and methodological departure from the conventional theory is needed to avoid inconsistencies of both physical and mathematical nature. As recalled earlier in this table, the conventional transformation theory is centered on the notion of preservation of the value of the Lie brackets. In the transition to Lie-admissible formulations one would then predictably attempt the construction of the transformation theory based on the preservation of the value of the Lie-admissible brackets, i.e., a transformation theory of the type

$$S^{\mu\nu}(b) \longrightarrow S'^{\mu\nu}(b') = \frac{\partial b'^{\mu}}{\partial b^{\rho}} S^{\rho\sigma}(b(b')) \frac{\partial b'^{\nu}}{\partial b^{\rho}} = S^{\mu\nu}(b'). \tag{3.5.6}$$

Unpredictably, transformations of this type are inconsistent on both physical and mathematical grounds. It is appropriate here to indicate that my early attempts at the construction of a transformation theory for Lie-admissible equations were based precisely on this approach. However, the inconsistencies I encountered in practical applications (e.g., the physical (non-conservative) spinning top under gravity) have been so severe to force me in 1973 into the laborious study of the Inverse Problem, as indicated in Section 1.

Predictably, the physical origin of the inconsistency rests on the physical nature of the systems considered. Consider a conservative system with conservation laws

$$\dot{X}_i = [X_i, H]_{(b)} = 0. \tag{3.5.7}$$

In order for any transformation theory to be physically consistent, it must be able to preserve the conserved nature of the X_i quantitities. This is the idea which is intended to express with the notion of "Lie identity isotopic transformation". Specifically, of utmost physical significance is that the value zero of brackets (3.5.7) is preserved, i.e.,

$$\dot{X}_i(b) = [X_i(b), H(b)]_{(b)} = 0 \longrightarrow \dot{X}'_i(b') = [X'_i(b'), H'(b')]_{(b')} = 0. \tag{3.5.8}$$

In the transition to nonconservative systems the physical profile is profoundly altered. And indeed, if nonconservative forces are added to the system with properties (3.5.7), the net effect is now turning them into nonconservation laws

$$\dot{X}_i(b) = (X_i, H)_{(b)} \neq 0, \qquad (3.5.9)$$

where, as typical of our Hamilton-admissible equations, the Hamiltonian, the base manifold and the physical quantities are unchanged by construction in such a transition, and the forces not derivable from a potential are embedded into the structure of the dynamical brackets. The net effect is that the preservation of the value of brackets (3.5.9) by the transformation theory would be, in general, physically inconsistent. The reason is that laws (3.5.9) now express the rate of variation of a physical quantity in time and such rate is not necessarily constant under the transformation theory. A typical case is that of the nonconservation law of the energy, that is, the necessary condition to ensure the existence of a nonconservative system. In this case, particularly when applied forced are included, the energy can arbitrarity vary in time. Thus, its rate of variation at one value of time is generally different than the corresponding rate at another value of time.

On mathematical grounds the canonical and canonical admissible equations can represent the same system, although in different coordinates. This means that there exists a transformation $a \rightarrow b(a)$ mapping the former equations into the latter, i.e., (Table 3.4)

$$\omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial H^{q}}{\partial a^{\mu}} = \frac{\partial b^{\rho}}{\partial a^{\mu}}\left(S_{\rho\sigma}\dot{b}^{\sigma} - \frac{\partial H^{c}}{\partial b^{\rho}}\right), \quad H^{q} = \tilde{\rho} H^{c} + \tilde{\sigma}. \qquad (3.5.10)$$

In turn, this means that it is possible to construct the transformation theory of the canonical-admissible equations as an "image" of that of the canonical equations,[5b] i.e.,

$$\omega_{\mu\nu}\dot{a}'^{\nu} - \frac{\partial H'^{q}}{\partial a'^{\mu}} = \frac{\partial b'^{\rho}}{\partial a'^{\mu}}\left(S'_{\rho\sigma}\dot{b}'^{\sigma} - \frac{\partial H'^{c}}{\partial b'^{\rho}}\right), \quad H'^{q} = \tilde{\rho}' H'^{c} + \tilde{\sigma}'. \qquad (3.5.11)$$

The mathematical inconsistency we are here referring to is constituted by the fact that if in the transition from Eqs. (3.5.10) to their transformed form (3.5.11) (where we have ignored the Jacobian of the transformation) one imposes property (3.5.6) for the $b \rightarrow b'(b)$ case, the corresponding transformations $a \rightarrow a'(a)$ are generally noncanonical. Viceversa, if the transformation $a \rightarrow a'(a)$ is canonical, the image $b \rightarrow b'(b)$ constructed with the above rule does preserve the form of the canonical-admissible equations and, most importantly, its Lie-admissible character, but rule (3.5.6) is generally violated.

We shall therefore define the canonical-admissible transformations as the Lie-admissible

isotopic transformations of the tensor $S^{\mu\nu} = \partial b^{\mu}/\partial R_{\nu}$. The preservation of the algebraic character of the canonical-admissible equations is in this way ensured, but not that of the value of the Lie-admissible brackets.

DEFINITION 3.5.1: A class C^{∞} and invertible transformation $b \rightarrow b'(b)$ is a canonical-admissible transformation when all the following equations

$$S'^{\mu\nu} = \frac{\partial b'^{\mu}}{\partial R'_{\nu}}\frac{\partial H'^{c}}{\partial b'^{\nu}} = \omega^{\mu\nu}\frac{\partial H'^{c}}{\partial b'^{\nu}} + F^{\mu}(b'), \qquad (3.5.12)$$
$$\{F^{\mu}\} = \{0, F_{ka}(a)\},$$

are identically satisfied.

Notice that in Eqs. (3.5.12) the forces not derivable from a potential preserve their functional form and are simply computed in the new system of coordinates. In different terms, conditions (3.5.12) ensure that the transformed canonical-admissible equations coincide with the transformed equations of motion up to the Jacobian of the transformation. For explicit examples see ref.[5b]

In conclusion, the transformation theory of the canonical-admissible equations appears to be considerably broader than that of the canonical equations. In particular, it is not an identity isotopic theory, as it can now be seen from the inhomogeneous nature of system (3.5.12) (compared to the homogeneous nature of the corresponding systems for canonical equations). As a result, a canonical-admissible transformation is not, in general, either canonical or generalized canonical and viceversa.

The covering nature of the canonical-admissible over the conventional canonical transformation is then indicated by the fact that, at the limit of null forces not derivable from a potential the Lie-admissible tensor $S^{\mu\nu}$ reduced to the canonical tensor $\omega^{\mu\nu}$, conditions (3.5.12) reduce to a form equivalent to (3.5.3) and the conventional transformation theory is recovered identically. This covering notion is further elaborated by the following property[5b]

LEMMA 3.5.4: The set of all canonical-admissible transformations forms a group.

The practical construction of the canonical-admissible transformations can be conducted by using that of the canonical transformations and then constructing their image of Lie-admissible type via equations of type (3.5.10) and (3.5.11). See ref.[5b] for details and examples. In the final analysis, this is an illustration of the complementary nature of the Inverse Problem and of the Lie-admissible problem. Similarly, we refer the reader to ref.[5b] for the study of

the integrability conditions for the existence of a new Hamiltonian as well as for other topics (e.g., the canonical-admissible covering of the so-called canonical inversion formulae, conjugate quantities, variational principles and Hamilton-Jacobi theory).

For completeness, permit me here to outline a step crucial for the subsequent analysis of this paper, the Lie-admissible covering of the infinitesimal canonical transformations.

Recall that an infinitesimal transformation

$$a^\mu \longrightarrow a'^\mu = a^\mu + \delta\theta\, G^\mu(a), \qquad (3.5.13)$$

is canonical, that is, identity isotopic with respect to $\omega^{\mu\nu}$, when the following conditions

$$\omega^{\mu\rho}\frac{\partial G^\nu}{\partial a^\rho} + \frac{\partial G^\mu}{\partial a^\rho}\omega^{\rho\nu} = 0, \qquad (3.5.14)$$

are verified. The study of this system within the context of the converse of the Poincaré Lemma then yields a solution

$$G^\mu = \omega^{\mu\nu}\frac{\partial G}{\partial a^\nu}, \quad \delta a^\mu = \delta\theta\,[a^\mu, G], \qquad (3.5.15)$$

as well as the integrability conditions which can be written

$$[a^\mu, [a^\nu, G]] + [a^\nu, [G, a^\mu]] + [G, [a^\mu, a^\nu]] = 0. \qquad (3.5.16)$$

This indicates the deep link of infinitesimal canonical transformations and Lie algebras in the sense that the algebraic laws enter into the integrability conditions for the transformations.

In the transition to generalized canonical transformations the situation is methodologically equivalent in the sense that transformations (3.5.13) are generalized canonical if they are identity isotopic with respect to $\Omega^{\mu\nu}$. Instead of Eqs. (3.5.14) this yields

$$\Omega^{\mu\rho}\frac{\partial G^\nu}{\partial a^\rho} + \frac{\partial G^\mu}{\partial a^\rho}\Omega^{\rho\nu} + G^\rho\frac{\partial \Omega^{\mu\nu}}{\partial a^\rho} = 0, \qquad (3.5.17)$$

with a solution

$$G^\mu = \Omega^{\mu\nu}\frac{\partial G}{\partial a^\nu}, \quad \delta a^\mu = \delta\theta\,[a^\mu, G]^*, \qquad (3.5.18)$$

and the integrability conditions

$$[a^\mu, [a^\nu, G]^*]^* + [a^\nu, [a^\mu, G]^*]^* + [G, [a^\mu, a^\nu]^*]^* = 0, \qquad (3.5.19)$$

of course now expressed in terms of the brackets of Birkhoff's equations. In this way we continue to illustrate the equivalence of Birkhoff's and Hamilton's equations up to the point that conventional transformations, such as translations in time, translations in space, rotations, etc., can be

fully defined within the context of Birkhoff's equations at both the finite and infinitesimal levels. In particular, the "Birkhoffian H^B is indeed, the generator of translations in time, i.e.,

$$\delta A = \delta t\,[A, H^B]^*, \qquad (3.5.20)$$

but now referred to the generalized Poisson brackets. Similarly, the $y_{ka_{ka}}$ component of the variables of Birkhoff's equations is the generator of translations in the r^{ka} direction, i.e.,

$$\delta A = \delta r^{ka}\,[A, y_{ka}]^* \text{ (no sum)}, \quad \{a^\mu\} = \{r^{ka}, y_{ka}\}, \qquad (3.5.21)$$

and the Birkhoffian "angular momentum" along an axis of unit vector $\underset{\sim}{n}$, $\underset{\sim}{M}^B\cdot\underset{\sim}{n} = (\underset{\sim}{r}\times\underset{\sim}{y})\cdot\underset{\sim}{n}$, is the component of rotations around $\underset{\sim}{n}$, i.e.,

$$\delta A = \delta\alpha\,[A, \underset{\sim}{M}^B\cdot\underset{\sim}{n}]^*. \qquad (3.5.22)$$

This illustrates again the remarks of Section 2 (particularly Table 2.14) to the effect that within the context of the Lie treatment of Newtonian systems, "physical transformations", that is, transformations of direct physical significance (translations in time, translations in space, rotations, etc.) can be characterized by "nonphysical quantities" as generators, that is, quantities without the conventional direct physical significance (energy, linear momentum, angular momentum, etc.). This situation does not occurs within the context of conventional treatments of trivial (conservative) systems, but it does occur within the context of unconventional treatments (e.g., Birkhoffian) of conservative systems or conventional treatments (that is, Hamiltonian) of nonconservative systems.

Equivalently, the Lie approach to the transformation theory of nonconservative systems implies the loss of the conventional aspects of the generators of conservative mechanics: to induce a physical transformation and to directly represent a physical quantity. This has a number of quite delicate implications at a classical as well as quantum mechanical level.[5c]

<u>One of the central objectives of the Lie-admissible formulations is that of restoring this symbiotic meaning of the generators for nonconservative mechanics too,</u> that is, the generators of physical transformations (translations in time, translations in space, rotations, etc.) are physical quantities (the physical energy, linear momentum, angular momentum, etc. respectively). It appears that this is an essential prerequisite to extract physical informations in a form as direct as possible at both the classical and the quantum mechanical level.

In particular, the above objective implies that the covering relativity which will be conjectured

in the next section is based on the preservation of the generators of the conventional Galilei (symmetry) Lie algebra of conservative systems. The action of the nonconservative forces is represented by the transition from Lie to Lie-admissible algebras.

By keeping in mind these objectives, we shall say that the transformations

$$b^\mu \longrightarrow b'^\mu = b'^\mu + \delta\theta\, G^\mu(b), \qquad (3.5.23)$$

are infinitesimal canonical-admissible transformations when they are Lie-admissible isotopic, that is, the fundamental dynamical brackets (3.4.25) are transformed into the new form

$$(b'^\mu, b'^\nu)_{(b)} = S'^{\mu\nu}(b'), \qquad (3.5.24)$$

which is still Lie-admissible, i.e.,

$$(S'^{\mu\rho} - S'^{\rho\mu}) \frac{\partial}{\partial b'^\rho} (S'^{\nu\tau} - S'^{\tau\nu}) \qquad (3.5.25)$$
$$+ (S'^{\nu\rho} - S'^{\rho\nu}) \frac{\partial}{\partial b'^\rho} (S'^{\tau\mu} - S'^{\mu\tau}) + (S'^{\tau\rho} - S'^{\rho\tau}) \frac{\partial}{\partial b'^\rho} (S'^{\mu\nu} - S'^{\nu\mu}) = 0.$$

It is possible to show that a solution is of the type

$$G^\mu = S_G^{\mu\nu} \frac{\partial G}{\partial b^\nu}, \quad \delta b^\mu = \delta\theta (b^\mu, G)_{(b,G)}, \qquad (3.5.26)$$

with the integrability conditions

$$((b^\mu, b^\nu), G) + ((b^\nu, G), b^\mu) + ((G, b^\mu), b^\nu) + (G, (b^\nu, b^\mu))$$
$$+ (b^\nu, (b^\mu, G)) + (b^\mu, (G, b^\nu)) = (b^\mu, (b^\nu, G)) + (b^\nu, (G, b^\mu)) \qquad (3.5.27)$$
$$+ (G, (b^\mu, b^\nu)) + ((G, b^\nu), b^\mu) + ((b^\nu, b^\mu), G) + ((b^\mu, G), b^\nu).$$

In this way, the Lie-admissibility condition again enters into the construction of the infinitesimal transformations. This was, after all, expected, from the covering nature of the approach. And indeed, integrability conditions (3.5.16) or (3.5.19) are Lie-admissible, but only expressed in the case of an anticommutative algebra (Table 3.3).

The first major new occurrence of this broader approach is the lack of uniqueness of the Lie-admissible tensor $S_G^{\mu\nu}$ for all possible generators, contrary to the uniqueness of the $\omega^{\mu\nu}$ tensor for all generators of canonical transformations. Rather than being a drawback, this appears to be a necessary condition for consistency, as well as an illustration of the capabilities of the approach.

A simple physical argument can be presented as follows. Nonconservative forces guarantee the nonconservation of the energy, but not necessarily that of other physical quantities, e.g.,

the angular momentum. This implies that the Galilei symmetry can only be partially broken by nonconservative systems. The net result is that a full embedding of the Galilei algebra into a covering Galilei-admissible algebra (see Tables 3.6 and 3.7) would in this case be physically inconsistent because it automatically implies the nonconservation of all physical quantities, contrary to assumption.

An intriguing aspect of the theory of Lie-admissible algebras is that these algebras can be Lie as a particular case (see Theorems 4 and 7 of ref.[26e]). At the level of classical realizations (in the sense of Table 3.3) this implies that the Lie-admissible tensor $S_G^{\mu\nu}$ can be the Lie tensor $\omega^{\mu\nu}$ for particular generators. Therefore, when a physical generator (say, the angular momentul) is conserved, it is expected that Eqs. (3.5.26) and (3.5.27) coincide with Eqs. (3.5.18) and (3.5.19), respectively.

To restate this situation in different terms, in order for any possible covering of the Galilei relativity to be physically consistent in nonconservative mechanics it must also be able to characterize a partial breaking of the Galilei relativity.

TABLE 3.6: LIE-ADMISSIBLE COVERING OF LIE'S THEORY.

Table 3.5 essentially indicates that the transformation theory of our canonical-admissible equations is such to preserve a group structure for the case of finite transformations, while exhibiting a Lie-admissible algebraic character for the case of infinitesimal transformations. Without any doubt, this has been for me one of the most intriguing features of the Lie-admissible formulations, because they clearly give hopes for the existence of a consistent Lie-admissible generalization of Lie's theory. In turn, this problem results to be the true, first, technical problem for the construction of a possible covering of the Galilei (and Einstein[5b]) relativity.

My efforts in the identification and study of this problem are summarized below, with the understanding that they are rudimentary as well as in need of inspection and implementation by independent researchers. The reader should be aware that the terms "Lie's theory" nowadays refer to a rather vast, articulated and sophysticated body of methodological tools encompassing a number of diversified disciplines. I will have achieved my objective in its entirety if I succeed in only indicating the existence of realistic hopes in achieving, in due time, a Lie-admissible covering of Lie's theory.

Let me begin with the problem of a Lie-admissible covering of Lie's first, second and third theorems[29], as an abstract version of the canonical-admissible theory of the preceding Table.

For the sake of notational convenience, let me recall that an n-parameter connected Lie transformation

$$a'^{\mu}(\theta) = f^{\mu}(a;\theta) = f^{\mu}(a;\theta^1,\dots,\theta^n), \quad \mu = 1,2,\dots,6N, \tag{3.6.1}$$

can be written in the neighborhood of the identity

$$da^{\mu} = U^{\mu}{}_{k}(a)\,\lambda^{k}{}_{i}(\theta)\,d\theta, \tag{3.6.2a}$$

$$U^{\mu}{}_{k}(a) = \left[\frac{\partial f^{\mu}}{\partial \theta^{k}}\right]_{\theta=0}, \tag{3.6.2b}$$

yielding Lie's first theorem.[29g]

THEOREM 3.6.1: If the transformations $a'^{\mu}(\theta) = f^{\mu}(a;\theta)$ form an n-dimensional connected Lie group, then

$$\frac{\partial a'^{\mu}}{\partial \theta^{i}} = U^{\mu}{}_{k}(a)\,\lambda^{k}{}_{i}(\theta), \tag{3.6.3}$$

where the functions $U^{\mu}{}_{k}(a)$ are analytic.

Before entering into the problem of a Lie-admissible generalization of this theorem, it is advisable to study its "Lie's covering" that is, the generalization related to the transformation theory of Birkhoff's equations which, to the best of my knowledge, has not been studied in the available literature. In turn, this is intimately linked to the problem of symmetries and first integrals and, specifically, to the nonuniqueness of a Lie symmetry for the characterization of the same first integral via Noether's theory (Table 2.12). I am here referring to the notion of Lie algebra isotopy, e.g., Eqs. (2.13.4). Clearly, for this notion to be fully realized, it needs the corresponding notion of Lie group isotopy. An example is soon given by using the methods of Table 3.5 for case (2.13.4). In correspondence to the isotopically related Lie algebras SO(3) and SO(2.1) for the characterization of the angular momentum conservation laws, one can construct the canonical and generalized canonical infinitesimal transformations, respectively, and, after integration to a finite series, reach the structures

$$SO(3):\ a'^{\mu} = \exp\left\{\theta^{i}\,\omega^{\alpha\beta}\frac{\partial M_i}{\partial a^{\beta}}\frac{\partial}{\partial a^{\alpha}}\right\} a^{\mu}, \tag{3.6.4a}$$

$$SO(2.1):\ a^{*\mu} = \exp\left\{\theta^{i}\,\Omega^{\alpha\beta}\frac{\partial M_i}{\partial a^{\beta}}\frac{\partial}{\partial a^{\alpha}}\right\} a^{\mu}, \tag{3.6.4b}$$

where $\Omega^{\mu\nu}$ is the generalized Lie tensor (2.13.4c). We shall say that the realization of the

SO(2.1) group so constructed is an isotope of SO(3). Notice the central feature of the notion of isotopy: it realizes a group in terms of the base manifold, generators and parameters of a generally nonisomorphic group.

DEFINITION 3.6.1: Consider an n-paremeter connected Lie group G of transformations $a'^{\mu} = f^{\mu}(a;\theta)$. A Lie isotopic image (or simply a Lie isotope) of G is a connected, n-parameter Lie group G* of transformations

$$a^{*\mu} = g^{\mu}{}_{\nu}(a;\theta)\,f^{\nu}(a;\theta) = f^{*\mu}(a;\theta), \tag{3.6.5}$$

characterized by $36N^2$ factor functions $g^{\mu}{}_{\nu}(a;\theta)$, called isotopic functions, which is such to to admit a Lie algebra structure in the neighborhood of the identity when expressed in terms of the base manifold (the a-variables), generators (say, the quantities X_i) and the parameters (θ^i) of the original Lie group G.

This immediately yields the following Lie covering of Lie's first theorem.

THEOREM 3.6.2: If the transformations $a^{*\mu} = f^{*\mu}(a;\theta) = g^{\mu}{}_{\nu}(a;\theta)f^{\nu}(a;\theta)$ characterize an isotope G* of a connected n-dimensional Lie group G with transformations $a'^{\mu} = f^{\mu}(a;\theta)$, then there exist isotopic functions $g^{*}{}_{k}{}^{i}(a)$ such that

$$\frac{\partial a^{*\mu}}{\partial \theta^{k}} = g^{*}{}_{i}{}^{j}(a)\,U^{\mu}{}_{j}(a)\,\lambda^{i}{}_{k}(\theta), \tag{3.6.6}$$

where the functions $U^{\mu}{}_{j}(a)$ and $g^{*}{}_{i}{}^{j}(a)$ are analytic.

In essence, in this case we have, instead of Eqs. (3.6.2b)

$$U^{*\mu}{}_{k}(a) = \left[\frac{\partial}{\partial \theta^{k}}\, g^{\mu}{}_{\nu}\, f^{\nu}\right]_{\theta=0}. \tag{3.6.7}$$

The functions $g^{*}{}_{i}{}^{j}$, therefore, are (uniquely) characterized by the factorization into of Eqs. (3.6.6).

Now, the original group G can be subjected to the familiar realization in the neighborhood of the origin[29g, 5b]

$$U^{\nu}_{i} \frac{\partial}{\partial a^{\nu}} U^{\mu}_{j} - U^{\nu}_{j} \frac{\partial}{\partial a^{\nu}} U^{\mu}_{i} = C^{k}_{ij} U^{\mu}_{k} , \qquad (3.6.8a)$$

$$C^{k}_{ij} = \mu^{r}_{i} \mu^{s}_{j} \left\{ \frac{\partial \lambda^{k}_{r}}{\partial \theta^{s}} - \frac{\partial \lambda^{k}_{s}}{\partial \theta^{r}} \right\} , \qquad (3.6.8b)$$

$$[X_i, X_j]_A = X_i X_j - X_j X_i = C^{k}_{ij} X_k , \qquad (3.6.8c)$$

$$X_i = U^{\mu}_{i}(a) \frac{\partial}{\partial a^{\mu}} , \qquad (3.6.8d)$$

which we shall refer to in this paper as the standard realization .

For the isotope G* we have a dual possibility. First, since G* is a connected Lie group, it can be subjected to the standard realization and we write

$$U^{*\nu}_{i} \frac{\partial}{\partial a^{\nu}} U^{\mu}_{j} - U^{*\nu}_{j} \frac{\partial}{\partial a^{\nu}} U^{*\mu}_{i} = C^{*k}_{ij} U_{k} , \qquad (3.6.9a)$$

$$C^{*k}_{ij} = \mu^{*r}_{i} \mu^{*s}_{j} \left\{ \frac{\partial \lambda^{*k}_{r}}{\partial \theta^{s}} - \frac{\partial \lambda^{*k}_{s}}{\partial \theta^{r}} \right\} , \qquad (3.6.9b)$$

$$[X^{*}_{i}, X^{*}_{j}]_A = X^{*}_{i} X^{*}_{j} - X^{*}_{j} X^{*}_{i} = C^{*k}_{ij} X^{*}_{k} , \qquad (3.6.9c)$$

$$X^{*}_{k} = U^{*\mu}_{k}(a) \frac{\partial}{\partial a^{\mu}} , \qquad (3.6.9d)$$

However, in order to realize G* as an isotope of G, we must express it in terms of the generators of G. This essentially implies a redefinition of the associative product $X^{*}_{i}X^{*}_{j}$ of Lie's fundamental rule (3.6.9c) according to

$$A(X^{*}) : X^{*}_{i} X^{*}_{j} \longrightarrow A^{*}(X) : X_i * X_j . \qquad (3.6.10)$$

In other words, to realize G* as an isotope of G, we must change the standard basis of G* into the standard basis of G and, jointly, modify the associative product of rule (3.6.9c) into a form which, after exponentiation, yields a group isomorphic to G* and not to G. This is exactly the occurrence of example (3.6.4).

We reach in this way a crucial point of our analysis. The notion of Lie group isotopy goes at the very foundation of Lie's theory, the universal enveloping associative algebra induced by product X_iX_j of rule (3.6.9c) (see Table 3.7). A study of this problem indicates that the notion can be realized via an isotopy of such enveloping algebra, that is, an associativity preserving mapping of the product X_iX_j of the type

$$A^{*}(X) : X_i * X_j = g^{*}{}_{i}{}^{r} X_r \, g^{*}{}_{j}{}^{s} X_s , \quad X_r = U^{\mu}_{r}(a) \frac{\partial}{\partial a^{\mu}} . \qquad (3.6.11)$$

This enveloping aspect will be outlined in more details in Table 3.7 . At this point let us recall for notational convenience Lie's second theorem .[29g]

THEOREM 3.6.3: If $X_i = U^{\mu}_{i}(a) \frac{\partial}{\partial a^{\mu}}$ are the generators of an n-dimensional connected Lie group, they satisfy the closure relations

$$[X_i, X_j]_A = X_i X_j - X_j X_i = C^{k}_{ij} X_k , \qquad (3.6.12)$$

where the quantities C^{k}_{ij} are constants (Lie's structure constants).

As indicated in Table 2.13 , for the Lie algebra $\underline{G}^{*}$ of G* to be an isotope of $\underline{G}$, it must be closed with the generators of $\underline{G}$ (and generalized brackets). This implies that the necessary and sufficient conditions for the functions $g^{*}{}_{i}{}^{j}$ to be isotopic can be written[5b]

$$g^{*k}_{i} U^{\nu}_{k} \frac{\partial}{\partial a^{\nu}} g^{*e}_{j} - g^{*k}_{j} U^{\nu}_{k} \frac{\partial}{\partial a^{\nu}} g^{*e}_{i} = g^{*r}_{j} g^{*s}_{i} C^{e}_{rs} + C^{*r}_{ij} g^{*e}_{r} , \qquad (3.6.13)$$

under which the standard realization of G*, Eqs. (3.6.9c) is turned into the isotopic realization[5b]

$$U^{\nu}_{i} \frac{\partial}{\partial a^{\nu}} * U^{\mu}_{j} - U^{\nu}_{j} \frac{\partial}{\partial a^{\nu}} * U^{\mu}_{i} = \tilde{C}^{k}_{ij} U^{\mu}_{k} , \qquad (3.6.14a)$$

$$\tilde{C}^{k}_{ij} = C^{*r}_{ij} g^{*k}_{r}(a) . \qquad (3.6.14b)$$

This yields the following Lie covering of Lie's second theorem .[5b]

THEOREM 3.6.4: If $X_i = U^{\mu}_{i}(a) \frac{\partial}{\partial a^{\mu}}$ are the generators of a n isotope G* of an n-parameter connected Lie group G, they satisfy the closure relations

$$[X_i, X_j]_{A*} = X_i * X_j - X_j * X_i = \tilde{C}^{k}_{ij}(a) X_k , \qquad (3.6.15)$$

where the quantities $\tilde{C}^{k}_{ij}(a)$, here called Lie's structure functions, are generally dependent on the local coordinates of the base manifold of G.

As is well know, the use of the Lie algebra laws in rule (3.6.8c) yields Lie's third theorem [29g]

THEOREM 3.6.5: The structure constants of the standard realization of an n-dimensional connected Lie group G satisfy the identities

$$C_{ij}^{k} + C_{ji}^{k} = 0, \tag{3.6.16a}$$

$$C_{ij}^{k} C_{ke}^{r} + C_{je}^{k} C_{ki}^{r} + C_{ei}^{k} C_{kj}^{r} = 0. \tag{3.6.16b}$$

One of the crucial requirements of the notion of Lie isotopy is that of preserving the Lie algebra structure in the neighborhood of the identity. The use of Eqs. (3.3.10) for rule (3.6.15) then yields the following Lie covering of Lie's third theorem. [5b]

THEOREM 3.6.6: The structure functions of the isotopic realization of an n-dimensional connected Lie group satisfy the identities

$$\tilde{C}_{ij}^{k} + \tilde{C}_{ji}^{k} = 0, \tag{3.6.17a}$$

$$\tilde{C}_{ij}^{k} \tilde{C}_{ke}^{r} + \tilde{C}_{je}^{k} \tilde{C}_{ki}^{r} + \tilde{C}_{ei}^{k} \tilde{C}_{kj}^{r} + [\tilde{C}_{ij}^{r}, X_e]_{A^*} + [\tilde{C}_{je}^{r}, X_i]_{A^*} + [\tilde{C}_{ei}^{r}, X_j]_{A^*} = 0. \tag{3.6.17b}$$

In essence, the constancy of the quantities C_{ij}^{k} of Lie's fundamental rule appears to be linked to the use of the standard realization. If an isotopic realization is instead assumed, these quantities can indeed acquire an explicit dependence on the base manifold, but in such a way to preserve the Lie algebra laws. This is the meaning of Theorems 3.6.4 and 3.6.6.

For completeness we must now touch on the question of exponentiation to a finite transformation. For the case of the standard realization we have the familiar exponential law [29g]

$$a'^{\mu} = e^{\theta^i X_i} a^{\mu}, \tag{3.6.18}$$

(under the assumption of all necessary convergence conditions). It is an instructive exercise for the interested reader to see that such exponential mapping carries over to the isotopic realization yielding the isotopically mapped exponential law [5b]

$$a^{*\mu} = e^{\theta^i g_i^{*j} X_j} a^{\mu}, \tag{3.6.19}$$

here again written under the assumption of all necessary convergence conditions, as well as that the functions g_i^{*j} satisfy conditions (3.6.13).

A final essential aspect to be inspected for the study of the consistency of the notion of Lie isotopy is that of the composition law. For the standard realization we have

$$e^{X_\beta} e^{X_\alpha} = e^{X_\gamma}, \tag{3.6.20}$$

where the new element is given by the Baker-Campbell-Hausdorff formula [29g]

$$X_\gamma = X_\alpha + X_\beta + \frac{1}{2}[X_\alpha, X_\beta]_A + \frac{1}{2}[(X_\alpha - X_\beta), [X_\alpha, X_\beta]_A]_A + \cdots. \tag{3.6.21}$$

For the isotope G* we have instead

$$e^{X^*_\beta} e^{X^*_\alpha} = e^{X^*_\gamma}, \quad X^* = g^* X, \tag{3.6.22}$$

where the new element is now given by the isotopically mapped Baker-Campbell-Hausdorff formula [5b]

$$X^*_\gamma = X^*_\alpha + X^*_\beta + \frac{1}{2}[X_\alpha, X_\beta]_{A^*} + \frac{1}{2}[(X_\alpha - X_\beta), [X_\alpha, X_\beta]_{A^*}]_{A^*} + \cdots. \tag{3.6.23}$$

Finally, we must touch on the question of the realization of the standard and isotopic canonical realizations of Lie groups in Newtonian mechanics. As is well known, the realization of the former is given by the transformation theory of Hamilton's equations (that is, canonical transformations), according to

$$u^{\mu}_{k}(a) \longrightarrow \omega^{\mu\nu} \frac{\partial G_k}{\partial a^\nu}, \tag{3.6.24a}$$

$$X_k(a) \longrightarrow \omega^{\mu\nu} \frac{\partial G_k}{\partial a^\nu} \frac{\partial}{\partial a^\mu}, \tag{3.6.24b}$$

$$\exp(\theta^i X_i) \to \exp\left(\theta^i \omega^{\mu\nu} \frac{\partial G_i}{\partial a^\nu} \frac{\partial}{\partial a^\mu}\right), \tag{3.6.24c}$$

$$[X_i, X_j]_A \longrightarrow [G_i, G_j]_{(a)} = \frac{\partial G_i}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial G_j}{\partial a^\nu}, \tag{3.6.24d}$$

where the last identity must be intended up to neutral elements of the universal enveloping associative algebra.

It has been rewarding for me to see that the corresponding realization of the notion of isotopically mapped Lie group is given by the transformation theory of Birkhoff's equations (that is, the generalized canonical transformations) according to

$$u^{*\mu}_{k} = g^{*i}_{k}\, u^{\mu}_{i} \longrightarrow \Omega^{\mu\nu}(a)\, \frac{\partial G_k(a)}{\partial a^{\nu}}, \qquad (3.6.25a)$$

$$X^{*}_{k} = g^{*i}_{k}\, u^{\mu}_{i} \longrightarrow \Omega^{\mu\nu}(a)\, \frac{\partial G_k(a)}{\partial a^{\nu}} \frac{\partial}{\partial a^{\mu}}, \qquad (3.6.25b)$$

$$\exp(\theta^i X^{*}_{k}) \longrightarrow \exp\left(\theta^i \Omega^{\mu\nu} \frac{\partial G_k}{\partial a^{\nu}} \frac{\partial}{\partial a^{\mu}}\right), \qquad (3.6.25c)$$

$$[X_i, X_j]_{A^*} \longrightarrow [G_i, G_j]^{*}_{(a)} = \frac{\partial G_i}{\partial a^{\mu}} \Omega^{\mu\nu} \frac{\partial G_j}{\partial a^{\nu}}, \qquad (3.6.25d)$$

with the same understanding for the last equation as that of Eqs. (3.6.24d). In conclusion, the property that Birkhoff's equations characterize a Lie algebra isotopy of Hamilton's equations carries over to the notion of groups via the transformation theory. As we shall see, this is a crucial intermediate step for our attempted covering relativity of Section 4.

We are now sufficiently equipped to outline the Lie-admissible covering of Lie's theorems. It is at this point where a generalization of the notion of isotopy is useful, particularly on intuitional grounds. By returning for a moment to the language of Abstract Algebras of Table 3.3, let U be an algebra with elements a, b, c, . . . , and product ab over a field F. A genotopic mapping of the product is any invertible mapping ab $\longrightarrow$ a $\circ$ b which violates the algebraic laws of ab[5b] (e.g, if ab is associative or Lie, a $\circ$ b is nonassociative or non-Lie, respectively). The algebra U which is the same vector space as U but equipped with the product a $\circ$ b and now satisfying different algebraic laws is called a genotope of U.[5b] Thus, on a comparative ground, the algebraic isotopy is based on the preservation of the laws of the original algebra, while the algebraic genotopy is based on the violation of these original algebraic laws.

We reach in this way a crucial notion for the attempted covering relativity of Section 4. And indeed, our conjectured Galilei-admissible relativity is a genotope of the Galilei relativity. It is therefore of some relevance to identify the various stages of realization of the notion of genotopy for nonconservative mechanics.

First of all, the notion of genotopy, still at an abstract algebraic level, can be interpred as a mapping which (a) preserves the original algebra U as vector space, (b) changes the product in an invertible form, but in such a way to (c) induce a desired algebraic structure, according to

the meaning[3c, 5b]

"γενναίω" "τόπος" = "produce" "configuration".

We shall therefore use the notion of algebraic genotopy as an invertible algebra inducing mapping. Of course, the mapping in which we are interested most is that inducing a Lie-admissible algebra at a classical level. We reach in this way our first step, the realization in Newtonian Mechanics of the Lie-admissible genotopic mapping of the Poisson brackets, that is, the mapping of the Poisson brackets which violates the Lie algebra laws by assumption, but it is such to induce a Lie-admissible algebra. This mapping can be realized in terms of functions $\frac{\partial b^{\nu}}{\partial g^{\mu}}$ on the base manifold and, from realization (3.4.2) of the canonical-admissible equations, can be written[5b]

$$[A, B]_{(b)} = \frac{\partial A}{\partial b^{\mu}}\, \omega^{\mu\nu} \frac{\partial B}{\partial b^{\nu}} \longrightarrow (A, B)_{(b)} = \frac{\partial A}{\partial b^{\mu}}\, S^{\mu\nu}_{(b)} \frac{\partial B}{\partial b^{\nu}}. \qquad (3.6.26)$$

Thus, the mapping from the Poisson to our dymanical brackets is an example of Lie-admissible genotopy. This is clearly a natural extension of the notion of Lie isotopy as the mapping from the conventional to the generalized Poisson brackets.

The second step is that of identifying the analytic origin of this Newtonian notion of Lie-admissible genotopy. It is clearly given by the transition from Hamilton's to our Hamilton-admissible equations. In turn, this provides a first algebraic characterization of nonconservation laws for systems with forces not derivable from a potential as the Lie-admissible genotopic mapping of conservation laws[5b]

$$\dot{X}_i = [X_i, H]_{(b)} = 0 \longrightarrow \dot{X}_i = (X_i, H)_{(b)} \neq 0, \qquad (3.6.27)$$

where the forces responsible for the nonconservations are embedded into the Lie-admissible product.

The mechanics of this mapping should be kept in mind. The starting ground is that of a conservative system in the physical variables $\{b^{\mu}\} = \{r^{ka}, p_{ka}\}$ (in the sense of Theorem 3.4.1) with Hamiltonian H^c representing the physical energy, and the quantities X_i representing physical conservation laws. This setting in then implemented with forces not derivable from a potential. This does not affect the definition of physical quantitives or, if you prefer, their explicit functional form in the space of the b-variables, but only their character which now is of nonconserved nature. This results in nonconservation laws. Our central objective is to achieve an algebraic characterization of these nonconservation laws (a) without changing the space of the physical variables b^{μ} and (b) without changing the explicit functional

dependence of the physical quantities in the b-variables. Such an objective cannot be realized within the context of Lie algebras and their isotopies. The notion of Lie-admissible genotopy (3.6.27), instead, does satisfy these requirements by emerging at the same time rather natural.

In conclusion, the content of Tables 3.3 and 3.4 can be reinterpreted by saying that, first, the Lie-admissible algebras emerge as a genotope of the Lie algebra (algebraic profile); secondly, the Hamilton-admissible equations can be interpreted also as a genotope of Hamilton's equations (analytic profile) and; thirdly, nonconservation laws can be equally interpred as a genotope of the corresponding conservation laws at the limit of null nonconservative forces (dynamical profile).

In order to achieve the rudiments of a notion of genotopic mappings in Newtonian Mechanics which is sufficiently diversified to allow the conjecture of a covering relativity, several additional aspects must be investigated. In this section we are interested to see whether the notion of isotopy of Lie's transformation theory indicated by Theorems 3.6.2, 3.6.4 and 3.6.6 admits a consistent generalization of genotopic nature. Of course, this implies, in particular, the study of the notion in the neighborhood of the identity (infinitesimal genotopy) as well as for finite transformations (finite genotopy). Predictably, these two aspects turn out to be deeply interrelated. As we shall outline later on, the hope is then that of achieving an algebraic-group theoretic characterization of broken symmetries.

5b

DEFINITION 3.6.2: Consider an n - parameter connected Lie group G of transformations $b'^{\mu} = f^{\mu}(b;\theta)$. A Lie-admissible genotopic image (or simply a genotope) of G is an n-parameter, connected Lie group $\hat{G}$ of transformations

$$\hat{b}^{\mu} = \hat{g}^{\mu}_{\nu}(b;\theta)\, f^{\nu}(b;\theta) = \hat{f}^{\mu}(b;\theta), \qquad (3.6.28)$$

characterized by $36N^2$ factor functions $\hat{g}^{\mu}_{\nu}(b;\theta)$, called genotopic functions, which is such to admit a Lie-admissible algebra in the neighborhood of the origin when expressed in terms of the base manifold (the b-variables), the generators (X_i) and the parameters (θ^i) of the original Lie group G.

On more explicit terms, the objective of the above definition is to attempt the characterization of transformations (3.6.28) which are such that (in canonical generators G_i rather than standard abstract generators X_i)

$$\hat{b}^{\mu} \approx b^{\mu} + \theta^i (b^{\mu}, G_i), \qquad (3.6.29)$$

where now, by central condition, the product (b^{μ}, G_i) is non-Lie, although Lie-admissible. This remark is sufficient to indicate that the theory we are looking for is based on a rather profound departure from Lie's theory. The hopes for the existence of such generalized context is provided by the infinitesimal canonical-admissible transformations of Table 3.5 which are precisely of type (3.6.29).

The following property is useful to identify the nature of the transition from a Lie group G to its isotope G* and to its genotope $\hat{G}$. 3d, 19c, 5b

LEMMA 3.6.1: Under the assumption that an n-parameter connected Lie group G admits an isotope G* and a genotope $\hat{G}$, the groups G, G* and $\hat{G}$ are generally nonisomorphic amoung themselves. In particular, both the isotopic and the genotopic mappings do not, in general, preserve the compact or noncompact, Abelian or non-Abelian and semisimple or nonsemisimple character of the original group.

For instance, a three-dimensional, connected, Abelian Lie group can be a genotope of SO(3), the group of rotations. Notice that each Lie group can admit, at least in principle, a family of isotopes and genotopes.

The genotope $\hat{G}$ is a Lie group by assumption. Thus, it can be subjected to the standard realization and we shall write

$$d\hat{a}^{\mu} = \hat{u}^{\mu}_{k}(b)\, \hat{\lambda}^{k}_{i}\, d\theta^i, \quad \hat{u}^{\mu}_{k}(b) = \left[\frac{\partial}{\partial\theta^k}\hat{f}^{\mu}(b;\theta)\right]_{\theta=0}, \qquad (3.6.30a)$$

$$\hat{u}^{\nu}_{i}\frac{\partial}{\partial b^{\nu}}\hat{u}^{\mu}_{j} - \hat{u}^{\nu}_{j}\frac{\partial}{\partial b^{\nu}}\hat{u}^{\mu}_{i} = \hat{C}^{k}_{ij}\,\hat{u}^{\mu}_{k}, \qquad (3.6.30b)$$

$$\hat{C}^{k}_{ij} = \hat{\mu}^{z}_{i}\,\hat{\mu}^{s}_{j}\left\{\frac{\partial\hat{\lambda}^{k}_{z}}{\partial\theta^s} - \frac{\partial\hat{\lambda}^{k}_{s}}{\partial\theta^z}\right\}, \qquad (3.6.30c)$$

$$[\hat{X}_i, \hat{X}_j]_A = \hat{X}_i\hat{X}_j - \hat{X}_j\hat{X}_i = \hat{C}^{k}_{ij}\,\hat{X}_k, \qquad (3.6.30d)$$

$$\hat{X}_i = \hat{u}^{\mu}_{i}(b)\frac{\partial}{\partial b^{\mu}}.$$

Our problem is now that of turning this realization which is strictly Lie in algebraic character into a new realization in terms of the generators X_i which is, instead, of Lie-admissible nature. A study of this problem indicates the need in this case of performing a genotopic mapping of the universal enveloping associative algebra

$$A(\hat{X}):\ \hat{X}_i \hat{X}_j \longrightarrow U(X):\ X_i \circ X_j\,, \qquad (3.6.31)$$

that is a mapping which, unlike that of Eqs. (3.6.10) now is such to violate the associativity of the product, although such to preserve the Lie-admissibility. This aspect will be considered in more details in Table 3.7. At this point we are interested in the generalization of the Lie isotopy considered in the first part of this table.

Introduce the following realization of the $\hat{u}^\mu_i$ functions

$$\hat{u}^\mu_i = [\alpha^r_i(b) + \beta^r_i(b)]\, u^\mu_r(b) \qquad (3.6.32)$$

subject to the subsidiary conditions

$$\alpha^r_i u^\nu_r \frac{\partial}{\partial b^\nu}(\beta^s_j u^\mu_s) + \beta^r_i u^\nu_r \frac{\partial}{\partial b^\nu}(\alpha^s_j u^\mu_s) - \alpha^r_j u^\nu_r \frac{\partial}{\partial b^\nu}(\beta^s_i u^\mu_s) - \beta^r_j u^\nu_r \frac{\partial}{\partial b^\nu}(\alpha^s_i u^\mu_s) = 0 \qquad (3.6.33)$$

which eliminates the free functions in (3.6.32). Then rule (3.6.30b) becomes[5b]

$$\left[\alpha^r_i u^\nu_r \frac{\partial}{\partial b^\nu}(\alpha^s_j u^\mu_s) - \beta^r_j u^\nu_r \frac{\partial}{\partial b^\nu}(\beta^s_i u^\mu_s)\right] - \left[\alpha^r_j u^\nu_r \frac{\partial}{\partial b^\nu}(\alpha^s_i u^\mu_s) - \beta^r_i u^\nu_r \frac{\partial}{\partial b^\nu}(\beta^s_j u^\mu_s)\right] = \hat{C}^k_{ij}(\alpha^s_k + \beta^s_k)\, u^\mu_s\,. \qquad (3.6.34)$$

This yields the product

$$X_i \circ X_j = \alpha^r_i u^\nu_r \frac{\partial}{\partial b^\nu}\left(\alpha^s_j u^\mu_s \frac{\partial}{\partial b^\mu}\right) - \beta^r_j u^\nu_r \frac{\partial}{\partial b^\nu}\left(\beta^r_i u^\mu_r \frac{\partial}{\partial b^\mu}\right) \qquad (3.6.35)$$

which, as desired, is nonassociative and Lie-admissible. Rule (3.6.30d) can now be written[5b]

$$[X_i, X_j]_U = X_i \circ X_j - X_j \circ X_i = \hat{C}^{*k}_{ij} X_k\,, \qquad (3.6.36a)$$

$$\hat{C}^{*k}_{ij} = \hat{C}^r_{ij}[\alpha^k_r(b) + \beta^k_r(b)]\,, \qquad (3.6.36b)$$

and represents, to the best of my knowledge, the broadest possible generalization of the fundamental Lie's rule (3.6.8c) capable of still characterizing a Lie algebra (in the sense that if rule (3.6.36) is realized with any algebra other than a Lie-admissible algebra, the Lie content of the theory is lost).

To restate these findings in different terms, we can say that the notion of Lie-admissibility is at the very foundation of Lie's theory, only expressed in its simplest possible form, the associative algebra A with Lie content A^-. A central technical aspect of this study thus consists

in attempting a generalization of this rule via the use of nonassociative but Lie-admissible algebras. If such generalization exists, it is expected to yield the desired dual profile, a Lie-admissible behaviour in the neighborhood of the identity while preserving the global structure of connected finite transformations.

The Lie-admissible covering of Lie's first theorem is now trivial and can be formulated as follows.[5b]

THEOREM 3.6.7: If the transformations

$$\hat{b}^\mu = \hat{g}^\mu_\nu(b;\theta) f^\nu(b;\theta) = \hat{f}^\mu(b;\theta) \qquad (3.6.37)$$

characterize a Lie-admissible genotopic image $\hat{G}$ of an n-dimensional connected Lie group G of transformations $b' = f^\mu(b;\theta)$, then

$$\frac{\partial \hat{b}^\mu}{\partial \theta^i} = [\alpha^r_i(b) + \beta^r_i(b)]\, u^\mu_r(b)\,, \qquad (3.6.38a)$$

$$\alpha^r_i u^\nu_r \frac{\partial}{\partial b^\nu}(\beta^s_j u^\mu_s) + \beta^r_i u^\nu_r \frac{\partial}{\partial b^\nu}(\alpha^s_j u^\mu_s) - \alpha^r_j u^\nu_r \frac{\partial}{\partial b^\nu}(\beta^s_i u^\mu_s) - \beta^r_j u^\nu_r \frac{\partial}{\partial b^\nu}(\alpha^s_i u^\mu_s) = 0\,, \qquad (3.6.38b)$$

where the functions $\alpha^r_i(b)$, $\beta^r_i(b)$ and $u^\mu_r(b)$ are analytic.

The integrability conditions on the functions α^r_i and β^r_i to be "genotopic functions" can be written

$$\alpha^r_i u^\nu_r \frac{\partial \alpha^k_j}{\partial b^\nu} - \alpha^r_j u^\nu_r \frac{\partial \alpha^k_i}{\partial b^\nu} - \beta^r_j u^\nu_r \frac{\partial \beta^k_i}{\partial b^\nu} + \beta^r_i u^\nu_r \frac{\partial \beta^k_j}{\partial a^\nu} = \hat{C}^{*k}_{ij} - (\alpha^r_i \alpha^s_j + \beta^r_i \beta^s_j) C^k_{rs}\,, \qquad (3.6.39)$$

where the C's are the structure constants of the original group G and the $\hat{C}^*$'s are the structure constants of the isotope $\hat{G}^*$ of G as originating from rule (3.6.36). Notice that the genotope $\hat{G}$ and the isotope $\hat{G}^*$ are not, in general, isomorphic. This indicates that the analysis for $\hat{G}$ can be carried out by considering the product $X_i \circ X_j$ alone, rather than the Lie product $[X_i, X_j]_U$. By writing

$$X_i \circ X_j = \alpha^r_i u^\nu_r \frac{\partial}{\partial b^\nu}(\alpha^s_j u^\mu_s)\frac{\partial}{\partial b^\mu} + \alpha^r_i \alpha^s_j u^\nu_r u^\mu_s \frac{\partial^2}{\partial b^\mu \partial b^\nu} \qquad (3.6.40)$$

$$= (f^k_{ij} + g^{ke}_{ij} X_e) X_k \equiv U^k_{ij}(b, x) X_k$$

we then reach the following Lie-admissible covering of Lie's second theorem[5b]

THEOREM 3.6.8: The generators $X_i = U^{\mu}_{i}(b) \frac{\partial}{\partial b^{\mu}}$ of a genotope $\hat{G}$ of a Lie Lie group G satisfy the relations

$$X_i \circ X_j = U^{k}_{ij}(b, X)\, X_k , \qquad (3.6.41)$$

where the quantities $U^{k}_{ij}(b,X)$ (here called Lie-admissible structure quantities) are generally dependent on the base manifold as well as the generators of the original group.

The covering nature of the approach can be indicated with the following[5b]

COROLLARY 3.6.8.A: The Lie-admissible structure quantities satisfy the identities

$$U^{k}_{ij}(b,X) - U^{k}_{ji}(b,X) = \hat{C}^{*k}_{ij}(b), \qquad (3.6.42)$$

where the $\hat{C}^{*}$'s are the Lie structure functions of the isotope $\hat{G}^{*}$.

COROLLARY 3.6.8.B: Under the limit

$$\alpha^{j}_{i} \to \delta^{j}_{i} \quad , \beta^{j}_{i} \to 0 , \qquad (3.6.43)$$

the Lie-admissible nonassociative product $X_i \circ X_j$ becomes associative in which case the genotopic mapping $G \to \hat{G}$ is the identity.

COROLLARY 3.6.8.C: Under the limit

$$\alpha^{j}_{i} \quad , \beta^{j}_{i} \to \delta^{j}_{i} , \qquad (3.6.44)$$

the Lie-admissible product $X_i \circ X_j$ becomes Lie's product, in which case the structure functions U^{k}_{ij} reduce to the structure constants of the original group in standard realization.

By using the general Lie-admissibility conditions (3.3. 8), we finally reach the following Lie-admissible covering of Lie's third theorem.[5b]

THEOREM 3.6.9: The structure quantities of a genotope $\hat{G}$ of a Lie group G satisfy the properties

$$(U^{k}_{ij} - U^{k}_{ji})(U^{r}_{ke} - U^{r}_{ek}) + (U^{k}_{je} - U^{k}_{ej})(U^{r}_{ki} - U^{r}_{ik}) + (U^{k}_{ei} - U^{k}_{ie})(U^{r}_{kj} - U^{r}_{jk}) \qquad (3.6.45)$$
$$+ [(U^{r}_{ij} - U^{r}_{ji}), X_e]_U + [(U^{r}_{je} - U^{r}_{ej}), X_i]_U + [(U^{r}_{ei} - U^{r}_{ie}), X_j]_U = 0 .$$

Again, the following properties indicate the covering nature of the approach.

COROLLARY 3.6.9. : Under limit (3.6.43) the Lie-admissible identities (3.6.45) recover the Lie identities (3.6.16) identically. Under limit (3.6.44) identities (3.6.45) become twice the Lie identities (3.6.16).

The exponential mapping can also be reached (under all the necessary convergence conditions) and we shall write it in the form

$$\hat{b}^{\mu} = e^{\theta^{i}[\alpha^{j}_{i}(b) + \beta^{j}_{i}(b)]\, X_j(b)}\, b^{\mu} , \qquad (3.6.46)$$

here called genotopically mapped exponential law,[5b] where the functions α^{j}_{i} and β^{j}_{i} satisfy subsidiary conditions (3.6.38b) as well as the integrability conditions (3.6.39).

The composition law of the genotope can then be written

$$e^{\hat{X}_{\beta}}\, e^{\hat{X}_{\alpha}} = e^{\hat{X}_{\gamma}} , \quad \hat{X} \Rightarrow \theta\,[\alpha + \beta]\, X , \qquad (3.6.47)$$

where

$$\hat{X}_{\gamma} = \hat{X}_{\alpha} + \hat{X}_{\beta} + \frac{1}{2}[X_{\alpha}, X_{\beta}]_{A^*} + \frac{1}{12}[(X_{\alpha} + X_{\beta}), [X_{\alpha}, X_{\beta}]_{A^*}]_{A^*} , \qquad (3.6.48)$$

is, again, the isotopically mapped Baker, Campbell-Hausdorff formula (3.6.23).[5b]

As a reinterpretation of Definition 3.6.2, we now introduce the following[5b]

DEFINITION 3.6.3: A Lie-admissible group of transformations is the set $\hat{G}$ of n-parameter connected transformations

$$\hat{b}^{\mu} = \hat{g}^{\mu}{}_{\nu}(b;\theta)\, \hat{f}^{\nu}(b;\theta) , \qquad (3.6.49)$$

acting in the base manifold. of a generally nonisomorphic group G in the same parameters which possesses:

(a) a Lie-admissible algebra in the neighborhood of the identity when realized in terms of the generators of G, according to rule (3.6.29);

(b) a genotopically mapped exponemtial law according to rule (3.6.46); and

(c) an isotopically mapped composition law according to rule (3.6.48).

In essence, the above definition is meant to attempt the identification of a group theoretic image of the algebraic notion of Lie-admissibility. A Lie-admissible group $\hat{G}$ is Lie by assumption and it is simply realized in an "unconventional" way. Neverthless, such realization is such to render the group $\hat{G}$ Lie-admissible in a double meaning. First, $\hat{G}$ admits a nonisomorphic Lie group $\hat{G}^*$ via the isotopy rule (3.6.36). Secondly, the Lie-admissible group $\hat{G}$ is capable of recovering the Lie group G identically under limit (3.6.43). These features are clearly promising for the problem of a covering of the Galilei relativity.

By looking in retrospective the reader can now see the methodological function of the the notion of Lie isotopy whose analytic origins lies withon the context of Birkhoff's equations. And indeed, starting from a Lie algebra $\underline{G}$ with generators X_i, the Lie content $\hat{\underline{G}}^-$ of the Lie-admissible generalization $\hat{\underline{G}}$ is isomorphic to $\hat{\underline{G}}^*$ and not to $\underline{G}$. This point will be somewhat refined in the next table.

It is easy to see that the theory of the canonical-admissible transformations provides a classical realization of the Lie-admissible covering of Lie's theory indicated in this table. And indeed, we can write[5b]

$$\hat{b} \approx b^\mu + \theta^i X_i \circ b^\mu \longrightarrow \hat{b} \approx b^\mu + \theta^i (X_i, b^\mu), \quad (3.6.50a)$$

$$[\alpha_i^r + \beta_i^r] u^\mu_r \longrightarrow S^{\mu\nu} \frac{\partial G_i}{\partial b^\nu}, \quad (3.6.50b)$$

$$[\alpha_i^r + \beta_i^r] u^\mu_r \frac{\partial}{\partial b^\mu} \longrightarrow S^{\mu\nu}_{(i)} \frac{\partial G_i}{\partial b^\nu} \frac{\partial}{\partial b^\mu}, \quad (3.6.50c)$$

$$\exp\{\theta^i (\alpha_i^j + \beta_i^j) X_j\} \longrightarrow \exp\left\{\theta^i S^{\mu\nu}_{(i)} \frac{\partial G_i}{\partial b^\nu} \frac{\partial}{\partial b^\mu}\right\}, \quad (3.6.50d)$$

$$X_i \circ X_j \longrightarrow (X_i, X_j). \quad (3.6.50e)$$

This confirms the existence of a realization of Lie-admissible character in the neighborhood of the identity, while preserving a global Lie structure of finite-connected transformations, but of Lie-admissible type. In particular, the limit which reduces the Lie-admissible group $\hat{G}$ to G is given by the null value of the external forces

$$\lim_{F_{ka} \to 0} S^{\mu\nu}_{(i)} = \omega^{\mu\nu}. \quad (3.6.51)$$

The exponential law (3.6.50d) can be recovered via the generalization of conventional procedures. For instance, by writing Eqs. (3.5.25) in the form

$$\frac{\partial \hat{b}^\mu}{\partial \theta^i} = (\hat{b}^\mu, \hat{G}_i)^*_{(\hat{b})}, \quad (3.6.52)$$

and by performing a Lie-admissible isotopic transformation we can write the expressions

$$\frac{\partial \hat{b}^\mu}{\partial \theta^i} = (\hat{b}^\mu, G_i)_{(b)}, \quad (3.6.53)$$

which can be interpreted as a system of differential equations in the unknown functions $\hat{b}^\mu$ subject to the initial conditions $\hat{b}(\theta)|_{\theta=0} = b^\mu$. A formal power series solution can then be written

$$\hat{b}^\mu = b^\mu + \frac{\theta^i}{1!}(b^\mu, G_i) + \frac{\theta^i \theta^j}{2!}((b^\mu, G_i), G_j) + \cdots, \quad (3.6.54)$$

yielding the exponential law of type (3.6.50d).

Almost needless to say, the known possibility of lack of verification[29h] of the integrability conditions[29i] for the existence of a Lie group can have a corresponding occurrence at the Lie-admissible level. The study of this aspect is left to the interested reader.

A rather peculiar property of the Lie-admissible groups is that, once interpreted as topological transformation groups,[29f] their action on the group manifold is non-geodesic.[5b] In other words, the Lie-admissible extension (3.6.50d) of Lie's exponential law (3.6.24c) (when it exists) is generally nongeodesic in character. Rather than considering this occurrence as a drawback, I consider it most attractive, particularly on relativity grounds. It is known that the action of the Galilei group in its topological manifold is geodesic. The covering relativity I am interested in, is specifically intended to be nongeodesic in character. This attitude is motivated by the need that, as indicated in Section 1, any covering relativity, to be effective for nonconservative systems, must represent such nonconservative character in its entirety. The indicated departure from conventional geodesic characterization is intended precisely as one way of characterizing nonconservative systems. And indeed, such systems are basically nongeodesic in nature, in the sense, for instance, that their trajectories in the carrier space is never of geodesic nature even when all forces derivable from a potential are null.

A few simple examples are here in order. Eqs. (3.6.4) provide an example of SO(2.1) as an isotope of SO(3) in canonical realization (3.6.25c). For another example in abtract formalism, consider the group of dilations in one dimension, D(1),

$$x' = f(x;\theta) = e^{\theta} x . \tag{3.6.55}$$

The standard generator is

$$X = x \frac{\partial}{\partial x} , \tag{3.6.56}$$

with exponentiall law

$$e^{\theta x \frac{\partial}{\partial x}} x = \left[1 + \frac{\theta}{1!}\left(x \frac{\partial}{\partial x}\right) + \frac{\theta^2}{2!}\left(x \frac{\partial}{\partial x}\right)^2 + \dots \right] x = e^{\theta} x . \tag{3.6.57}$$

The compositionl law is trivial and reads

$$\theta'' = \theta + \theta' , \quad x'' = f(x';\theta') = f(x;\theta+\theta') . \tag{3.6.58}$$

An isotope D*(1) of D(1) is given by[56]

$$x^* = \frac{x}{1-\theta x} = g^*(x;\theta)\, f(x;\theta) , = f^*(x;\theta) \tag{3.6.59a}$$

$$g^* = \frac{e^{\theta}}{1-\theta x} , \tag{3.6.59b}$$

and it is induced by the isotopic function x. To see it, the computation of law (3.6.19) yields

$$e^{\theta x^2 \frac{\partial}{\partial x}} = \left[1 + \frac{\theta}{1!}\left(x^2 \frac{\partial}{\partial x}\right) + \frac{\theta^2}{2!}\left(x^2 \frac{\partial}{\partial x}\right)^2 + \dots \right] x \tag{3.6.60}$$

$$= \left[1 + \theta x + \theta^2 x^2 + \dots \right] x = \frac{x}{1-\theta x} .$$

The composition law now reads

$$x^{**} = f^*(x^*;\theta') = f^*(x;\theta+\theta') \tag{3.6.61a}$$

$$x^{**} = \frac{x^*}{1-\theta' x^*} = \frac{\frac{x}{1-\theta x}}{1-\theta' \frac{x}{1-\theta x}} = \frac{x}{1-(\theta+\theta')x} . \tag{3.6.61b}$$

Thus, we have a case of analytic isomorphy of the composition law, the case being of trivial one-dimensionality (to have genuine nonisomorphisms $G \not\approx G^*$ more than one dimension is needed).

For a Lie-admissible group, consider the canonical realization of SO(2)

$$J = r_x p_y - r_y p_z , \tag{3.6.62a}$$

$$b'^{\mu} = e^{\theta\, \omega^{\alpha\beta} \frac{\partial J}{\partial b^{\beta}} \frac{\partial}{\partial b^{\alpha}}} b^{\mu} , \quad (b^{\mu}) = \begin{pmatrix} r_x \\ r_y \\ p_x \\ p_y \end{pmatrix} , \tag{3.6.62b}$$

explicitly given by

$$SO(2)\left\{ \begin{aligned} \begin{pmatrix} r'_x \\ r'_y \end{pmatrix} &= \begin{pmatrix} r_x \\ r_y \end{pmatrix} + \frac{\theta}{1!}\begin{pmatrix} [r_x, J] \\ [r_y, J] \end{pmatrix} + \frac{\theta^2}{2!}\begin{pmatrix} [[r_x, J], J] \\ [[r, J], J] \end{pmatrix} + \dots \\ &= \begin{pmatrix} r_x \cos\theta - r_y \sin\theta \\ r_x \sin\theta + r_y \cos\theta \end{pmatrix} , \\ \begin{pmatrix} p'_x \\ p'_y \end{pmatrix} &= \begin{pmatrix} p_x \\ p_y \end{pmatrix} + \frac{\theta}{1!}\begin{pmatrix} [p_x, J] \\ [p_y, J] \end{pmatrix} + \frac{\theta^2}{2!}\begin{pmatrix} [[p_x, J], J] \\ [[p_x, J], J] \end{pmatrix} + \dots \\ &= \begin{pmatrix} p_x \cos\theta - p_y \sin\theta \\ p_x \sin\theta + p_y \cos\theta \end{pmatrix} . \end{aligned} \right. \tag{3.6.6 } \tag{3.6.63}$$

Suppose that, as a result of nonconservative forces, the algebraic tensor of the representation is mapped into the Lie-admissible form

$$S_J^{\mu\nu} = \begin{pmatrix} 0_{2\times 2} & \lambda(t) 1_{2\times 2} \\ -\mu(t) 1_{2\times 2} & 0_{2\times 2} \end{pmatrix} , \quad \lambda(t) \neq \mu(t) . \tag{3.6.64}$$

This is, essentially, an algebraic representative of the nonconservation of J. The canonical-admissible exponential law (3.6.50c) now yields the Lie-admissible covering group[56]

$$\hat{SO}(2)\left\{ \begin{aligned} \begin{pmatrix} \hat{r}_x \\ \hat{r}_y \end{pmatrix} &= \begin{pmatrix} r_x \\ r_y \end{pmatrix} + \frac{\theta}{1!}\begin{pmatrix} (r_x, J) \\ (r_y, J) \end{pmatrix} + \frac{\theta^2}{2!}\begin{pmatrix} ((r_x, J), J) \\ ((r_y, J), J) \end{pmatrix} + \dots \\ &= \begin{pmatrix} r_x \cos(\lambda(t)\theta) - r_y \sin(\lambda(t)\theta) \\ r_x \sin(\lambda(t)\theta) + r_y \cos(\lambda(t)\theta) \end{pmatrix} , \\ \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \end{pmatrix} &= \begin{pmatrix} p_x \\ p_y \end{pmatrix} + \frac{\theta}{1!}\begin{pmatrix} (p_x, J) \\ (p_x, J) \end{pmatrix} + \frac{\theta^2}{2!}\begin{pmatrix} ((p_x, J), J) \\ ((p_y, J), J) \end{pmatrix} + \dots \\ &= \begin{pmatrix} p_x \cos(\mu(t)\theta) - p_y \sin(\mu(t)\theta) \\ p_x \sin(\mu(t)\theta) + p_y \cos(\mu(t)\theta) \end{pmatrix} , \end{aligned} \right. \tag{3.6.65a} \tag{3.6.65b}$$

$$\lim_{\lambda,\mu \to 1} \hat{SO}(2) \equiv SO(2) . \tag{3.6.65c}$$

which is a Lie-admissible group in our terminology because constructed with the base manifold, generator and parameter of another Lie group. Also $\hat{SO}$ (2) is capable of recovering SO(2) identically at the conservative limit $\lambda, \mu \to 1$.

As another example of Lie-admissible groups, consider first the conventional canonical realization of the one-dimensional group of translations in time, say, for the harmonic oscillator $\ddot{r} + r = 0$ $(m = k = 1)$

$$H = \frac{1}{2}(p^2 + r^2), \quad p = \dot{r}, \quad (b^\mu) = \begin{pmatrix} r \\ p \end{pmatrix}, \qquad (3.6.66a)$$

$$T_1(t): \; b'^\mu = e^{t\,\omega^{\alpha\beta}\frac{\partial H}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}}\, b^\mu, \qquad (3.6.66b)$$

$$\begin{pmatrix} r' \\ p' \end{pmatrix} = \begin{pmatrix} r \\ p \end{pmatrix} + \frac{t}{1!}\begin{pmatrix} [r, H] \\ [p, H] \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} [[r, H], H] \\ [[p, H], H] \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} r\cos t + p\sin t \\ -r\sin t + p\cos t \end{pmatrix}. \qquad (3.6.66c)$$

The addition of a constant force (for simplicity) to the equations of motion, $\ddot{r} + r + F = 0$, can be represented with the canonical-admissible equations in terms of the same Hamiltonian H and the Lie-admissible tensor

$$S^{\mu\nu}_H = \begin{pmatrix} 0 & 1 \\ s & 0 \end{pmatrix}, \quad s = -\left(1 + \frac{F}{r}\right), \qquad (3.6.67)$$

yielding the Lie-admissible covering group $\hat{T}_1(t)$ of the Lie group $T_1(t)$ [5b]

$$\hat{T}_1(t): \begin{pmatrix} \hat{r} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} r \\ p \end{pmatrix} + \frac{t}{1!}\begin{pmatrix} (r, H) \\ (p, H) \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} ((r, H), H) \\ ((p, H), H) \end{pmatrix} + \cdots \qquad (3.6.68a)$$

$$= \begin{pmatrix} -F + (r + F)\cos t + p\sin t \\ -(r + F)\sin t + p\cos t \end{pmatrix} = e^{t\,S^{\mu\nu}_H \frac{\partial H}{\partial b^\nu}\frac{\partial}{\partial b^\mu}}\begin{pmatrix} r \\ p \end{pmatrix},$$

$$\lim_{F \to 0} \hat{T}_1(t) \equiv T_1(t), \qquad (3.6.68b)$$

$$(A, B) = \frac{\partial A}{\partial r}\frac{\partial B}{\partial p} - \left(1 + \frac{F}{r}\right)\frac{\partial A}{\partial p}\frac{\partial B}{\partial r}. \qquad (3.6.68c)$$

TABLE 3.7: THE NOTION OF LIE-ADMISSIBLE ALGEBRA AS ENVELOPING NONASSOCIATIVE ALGEBRA.

Table 3.6 essentially indicates the quite peculiar situation of Lie-admissible formulations according to which a Lie algebra structure in the neighborhood of the identity is lost by central requirement, but a Lie group structure persists! This situation clearly demands a more detailed inspection to see whether it is actually consistent.

We reach in this way the part of this analysis which I consider the true, ultimate characterization of the notion of Lie-admissibility, that via the enveloping algebra of a Lie algebra.

Permit be to recall that the notion of universal enveloping associative algebra A(G) of a Lie algebra G is truly crucial in Lie's theory on both physical and mathematical grounds. It is equally crucial for relativity considerations. For instance, if G is the Galilei Lie algebra, the computation of quantities, such as, the square of the angular momentum, necessarily demands the use of A(G) (trivially, because the square of all quantities are identically null within the context of a Lie algebra, the product being anticommutative). On the contrary, at the level of A(G) such quantities are fully definable because its product is associative. Similarly, if G is the SU (3) Lie algebra, the computation of the Gell-Mann-Okubo mass formula is often done with the use of the enveloping algebra A(G). It is therefore tempting to state that, without the universal enveloping algebras, Lie's theory would have little, if any, physical relevance. This aspect does not appear to be sufficiently emphasized in some of the existing physical literature.

On mathematical grounds, the algebras A(G) are equally crucial. First of all, they permit the construction of the exponential mapping, trivially, because from the second term on all elements of the expansion

$$e^{\theta X} = 1 + \frac{\theta X}{1!} + \frac{\theta^2 X^2}{2!} + \cdots \qquad (3.7.1)$$

are outside of the Lie algebra G and only definable in A(G). Also, the algebras A(G) play a fundamental role for the construction of the representation theory of G, and so on. For a rigorous account on this profile (which is also often neglected in physical literature) see N. JACOBSON[29e].

Let me therefore state in simplistic terms that the universal enveloping associative algebras of Lie algebras are the true representative of the dual algebraic-group theoretic aspects of Lie's theory. And indeed, first of all they contain Lie algebras via the isomorphism $[A(G)]^- \approx G$ and, besides, are constructed with the basis of G. Secondly, they express Lie groups via expansion (3.7.1). The net effect is that the algebras A(G) play a crucial methodological function for the characterization of the Galilei (as well as Einstein special) relativity.

The intended covering of the Galilei relativity of Section 4 now begins to take shape: it is conceived as a Lie-admissible covering of the universal enveloping associative algebra of the Galilei algebra. It is such an approach which allows the compliance with numerous requirements. For instance, it ensures the capability of recovering the conventional Galilei relativity identically at the limit of null relativity breaking forces. It ensures the representation of the nonconservative character of the systems via the Lie-admissible behaviour in the neighborhood of the identity, e.g., according to rule (3.6.5a). It allows the study of the preservation of a group for the replacement of the Galilei group as the invariance group of nonlinear, essentially nonselfadjoint, and explicitly time dependent equations of motion, etc.

Let me begin by recalling, for notational advantages, the notion of universal enveloping associative algebra.

DEFINITION 3.7.1:[29e] The universal enveloping associative algebra of a Lie algebra $\underline{G}$ is the set $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is an associative algebra and τ a homomorphism of $\underline{G}$ into the attached algebra $\mathcal{A}^-$ of $\mathcal{A}$ satisfying the following property. If $\mathcal{A}'$ is another associative algebra and τ' a homomorphism of $\underline{G}$ into $\mathcal{A}'^-$, there exists a unique homomorphism γ of $\mathcal{A}$ into $\mathcal{A}'$ such that $\tau' = \tau\gamma$; i.e., the following diagram is commutative.

$$\begin{tikzcd} {[\mathcal{A}]}^- \arrow[rr, "\gamma"] & & {[\mathcal{A}']}^- \\ & \underline{G} \arrow[ul, "\tau"] \arrow[ur, "\tau'"'] & \end{tikzcd} \qquad (3.7.2)$$

(it should be here recalled that all algebras and fields have characteristic zero and $\underline{G}$ is finite dimensional).

In essence, the definition stresses the uniqueness of the universal enveloping associative algebra of a Lie algebra, up to local isomorphisms.

The construction of $\mathcal{A}$ is usually conducted by first identifying the most general associative tensor algebra which can be constructed with $\underline{G}$ as vector space, i.e.,[29e]

$$\mathcal{T} = F \oplus \underline{G} \oplus \underline{G} \otimes \underline{G} \oplus \cdots, \qquad (3.7.3a)$$

$$\underline{G} \otimes \underline{G} \; : \; X_{i_1} \otimes X_{i_2} \qquad i_1, i_2 = 1, 2, \ldots, n, \qquad (3.7.3b)$$

where the product $\otimes$ is associative, the basis of $\underline{G}$ is ordered, i.e., $\{X_i \mid i \in I = \text{ordered set}\}$, and the right hand side of Eqs. (3.7.3b) is the product of $\underline{G} \otimes \underline{G}$.

Let $\mathcal{R}$ be the ideal of $\mathcal{T}$ generated by the elements

$$[X_i, X_j] - (X_i \otimes X_j - X_j \otimes X_i), \qquad (3.7.4)$$

where $[X_i, X_j]$ is the product in $\underline{G}$.

The universal enveloping associative algebra $\mathcal{A}(\underline{G})$ of $\underline{G}$ is then given by (or can be equivalently defined by) the quotient algebra[29e]

$$\mathcal{A}(\underline{G}) = \mathcal{T} / \mathcal{R}. \qquad (3.7.5)$$

The basis of $\mathcal{T}$ is given by the so-called standard monomials

$$M^s_m = X_{i_1} \otimes X_{i_2} \otimes \cdots \otimes X_{i_m}, \qquad (3.7.6)$$
$$i_1 \leq i_2 \leq \cdots \leq i_m.$$

A number of technical steps then yield a fundamental theorem of enveloping algebras, the Poincaré-Birkhoff-Witt Theorem, which can be formulated as follows.[29e]

THEOREM 3.7.1: The cosets of 1 and the standard monomials form a basis of the universal enveloping associative algebra $\mathcal{A}(\underline{G})$ of a Lie algebra $\underline{G}$.

This algebra $\mathcal{A}(G)$, being of tensorial type, is not used in practical applications (particularly in physics), where the ordinary associative algebra A(G) with product $X_i X_j$ is used instead. Neverthless, it is possible to prove that there exists a (linear) mapping ℓ of $\mathcal{A}(G)$ into A(G) such that $\mathcal{R}\ell = 0$, and $\{$cosets of 1 and standard monomials of $\mathcal{A}(G)\} \to \{1$ and elements of A(G)$\}$. The net effect is that a basis of A(G) is provided by

$$1, \; X_i, \; X_{i_1} X_{i_2}, \; X_{i_1} X_{i_2} X_{i_3}, \ldots, \qquad (3.7.7)$$

while an arbitrary element of A(G) can be written as a linear combination of

$$X_{i_1}^{k_1} X_{i_2}^{k_2} \cdots X_{i_m}^{k_m} \qquad k_1, k_2, \ldots k_m, = 0, 1, 2, \ldots. \qquad (3.7.8)$$

It is precisely this structure which renders Lie's theory useful for practical calculations (at both classical and quantum mechanical levels). And indeed, A(G) not only characterizes the basis X_i, but also the Casimir invariants and, more generally, any desired (associative) power of X_i. Also, any representation of A(G) yields a representation of G via Lie's rule (3.6.8c). Thus, a number of theorems on the representation theory of Lie algebras (e.g., the existence of faithful representations, Ado's theorem, etc.) demands their treatement at the level of A(G).

Finally, the associative nature of A(G) allows the existence of both linear as well as nonlinear representations.

Before passing to the Lie-admissible generalization of these ideas, it is recommendable to outline first the intermediate step of Lie isotopy. Specifically, if generalized exponential mappings of type (3.6.25c) exist with a Lie behaviour in the neighborhood of the origin, this can only be accounted for, to the best of my knowledge, via an isotope of A(G).

DEFINITION 3.7.2: [5b] The isotopically mapped universal enveloping associative algebra of a Lie algebra $\underline{G}$ is the set $[(\mathcal{A}, \tau)\mathcal{A}^*, i, \tau^*]$ where

- $(\mathcal{A}, \tau)$ is the universal enveloping associative algebra according to Definition 3.7.1;
- i is an isotopic mapping of $\underline{G}$, $i\underline{G} = \underline{G}^*$;
- $\mathcal{A}^*$ is an associative algebra generally nonisomorphic to $\mathcal{A}$; and
- τ^* is a homomorphism of $\underline{G}^*$ into $[\mathcal{A}^*]^-$,

satisfying the following properties. If $\mathcal{A}^{*\prime}$ is still another associative algebra and $\tau^{*\prime}$ a homomorphism of $\underline{G}^*$ into $[\mathcal{A}^{*\prime}]^-$, there exists a unique homomorphism γ^* of $\mathcal{A}^*$ into $\mathcal{A}^{*\prime}$, $\tau^* = \gamma^*\tau^{*\prime}$, and a unique isotopy of $\mathcal{A}$ into $\mathcal{A}^*$, $\hat{i}\mathcal{A} = \mathcal{A}^*$, such that the following diagram is commutative.

$$\begin{tikzcd}
{[\mathcal{A}^*]} \arrow[rr, "\gamma^*"] & & {[\mathcal{A}^{*\prime}]^-} \\
{[\mathcal{A}]^-} \arrow[rr, "\gamma"] & \underline{G}^* \arrow[ul, "\tau^*"] \arrow[ur, "\tau^{*\prime}"] \arrow[u, "\hat{i}"] & {[\mathcal{A}']^-} \arrow[u, "\hat{i}'"'] \\
& \underline{G} \arrow[ul, "\tau"] \arrow[ur, "\tau'"'] \arrow[u, "i"] &
\end{tikzcd} \qquad (3.7.9)$$

The practical realization of this notion is as follows. It is essentially induced by an isotopic mapping of the tensorial product $\otimes$, i.e., an invertible, associativity preserving mapping

$$\mathcal{A} : X_i \otimes X_j \longrightarrow \mathcal{A}^* : X_i * X_j , \qquad (3.7.10)$$

under which we have the isotopically mapped (associative) tensorial algebra $\mathcal{T}^*$ [5b]

$$\mathcal{T}^* = F \oplus \underline{G} \oplus \underline{G} * \underline{G} \oplus \cdots . \qquad (3.7.11)$$

The isotopically mapped ideal $\mathcal{R}^*$ of $\mathcal{T}^*$ is then generated by elements of the type (the product in $\underline{G}^*$ is now denoted with $[X_i, X_j]^*$)

$$[X_i, X_j]^* - (X_i * X_j - X_j * X_i). \qquad (3.7.12)$$

Then, the isotopically mapped universal enveloping associative algebra is given by (or, equivalently, can be defined by) [5b]

$$\mathcal{A}^*(\underline{G}) = \mathcal{T}^* / \mathcal{R}^* . \qquad (3.7.13)$$

The generic elements of $\mathcal{T}^*$ are now reducible to (linear) combinations of the isotopically mapped standard monomials

$$M^{*s}_m = X_{i_1} * X_{i_2} * \cdots * X_{i_m} , \quad i_1 \le i_2 \le \cdots \le i_m . \qquad (3.7.14)$$

A study of the problem reveals that the other pertinent aspects of the conventional case extends to the isotopically mapped case. [5b] We reach in this way the following Lie covering of the Poincaré-Birkhoff-Witt Theorem .

THEOREM 3.7.2: The cosets of 1 and the standard isotopically mapped monomials form a basis of the isotopically mapped universal enveloping associative algebra $\mathcal{A}^*(\underline{G})$ of a Lie algebra $\underline{G}$.

The nontriviality of this theorem is represented by the fact that, starting from the isotopy (3.7.10), we reach an envelop whose Lie content $[\mathcal{A}^*(\underline{G})]^-$ is not, in general, isomorphic to $\underline{G}$, even though the algebra $\mathcal{A}^*(\underline{G})$ has been constructed in terms of the basis of $\underline{G}$, i.e.,

$$\underline{G}^* \simeq [\mathcal{A}^*(\underline{G})]^- \not\simeq \underline{G} \simeq [\mathcal{A}(\underline{G})], \qquad (3.7.15)$$

Again, the basis can be written

$$1 , X_i , X_{i_1} * X_{i_2} , X_{i_1} * X_{i_2} * X_{i_3} , \ldots , \qquad (3.7.16)$$

and the general elements are of the type

$$X_{i_1}^{k_1} * X_{i_2}^{k_2} * \cdots * X_{i_m}^{k_m} , \quad k_1, k_2, \ldots, k_m = 0, 1, 2, \ldots , \qquad (3.7.17)$$

where now powers are in $\mathcal{A}^*$. The reduction to a nontensorial form is inessential for Theorem 3.7.2 because a case of associative isotopy is precisely the mapping $\mathcal{A}(G) \rightarrow A(G)$ which is needed for practical applications . In order words, in the product isotopy

one can already incorporate the provision for the product actually used in practical calculations.

Notice that the isotope $A^*(\underline{G})$ of $A(\underline{G})$ is not unique, in the sense that there can exist a family of nonisomorphic isotopes $A^*(\underline{G})$, $A'^*(\underline{G})$, $A''^*(\underline{G})$, ... all realized in terms of the basis of $\underline{G}$, but via different mappings (3.7.10), which are such that the attached algebras $[A^*(\underline{G})]^-$, $[A'^*(\underline{G})]^-$, $[A''^*(\underline{G})]^-$, ..., are nonisomorphic among themselves. This is not in contradiction with the uniqueness of the associative envelope (up to isomorphisms) because, say, for the case of $A^*(\underline{G})$ we can construct its Lie content $\underline{G}^*$ in the standard form with corresponding conventional envelop $A(\underline{G}^*)$, and, thus.

$$\underline{G}^* \approx [A^*(\underline{G})]^- \approx [A(\underline{G}^*)]^- \not\approx \underline{G}. \qquad (3.7.18)$$

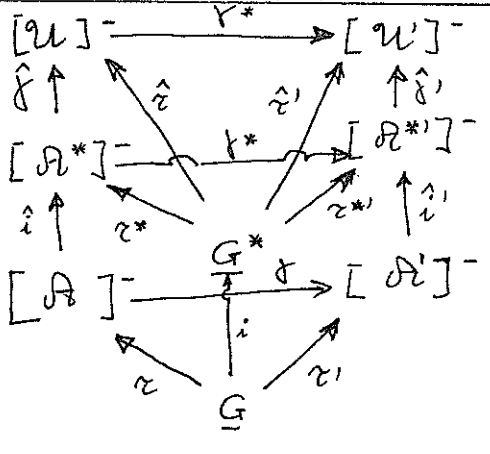
We are now equipped to introduced the intended notion of Lie-admissibility (see ref.[5b] for details).

DEFINITION 3.7.3: A Lie-admissible genotopically mapped universal enveloping associative algebra of a Lie algebra $\underline{G}$ is the set

$$\{[(\mathcal{A}, \tau), \mathcal{A}^*, i, \tau^*], \mathcal{U}, \hat{\tau}, \hat{\gamma}\}$$ where

- $(\mathcal{A}, \tau)$ is the universal enveloping associative algebra of Definition 3.7.1,
- $[(\mathcal{A}, \tau), \mathcal{A}^*, i, \tau^*]$ is the isotopically mapped associative algebra according to Definition 3.7.2,
- $\mathcal{U}$ is a Lie-admissible algebra,
- $\hat{\tau}$ is a homomorphism of $\underline{G}^*$ into $\mathcal{U}^-$,
- $\hat{\gamma}$ is an isomorphism of $[\mathcal{A}^*]^-$ into $\mathcal{U}^-$,

such that the following property holds. If $\mathcal{U}'$ is another Lie-admissible algebra and $\hat{\tau}'$ a homomorphism of $\underline{G}^*$ into $[\mathcal{U}']^-$, there exists a homomorphism γ^* of $\mathcal{U}^-$ into $\mathcal{U}'^-$, i.e., the following diagram is commutative.



(3.7.19)

The idea which is attempted with the above definition is that the envelop of a Lie algebra is not unique in the sense that algebras characterized by different laws can characterize the same Lie algebra, provided that they are Lie-admissible. And indeed, the classification of the different classes of Lie-admissible algebras of Table 3.3 allows the following possibilities.

(1) $\mathcal{U}$ is an associative algebra. In this case diagram (3.7.19) recovers (3.7.9) identically.

(2) $\mathcal{U}$ is a Lie algebra. Then diagram (3.7.19) reduces to (3.7.2) in the sense that no nonassociative envelop is characterized by an anticommutative algebra.

(3) $\mathcal{U}$ is a flexible Lie-admissible algebra, e.g., the mutation algebras (3.3.5). This yields a first possibility of costructing a genuine nonassociative envelop.

(4) $\mathcal{U}$ is a general Lie-admissible algebra. This is clearly a second possibility for a nontrivial nonassociative envelope.

In conclusion, there are three classes of Lie-admissible algebras which are significant for the envelope of a Lie algebra: the associative, the flexible Lie-admissible and the general Lie-admissible. In principle the same Lie algebra $\underline{G}$ can be homomorphic to the attached algebra of one algebra per each of these three classes and it is in this sense that the envelop of a Lie algebra is here intended to be nonunique. Of course, if one imposes that the envelop be associative, that the uniquess of Definition 3.7.1 is recovered. But the envelope $\mathcal{U}$, in general, is not "universal".

The first studies on the construction of a nonassociative, but flexible and Lie-admissible envelops of a Lie algebras have been conducted by C. N. KTORIDES,[24a] to the best of my knowledge. In the following we shall closely follow the analysis by this author, with only the necessary implementation into the case of the general Lie-admissible algebra, as requested by the fact that these algebras actually emerge in Newtonian Mechanics (Table 3.4).

The first step in the construction of a genotope of $\mathcal{A}$ is given by the Lie-admissible genotopic mapping of the tensorial algebra

$$\mathcal{A}: X_i \otimes X_j \longrightarrow \mathcal{U}: X_i \circ X_j, \qquad (3.7.20)$$

where $X_i \circ X_j$ can now be interpreted, say, as in Eqs. (3.6.35). This yields the Lie-admissible genotope of the associative tensorial algebra

$$\hat{\mathcal{L}} = F \oplus \underline{G} \oplus \underline{G} \circ \underline{G} \oplus \cdots \qquad (3.7.21)$$

under the proviso that now the symbol $\odot$ represents the enclosure of all possible different associations, e.g.,

$$X_i \odot X_j \odot X_k = \{ (X_i \circ X_j) \circ X_k \,,\; X_i \circ (X_j \circ X_k) \} \,, \qquad (3.7.22)$$

as requested by the nonassociative nature of the product.

For the ideal notice that one can select the isotope in such a way that

$$[X_i, X_j]_{\mathcal{U}} = X_i \circ X_j - X_j \circ X_i \equiv X_i * X_j - X_j * X_i \equiv [X_i, X_j]_{\mathcal{A}^*} \,. \qquad (3.7.23)$$

Thus, the genotope $\hat{\mathcal{R}}$ of the ideal $\mathcal{R}$ coincides with the isotope $\mathcal{R}^*$, i.e., it is generated by elements of the type

$$[X_i, X_j]^* - (X_i * X_j - X_j * X_i) \,. \qquad (3.7.24)$$

The Lie-admissible genotopic mapping of a universal enveloping associative algebra can then be written or(or, equivalently, be defined by)

$$\mathcal{U}(G) = \hat{\mathcal{T}} / \hat{\mathcal{R}} \,. \qquad (3.7.25)$$

The Lie algebra content of $\mathcal{U}$ is then given by

$$\mathcal{U}^- \approx \underline{G}^* \not\approx \underline{G} \,. \qquad (3.7.26)$$

The study of the basis of $\mathcal{U}$ turn out to be more involved than that of $\mathcal{A}^*$ because of the nonassociative nature of the product. Neverthless one can define the standard genotopically mapped monomials as the union of all independent standard monomials in $\hat{\mathcal{T}}$ with different associations, i.e.,

$$\hat{M}^{s}_{m} = X_{i_1} \odot X_{i_2} \odot \cdots \odot X_{i_m} \,, \qquad (3.7.27)$$
$$i_1 \leq i_2 \leq \cdots \leq i_m \,.$$

It is easy to see that this set is not necessarily a basis for $\mathcal{U}$ because an arbitrary monomial now cannot be necessarily reduced to an F-linear combination of monomials (3.7.27). The study of this problem indicates the emergence in this reduction of the standard isotopically mapped monomials. Thus, a basis of $\mathcal{U}$ is expected to be constituted by both, genotopically and isotopically mapped monomials. After all, this feature is not surprising. And indeed, the need of the isotopically mapped monomials can be seen already from the composition law of Lie-admissible groups, Eqs. (3.6.47) and (3.6.48). In turn, this is crucial for attempting the construction

of a nonuniversal, generally nonassociative covering of the associative envelope $\mathcal{A}(G)$ of $\underline{G}$.

Upon a number of technical steps, we have the following Lie-admissible covering of the Poincaré-Birkhoff-Witt theorem, which, owing to the contribution by C. N. KTORIDES, we shall call the Poincaré-Birkhoff-Witt-Ktorides theorem.[24a, 5b]

THEOREM 3.7.3: The cosets of 1 and the union of the F*-linearly independent standard genotopically mapped and standard isotopically mapped monomials form a basis of a Lie-admissible nonassociative genotope $\mathcal{U}$ of a universal enveloping associative algebra $\mathcal{A}$ of a Lie algebra $\underline{G}$.

The terms "F*-linearly independent" are referred to the fact that combinations of the basis of $\underline{G}$ generally occurs within $\mathcal{U}$ with functions of the base manifolds as coefficients. For details, see ref.[5b] The above theorem is in essence a simple generalization to general Lie-admissible algebras of Theorem 2.1 by C. N. KTORIDES on flexible Lie-admissible algebras.[24a] The interested reader is here urged to inspect the example by this latter author with $A(\lambda,\mu)$ mutation algebras and their application to the construction of the Gell-Mann-Okubo mass formula (see in this respect also Table 3.9).

Theorem 3.7.3 essentially identifies the basis as being of the type

$$\alpha 1 \,,\; \beta X_i \,,\; \gamma X_i \circ X_j \,,\; \delta X_i * X_j \,, \qquad (3.7.28)$$
$$\rho X_i \odot X_j \odot X_k \,,\; \sigma X_i * X_j * X_k \,, \ldots$$

where the coefficients $\alpha, \beta, \gamma, \ldots$ are, in general, functions of the variables of the base manifold (the b^μ variables of Tables 3.4, 3.5 and 3.6). The actual construction of the basis demands the explicit form of the Lie-admissible product which, as by now familiar, may vary from generator to generator. Neverthless, structure (3.7.28) is sufficient for the objectives of this paper. The studies of the general methods for the construction of the basis of $\mathcal{U}$ is here left to the interested reader.

One of the most intriguing properties of the Lie-admissible algebras $\mathcal{U}$ is that their only admissible representations are, in general, nonlinear,[24a] owing to the nonassociative nature of the product. As a result, recent studies on nonlinear representations of Lie's groups[30] might be significant, upon due technical implementations, for the study of the representations of Lie-admissible algebras and groups. On physical grounds this is perhaps one of the potentially most significant possibilities for a differentiation between the electromagnetic and the strong interactions, as we shall indicate in a subsequent paper.

An example may be here useful to illustrate the objectives of the analysis of this table. Consider a conservative Newtonian system in a 2-dimensional Euclidean space possessing the exact symmetry under the group of rotations SO(2) with canonical realization (3.6.62). The Lie algebra $\underline{SO}(2)$ is in this case one-dimensional with generator J. The basis of the universal enveloping associative algebra $\mathcal{A}$(SO(2)) is now given, from Eq. (3.7.7), by

$$\underline{SO}(2): \quad 1, J, J\otimes J, J\otimes J\otimes J, \ldots, \quad J\otimes J = \text{Ass.}, \quad J = r_x p_y - r_y p_x. \tag{3.7.29}$$

Let us recall that it is the existence of this basis which allows the exponentiation of the Lie algebra $\underline{SO}(2)$ into the Lie group (3.6.63).

Suppose now that this SO(2) symmetry is broken by nonconservative forces (and thus, J is nonconserved). Suppose also that the broken SO(2) context admits a Lie-admissible characterization in terms of the tensor (3.6.64) with corresponding Lie-admissible group (3.6.65). Our problem is that of identifying the algebraic envelope which necessarily underlays the transition from the Lie-admissible algebra in J and the Lie-admissible group $\hat{SO}(2)$, i.e., Eqs. (3.6.65), under the assumption of the preservation of the generator, parameter and base manifold of SO(2). The reader should be aware that this last assumption is simply uncompromisable for the objective of this paper, because its relaxation would render virtually impossible the attempt of identifying a generalization of the Galilei group capable of recovering this latter group identically at the limit of null symmetry breaking forces. The only possibility of achieving the objective considered under the assumption considered known to me is by performing the Lie-admissible genotopic mapping of the basis (3.7.29)

$$\hat{\underline{SO}}: \quad 1, J, J\circ J, J\circ J\circ J, \ldots, \quad J\circ J = \text{Nonass.}, \quad J = r_x p_y - r_y p_x, \tag{3.7.30}$$

where now the product $\circ$ is nonassociative by central requirement, but Lie-admissible, i.e., it constitutes the abstract characterization of the Lie-admissible product in expansion (3.6.65).

In turn, this necessarily implies, for the proper treatement, the study of the isotopically mapped basis

$$\underline{SO}^*: \quad 1, J, J*J, J*J*J, \ldots, \quad J*J = \text{Assoc.}, \quad J = r_x p_y - r_y p_x, \tag{3.7.31}$$

because the Lie algebra content of $\mathcal{U}(\underline{SO}(2))$ does not coincide with $\underline{SO}(2)$. Instead, it coincides with the isotope $\underline{SO}^*(2)$ induced by J, but now in terms of the generalized Poisson brackets with Lie tensor $\Omega^{\mu\nu} = S^{\mu\nu} - S^{\nu\mu}$ attached to the tensor (3.6.64). This is equivalent to assume (3.7.23).

In conclusion, the indicaled Lie-admissible approach necessarily implies three layers. (A) The conventional Lie approach which (according to our uncompromisable condition) is identically recovered at the limit of nonconservative forces. (B) The covering Lie-admissible approach of this section. And (C) the intermediate Lie covering of the conventional approach induced by the algebraic isotopy. The emerging notion of Lie-admissible envelope is then nonintrinsic by construction, although studies of a possible intrinsic approach (i.e., that without the notion of genotopic mapping) are strongly encouraged.

TABLE 3.8: SYMPLECTIC-ADMISSIBLE COVERING OF THE SYMPLECTIC GEOMETRY.

Without doubt, the symplectic geometry[6] is one of the most fascinating, mature and rigorous disciplines for the reduction of physical laws to primitive geometrical notions. It was, therefore, for me reason of considerable surprise the identification of a number of difficulties in the classical and quantum mechanical use of the symplectic geometry for the study of nonconservative systems. The doubt that this was only the result of my largely insufficient knowledge of differential geometry persisted for a considerable period of time (by delaying the presentation of my efforts) and still persists as of today. Neverthless, since I have been unable to resolve this doubt and, as a matter of fact, the difficulties indicated have increased in time, I think that an unpedagogical report of my studies of this profile might be of some value for the receptive and open minded expert in differential geometry, in the hope that they can be subjected to a scrutiny, assessement and technical finalization.

The difficulties which I have encountered in the use of the symplectic geometry for the study of essentially nonselfadjoint systems can be reduced to the following three aspects.

(A) Difficulties for relativity considerations. Apparently, one of the central problems for the relativity which is applicable to the systems considered is the identification of a nonmanifest, connected, Lie symmetry for the form-invariance of the equations of motion, capable of satisfying our, by now familiar, uncompromisable requirement, that is, the capability of recovering the Galilei group identically at the limit of null relativity breaking forces. By recalling that the equations considered are nonconservative, nonlinear and explicitly dependent on time, this is not an easy task. Despite my best efforts, I have been unable to even partially confront this problem by using the symplectic geometry for a number of reasons I shall outline below. The use instead, of the covering geometry which appears to be suggested by this line of study, here tentatively called symplectic-admissible geometry, seems to offer some genuine hope of attacking the problem and eventually solving it, as I shall indicate in details in Section 4. It should be stressed that the solution, to have any pragmatic value for physicists, must be able to produce rules for the explicit construction of the desired nonmanifest symmetry for given forces not derivable from a potential. It should also be stressed that by no means I intend to deny the possible existence of a geometrically equivalent solution within the context of the symplectic geometry. However, to have any pragmatic value for physical applications, that solution must hold for the coordinate systems actually used in experiments which, as we shall see in a moment, appears to be the source of the difficulties.

(B) Difficulties of quantum mechanical nature. As is known, the problem of quantization of forces not derivable from a potential is unsolved as of today, irrespective of whether conventional or geometrical methods[6c] are used. My difficulties in attempting the quantization

of nonconservative systems as globally Hamiltonian vector fields are not of formal mathematical treatment, but instead of consistency on physical grounds as far the physical interpretation of the algorithms at hand are concerned. For instance, the customary interpretation of the expectation values of the operator "p" under the indicated characterization (canonical momentum for a nonconservative system in globally Hamiltonian form) as those of the physical linear momentum have met with virtually unsurmontable inconsistency problems. This aspect will be treated in a subsequent paper. Again, it should be stressed that, by no means, I here exclude a "symplectic quantization", because the problem, as indicated earlier, is mainly of physical, rather than mathematical nature.

(C) Difficulties of algebraic origin. These difficulties are independent of the preceding ones (at least at a first inspection) and more closely related to the content of the preceding parts of this paper. In few nontechnical terms, the symplectic geometry is known to be fully compatible with the Lie algebras, to the point of achieving a symbiotic geometrical-algebraic duality. In the transition to the covering Lie-admissible algebras I have encountered severe problems of geometrical consistency if I insisted in the preservation of the symplectic geometry as currently known. The reason is essentially due to the nature of the Lie-admissible product which is neither symmetric nor antisymmetric and the inability of the symplectic geometry of producing a technical characterization of the symmetric part. Perhaps greater problems of geometrical consistency I have found in the attempt of using the Riemannian geometry as currently known, this time, because of the essential antisymmetric part of the Lie-admissible product. As a matter of fact, these difficulties have been so great so force me into my rudimentary attempts at constructing a covering geometry. But, again, it should be stressed that perhaps these difficulties are due to my insufficient knowledge of these established geometries, rather than the geometries themselves.

With an open mind on these issues, permit me to summarize my argument. For all necessary details the interested reader is suggested to consult ref.[5b]

First, to avoid misrepresentations of the speculative spirit of this table, I would like to stress the conceptual, physical and geometrical consistency of the symplectic geometry for conservative systems. In my unpedagogical terms, conservative systems can be trivially represented with Hamilton's equations in the variables r^{ka} and p_{ka} where r^{ka} represents the Cartesian coordinates actually used in the experimental set up and p_{ka} represents the physical linear momentum, that is, $m_k \dot{r}_{ka}$. This trivially provides a symplectic characterization of the systems (as recalled in Table 2.8) in the assumed coordinates. This restrictive character of the local coordinates is then entirely removed by the proper, geometrical, coordinate-free treatment. The emerging context is not only mathematically

and physically consistent, but constitutes one of the most effective ways of characterizing the Galilei relativity in its own arena of "unequivocal applicability", as we shall indicate in Sec.4. According to our findings, the reason for this consistency apparently relies on the fact that the family of all admissible local coordinates admits the coordinates r^{ka} and p_{ka} of direct physical significance. For practical computations of physical significance, however, care must be used at both the classical and quantum mechanical levels in relation to these degrees of freedom of the local coordinates to avoid inconsistency. For instance, new admissible space coordinates $r'^{ka}(r,p)$, even though mathematically consistent, can be noninertial and non-realizable in actual experiments (because of the nonlinear dependence on the p's). Similarly, new admissible conjugate momenta p'^{ka} may produce substantial consistency problems on physical grounds.

It this in this latter respect that the reader is urged to work out specific examples. For instance, the conventional harmonic oscillator

$$\left(m\ddot{r} + kr\right)^{\substack{C\infty, R \\ SA}} = 0, \quad m = 1, k = 1, \tag{3.8.1}$$

can be lifted to T'*M (rather than to T*M with local coordinates r and $p = m\dot{r}$) as a global Hamiltonian vector field in the Hamiltonian (see Section 2.10)

$$\dot{a}^\mu - \Xi^\mu(a) = 0, \mu = 1, 2, \quad \{a^\mu\} = \{t, p'\}, \tag{3.8.2a}$$

$$\Xi^\mu = \omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}, \quad H = 2 \ln \left|2 \sec \frac{1}{2} q p'\right| \tag{3.8.2b}$$

This is, geometrically, a fully admissible characterization of the harmonic oscillator. However, the reader is urged to quantize system (3.8.2b) and "touch with hand", so to say, the consistency problems of such a quantum mechanical system with respect to the established quantum mechanical oscillator. According to our findings, to be indicated in details in a subsequent paper, the reason of the difficulties lies on the crucial property that the algorithm "p' " for system (3.8.2b) by no means represents the physical linear momentum. However, for conservative system this is not a deficiency of the symplectic geometry, because the representation of oscillator (3.8.1) as a globally Hamiltonian vector field in the variables r and $p = m\dot{r}$ is fully admissible. The quantization in this system of coordinates of direct physical meaning is then mathematically and physically consistent.

In the transition to nonconservative systems the situation appears to be fundamentally different. First of all the reader should be aware that the Inverse Problem provides a distinction of nonconservative systems versus their primitive association with nonautonomous systems in the sense that Hamiltonians without an explicit dependence on time can be representative of genuine nonconservative systems (to stress the point, we shall use below only this type of

Hamiltonians). Secondly, one of the reasons which suggested my laborious involvement with the Inverse Problem was to provide a proof understandable by a broader segment of our community that nonconservative systems are indeed treatable with the symplectic geometry. This is, in essence, the spirit of the Theorem of Indirect Universality of the Inverse Problem (Section 2.9). However, the methodology of the Inverse Problem also provides a specific basis for the study of possible physical limitations of such geometrical setting. And indeed, it emerges that a necessary condition for the characterization of nonconservative (nonessentially or essentially nonselfadjoint) Newtonian systems as globally Hamiltonian vector fields is that the family of all admissible local coordinates does not admit the coordinates r^{ka} and $p_{ka}=m_k\dot{r}_{ka}$ of direct physical significance.

Again, the reader is here urged to work out explicit examples and "touch with hand" the underlying difficulties of physical consistency. There is no need of working out complicated systems. Instead, the simplest possible nonconservative extension of the harmonic oscillator (3.8.1), the damped oscillator

$$\left[(\ddot{r}+r)^{C^\infty,R}_{SA}+\gamma\dot{r}\right]^{C^\infty,R}_{NSA}=0, \qquad (3.8.3)$$

is fully sufficient for the purpose. The methods of the Inverse Problem yield the autonomous solution

$$L=\frac{2\dot{r}+\gamma r}{2r\omega}\,tg^{-1}\left(\frac{2\dot{r}+\gamma r}{2r\omega}\right)-\frac{1}{2}\ln\left(\dot{r}^2+\gamma r\dot{r}+r^2\right), \quad \omega^2=1-\gamma^2/4, \qquad (3.8.4)$$

which, in turn, provide the symplectic characterization of the damped oscillator as the globally Hamiltonian vector field

$$\dot{a}^\mu=\hat{\dot{=}}^\mu(a)=0, \quad \{a^\mu\}=\{r,p'\}, \quad p'=\partial L/\partial\dot{r}, \qquad (3.8.5a)$$

$$\hat{\dot{=}}^\mu(a)=\omega^{\mu\nu}\,\partial H/\partial a^\nu, \quad H=\ln r-\ln\left[\cos(\omega r p')\right]-\frac{1}{2}\gamma r p', \qquad (3.8.5b)$$

This geometrical characterization of system (3.8.3) is, of course, fully consistent on mathematical grounds. However, the reader is urged to work out, for instance, the quantization of vector field (3.8.5) and compare the results with those of the quantization of vector field (3.8.2). He will then discover a number of problems of consistencies of physical nature, such as the computation and meaning of the expectation values of the quantum mechanical algorithm "p' ", the inability to recover the conventional quantum mechanical oscillator at the limit $\gamma\to 0$, etc. In full analogy with the case of the harmonic oscillator, the difficulties appear to be linked to the fact that the (classical) symbol "p' ", by no means, is directly representative of a physical quantity. But, unlike the conservative case, a globally Hamiltonian characterization of the linearly damped oscillator in the variables r and $p = m\dot{r}$ does not exists, as it can be proved via the methods of the Inverse Problem. This is, in

essence, the difference between the conservative and nonconservative systems referred earlier.

When I became aware of this point I then entered into the study of Birkhoff's equations. In essence, the above restriction occurs only for globally Hamiltonian vector field characterization of nonconservative systems. If locally Hamiltonian vector fields are instead admitted, the local variables can indeed coincide with what I refer to as the variables r^{ka} and p_{ka} of direct physical significance (as it is possible to prove by using theorems of symplectic geometry). However, the pragmatic need of physicists is to compute time evolution laws, etc. Birkhoff's equations then emerged as potentially crucial, because capable of preserving the methodological significance of Hamilton's equations in full, while lifting the indicated restriction on the local variables, and while providing a genuine characterization of locally Hamiltonian vector fields (Section 2.8). However, my initial enthusiasm and hopes of preserving the symplectic geometry for nonconservative systems soon met with severe technical difficulties. The representation of the damped oscillator (3.8.3) via Birkhoff's equations (2.8.4) demands the solution of the equations

$$\mathcal{R}_{\mu\nu}\,\hat{\dot{=}}^\nu(b)=\left(\frac{\partial R_\mu}{\partial b^\nu}-\frac{\partial R_\nu}{\partial b^\mu}\right)\hat{\dot{=}}^\nu=\hat{\dot{=}}_\mu=\frac{\partial H^B}{\partial b^\mu}, \qquad (3.8.6a)$$

$$\{b^\mu\}=\{r,p\}, \quad p=m\dot{r}=\dot{r}, \quad (m=1). \qquad (3.8.6b)$$

Unfortunately, these equations, even though consistent (as guaranteed by the existence theorems), admit solutions of quite difficult computation in a closed form (admitting that such a form exists). In conclusion, the explicit computation of locally Hamiltonian characterization of nonconservative systems turned out to be extremely difficult, even for one-dimensional systems such as (3.8.3).

The reader is urged to verify that these difficulties are magnified when considering the class of nonconservative systems of true interest for this paper, that of essentially nonselfadjoint systems in arbitrary (finite) dimensions, under the for us uncompromisable condition that any admissible geometrical treatment allows the use of the variables r^{ka} and p_{ka} identified earlier.

Owing to these difficulties, we put the Lie-admissible formulations at work. The first objective, that of achieving a simple, direct, and immediate analytic representation of the vector fields for the damped oscillator

$$\left(\hat{\dot{=}}^\mu\right)=\begin{pmatrix}p\\-\gamma p-r\end{pmatrix}, \quad p=\dot{r}, \quad (m=1), \qquad (3.8.7)$$

without redefinition of the variables, is trivially made possible by our Hamilton-admissible equations for which

$$\hat{\dot{=}}^\mu=S^{\mu\nu}\frac{\partial H}{\partial b^\nu}, \quad (S^{\mu\nu})=\left(\frac{\partial b^\mu}{\partial R_\nu}\right)=\begin{pmatrix}0&1\\-1&-\gamma\end{pmatrix}, \qquad (3.8.8a)$$

$$H=\frac{1}{2}\left(p^2+r^2\right), \quad \{R_\mu\}=\{(-p-\gamma r),\,r\}, \qquad (3.8.8b)$$

with corresponding covariant form

$$\left(S_{\mu\nu}\,\hat{\dot{-}}^{\nu}\right) = \left(\frac{\partial R_\mu}{\partial b^\nu}\,\hat{\dot{-}}^{\nu}\right) = \begin{pmatrix} -\gamma & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ -\gamma p - r \end{pmatrix} = \begin{pmatrix} r \\ p \end{pmatrix} = \left(\dot{-}_\mu\right) = \frac{\partial H}{\partial b^\mu}, \qquad (3.8.9)$$

in which, according to the fundamental equations (3.4.4), the contraction of the tensor $S_{\mu\nu}$ with the nonconservative vector field $\hat{\dot{-}}^{\nu}$ produces the covariant conservative form $\dot{-}_\mu$.

As by now familiar, equations (3.8.8) and (3.8.9) imply the abandonment of Lie algebras as the underlying algebraic structure in favor of the Lie-admissible algebra. This creates the problem of serching for the possible existence of a covering of the symplectic geometry specifically conceived for Lie-admissible algebras, that is, for nonconservative systems. I report below the central aspects of my rudimentary studies in this truly intriguing problem with an understanding they they are the efforts by a physicist with a grossly insufficient knowledge of differential geometry and, thus, they are in need of a severe inspection by experts. Permit me to also indicate that I perform this disclosure simply because forced into it: to the best of my understanding it is extremely difficult, if not impossible, to arrive at a first solution of the relativity laws of nonconservative systems by using the conventional symplectic (or Riemannian) geometry. Since I have been unable to identify any treatment of the geometry which is applicable to the Lie-admissible algebra, I have been simply forced into a study which, on strict grounds, is a job for pure mathematicians. I therefore hope that the receptive and understanding reader takes into consideration the main ideas, rather than technical details of pure mathematical nature which do not effect the study of Section 4.

A tentative statement of the problem can therefore be formulated as follows: it consists of the identification of a geometry which is capable of characterizing the Lie-admissible algebras and the Hamilton-admissible equations in exactly the same way as symplectic geometry characterizes Lie algebras and Hamilton's equations. In particular, as it will be selfevident in Section 4, a geometrical interpretation of Eqs. (3.8.9) appears to be crucial for relativity considerations.

One of the basic difficulties in the problem under consideration rests with one of the basic methodological tools of symplectic geometry, the calculus of exterior forms. It appears to be simply incompatible with Lie-admissible formulations on a number of counts. First of all, the calculus considered is based on the antisymmetry property $db^\mu \wedge db^\nu = -db^\nu \wedge db^\mu$ which, while crucial for Lie algebras (owing to the antisymmetric nature of their product), is inconsistent with Lie-admissible algebras (because their product is neither totally symmetric not totally antisymmetric). In turn, this has a number of technical difficulties. But perhaps the most direct way to indicate the incompatibility of the calculus considered with Lie-admissible formulations is by noting that the computation of an exterior two-form with the Lie-admissible tensor $S_{\mu\nu}$

produces the attached Lie tensor $\mathcal{L}_{\mu\nu} = \frac{1}{2}(S_{\mu\nu} - S_{\nu\mu})$ because of the antisymmetric nature of the exterior product, i. e.

$$S_2 = S_{\mu\nu}\, db^\mu \wedge db^\nu = \frac{1}{2}(S_{\mu\nu} - S_{\nu\mu})\, db^\mu \wedge db^\nu. \qquad (3.8.10)$$

The net effect is that the calculus considered is capable of characterizing, via an exterior two-form, only the antisymmetric part of the Lie-admissible tensor $S_{\mu\nu}$. This implies the loss of the Lie-admissible formulations and their reduction to the Lie content only.

It therefore appears that an implementation of the calculus of exterior forms is needed for the proper characterization of Lie-admissible formulations.

The calculus in which I have conducted my rudimentary studies is based on the product [5b]

$$db^\mu \circ db^\nu \Rightarrow db^\mu \times db^\nu + db^\mu \wedge db^\nu, \qquad (3.8.11a)$$

$$db^\mu \times db^\nu = db^\nu \times db^\mu, \quad db^\mu \wedge db^\nu = -db^\nu \wedge db^\mu, \qquad (3.8.11a)$$

which I have called exterior-admissible product in the sense that its attached (that is, antisymmetric) part is exterior. Next, I have considered the exterior-admissible p-forms [5b]

$$\hat{A}_0 = p(b), \qquad (3.8.12a)$$

$$\hat{A}_1 = A_\mu\, db^\mu, \qquad (3.8.12b)$$

$$\hat{A}_2 = A_{\mu\nu}\, db^\mu \circ db^\nu = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})\, db^\mu \times db^\nu + \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})\, db^\mu \wedge db^\nu, \text{ etc.} \qquad (3.8.12c)$$

(i.e., the product $\circ$ can also be interpreted as the ordinary tensor product).

These forms are exterior-admissible in a double meaning fully parallel to that of Lie-admissible algebras. First of all they admit the conventional exterior forms at the limit when the A-tensors become totally antisymmetric in their indeces, and, secondly, they admit the conventional exterior forms in the attached form, i. e.,

$$\hat{A}_2 - \hat{A}_2^T = A_2 = A_{\mu\nu}\, db^\mu \wedge db^\nu. \qquad (3.8.13)$$

The exterior-admissible sum of forms (3.8.12) is the conventional sum, while the exterior-admissible product of forms (3.8.12) is done via rules of type (3.8.11). Next, I introduce the (left) exterior admissible derivative [5b]

$$\hat{d}\,\hat{A}_0 = (\partial A_0 / \partial b^\mu)\, db^\mu, \qquad (3.8.14a)$$

$$\hat{d}\,\hat{A}_1 = \frac{\partial A_\mu}{\partial b^\nu}\, db^\nu \circ db^\mu, \qquad (3.8.14b)$$

$$\hat{d}\,\hat{A}_2 = \frac{\partial A_{\mu\nu}}{\partial b^\tau}\, db^\tau \circ db^\mu \circ db^\nu, \qquad (3.8.14c)$$

etc.

with an understanding that, in a way fully parallel to the corresponding occurrence for the

for the conventional exterior derivative, we have

$$\hat{d} A_1 = \frac{\partial A}{\partial b^\nu} d b^\nu \circ d b^\mu \neq \hat{d}' A_1 = \frac{\partial A_\mu}{\partial b^\nu} d b^\mu \circ d b^\nu . \qquad (3.8.15)$$

The reader should be aware that the operation " $\hat{d}$ " is not a derivative as commonly understood because, in general,

$$\hat{d} \ (A_p B_p) \neq (\hat{d} A_p) B_p + A_p (\hat{d} B_p) . \qquad (3.8.16)$$

But then, this is precisely the reason for its possible relevance for a direct geometrical characterization of nonconservation laws (or nongeodesic trajectories) in a way admitting the conventional geometrical characterization of conservation laws as the limit for null values of nonconservative forces. And indeed, as we shall see below, at this limit the tensors characterizing the differential structure become totally antisymmetric, their experior-admissible forms become ordinary exterior forms and, thus, the "derivative" $\hat{d}$ becomes the ordinary derivative of exterior forms.

The significance of the exterior-admissible derivative is that it allows the formulation of the notion of the (left) exact exterior-admissible forms,[5b] i.e.,

$$\hat{A}_1 = \hat{d} \hat{A}_0 , \qquad A_\mu = \frac{\partial f}{\partial b^\mu} , \qquad (3.8.17a)$$

$$\hat{A}_2 = \hat{d} \hat{A}_1 , \quad A_{\mu\nu} = \frac{\partial A_\mu}{\partial b^\nu} . \qquad (3.8.17b)$$

In turn, this will be crucial for the covering of the notion of globally Hamiltonian vector field.

We call exterior-admissible calculus[5b] that of forms (3.8.12) with the outlined operations. One of its central features is the lack of the concept of closure. This can be seen from the fact that, since $\hat{d}$ is not an ordinary (exterior) derivative, $\hat{d}(\hat{d} A_n) \neq 0$. Thus, the calculus considered is such that it does not admit a direct, consistent generalization of both the Poincaré Lemma (of the calculus of exterior forms) and its converse. The reader should be aware at this point that the notion of closure is at the basis of the symplectic geometry, as recalled in Table 2.8.

Our next step is the representation of essentially nonselfadjoint Newtonian systems as vector fields on manifolds. The idea (see ref.[5b] for details) is to use the noncanonical method of Table 2.7 for the construction of an equivalent system of 6N first-order ordinary differential equations in the covariant general form

$$\left[S_{\mu\nu}(b) \dot{b}^\nu - \Gamma_\mu(b) \right]_{NSA} \overset{C^\infty, R}{=} 0 , \qquad (3.8.18)$$

here assumed to be of autonomous type.

We then characterize the exterior-admissible two-form

$$\hat{S}_2 = S_{\mu\nu} d b^\mu \circ d b^\nu \qquad (3.8.19)$$
$$= \frac{1}{2} (S_{\mu\nu} + S_{\nu\mu}) d b^\mu \times d b^\nu + \frac{1}{2} (S_{\mu\nu} - S_{\nu\mu}) d b^\mu \wedge d b^\nu ,$$

via the covariant tensor $S_{\mu\nu}$ of Eqs. (3.8.18). This yields a (Hausdorff, second countable, ∞-differentiable, 6N-dimensional) manifold $M(b, \hat{S}_2)$ in local coordinate b^μ equipped with the differentiable structure $\hat{S}_2$, where "differentiability" can be interpreted in both, ordinary meaning for manifolds and exterior-admissible meaning. The reader should be aware that this implies a number of consequences, such as the crucial property that the tensor $S_{\mu\nu}$ is the tensor for the lowering of the indeces in $M(b, \hat{S}_2)$, or for the mapping from TM to T*M, etc. In general, all notions of the theory of manifolds which hold for arbitrary (that is, generally nonsymplectic) structures extend to manifolds whose structure is of exterior-admissible type. Notice that, as by now familiar, the symbol " R " in Eqs. (3.8.18) stands to represent the nondegeneracy of the matrix $(S_{\mu\nu})$. In turn, this can be technically implemented into to nondegeneracy of the exterior-admissible two-form (3.8.19) (see ref.[5b] for details). This implies the characterization of the (unique) contravariant tensor $S^{\mu\nu}$, $(S^{\mu\nu}) = (S_{\mu\nu})^{-1}$, and of the co-exterior-admissible two-form

$$\hat{S}^2 = S^{\mu\nu} \frac{\partial}{\partial b^\mu} \circ \frac{\partial}{\partial b^\nu} \qquad (3.8.20)$$
$$= \frac{1}{2} (S^{\mu\nu} + S^{\nu\mu}) \frac{\partial}{\partial b^\mu} \times \frac{\partial}{\partial b^\nu} + \frac{1}{2} (S^{\mu\nu} - S^{\nu\mu}) \frac{\partial}{\partial b^\mu} \wedge \frac{\partial}{\partial b^\nu} .$$

To summarize, our starting point is the most general possible form of a class C^∞, regular, unconstrained Newtonian system, the essentially nonselfadjoint form. The lifting to T*M is done under the condition that the variables be the coordinates of the frame used for the detection of the system and the physical (rather than canonical) linear momentum. Thus, out starting point is not only local, but actually unique as far the coordinates are concerned. This attitude is motivated by relativity considerations to be indicated in Section 4 (as well as quantum mechanical consideration to be treated in a subsequent paper). Of course, the uniqueness of the coordinate system will be removed, but after the geometry for the characterization of the assumed systems in the assumed coordinates has been identified.

These two central features (essential nonselfadjointness of Newton's equations and physical nature of the b^μ variables) implies that the covariant general forms of the systems are nonselfadjoint . In turn, this implies that the tensor $S_{\mu\nu}$ of forms (3.8.18) is neither totally symmetric not totally antisymmetric in its indeces. In turn, this implies the insufficiency of the conventional calculus of exterior forms to characterize such tensor in its entirety.

These occurrences lead in a rather natural way to what we have called the calculus of exterior-admissible forms. in which a number of notions of conventional canonical formulations can be formulated, although in a generalized form (e.g., the notion of exactness) but, most importantly, the notion of closure is not definable.

The geometrical problem under consideration can now be be better identified if one realizes that the indicated representation of vector fields on manifolds is the most direct possible representation of nonconservative systems and that, from Universality Theorem 3.4.1, systems (3.8.18) coincide with our canonical-admissible equations. Thus, the tensor $S^{\mu\nu}$ characterizes a Lie-admissible algebra. The problem then consists in the identification of the geometry capable of characterizing such framework. This geometry is expected to be a covering of the symplectic geometry because, by construction, the exterior-admissible form (3.8.18) recovers a symplectic form identically at the limit of null forces not derivable from a potential, while the crucial notion of closure is not even definable at the full Lie-admissible level.

I have tentatively called this covering geometry the symplectic-admissible geometry because as by now also familiar, the term "admissible" stands to indicate that the conventional symplectic geometry can be recovered from its symplectic-admissible covering in a dual way (typical of all Lie-admissible formulations): via a limit of precise physical meaning (null forces not derivable from a potential)

$$\lim_{F_{k\alpha}\to 0} \hat{S}_2 \equiv \omega_2 = \omega_{\mu\nu}\, db^\nu db^\mu, \qquad (3.8.21)$$

and via the attached rule

$$\hat{S}_2 - \hat{S}_2^T = \Omega_2 = \frac{1}{2}\left(\frac{\partial R_\mu}{\partial b^\nu} - \frac{\partial R_\nu}{\partial b^\mu}\right) db^\mu \wedge db^\nu, \qquad (3.8.22)$$

which emerges in this way as the geometrical counterpart of the algebraic rule of Lie-admissibility, Eq. (3.3.4). This yields a rather crucial result for our analysis, namely, that the notion of Lie-admissibility admits consistent realizations at all the three levels which are essential for relativity considerations, the analytic, algebraic and geometrical levels.

The identification of a symplectic-admissible manifold can be done in several ways. Here, let me indicate two approaches, the first which is more algebraic in inspiration, and the second which is more geometrical in contemporary standards.

Approach I. By generalizing the treatement of symplectic manifolds by R. JOST,[6h] we see that the co-exterior-admissible two-form induces a bilinear composition law in a way quite symilar to the conventional case

$$\hat{S}^{(2)}(df, dg) = f \circ g = \frac{\partial f}{\partial b^\mu} S^{\mu\nu} \frac{\partial g}{\partial b^\nu}. \qquad (3.8.23)$$

The manifold $M(b, \hat{S}^2)$ in the local coordinates $b^\mu, \mu = 1, 2, \ldots, 6N$, equipped with the nowhere degenerate two-form (3.8.20) is called a co-symplectic-admissible manifold[5b] when brackets (3.8.23) satisfy the conditions of Lie-admissibility, i.e.,

$$J(f, g) = [f, g]_u - [g, f]_u = 0, \qquad (3.8.24a)$$

$$J(f, g, b) = [[f, g]_u, h]_u + [[g, h]_u, f]_u + [[h, f]_u, g]_u = 0, \qquad (3.8.24b)$$

$$[f, g]_u = f \circ g - g \circ f, \qquad (3.8.24c)$$

or, equivalently, when the attached brackets

$$[f, g]_u = \frac{\partial f}{\partial b^\mu} \Omega^{\mu\nu} \frac{\partial g}{\partial b^\nu}, \qquad (3.8.25)$$

are (nondegenerate) generalized Poisson brackets.

The nowhere degeneracy of $\hat{S}^2$ allows the construction of a (unique) form $\hat{S}_2$, Eq. (3.8.19) which, in turn, can characterize the brackets

$$\hat{S}_2(df, dg) = \frac{\partial b^\mu}{\partial f} S_{\mu\nu} \frac{\partial b^\nu}{\partial g} = f \square g, \qquad (3.8.26)$$

again, in full similarity with the symplectic treatment by R. JOST. The manifold $M(b, \hat{S}_2)$ in the local coordinates b^μ and now equipped with the nowhere degenerate two-form (3.8.19') is called a symplectic-admissible manifold when the brackets attached to (3.8.26)

$$f \square g - g \square f = \frac{\partial b^\mu}{\partial f} \Omega_{\mu\nu} \frac{\partial b^\nu}{\partial g}, \qquad (3.8.27)$$

are (nondegenerate) generalized Lagrange brackets.

This approach is clearly algebraic in inspiration because it makes direct use of the conditions of Lie-admissibility for the characterization of the (co) symplectic-admissible manifolds.

Approach II. Let $M(b, \hat{S}_2)$ be a (Hausdorff, second countable, ∞ - differentiable, 6N-dimensional) manifold in the local coordinates b^μ equipped with a nowhere degenerate exterior-admissible two-form (3.8.19). $M(b, \hat{S}_2)$ is called a symplectic-admissible manifold[5b] when the attached two-form

$$\Omega_2 = \hat{S}_2 - \hat{S}_2^T, \qquad (3.8.28)$$

is symplectic (that is, nowhere degenerate and closed). The form $\hat{S}_2$ ($\hat{S}^2$) will then be called symplectic-admissible form (co-symplectic-admissible form). Its central properties can be written

$$\hat{d}\,\hat{S}_2 \neq 0, \qquad (3.8.29\,a)$$

$$d\,(\hat{S}_2 - \hat{S}_2^T) = 0. \qquad (3.8.29\,b)$$

Clearly, any symplectic manifold is simplectic-admissible. This is the geometrical characterization of the property that any Lie algebra is Lie-admissible. On similar grounds, we can say that any closed symplectic-admissible form is symplectic. This is the geometrical characterization of the property that any anticommutative Lie-admissible product is Lie. However, a symplectic-admissible manifold is not necessarily symplectic. In essence, when the tensor $S_{\mu\nu}$ is totally antisymmetric, the symmetric part of the structure (3.8.19) is automatically eliminated and one recovers the conventional symplectic setting.. But, when the tensor $S_{\mu\nu}$ is neither totally antisymmetric not totally symmetric the full exterior-admissible structure applies, and a nontrivial generalization of the symplectic geometry emerges.

Next, by using the classification of Lie-admissible algebras (ref.[4d]) we can classify the symplectic-admissible manifolds as follows.

- GENERAL SYMPLECTIC-ADMISSIBLE MANIFOLDS. They occur when the tensor $S^{\mu\nu}$ satisfies the conditions of Theorem 3.3.1.
- FLEXIBLE SYMPLECTIC-ADMISSIBLE MANIFOLDS. They occur when the tensor $S^{\mu\nu}$ satisfies Theorem 3.3.2.
- SYMPLECTIC- MANIFOLDS. They occurs when the tensor $S^{\mu\nu}$ satisfies Theorem 3.3.3.

A nontrivial symplectic-admissible manifold is a symplectic-admissible manifold of either general or of flexible type.

The reader should be again reminded that the symplectic-admissible manifolds have been here identified not only in local coordinates, but actually in terms of a unique system of coordinates. This means that the tensor $S^{\mu\nu}$ has the specific structure of Eqs. (3.4.2). This is contrary to the conventional coordinate-free treatement of geometry. The point is that the geometrical, coordinate-free treatement of relativity problems appears to be physically consistent provided that the the family of all admissible local coordinates admit the coordinates actually used in experiments..

Now that the notion of symplectic-admissible manifold has been identified within the coordinate

system of our primary interest we can pass to the study of the coordinate-free generalization. The point is that in this way we are sure that the family of admissible coordinates does indeed contain that of direct physical significance, by restoring in this way a rather subtle property of the conventional symplectic geometry for the characterization of conservative systems.

The coordinate-free formulation of the symplectic-admissible manifolds can be studied in sequential steps and with different approaches (for details see ref.[5b]). The idea is that of generalizing the existing theorems of symplectic geometry, that is, Pauli's theorem[6h] (where the emphasis is more in the transformation theory) and Darboux's theorem[6a] (where the emphasis is more on the geometrical treatement) to the symplectic-admissible context. The conceptual attitude in the use of these generalization is however the opposite of the conventional one. Typically in symplectic geometry one starts from an arbitrary symplectic form and then uses these theorems for its reduction to the fundamental symplectic form (2.8.24). In our case the situation is the opposite. Our uncompromisable point has been the identification of the form (3.8.19) which is related to actual experiments, i.e.,

$$\hat{S}_2 = S_{\mu\nu}\, d b^\mu \circ d b^\nu, \qquad (3.8.30\,a)$$

$$\{b^\mu\} = \{r^{ka}, p_{ka}\},\ p_{ka} = m_k \dot{r}_{ka},\ (S_{\mu\nu}) = \left(\frac{\partial R_\mu}{\partial b^\nu}\right),\ \{R_\mu\} = \{R_{ka}, r^{ka}\}, \qquad (3.8.30\,b)$$

which we shall call fundamental symplectic-admissible form in the physical coordinates b^μ. Then we use form (3.8.30) as a "germ" to construct the family of all admissible coordinates and then, as a ultimate geometrical treatement, its coordinate free formulation.

The first step is provided by the following symplectic-admissible covering of Pauli's theorem[5b]

THEOREM 3.8.1: Given a fundamental symplectic-admissible form $\hat{S}_2$ on a manifold $M(b,\hat{S}_2)$ with local coordinates b^μ, $\mu = 1,2,\ldots,6N$, then there exist an infinite number of diffeomorphisms $\ell : M(b,\hat{S}_2) \longrightarrow M(b',\hat{S}'_2)$ realizable through class C^∞, everywhere invertible transformations $b \rightarrow b'(b)$ under which the fundamental form $\hat{S}_2$ transforms into an arbitrary symplectic-admissible form $\hat{S}'_2$. Viceversa, given an arbitrary symplectic-admissible form $\hat{S}'_2$ in the local coordinates b', there always exist a (class C^∞, everywhere invertible) transformation $b' \rightarrow b(b')$ which reduces $\hat{S}'_2$ to the fundamental symplectic-admissible form $\hat{S}_2$ in b.

The reader should be aware that, since a symplectic manifold is symplectic-admissible, the transformations of the above theorem imply, as a particular case, the mapping of a

symplectic-admissible form into a conventional symplectic form. Thus, on analytic grounds, the theorem is inclusive of the direct representation of nonconservative vector fields via our canonical-admissible equations and their indirect representation via the conventional canonical equations.

Interpret now a conventional symplectic form Ω_2 as the symplectic content of a symplectic-admissible form $\hat{S}_2$ according to rule (3.8.28). Then the proof of Theorem 2.8.1 applies to the form $\Omega_2 = \hat{S}_2 - \hat{S}_2^T$ and it is sufficient to imply for $\hat{S}_2$ the following symplectic-admissible covering of Darboux-Weinstein theorem (see, again, ref [5b] for details).

THEOREM 3.8.2: Let M_1 be a submanifold of a manifold M and let $\hat{S}_2$ and $\hat{S}_2'$ be two-symplectic-admissible forms such that $\hat{S}_2|_{M_1} = \hat{S}_2'|_{M_1}$. Then there exists a neighborhood $N(M_1)$ of M_1 and a diffeomorphism $f: N(M_1) \longrightarrow M$ such that

(a) $f(m) = m$ for all $m \in M_1$ and

(b) $f^*\hat{S}_2 = \hat{S}'_2$.

We finally remain with a problem which, as we shall see in Section 4, appears to be crucial for the construction of a group of transformations leaving form-invariant nonconservative, non-linear systems. We are here referring to the generalization of the notion of globally Hamiltonian vector field (Table 2.8) which is needed in the symplectic-admissible geometry.

We here define the (left) inner-admissible product of a contravariant vector field $\hat{\dot{-}}^\mu$ with a symplectic-admissible structure the quantity [5b]

$$\hat{\dot{-}}\,\hat{S}_{2\,1} = \hat{\dot{-}} \boxtimes \hat{S}_2 = S_{\mu\nu}\,\hat{\dot{-}}^\nu\, db^\mu = \dot{-}_\mu\, db^\mu. \qquad (3.8.31)$$

Again, the above product is inner-admissible in a dual sense. First, the ordinary inner product is recovered identically at the limit

$$\lim_{F_{ka}\to 0} \hat{\dot{-}} \boxtimes \hat{S}_2 = \dot{-} \rfloor \omega_2 = \dot{-}_1\,\omega_{2\,1} \qquad (3.8.32)$$

and, secondly, we can recover it in the attached form

$$\dot{-}\,\Omega_{2\,1} = \frac{1}{2}\left(\dot{-} \boxtimes \hat{S}_2 - \dot{-} \boxtimes \hat{S}_2^T\right) = \dot{-} \rfloor \Omega_2 = \dot{-}_1\,\Omega_{2\,1}. \qquad (3.8.33)$$

A vector field $\hat{\dot{-}}^\mu$ on $M(b,\hat{S}_2)$ is called Hamiltonian-admissible [5b] when the one-form $\hat{\dot{-}}\,\hat{S}_{2\,1}$ is exact, i.e., at a point $m \in M(b,\hat{S}_2)$ there exists a neighborhood N(m) and a function H(b), the Hamiltonian, on N(m) such that

$$\hat{\dot{-}}\,\hat{S}_{2\,1} = S_{\mu\nu}\,\hat{\dot{-}}^\nu\, db^\mu = \dot{-}_\mu\, db^\mu = dH = \frac{\partial H}{\partial b^\mu}\, db^\mu. \qquad (3.8.34)$$

This is the desired geometrical characterization of Eqs. (3.8.9) which, as we shall see in Section 4, appears to be crucial for relativity considerations. The reader should be aware that we are essentially referring to the fundamental equations (3.4.4) of the Lie-admissible formulations.

We cannot close this table without few highly conjectural considerations related to the geometrical characterization of nonconservation laws. This inevitably brings into focus the problem of the possible existence of a covering of the notion of Lie derivative for Lie-admissible formulations. Let us recall that (Section 2.8) two layers of characterization of the Lie derivative can be identified within the context of the symplectic geometry, i.e.,

$$\mathcal{L}_{\dot{-}} F = \lim_{t\to 0} \frac{F\circ G_b(t) - F\circ G_b(0)}{t} = \left(\omega^{\mu\nu}\frac{\partial H}{\partial b^\nu}\frac{\partial}{\partial b^\mu}\right)F = [F, H], \qquad (3.8.35a)$$

$$\mathcal{L}_{\dot{-}^*} F = \lim_{t\to 0} \frac{F\circ G^*_b(t) - F\circ G^*_b(0)}{t} = \left(\Omega^{\mu\nu}\frac{\partial H^B}{\partial b^\nu}\frac{\partial}{\partial b^\mu}\right)F = [F, H^B]^* \qquad (3.8.35b)$$

where $G_b(t)$ is a one-parameter connected Lie group and $G^*_b(t)$ one of its isotopes (Table 3.7).

Eq. (3.8.35a) is the conventional form and Eq. (3.8.35b) is the generalization we have attempted for Birkhoff's equations.

The third layer of the notion considered is here called Lie-admissible derivative [5b] and it is given by

$$\mathcal{L}_{\hat{\dot{-}}} F = \lim_{t\to 0} \frac{F\circ \hat{G}_b(t) - F\circ \hat{G}_b(0)}{t} = \left(S_H^{\mu\nu}\frac{\partial H}{\partial b^\nu}\frac{\partial}{\partial b^\mu}\right)F, \qquad (3.8.36)$$

where now $\hat{G}_b(t)$ is a genotope of $G_b(t)$, i.e., a one-dimensional, connected, Lie-admissible group according to Definition 3.7.3. Again, the above operation is not a "derivative" in the conventional sense. But this is precisely its advantage because, if applied to the generators X_i of a (Lie or) Lie-admissible algebra, yields the following symplectic-admissible characterization of nonconservation laws

$$\mathcal{L}_{\hat{\dot{-}}} X_i = \left(S_H^{\mu\nu}\frac{\partial H}{\partial b^\nu}\frac{\partial}{\partial b^\mu}\right)X_i = (X_i, H) \neq 0. \qquad (3.8.37)$$

Notice that in the transition from laws (3.8.35) to their covering (3.8.36) we have not changed, by central requirement, the base manifold, the Hamiltonian and the parameter of the time evolution group. The action of the nonconservative forces is then represented by the departure of the generalized formulations from the conventional ones, which is at the basis of the Lie-admissible formulations. Notice that these nonconservative forces are present at several levels, such as the structure of the genotopically mapped Lie group $\hat{G}_b(t)$, the symplectic-admissible structure $\hat{S}_2$, the Lie-admissible derivative, etc.. Notice also

that the notion of Lie-admissible derivative is ensured by the property that a Lie-admissible group is a Lie group, only written in an unconventional form. And indeed, limit (3.8.36) can be first established in terms of the standard realization of $\hat{G}_b(t)$. This yields the conventional Lie derivative (3.8.35a), but now expressed in terms of another generator. Form (3.8.36) is then established by reformulating the realization of the group $\hat{G}_b(t)$ in terms of the generator of $G_b(t)$.

For completeness let me indicate that the analysis outlined until now for autonomous systems and symplectic-admissible geometry appears to carry over to the nonautonomous systems, yielding what we have called in ref.[5b] contact-admissible manifolds, that is, 6N+1 dimensional manifolds $M(\bar{b}, \hat{\bar{S}}_2)$ in the local coordinates $\bar{b}^i$, $i = 0,1,2,\ldots,6N$ equipped with an exterior-admissible two-form of maximal rank

$$\hat{\bar{S}}_2 = \bar{S}_{ij}\, db^i \circ db^j, \quad b^0 = t, \tag{3.8.38}$$

which is such that its attached form

$$\bar{\Omega}_2 = \hat{\bar{S}}_2 - \hat{\bar{S}}_2^T = \bar{\Omega}_{ij}\, db^i \wedge db^j, \tag{3.8.39}$$

is a contact form. For brevity, we here refer the interested reader to ref.[5b] for more details.

This concludes our review of methodological tools which will be used in Section 4 to attempt the construction of a covering of the Galilei relativity for nonconservative systems.

By looking in retrospective, it is rather tempting to conclude that

(A) The conventional canonical, Lie and symplectic formulations appear to admit consistent covering formulations of canonical-admissible, Lie-admissible and symplectic-admissible type, respectively.

(B) The deep interrelation, complementary and compatibility of the analytic, algebraic and geometrical aspects of the conventional formulations appear to carry over to their coverings in their entirety, and

(C) The covering formulations are conceptually, technically and methodologically different that the conventional formulations. Neverthless, they are capable of recovering the latter in their entirety at the limit of null nonconservative forces as well as via the attached rule of Lie-admissibility.

It is again appropriate here to stress that what we have attempted in this section is the indication of the existence of the indicated covering with properties (A), (B) and (C). Their actual construction in all the necessary technical details will predictably demand the contributions from a significant number of independent researchers.

TABLE 3.9: SOME POSSIBLE APPLICATIONS OF LIE-ADMISSIBLE FORMULATIONS IN PHYSICS.

The possible application for which the Lie-admissible formulations have been conceived is the study of the breaking of the fundamental space-time symmetries in Newtonian mechanics and the hope that their non-Newtonian (relativistic and quantum mechanical) extensions can, in due time, be identified and result to be physically significant for the problem of the hadronic structure under the assumption that the strong hadronic forces are structurally more general than the atomic and the nuclear forces (essentially nonselfadjoint strong hadronic forces).

As by now familiar, this paper is solely devoted to the relativity aspect of Newtonian Mechanics. Neverthless, it appears advisable to outline the intended use of Lie-admissible formulations for the study of broken Lie symmetries in general, as well as for other aspects of current relevance in theoretical physics.

In essence, the Lie-admissible formulations appear to provide an algebraic-group theoretic characterization of broken symmetries and nonconservation laws as a covering of exact Lie symmetries and conservation laws. In the following we would like to outline the mechanics of the use of these broader formulations as well as their dual nature of ensuring, on one side, that the conventional Lie context is indeed brooken and of providing, on the other side, methods for the treatement of the broken context. It appears that there is the need of both these profiles.

For example, consider the familiar case of the Gell-Mann-Okubo mass formula and the SU(3) symmetry. In order to avoid equal mass multiplets, the SU(3) symmetry must be broken. On the other side, as stressed in Table 3.7, the Gell-Mann-Okubo mass formula is undefinable within the context of a Lie algebra and necessarily demands the use of an enveloping algebra to properly characterize powers of SU(3) generators. The conventional derivation of the formula is conducted, as well known, within the context of the universal enveloping associative algebra of SU(3), the algebra A(SU(3)). But then a possible fundamental inconsistency arises. As also stressed in Table 3.7, the algebra A(SU(3)) is the true representative of the exact SU(3) symmetry, both algebraically and group theoretically. The net effect is that, even though the SU(3) symmetry can be semiempirically broken at the level of semiphenomenological models, it is still exact at the algebraic level. In other words, the use of the algebra A(SU(3)) by no means guarantees that the SU(3) symmetry is broken. Instead, it constitute the most rigorous way to technically characterize the exact SU(3) symmetry.

These remarks are here introduced to illustrate the first aspect of broken symmetries which is relevant for my objective. When studying any broken symmetry, the first problem is to ascertain that the used tools do indeed, algebraically characterize a broken Lie symmetry.

If this crucial requirement is not ensured, there is the possibility of working with incompatible tools (e.g., a semiphenonenological Lagrangian breaking of SU(3) and an exact associative envelope A(SU(3)) of SU(3)). The depth of the physical insight of the approach is then in question.

The use of the Lie-admissible formulations for the characterization of broken symmetries clearly removes any shadow of doubt in respect to this issue: not only the original Lie symmetry algebra is broken, but actually it is undefinable jointly with the related universal associative algebra because, e.g., the analytic equations are non-Lie in algebraic character.

But, to ensure that a Lie symmetry is indeed broken, is "per sé" purely formal, particularly on physical grounds. This naturally brings into focus the second aspect of the issue, the need of methods for the treatement of the broken context. I am here of course referring to the identification of methods capable of producing specific physical predictions via a mathematical process. It is in this second respect that my hopes for the Lie-admissible formulations rest, because they constitute a covering of the Lie formulations. This means that the broken context is not left algebraically and group theoretically undefined. Instead it is treated with methods fully equivalent, although generalized, than those of the exact symmetry.

This is not the place to recall the physical relevance of the Gell-Mann-Okubo formula. The above remarks were, therefore, solely devoted to the derivation of this formula as currently conducted. If the broken SU(3) symmetry is truly realized on algebraic grounds this means the nonapplicability of the envelop A(SU(3)). But then the question which immediately arises is: how we construct the Gell-Mann-Okubo mass formula if the associative envelop cannot be used? It is at this point that the potential physical relevance of Lie-admissible algebras as universal enveloping nonassociative algebra of a Lie algebra emerges in full. And indeed, if A(SU(3)) is replaced by the genotope U(SU(3)) (Theorem 3.7.3) the following picture emerges. First of all, the generators, the parameters and the base manifolds are preserved according to the construction of U(SU(3)). Secondly, the Lie algebra SU(3) emerges as broken because the algebra acting in the neighborhoof of the identity is non-Lie, although Lie-admissible. Thirdly the breaking of SU(3) is truly ensured by the fact that the attached algebra $[U(SU(3))]^-$ is non-isomorphic to SU(3). Fourthly, The SU(3) algebra is recoverable in full at the limit when the nonassociative envelop recovers the conventional associative envelop. Fifthly, the approach enjoys an analytic and geometrical backing by therefore removing any inconsistency between different methodological approaches to the same broken context. Sixtly, The covering envelop U(SU(3)) is fully capable of producing the Gell-Mann-Okubo mass formula identically, that is, as currently known, under a suitable selection of U. And last, but not least, the departure of the Lie-admissible over the Lie formulations is representative of the symmetry breaking forces by therefore opening the possibility of obtaining some informations on the dynamical origin of the

parameters of the Gell-Mann-Okubo mass formula, that is, their link to the symmetry breaking forces (these parameters are null for exact symmetry). In turn, this last aspect appears to have some quite intriguing possibilities for a truly central problem in hadron physics: the nature of the strong hadronic forces.

I worked out the rudiments of this Lie-admissible approach for the construction of the Gell-Mann-Okubo mass formula during my stay at the University of Miami in Coral Gables, Fla, in 1967-1968 and then presented the approach at the Indiana Conference of 1968 (see the proceedings [4d]). Subsequently, the approach was reinspected by C. N KTORIDES in 1975 [24a] in great details. More recently, the approach can benefit of various progresses in Lie-admissible formulations. For an outline of the current status of the art on this issue see ref. [5b]

After these introductory remarks, let me outline the mechanics of the intended use of the Lie admissible formulations for the case of broken space-time symmetries (the extension to non-space-time symmetries being trivial) [5b]. Other possibilities will be indicated at the end of the table.

(I) EXACT LIE SYMMETRIES AND CONSERVATION LAWS. The starting ground is, of course, the established ground. Consider a conservative (essentially selfadjoint) Newtonian system represented with a Hamiltonian H and suppose that the system exhibits an exact, manifest, n-dimensional, connected, space-time Lie symmetry G. We then write in canonical formulations the exact symmetry (ES) as follows

$$\left[\dot{b}^{\mu} - \omega^{\mu\nu}\frac{\partial H(b)}{\partial b^{\nu}}\right]^{ES}_{SA} = 0, \qquad (3.9.1)$$

with underlying conservation laws

$$\dot{X}_i(b) = \frac{\partial X_i}{\partial b^{\mu}}\omega^{\mu\nu}\frac{\partial H}{\partial b^{\nu}} = [X_i, H] = 0. \qquad (3.9.2)$$

This is the well known reduction of physical laws to primitive Lie notions. It is however appropriate to stress that the direct physical effectiveness of the Lie formulations for the characterization of physical laws is crucially dependent on the fact that the mathematical algorithms "r", "p" and "H" of Eqs. (3.9.1) and (3.9.2) are not abstruse quantities such as Eqs. (2.10.5e) or (2.12.14). Instead, they represent the Cartesian coordinates of the experimental detection of the system, the physical linear momentum (which in this case coincide with the canonical momentum) and the physical Hamiltonian (which in this case coincide with the canonical Hamiltonian).

(II) BROKEN SYMMETRIES AND NONCONSERVATION LAWS. Suppose now that, according to experimental evidence, the G symmetry of the system is in actuality broken as a result of additive forces. This is the case of the systems of our everydays life. For instance, the

following case

$$T_1(t): \quad \left[\left(m\ddot{\underset{\sim}{r}}\right)_{SA}^{ES} + \gamma t \dot{\underset{\sim}{r}}\right]_{NSA}^{BS} = 0, \qquad (3.9.3)$$

represents an example of the breaking of the exact symmetry of a free particle under translations in time due to drag forces produced by the medium in which the motion occurs. The other case

$$SO(2): \quad \left[\left(I\ddot{\theta}\right)_{SA}^{ES} - T(\theta,\dot{\theta})\right]_{NSA}^{BS} = 0, \qquad (3.9.4)$$

provides an example of the breaking of the exact symmetry under rotation of the conservative abstraction of the spinning top under gravity (here assumed for simplicity with only one degree of rotational freedom). Perhaps more significative on methodological grounds is the case of charged particles under nonconservative forces also produced by the medium in which the motion occurs

$$G(3.1): \quad \left\{\left[m_k \ddot{\underset{\sim}{r}} - \underset{\sim}{f}^{Coul}(\underset{\sim}{r})\right]_{SA}^{ES} - \underset{\sim}{F}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})\right\}_{NSA}^{BS} = 0, \qquad (3.9.5)$$

which constitutes an example of the breaking of the full Galilei symmetry.

Customarily, symmetries are broken in classical mechanics by additing a symmetry breaking term to the Lagrangian or Hamiltonian. This breaking, which we have called selfadjoint breaking (Table 2.14), is highly insufficient for our objectives. We therefore assume that the forces responsible for the G-symmetry breaking are the most general, local, class C^{∞}, regular, Newtonian forces, i.e., we assume an essentially nonselfadjoint breaking of the G-symmetry which is inclusive of the subclasses of canonical breaking and semicanonical breaking. This broader broken symmetry (BS) context will be written

$$\left\{\left[\dot{b}^{\mu} - \omega^{\mu\nu}\frac{\partial H(b)}{\partial b^{\nu}}\right]_{SA}^{ES} - F^{\mu}(t,b)\right\}_{NSA}^{BS} = 0, \qquad (3.9.6)$$

with consequential nonconservation laws

$$\dot{X}_i = [X_i, H] + \frac{\partial X_i}{\partial b^{\mu}} F^{\mu} \neq 0. \qquad (3.9.7)$$

A point of crucial physical and methodological significance is that in the transition from the exact symmetry (3.9.1) to the broken symmetry (3.9.6) the physical quantities remain unaffected. Typically, when one adds a damping velocity dependent term to the harmonic oscillator equation, this leaves the expression of the energy unchanged. The problem is then simply shifted to the computation of the variation of this energy in time, i.e., nonconservation law (3.9.7). Similarly, the physical angular momentum of a system is $\underset{\sim}{M} = \underset{\sim}{r}^k \times \underset{\sim}{p}_k = \underset{\sim}{r}^k \times m\dot{\underset{\sim}{r}}_k$. This quantity holds irrespective of whether there are forces not derivable from a potential or

not, e.g., for the nonconservative Coulomb system (3.9.5).

This physical property is at the very foundations of the Lie-admissible formulations. And indeed, all efforts are focused in preserving the algorithms "r", "p" and "H" of direct physical significance and changing instead the formulations for their treatement.

(III) CANONICAL-ADMISSIBLE CHARACTERIZATION OF BROKEN SYMMETRIES AND NONCONSERVATION LAWS. Hamilton's equations with external terms, Eqs. (3.9.6), do not appear to be promising (on grounds of my current mathematical knowledge) for the objective at hand because they do not characterize an algebra via the brackets of the time evolution law (Table 3.1). Jointly, formulations which are Lie in algebraic character are strictly excluded from our approach to ensure the maximal possible breaking of the Lie symmetries according to the remarks at the beginning of this table. This leads to the canonical-admissible characterization of broken symmetries

$$\left[\dot{b}^{\mu} - S^{\mu\nu}(t,b)\frac{\partial H}{\partial b^{\nu}}\right]_{NSA}^{BS} = 0, \qquad (3.9.8)$$

and the canonical-admissible characterization of the nonconservation laws

$$\dot{X}_i(b) = \frac{\partial X_i}{\partial b^{\mu}} S^{\mu\nu}(t,b)\frac{\partial H}{\partial b^{\nu}} = (X_i, H) \neq 0, \qquad (3.9.9)$$

in which the departure of the analytic equations from the conventional Hamiltonian form is a representative of the symmetry breaking forces, e.g.,

$$\left(S^{\mu\nu} - \omega^{\mu\nu}\right)\frac{\partial H}{\partial b^{\nu}} = F^{\mu}_{SB}. \qquad (3.9.10)$$

At the risk of being pedantic, the difference of this approach with current trends in classical symmetry breaking must be reemphasized. The virtual totality of established physical models are based on the conventional structure of a Hamiltonian, $H_{tot} = H_{free} + H_{int}$. A symmetry of this Hamiltonian is customarily broken by adding a further term, which this time is responsible of the symmetry breaking, and one writes $H_{tot}^{BS} = (H_{free} + H_{int})^{ES} + H_{int}^{BS}$. A part from the fact that this breaking is highly restrictive and precludes several additional classes of more physically significant breakings (Table 2.14), there is one aspect which, unless properly treated, can lead to inconsistencies. The virtual totality of established physical models sees its analytic origin on what we have called in Section I the "truncated" Hamilton's equations or Lagrange's

equations, that is, those without external terms. In the transition to the case of broken symmetry the Hamiltonian is modified, but the analytic equations are left unchanged. This implies that in the conventional classical treatement of broken Lie symmetries the Hamiltonian breaks the symmetry while the underlying analytic equations remain strictly Lie in algebraic character. The net effect is the situation recalled earlier in this table according to which the analytic level is representative of SU(3) breaking, but the exact A(SU(3)) algebra is used for calculations. The attitude implemented in the Lie-admissible formulations is exactly the opposite of the above. In the Lie-admissible treatement of broken Lie symmetries the Hamiltonian remains fully invariant, while the underlying analytic equations are strictly non-Lie in algebraic character. It is this point which ensures the nonapplicability of Lie algebras "ab initio" as a methodological tool for the broken context.

Once the broken symmetry equations are represented with the canonical-admissible equations, the remaining tools of the analytic covering are applicable, if needed. I am here referring to the canonical-admissible transformations (Table 43.4) or to other tools (such as variational principles, Hamilton-Jacobi theory, canonical-admissible perturbation theory, etc.) which we have not indicated in this paper for brevity, but which appear to exist.

In conclusion, there is hope that the canonical-admissible equations characterizing the broken symmetry context can indeed be equipped, in due time, with a covering of the conventional canonical formulations of canonical-admissible type. This is crucial for the objective of providing the broken context with as many as possible methodological tools (for the study of any specific topic of interest)up to a possible future point of full methodological equivalence with the case of the exact symmetry.

(IV) LIE-ADMISSIBLE CHARACTERIZATION OF BROKEN SYMMETRIES AND NONCONSERVATION LAWS. The non-Lie algebra character of the analytic equations, has a number of rather deep methodological implications. It essentially implies that the universal enveloping associative algebra of the original exact symmetry algebra $\underline{G}$, that is, $\mathbf{A}(\underline{G})$ is replaced by the Lie-admissible envelop of the broken Lie algebra $\underline{G}$, i.e., (Table 3.7)

$$\mathcal{A}(\underline{G}) \rightarrow \mathcal{U}(\underline{G}) = \hat{\mathcal{T}} / \hat{\mathcal{R}} \quad . \qquad (5.9.11)$$

In turn, this implies a Lie-admissible algebra in the neighborhood of the identity, i.e.

$$\hat{b}^{\mu} \approx b^{\mu} + \theta^{i} (b^{\mu}, G_i), \qquad (5.9.12)$$

as well as a Lie-admissible group of finite,connected,transformations, i.e.,

$$\hat{b}^{\mu} = e^{\theta^{i} S^{\alpha\beta}_{(i)} \frac{\partial G_i}{\partial a^{\beta}} \frac{\partial}{\partial a^{\alpha}}} b^{\mu} . \qquad (3.9.13)$$

For the sake of clarity, permit me to stress that it is not the original algebra $\underline{G}$ which is embedded into $U(\underline{G})$, but insted $A(\underline{G})$. In other words, the Lie-admissible profile brings into full focus the elemental nature of the enveloping algebras for the Lie characterization of exact symmetry. This elemental nature simply persists in the Lie-admissible covering. Thus, the algebra was infinite-dimensional to begin with already at the level of the exact symmetry and remains infinite-dimensional at the covering level. Of course we are here referring to the infinite number of elements (3.7.7) for the Lie case, and (3.7.28) for the Lie-admissible covering, all induced by the same basis, the generators X_i of $\underline{G}$.

Of course in this paper we have focused our attention on only the central aspects (3.9.11), (3.9.12) and (3.9.13). It is hoped, however, that the "Lie-admissible theory" can be, in due time, brought up to the level of the "Lie's theory" at least in sufficiently effective way. When this is accomplished, then the algebraic-group theoretic tools for the characterization of broken Lie symmetries would be equivalent to those for the exact Lie symmetry as currently established. This is what we referred to as the methodological characterization of broken symmetry. Notice that, if this program is truly realized in due time, each aspect of the exact symmetry is replaced by a covering aspect. For instance,the" broken symmetry Lie algebra" is not left as an algebraically undefined entity. Instead, it is replaced by a broader, but fully defined Lie-admissible algebra. Most intriguing (particularly for relativity considerations, see next section) is the fact that "the broken symmetry Lie group" is not left also undefined, but instead it is replaced by a broader group structure which we have called Lie-admissible group. The point is that, again the broken and as such, unusable associative envelop $A(\underline{G})$ is replaced by an acting, and thus usable, Lie-admissible covering $U(\underline{G})$, and similarly, the broken and thus unusable group G is replaced by an acting Lie-admissible covering $\hat{G}$.

(V) SYMPLECTIC-ADMISSIBLE CHARACTERIZATION OF BROKEN SYMMETRIES AND NONCONSERVATION LAWS.

The methodological characterization of broken symmetries would be highly deficient, particularly for relativity considerations, without the inclusion of geometrical methods. The canonical-admissible and Lie-admissible formulations, however, are incompatible with the symplectic or contact geometry. This necessarily demands the identification of a covering geometry which I have tentatively called symplectic-admissible or contact-admissible. One of the primary functions

of this broader approach (according to the best of my knowledge at this time) is to reconstruct the covariant vector field of the exact symmetry case via the inner-admissible product with the two-form $\hat{S}_2$, i.e., the Hamiltonian-admissible characterization of broken symmetries

$$\hat{\dot{\underline{\ }}} \boxtimes \hat{S}_2 = S_{\mu\nu} \underset{\substack{\uparrow \\ BS}}{\hat{\dot{\underline{\ }}}^{\nu}} db^{\mu} = \underset{\substack{\uparrow \\ ES}}{\dot{\underline{\ }}_{\mu}} db^{\mu} = \underset{\substack{\uparrow \\ ES}}{dH}, \qquad (3.9.14)$$

with symplectic-admissible characterization of the nonconservation laws

$$\mathcal{L}_{\hat{\dot{\underline{\ }}}} X_i \neq 0. \qquad (3.9.15)$$

The reason for the interest in structure (3.9.14) is that it gives hopes for the identification of a nonmanifest symmetry group which leaves form-invariant the broken symmetry equations. And indeed, the Hamiltonian of the approach is fully invariant under the original symmetry, while generators, base manifold and parameters of the original symmetry are left unchanged in the construction of the Lie-admissible group (3.9.13) by construction. Contraction (3.9.14) then indicates a possible form-invariance of the equations of motion under the covering group (3.9.13).

Again, it is hoped that, in due time, this "unconventional" geometrical approach (based on a calculus which does not admit conventional notions, such as derivative and closure) may be sufficiently developed to the point of being effective for practical problems. If this will be the case, then the geometrical methods for the treatement of broken symmetries would be equivalent to the methods currently available for exact Lie symmetries.

This concludes our review of the intended use of Lie-admissible formulations. Notice that this use, again, is primarily intended for the breaking of space-time symmetries. Neverthless their applicability to other symmetry breakings should not be overlooked. As a matter of fact, the method appears to be applicable also to the current approach to symmetry breaking, that is,

$$H^{ES}_{tot} \rightarrow H^{BS}_{tot} = H^{ES}_{tot} + H^{BS}_{Int}, \qquad (3.9.16)$$

via the reinterpretation

$$\left\{ \left[\dot{b}^{\mu} - \omega^{\mu\nu} \frac{\partial H}{\partial b^{\nu}} \right]^{ES}_{SA} - F^{\mu} \right\}_{SA} = 0, \quad F^{\mu} = \omega^{\mu\nu} \frac{\partial H^{BS}_{Int}}{\partial b^{\nu}} \qquad (3.9.17)$$

In other words, the symmetry breaking forces F^{μ} used in steps (II) through (V) are, by no means, restricted to be nonderivable from a potential and the methods are fully applicable even

when they are indeed derivable from a potential. Our emphasis on forces not derivable from a potential is inspired by relativity arguments and it is also intended to express the fact that the methods at hand do indeed apply for arbitrary (but local and of class C^{∞}) Newtonian forces.

This latter aspect is important because it indicates that Lie-admissible formulations can be applied to the breaking of internal symmetries without affecting the physical framework of the conventional space-time symmetries. To be specific, symmetry breaking (3.9.17) may be referred to only an internal symmetry group G, while the full Hamiltonian H^{BS}_{tot} may remain invariant, say, under the Poincaré group. In this case approach (3.9.17) yields the breaking of only the internal symmetry G because the Poincaré symmetry is recoverable in full via the conventional Hamiltonian formulations in H^{BS}_{tot}. Other aspects of this intriguing situation will be considered in a subsequent paper.

In conclusion, the Lie-admissible formulations appear to be promising for the characterization of broken symmetries. The characterization is of dual nature in the sense that it is a

- LIE-ADMISSIBLE BREAKING OF LIE SYMMETRIES, because, for instance, the attached algebra of U(G) is not isomorphic to G, i.e.,

$$[U(\underline{G})]^{-} \not\approx \underline{G}, \qquad (3.9.18)$$

as well as a

- LIE-ADMISSIBLE COVERING OF LIE SYMMETRIES, in the sense that the formulations are nontrivially different, but capable of recovering the conventional exact symmetry context identically at the limit of null symmetry breaking forces, e.g.,

$$\begin{pmatrix} S^{\mu\nu} \\ U(\underline{G}) \\ \hat{\underline{G}} \\ \hat{G} \\ \hat{S}_2 \\ \mathcal{L}_{\hat{\dot{\underline{\ }}}} \end{pmatrix}_{\text{LIE-ADMISSIBLE}} \xrightarrow{\text{SYMM. BREAK. FORCES} \to 0} \begin{pmatrix} \omega^{\mu\nu} \\ A(\underline{G}) \\ \underline{G} \\ G \\ \omega_2 \\ \mathcal{L}_{\dot{\underline{\ }}} \end{pmatrix}_{\text{LIE}}. \qquad (3.9.19)$$

We shall therefore refer to the formulations considered as the Lie-admissible covering-breaking of Lie symmetries.

The question which we would like to touch in closing this section is whether Lie-admissible formulations may be significant beyond the case of broken Lie symmetries.

A presentation of my current knowledge on this aspect is provided in ref.[5a,b,c]. Here, let me recall three aspects as potentially promising.

(A) Lie-admissible covering of the deformation theory. As is well known, the deformation theory is based on a modification of the product of the type

$$[X_i, X_j](\varepsilon) = [X_i, X_j]_A + \varepsilon F_1(X_i, X_j) + \varepsilon^2 F_2(X_i, X_j) + \dots \quad (3.9.20)$$

which, however, remains strictly Lie in algebraic character. This theory can be subjected to the Lie-admissible covering, that is, an expansion of the type (3.9.20) which now satisfies the Lie-admissibility laws, rather than the Lie laws. This approach has been proposed by N. KÕIV and J. LÕHMUS[26,9] who have also worked out the Lie-admissible deformation theory of the spin 1/2 Pauli algebra. Their results are essentially equivalent to the embedding of the spin 1/2 Pauli algebra into the $A(\lambda, \mu)$ mutation algebra proposed by R. M. SANTILLI[4a], as it is shown in ref.[5b] Neverthless, they are significant to indicate that deformation-type methods[31] are consistent, as well as particularly useful for Lie-admissible formulations.

In my opinion, this line of study deserves a close scrutiny by independent researchers because it touches on a number of fundamental physical problems which will be presented in a proper light in subsequent papers.

To have an indication, consider the case of a first-order Lie-admissible deformation of the spin 1/2 case. This literally means that the spin SU(2) Lie symmetry is broken, although in this case in an infinitesimal way. This has bound to have statistical implications (Lie-admissible algebras, being neither totally symmetric not totally antisymmetric are incompatible with Bose-Einstein and Fermi-Dirac statistics). In turn, this has direct bearing with Pauli exclusion principle.[32] In essence, upon a number of technical steps (see refs.[33a, 5c]) a first order Lie-admissible deformation of the SU(2)-spin algebra implies the nonapplicability of Pauli's principle, although in an infinitesimal measure depending on the structure of the selected deformation.

The aspect which is physically relevant is whether such nonapplicability of Pauli principle has any ground of plausibility.

Clearly, at the atomic level such inapplicability of Pauli principle is conceptually and physically inconsistent. The validity of Pauli principle in an exact form is here established by a rather large amount of incontrovertible experimental evidence (Pauli principle is crucial for the interpretation of several central features of the Mendèleyèv table, such as the existence of the long periods containing the iron, platinum and palladium groups, and even those of the 14 rare earths).

In the transition to the nuclear level considerable scientific caution must be exercised to avoid prejudices. What we can safely state is that the use of Pauli principle in nuclear physics

produces an excellent agreement with the experimental data. Neverthless, on grounds of our current experimental and theoretical knowledge we cannot state that Pauli principle is exactly valid in nuclear physics or that it is valid in the same measure as that of atomic physics.

And indeed, the question which was submitted by R. M. SANTILLI[33a], is whether our current knowledge on the validity of Pauli exclusion principle for the nuclear structure is quantitatively comparable to the current knowledge of the PCT symmetry in particle physics or it is at a stage prior to the discovery of parity violation.

In the same paper[33a] the experimental resolution of the issue was then advocated. At the theoretical level it appears to be rather difficult to go beyond the level of personal viewpoints, or opinions, or conjectures which, in any case, remain far from a scientific truth. As a matter of fact, we here have a situation in which opposite viewpoints could be equally plausible because of different reasons. The argument in favor of an exact validity of Pauli principle in nuclear physics in known. See, for instance, ref.[32]. An opposite viewpoint, in which, of course, an infinitesimal deviation is advocated, is presented in ref.[5a,b,c] The epistemological argument is quite simple. The atomic structure exhibits dimensions which are substantially greater than the charge diameter of the constituents. Within such a setting, it is fully plausible on conceptual grounds that the constituents preserve the value of their spin (and thus, their statistical character, and thus, the applicability of Pauli principle) during the life of the system. In the transition to the nuclear structure the situation is different. Here, according to experimental evidence, the charge volume of the nucleus is (approximately) proportional to the number of nucleons. This means that, at a primitive view, "nucleons are very close to each other". But then the idea that a nucleon preserves exactly unchanged its spin (and statistical character, and verification of Pauli principle) during the entire life of the system becomes rather unappealing on conceptual grounds. And indeed mutual interferences originated by the extremely close distances might well render (at some deeper future treatment) unrealizable such "perennian value of the spin". The plausibility of an infinitesimal deviation then creeps in in a rather natural way.

In the transition to the hadronic level the situation is drastically different and, in this case, extreme scientific caution must be exercised in the traditional spirit of unsolved physical problems. This problem is the subject of study of refs.[5]. The epistemological argument is, again, quite simple. Consider a massive, charged and spinning particle under electromagnetic interactions moving in vacuum. The conventional quantization of the spin, the preservation of its value during the life of the particle and the complyance with Pauli principle are again unequivocal for much of the same reasons as applying for the atomic structure. Suppose now that this particle penetrates a hadronic structure and, by doing so, preserves its identity for a sufficiently long

period of time (at the hadronic time scale). In this case, quite candidly, it is for me extremely difficult to accept the idea that the particle preserves the value of its spin unchanged during its life within the hadronic structure. It would be the same as asking that an electron produced in the core of, say, a neutron star undergoing phase transition to the hadronic constituents has exactly the same value of the spin (and thus obeys exactly the same laws) of the same particle but belonging to the atomic structure, despite the extreme density of the hadronic medium.

The net effect is that now an infinitesimal departure from Pauli principle becomes unplausible, but in favor of a full finite departure.[33c] In different terms, an infinitesimal, first-order, Lie-admissible deformation of the spin 1/2 SU(2) Lie algebra could be relevant for the nuclear structure, but in the transition to the hadronic structure a full Lie-admissible embedding of A(SU(2))-spin into U(SU(2)) might be conceivable.

Again, we are facing a case in which opposite viewpoints could be equally plausible for different reasons. The argument in favor of the validity of Pauli exclusion principle within the hadronic structure is well known and treated within the context of the recent color unitary models of hadronic structure. An opposite argument is presented in ref.[5]. Clearly, the issue demands an experimental resolution, as proposed by R. M. SANTILLI[5c,33a].

In conclusion, Lie-admissible algebras applied to one of the central methodological tools of contemporary theoretical physics, the SU(2)-spin group, in either a first-order deformation form or for a full breaking, might be of some relevance for the study of the validity or invalidity of Pauli principle under strong interactions.

(B) Lie-admissible covering of supersymmetries. As stressed in this section, the Lie-admissible algebras admit products which are neither totally symmetric nor totally anti-symmetric. As a result, these products can be resolved into a mixture of commutator and anti-commutators, e.g.

$$X_i \circ X_j = \alpha_{ij}^{rs} [X_r, X_s] + \beta_{ij}^{rs} \{X_r, X_s\}, \quad \alpha_{ij}^{rs} = -\alpha_{ji}^{rs}, \quad \beta_{ij}^{rs} = \beta_{ji}^{rs}, \quad i,j = 1,2,\dots,n \tag{3.9.21}$$

It is then trivial to see that the Lie admissible algebras admit as a particular case the graded Lie algebra of supersymmetric models[34]

$$[X_i, X_j] = C_{ij}^{k} X_k, \quad i,j,k = 1,2,\dots,m, \tag{3.9.22a}$$

$$[X_i, X_j] = D_{ij}^{k} X_k, \quad i = 1,2,\dots,m, \quad j,k = m+1, m+2, \dots, n, \tag{3.9.22b}$$

$$\{X_i, X_j\} = E_{ij}^{k} X_k, \quad i,j,k = m+1, m+2, \dots, n. \tag{3.9.22c}$$

This is tantamount to saying that the Lie-admissible algebra are a covering not only of Lie-algebras and their deformation theory, but also of the graded Lie algebra.[5b]

This point is not of marginal significance, particularly on methodological grounds. And indeed as a result of recent studies on supersymmetric models, several quite valuable methodological insights have been gained for graded algebras.[34] As it was the case for the studies on nonlinear representations[30] indicated in Table 3.7, these insights appear to be particularly valuable for Lie-admissible formulations because conceivably extendable to the broader context considered. And indeed, the graded structures (3.9.22) are clearly representative of an intermediate layer between the strictly Lie structure and the Lie-admissible structure.

Equally intriguing, Lie-admissible formulations appear to provide a covering-breaking characterization of supersymmetries.[5b] In other words, I am here referring to the property that Lie-admissible formulations can apparently characterize not only broken Lie symmetries as indicated earlier in this table, but also their sypersymmetric extensions, according to precisely the same lines (I)-(V) given above.

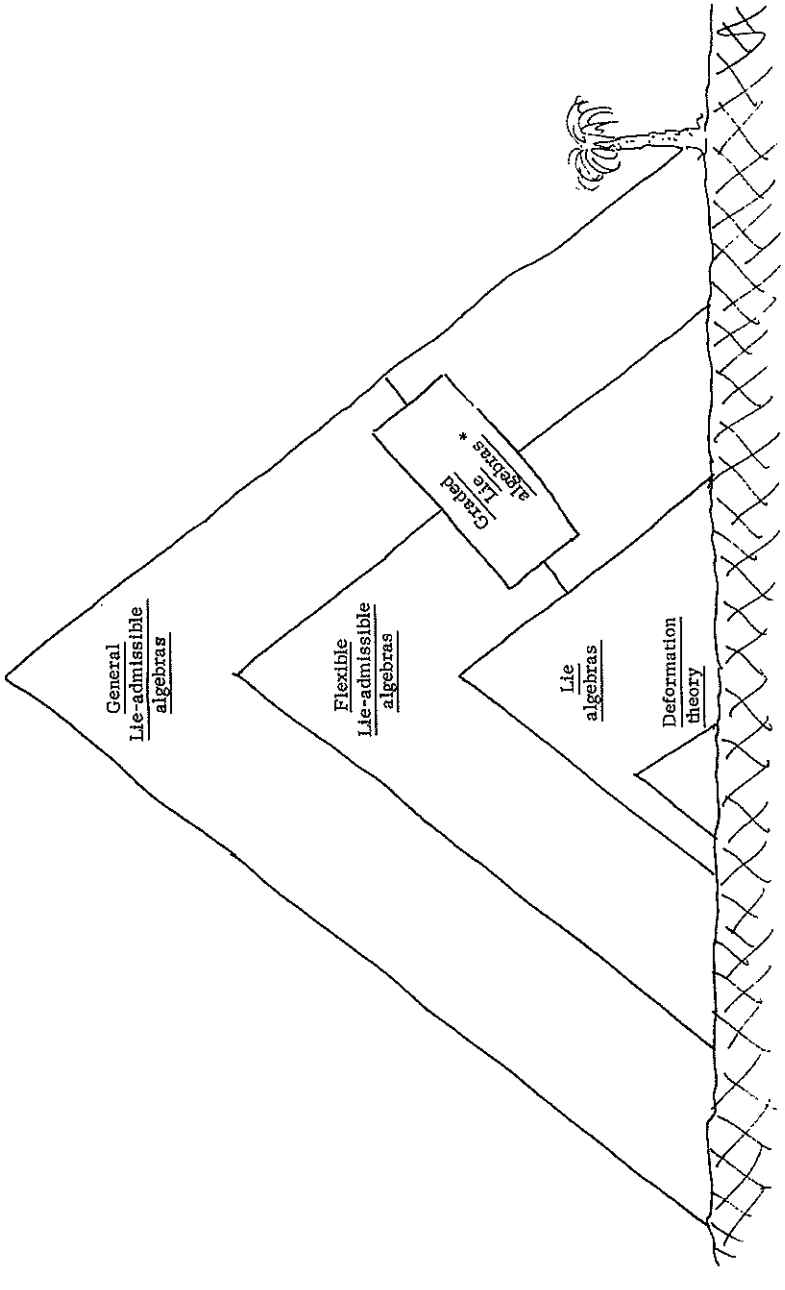
One aspect, however, deserves particular care. We here reach the essence of this table. The breaking of Lie symmetries produced by Lie-admissible algebras is so effective, that may inevitably imply the breaking of space-time symmetries, unless adequately treated. In relation to supersymmetries it is here appropriate to recall that the conventional statistical (or parastatistical) character is preserved by graded algebras (3.9.22), basically in view of the "decoupled" nature of the supersymmetric product. In the transition to a Lie-admissible characterization of their breaking, a number of technical aspect should be considered if one intends to preserve the indicated statistical character (notice that these precautions are mostly abstent in the conventional Lie case owing to the lack of presence of the symmetric part of the product). Restated in different terms, the graded Lie algebras are more genuinely Lie-admissible than the Lie algebras, to the point that the explicit form of their product assumes different structures for different generators, as typical of the Lie-admissible algebras (Tables 3.4 and 3.7). The natural embedding-breaking of these algebras into Lie-admissible structure is that of Eqs. (3.9.21). But then the indicated statistical character is generally lost,[4f] unless the problem is adequately treated.

(C) Lie-admissible approach to the hadronic structure. This is, again, the line of study of ref.[5]. Permit me here to outline the epistemological argument.

One of the most intriguing experimental data on hadrons is that, unlike the corresponding case of nuclei, their charge radius does not sensibly increase with the mass and it is of the same order of magnitude of any other experimentally known massive, charged, particle (~1F).

If the hadronic constituents are assumed to be physical (that is, non-point-like), massive, charged, particles, then a picture of the hadronic structure which is substantially different than the atomic and the nuclear structure emerges. Each constutuent is bounded to move within the charge radius of the other. In other words, starting from the very large distances (as compared to the charge radius of the constituents) of the atomic structure, and passing through the intermediate nuclear structure of extremely small distances between the charge valumes of the constituents, we reach a hadronic state of penetration of the charge volume of each constituent with that of the others. We do not know at this time whether such a picture is plausible. But if it is, it will inevitably demand profound methodological departures from available techniques for its proper treatement. In particular, the acting forces are likely to be nonlocal. Neverthless, it is known that local forces not derivable from a potential constitute a rather good approximation of nonlocal forces. This yields the idea of strong hadronic couplings as not derivable from a potential, that is, a class of models which, at the primitive Newtonian level, is exactly given by models (3.9.6). The potential significance of the Lie-admissible formulations is then self-evident.

In my opinion, however, none of these applications will reach a physical depth of any significance unless the problem of the applicable relativity for forces not derivable from a potential is first solved.



*) Secret passage to bigger pyramids

SECTION 4: THE CONJECTURE OF A LIE-ADMISSIBLE COVERING OF THE GALILEI RELATIVITY IN NEWTONIAN MECHANICS.

As a result of the laborious journey outlined in Sections 2 and 3, I have finally been in a position to confront the problem of

the relativity laws of Newtonian Mechanics.

My efforts can be summarized as follows.

CONJECTURE 4.1: Consider a local, class C^∞, regular, unconstrained, conservative (selfadjoint-SA), Newtonian system of N particles in the physical space

$$\{b^\mu\} = \{r^{ka}, p_{ka}\}, \quad \begin{matrix}\mu = 1,2,\dots,6N, \\ k = 1,2,\dots,N, \\ a = x,y,z,\end{matrix} \tag{4.1}$$

of the Cartesian coordinates of the reference frame of its experimental detection and linear momenta with equations of motion in the (unique) contravariant normal form

$$\left[\dot{b}^\mu - \Xi^\mu(b)\right]^{C^\infty, R}_{SA} = 0, \tag{4.2a}$$

$$\left(\Xi^\mu\right) = \begin{pmatrix} p_{ka} \\ f_{ka} \end{pmatrix}, \quad f_{ka} = -\frac{\partial V}{\partial r^{ka}}, \tag{4.2b}$$

and physical, conserved quantities

$$H_{tot} = T + V = X_1, \tag{4.3a}$$

$$p_{\mu\, tot} = \sum_{k=1}^{N} p_{\mu k} = \sum_{k=1}^{N} m_k \dot{r}_{\mu k} = \{X_2, X_3, X_4\}, \tag{4.3b}$$

$$M_{\mu\, tot} = \sum_{k=1}^{N} r_{\mu k} \times p_{\mu k} = \{X_5, X_6, X_7\}, \tag{4.3c}$$

$$G_{\mu\, tot} = \sum_{k=1}^{N} m_k r_{\mu k} - t\, p_{\mu\, tot} = \{X_8, X_9, X_{10}\}, \tag{4.3d}$$

Then the applicable relativity, the GALILEI RELATIVITY, is characterizable in terms of the following formulations.

(A) ANALYTIC FORMULATIONS, essentially consisting of the representation of the equations of motion with the conventional Hamilton's equations

$$\dot{b}^\mu - \Xi^\mu(b) \equiv \dot{b}^\mu - \omega^{\mu\nu} \frac{\partial H_{tot}(b)}{\partial b^\nu}, \tag{4.4a}$$

$$\left(\omega^{\mu\nu}\right) = \begin{pmatrix} 0_{3N\times 3N} & 1_{3N\times 3N} \\ -1_{3N\times 3N} & 0_{3N\times 3N} \end{pmatrix}, \tag{4.4b}$$

the canonical characterization of the physical conservation laws

$$\dot{X}_i = [X_i, H_{tot}] + \frac{\partial X_i}{\partial t} = \frac{\partial X_i}{\partial b^\mu} \omega^{\mu\nu} \frac{\partial H_{tot}}{\partial b^\nu} + \frac{\partial X_i}{\partial t} = 0, \quad i = 1,2,\dots,10, \tag{4.5}$$

and related canonical formulations (canonical transformations, etc.).

(B) ALGEBRAIC FORMULATIONS, essentially consisting of the universal enveloping associative algebra $\mathfrak{A}(G(3.1))$ of the Lie algebra $G(3.1)$ of the Galilei group $G(3.1)$

$$\mathfrak{A}(G(3.1)) = \mathcal{T} / \mathcal{R}, \tag{4.6a}$$

$$\mathcal{T} = F \oplus G \oplus G \otimes G \oplus \cdots, \quad \otimes = \text{Assoc.}, \tag{4.6b}$$

$$\mathcal{R}: \ [X_i, X_j] - (X_i \otimes X_j - X_j \otimes X_i), \tag{4.6c}$$

the Galilei Lie algebra $G(3.1)$ in the neighborhood of the identity transformations

$$G(3.1) \approx \left[\mathfrak{A}(G(3.1))\right]^- : [X_i, X_j] = C^k_{ij} X_k, \tag{4.7}$$

the Lie group of connected, finite, canonical realization of the Galilei transformations

$$G(3.1): \ b'^\mu = e^{\theta^i \omega^{\alpha\beta} \frac{\partial X_i}{\partial b^\beta} \frac{\partial}{\partial b^\alpha}} b^\mu, \tag{4.8a}$$

$$\{\theta^i\} = \{t_0; r_0; \alpha_0, \beta_0, \gamma_0; v_0\}, \tag{4.8b}$$

and the use of the Lie's theory (representation theory, etc.).

(C) GEOMETRICAL FORMULATIONS, essentially consisting of the characterization of the equations of motion as a globally Hamiltonian vector field (for autonomous cases)

$$\Xi \lrcorner\, \omega_2 = \omega_{\mu\nu} \Xi^\nu d b^\mu = \Xi_\mu d b^\mu = d H_{tot} = \frac{\partial H}{\partial b^\mu} d b^\mu, \tag{4.9}$$

with respect to the fundamental symplectic structure

$$\omega_2 = \omega_{\mu\nu}\, d b^\nu \wedge d b^\mu, \tag{4.10}$$

for the autonomous case (with a contact extension for the nonautonomous case), the

characterization of the physical conservation laws in terms of the Lie derivative (for autonomous generators)

$$\mathcal{L}_{\underline{\dot{\Xi}}} X_i = 0, \qquad (4.11)$$

and the use of the symplectic (or contact) geometry.

Consider now a nonconservative (nonselfadjoint-NSA) extension of the system due to local, class C^{∞}, regular, additive, Newtonian forces not derivable from a potential with equations of motion.

$$\left[\dot{b}^{\mu} - \hat{\dot{\Xi}}^{\mu}(t,b)\right]^{c^{\infty},R}_{NSA} = \left\{\left[\dot{b}^{\mu} - \dot{\Xi}^{\mu}(b)\right]^{c^{\infty},R}_{SA} - F^{\mu}(t,b)\right\}^{c^{\infty},R}_{NSA} = 0, \qquad (4.12)$$

where

$$\left(\hat{\dot{\Xi}}^{\mu}\right) = \left(\dot{\Xi}^{\mu}\right) + \left(F^{\mu}\right) = \begin{pmatrix} p_{ka} \\ f_{ka} \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{F}_{ka}(t,b) \end{pmatrix}, \quad \tilde{F}(t,\underline{\xi},\underline{p}) = F(t,\underline{\xi}_{\mu},\underline{\dot{\xi}}),$$

$$f_{ka} = -\frac{\partial V}{\partial \xi^{ka}}, \quad F_{ka} \neq -\frac{\partial U}{\partial \xi^{ka}} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\xi}^{ka}}, \qquad (4.13)$$

in which the original system (4.2a) is the maximal associated selfadjoint system. Then the relativity which is conjectured as applicable in the physical space of the Cartesian coordinates r^{ka} of the experimental verification of the system and the linear momenta $p_{ka} = m_k \dot{r}_{ka}$, here called GALILEI-ADMISSIBLE RELATIVITY, is characterizable in terms of the following covering formulations.

(A') COVERING ANALYTIC FORMULATIONS, essentially consisting of the representation of the equations of motion with the Hamilton-admissible equations

$$\dot{b}^{\mu} - \hat{\dot{\Xi}}^{\mu}(t,b) \equiv \dot{b}^{\mu} - S^{\mu\nu}(t,b)\frac{\partial H_{tot}(b)}{\partial b^{\nu}}, \qquad (4.14a)$$

$$S^{\mu\nu}\frac{\partial H_{tot}}{\partial b^{\nu}} \equiv \omega^{\mu\nu}\frac{\partial H_{tot}}{\partial b^{\nu}} + F^{\mu}, \quad \det(S^{\mu\nu}) \neq 0, \qquad (4.14b)$$

$$\left(S^{\mu\nu}\right) = \left(\frac{\partial b^{\mu}}{\partial T_{\nu}}\right) = \begin{pmatrix} 0_{3N\times 3N} & 1_{3N\times 3N} \\ \left(\frac{\partial \xi^{ia}}{\partial T^{jb}}\right) & \left(\frac{\partial p^{ia}}{\partial T^{jb}}\right) \end{pmatrix}, \qquad (4.14c)$$

the canonical-admissible characterization of the physical nonconservation laws

$$\dot{X}_i = \left(X_i, H_{tot}\right) + \frac{\partial X_i}{\partial t} = \frac{\partial X_i}{\partial b^{\mu}} S^{\mu\nu}\frac{\partial H_{tot}}{\partial b^{\nu}} + \frac{\partial X_i}{\partial t} \neq 0, \qquad (4.15)$$

and the canonical-admissible formulations (canonical-admissible transformation theory, etc.).

(B') COVERING ALGEBRAIC FORMULATIONS, essentially consisting of the general, Lie-admissible, genotopic mapping of the universal enveloping associative algebras of the Galilei Lie algebras induced by the nonconservative forces

$$\mathcal{U}(\underline{G}(3.1)) = \hat{\mathcal{T}} / \hat{\mathcal{R}}, \qquad (4.16a)$$

$$\hat{\mathcal{T}} = F \oplus \underline{G} \oplus \underline{G} \odot \underline{G} \oplus \cdots, \quad \odot = \text{NonAssoc.}, \qquad (4.16b)$$

$$\mathcal{R} : [X_i, X_j]_{\mathcal{A}^*} - (X_i \circ X_j - X_j \circ X_i), \qquad (4.16c)$$

the Lie admissible genotope of the Galilei Lie algebra in the neighborhood of the identity

$$\hat{\underline{G}}(3.1) : (X_i, X_j)_{(b,\dot{b})} = \frac{\partial X_i}{\partial b^{\mu}} S^{\mu\nu}_{(j)} \frac{\partial X_j}{\partial b^{\nu}} = \hat{u}_{ij}^{\ k}(t,b,X) X_k, \qquad (4.17)$$

with attached isotope

$$\underline{G}^*(3.1) = [\mathcal{U}(\underline{G}(3.1))] : [X_i, X_j]_{\mathcal{U}} = \frac{\partial X_i}{\partial b^{\mu}} \Omega^{\mu\nu}_{(j)} \frac{\partial X_j}{\partial b^{\nu}} = \tilde{C}_{ij}^{\ k}(b) X_k, \quad \Omega^{\mu\nu}_{(j)} = S^{\mu\nu}_{(j)} - S^{\nu\mu}_{(j)}, \quad \underline{G}^*(3.1) \not\simeq \underline{G}(3.1), \qquad (4.18)$$

the Lie-admissible covering group of the canonical realization of the Galilei group

$$\hat{G}(3.1) : \hat{b}^{\mu} = e^{\theta^i S^{\alpha\beta}_{(i)}(t,b)\frac{\partial X_i}{\partial b^{\beta}}\frac{\partial}{\partial b^{\alpha}}} b^{\mu}, \qquad (4.19a)$$

$$\{\theta^i\} = \{t_0, \underline{\xi}_0, \alpha, \beta, \gamma_0, \underline{v}_0\}, \qquad (4.19b)$$

with attached isotope

$$G^*(3.1) : b^{*\mu} = e^{\theta^i \Omega_{(i)}(t,b)\frac{\partial X_i}{\partial b^{\beta}}\frac{\partial}{\partial b^{\alpha}}} b^{\mu}, \qquad (4.20)$$

$$G^*(3.1) \not\simeq G(3.1) \not\simeq \hat{G}(3.1),$$

and the Lie-admissible formulations (representation theory, etc.).

(C') COVERING GEOMETRICAL FORMULATIONS, essentially consisting of the characterization of the nonconservative vector fields as globally Hamiltonian-admissible

$$\hat{\dot{\Xi}} \boxtimes \hat{S}_2 = S_{\mu\nu} \hat{\dot{\Xi}}^{\nu} \, d b^{\mu} = \dot{\Xi}_{\mu} \, d b^{\mu} = d H_{tot} = \frac{\partial H_{tot}}{\partial b^{\mu}} \, d b^{\mu}, \qquad (4.21)$$

with respect to the general, fundamental, symplectic-admissible structure for the autonomous case (or contact-admissible structure for the nonautonomous case)

$$\hat{S}_2 = \frac{1}{2}\left(S_{\mu\nu} + S_{\nu\mu}\right) d b^{\mu} \times d b^{\nu} + \frac{1}{2}\left(S_{\mu\nu} - S_{\nu\mu}\right) d b^{\mu} \wedge d b^{\nu}, \qquad (4.22)$$

the characterization of the physical nonconservation laws via the Lie-admissible derivative (for autonomous generators)

$$\hat{\mathcal{L}}_{\dot{\Xi}} X_i \neq 0, \qquad (4.23)$$

and the use of the symplectic-admissible (or contact-admissible) geometry.

A few comments are here in order. Let me recall that one of the first meanings of the terms "relativity" is that of referring to a form-invariant description of physical reality. And indeed, one of the crucial properties of the Galilei relativity within its "arena of unequivocal applicability" is that of the form-invariance of the equations of motion under the Galilei transformations,

$$G(3.1): \left[\mu_k \ddot{r}_{ka} - f_{ka}(r_{\mu})\right]_{SA}^{C^{\infty},R} = 0 \rightarrow \left[\mu_k \ddot{r}'_{ka} - f_{ka}(r'_{\mu})\right]_{SA}^{C^{\infty},R} = 0. \qquad (4.24)$$

As by now familiar, this property of the Galilei relativity fails to apply for the considered class of broader systems which now are form-noninvariant under the Galilei transformations

$$G(3.1): \left\{\left[\mu_k \ddot{r}_{ka} - f_{ka}(r_{\mu})\right]_{SA}^{C^{\infty},R} - F_{ka}(t, r_{\mu}, \dot{r}_{\mu})\right\}_{NSA}^{C^{\infty},R} = 0 \rightarrow \left\{\left[\mu_k \ddot{r}'_{ka} - f_{ka}(r'_{\mu})\right]_{SA}^{C^{\infty},R} - F'_{ka}(t', r'_{\mu}, \dot{r}'_{\mu})\right\}_{NSA}^{C^{\infty},R} = 0. \qquad (4.25)$$

This is, in essence, one of the arguments for the need of reinspecting the problem of the relativity laws in Newtonian Mechanics.

Neverthless, the need of a form-invariant description of physical reality persists for any possible relativity. But the symplectic-admissible contration of the contravariant nonconservative vector field yields the covariant conservative form of the original system, Eq. (4.21). This suggests the following

SUBCONJECTURE 4.1.A: The Galilei-admissible transformations (4.19) leave form-invariant nonconservative, nonlinear, explicitly time dependent equations of motion

$$\hat{G}(3.1): \left\{\left[\mu_k \ddot{r}_{ka} - f_{ka}(r_{\mu})\right]_{SA}^{C^{\infty},R} - F_{ka}(t, r_{\mu}, \dot{r}_{\mu})\right\}_{NSA}^{C^{\infty},R} = 0 \rightarrow \left\{\left[\mu_k \ddot{r}'_{ka} - f_{ka}(r'_{\mu})\right]_{SA}^{C^{\infty},R} - F_{ka}(t', r'_{\mu}, \dot{r}'_{\mu})\right\}_{NSA}^{C^{\infty},R} = 0. \qquad (4.26)$$

It is here essential to assess the plausibility of this subconjecture with explicit examples. In turn, these examples will be useful for the subsequent considerations of this section. In order not to obfuscate the primitive concepts with unnecessarily complex algorithms, I shall consider some of the simplest possible examples.

Consider the case of the free, one dimensional motion (in vacuum). Its contravariant normal form (4.2) is given by

$$\left(\dot{b}^{\mu}\right) - \left(\dot{\Xi}^{\mu}\right) = \begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} - \begin{pmatrix} p \\ 0 \end{pmatrix} = 0, \quad m = 1. \qquad (4.27)$$

The canonical formulations of the group of translations in time, the $T_1(t)$ subgroup of group (4.8), explicitly reads

$$r' = r + \frac{t_0}{1!}[r, H] + \frac{t_0^2}{2!}[[r, H], H] + \cdots, \qquad H = \frac{1}{2} p^2, \qquad (4.28)$$
$$p' = p + \frac{t_0}{1!}[p, H] + \frac{t_0^2}{2!}[[p, H], H] + \cdots,$$

yielding the canonical version of the translations in time

$$T_1(t): \begin{cases} r' = r + t_0 p, & (4.29a) \\ p' = p, & (4.29b) \end{cases}$$

where now r and p are (costant) initial values. The derivative with respect to (the new) time t_0 then establishes the form-invariance of Eqs. (4.27) under transformations (4.29) because, trivially in this case,

$$\begin{cases} \dot{r}' = p = p' \\ \dot{p}' = 0. \end{cases} \qquad (4.30)$$

We now break the $T_1(t)$ symmetry of Eqs. (4.27) via the motion in a physical medium which results in dissipative forces and no force derivable from a potential (under the assumption that Eqs. (4.2) are the maximal associated selfadjoint subsystem of the more general nonconservative system). This latter point is mainly due to the advantage of avoiding the redefinition of the Hamiltonian and can be easily disposed of.

A significant point is the nature of the breaking of the $T_1(t)$ symmetry. As indicated in Table 2.14, the essentially nonselfadjoint breaking is in this case unrealizable owing to the insufficient dimensionality. We then remain with the canonical and semicanonical breakings. We clearly select the canonical breaking because it implies an explicit dependence of the dissipative force in time. The selected nonselfadjoint extension of Eqs. (4.27) reads

$$\left(\dot{b}^\mu\right) - \left(\hat{\dot{=}}^\mu\right) = \left(\dot{b}^\mu\right) - \left(\dot{=}^\mu\right) - \left(F^\mu\right) = \begin{pmatrix}\dot{r}\\ \dot{p}\end{pmatrix} - \begin{pmatrix}p\\ 0\end{pmatrix} - \begin{pmatrix}0\\ \gamma t p\end{pmatrix} = \begin{pmatrix}\dot{r} - p\\ \dot{p} + \gamma t p\end{pmatrix} = 0, \tag{4.31}$$

with a manifest breaking of the Galilei subsymmetry under translations in time.

Now we put the Lie-admissible formulations at work. First, we must construct a representation of Eqs. (4.31) in terms of the Hamilton-admissible equations via Theorem 3.4.1. This is easily accomplished by solving Eqs. (3.4.4). The desired representation can be written

$$\frac{\partial R_\mu}{\partial b^\nu}\dot{b}^\nu - \frac{\partial H}{\partial b^\mu} = 0, \quad \{R_\mu\} = \{(-p - \gamma t r), r\}, \quad H = \frac{1}{2}p^2, \tag{4.32a}$$

$$\left(S_{\mu\nu}\right) = \left(\frac{\partial R_\mu}{\partial b^\nu}\right) = \begin{pmatrix}-\gamma t & -1\\ 1 & 0\end{pmatrix}. \tag{4.32b}$$

Although not essential, it is instructive to verify that the inner-admissible product (4.21) does indeed reproduce the covariant form of the original conservative system. And indeed, we simply have in this case

$$\left(S_{\mu\nu}\hat{\dot{=}}^\nu\right) = \begin{pmatrix}-\gamma t & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}p\\ -\gamma t p\end{pmatrix} = \begin{pmatrix}0\\ p\end{pmatrix} = \left(\dot{=}_\mu\right) = \left(\frac{\partial H}{\partial b^\mu}\right). \tag{4.33}$$

This gives hope of identifying covering transformations via a Lie-admissible embedding of the Galilei transformation (the Hamiltonian H of Eqs. (4.32) remains fully invariant under Galilei transformations).

It is here simple to see whether this is indeed the case. The Galilei-admissible covering of transformations (4.29), from Eqs. (4.19), are given by

$$\begin{cases} \hat{r} = r + \frac{t_0}{1!}(r, H) + \frac{t_0^2}{2!}((r, H), H) + \cdots, & (4.34a)\\ \hat{p} = p + \frac{t_0}{1!}(p, H) + \frac{t_0^2}{2!}((p, H), H) + \cdots, & (4.34b)\end{cases}$$

where now, of course, the expansion is in terms of the Lie-admissible product

$$(A, B) = \frac{\partial A}{\partial b^\mu} S^{\mu\nu} \frac{\partial B}{\partial b^\nu} = \frac{\partial A}{\partial r}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial r} - \gamma t \frac{\partial A}{\partial p}\frac{\partial B}{\partial p}. \tag{4.35}$$

Simple calculations then yield the finite, connected, one-parameter transformations

$$\hat{T}_1(t): \begin{cases} \hat{r} = r - \frac{p}{\gamma t}\left(e^{-\gamma t t_0} - 1\right), & (4.36a)\\ \hat{p} = p\, e^{-\gamma t t_0}, & (4.36b)\end{cases}$$

which constitute <u>an example of a Lie-admissible group</u> (Definition 3.6.3) because they are constructed in terms of the same base manifold (b^μ), the same generator (H) and the same parameter (t_0) of the Galilei subgroup $T_1(t)$.

By performing the derivative with respect to time in exactly the same way as per Eqs. (4.30), we have

$$\begin{cases} \dot{\hat{r}} = p\, e^{-\gamma t t_0} = \hat{p}, & (4.37a)\\ \dot{\hat{p}} = -\gamma t p\, e^{-\gamma t t_0} = -\gamma t \hat{p}. & (4.37b)\end{cases}$$

Thus, <u>the Galilei-admissible covering $\hat{T}_1(t)$ of the Galilei group of translations in time $T_1(t)$ leaves form-invariant the explicitly time-dependent equations of motion (4.31)</u>.

The connectivity of transformations (4.36) is selfevident. We then remain with the uncompromisable consistency requirement that the covering transformations (4.36) must recover the Galilei transformations (4.29) identically at the limit of null symmetry breaking forces. This is easily established by the property

$$\lim_{\gamma \to 0} \hat{T}_1(t) = \left\{\begin{matrix}\hat{r} = r + t_0 p\\ \hat{p} = p\end{matrix}\right\} \equiv T_1(t). \tag{4.38}$$

As a result, the Galilei-admissible transformations (4.36) do constitute a consistent covering of the Galilei transformations (4.29) for the form-invariance of the system considered.

Eqs. (4.31) also provide a canonical breaking of the symmetry under the Galilei boosts. It is therefore instructive to study this case too. First, to avoid possible misrepresentations, we must reinspect the case of the exact symmetry, that is, Eqs. (4.27). This is symply done by computing Eqs. (4.8) for the generator G = r -tp (m = 1)

$$r' = r + \frac{p_0}{1!}[r, G] + \frac{p_0^2}{2!}[[r, G], G] + \cdots , \qquad (4.39)$$
$$p' = p + \frac{p_0}{1!}[p, G] + \frac{p_0^2}{2!}[[p, G], G] + \cdots ,$$

yielding the one-parameter group of Galilei boosts

$$G_1(v) : \begin{cases} r' = r - p_0 t , \\ p' = p - p_0 , \end{cases} \qquad p_0 = v_0 . \qquad (4.40)$$

The derivative in time (r now varies) yields

$$\begin{cases} \dot{r}' = \dot{r} - p_0 = p - p_0 = p' , \\ \dot{p}' = \dot{p} = 0 , \end{cases} \qquad (4.41)$$

by therefore establishing the form-invariance of Eqs. (4.27) under the group $G_1(v)$.

In the transition to the nonconservative extension (4.31) this symmetry is manifestly broken. We therefore again put the Lie-admissible formulations at work. An additional technical point however must now be taken into account. It is constitutedb by the fact that the Lie-admissible tensor $S^{\mu\nu}_{(i)}$ generally varies with the generators. Since the generator is now (r-tp), a new Lie-admissible tensor must be computed. With an understanding that the techniques for this computation are highly rudimentary at this point, a solution is given by

$$S^{\mu\nu}_{(G)} = \begin{pmatrix} 0 & \frac{\alpha}{t} \\ \beta & 0 \end{pmatrix} , \quad \alpha = \int_0^t \beta(t)\, dt , \quad \beta = e^{-\frac{1}{2}\gamma t^2} , \qquad (4.42a)$$

$$\hat{G}_1(v) : \begin{cases} \hat{r} = r - p_0 \alpha , \\ \hat{p} = p - p_0 \beta , \end{cases} \quad \lim_{\gamma \to 0} \hat{G}_1(v) \equiv G_1(v) , \qquad (4.42b)$$

$$\begin{cases} \dot{\hat{r}} = \dot{r} - p_0 \dot{\alpha} = p - p_0 \beta = \hat{p} , \\ \dot{\hat{p}} = \dot{p} - p_0 \dot{\beta} = -\gamma t (p - p_0 \beta) = -\gamma t \hat{p} . \end{cases} \qquad (4.42c)$$

This example is indicative because it presents a transcendental function, from Eqs. (4.42a), in the transformation law. This indicates that the Galilei-admissible transformations can be rather involved even for simple systems. This was, after all, expected, because it is a feature common to all nonmanifest symmetries, while the covering Galilei transformations which are needed for the form-invariance of the systems considered must be nonmanifest by central requirement.

Eqs. (4.31) also provide a semicanonical breaking of the group of translations in space. The equations of motion are form-invariant under these transformations, but this symmetry, by no means, is in this case representative of the conservation of the linear momentum. The study of this case is left to the interested reader. We hope to treat in a separate study the case of the Lie-admissible covering of the group of rotations.

For subsequent needs, the reader should keep in mind that all our efforts have been centered in attempting the construction of covering transformations, that is, transformations which apply to a broader physical context (form-invariance of nonconservative systems) while are capable of recovering the conventional Galilei transformations identically at the limit of null Galilei relativity breaking forces (this is the aspect which we have referred to in Section 3 as our uncompromisable condition of compatibility). Thus, the Galilei relativity is not "destroyed" but simply embedded in a broader context. As a matter of fact, the Galilei-admissible relativity, as presented in Conjecture 4.1, cannot even be constructed without the use of the conventional Galilei relativity as a foundation.

Despite this compatibility of the Galilei relativity with its conjectured Lie-admissible covering, the latter relativity appears to produce a rather profound conceptual departure from the former relativity, as we indicate below.

As is well known, one of the conceptual foundations of the Galilei (as well as Einstein) relativity is the lack of existence of a previledged frame of reference. But this relativity refers to the motion of particles in vacuum with action-at-a-distance forces derivable from a potential. The physical context we are here considering is profoundly different than that. In particular, one of the conceptual foundations of the theory of nonconservative systems is the existence of a medium which is responsible for the energy dissipation. Thus, in the transition from the Galilei relativity to the conjectured Galilei-admissible covering, the conceptual profile is shifted from motion-in-vacuum with action-at-a-distance forces to motion-in-a-physical medium with action-at-a-distance and contact forces.

Our problem is to see whether the lack of existence of a privileged frame can be preserved within such a broader setting. It is at this point where the joint study of the Newtonian framework

under consideration and its possible relativistic extension is effective because difficulties of the former can be magnified in such a relativistic transition. The results of my efforts in this issue can be presented with the following

SUBCONJECTURE 4.1.B: The Galilei-admissible relativity characterizes one class of reference frames, those at rest with respect to the medium in which the motion occurs, as priviledged with respect to any other frame.

Consider the motion of a particle in a medium, say, our earthly atmosphere. The "natural" reference frame which is customarily used in practical measurements is that which is at rest with respect to the medium. The term "priviledged" in Subconjecture 4.1.B is intended to express the fact that (a) the equations of motion are not form-invariant under ordinary Galilei transformations to other admissible frames, (b) the frames induced by Galilei-admissible transformations are not expected to be practically realizable with experimental set ups (see below), and, thus, (c) the reference frame at rest with respect to the medium possesses a unique character for direct experimental measurements.

Admittedly, the indicated conjecture appears not entirely justified at a Newtonian level, owing to extended practice of use of velocity transformations. It is therefore of some possible significance to indicate the corresponding occurrence at a field theoretical level. The field equations are now of the type (for the second-order case)

$$\left\{ \left[(\Box + m^2)\varphi(x) - j^\mu(\varphi, \partial\varphi/\partial x^\alpha) \right]_{SA}^{C^\infty, R} - J^\mu(x^\alpha, \varphi, \partial\varphi/\partial x^\alpha) \right\}_{NSA}^{C^\infty, R} = 0, \quad \alpha, \mu = 0,1,2,3 \qquad (4.43)$$

and are subjected to the following interpretation: (I) Eqs. (4.43) are assumed as constituting an approximation of the motion of a particle (the field φ) within a hadronic medium (a hadron or a neutron star); (II) the variables x^α of Eqs. (4.43) are the coordinates of a Minkowski frame whose space components are at rest with the hadronic medium, and (III) strong interactions are given by couplings derivable as well as not derivable from a Lagrangian density, the latter being representative of the the motion of a finite, nonull, charge volume within the medium considered (see also the remarks at the end of Table 3.9). These aspects are discussed in details in ref.[33c]. At this point I would like simply to indicate that, on grounds of my current knowledge, I am unable to compute Eqs. (4.43) in any other frame related to x by the conventional Poincaré transformations $x' = \Lambda x + a$, because the field φ does not (necessarily) transform covariantly under the Poincaré group, as indicated in ref.[33b] with a linearization process. As we shall see in more details in ref.[33c], the major technical difficulties appear to be related to the Lorentz boosts which are precisely the relativistic extensions of the Galilean velocity transformations.

Another crucial aspect of the problem of the relativity laws of nonconservative mechanics is that related to inertial characterizations. The Galilei relativity, within its own arena of applicability, does indeed provide an inertial characterization of physical reality in the sense that it is applicable to the reference frame used in experiments and conventionally assumed as inertial, while all other frames induced by the family of Galilei transformations preserve such inertial character. As a matter of fact, the lack of existence of a priviledged reference frame is a consequence of these properties and of the form-invariance of the equations of motion.

In the transition to nonconservative systems the situation appears to be considerably different. First of all, the medium in which the motion occurs is not, in general, in inertial conditions. This is typically the case for the earthly atmosphere (or a hadronic medium). Also, the transformed reference frame under the broader relativity group which leaves form-invariant the nonconservative equations of motion is expected to be generally noninertial irrespective of whether the original frame was inertial or not. This is due, for instance, to the functional dependence of the variables $\hat{b}^\mu$ in Eqs. (4.19) on the old variables b^μ; the generally nonlinear nature of the representations of the nonassociative envelope $\mathcal{U}(G(3.1))$; the generally nongeodesic character of the motion, etc.

This situation (which appears to be, again, better focused when studying motion of hadronic constituents under the assumption that they are physical particles-that is, non-point-like[5, 33c]) suggests the following

SUBCONJECTURE 4.1.C: The Galilei-admissible relativity provides a generally noninertial characterization of nonconservative and Galilei form non-invariant systems.

Here the term "noninertial" is referred to the character of both, the original and the transformed systems under the Galilei-admissible transformations.

In essence, the inertial frames of the Galilei (and Einstein) relativity are a conceptual abstraction because no experiment in an inertial frame has been actually conducted to date and it will not be conducted until a sophisticated interplanetary (or interstellar) technology is available. Thus, by looking in retrospective, my efforts were aimed at the construction of a possible relativity which is noninertial by central conception and, thus, usable in actual Earthly experiments, while admitting the conventional inertial formulation at the limit of null dissipative medium.

Almost needless to say, the number of problems I am here leaving open is too large to suggest an outline. In any case, they will not escape to the attentif reader. Some of the open problems deserve a special mention.

(A) Scalar extensions. Undoutedly, a more rigorous treatment of any applicable relativity for nonconservative systems will necessarily demand the use of the scalar extension of the Galilei group.

(B) Integrability conditions. The conditions under which exponentials (4.19) exist demand a specific study.

(C) Desired "relativity". There exists an epistemological aspect which also deserves attention. Owing to the unexplored and unresolved nature of the topic, different researchers may have basically different objectives which, in turn, could imply basically different meanings of the word "relativity". Some researcher may demand so stringent conditions that no "relativity" is admitted for nonconservative systems. Other may demand so few conditions that the emerging "relativity" has no physical effectiveness comparable to that of the Galilei relativity. Yet other researchers may reject any final, future "relativity" because excessively different than that characterized by decades of familiar use, the conventional "Galilei relativity". This is not a mere question of semantic. Instead, it is additional indication of the fact that the problem of the relativity laws of Newtonian mechanics is still unresolved as of today. The problem of the identification of the applicable relativity for physical systems more general than the conservative and Galilei form-invariant ones, however, persists. The mental attitude which is recommended is that, in any case, the researcher should expect a prodound departure from conventional relativity ideas because of the profound physical departure from conservative settings represented by nonconservative mechanics. The epistemological attitude which is suggested to avoid an unecessary controversy is the identification of the used term "relativity" in a way as precise as possible. The technical attitude which is advocated is to put primary emphasis on the methodological tools for the study of forces (or currents) not derivable from a potential. The emerging relativity is then conceivably sequential.

Within the context of this paper the term "relativity" is referred to a "form-invariant, noninertial characterization of local, class C^{∞}, regular, unconstrained Newtonian systems with arbitrary forces via a Lie-admissible covering of the Galilei relativity, within the reference frame at rest with respect to the medium in which the motion occurs." This is equivalent to saying the the term "relativity" of this section is defined by Conjecture 4.1 and its subconjectures.

In closing, permit me to emphasize the use of the terms CONJECTURE and SUBCONJECTURES in the presentation of my relativity efforts. With this, I intend to stress the fact that the verification of the validity, invalidity or need of implementations of my studies is entirely left to interested, independent researchers.

Let me also indicate that, when the relativity which is applicable to systems (4.12) is finally identified (irrespective of whether it will be of Lie-admissible type or not), by no means should this broader relativity be considered as the "terminal relativity" of Newtonian Mechanics. For instance, my studies exclude the case of nonlocal forces (even though these forces possess some degree of implementation in Lie-admissible formulations via solution of the crucial Equations(3.4.4) of integrodifferential nature). This is a first illustration of my belief indicated in Section I according to which Theoretical Physics will never admit terminal disciplines.

5: CONCLUDING REMARKS ON THE CURRENT STATUS OF RELATIVITY IDEAS.

It might be of some significance to biefly touch on the potential implications in non-Newtonian frameworks of possible relativity implementations at the Newtonian level. This problem is opened by the truly elemental nature of Newtonian Mechanics for theoretical physics.

Let me first indicate one "arena of unequivocal validity" of the Galilei relativity, the Einstein special relativity and the general theory of gravitation and then pass to the speculative part of the issue.

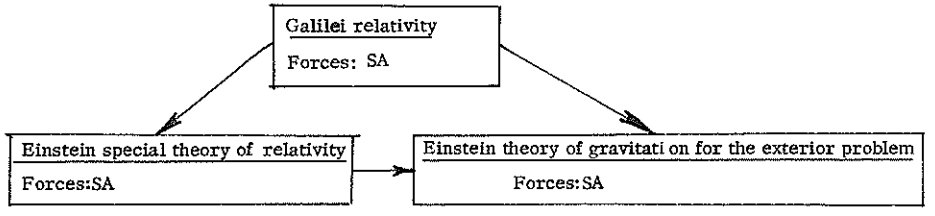
The validity of the Galilei relativity (as currently known) for the motion in vacuum of particles under forces derivable from a potential and form-invariant under the Galilei transformations is simply unequivocal. Thus, the Galilei relativity, within such an arena, can indeed be qualified as constituting a scientific truth.

Neverthless the limitations of the Galilei relativity for other arenas of physical systems was known since the earlier stages of the electromagnetic theory. These limitations motivated the conception of a covering relativity for the electromagnetic interactions. The validity of the Einstein special relativity for the relativistic motion of charged particles in vacuum under ut most electromagnetic interactions is also unequivocal. Thus, the Einstein relativity too, within its own arena, can be qualified as constituting a scientific truth.

In turn, the special theory of relativity was known to possess limitations at the very time of its inception. These limitations motivated the conception of a still broader relativity for the inclusion of the gravitational phenomenology. The validity of the Einstein general theory of relativity for the exterior problem appears to be also established on rather solid experimental grounds. Thus, the relativity indicated, in its own arena, can also be qualified as constituting a scientific truth.

Intriguingly, the indicated three arenas of applicability of the respective chain of coverings appear to admit a unified characterization within the context of the Inverse Problem in Euclidean space, Minkonski space and Riemannian space, respectively. In essence, all admitted models are derivable from a variational principle in the respective carrier space. This implies that the models are variationally selfadjoint. A closer inspection then indicates that the admitted forces are variationally selfadjoint. This is typical the case of the electromagnetic force in Newtonian mechanics, special and general theory of relativity.

Pending independent verification by interested researchers, in the diagram below I attempt the characterization of one arena of unequivocal applicability of the relativities considered via the variational selfadjointness of the admitted forces in the underlying carrier space. The diagram is intended as a complement of the diagram of Section 1.



But this implies that all admitted forces have the primitive Newtonian form

$$\underset{\sim}{f} = -\frac{\partial U}{\partial \underset{\sim}{r}} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\underset{\sim}{r}}} . \qquad (5.1)$$

It is known that these forces, by no means, exhaust the forces of physical reality. This creates the problem of the applicable relativities for broader, or better, the broadest conceivable forces.

Before presenting few conjectural remarks in this respect, let me recall that theoretical physics is a science with an absolute standard of value: the physical reality. Until theoretical ideas have been experimentally proved in unequivocal terms, they constitute conjectures, not scientific truths. This is not intended to diminish the value of conjectures for the nowadays established scientific process (presentation of ideas, critical inspections by independent researchers, and experimental verification). Neverthless, too often in the history of physics the behaviour of originators of new insights has been genuinely scientific because critical of experimentally unverified knowledge, while the behaviour of their followers has been strictly antiscientific because inspired by an unlimited belief of unlimited applications. This is not the place to recall the historical inapplicability of previously established knowledge for the problem of the atomic structure or the more recent, but equally historical, discovery of parity violation.

With an open mind on these issues and with a firm belief of the limitations of our knowledge as compared to the complexities of the physical universe, let me pass to the speculative comments.

The central objective of this paper was to indicate that the problem of the relativity laws of the nonrelativistic motion of particles is still open as of today on theoretical grounds.[52a] This is so irrespective of my personal, conjectural efforts. Notice that the experimental aspect has been excluded in the above statement because established by centuries of knowledge on Newtonian forces.

In the transition to the case of the relativistic motion, the situation becomes considerably more nebulous, delicate and insidious. The unequivocal validity of the Einstein special relativity for the motion of hadrons under electromagnetic interactions is, by no means, evidence of the validity of the same relativity for the hadronic constituents. It is true that the virtual totality of our theoretical knowledge of hadron physics is based on the Einstein special relativity. But this,

besides indicating undeniable plausibility and scientific values, strictly speaking, does not constitute evidence of the validity of the special theory of relativity for the hadronic structure. Again, Einstein' special relativity can be claimed as constituting a scientific truth within a hadron only when experimentally proved in unequivocal terms. Lacking this verification, the relativity considered in the arena considered is only a conjecture.

In the recent paper[33b], R. M. SANTILLI has proposed the experimental verification of the validity or invalidity of Einstein' special relativity for the hadronic constituents. The epistemological argument is essentially the following. If the strong hadronic forces are analytically equivalent to the electromagnetic forces, that is, derivable from a potential (in the sense, e.g., of refs.[35]), then Einstein's relativity is "expected" to apply. However, if the strong hadronic forces are structurally nonequivalent to the electromagnetic forces, that is, not derivable from a potential (in the sense of ref.[5]), then Einstein' special relativity is "expected" to be invalid. It should be stressed that the term "expected" has exactly the same implications in both occurrences. In the former the relativity could, in the final analysis, be violated for reasons unrelated to the structure of the acting forces and unknown at this time. In the latter the relativity could, instead, apply irrespective of the nature of the acting forces. The net effect is that the issue considered does not appear to be resolvable at the theoretical level only. Intriguingly, the problem of the relativity laws for the hadronic constituents appears to be linked to that of the nature of the strong hadronic forces.[5, 33b]

As an incidental remark, forces not derivable from a potential can be extended to a relativistic context in more than one way. This opens two possibilities of studies which are opposite in conceptual attitude: the compatibility and the incompatibility of Einstein' special relativity for forces not derivable from a potential. Clearly, both possibilities must be studied and subjected to a comparative confrontation with physical reality. As it was the case for the Galilei relativity, the studies of a possible compatibility of the relativity considered for the forces considered will be left to the interested researchers. In the forthcoming paper[33c] I present few conjectural arguments related to the opposite line of studies. In any case, the truly fundamental aspect appears to be of Newtonian, rather than relativistic nature. And indeed, if a covering of the Galilei relativity will result to be needed at the Newtonian level, this will conceivably imply a subsequent, necessary modification of Einstein' special relativity (e.g., the reader is urged to verify that Einstein' special relativity is incompatible with a possible Galilei-admissible relativity because the first of strict Lie algebraic character while the second of strict non-Lie algebraic character).

In conclusion, what we can state at this moment on grounds of necessary scientific caution is that the problem of the relativity laws of the hadronic constituents is open on both theoretical and experimental grounds.[52a] For a more detailed study of this occurrence (as well as of the spirit of an open, scientifically productive debate for which it is intended) see also ref.[33c].

The last issue, the problem of the Einstein general theory of relativity for the interior problem, is even more delicate and more likely subjectable to opposite personal viewpoints. On experimental grounds one could attempt a semplistic resolution of the issue by saying that all available experimental verifications of the general theory of relativity are for the exterior problem because no clear experimental test exists for, say, the interior of a star. The net effect is that the validity of the theory for the exterior problem, by no means, should be considered as evidence of the validity of the corresponding theory for the interior problem.

Here a subtle but potentially significant parallelism with the hadronic case occurs. By ignoring gravitational considerations, the validity of the special theory of relativity for the "exterior behaviour" of hadrons under (ut most) electromagnetic interactions is established on solid experimental grounds, while the validity of the same relativity for the "interior problem" of the hadrons, that is, the structure problem, is not established and, as a matter of fact, questionable. With the inclusion of gravitational considerations the situation becomes considerably more involved on technical grounds, but conceptually equivalent. The geodesic behaviour of test particles in the Riemannian characterization of the exterior problem of, say, a start, appears to be established on solid grounds. In the transition to the interior problem the situation is different and opposite attitudes can be, again, implemented. The first attitude is that of attempting the compatibility of Einstein' relativity ideas for the interior problem with possible generalized forces which are conceivable for the hadronic structure. This line of study is, of course, valuable and recommendable. An opposite line of study is instead attempted in refs.[5]. In essence, the forces not derivable from a potential, upon implementation into a gravitational context, do not appear to be necessarily compatible with Einstein' general theory for the interior problem (only) on numerous technical and conceptual grounds, such as, the emerging equations of motion for the interior problem are nonderivable from a variational principle by central assumption, there is the lack of curvature as geodesic deviation, there is the lack of conservation laws, etc. To account for the available experimental evidence, the gravitational model which is attempted is that based on Einstein's equations for the exterior problem, but interpreted as subsidiary constraints to a more general, nongeodesic, non-Riemannian model for the interior problem.

In conclusion, what we can state at this moment on grounds, again, of scientific caution, is that the interior problem of the theory of gravitation is open on both theoretical and experimental grounds.[5a]

My personal belief is that the problem of the structure of the hadrons is of a complexity beyond our most vivid imagination, the latter being that materializable in terms of our knowledge

on the relatively simpler atomic and nuclear structures. In turn, the interior problem of the theory of gravitation is of relatively much greater complexity because clearly inclusive of the problem of the hadronic structure with additional gravitational considerations. Once mass terms in gravitational equations are recognized as technical expedients to overcome our ignorance on the structure problem, the complexity of the interior problem appears in the proper light. In the language of ref.[5] it is the problem of the "origin" of the gravitational field.

Therefore, our current knowledge on the relativity laws of the physical universe, rather than having reached a terminal stage, appears to be potentially open to new, intriguing horizons.

In closing, let me stress that this is only my personal viewpoint and, as such, it should in no way be related to the current viewpoint of the physics community as a whole. Also, permit me to stress the spirit with which this viewpoint is presented. It is essentially based on the hope of stimulating a scientifically productive debate in the traditional spirit of unsolved physical problems. For instance, the hope of the forthcoming paper[33c] is that of stimulating the awareness of the physics community on the need of confronting the problem of the relativity laws of the hadronic constituents, which is nowadays virtually ignored in available literatures, to the best of my knowledge. Hopefully, this awareness, will stimulate a possible future resolution. For instance (and this is the essence of my speculative remarks) it is conceivable that a simple reinspection of available experimental data could establish the validity of Einstein special relativity within a hadron in the needed incontrovertible form. The point is that (and this is the spirit of my personal efforts) this job must be explicitly done because the respect for Einstein special relativity which is so rooted in all of us is not sufficient alone (that is, without explicit, direct and incontrovertible experimental evidence) to render the relativity considered in the arena considered a scientific truth. The aspect which is uncompromisable is the "incontrovertible" nature of the proof prior to claiming a final resolution of the issue. After all, irrespective of my personal efforts on the nature of the strong hadronic forces, massive objects are basically an aggregate of hadrons. But then one could expect a violation of the special relativity within a hadron (or within the core of, say, a neutron star) in favor of a more general possible relativity inclusive of gravitation.

Almost needless to say, in the traditional spirit of scientifically productive debates, the presentation of different, complementary or opposite viewpoints is not only recommended, but actually encouraged.

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Almost needless to say, I am solely responsible for the content of this paper also in view of the so many modifications implemented in the final version.

NOTE ADDED IN PROOF. It is a pleasure to report that, just while releasing this manuscript for printing, I received copy of the paper

CLASSICAL ALGEBRAIC CHOMODYNAMICS, by S. L. ADLER,
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which, in addition to other contributions, presents (apparently for the first time) a Lie-admissible, color, charge algebra.

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Addendum to: On a possible Lie-admissible covering of the Galilei relativity in Newtonian Mechanics for nonconservative and Galilei form-noninvariant systems

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Abstract

In this note I first present a needed clarification on the realization of Lie-admissible algebras for nonconservative Newtonian systems considered in a recent paper. The following aspects are then treated. (A) I introduce a new realization which is directly applicable to all nonconservative systems of the admitted class (local, class C^∞ and regular) via the solution of simple algebraic systems. (B) I indicate that such a realization yields compatible generalized formulations of Hamilton-admissible, Lie-admissible and symplectic-admissible type. (C) I show in details that such a realization is applicable to the construction of the Lie-admissible generalization of the Galilei relativity I proposed for nonconservative systems, and I work out a number of examples. (D) I then prove that, for the subclass of autonomous systems, the time-component of the proposed generalized relativity is indeed universal, that is, it provides the form-invariant description of all systems considered, while being able to recover the conventional time-component of the Galilei relativity at the limit of the null value of the Galilei symmetry-breaking forces. And, finally, (E) I prove that the dual Lie-admissible algebraic and symplectic-admissible geometrical character of the proposed generalized relativity is independent from the selected system of local coordinates, by therefore confirming the hope for a coordinate free-globalization of the symplectic-admissible geometry and relativity. This last objective is achieved by first recalling the property at the basis of the globalization of the symplectic geometry, namely, that symplectic forms preserve their symplectic character under arbitrary (class C^∞ and regular) transformations of the local coordinates. It is then proved that this property extends to the symplectic-admissible forms in its entirety, and admits a joint Lie-admissible algebraic image under a suitable restriction of the forms admitted. A number of open mathematical and physical problems are also identified.

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ADDENDUM TO THE PAPER:

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ON A POSSIBLE LIE-ADMISSIBLE COVERING OF THE GALILEI RELATIVITY IN NEWTONIAN MECHANICS FOR NONCONSERVATIVE AND GALILEI FORM-NON-INVARIANT SYSTEMS

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In the recent paper[1] I have attempted the construction of a conceivable generalization of the Galilei relativity for nonconservative and Galilei form-noninvariant Newtonian systems. The class of systems considered is given by all local[2], class C^{∞}, regular[3], Newtonian systems in a three-dimensional Euclidean space with local coordinates r^{ka}, $k=1,2,\ldots,n$, $a = x,y,z$. These systems are generally nonconservative, nonlinear (in the coordinates as well as the velocities) and explicitly dependent on time. In the contravariant vector field form, they can be written[4]

$$\dot{b}^{\mu} - \tilde{\dot{\underline{\ }}}^{\mu}(t,b) = 0, \quad \mu = 1,2,\ldots,6n, \tag{1a}$$

$$(b^{\mu}) = \begin{pmatrix} \underset{\sim}{r} \\ \underset{\sim}{p} \end{pmatrix}, \quad \left(\tilde{\dot{\underline{\ }}}^{\mu}\right) = \begin{pmatrix} \frac{1}{m}\underset{\sim}{p} \\ \hat{\underset{\sim}{f}}^{SA}(\underset{\sim}{r}) + \underset{\sim}{F}^{NSA}(t,\underset{\sim}{r},\underset{\sim}{p}) \end{pmatrix}, \tag{1b}$$

$$\tilde{\dot{\underline{\ }}}^{\mu} \overset{def}{=} \dot{\underline{\ }}^{\mu} + F^{\mu}, \quad \left(\dot{\underline{\ }}^{\mu}\right) = \begin{pmatrix} \frac{1}{m}\underset{\sim}{p} \\ \hat{\underset{\sim}{f}}^{SA} \end{pmatrix}, \quad (F^{\mu}) = \begin{pmatrix} 0 \\ \underset{\sim}{F}^{NSA} \end{pmatrix}, \tag{1c}$$

where $\underset{\sim}{f}^{SA}$ represents all forces which are form-invariant under the conventional Galilei transformations and verify the integrability

conditions for the existence of a potential (variationally self-adjoint forces); $\underset{\sim}{F}^{NSA}$ represents all forces which break the Galilei symmetry as well as, in general, the integrability conditions for the existence of a potential (variationally nonselfadjoint forces); and the $\underset{\sim}{p}$'s are the physical linear momenta ($m\dot{\underset{\sim}{r}}$).

By recalling that a central objective of any relativity is to provide a form-invariant description of nature, a primary objective of paper[1] was to attempt the construction of a 10-parameter group of transformations $\hat{G}(3.1)$ which leaves form-invariant systems (1a)

$$\hat{G}(3.1): \quad (t,b) \longrightarrow (\hat{t},\hat{b})_{(t,b)}, \tag{2a}$$

$$\frac{db^{\mu}}{dt} - \tilde{\dot{\underline{\ }}}^{\mu}(t,b) = 0 \longrightarrow \frac{d\hat{b}^{\mu}}{d\hat{t}} - \tilde{\dot{\underline{\ }}}^{\mu}(\hat{t},\hat{b}) = 0 \tag{2b}$$

while being jointly capable of recovering the conventional Galilei group $G(3.1)$ at the limit of null symmetry breaking forces

$$\lim_{F^{NSA} \to 0} \hat{G}(3.1) = G(3.1). \tag{3}$$

The transformations verifying these requirements were called in paper[1] Galilei-admissible transformations, in the sense that they admit the Galilei transformations under limit (3). The group $\hat{G}(3.1)$ was called a "covering of the Galilei group" not in a topological sense, but rather in the physical sense of providing the form-invariance for a broader class of physical systems, while being able to recover the latter group under limit (3). The relativity characterized by $\hat{G}(3.1)$, called Galilei-admissible relativity[1], was then interpreted as a covering of the conventional Galilei relativity.

The reader should recall that the systems considered, being generally nonconservative, nonlinear and explicitly time dependent, do not exibit, in general, manifest symmetries at all. Therefore, cases such as the simple identification of the manifest Galilei symmetry for the Kepler system in vacuum, do not generally occur within the framework considered. Instead, the symmetries providing form-invariance (2) are, in general, highly nonmanifest, that is, of rather involved functional dependence and, as such, of nontrivial identification.

This primary characteristic of nonconservative Newtonian systems created a dual methodological problem. First, there was the need of identifying methods for the explicit construction of nonmanifest symmetries of the equations of motion and, second, these methods must be such to identify that particular 10-parameter symmetry which verifies the crucial limit (3). The verification of this limit was clearly essential to achieve the intended notion of covering of the Galilei relativity, including the compatibility of the generalized with the conventional relativity.

After the study of a number of conceivable alternatives, the methods selected in paper[1] to achieve properties (2) and (3) were those of the so-called Lie-admissible approach to nonconservative systems, and consisted of three major, interrelated aspects: (A) an analytic, (B) an algebraic and (C) a geometrical aspect. These methodological profiles were implemented via the solution of the so-called fundamental system of the Lie-admissible formulations. It consists of the construction of a covariant tensor $S_{\mu\nu}$ such that

$$S_{\mu\nu}\,\tilde{\dot{x}}^{\nu} = \bar{\dot{x}}_{\mu}\,, \quad \det(S_{\mu\nu}) \neq 0, \qquad (4)$$

that is, such that its contraction with the nonconservative contravariant vector field $\tilde{\dot{x}}^{\mu}$ produces the conservative covariant vector field $\bar{\dot{x}}_{\mu}$ without the Galilei symmetry-breaking forces, under the following properties.

(A) Generalized analytic profile. The nonconservative equations of motion are correctly represented by the following (covariant and contravariant) generalization of Hamilton's equations

$$\left(S_{\mu\nu}\dot{b}^{\nu} - \frac{\partial H}{\partial b^{\mu}}\right)^{C^{\infty},R}_{NSA} = 0\,, \quad \dot{b}^{\mu} - S^{\mu\nu}\frac{\partial H}{\partial b^{\nu}} = 0\,, \qquad (5a)$$

$$H = T(\underset{\sim}{p}) + V(\underset{\sim}{r})\,, \quad \dot{\underset{\sim}{p}}^{SA} = -\frac{\partial V}{\partial \underset{\sim}{r}}\,, \quad (S^{\mu\nu}) = (S_{\mu\nu})^{-1}, \qquad (5b)$$

which are called Hamilton-admissible equations because capable of recovering, by construction, the conventional Hamilton's equations at the limit of null $\underset{\sim}{F}^{NSA}$ - forces

$$\lim_{\underset{\sim}{F}^{NSA}\to 0}\left(S_{\mu\nu}\dot{b}^{\nu} - \frac{\partial H}{\partial b^{\mu}}\right)^{C^{\infty},R}_{NSA} = \left(\omega_{\mu\nu}\dot{b}^{\nu} - \frac{\partial H}{\partial b^{\mu}}\right)^{C^{\infty},R}_{SA} \qquad (6a)$$

$$\lim_{\underset{\sim}{F}^{NSA}\to 0}\left(\dot{b}^{\mu} - S^{\mu\nu}\frac{\partial H}{\partial b^{\nu}}\right) = \dot{b}^{\mu} - \omega^{\mu\nu}\frac{\partial H}{\partial b^{\nu}} \qquad (6b)$$

$$(\omega_{\mu\nu}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (\omega^{\mu\nu})^{-1} \qquad (6c)$$

(B) Generalized algebraic profile. The generalized brackets of the time evolution law characterized by Eqs. (5)

$$\dot{A}(b) = \frac{\partial A}{\partial b^\mu}\,\dot{b}^\mu = \frac{\partial A}{\partial b^\mu}\,S^{\mu\nu}\,\frac{\partial H}{\partial b^\nu} \stackrel{\text{def}}{=} (A,H) \qquad (7)$$

constitute a (regular) realization in Newtonian Mechanics of the (non-associative, non-Lie) Lie-admissible algebras i.e., they verify the general Lie-admissible law

$$\begin{aligned}&((A,B),C)+((B,C),A)+((C,A),B)+(C,(B,A))+(B,(A,C))+(A,(C,B))\\ &=(A,(B,C))+(B,(C,A))+(C,(A,B))+((C,B),A)+((B,A),C)+((A,C),B),\end{aligned} \qquad (8)$$

or, equivalently (Theorem 3.3.1 of ref.[1]), the contravariant tensor $S^{\mu\nu}$ of Eq. (7) satisfies the system of partial differential equations

$$\begin{aligned}&(S^{\mu\rho}-S^{\rho\mu})\frac{\partial}{\partial b^\rho}(S^{\nu\tau}-S^{\tau\nu})\\ &+(S^{\nu\rho}-S^{\rho\nu})\frac{\partial}{\partial b^\rho}(S^{\tau\mu}-S^{\mu\tau})\\ &+(S^{\tau\rho}-S^{\rho\tau})\frac{\partial}{\partial b^\rho}(S^{\mu\nu}-S^{\nu\mu})=0.\end{aligned} \qquad (9)$$

The Lie-admissible algebras constitute a nontrivial generalization of the conventional Lie algebras, as costumarily used in conservative mechanics. In particular, they are "Lie-admissible" in a dual meaning, that is, they admit the Lie brackets first via the attached brackets

$$(A,H) - (H,A) = [A,H]^* = \qquad (10)$$

= GENERALIZED POISSON BRACKETS

and, second, via the limit brackets

$$\lim_{F^{NSA}\to 0} (A,H) = [A,H] = \qquad (11)$$

= CONVENTIONAL POISSON BRACKETS

Thus, Lie algebras constitute a particular case of the algebras characterized by law (8), that is, Lie algebras are Lie-admissible (but the converse is not necessarily true).

(C) Generalized geometrical profile. The (neither antisymmetric nor symmetric) covariant tensor $S_{\mu\nu}$ characterizes the symplectic-admissible forms (or structures)[5]

$$S_2 = S_{\mu\nu}\, db^\mu \otimes db^\nu \qquad (12)$$

$$= \frac{1}{2}(S_{\mu\nu}-S_{\nu\mu})\, db^\mu \wedge db^\nu + \frac{1}{2}(S_{\mu\nu}+S_{\nu\mu})\, db^\mu \times db^\nu$$

i.e., it is such that the attached tensor

$$S_{\mu\nu} - S_{\mu\nu} = \Omega_{\mu\nu}, \qquad (13)$$

verifies the necessary and sufficient conditions

$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0, \qquad (14a)$$

$$\frac{\partial\Omega_{\mu\nu}}{\partial b^\tau} + \frac{\partial\Omega_{\nu\tau}}{\partial b^\mu} + \frac{\partial\Omega_{\tau\mu}}{\partial b^\nu} = 0, \qquad (14b)$$

for the (nondegenerate) exterior, two-form

$$\Omega_2 = \Omega_{\mu\nu}\, db^\mu \wedge db^\nu \qquad (15)$$

to be symplectic (i.e., closed). More generally, it is expected that the applicable geometry is the so-called <u>symplectic-admissible geometry</u>[1], i.e., the geometry of the (nondegenerate) symplectic-admissible structures (12) on a (Hausdorff, second-countable, ∞-differentiable, 6n-dimensional) manifold. In analogy with Lie-admissible algebras, this geometry is "symplectic-admissible" in a dual meaning, that is, it admits the symplectic geometry first in the sense that the <u>attached exterior form</u>

$$S_2 - S_2^T = \Omega_2 = \underline{\text{ARBITRARY SYMPLECTIC FORM}} \qquad (16)$$
$$S_2 = S_{\mu\nu}\, db^\mu \otimes db^\nu, \quad S_2^T \overset{def}{=} S_{\nu\mu}\, db^\mu \otimes db^\nu,$$

is an arbitrary symplectic form and, second, in the sense that the <u>limit exterior form</u>

$$\lim_{F^{NSA} \to 0} S_2 = \omega_{\mu\nu}\, db^\mu \wedge db^\nu = \omega_2 \qquad (17)$$
$$= \underline{\text{FUNDAMENTAL SYMPLECTIC FORM}}$$

is the fundamental symplectic form, with a selfevident correspondence with the algebraic counterparts (10) and (11).

The generalized formulations (A), (B) and (C) were interpreted in paper[1] as coverings of the conventional analytic, Lie and symplectic formulations, in the sense that

(a) they apply for the direct treatment of a broader physical context (nonconservative mechanics) without redefinition of the local variables r^{ka} and p_{ka} (which would be necessary if a Hamiltonian description of the same nonconconservative systems is desired via the use of the Inverse Problem[1]);

(b) the generalized formulations recover the conventional formulations identically under a limit of clear physical meaning, the null value of the forces nonderivable from a potential, Eqs. (6), (11) and (17); and

(c) the generalized analytic, algebraic and geometrical formulations are interrelated and compatible in the same measure as that of the conventional analytic, Lie and symplectic formulations.

It should be here also recalled that the latter aspect is essentially due to the property that the Lie-admissible algebras originate via the brackets of the time evolution law of the generalized analytic equations, and are characterized by a contravariant (regular) tensor $S^{\mu\nu}$ whose covariant form $S_{\mu\nu}$ is symplectic-admissible, and viceversa. Thus, <u>a crucial condition for this compatibility is the joint Lie-admissible algebraic/symplectic-admissible geometrical character</u>[6] via the rule $(S^{\mu\nu}) = (S_{\mu\nu})^{-1}$.

By here restricting the analysis to only autonomous systems (nonconservative systems without an explicit dependence on time) and to only the time-component $\hat{G}(\hat{t}) \in G(3.1)$, the Lie-admissible formulations (A), (B) and (C) outlined above allowed the construction in paper[1] of the following Galilei-admissible covering of the canonical realization of the Galilei group of translations in time

$$G(\hat{t}) : e^{\hat{t}\, \omega^{\alpha\beta} \frac{\partial H}{\partial b^\beta} \frac{\partial}{\partial b^\alpha}} \rightarrow \hat{G}(\hat{t}) : e^{\hat{t}\, S^{\alpha\beta} \frac{\partial H}{\partial b^\beta} \frac{\partial}{\partial b^\alpha}} \qquad (18)$$

<u>LIE TENSOR</u> <u>LIE-ADMISSIBLE TENSOR</u>

obtained via the sole replacement of the Lie (or cosymplectic) tensor $\omega^{\alpha\beta}$ with the Lie-admissible (or cosymplectic-admissible) tensor $S^{\alpha\beta}$ without changing the Hamiltonian, the parameter $\hat{t}$ and the base manifold

with local coordinates b.

The structure $\hat{G}(\hat{t})$ emerged form the analysis of paper[1] as being a rather peculiar type of connected Lie groups. First of all, from the assumed continuity conditions, the connectivity property

$$\lim_{\hat{t}\to 0} \hat{b}^{\mu} = b^{\mu}, \tag{19a}$$

$$\hat{b}^{\mu} = \exp\left\{\hat{t}\, S^{\alpha\beta} \frac{\partial H}{\partial b^{\beta}} \frac{\partial}{\partial b^{\alpha}}\right\} b^{\mu}, \tag{19b}$$

and the composition law

$$\hat{g}(\hat{t})\, \hat{g}(\hat{t}') = \hat{g}(\hat{t}+\hat{t}') \quad , \quad g(\hat{t}) \in \hat{G}(\hat{t}), \tag{20}$$

we easily infer that $\hat{G}(\hat{t})$ is a connected Lie group. However, such group $\hat{G}(\hat{t})$ is <u>non-Lie in the neighborhood of the identity</u> because, by construction,

$$\hat{b}^{\mu} = \exp\left\{\hat{t}\, S^{\alpha\beta} \frac{\partial H}{\partial b^{\beta}} \frac{\partial}{\partial b^{\alpha}}\right\} b^{\mu} \cong b^{\mu} + \hat{t}\,(b, H), \tag{21}$$

$(b, H) =$ <u>NON-LIE, LIE-ADMISSIBLE PRODUCT</u>

Also, $\hat{G}(\hat{t})$ is constructed in terms of the generator H, the parameter $\hat{t}$ and the base manifold M(b) of a <u>different</u> Lie group, the limit group $G(\hat{t})$. Owing to these rather peculiar features, the structures of the type $\hat{G}(\hat{t})$ were called in paper[1] <u>Lie-admissible groups</u> in the sense of

(a) being ordinary Lie groups, although realized by means of the generators, parameters and base manifold of different (generally nonisomorphic) Lie groups (without prohibiting the existence of the customary, standard realization);

(b) admitting a non-Lie, but Lie-admissible algebra in the neighborhood of the identity; and, most importantly for limit (3),

(c) admitting different (generally nonisomorphic) groups in the standard realization under the limit $S \to \omega$.

In particular, all these features of the Lie-admissible groups were conceived for the specific objective to verify our fundamental limit (3). The hope for the joint achievement of the form-invariance (2) was supported by the crucial property of Eqs. (4) according to which the tensor $S_{\mu\nu}$ is constructed in such a way to eliminate the forces which break the original symmetry $G(\hat{t})$, while the Hamiltonian H is invariant under $G(\hat{t})$ and remains so when used as a generator of $\hat{G}(\hat{t})$.

A number of examples were worked out in paper[1] as well as in the subsequent paper[7]. In these cases the Lie-admissible formulations did perform the desired function, that is, the pragmatic identifications of means for the explicit construction of nonmanifest symmetries verifying properties (2) and (3). However, the problem whether these Lie-admissible methods are "universal", that is, capable of producing a covering form-invariant description for <u>all</u> Newtonian systems of the class considered, was left open. Numerous other technical problems were also left open, and the analysis was presented in the intended spirit: a mainly conjectural first step.[8,9]

In this note I shall

- <u>Present a needed clarification on the realization of Lie-admissible algebras in Newtonian Mechanics</u>. This will be done by inspecting the realization of paper[1] and by presenting an alternative, regular, realization. As an incidential remark, a realization of Lie-admissible algebras via singular brackets will be also presented.[10]

- <u>Indicate that, for the case of autonomous systems, the Galilei-admissible transformations $\hat{G}(\hat{t})$ are indeed universal</u>, that is, they provide the form-invariance for all the systems of the class considered

while they satisfy limit (3). A number of examples will be also worked out.

- Outline the reasons for my hope for a possible, future, coordinate free-globalization of the symplectic-admissible geometry. For this objective I shall first recall the known property at the basis of the globalization of the symplectic geometry, that is, symplectic forms (15) preserve their symplectic character under arbitrary (but of class C^{∞} and regular) transformations of the local coordinates. Then, I shall indicate that this property extends to the symplectic-admissible forms (12) in its entirety. The problem of the possible globalization of the Galilei-admissible relativity will be then briefly considered.

* * *

The formulations of paper[1] were based on the realization of system (4) via partial differential equations of the type

$$\frac{\partial R_\mu}{\partial b^\nu} \tilde{\dot{=}}^{\nu} = \dot{=}_\mu , \quad \det\left(\partial R_\mu / \partial b^\nu\right) \neq 0, \tag{22a}$$

$$\{R_\mu\} = \{ R_{ka}(\underline{t}, \underline{b}), \underline{r}^{ka} \}, \tag{22b}$$

in the 3n unknown functions R_{ka} (the last 3n functions of the set $\{R_\mu\}$ are assumed to coincide with the Cartesian coordinates r^{ka} to preserve the direct physical significance of the momenta p_{ka} in Eqs. (5)). This yields a realization of the tensor $S_{\mu\nu}$,

$$S_{\mu\nu} = \frac{\partial R_\mu}{\partial b^\nu} \tag{23}$$

which is indeed symplectic-admissible because the attached tensor

$$S_{\mu\nu} - S_{\nu\mu} = \Omega_{\mu\nu} = \frac{\partial R_\mu}{\partial b^\nu} - \frac{\partial R_\nu}{\partial b^\mu} \tag{24}$$

verifies the antisymmetry and closure conditions (14), by therefore characterizing a symplectic form (15). As a result, realization (22) of our fundamental equations (4) does verify our geometric requirements in their entirety.

However, the associated algebraic profile needs a clarification. The contravariant form of tensor (23) is given by

$$S^{\mu\nu}(b) = \frac{\partial b^\nu}{\partial R_\mu}(R(b)) \tag{25}$$

and characterize the brackets

$$(A,B)_{(b,R)} = \frac{\partial A}{\partial b^\mu}\frac{\partial B}{\partial R_\mu}, \qquad \begin{array}{l} A = A(b,R), \\ B = B(b,R), \end{array} \tag{26}$$

which are formally Lie-admissible in the sense that the attached brackets

$$(A,B)_{(b,R)} - (B,A)_{(b,R)} = [A,B]_{(b,R)} = \frac{\partial A}{\partial b^\mu}\frac{\partial B}{\partial R_\mu} - \frac{\partial B}{\partial b^\mu}\frac{\partial A}{\partial R_\mu}, \tag{27}$$

have the structure of the Poisson brackets when realized in the (double dimensional) space of the variables (b,R).

The point which needs clarification is that brackets (26) are not Lie-admissible when reduced to the sole dependence on the b-variables. This is due to the fact that, when the attached brackets are written in such a form

$$(A,B)_{(b)} - (B,A)_{(b)} = \frac{\partial A}{\partial b^\mu}\frac{\partial b^\nu}{\partial R_\mu}\frac{\partial B}{\partial b^\nu} - \frac{\partial B}{\partial b^\mu}\frac{\partial b^\nu}{\partial R_\mu}\frac{\partial A}{\partial b^\nu} \stackrel{def}{=} A \circ B, \quad (28)$$

$$A = A(b) \quad , \quad B = B(b)$$

they do not verify the Jacobi law, *i.e.*,

$$(A \circ B) \circ C + (B \circ C) \circ A + (C \circ A) \circ B \neq 0 \quad (29)$$

because of the lack of commutativity of the second-order derivatives in the b- and R-quantities[11]

$$\frac{\partial^2 A}{\partial b^\mu \partial R_\nu} = \frac{\partial}{\partial b^\mu}\left(\frac{\partial A}{\partial b^\alpha}\frac{\partial b^\alpha}{\partial R_\nu}\right) \neq \frac{\partial^2 A}{\partial R_\nu \partial b^\mu} = \frac{\partial b^\alpha}{\partial R_\nu}\left(\frac{\partial^2 A}{\partial b^\alpha \partial b^\mu}\right), \quad (30)$$

as the reader can verify with a simple but tedious inspection.

In this note I shall however indicate that, when Lie-admissibility in the sole b-variables is desired, it can be consistently formulated. The net effect is that this situation, rather than being here considered as a drawback, creates instead the rather intriguing problem of the "degrees of freedom" of the Lie-admissible formulations in general and of the Galilei-admissible relativity, in particular; that is, the study of the family of solutions $(S_{\mu\nu})$ of Eqs. (4) as well as of their algebraic and geometrical character.

For this purpose, I shall call the brackets (A,H) of the generalized time evolution law (7)

- formally Lie-admissible,[12] when they are Lie-admissible in the formal 12-dimensional space of the variables (b,R), as presented in paper[1], and
- strictly Lie-admissible, when they are Lie-admissible in the dynamical space of the b-variables, as I originally introduced them in my earlier studies on Lie-admissibility[13].

In this note I would like to indicate that the reformulation of the content of paper[1] in terms of the notion of strict Lie-admissibility is indeed possible. It is advantageous to consider separately the most relevant profiles under consideration.

1. ANALYTIC PROFILE. Permit me to recall the algebraic motivation for my efforts in relation to the construction of a generalization of the "natural equations" for nonconservative systems, Hamilton's equations with external terms

$$\begin{cases} \dot{r}^{ka} = \dfrac{\partial H}{\partial p_{ka}}, & (31a) \\ \dot{p}_{ka} = -\dfrac{\partial H}{\partial r^{ka}} + F_{ka}, & (31b) \end{cases} \qquad k = 1,2,\ldots,n, \quad a = x,y,z,$$

It is given by the fact that the induced time evolution law

$$\dot{A}(\underset{\sim}{r}, \underset{\sim}{p}) = \frac{\partial A}{\partial r^{ka}}\frac{\partial H}{\partial p_{ka}} - \frac{\partial A}{\partial p_{ka}}\frac{\partial H}{\partial r^{ka}} + \frac{\partial A}{\partial p_{ka}} F_{ka} \stackrel{def}{=} A * H \quad (32)$$

even though capable of providing a consistent characterization of the dynamics, does not characterize a consistent algebra, as commonly understood, because the brackets $A * H$ violate the right distributive and scalar laws (see ref.[1], Sec. 3.1 for details).

It should be stressed that nonconservative systems can be and have been studied without the backing of an algebraic structure for the characterization of the complete systems (i.e., inclusive of the nonconservative forces and not only restricted to its conservative part).

However, it should be equally stressed that the lack of such an algebraic backing constitutes a rather significance methodological drawback at numerous levels of study. The reader may appreciate the point by attempting the study of physical issues such as, the Galilei relativity, the gauge symmetries, the notion of spin, etc., without the use of any algebra at all, whether Lie or not.

In conclusion, Hamilton's equations with external terms, even though analytically consistent, exibit a major methodological deficiency on a comparative basis with the same equations without external terms, the lack of a consistent algebraic structure.

The simplest way known to me for removing this deficiency and transforming Eqs. (31) into an algebraically consistent form, is given by the following generalized equations

$$\begin{cases} \dot{r}^{ka} = \dfrac{\partial H}{\partial p_{ka}}, & (33a) \\ \dot{p}_{ka} = -\dfrac{\partial H}{\partial r^{ka}} - s_{kajc}\dfrac{\partial H}{\partial p_{jc}}, \quad s = (s) = s^T, & (33b) \end{cases}$$

where the s-matrix is a solution of the system (for given conservative Hamiltonian H and nonconservative forces $F = F^{NSA}$)

$$s_{kajc}\frac{\partial H}{\partial p_{jc}} = -F^{NSA}_{ka}, \quad s = s^T \qquad (34)$$

and where the symmetric condition on the s-matrix is imposed for later needs.

Indeed, Eqs. (33) now induce the time evolution law

$$\dot{A}(\underset{\sim}{r}, \underset{\sim}{p}) = \frac{\partial A}{\partial r^{ka}}\frac{\partial H}{\partial p_{ka}} - \frac{\partial A}{\partial p_{ka}}\frac{\partial H}{\partial r^{ka}} - \frac{\partial A}{\partial p_{ia}} s_{iajc}\frac{\partial H}{\partial p_{jc}} \overset{def}{=} (A,H), \qquad (35)$$
$$s = s^T$$

whose brackets (A,H) do verify the right and left distributive and scalar laws (in the space of the independent $(\underset{\sim}{r},\underset{\sim}{p})$-variables). As a result, Eqs. (33) do characterize a consistent algebra for the treatment of (complete) nonconservative systems (i.e., inclusive of the nonconservative forces). Also, Eqs. (33) trivially recover the conventional Hamilton's equations (without external terms) at the limit of null external forces

$$\lim_{\underset{\sim}{F}^{NSA} \to 0} \begin{Bmatrix} \dot{r}^{ka} = \dfrac{\partial H}{\partial p_{ka}} \\ \dot{p}_{ka} = -\dfrac{\partial H}{\partial r^{ka}} - s_{kajc}\dfrac{\partial H}{\partial p_{jc}} \end{Bmatrix} = \begin{Bmatrix} \dot{r}^{ka} = \dfrac{\partial H}{\partial p_{ka}} \\ \dot{p}_{ka} = -\dfrac{\partial H}{\partial r^{ka}} \end{Bmatrix} \qquad (36)$$

In conclusion, Eqs. (33) are Hamilton-admissible equations in the sense of paper[1], that is, they

- represent nonconservative systems;
- characterize a consistent algebra via the brackets of the time evolution law; and
- recover the conventional Hamilton's equations at the limit of null value of the nonconservative forces.[14]

2. ALGEBRAIC PROFILE. It is easy to see that the brackets (A,H) of the generalized time evolution law (35) are strictly Lie-admissible (as those of refs.[13]). Owing to the symmetric character of the s-matrix, the attached brackets

$$(A,H) - (H,A) = 2\,[A,H] \tag{37}$$
$$= 2\left(\frac{\partial A}{\partial r^{ka}}\frac{\partial H}{\partial p_{ka}} - \frac{\partial A}{\partial p_{ka}}\frac{\partial H}{\partial r^{ka}}\right) = LIE$$

are Lie in the 6n-dimensional space of the independent variables $(\underset{\sim}{r},\underset{\sim}{p})$ (in particular, they are simply twice the conventional Poisson brackets). As a consequence, Hamilton-admissible equations (33) are strictly Lie-admissible in algebraic character.

This algebraic property can also be seen by noting that the brackets (A,H) verify the general Lie-admissible law (8) or, equivalently, by writing Eqs. (33) in the unified tensorial notation

$$\dot{b}^{\mu} - S^{\mu\nu}\frac{\partial H}{\partial b^{\nu}} = 0, \tag{38a}$$

$$(b^{\mu}) = \begin{pmatrix} \underset{\sim}{r} \\ \underset{\sim}{p} \end{pmatrix}, \quad (S^{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ -1 & -S \end{pmatrix}, \quad H = T(\underset{\sim}{p}) - V(\underset{\sim}{r}), \quad S = S^{T} \tag{38b}$$

and by noting that the tensor $S^{\mu\nu}$ is a solution of system (9) in the strict sense (i.e., in the space of the b-variables only, rather than in the space of the (b,R)-variables).

Most intriguingly, Hamilton-admissible equations (33) and their strict Lie-admissible character are <u>directly universal</u> for nonconservative Newtonian systems, in the sense that they apply to <u>all</u> nonconservative systems (1) <u>without</u> redefinition of the Cartesian coordinates r^{ka} used in the experimental verification and the physical linear momenta $p_{ka} = m_k \dot{r}_{ka}$.

Equally significant is the fact that such universality originates from the consistency of system (34) which is of <u>algebraic</u>, rather than

of partial differential nature. In particular, under the assumed regularity properties, $\partial H/\partial p_{jc} \neq 0$ for all j = 1,2,...,n and c=x,y,z and the following trivial, diagonal solutions

$$(S_0^{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ -1 & -S_0 \end{pmatrix}, \quad (S_0)_{ia\,jb} = \frac{F_{ia}}{\partial H/\partial p_{ia}}\,\delta_{ij}\,\delta_{ab} \ \text{(NO SUM)} \tag{39}$$

is always possible and consistent with the desired strict Lie-admissibility. As a result, <u>our Hamilton-admissible equations are not only directly universal for nonconservative mechanics, but their construction is truly easy for arbitrary nonconservative forces</u> $F^{NSA}(t,\underset{\sim}{r},\underset{\sim}{p})$.

Permit me to confess that this type of direct universality of the Lie-admissible algebras in nonconservative mechanics is beyond the most optimistic expectation of paper[1]. This point can be appreciated by noting that the formal Lie-admissible algebras of paper[1] are also directly universal, but via the solution of system (22) of <u>partial</u> differential equations. Such a system is consistent under the assumed continuity and regularity properties and, therefore, the existence of formal Lie-admissible brackets is guaranteed by the existence theorems of partial differential equations. Neverthless, the practical computation of a solution R_{μ} becomes nontrivial for functionally involved forces F^{NSA}, assuming that such a solution can be practically reduced to a closed form. The complexity of these computations for nontrivial forces F^{NSA} for the case of formal Lie-admissibility should be then compared with the triviality of solution (39) in the same forces F^{NSA} for the case of strict Lie-admissibility.

The algebraically inclined reader might be interested to know that, as it is the case for brackets (26), the strict Lie-admissible brackets

(35) also violate the flexibility law, the power-associativity law and the Jordan-admissible law.

3. GEOMETRICAL PROFILE. It is also easy to see that our Hamilton-admissible equations (33) are symplectic-admissible in geometrical character. Indeed, the covariant version of the Lie-admissible tensor (39)

$$(S_{o\mu\nu}) = \begin{pmatrix} -s_o & -1 \\ 1 & 0 \end{pmatrix} = (S_o^{\mu\nu})^{-1}, \qquad (40)$$

$$(S_o)_{iajb} = \delta_{ij}\,\delta_{ab}\,F_{ia}/\partial H/\partial p_{ia}$$

is not symplectic, owing to the "symplectic-geometry-breaking" terms $-s_o$ (which are also the "Lie-algebra-breaking" terms). Neverthless, form (40) is indeed symplectic-admissible because the attached form

$$S_{o2} - S_{o2}^T = S_{o\mu\nu}\, db^\mu \otimes db^\nu - S_{o\nu\mu}\, db^\mu \otimes db^\nu$$
$$= 2\,\omega_{\mu\nu}\, db^\mu \wedge db^\nu = -4\, dp_{ka} \wedge dr^{ka} \qquad (41)$$

is symplectic (a multiple of the fundamental symplectic form).

The covering character of the generalized formulations under consideration with respect to the conventional analytic-Lie-symplectic formulations is selfevident. In particular, the covering of the Lie-admissible algebras/symplectic-admissible geometry over the conventional Lie algebras/symplectic geometry is three-fold, in the sense that the latter formulations can be recovered from the former

(I) according to the attached rules of Lie-admissibility, Eq. (10), and symplectic-admissibility, Eq. (16);

(II) as a particular case of the generalized formulations, because Lie algebras are Lie-admissible and symplectic forms are symplectic-admissible; and, perhaps more significantly from a physical profile,

(III) at the limit of null nonconservative (and Galilei symmetry-breaking) forces

$$\lim_{F^{NSA} \to 0} \left\{ \begin{array}{c} \text{HAMILTON-ADMISSIBLE EQUATIONS} \\ \text{LIE-ADMISSIBLE ALGEBRAS} \\ \text{SYMPLECTIC-ADMISSIBLE GEOMETRY} \end{array} \right\} = \left\{ \begin{array}{c} \text{HAMILTON'S EQUATIONS} \\ \text{LIE ALGEBRAS} \\ \text{SYMPLECTIC GEOMETRY} \end{array} \right\} \qquad (42)$$

The interrelation and compatibility of the three generalized approaches is also selfevident. The Lie admissible algebras verify the "uncompromisable dynamical origin" of papers[1,7], that is, they originate via the brackets of the time evolution law of the analytic equations, and they are characterized by a tensor $S_o^{\mu\nu}$ whose covariant form $(S_{o\mu\nu})$, $(S_{o\mu\nu}) = (S_o^{\mu\nu})^{-1}$, is symplectic-admissible, and viceversa.

Finally, the formulations considered are strictly Lie-admissible, as desired.

4. RELATIVITY PROFILE. In order to see that these formulations are indeed applicable to the construction of the Galilei-admissible relativity, we need only to verify the following

PROPOSITION 1: Form (40), here called fundamental symplectic-admissible form, provides an algebraic solution to system (4).

PROOF.

$$(S_0)\begin{pmatrix}\tilde{\dot{r}}\\ \tilde{\dot{p}}\end{pmatrix} = \begin{pmatrix}-S_0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\left(\frac{\partial H}{\partial p}\right)\\ \left(-\frac{\partial H}{\partial r}+F^{NSA}\right)\end{pmatrix}$$

$$= \begin{pmatrix}-(S_0)\left(\frac{\partial H}{\partial p}\right)+\left(\frac{\partial H}{\partial r}-F^{NSA}\right)\\ \left(\frac{\partial H}{\partial p}\right)\end{pmatrix} = \begin{pmatrix}\left(\frac{\partial H}{\partial r}\right)\\ \left(\frac{\partial H}{\partial p}\right)\end{pmatrix} = \begin{pmatrix}\dot{r}\\ \dot{p}\end{pmatrix} \qquad (43)$$

Q.E.D

This produces the desired effect, but now in a strictly Lie-admissible sense and via the solution of an algebraic system. I am here referring to the mechanism of elimination of the Galilei symmetry-breaking forces F^{NSA} via the contraction of the symplectic-admissible tensor $S_{0\mu\nu}$ with the nonconservative vector field $\tilde{\dot{r}}^{\nu}$.

A few examples are here in order. First, let me ignore the presence of conservative and Galilei form-invariant forces because inessetial for our analysis.Also, I consider for simplicity the case of only one space-dimension. Thus, the subsystem verifying the Galilei relativity is the trivial system of one free particle in one dimension. Then I implement this system with nonconservative forces which do not depend explicitly on time (autonomous systems). The subclass of system (1) I am considering is therefore of the type

$$\underset{(m=1)}{\left(\dot{b}^{\mu}\right)} - \left(\tilde{\dot{r}}^{\mu}\right) = \left(\dot{b}^{\mu}\right) - \begin{pmatrix}p\\ F^{NSA}(r,p)\end{pmatrix} = 0 \qquad (44)$$

where the reader will recall from paper[1] that nonselfadjoint forces

are considered as a class and, thus, inclusive as a particular case of forces derivable from a potential (but Galilei form-noninvariant).

The objective is here to illustrate how the strictly Lie-admissible formulations considered in this note produce the desired nonmanifest symmetries of systems (44) as a covering of Galilei's translations in time.

Although not essential, it is instructive to first identify the original manifest symmetry of the conservative subsystem. For the case considered it is given by the canonical realization of $G(\hat{t})$

$$G(\hat{t}):\ (b'^{\mu}) = \exp\left\{\hat{t}\,\omega^{\alpha\beta}\frac{\partial H}{\partial b^{\beta}}\frac{\partial}{\partial b^{\alpha}}\right\}(b^{\mu})$$

$$= \begin{pmatrix}r'\\ p'\end{pmatrix} = \begin{pmatrix}r+\frac{\hat{t}}{1!}[r,H]+\frac{\hat{t}^2}{2!}[[r,H],H]+\cdots\\ p+\frac{\hat{t}}{1!}[p,H]+\frac{\hat{t}^2}{2!}[[p,H],H]+\cdots\end{pmatrix} = \begin{pmatrix}r+\hat{t}p\\ p\end{pmatrix} \qquad (45)$$

$$H=\frac{1}{2}p^2,\quad m=1,\quad p=\dot{r},$$

with trivial form-invariance

$$\frac{dr'}{d\hat{t}} = p' \quad , \quad \frac{dp'}{d\hat{t}} = 0 \qquad (46)$$

The point I would like to bring to the reader's attention is that, unlike the conventional form

$$t \longrightarrow \hat{t} = t + \text{const.} \qquad (47)$$

the canonical group of translations in time varies with the Hamiltonian. Thus, different conservative and Galilei form-invariant forces f^{SA} ($\neq 0$) imply different explicit forms of the canonical group $G(\hat{t})$. Also,

these explicit forms are sometimes of rather involved practical computation (depending on the explicit form of f^{SA}) <u>already at the level of the conventional Galilei relativity for conservative mechanics</u>. It is of course expected that a similar situation occurs for the explicit computation of the covering transformations $\hat{G}(\hat{t})$ which now vary, not only with f^{SA}, but also with F^{NSA}. However, for the class of examples considered below, f^{SA} is always null, the original symmetry (45) remains unchanged, and $\hat{G}(\hat{t})$ varies only with F^{NSA}.

By using now familiar rules, the Galilei-admissible covering $\hat{G}(\hat{t})$ of the group (45) is constructed, for each given F^{NSA}, via the expansions

$$\hat{G}(\hat{t}):\ (\hat{b}^\mu) = \exp\left\{\hat{t}\, S_0^{\alpha\beta}\frac{\partial H}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}\right\}(b^\mu)$$

$$= \begin{pmatrix}\hat{r}\\ \hat{p}\end{pmatrix} = \begin{pmatrix} r + \frac{\hat{t}}{1!}(r,H) + \frac{\hat{t}^2}{2!}((r,H),H) + \cdots \\ p + \frac{\hat{t}}{1!}(p,H) + \frac{\hat{t}^2}{2!}((p,H),H) + \cdots \end{pmatrix} \qquad (48)$$

$$H = \frac{1}{2}p^2,\quad m = 1,\quad p = \dot{r}$$

where the brackets (A,H) are strictly Lie-admissible and given explicitly by

$$(A,H) = \frac{\partial A}{\partial r}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial r} - \frac{\partial A}{\partial p} S_0 \frac{\partial H}{\partial p}, \qquad (49)$$

$$-S_0\,\partial H/\partial p = F^{NSA}.$$

<u>EXAMPLE</u> 1: <u>Particle with constant external force.</u>

$$\begin{pmatrix}\tilde{\cdot}\,\mu\\ \cdot\end{pmatrix} = \begin{pmatrix} p \\ -F\end{pmatrix},\quad F = \text{const.} \qquad (50)$$

The Lie-admissible tensor in this case is given by

$$(S_0^{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{F}{p}\end{pmatrix}. \qquad (51)$$

Expansion (48) then yields the desired explicit form of the <u>Galilei-admissible translations in time for the particle with a constant force</u>

$$\begin{cases} \hat{r} = r + \hat{t}\,p + \frac{1}{2}\hat{t}^2 F, & (52a) \\ \hat{p} = p + \hat{t}\,F, & (52b) \end{cases}$$

which do produce the desired form-invariance (2), i.e.,

$$\begin{cases} \dfrac{d\hat{r}}{d\hat{t}} = p + \hat{t}\,F = \hat{p}, \\ \dfrac{d\hat{p}}{d\hat{t}} = F, \end{cases} \qquad (53)$$

as a covering of the conventional Galilei transformations, limit (3), i.e.,

$$\lim_{F\to 0}\left\{\begin{matrix}\hat{r} = r + \hat{t}\,p + \frac{1}{2}\hat{t}^2 F \\ \hat{p} = p + \hat{t}\,F\end{matrix}\right\} = \left\{\begin{matrix}\hat{r} = r + \hat{t}\,p \\ \hat{p} = p\end{matrix}\right\}. \qquad (54)$$

The last property to verify is that transformations (52), in realization (48), form a Lie-admissible group. The connectivity property, Eq. (19), is trivially verified. The composition law, Eq. (20), is also trivially verified,

$$\hat{\hat{r}} = \hat{r} + \hat{t}'\hat{p} + \frac{1}{2}\hat{t}'^2 F = (r + \hat{t}\,p + \frac{1}{2}\hat{t}^2 F) + \hat{t}'(p + \hat{t}\,F) + \frac{1}{2}\hat{t}'^2 F$$

$$= r + (\hat{t} + \hat{t}')p + \frac{1}{2}(\hat{t} + \hat{t}')^2 F\ ;\quad \hat{\hat{p}} = \hat{p} + \hat{t}'F = p + (\hat{t} + \hat{t}')F, \qquad (55)$$

and, thus, transformations (52) form a connected Lie group. However, the group is realized in terms of the generator H, the parameter $\hat{t}$ and

the base manifold M(b) of a different group, the limit group (45), while it is of non-Lie, but Lie-admissible character in the neighborhood of the identity, Eq. (21). Thus, structure (52) characterizes a Lie-admissible group and this concludes the example.

EXAMPLE 2: Particle with a linear, velocity-dependent drag force

$$\left(\tilde{\dot{\mu}}\right) = \begin{pmatrix} p \\ -\gamma p \end{pmatrix} \tag{56}$$

The (strictly) Lie-admissible tensor is in this case

$$\left(S_0^{\mu\nu}\right) = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix} \tag{57}$$

yielding the Galilei-admissible translations in time for the particle with a linear, velocity-dependent drag force[15]

$$\begin{cases} \hat{r} = r - \frac{1}{\gamma} p \left(e^{-\gamma \hat{t}} - 1\right), & (58a) \\ \hat{p} = p\, e^{-\gamma \hat{t}}, & (58b) \end{cases}$$

These transformations constitute a nonmanifest symmetry of the equations of motion as a covering of the conventional Galilei symmetry for the free case, in the sense that they are of nontrivial structure and (thus) identification (see the next example for a better appreciation of this point); they verify the requirement of form-invariance (2);

$$\begin{cases} \frac{d\hat{r}}{d\hat{t}} = p\, e^{-\gamma \hat{t}} = \hat{p}, & (59a) \\ \frac{d\hat{p}}{d\hat{t}} = -\gamma p\, e^{-\gamma \hat{t}} = -\gamma \hat{p}. & (59b) \end{cases}$$

jointly with the verification of the limit (3),

$$\lim_{\gamma \to 0} \begin{cases} \hat{r} = r - \frac{1}{\gamma} p \left(e^{-\gamma \hat{t}} - 1\right) \\ \hat{p} = p\, e^{-\gamma \hat{t}} \end{cases} = \begin{cases} \hat{r} = r + \hat{t} p \\ \hat{p} = p \end{cases} \tag{60}$$

The verification of the property that transformations (58) in realization (48) form a (connected) Lie-admissible group is left to the interested reader.

EXAMPLE 3: Particle with quadratic (nonlinear) velocity-dependent drag force

$$\left(\tilde{\dot{\mu}}\right) = \begin{pmatrix} p \\ -\gamma p^2 \end{pmatrix} \tag{61}$$

The Lie-admissible tensor is now given by

$$\left(S_0^{\mu\nu}\right) = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma p \end{pmatrix} \tag{62}$$

yielding, via expansion (48), the Galilei-admissible translations in time for the particle with quadratic drag force

$$\begin{cases} \hat{r} = r + \frac{1}{\gamma} \ln\left(1 + \gamma \hat{t} p\right), & (63a) \\ \hat{p} = \dfrac{p}{1 + \gamma \hat{t} p}, & (63b) \end{cases}$$

which is a highly nonmanifest symmetry of the equations of motion jointly with being a covering of Galilei's transformations (45). Indeed, form-invariance (2) is now verified according to

$$\begin{cases} \dfrac{d\hat{r}}{d\hat{t}} = \dfrac{p}{1+\gamma\hat{t}p} = \hat{p}\,, & (64a) \\[2ex] \dfrac{d\hat{p}}{d\hat{t}} = \dfrac{-\gamma p^2}{(1+\gamma\hat{t}p)^2} = -\gamma\hat{p}^2\,. & (64b) \end{cases}$$

The verification of limit (3) as well as of the property for transformations (63) to characterize a Lie-admissible group, is here left to the interested reader.

The above examples illustrate the following aspect.

> A PRIMARY FUNCTION OF THE GALILEI-ADMISSIBLE RELATIVITY IS:
> To identify, among all reference frames admissible via arbitrary (but of class C^∞ and regular) transformations, those which leave form-invariant each given nonconservative system (that is, each given nonconservative and Galilei-form-noninvariant force), in such a way to verify law (3).

Equivalently, the above examples illustrate the covering form-invariant description of physical systems provided by the Galilei-admissible relativity for nonconservative mechanics.

Although not essential for the content of this note, the reader should keep in mind the departures implied by such covering description over the conventional form-invariant description of conservative systems, as emphasized in papers[1,7]. I am here referring to the non-Lie, non-geodesic, non-inertial, non-symplectic and nonconservative character of the Galilei-admissible relativity. Apparently, these features of the relativity under study have a precise function, upon quantization, for the characterization of the interactions of extended, charged and massive particles at mutual

distances equal or smaller than their charge diameter with consequential forces more general than f^{SA} (nonselfadjoint); for the study of strong nonselfadjoint interactions, in which case f^{SA} represents the electromagnetic interactions and F^{NSA} represents the strong; for the identification of the dynamical effects of these short range, broader forces, such as the breaking of the SU(2)-spin symmetry for the motion of a charged, extended particle within dense hadronic matter (e.g., the core of a neutron star); the experimental resolution of the presence or absence of predicted small deviations from Pauli's Exclusion Principle in nuclear physics[7]; etc.

5. UNIVERSALITY PROFILE. The problem under consideration is whether the Galilei-admissible transformations $\hat{G}$ ($\hat{t}$) are "universal", that is, capable of producing the indicated covering form-invariance description for all nonconservative systems considered (local, class C^∞ and regular).

From Eqs. (48) we see that the universality of the $\hat{r}$ -transformations is trivial because

$$\begin{aligned} \frac{d\hat{r}^{ka}}{d\hat{t}} &= (r^{ka}, H) + \frac{\hat{t}}{1!}((r^{ka}, H), H) + \frac{\hat{t}^2}{2!}(((r^{ka}, H), H), H) + \cdots \\ &= p^{ka} + \frac{\hat{t}}{1!}(p^{ka}, H) + \frac{\hat{t}^2}{2!}((p^{ka}, H), H) + \cdots = \hat{p}^{ka} \end{aligned} \tag{65}$$

and this property holds for all Hamiltonians of the class admitted, that is, $H = T(\underset{\sim}{p}) + V(\underset{\sim}{r})$, as well as, of course, for all F^{NSA}-forces.

The universality of the $\hat{p}$ -transformations is not trivial, because it implies the identities

$$\frac{d\hat{p}_{ka}}{d\hat{t}} = (p_{ka}, H) + \frac{\hat{t}}{1!}((p_{ka}, H), H) + \frac{\hat{t}^2}{2!}(((p_{ka}, H), H), H) + \cdots$$
$$= f^{SA}_{ka}(\hat{z}_\mu) + F^{NSA}_{ka}(\hat{z}_\mu, \hat{p}_\mu) \qquad (66)$$

where, as the reader should keep in mind, the product (A,H) is non-Lie.

A formal solution of the problem considered is the following. The issue can be essentially reduced to the problem whether the transforms of the functions $\tilde{\dot{-}}^{\mu}(b)$ are the functions $\tilde{\dot{-}}^{\mu}(\hat{b})$ of the transformed variables $\hat{b}$, i.e.,

$$e^{\hat{t}\hat{H}}\,\tilde{\dot{-}}^{\mu}(b) = \tilde{\dot{-}}^{\mu}(e^{\hat{t}\hat{H}}b) = \tilde{\dot{-}}^{\mu}(\hat{b}), \qquad (67a)$$

$$\hat{G}(\hat{t}): e^{\hat{t}\hat{H}} = e^{\hat{t}\,S_0^{\alpha\beta}\frac{\partial H}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}}. \qquad (67b)$$

It is possible to prove that this is always the case (for class C^∞ functions $\tilde{\dot{-}}^{\mu}$ in the neighborhood of a regular point b), because property (67a) holds for connected Lie groups, while the Galilei-admissible group $\hat{G}(\hat{t})$ is indeed a group of this type, despite its "unconventional" realization (67b). This concludes our argument for the "universality" of the component $\hat{G}(\hat{t})$ of the Galilei-admissible relativity.

It is understood that the explicit computation of the transformations $\hat{G}(\hat{t})$ may be rather involved in practice. Also, these transformation may not exist in a closed form. But this is already the case for the conventional canonical treatment of the Galilei relativity in costumary conservative mechanics. In any case, permit me to confess that

this universality is also beyond my most optimistic expectation at the time of paper[1].

The extension of the analysis to the case of nonautonomous nonconservative systems is expected to require the transition from the symplectic-admissible to the contact-admissible forms[1]. Presumably, this should produce a form-invariance also for the case of an explicit dependence on time.

Considerably more complex remains the problem of the explicit construction of the remaining nine components of the Galilei-admissible transformations (Lie-admissible covering of rotations, translations in space and Galilei boosts), and of the expected scalar extension. As stressed in paper[1], the study of this problem demands the prior knowledge of sufficiently well established Lie-admissible formulations, with particular reference to (i) the transformation theory of the Hamilton-admissible equations,[16] (ii) the structure and representation theory of Lie-admissible algebras,[17] and (iii) the symplectic-admissible geometry.[18]

6. UNIQUENESS PROFILE. As stressed in paper[1], the Lie-admissible formulations are not intended nor expected to be the only formulations which are applicable for the treatment of nonconservative systems.[19] As a matter of fact, one of the primary functions of the Inverse Problem[20] is that of indicating that, under certain technical conditions, all local, class C^∞ and regular nonconservative systems can be treated via conventional Lie formulations (Hamilton's equations, Lie algebras and symplectic -or-contact geometry for autonomous-or-nonautonomous systems) .

This is essentially realized via, first, the representation of systems (1) in terms of the SARLET-CANTRIJN extension of BIRKHOFF's

equations[21] in the original $(\underset{\sim}{r},\underset{\sim}{p})$ = (b) variables

$$\left[\left(\frac{\partial R_\mu}{\partial b^\nu}-\frac{\partial R_\nu}{\partial b^\mu}\right)\frac{db^\nu}{dt}-\frac{\partial H^B}{\partial b^\mu}+\frac{\partial R_\mu}{\partial t}\right]_{SA}^{C^\infty,R}=0, \qquad (68)$$

and then their reduction to a conventional Hamiltonian form, but in new coordinates,

$$\left[\omega_{\mu\nu}\frac{db'^\nu}{dt'}-\frac{\partial H'}{\partial b'^\mu}\right]_{SA}^{C^\infty,R}=0, \qquad (69)$$

via Pauli-Darboux transformations.

As a result, our Galilei-admissible relativity is expected to have an "image" within the context of the conventional Lie formulations, and studies to this effect are solicited.

A few comments are however in order.

Recall that mechanism (67) of form-invariance is Lie in character. Thus, nonmanifest symmetries of nonconservative systems can indeed be constructed by using standard Lie techniques. However, the construction of the Galilei-admissible relativity demands the joint fulfillment of the crucial limit (3). It is in this dual respect (nonmanifest form-invariance and covering character over the Galilei symmetry) that I have encountered considerable difficulties with the use of Lie techniques, as indicated in paper[1] (end of Sec. 2), while the identification and use of the Lie-admissible techniques resulted more promising for a first study of the problem. The use of algebras other than Lie-admissible algebras had to be excluded owing to the loss, in this case, of a number of crucial methodological aspects (e.g., the enveloping nonassociative algebra) which play a subtle, but crucial role in achieving the desired form invariance of nature in a closed form. The use of formulations

without a consistent algebraic structure (see the remarks in this respect in the Section 1 for the Analytic Profile) had to be excluded for a first study of the problem, howing to the difficulties of the methods, in this case, to achieve a covering of each aspect of the Galilei relativity (group, algebra and enveloping algebra). The net effect is that the Lie-admissible formulations emerged as possessing a rather unique capability for a first construction of the Galilei-admissible relativity. After all, they have been conceived to verify limit (3) by construction.

Another aspect which deserves some attention, particularly from a physical profile, is that the use of formulations other than those of Lie-admissible type generally demand the abandonment of the direct physical significance of the algorithms at hand. I am here referringg to the fact that, when Hamilton-admissible equations (33) are used,

(a) the $\underset{\sim}{r}$'s are the coordinates of the experimental verification of the systems (which is always <u>non</u>inertial for the available technology);

(b) the $\underset{\sim}{p}$'s are the physical linear momenta $m\,\dot{\underset{\sim}{r}}$;

(c) H is the total mechanical energy (which is <u>not</u> conserved);

(d) $\underset{\sim}{M} = \underset{\sim}{r} \times \underset{\sim}{p}$ is the physical angular momentum (also nonconserved);

etc.

In the transition to the use of Lie techniques for the treatment of nonconservative systems, these features are generally lost. First of all, in the transition from Hamilton-admissible to Sarlet-Catrijn equations (68) the function H^B loses its direct physical meaning as the energy (and a similar situation occurs for all other generators of physically relevant transformations[22]), even though the variables can be $(\underset{\sim}{r},\underset{\sim}{p})$ as for Eqs. (33). In the further reduction of Eqs. (68) to the Hamiltonian form (69), there is a further loss of the direct

physical significance of the algorithms at hand because the local coordinates b'^{μ} reached via a Pauli-Darboux transformation, besides being generally noninertial, are generally nonrealizable with experiments (owing to a generally nonlinear dependence on the physical coordinates and velocities).

In conclusion, when the condition of direct physical significance of all the algorithms at hand (coordinates, generators of physically relevant transformations, etc.) is imposed, the Lie-admissible formulations again emerge as being rather unique . The reader will recall from ref.[7] the numerous and delicate physical problems which are created by the quantum mechanical treatment of nonconservative forces without a direct physical significance of the algorithms at hand (e.g., the risk of comparing the expectation values of the operators $i\hbar\,\partial/\partial t$, $\frac{\hbar}{i}\nabla_{\mu}$, ... with experimental data on energy, linear momentum,.... when these operators do not represent the energy, the linear momentum, ..., as a necessary condition for the existence of a Schrodinger-type representation).

But perhaps, the most significant argument in favor of the study of Lie-admissible formulations and of their preference over Lie formulations for the treatment of nonconservative systems is as pragmatic as conceivable possible. As indicated earlier, when a nonconservative system (1) is assigned, the construction of their Hamilton-admissible representation is trivial, and done via the solution of the algebraic system (34) .If one intends to study the same system via Lie techniques the situation is profoundly different. In this case, even though the existence of a conventional Hamiltonian representation is guaranteed (under the assumed continuity and regularity conditions), one must solve first a system of linear partial differential equations to reach a

representation via Eqs. (68) , and then an additional system of nonlinear partial differential equations to reach a Hamiltonian. Some times, even for the case of simple nonconservative forces and low dimensionality, these systems are so complex to discourage the most devoted Lie scolar (assuming that their solution exists in the needed closed form).

In closing this uniqueness profile, permit me to indicate that, even within the context of Lie-admissible formulations, the explicit form of the analytic equations, the tensor $S^{\mu\nu}$, etc. is not unique. A few comments on the possible relationship between the strictly and formally Lie-admissible formulations of regular type (i.e., realized via nondegenerate symplectic-admissible forms with nondegenerate attached symplectic forms) will be given at the end of this note.Here I would like to point out the existence in nonconservative Newtonian mechanics of degenerate, strictly Lie-admissible formulations.

Suppose that a nonconservative system (1) is assigned in the (b) = $(\underset{\sim}{r},\underset{\sim}{p})$ coordinates and performs the class C^{∞} , regular transformations

$$\underset{\sim}{r}^{k} \rightarrow \underset{\sim}{r}'^{k} = \underset{\sim}{r}^{k} \;, \quad \underset{\sim}{p}_{k} \rightarrow \underset{\sim}{w}_{k} = \underset{\sim}{w}_{k}(t, \underset{\sim}{r}, \underset{\sim}{p}). \qquad (70)$$

The problem under consideration is that of constructing a representation of the system considered in the new coordinates $(\underset{\sim}{r},\underset{\sim}{w})$, Hamiltonian $\bar{H} = \bar{H}(t,\underset{\sim}{r},\underset{\sim}{w}) = H(\underset{\sim}{r},\underset{\sim}{p}(t,\underset{\sim}{r},\underset{\sim}{w}))$ and generalized equations

$$\begin{cases} \dot{r}^{ia} = A^{ia\,jb}\dfrac{\partial \bar{H}}{\partial r^{jb}} + B^{ia}{}_{jb}\dfrac{\partial \bar{H}}{\partial w_{jb}} \,, \\ \dot{w}_{ia} = C_{ia}{}^{jb}\dfrac{\partial \bar{H}}{\partial r^{jb}} + D_{ia\,jb}\dfrac{\partial \bar{H}}{\partial w_{jb}} \,, \end{cases} \qquad (71)$$

under the conditions that the brackets of the time evolution law

$$\dot{\bar{A}} = (\bar{A}, \bar{H})_{(\underset{\sim}{r}, \underset{\sim}{w})}$$
$$= \frac{\partial \bar{A}}{\partial r^{ia}} A^{ia;jb} \frac{\partial \bar{H}}{\partial r^{jb}} + \frac{\partial \bar{A}}{\partial r^{ia}} B^{ia}{}_{jb} \frac{\partial \bar{H}}{\partial w_{jb}} \tag{72}$$
$$+ \frac{\partial \bar{A}}{\partial w_{ia}} C_{ia}{}^{jb} \frac{\partial \bar{H}}{\partial r^{jb}} + \frac{\partial \bar{A}}{\partial w_{ia}} D_{ia;jb} \frac{\partial \bar{H}}{\partial w_{jb}}$$

are strictly Lie-admissible but singular.

A solution of this latter problem is given by the following generalized equations (where upper bar denotes computation in the $(\underset{\sim}{r}, \underset{\sim}{w})$-variables)

$$\begin{cases} \dot{r}^{ia} = \overline{\dfrac{\partial w_{jb}}{\partial p_{ia}}} \dfrac{\partial \bar{H}}{\partial w_{jb}}, & (73a) \\ \dot{w}_{ia} = -\overline{\dfrac{\partial w_{ia}}{\partial p_{kc}}} \overline{\dfrac{\partial w_{jb}}{\partial r^{kc}}} \dfrac{\partial \bar{H}}{\partial w_{jb}}, & (73b) \end{cases}$$

which I here call a <u>singular, strictly Lie-admissible generalization of Hamilton's equations</u>. The singular character of these equations is selfevident because the matrix of their characteristic tensor is singular,

$$\det(S^{*\mu\nu}) = \det \begin{pmatrix} 0 & \left(\overline{\dfrac{\partial w_{jb}}{\partial p_{ia}}}\right) \\ 0 & -\left(\overline{\dfrac{\partial w_{ia}}{\partial p_{kc}}} \overline{\dfrac{\partial w_{jb}}{\partial r^{kc}}}\right) \end{pmatrix} = 0. \tag{74}$$

The strictly Lie-admissible character can be proved in the following

way. The generalized brackets are now given by

$$\dot{\bar{A}} = (\bar{A}, \bar{H})_{(\underset{\sim}{r}, \underset{\sim}{w})}$$
$$= \frac{\partial \bar{A}}{\partial r^{ia}} \overline{\frac{\partial w_{jb}}{\partial p_{ia}}} \frac{\partial \bar{H}}{\partial w_{jb}} - \frac{\partial \bar{A}}{\partial w_{ia}} \overline{\frac{\partial w_{ia}}{\partial p_{kc}}} \overline{\frac{\partial w_{jb}}{\partial r^{kc}}} \frac{\partial \bar{H}}{\partial w_{jb}} \tag{75}$$

The Lie character of the attached brackets can then be seen by performing an inverse transform to the original variables, i.e.,

$$(\bar{A}, \bar{H})_{(\underset{\sim}{r}, \underset{\sim}{w})} - (\bar{H}, \bar{A})$$
$$= \frac{\partial \bar{A}}{\partial r^{ia}} \overline{\frac{\partial w_{jb}}{\partial p_{ia}}} \frac{\partial \bar{H}}{\partial w_{jb}} - \frac{\partial \bar{A}}{\partial w_{ia}} \overline{\frac{\partial w_{ia}}{\partial p_{jb}}} \frac{\partial \bar{H}}{\partial r^{jb}}$$
$$- \frac{\partial \bar{A}}{\partial w_{ia}} \left(\overline{\frac{\partial w_{ia}}{\partial p_{kc}}} \overline{\frac{\partial w_{jb}}{\partial r^{kc}}} - \overline{\frac{\partial w_{ia}}{\partial r^{kc}}} \overline{\frac{\partial w_{jb}}{\partial p_{kc}}} \right) \frac{\partial \bar{H}}{\partial w_{jb}} \tag{76}$$
$$= \overline{\frac{\partial \bar{A}}{\partial b'}} \left(\frac{\partial b'}{\partial b}\right) (\omega^{\mu\nu}) \left(\frac{\partial b'}{\partial b}\right)^T \left(\frac{\partial \bar{H}}{\partial b'}\right)$$
$$= \frac{\partial A}{\partial r^{ka}} \frac{\partial H}{\partial p_{ka}} - \frac{\partial A}{\partial p_{ka}} \frac{\partial H}{\partial r^{ka}}, \quad (b') = (\underset{\sim}{r}, \underset{\sim}{w}), \quad A = \bar{A}$$
$$(b) = (\underset{\sim}{r}, \underset{\sim}{p}), \quad H = \bar{H}$$

namely, the attached brackets <u>coincide</u> with the conventional Poisson brackets (for the case of Eqs. (33) the attached brackets-via the same rule-are twice the conventional Poisson brackets).[23] This illustrates the nonuniqueness of the generalization of Hamilton's equations of Lie-admissible algebraic character.

As a simple example, the nonconservative system

$$m\ddot{r} + \gamma r \dot{r}^2 = 0 \tag{77}$$

can be represented with Eqs. (73) via the values

$$\bar{H} = \frac{1}{2} w^2, \qquad w = \alpha p z, \qquad (78)$$
$$p = m \dot{z}, \quad m^2 \alpha^2 (1+\alpha) = \gamma.$$

Almost needless to say, the regular Hamilton-admissible equations (33) are preferable over the singular form (73) for numerous reasons, such as the fact that these latter equations lose a consistent geometrical backing. Also, the construction of an analytic representation of systems (1) via singular equations (73) is nontrivial.

Finally, these singular Lie-admissible equations do not appear to be relevant for the construction of the Galilei-admissible relativity for regular nonconservative Newtonian systems.

Neverthless, singular equations (73) are useful to illustrate the <u>variety of realizations of Lie-admissible algebras which are possible in Newtonian Mechanics</u>.

7. <u>GLOBALIZATION PROFILE</u>. Permit me to recall, in the rudimentary language of local coordinates, the fundamental property of symplectic forms, say, in the representation (15), which is at the basis of the coordinate free-globalization of the symplectic geometry. It can be expressed by saying that <u>a symplectic tensor $\Omega_{\mu\nu}$ remains symplectic under arbitrary (but of class C^∞ and regular) transformations</u>.[24] Explicitly, given a tensor $\Omega_{\mu\nu}$ which verifies the nondegeneracy condition and the closure properties (14), all possible transformed tensors

$$b^\mu \to b'^\mu = b'^\mu(b); \quad \Omega_{\mu\nu} \to \Omega'_{\mu\nu} = \frac{\partial b^\rho}{\partial b'^\mu} \Omega_{\rho\sigma}(b(b')) \frac{\partial b^\sigma}{\partial b'^\nu} \qquad (79)$$

verify the same nondegeneracy and closure properties. Indeed, the nondegeneracy of $\Omega'_{\mu\nu}$ is ensured by that of $\Omega_{\mu\nu}$ and of the transformation $b \to b'$, while the closure properties in b'

$$\Omega'_{\mu\nu} + \Omega'_{\nu\mu} = \frac{\partial b^\rho}{\partial b'^\mu} \left(\Omega_{\rho\sigma} + \Omega_{\sigma\rho} \right) \frac{\partial b^\sigma}{\partial b'^\nu} \equiv 0, \qquad (80a)$$

$$\frac{\partial \Omega'_{\mu\nu}}{\partial b'^\tau} + \frac{\partial \Omega'_{\tau\nu}}{\partial b'^\mu} + \frac{\partial \Omega'_{\tau\mu}}{\partial b'^\nu} =$$
$$= \left[\frac{\partial}{\partial b'^\tau} \left(\frac{\partial b^\rho}{\partial b'^\nu} \frac{\partial b^\sigma}{\partial b'^\mu} \right) + \frac{\partial}{\partial b'^\mu} \left(\frac{\partial b^\rho}{\partial b'^\tau} \frac{\partial b^\sigma}{\partial b'^\nu} \right) + \frac{\partial}{\partial b'^\nu} \left(\frac{\partial b^\rho}{\partial b'^\tau} \frac{\partial b^\sigma}{\partial b'^\mu} \right) \right] \Omega_{\rho\sigma}$$
$$+ \left[\frac{\partial b^\rho}{\partial b'^\mu} \frac{\partial b^\sigma}{\partial b'^\nu} \frac{\partial b^\alpha}{\partial b'^\tau} + \frac{\partial b^\rho}{\partial b'^\nu} \frac{\partial b^\sigma}{\partial b'^\tau} \frac{\partial b^\alpha}{\partial b'^\mu} + \frac{\partial b^\rho}{\partial b'^\tau} \frac{\partial b^\sigma}{\partial b'^\mu} \frac{\partial b^\alpha}{\partial b'^\nu} \right] \frac{\partial \Omega_{\rho\sigma}}{\partial b^\alpha}$$
$$\equiv 0, \qquad (80b)$$

are ensured by Eqs. (14).

Along similar lines one can prove the "algebraic image" of this property, namely, that if the tensor $\Omega^{\mu\nu}$ is (nondegenerate and) Lie, that is, it verifies the properties

$$\Omega^{\mu\nu} + \Omega^{\nu\mu} = 0, \qquad (81a)$$

$$\Omega^{\mu\rho} \frac{\partial \Omega^{\nu\tau}}{\partial b^\rho} + \Omega^{\nu\rho} \frac{\partial \Omega^{\tau\mu}}{\partial b^\rho} + \Omega^{\tau\rho} \frac{\partial \Omega^{\mu\nu}}{\partial b^\rho} = 0, \qquad (81b)$$

then, all possible transformed tensors

$$b^\mu \to b'^\mu(b)\ ;\ \Omega^{\mu\nu} \to \Omega'^{\mu\nu} = \frac{\partial b'^\mu}{\partial b^\rho}\,\Omega^{\rho\sigma}(b(b'))\,\frac{\partial b'^\nu}{\partial b^\sigma} \qquad (82)$$

are also (nondegenerate and) Lie, i.e.,

$$\Omega'^{\mu\nu} + \Omega'^{\nu\mu} = 0, \qquad (83a)$$

$$\Omega'^{\mu\rho}\frac{\partial \Omega'^{\nu\tau}}{\partial b'^\rho} + \Omega'^{\nu\rho}\frac{\partial \Omega'^{\tau\mu}}{\partial b'^\rho} + \Omega'^{\tau\rho}\frac{\partial \Omega'^{\mu\nu}}{\partial b'^\rho} = 0 \qquad (83b)$$

The interrelation between the geometrical and the algebraic profile is then established by the fact that, under the familiar rules

$$(\Omega_{\mu\nu}) = (\Omega^{\mu\nu})^{-1},\quad (\Omega'_{\mu\nu}) = (\Omega'^{\mu\nu})^{-1} \qquad (84)$$

the symplectic properties (14) are equivalent to the Lie properties (81), and viceversa,[25] and this equivalence is preserved by arbitrary (class C^∞, regular) transformations. In turn, the preservation of this dual symplectic/Lie character is at the basis of the globalization of the symplectic geometry.

The remarks I intended for this note is that the above properties extend to symplectic-admissible/Lie-admissible formulations. As a matter of fact, the extension is such to <u>apparently</u> indicate the existence of a covering geometry.

The property under consideration can be expressed in terms of the following lemma, where we deal with the more conventional geometrical notion of differentiable structure on manifold.

<u>LEMMA</u> 1: <u>A (nondegenerate) symplectic-admissible structure $S_2 = (S_{\mu\nu}(b))$ in a (Hausdorff, second countable, ∞-differentiable, 6n-dimensional) manifold M with local coordinates $b = (\underset{\sim}{r}, \underset{\sim}{p})$ preserves its (nondegenerate) symplectic-admissible character under arbitrary (class C^∞, invertible) transformations $b \to b' = b'(b)$.</u>

<u>PROOF</u>. Let $S_{\mu\nu}$ be symplectic-admissible, that is, such that the attached structure $S_{\mu\nu} - S_{\nu\mu} = \Omega_{\mu\nu}$ is symplectic. Then, all possible transformed structures

$$b^\mu \to b'^\mu = b'^\mu(b), \qquad (85a)$$

$$S_{\mu\nu} \to S'_{\mu\nu} = \frac{\partial b^\rho}{\partial b'^\mu}\,S_{\rho\sigma}(b(b'))\,\frac{\partial b^\sigma}{\partial b'^\nu}, \qquad (85b)$$

are always symplectic-admissible under the transformations admitted, because the attached structures

$$\begin{aligned} &S'_{\mu\nu} - S'_{\nu\mu} \\ &= \frac{\partial b^\rho}{\partial b'^\mu}\left(S_{\rho\sigma} - S_{\sigma\rho}\right)\frac{\partial b^\sigma}{\partial b'^\nu} \qquad (86) \\ &= \frac{\partial b^\rho}{\partial b'^\mu}\,\Omega_{\rho\sigma}\,\frac{\partial b^\sigma}{\partial b'^\nu} = \Omega'_{\mu\nu} \end{aligned}$$

are the transformed symplectic structures which, as such, are symplectic. The preservation of the nondegenerate character of the structure $S_{\mu\nu}$ and of the attached symplectic structure $\Omega_{\mu\nu}$ is selfevident and this completes the proof. <u>Q.E.D</u>.

The covering character of the property of Lemma 1 over the corresponding property of symplectic structures is selfevident. It was this property which suggested my conjecture in paper[1], Sec. 3.8, of a symplectic-admissible geometry, despite the inapplicability of numerous methodological tools of the symplectic geometry, such as, the calculus of exterior forms, the notion of closure, the Poincaré lemma and its converse, the Lie algebras, etc. For more details see ref.[9]

To study the algebraic counterpart of Lemma 1 we now define as special symplectic-admissible structures, the (nondegenerate) symplectic-admissible structures $S_2(b)$ whose contravarian- form $S^2(b)$ characterizes a Lie-admissible algebra in the strict sense. The proof of the following property is then trivial

LEMMA 2. The preservation of the special symplectic-admissible character of a structure $(S_{\mu\nu})$ under coordinates transformations (of the admitted class) is equivalent to the preservation of the strict Lie-admissible character of the associated contravariant form $(S^{\mu\nu}) = (S_{\mu\nu})^{-1}$, and viceversa.

We shall then symbolically say that the following diagram is closed and invertible[26]

$$\begin{array}{ccccc}
\text{special symplectic-admissible} & (S_{\mu\nu}) & \longleftrightarrow & (S'_{\mu\nu}) & \text{special symplectic-admissible} \\
(S_{\mu\nu}) = (S^{\mu\nu})^{-1} & \updownarrow & b \longleftrightarrow b' & \updownarrow & (S'_{\mu\nu}) = (S'^{\mu\nu})^{-1} \\
\text{strictly Lie-admissible} & (S^{\mu\nu}) & \longleftrightarrow & (S'^{\mu\nu}) & \text{strictly Lie-admissible}
\end{array} \qquad (87)$$

We are now equipped to reinspect our Galilei-admissible relativity

from a new profile. The following remarkable property holds as a consequence of those of Lemmas 1 and 2.

LEMMA 3. The special symplectic-admissible character of the Galilei-admissible relativity with respect to the solution

$$(S_{0\mu\nu}) = \begin{pmatrix} -s_0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (s_0)_{ia\,jb} = (F_{ia}/\partial H/\partial p_{ia})\,\delta_{ij}\,\delta_{ab} \qquad (88)$$

of the fundamental equations

$$\tilde{\dot{\underline{\ }}}_{(1)} = S_{\mu\nu}\,\tilde{\dot{\underline{\ }}}^{\nu}\,db^{\mu} = \dot{\underline{\ }}_{\mu}\,db^{\mu} = dH, \qquad (89)$$

$$\left(\tilde{\dot{\underline{\ }}}^{\nu}\right) = \begin{pmatrix} p_{\mu}/m \\ f^{SA}_{\mu} + F^{NSA}_{\mu} \end{pmatrix}, \quad (b) = \begin{pmatrix} r_{\mu} \\ p_{\mu} \end{pmatrix}, \quad H = T(p_{\mu}) + V(r_{\mu})$$

and the strict Lie-admissible character of the associated contravariant form

$$(S_0^{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ -1 & -s_0 \end{pmatrix} = (S_{0\mu\nu})^{-1} \qquad (90)$$

are independent from the selected local coordinates.

Permit me to confess that this property too is beyond my most optimistic hope at the time of paper[1]. Clearly, this property offers realistic possibilities of reaching a globalization of the Galilei-admissible relativity as a covering of the conventional globalization of the Galilei relativity.[27,28]

* * *

I cannot close this note without a few remarks on the degrees of freedom of the Lie-admissible formulations and an outline of the theorems of Section 3.8 of paper[1] on the symplectic-admissible geometry.

The symplectic limit of our crucial equations (4) yields the familiar Hamiltonian case

$$\Xi_1 = \omega \rfloor \Xi = \omega_{\mu\nu} \Xi^\nu db^\mu = \Xi_\mu db^\mu = dH \qquad (91)$$

which, within a fixed system of equations of motion, Hamiltonian H and local coordinates b, admits a unique solution, the fundamental symplecticf form ω.

The corresponding situation for the symplectic-admissible geometry is fundamentally different. In this case Eqs. (4) admit a family of different solutions $S_{\mu\nu}$ within a fixed system of equations of motion, for a fixed Hamiltonian and within a fixed system of local variables.

Indeed, a first class of different, but equivalent solutions is provided via the algebraic system (43), and consists of the class of symmetric solutions of Eqs. (34) (with s_0 either diagonal or not). A second class of solutions is provided by the system (22) of partial differential equations, with related functional degrees of freedom in the R_μ functions. The existence of additional classes of solutions is then conceivable.

All these different symplectic-admissible tensors $S_{\mu\nu}$ produce the same equations of motion in the same Hamiltonian H and within the same coordinate systems, when used for our Hamilton-admissible equations (33). As such, they constitute a truly intriguing degree of freedom of the Lie-admissible formulations which does not appear to have a direct counterpart for the Lie formulations.

These degrees of freedom are a particular form of geometric isotopy as presented in paper[1] and, of course, realized for

the case of symplectic-admissible isotopy.

As for the case of the symplectic isotopy (ref.[1], page 290), a first, rudimentary characterization of the symplectic-admissible isotopy within a fixed system of local coordinates is provided by any invertible application

$$S_{\mu\nu}(b) \longrightarrow S'_{\mu\nu}(b) \qquad (92)$$

which preserves the symplectic-admissible character of the forms. The preservation of the regularity of $S_{\mu\nu}$ as well as of the attached $\Omega_{\mu\nu}$ is understood.

There exist a number of ways to refine this idea up to a mathematically rigorous formulation . I mention, in particular, the approach to geometric isotopy by A. Banyaga[29] which has been recently brought to my attention. These refinements were not considered in paper[1] and will not be considered at this time.

The problem of the possible existence of a conceivable extension of Darboux Theorem to the symplectic-admissible geometry was investigated in paper[1] via the joint use of the notion of coordinate transformations (diffeomorphisms) and isotopic applications.

Permit me first to recall, in the rudimentary language of local coordinates, that Darboux Theorem, in essense, ensures the existence of a coordinate transformation under which two arbitrarily given symplectic forms can be interpreted as one the transformed of the other. If one of the two forms is the fundamental symplectic form, Darboux Theorem ensures that an arbitrary symplectic structure $\Omega_{\mu\nu}(b)$ can always be reduced to the fundamental symplectic structure, and we shall symbolically write

DARBOUX THEOREM FOR THE SYMPLECTIC GEOMETRY

$$\left(\begin{array}{l}\text{Arbitrary}\\ \text{symplectic}\\ \text{form}\end{array}\right) \longrightarrow \boxed{\text{Diffeomorphism}} \longrightarrow \left(\begin{array}{l}\text{fundamental}\\ \text{symplectic}\\ \text{form}\end{array}\right) \qquad (93)$$

For a rigorous geometrical treatment, see the recent analysis by V. Guillemin and S. Sternberg[28].

In the transition to the symplectic-admissible geometry, Darboux Theorem does not apply as commonly understood, that is, via the sole use of coordinate transformations (diffeomorphisms). This can be easily seen by decomposing a symplectic-admissible structure into a symplectic and a symmetric part

$$S_{\mu\nu} = \Omega_{\mu\nu} + \tilde{S}_{\mu\nu} . \qquad (94)$$

Then the symmetric part prohibits, in general, the existence of a coordinate transformation under which two arbitrarily given symplectic-admissible structures can be considered as one the transformed of the other. In conclusion, an arbitrary symplectic-admissible structure cannot, in general, be reduced to the fundamental structure, say, form (40), via the sole use of coordinate transformations.

However, such a reduction is always possible for the attached symplectic part. The remaining component which cannot be reduced in this way can be seen as belonging to the class of symplectic-admissible isotopy. This is, in essence, the extension of Darboux Theorem to the symplectic-admissible geometry proposed in Section 3.8 of paper[1]. It can be stated by saying that there always exists a coordinate transformation and an isotopic application under which two arbitrarily given symplectic-admissible forms can be transformed one into the other. Equivalently, an arbitrary symplectic-admissible form can always be reduced to the fundamental symplectic-admissible form via a coordinate transformation and an isotopic application. The proposal of paper[1] can be therefore schematically written[30]

EXTENSION OF DARBOUX THEOREM
TO THE SYMPLECTIC-ADMISSIBLE GEOMETRY

$$\left(\begin{array}{l}\text{Arbitrary}\\ \text{symplectic-}\\ \text{admissible}\\ \text{form}\end{array}\right) \longrightarrow \boxed{\begin{array}{l}\text{diffeomorphism}\\ \text{plus}\\ \text{isotopy}\end{array}} \longrightarrow \left(\begin{array}{l}\text{fundamental}\\ \text{symplectic-}\\ \text{admissible}\\ \text{form}\end{array}\right) \qquad (95)$$

Of course, this was intended as noting but an initial study of the problem with a trivial solution for the definition of isotopy given above. The intriguing geometrical problem which is opened by such a proposal is the identification of that notion of isotopy under which reduction (95) is mathematically nontrivial. This problem will not be considered at this time.

An equally intriguing mathematical problem, but this time within the context of the theory of abstract algebras, is opened by the content of this note and, in the final analysis, results to be deeply related to the isotopic degrees of freedom. I am here referring to the fact that the family of equivalent symplectic-admissible solutions $S_{\mu\nu}$ of our fundamental equations (4) do not exibit the same algebraic character, in the sense that their contravariant forms $S^{\mu\nu}$, when realized in the same coordinates b^{μ}, do not verify the same laws.

This is, in essence, the notion expressed by the strictly Lie-admissible algebras, the formal Lie-admissible algebras and, possibly, even additional algebras characterized by the contravariant form of the regular solutions of Eqs. (4).

The algebraic problem under consideration can then be formulated as consisting of the identification of which of the following two possibilities is correct.

ALTERNATIVE I: All the (nonassociative) algebras characterized by the contravariant tensors $S^{\mu\nu}$ associated to the symplectic-admissible structures $S_{\mu\nu}$ via the rule $(S^{\mu\nu}) = (S_{\mu\nu})^{-1}$ belong to the same fundamental class of Lie-admissible algebras and are a manifestation of the degrees of freedom of these algebras with particular reference to the mechanism of grading, the notion of isotopy and the deformation theory.[31]

ALTERNATIVE II: The symplectic-admissible structures $S_{\mu\nu}$ characterize nonequivalent algebras via the contravariant forms $(S^{\mu\nu}) = (S_{\mu\nu})^{-1}$, in the sense that there exist no means of reducing all of them to a unique fundamental class.[32]

The study of this problem is here also left to the interested reader and we content ourself with the direct universality of the Lie-admissible algebras for nonconservative mechanics as pointed out earlier. Neverthless, the solution of the problem here identified is clearly essential for the final identification of the algebraic structure of the symplectic-admissible geometry.

In conclusion, the study of the classical treatment of local nonconservative forces and the problem of their relativity apparently identifies a number of new, intriguing, open mathematical problems, ranging from the notion of symplectic-admissible isotopy

to the actual construction of the symplectic-admissible geometry; from the definition of the radical of the general Lie-admissible algebras to the identification of their classification and representation theory; from the construction of nonassociative envelopping algebras to the study of Lie group with Lie-admissible algebras in the neighborhood of the identity; etc. A corresponding number of additional, equally intriguing and open mathematical problems is identified by the study of the quantum mechanical treatment of nonselfadjoint forces. All these open problems are essentially created by the present lack of sufficient study of the Lie-admissible algebras in both mathematical and physical literatures.

It is appropriate here to recall that, when the Lie algebras made their first appearance in physics, their structure, classification and representation theory had been well developed by mathematicians. In turn, this mathematical study proved to be crucial for a number of physical issues, ranging from cristallography to nuclear physics. On a camparative basis, the Lie-admissible algebras emerge in rather unfavorable grounds. Indeed, now that these generalizations of Lie algebras are making their appearance in physics, their theory has been little developed by mathematicians, on a comparative basis with the corresponding situation for Lie algebras. In turn, this situation might have an adverse effect on the branches of physics, engineering and applied mathematics dealing with the arena of direct universality of Lie-admissible algebras: forces nonderivable from a potential or, equivalently, systems of differential equations which are not derivable from a (conventional) variational principle.

In conclusion, a primary hope of this note is that of stimulating studies on Lie-admissible algebras.

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REFERENCES AND FOOTNOTES

1. R.M.SANTILLI, Hadronic J. 1, 223 (1978).

2. With the term "local" we intend to restrict the admitted motions to those represented via systems of ordinary differential equations, and exclude those represented via systems of integro-differential equations.

3. With the term "regular" we here refer to the condition that the functional determinant of the Newtonian system of second-order differential equations is non-null as a function of the local variables. This allows the construction of an equivalent system of first-order differential equations, i.e., the form of Newtonian systems used in ref.[1] as well as in this note.

4. For simplicity we shall tacitly assume the same conventions and notations of paper[1], including the convention on the sum of repeated indeces.

5. We shall tacitly assume that the symplectic-admissible forms verify a double nondegeneracy condition, that of the tensor $S_{\mu\nu}$ as well as of the attached antisymmetric tensor $\Omega_{\mu\nu}$. Notice that the former does not necessarily imply the latter, and viceversa.

6. It should be here indicated that, in general, if a form $(S^{\mu\nu})$ verifies Eqs. (9), i.e., it is Lie-admissible, its inverse $(S_{\mu\nu})$ does not necessarily verify the condition for symplectic-admissibility, and viceversa. This statement, however, excludes the "degrees of freedom" of Lie-admissible algebras, such as those induced by grading, isotopy deformation, etc. As a matter of fact, one of the intriguing problems

pointed out at the end of this note is precisely that whether all forms $(s^{\mu\nu})$ characterized by the symplectic-admissible geometry can be reduced to a Lie-admissible form via the degrees of freedom indicated.

7. R.M.SANTILLI, Hadronic J. 1, 574 (1978).

8. Regrettably, the presentation of the Lie-admissible formulations of paper[1] contains a number of imperfections, some of which have been reported in the ERRATA-CORRIGE, Hadronic J. 1, 902 (1978). The reader interested in a more technical as well as more detailed presentation of the Lie-admissible formulations is urged to consult ref.[9]

9. R.M.SANTILLI, Lie-admissible approach to the hadronic structure, Vol. II, Hadronic Press, Nonantum, Ma (in press for 1979 distribution).

10. The simplest and most intriguing origin of the non-associative, non-Lie, but Lie-admissible algebras in Newtonian Mechanics of which I am aware lies within the context of the Poisson brackets. Indeed, the brackets

$$A \cdot B = \frac{\partial A}{\partial z^{ka}} \frac{\partial B}{\partial p_{ka}}$$

are nonassociative and non-Lie. Neverthless, they are Lie-admissible because the attached brackets

$$A \cdot B - B \cdot A = [A, B]_{CM} = \frac{\partial A}{\partial z^{ka}} \frac{\partial B}{\partial p_{ka}} - \frac{\partial B}{\partial z^{ka}} \frac{\partial A}{\partial p_{ka}}$$

are Lie (the conventional Poisson brackets). As a result,

we have the following property.

> LEMMA: The conventional Poisson brackets are the attached brackets of a general (i.e., nonassociative, non-Lie and non-flexible[1]) Lie-admissible algebra .

Equivalently, we can say that the general Lie-admissible algebras are at the foundations of the very structure of the conventional Poisson brackets. This property can be better appreciated (and understood, as far as its implications are concerned) by passing to the corresponding quantum mechanical context. Here the product is, of course, Lie and it is still constructed via the rule of Lie-admissibility, as it must be the case for all Lie algebras[1,7]. However, such a quantum mechanical Lie product is now constructed via the associative Lie-admissible product (of operators in a Hilbert space), i.e.,

$$AB - BA = [A, B]_{QM}$$

In different words, in the costumary quantization process of forces derivable from a potential

$$[A, B]_{CM} \longrightarrow \frac{1}{i\hbar} [A, B]_{QM}$$

we do preserve the Lie algebra, as well known. However, and this point does not appear to be sufficiently identified in the available mathematical and physical literature, in the transition from the classical to the quantum mechanical realizations,

we have a profound change in the methods for the construction of a Lie algebra , in the sense that

- CLASSICAL REALIZATIONS OF LIE ALGEBRAS are realized via nonassociative Lie-admissible algebras

$$\boxed{A\cdot B - B\cdot A = \text{LIE}, \quad A\cdot B = \text{NONASSOCIATIVE LIE-ADMISSIBLE};}$$

- QUANTUM MECHANICAL REALIZATIONS OF LIE ALGEBRAS are realized via associative Lie-admissible algebras

$$\boxed{AB - BA = \text{LIE}, \quad AB = \text{ASSOCIATIVE LIE-ADMISSIBLE}.}$$

I claim that this situation has a number of intriguing consequences (which have not been yet investigated, to my knowledge) already at the level of the theory of Lie algebras. This is due to the fact that the vast majority of mathematical studies of Lie algebras have been done in the costumary abstract treatment, in which case the above differentiation is lost. Also, any Lie algebra can be constructed via the rule of associative Lie-admissibility (Poincaré-Birkhoff-Witt Theorem), and this property might be the reason for the lack of emphasis on the Newtonian structure of the Lie algebra product.

The implications of the property identified in this footnote can be indicated as follows. The algebraic structure which is at the basis of numerous crucial aspects (construction of polinomials in the basis, such as, the square of the angular momentum; the representation theory; the transition from a Lie algebra to the corresponding Lie group; etc.) is the enveloping algebra of a Lie algebra, rather than the Lie algebra itself. Certain technical reasons then suggest that the product of such envelop is that of the Lie-admissible rule. Thus, the product of the enveloping algebras used in quantum mechanics

is associative Lie-admissible (the conventional product of operators). The corresponding situation for the classical realizations of the Lie algebras is altered by the property identified in this footnote. Indeed, the product of the "natural" enveloping algebra of the Poisson brackets realizations of Lie algebra is expected to be nonassociative Lie-admissible. This recovers in a natural way the primary idea of presentation of Lie-admissible algebras of paper[1], that of enveloping associative and nonassociative algebras of Lie algebras.

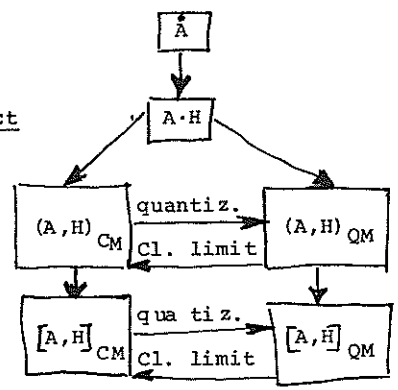
The indicated nonassociative Lie-admissible origin of the Poisson brackets open up a number of intriguing and apparently new problems, the most intriguing of which is the actual construction of the conventional canonical realization of the Galilei group via a nonassociative Lie-admissible envelop. This would then unify the algebraic structure at the basis of both, the Galilei relativity and its Galilei-admissible covering, in the sense that both relativities would result to be different realizations of the same nonassociative enveloping algebras. The relevance for such a possible unified view for classical formulations is selfevident.

It is here appropriate to point out that the indicated nonassociative Lie-admissible origin of the Poisson brackets appears to have implications of particular relevance also for the quantum mechanical profile. In essence, it provides a possible framework for the technical study of the Lie-admissible generalization of Heisenberg's equations proposed in ref.[7]

$$\dot{A} = \frac{1}{i\hbar}(A,H)_{QM} = \frac{1}{i\hbar}(ARH - HSA)$$

$$R \neq \pm S$$

as well as for the rather unpredictable property that these equations too admit an algebra preserving classical limit (ref.[7], page 747), in analogy with the corresponding Lie case. In essence, according to this view, the conventional classical and quantum mechanical treatments of Lie algebras for the case of forces derivable from a potential appear as a particular case of a more general classical and quantum mechanical context for the case of forces non-derivable from a potential, in which the dominant role is played by nonassociative Lie-admissibility; and, most importantly, such role is preserved at <u>both</u> the classical and quantum mechanical level, according to the schematic view below.



11. Of course, the partial derivatives in the b-and R variables commute in a number of particular cases, such as, for brackets in two-dimension, for a linear dependence of the R's in the b's, etc. In these cases brackets (26) are Lie-admissible in the b-variables too.

12. A comparative analysis of brackets (27) with Poisson brackets is here instructive. In essence, the brackets

$$A\cdot B = \frac{\partial A}{\partial z^{ka}}\frac{\partial B}{\partial p_{ka}} \quad , \quad (A,B) = \frac{\partial A}{\partial b^{\mu}}\frac{\partial B}{\partial R_{\mu}}$$

exibit a number of similarities as well as differences. The former are nonassociative Lie-admissible, as pointed out in footnote[10], while the latter have a formally similar structure. Also, for the attached brackets

$$A\cdot B - B\cdot A = \frac{\partial A}{\partial z^{ka}}\frac{\partial B}{\partial p_{ka}} - \frac{\partial B}{\partial z^{ka}}\frac{\partial A}{\partial p_{ka}},$$

$$(A,B) - (B,A) = \frac{\partial A}{\partial b^{\mu}}\frac{\partial B}{\partial R_{\mu}} - \frac{\partial B}{\partial b^{\mu}}\frac{\partial A}{\partial R_{\mu}},$$

we have that the former are Lie, and the latter have a similar structure. Finally, in relation to the variables of the brackets, we have that

$$p_{ka} = \frac{\partial L}{\partial \dot{z}^{ka}} = p_{ka}(t, z, \dot{z}),$$
$$R_{\mu} = R_{\mu}(b),$$

namely, the canonical momenta are related to the coordinates via the familiar canonical prescriptions, while the R-variables are related to the b-variables via the solutions of Eqs. (22). However, there exist crucial differences between the brackets considered. The relationship between the p's and the r's <u>is not</u> integrable to a form independent of the velocities. As a result, the p- and r-variables are independent, as well known. For the case of the (b,R) variables the situation is different. The imposition of the dynamical conditions (22) to brackets (A,H) implies a dependence between the R

and the b-variables. In turn, this has the following consequence. For the case of the Poisson brackets, the quantities A,B,... have a unique functional dependence in the (r,p)-variables. When the same functions A,B,... are reformulated in terms of the (b,R)-variables, such uniqueness is lost, in the sense that each quantity A,B,... can be written in a variety of ways as a function of (b,R). Again, this is due to the fact that, out of the 12n variables (b,R), only 6n are actually independent. However, if the lack of uniqueness of the functional dependence of the elements of the algebra is permitted, brackets (26) are bona fide Lie-admissible brackets, in the sense that the attached brackets are Lie for each selected functional dependence.

In conclusion, the terms "formally Lie-admissible brackets" can be interpreted in more than one way. First of all, one can interpret these brackets without the dynamical conditions (22), in which case they are strictly Lie-admissible because the b and R variables are independent (the formal aspect is then referred to the lack of dynamics). Alternatively, one can represent brackets (26) in the "hypersurface" of the (b,R)-variables under dynamical conditions (22) and accept the lack of unique functional dependence of the elements (in which case the formal character is referred to the lack of algebraic characterization in the dynamical space of the b-variables only). All this applies in the absence of the degrees of freedom of Lie-admissible algebras (grading, isotopy, deformation, etc.)

13. R.M.SANTILLI, Meccanica 1, 3 (1968).

14. The Lagrangian image of Eqs. (33) is also of some significance.

By recalling that, for the class of forces f^{SA} admitted, $\frac{\partial H}{\partial p^{jc}} = \frac{\partial L}{\partial \dot{z}^{jc}} = p_{jc}$ for all j=1,2,...,n and all all a=x,y,z, the configuration space image of the Hamilton-admissible equations (33), as induced by a conventional Legendre transform, can be written

$$\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{z}^{ka}} - \frac{\partial L}{\partial z^{ka}} + S_{kajc}\frac{\partial L}{\partial \dot{z}^{jc}}\right]_{NSA}^{C^\infty, R} = 0$$

The above equations are Lagrange-admissible equations in the sense of paper[1]. It is understood that, for more general functional dependences of the Hamiltonian, the above equations assume a more general form.

15. This example has been already considered in paper[1], Sec. 4, but with an explicit time dependence. We have recalled it here to stress the fact that transformations (58) apply, strictly speaking, for the autonomous case. The extension of the model to an explicit time-dependnece demands the use of the broader contact-admissible geometry.

16. It is an instructive exercise for the interested reader to verify that none of the Galilei-admissible transformations (54), (58) and (63) preserve the structure of the Lie-admissible tensor, that is, they belong to the class of non-identity isotopic transformations[1]

$$S^{\mu\nu}(b) \rightarrow S'^{\mu\nu}(b') = \frac{\partial b'^{\mu}}{\partial b^{\rho}} S^{\rho\sigma}(b(b')) \frac{\partial b'^{\nu}}{\partial b^{\sigma}} \neq S^{\mu\nu}(b') \qquad (*)$$

However, they are canonical-admissible transformations in the sense that they admit the conventional canonical transformations at the limit of null F^{NSA} forces, i.e.,

$$\lim_{F^{NSA} \to 0} S'^{\mu\nu} = \omega^{\mu\nu}. \qquad (**)$$

And indeed, the Galilei transformations are canonical transformations (identity isotopic transformations of the Lie tensor, in the language of paper[1]). Eqs. (*) and (**) can also be considered as the defining conditions for a canonical-admissible transformation.

17. For recent studies by a mathematician on Lie-admissible algebras see H.C.MYUNG, Hadronic J. 1, 169 (1978), and 1, 1021 (1978).

18. Besides the rudimentary treatment of paper[1] and of this note, this geometrical profile does not appear to have been investigated by mathematicians, to my best knowledge. It is hoped that experts of differential geometry will consider the problem because it appears to be relevant (if not crucial) for a fundamental open problem of theoretical physics, namely whether

- the strong interactions are indeed as currently represented, i.e., analytically equivalent to the electromagnetic interactions (local and variationally selfadjoint) by therefore obeing the same relativity and quantum mechanical laws of these latter interactions; or

- the strong interactions are analytically nonequivalent to the electromagnetic interactions (local and variationally nonselfadjoint as an approximation of nonlocal nonderivable from a potential), in which case the problem of the applicable relativity and quantum mechanical laws is open on both theoretical and experimental grounds[7].

In turn, the resolution of this issue is clearly crucial for the

problem of the structure of the strongly interacting particles.

19. Nonconservative systems can be treated with a variety of formulations. With reference to the analytic equations, we here mention the following possibilities (without any claim of completeness).
    (A) Conventional, regular, Hamilton's equations without subsidiary constraints (Inverse Problem[20]);
    (B) Regular Hamilton-admissible equations (Lie-admissible Problem[9]);
    (C) Singular Hamilton-admissible equations (i.e., equations with singular Lie-admissible brackets);
    (D) Singular Hamilton's equations (i.e., conventional equations but with Dirac-type Lie brackets);
    (E) Conventional Hamilton's equations with external terms (i.e., those without an algebraic structure).

20. R.M.SANTILLI, Foundations of Theoretical Mechanics, I and II, Springer-Vergal, Heidelberg, W. Germany (in press for 1978 and 1979 distribution).

21. W.SARLET and F. CANTRIJN, Hadronic J. 1, 101 (1978).

22. See ref.[1], page 327.

23. Notice that the attached brackets are regular. This illustrates the possible occurrence for Lie-admissible formulations

$$\det(S) = 0, \qquad \det(S - S^T) \neq 0$$

as well as the reason for the assumption of double regularity of footnote[5].

24. Of course, this is a typical property for the geometrical treatment in closed form. See in this respect S. STERNBERG, Lectures in Differential Geometry, Prentice Hall, Englewood Cliff, N.J. (1964). For recent treatments of the property in local coordinates see ref.[21], as well as J. KOBUSSEN, Hadronic J. 1, 966 (1978).

25. For a detailed study of this equivalence, see ref.[20], Sections I-2.7 and I-2.9.

26. It should be indicated at this point that, in general, the attached structure ($\Omega_{\mu\nu}$) of the symplectic-admissible forms ($S_{\mu\nu}$) do not coincide with the inverse ($\hat{\Omega}^{\mu\nu}$)$^{-1}$ of the Lie structure attached to the Lie-admissible form ($S^{\mu\nu}$), i.e., the following diagram attached to (87) is open to the right

$$\begin{array}{lcl}
\text{Special symplectic-admissible} & (S_{\mu\nu}) \rightarrow (\Omega_{\mu\nu}) = (S_{\mu\nu} - S_{\nu\mu}) & \text{attached symplectic} \\
(S_{\mu\nu}) = (S^{\mu\nu})^{-1} & \updownarrow & (\Omega_{\mu\nu}) \neq (\hat{\Omega}^{\mu\nu})^{-1} \\
\text{strictly Lie-admissible} & (S^{\mu\nu}) \rightarrow (\hat{\Omega}^{\mu\nu}) = (S^{\mu\nu} - S^{\nu\mu}) & \text{attached Lie}
\end{array}$$

27. J.M.SOURIAU, Structure des Systèmes Dynamiques, Herman, Paris (1970).

28. V. GUILLEMIN and S. STERNBERG, Geometric Adymptotic, Amer. Math. Soc. Providence, R.I. (1977).

29. A. BANYAGA, Comment. Math. Helvetici 53, 174 (1978).

30. Permit me to recall the theorems of paper[1] to this effect, owing to a subsequent errata-corrige of rather crucial importance. In essence, I presented two theorems in Section 3.8 of paper[1] on the extension of Darboux theorem to symplectic-admissible geometry,

one in local and one in closed form. The presentation of a detailed treatment was deferred to ref.[9].

THEOREM 3.8.1: Given the fundamental symplectic-admissible form S_2 on a manifold $M(b,S_2)$ with local coordinates b^μ, $\mu = 1,2,\ldots,6n$, there exist an infinite number of diffeomorphisms $\varphi : M(b,S_2) \longrightarrow M(b',S'_2)$ realizable through class C^∞, everywhere invertible transformations $b \rightarrow b'(b)$ under which the fundamental form S_2 transforms into an arbitrary symplectic-admissible form S'_2. Viceversa, given an arbitrary symplectic-admissible form S'_2 in the local coordinates b', there always exists a (class C^∞, everywhere invertible) transformation $b' \longrightarrow b(b')$ which reduces S'_2 to the fundamental symplectic-admissible form S_2 in b up to local isotopy.

The first part of the above theorem, the direct transition $S_2 \longrightarrow S'_2$, is intended to express the content of Lemma 1 of this note. The inverse transition between two given forms $S'_2 \longrightarrow S_2$ is made possible by the joint use of transformations and isotopies, as elaborate in the text of this note.

We have recalled this theorem because, regrettably, the last words "up to local isotopy" were not printed in the original version of paper[1] (April issue of the Hadronic J., 1978), although they were communicated in the immediately subsequent issue of the Journal (June 1978), in the ERRATA-CORRIGE of page 902. As elaborated in this note, the inverse statement without isotopy is impossible.

The closed version of this theorem was a simple extension of the the presentation by V.Guillemin and S.Sternberg (Ref[28], page 109) of the Darboux-Weinstein theorem and reads as follows.

THEOREM 3.8.2. *Let M_1 be a submanifold of a manifold M and let S_2 and S'_2 be two symplectic-admissible forms such that $S_2|_{M_1} = S'_2|_{M_1}$. Then there exists a neighborhood $N(M_1)$ and of M_1 and a diffeomorphism $f: N(M_1) \longrightarrow M$ such that, up to isotopy,*

(a) $f(m) = m$ for all $m \in M_1$ and

(b) $f^*S_2 = S'_2$.

For the sake of clarity, it should be stressed that the above extensions of Darboux Theorem are *mathematically trivial* because based on a notion of isotopy not sufficiently restrictive. Neverthless, they were sufficient for the *physical* objectives of paper[1]. As indicated in the text, the problem of the proof of the above theorems under a more restrictive notion of isotopy is left to the interested readers.

31. In essence, we would like here to point out the possibility that the formal and strict Lie-admissible brackets are algebraically equivalent, because the latter can be transformed into the former, say, via a grading mechanism (i.e., a Lie-admissible generalization of the Lie mechanism of constructing supersymmetric algebras).

32. We here essentially would like to point out the possibility that the formal and strict Lie-admissible brackets are algebraically inequivalent when treated in the same local variables.