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Dedicated to the memory of Hanno Rund

ISONUMBERS AND GENONUMBERS OF DIMENSION 1, 2, 4, 8, THEIR ISODUALS AND PSEUDODUALS, AND "HIDDEN NUMBERS" OF DIMENSION 3, 5, 6, 7

Ruggero Maria Santilli*
Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
141980 Dubna, Moscow Region, Russia

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Abstract

In this paper we study: new numbers called isonumbers and genonumbers of dimension 1, 2, 4, 8, characterized by certain axiom-preserving liftings of the multiplication for normed algebras with multiplicative identity; the isodual isonumbers and isodual genonumbers of the same dimension, characterized by a certain antiautomorphic conjugation; the pseudoisonumbers, pseudogenonumbers and their isoduals characterized by the further lifting of the addition with loss of the distributive law; and submit the conjecture of "hidden numbers" of dimension 3, 5, 6, 7 which appear to be permitted by the pseudoisotopic and pseudogenotopic techniques, and present an explicit example of dimension 3. We show that the theory of isonumbers is at the foundation of the Lie-isotopic theory, which is a nonlinear-nonlocal-noncanonical, axiom-preserving lifting of of the conventional Lie theory, while the theory of genonumbers is at the foundation of the yet more general Lie-admissible theory. As such, the theories of isonumbers and genonumbers submitted in this paper emerge at the foundation of the ongoing studies of nonlocal interactions in various branches of physics, including nuclear, particle and statistical physics, superconductivity and other fields.

*) **Permanent address:** The Institute for Basic Research, P. O. Box 1577, Palm Harbor, FL, 34682, USA;Fax 1-813-934 9275; E-mail ibrrns@pinet.aip.org

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I: STATEMENT OF THE PROBLEM

As well known, the *theory of numbers* received momentous advances in the past century, thanks to the contributions of famed scholars such as Gauss [1], Abel [2], Hamilton [3], Cayley [4], Galois [5] and others (see review [6] in the early part of this century, and ref.s [7–9] for contemporary presentations).

Additional important advances in number theory were made during this century, including the axiomatic formulation, the theory of algebraic numbers, etc. (see, e.g., ref.s [10] and contributions quoted therein).

The "numbers" significant for this paper are the *real numbers, complex numbers, quaternions* and *octonions*. The topic is therefore the classification of all normed algebras with identity over the reals according to the studies, e.g., by Hurwitz [11], Albert [12] and (N.) Jacobson [13] (see also reviews [7,8]) which can be expressed via the following

THEOREM 1.1 (see, e.g, ref. [8], p. 122): All possible normed algebras with multiplicative unit over the field of real numbers are given by algebras of dimension 1 (real number), 2 (complex numbers), 4 (quaternions) and 8 (octonions).

During an talk at the conference *Differential Geometric Methods in Mathematical Physics* held in Clausthal, Germany, in 1980[1], this author submitted new numbers based on a certain axiom-preserving generalization of the multiplication, today known as *isotopic numbers* or *isonumbers* for short. The generalization is induced by the so-called *isotopies* of the conventional multiplication, with consequential generalization of the multiplicative unit, where the term "isotopy" was suggested from the Greek "ισοσ τοποσ", i.e., "same topology" [14,15]. The author subsequently submitted a new conjugation, under the name of *isoduality* [18–20] which yields an additional class of numbers, today known as *isodual isonumbers.*

These studies were motivated by specific physical needs outlined in this paper and were essentially restricted to the isotopies and isodualities of real and complex numbers. As such, the studies were conducted in the physical literature and do not appear to have propagated as yet to mathematical circles.

In this paper we present a systematic study of the isotopies and isodualities of normed algebras with multiplicative unit of dimensions 1, 2, 4 and 8, including a realization of *isoquaternions* and *isooctonions* and their isoduals in terms of the isotopies and isodualities of Pauli's matrices here presented apparently for the first time.

[1] Thanks to a kind invitation by Prof. H.–D. Doebner which is here gratefully acknowledged.

We then study a generalization of the isonumbers, here called *pseudoisonunmber*, and of their isoduals which are characterized by a certain lifting of the operation of addition, with loss this time of the distributive law.

We also submit a conjecture on the existence of "hidden numbers" of dimension 3, 5, 6, 7 as hidden in the operations of conventional numbers, and present an explicit illustration of dimension 3.

Finally, we introduce, apparently for the first time, an additional new class of numbers called *genonumbers* which are characterized by an axiom-preserving ordering of the isotopies. We then identify the pseudogenonumbers and their isoduals.

The mathematical nontriviality of these new numbers is indicated by the lack of unitary equivalence of isotopic and genotopic theories to conventional ones, the lack of applicability of conventional trigonometry and related Gauss plane in favor of covering notions, and other aspects.

The physical nontriviality stems from the fact that the novel *theory of isonumbers* introduced in this paper is at the foundations of the *Lie-isotopic theory*, which is a certain axiom-preserving isotopy of the conventional formulation of Lie theory for the study of nonlinear, nonlocal and nonhamiltonian systems. The more general *theory of genonumbers* results to be at the foundation of the still more general *Lie-admissible theory*, which is an axiom-inducing *genotopy* (from the Greek γενοσ τοποσ) of Lie's theory [14,15]. As such, the new theory of numbers studied in this paper is at the foundation of the current studies of nonlinear-nonlocal-nonhamiltonian systems in nuclear, particle and statistical physics, superconductivity and other fields.

In the main text of this paper we shall study isonumbers, pseudoisonumbers, "hidden numbers" and their isoduals. Genonumbers, pseudogenonumbers and their isoduals are studied in the appendix. A few introductory sections are presented to render the presentation self-sufficient.

The author would be grateful to any colleague who cares to bring to his attention contributions in the specialized mathematical literature in number theory which are directly or indirectly connected to topic of this paper.

2: PHYSICAL ORIGIN OF ISONUMBERS

The submission of isonumbers was made by this author for the specific physical need of a quantitative representation of the transition from:

a) the *exterior dynamical problem*, i.e., particles moving in the homogeneous and isotropic vacuum (empty space), with consequential local-differential and potential-canonical equations of motion, to

b) the *interior dynamical problem*, i.e., extended and therefore deformable particles while moving within an inhomogeneous and anisotropic physical medium, with consequential equations of motion of the most general known nonlinear, nonlocal-integral and nonpotential-noncanonical type.

Theoretical studies [14–27] (see also the independent reviews [28,31]) have shown that the above transition can be effectively represented via the isotopy of the conventional multiplication of numbers a, b (or functions or operators), from its simplest possible associative form a×b of current use, into the *isotopic multiplication*, or *isomultiplication* for short, introduced in ref. [14]

$$a \hat{\times} b \;:=\; a \times T \times b, \tag{2.1}$$

hereon denoted $\hat{\times} = \times T \times$, where T is a fixed and invertible quantity for all possible elements a, b called *isotopic element*.

The conventional (right and left) multiplicative unit I of current mathematical and physical theories, I×a = a×I ≡ a, is then lifted into the form

$$\hat{1} \hat{\times} a = a \hat{\times} \hat{1} \equiv a, \qquad \hat{1} := T^{-1}, \tag{2.2}$$

called the *multiplicative isounit*.

Under the condition that $\hat{1}$ preserves all the axioms of I (boundedness, smoothness, nowhere degeneracy, Hermiticity and positive-definiteness) the lifting I → $\hat{1}$ is an *isotopy*, that is, the conventional unit I and the isounit $\hat{1}$ (as well as the conventional product a×b and its isotopic form $a\hat{\times}b$) coincide at the abstract level by conception.

The *isonumbers* can be first introduced as the generalization of conventional numbers when characterized by isoproduct (2.1) with respect to the generalized isounit $\hat{1}$.

The consequences shown in ref.s [14–28] are that, for evident mathematical consistency, the isotopies of ordinary numbers imply compatible liftings of all mathematical structures used in physics [29,30]

isonumbers → isofields → → isospaces → isotransformations →
→ isoalgebras → isogroups → isosymmetries →
→ isorepresentations → isogeometries, etc.

The isotopic generalizations of classical [24,25] and quantum [26,27] Hamiltonian mechanics (with interconnecting isotopic quantization) are then consequential with the resulting capability to represent nonlinear, nonlocal and noncanonical systems.

In fact, the isounit $\hat{1}$ is generally assumed to be *outside* the original field, with the most general possible, axiom-preserving, integro-differential dependence on local coordinates x and their derivatives with respect to an

endent variable t of arbitrary order $\dot{x}$, $\ddot{x}$, ..., wavefunctions ψ(t,x), ψ†(t,x) and derivatives also of arbitrary order ∂ψ, ∂ψ†, ..., as well as any needed ional quantity to represent the physical media of the interior problem, such nsity μ, temperature τ, index of refraction n, etc.

$$\hat{1} = \hat{1}(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n, \ldots). \tag{2.3}$$

Conventional, local-potential systems are represented by only one quantity, :amiltonian H over the ordinary field R of real numbers, which implies the assumption of the trivial quantity I as basic unit. The more general ›cal-nonpotential systems are represented by the two independent quantities, ıme Hamiltonian H and generalized unit (2.3).

Stated in a nutshell, the isounit $\hat{1}$ can be interpreted as providing a ıetrization of the nonlinear, nonlocal and noncanonical, as well as nogeneous and anisotropic characters of physical media, in such a way to t the conventional geometrization of the homogeneous and isotropic vacuum ›articular case.

A mathematical presentation of the above ideas can be found in memoirs] (see also the independent review [31]).

This author briefly inspected the lifting of the addition in ref. [21]

$$+ \;\rightarrow\; \hat{+} = +\hat{K}+, \quad \hat{K} = K\times\hat{1} \tag{2.4}$$

consequential redefinition of the conventional additive unit

$$0 \rightarrow \hat{0} = -\hat{K}. \tag{2.5}$$

However, unlike the isotopy of the multiplication $\times \rightarrow \hat{\times}$, the lifting of the ion $+ \rightarrow \hat{+}$ implies the general loss of the right and left distributive laws (see 4). Thus, only the lifting of the multiplication continues to be used for ical applications at this time. The understanding is that the lifting of the ion is indeed mathematically intriguing and it will be studied in this paper at light.

3: PHYSICAL ORIGIN OF ISODUAL NUMBERS AND ISODUAL ISONUMBERS

The isodual isonumbers were introduced in ref.s [18–20] via the following ıgation of multiplication (2.1)

$$a \hat{\times} b \rightarrow a \hat{\times}^d b := a \times T^d \times b = - a \times T \times b = - a \hat{\times} b, T^d = - T, \quad (3.1)$$

under the name of *isoduality*. The isounit $\hat{1}$ is then no longer the (left and right) unit of the theory and must be lifted into the form

$$\hat{1}^d \hat{\times}^d a = a \hat{\times}^d \hat{1}^d \equiv a, \quad \hat{1}^d := - \hat{1}, \quad (3.2)$$

called *isodual isounit*.

The *isodual isonumbers* where first conceived as characterized by isodual multiplication (3.1) with respect to the the multiplicative isodual isounit $\hat{1}^d$.

Note that the notion of isoduality first applies to conventional numbers. In fact, the expressions

$$T^d = - 1, \quad \hat{1}^d = 1^d := - 1, \quad (3.3)$$

characterize *isodual numbers* consisting of *isodual reals, isodual complex, isodual quaternions and isodual octonions*. The isodual isonumbers then occurs for the most general possible isodual isomultiplication (3.1) and isodual isounit $\hat{1}^d = - \hat{1}$.

One can now see the necessity of lifting the product $\times \rightarrow \hat{\times}$ for the very conception of isodual numbers and isodual isonumbers. The restriction of the studies in number theory to the conventional multiplication $\times$ may therefore be a reason why isodual numbers have escaped detection until recently.

The isodual numbers and isodual isonumbers also emerged from quite specific physical needs according to the following general overview [21,22,25,27]:

1) conventional numbers are and will remain fundamental for the characterization of ordinary particles in vacuum (*exterior dynamical problem of particles*);

2) isonumbers are useful for the characterization of ordinary particles when moving within physical media (*interior dynamical problem of particles*);

3) isodual numbers are useful for the characterization of ordinary antiparticles in vacuum (*exterior dynamical problem of antiparticles*); and

4) isodual isonumbers are useful for the characterization of antiparticles moving within physical media (*interior dynamical problem for antiparticles*).

The treatment of antiparticles with isodual numbers emerged from a reinterpretation of the customary characterization of antiparticles via negative-energy solutions of Dirac's equations. As well known, such solutions behave in an unphysical way when conventionally interpreted, that is, interpreted with respect to the same numbers and unit 1 of particles, thus forcing physicists into various hypothetical assumption, such as postulating infinite seas of undetectable

particles which have left the characterization of antiparticles still resolved to this day.

On the contrary, the same negative energy solutions behaved in a fully physical way when reinterpreted as belonging to the field of isodual numbers, i.e., when reinterpreted as being defined with respect to the isodual multiplication $\times^d = \times (-1) \times$ and isodual unit $1^d = -1$. In particular, this reinterpretation implies no need of hypothesizing seas of undetectable particles.

The treatment of antiparticles with isodual numbers has rather intriguing geometrical implications. In fact, it permits the mathematical prediction of a hitherto unknown universe, called *isodual universe*, which is interconnected to our universe via isoduality, and identified by the isotopies of the Riemannian geometry and their isoduals [25,30,31].

In this paper we shall conduct a systematic study of the theory of isonumbers and their isoduals because they have a mathematical significance per se, irrespective of any possible physical application.

In closing these introductory words, the reader not familiar with isotopies should be alerted against the use of conventional mathematical thinking under isotopies because leading to (often undetected) inconsistencies. As an example, traditional statements of the type "two multiplied by two equals four" are, at best, mathematically incomplete because lacking the joint identification of the related unit, and they are inapplicable under isotopies. In fact, if we assume for multiplicative unit $\hat{1} = 3^{-1}$, "two multiplied two equal twelve".

Additional, often undetected inconsistencies occur in the preservation under isotopies of conventional operations on vector spaces and their completion, e.g., into Hilbert spaces, which have motivated the recent identification of a new branch of functional analysis under the name of *functional isoanalysis* [32].

As an example, the notion of exponentiation has no mathematical meaning under isotopy, evidently because of the lack of conventional multiplication needed for its definition as a power series expansion [14,15]; the notion of unitarity is also inapplicable because, again, referred to conventional products and units [11,13]; etc.

For these isotopic operations we refer the interested reader to ref.s [26,27]. Here we limit ourselves to recalling for later use that the notion of determinant of a matrix A is also inapplicable under isotopies because it does not preserve the basic axioms. We have instead the *isodeterminant* [16,21,26,27]

$$\hat{\text{Det}}\, A := [\, \text{Det}_F (A \times T)\,] \times \hat{1}, \quad (3.4)$$

where Det A represents the conventional determinant computed in the selected (ordinary) field F, which does preserve the axioms of Det A at the isotopic level because

$$\hat{Det}(A \hat{\times} B) = (\hat{Det} A) \hat{\times} (\hat{Det} B), \quad \hat{Det}(A^{-1}) = (\hat{Det} A)^{-1}. \quad (3.5)$$

The corresponding *isodual isodeterminant* is given by

$$\hat{Det}^d A := [\mathop{Det}_F (A \times T^d)] \times 1^d ; \quad (3.6)$$

Similar isotopic liftings occur for trace, Hermiticity, unitarity and all other operations [27].

4: ISOFIELDS, PSEUDOISOFIELDS AND THEIR ISODUALS

To render this paper minimally self-sufficient as well as for notational convenience, it appears recommendable to outline the essential background notions needed for the analysis.

Let us begin with the following definition of isofields [29].

DEFINITION 4.1: Let $F = F(a, +, \times)$ *be a "field", here defined as a ring with elements a, b, c, ..., which is commutative and associative under the operation of conventional addition + and (generally nonassociative but) alternative under the operation of conventional multiplication ×with corresponding additive unit 0 and multiplicative unit I. Then, the "isofields"* $\hat{F} = \hat{F}(\hat{n}, +, \hat{\times})$ *are given by elements* $\hat{a}, \hat{b}, \hat{c}, \ldots$ *characterized by one-to-one and invertible maps* $a \Rightarrow \hat{a}$ *of the original elements* $a \in F$ *equipped with two operations* $(+, \hat{\times})$*, the conventional addition + of F and a new multiplication* $\hat{\times}$ *called "isomultiplication", with corresponding conventional additive unit 0 and a generalized multiplicative unit* $\hat{1}$*, called "multiplicative isounit", which are such to verify all axioms of the original field F, i.e.:*

1) Axioms of addition:

1.A) The set $\hat{F}$ *is closed under addition,*

$$\hat{a} + \hat{b} \in \hat{F} \quad \forall\, \hat{a}, \hat{b} \in \hat{F}, \quad (5.1)$$

1.B) The addition is commutative for all elements $\hat{a}, \hat{b} \in \hat{F}$

$$\hat{a} + \hat{b} = \hat{b} + \hat{a}; \quad (4.2)$$

1.C) The addition is associative for all $\hat{a}, \hat{b}, \hat{c}, \in F,$

$$\hat{a} + (\hat{b} + \hat{c}) = (\hat{a} + \hat{b}) + \hat{c}; \quad (4.3)$$

1.D) There is an element 0, the "additive unit", such that for all elements $\hat{a} \in \hat{F}$

$$\hat{a} + 0 = 0 + \hat{a} \equiv \hat{a}; \quad (4.4)$$

1.E) For each element $\hat{a} \in \hat{F}$*, there is an element* $-\hat{a} \in \hat{F}$*, called the "opposite of* $\hat{a}$*", which is such that*

$$\hat{a} + (-\hat{a}) = 0 \quad (4.5)$$

2) Axioms of isomultiplication:

2.A) The set $\hat{F}$ *is closed under isomultiplication,*

$$\hat{a} \hat{\times} \hat{b} \in \hat{F}, \quad \forall\, \hat{a}, \hat{b} \in \hat{F}, \quad (4.6)$$

2.B) The multiplication is generally non-isocommutive, i.e., $\hat{a} \hat{\times} \hat{b} \neq \hat{b} \hat{\times} \hat{a}$*, but "isoalternative", i.e., it verifies the following left and right isoalternative laws for all elements* $\hat{a}, \hat{b}, \in \hat{F}$

$$\hat{a} \hat{\times} (\hat{b} \hat{\times} \hat{b}) = (\hat{a} \hat{\times} \hat{b}) \hat{\times} \hat{b}; \quad (\hat{a} \hat{\times} \hat{a}) \hat{\times} \hat{b} = \hat{a} \hat{\times} (\hat{a} \hat{\times} \hat{b}), \quad (4.7)$$

2.C) There exists a quantity $\hat{1}$*, the "multiplicative isounit", which is such that, for all elements* $\hat{a} \in \hat{F}$,

$$\hat{a} \hat{\times} \hat{1} = \hat{1} \hat{\times} \hat{a} \equiv \hat{a}, \quad (4.8)$$

2.D) For each element $\hat{a} \in \hat{F}$*, there is an element* $\hat{a}^{-1} \in \hat{F}$*, called the "isoinverse", which is such that*

$$\hat{a} \hat{\times} (\hat{a}^{-1}) = (\hat{a}^{-1}) \hat{\times} \hat{a} = \hat{1}. \quad (4.9)$$

3) Properties of joint addition and isomultiplication:

3.A) The set $\hat{F}$ *is closed under joint isomultiplication and addition,*

$$\hat{a} \hat{\times} (\hat{b} + \hat{c}) \in \hat{F}, \quad (\hat{a} + \hat{b}) \hat{\times} \hat{c} \in \hat{F}, \ \forall\, \hat{a}, \hat{b}, \hat{c} \in \hat{F}, \quad (4.10)$$

3.B) All elements $\hat{a}, \hat{b}, \hat{c} \in \hat{F}$ *verify the right and left "isodistributive laws"*

$$\hat{a} \hat{\times} (\hat{b} + \hat{c}) = \hat{a} \hat{\times} \hat{b} + \hat{a} \hat{\times} \hat{c}, \quad (\hat{a} + \hat{b}) \hat{\times} \hat{c} = \hat{a} \hat{\times} \hat{c} + \hat{b} \hat{\times} \hat{c}, \quad (4.11)$$

When there exists a least positive integer p *such that the equation*

$$p\,\hat{\times}\,\hat{a} = 0\,, \qquad (4.12)$$

admits solution for all elements $\hat{a} \in \hat{F}$, *then* $\hat{F}$ *is said to have "isocharacteristic p". Otherwise,* $\hat{F}$ *is said to have "isocharacteristic zero".*

The elements $\hat{a}$ *of isofields* $\hat{F}(\hat{a}+, \hat{\times})$ *are called "isonumbers".*

The reader is aware that there are various definitions of "fields" in the mathematical literature [7–10], with stronger or weaker conditions depending on the case at hand. Often, "fields" $F(a, +, \times)$ are assumed to be *associative under the multiplication* (see, e.g., ref. [8], p. 101)

$$a \times (b \times c) = (a \times b) \times c \quad \forall\, a, b, c, \in F\,, \qquad (4.13)$$

while in Definition 4.1 we have assumed "fields" to be alternative, i.e.,

$$a \times (b \times b) = (a \times b)\, b\,, \quad (a \times a) \times b = a \times (a \times b)\,, \ \forall\, a\,, b \in F, \qquad (4.14)$$

which is an evident generalization of associativity because every associative ring is also alternative, but an alternative ring is not necessarily associative (see ref. [8] for details).

Therefore, the "isofields" as per Definition 4.1 are not, in general isoassociative, i.e., they generally violate the *isoassociative law of the multiplication*

$$\hat{a}\,\hat{\times}\,(\hat{b}\,\hat{\times}\,) = (\hat{a}\,\hat{\times}\,\hat{b})\,\hat{\times}\,\hat{c} \quad \forall\ \hat{a}, \hat{b}, \hat{c} \in \hat{F}\,, \qquad (4.15)$$

and verify instead the weaker isoalternative laws (4.7).

The above assumptions are suggested by our need to reach a definition of "number" which is unified with the results of Theorem 1.1, and includes: the real numbers $R(n,+,\times)$, complex numbers $C(c,+,\times)$, quaternions $Q(q,+,\times)$ and octonions $O(o,+,\times)$. The corresponding "isofields" given by *isoreal numbers* $\hat{R}(\hat{n},+,\hat{\times})$, *isocomplex numbers* $\hat{C}(\hat{c},+,\hat{\times})$, *isoquaternions* $\hat{Q}(\hat{q},+,\hat{\times})$ and *isooctonions* $\hat{O}(\hat{o},+,\hat{\times})$, the latter being isoalternative but not isoassociative.

The realizations of the isonumbers, isomultiplication and related isounits used in this paper are those reviewed earlier, i.e.,

$$\hat{a} = a \times \hat{1}, \quad \hat{\times} = \times T \times, \quad \hat{1} = T^{-1}. \qquad (4.16)$$

where $\times$ is evidently the original multiplication in F, and $\hat{1}$ preserves all

properties of 1 (smoothness, boundedness, nondegeneracy, Hermiticity, and positive-definiteness).

Note that the lifting $\times \to \hat{\times} = \times T \times$ is an isotopy in the sense that it preserves the axioms verified by the original multiplication $\times$, i.e., if $\times$ is associative, $\hat{\times}$ is isoassociative, if $\times$ is alternative $\hat{\times}$ is isoalternative, etc. (see ref. [29], Sect. 5 for details).

Thus, "fields" and "isofields" as per Definition 4.1 coincide at the abstract level by conception, as a necessary condition to have an isotopy. In fact, all distinctions between the multiplications $\times$ and $\hat{\times}$ (as well as between the unit 1 and the isounit $\hat{1}$) cease to exist at the abstract level.

The liftings $a \to \hat{a}$, and $\times \to \hat{\times}$ can be used jointly or individually. The following properties are then important for our analysis.

PROPOSITION 4.1: Necessary and sufficient condition for the lifting (where the multiplication is lifted but the elements are not)

$$F(a, +, \times) \ \to \ \hat{F}(a, +, \hat{\times}), \quad \hat{\times} = \times T \times, \quad \hat{1} = T^{-1} \qquad (4.17)$$

to be an isotopy (that is, for $\hat{F}$ *to verify all axioms of the original field* F*) is that* T *is a non-null element of the original field* F.

In fact, the laws of addition are unchanged, while the multiplication and distributive laws can be readily verified to hold. The closure of the original set under the addition is evident because that operation is not changed. We then remain with the closure under the isomultiplication,

$$a\,\hat{\times}\,b \ = \ a \times T \times b \ \in \hat{F}\,, \ \ \forall\, a, b \in \hat{F}\,, \qquad (4.18)$$

which does indeed hold when $T \in \hat{F}$, by therefore establishing the sufficiency of the condition. Its necessity follows from simple contrary arguments.

PROPOSITION 4.2: The lifting (where both the multiplication and the elements are lifted)

$$F(a, +, \times) \ \to \ \hat{F}(\hat{a}, +, \hat{\times}), \ \hat{a} = a \times \hat{1}, \ \hat{\times} = \times T \times, \quad \hat{1} = T^{-1}, \qquad (4.19)$$

constitutes an isotopy even when the multiplicative isounit $\hat{1}$ *is not an element of the original field* F.

In fact, one can readily verify the validity of all axioms of a field, and closure under addition. Closure under multiplication readily holds because

$$\hat{a} \hat{\times} \hat{b} = (a \times b) \times 1 = c \times 1 = \hat{c} \in \hat{F}, \quad \forall a, b, c = a \times b \in F, \quad (4.20)$$

The mathematically simple Proposition 4.2 expresses the physically fundamental capability of generalizing Planck's unit $\hbar = 1$ of quantum mechanics into an integro-differential operator $\hat{1}$ for a quantitative treatment of nonlocal interactions [26,27].

A first application of the isotopies of numbers is the following. As well known, the set of purely imaginary numbers S = {in} *is not* a field, evidently because it is not closed under the multiplication, in×im = −nm ∉ S. However, the set of real numbers $\hat{S}(\hat{n},+,\hat{\times})$, $\hat{n} = ni$ with the purely imaginary isounit $\hat{1} = i$ is indeed an isofield, that is, it verifies all axioms of a field, including the closure under the isomultiplication, because $T = i^{-1}$, and we have $in \hat{\times} im = inm \in \hat{S}$.

This illustrates the possibility that, when a given set does not constitute a field, there may exist an isotopy under which it verifies all axioms for a field.

The following property illustrates the reasons for restricting the isotopies in Definition 5.1 to only those of the multiplication.

PROPOSITION 4.3: The lifting

$$F(a, +, \times) \rightarrow \hat{F}(\hat{a}, \hat{+}, \hat{\times}), \quad (4.21a)$$

$$\hat{a} = a \times 1, \quad \hat{+} = + \hat{K} +, \quad \hat{0} = -\hat{K} = -K \times \hat{1}, \quad \hat{\times} = \times T \times, \quad \hat{1} = T^{-1}, \quad (4.21b)$$

where K is an element of the original field F and T is arbitrary invertible quantity, is not an isotopy for all nontrivial values of the quantity K ≠ 0, because it preserves all axioms of Definition 4.1, except the distributive law (4.11)

.

In fact, all axioms (4.1)–(4.10) can be readily verified to be preserved under liftings (5.21). On the contrary, for the right distributive law we have

$$\hat{a} \hat{\times} (\hat{b} \hat{+} \hat{c}) = a \times (b + K + c) \times \hat{1} = (a \times b + a \times K + a \times c) \times \hat{1} \neq$$

$$\neq \hat{a} \hat{\times} \hat{b} \hat{+} \hat{a} \hat{\times} \hat{c} = (a \times b + K + a \times c) \times \hat{1}, \quad (4.22)$$

with similar lack of identities for the left isodistributive law. Note that the set $\hat{F}$ in lifting (4.21) is closed under isoaddition for K ∈ F (but not for K ∉ F), and, separately, under isomultiplication for an arbitrary isounit $\hat{1}$ outside the original set F. The same results hold for the lifting $F(a, +, \times) \rightarrow \hat{F}(a, \hat{+}, \hat{\times})$, $\hat{+} = + K +$, K ∈ F, K ≠ 0.

A central property expressed by Proposition 4.3 is that lifting (5.21) is not an isotopy because one of the original axioms is not preserved. We shall then use the term *"pseudoisotopy"* to denote the preservation of only part of the original axioms.

DEFINITION 4.2: Let $\hat{F}(\hat{a},+,\hat{\times})$ be an isofield as per Definition 4.1. Then, the "pseudoisofields" are given by the images of $\hat{F}(\hat{a},+,\hat{\times})$ under all possible further liftings of the addition $+ \rightarrow \hat{+} = + \hat{K} +$, with additive isounit $\hat{0} = -\hat{K} = -K \times \hat{1}$, K ≠ 0, in which case the elements $\hat{a}$ are called "pseudoisonumbers".

After having identified the notions of fields, isofields and pseudoisofields, we now study their *isotopic conjugation*, that is, their images under change of sign of the isounit

$$\hat{1} \rightarrow \hat{1}^d = -\hat{1}, \quad (4.23)$$

called *isoduality* [20–22].

DEFINITION 4.3: Let F(a,+,×) be a field as per Definition 4.1. Then the "isodual field" $F^d(a^d,+,\times^d)$ is constituted by elements called "isodual numbers"

$$a^d := a \times I^d = -a, \quad (4.24)$$

defined with respect to the "isodual multiplication" and related "isodual unit"

$$\times^d := \times I^d \times = -\times, \quad I^d = -I. \quad (4.25)$$

Let $\hat{F}(\hat{a},+,\hat{\times})$ be an isofield as per Definition 4.1. Then, the "isodual isofield" $\hat{F}^d(\hat{a}^d,+,\hat{\times}^d)$ is given by "isodual isonumbers"

$$\hat{a}^d := a^c \times \hat{1}^d = -a^c \times \hat{1}, \quad (4.26)$$

where a^c is the conventional conjugation of F (e.g., complex conjugation), defined in terms of the "isodual isomultiplication"

$$\hat{\times}^d := \times T^d \times = -\hat{\times}, \quad T^d = -T. \quad (4.27)$$

Finally, let $\hat{F}(\hat{a},\hat{+},\hat{\times})$ be a pseudoisofield as per Definition 4.2. Then the "isodual pseudoisofield" $\hat{F}^d(\hat{a}^d,\hat{+}^d,\hat{\times}^d)$ is given by the image of the original isofield under isodualities (4.25) and (4.26), plus the additional isoduality

$$\hat{0} \rightarrow \hat{0}^d = -0. \tag{4.28}$$

and its elements $\hat{a}^d$ *are called "isodual pseudoisonumbers".*

A few comments are now in order. All conventional operations with numbers depending on the multiplication are evidently altered under lifting to isofields. Let us consider first the isofields $\hat{F}(a, +, \hat{\times})$ of Proposition 4.1. Then, the "square" $a^2 = a \times a$ is lifted into the *isosquare* $a^{\hat{2}} = a \times T \times a$, with *n-th isopower*

$$a^{\hat{n}} = a \times T \times a \times T \times a \times \ldots\ldots \times T\, a \quad \text{(n times)} \tag{4.29}$$

Recall that the conventional square root can be defined as the quantity $a^{\frac{1}{2}}$ such that $(a^{\frac{1}{2}}) \times (a^{\frac{1}{2}}) = a$. Then, for the simple case in which T commutes with all elements $a \in \hat{F}$, the *isosquare root* is given by

$$a^{\hat{\frac{1}{2}}} = a^{\frac{1}{2}} \times \hat{1}^{\frac{1}{2}}, \quad a^{\hat{\frac{1}{2}}} \hat{\times} a^{\hat{\frac{1}{2}}} = a^{\hat{\frac{1}{2}}} \times T \times a^{\hat{\frac{1}{2}}} = a. \tag{4.30}$$

The *isoinverse*, from Eq. (4.9), is given by

$$a^{\hat{-1}} = \hat{1}\, a^{-1}\, \hat{1}. \tag{4.31}$$

The *isoquotient* can then be defined by

$$a \hat{/} b = c', \qquad c' \times T \times b = a. \tag{4.32}$$

The reader can then compute all other isooperations.

Note that the isounit $\hat{1}$ is idempotent of arbitrary (finite) order n as the original one

$$\hat{1}^{\hat{n}} = \hat{1} \times T \times \hat{1} \times T \times \ldots\ldots \times T \times \hat{1} \quad \text{(n times)} \equiv \hat{1} \tag{4.33}$$

the isosquare root of the isounit is the isounit itself,

$$\hat{1}^{\hat{\frac{1}{2}}} = \hat{1}, \tag{4.34}$$

and the isoquotient of the isounit by itself is the isounit,

$$\hat{1} \hat{/} \hat{1} \equiv \hat{1}, \tag{4.35}$$

thus confirming the isotopic nature of the lifting $1 \rightarrow \hat{1}$.

Fully equivalent expressions hold for the isofields $\hat{F}(\hat{a}, +, \hat{\times})$ of Proposition 5.2, for which we have

$$\hat{a}^{\hat{2}} = \hat{a} \hat{\times} \hat{a} \hat{\times} \ldots \hat{\times} \hat{a} = a^2 \times \hat{1}, \tag{4.36a}$$

$$\hat{a}^{\hat{\frac{1}{2}}} = a^{\frac{1}{2}} \times \hat{1}^{\frac{1}{2}}, \tag{4.36b}$$

$$\hat{a} \hat{/} \hat{b} = \hat{c} = c \times \hat{1}, \quad \hat{c} \hat{\times} \hat{b} = (c \times b) \times \hat{1} = \hat{a}. \tag{4.34c}$$

Note also that the number 1 may be an element of the isofield $\hat{F}(a,+,\hat{\times})$, although it is no longer the unit. Similarly, the number 0 may be an element of the pseudoisofield $\hat{F}(a,\hat{+},\times)$, but it is no longer the additive identity.

Kadeisvili [32] provided an important classification of isounits into five primary classes which is hereon tacitly assumed.. In this paper we shall only study two out of five classes, namely, Class I with $\hat{1} > 0$ for isofields and Class II with $\hat{1} < 0$ for isodual isofields, for the sole case of isocharacteristic zero. Among the remaining classes not studied in this paper for brevity, Class IV is particularly intriguing inasmuch as it deals with degenerate isotopic elements $T \rightarrow 0$ and corresponding singular isounits $\hat{1} \rightarrow \infty$ which can represent gravitational collapse into a singularity.

5: ISOSPACES, PSEUDOISOSPACES AND THEIR ISODUALS

Consider a metric or pseudo metric n-dimensional space $S(x,g,R(n,+,\times))$ with local coordinates x and (Hermitean) metric $g = g^\dagger$ over the reals $R(n,+,\times)$. Another notion needed for our analysis is given by the *isospaces* $\hat{S}(x,\hat{g},\hat{R}(n,+,\hat{\times}))$ over the isoreals $\hat{R}(n,+,\hat{\times})$ (see Proposition 4.1), first introduced in ref. [18] (see also ref.s [19,23,28]),

$$\hat{S}(x,\hat{g},\hat{R}(n, +, \hat{\times})), \quad \hat{g} = T \times g, \quad \hat{\times} = \times\, T\,, \quad \hat{1} = T^{-1}; \tag{5.1}$$

The *isoduals isospaces*, first introduced in ref. [20] (see also ref.s [21,23,28]) are then given by

$$\hat{S}^d(x,\hat{g}^d,\hat{R}^d(\hat{n}^d,+,\hat{\times}^d)): \hat{g}^d = T^d \times g = -T \times g, \ \hat{\times}^d = \times\, T^d \times = -\times T \times, \ \hat{1}^d = -\hat{1}. \tag{5.2}$$

Again, as it is the case for isotopies of fields, isospaces $\hat{S}(x,\hat{g},\hat{R})$ coincide, by construction with the conventional spaces $S(r,g,R)$ at the abstract, realization-free level, thus verifying the isomorphism $\hat{S}(x,\hat{g},\hat{R}) \approx S(x,g,\mathfrak{R})$. Nevertheless, the former have the most general known curvature and integral character owing to the arbitrariness in the isotopic element T. In fact, the isometrics $\hat{g} = T \times x$ have the most general possible, nonlinear, nonlocal and noncanonical dependence in all variables,

$$g = g(x) \rightarrow \hat{g} = T(t, x, \dot{x}, \ddot{x}, ...) \times g(x) \quad \hat{g}(t, x, \dot{x}, \ddot{x}, ...). \tag{5.3}$$

Similarly, isodual isospaces $\hat{S}^d(x,\hat{g},\hat{R}^d(n^d,+,\hat{\times}^d))$ are locally isomorphic to the *isodual spaces* $S^d(x,g^d,R^d(n^d,+,\times^d))$, which are conventional spaces although defined on the isodual real fields R^d.

The isospaces most important for physical and mathematical applications are the *isoeuclidean spaces* $\hat{E}(x,\hat{\delta},\hat{R})$, *isominkowski spaces* $\hat{M}(x,\hat{\eta},\hat{R})$ and *isoriemannian spaces* $\hat{R}(x,\hat{g},\hat{R})$, which are at the foundations of the representation of nonlinear, nonlocal and noncanonical interior systems in nonrelativistic, relativistic and gravitational interior problems, respectively (see ref.s [25,29] for details).

Note that in the above definition the local coordinates x and numbers n of an isospace $\hat{S}(x,\hat{g},\hat{R}(n,+,\hat{\times}))$ are not lifted into the forms $\hat{x} = x \times \hat{1}$, $\hat{n} = n \times \hat{1}$, which renders them the vector space equivalent of Proposition 4.1. Needless to say, the liftings $x \rightarrow \hat{x} = x \times \hat{1}$ and $n \rightarrow \hat{n} = n \times \hat{1}$ are indeed possible, implying the additional forms $\hat{S}(x,\hat{g},\hat{R}(\hat{n},+,\hat{\times}))$, primarily used in this paper, and $\hat{S}(\hat{x},\hat{g},\hat{R}(\hat{n},+,\hat{\times}))$.

Given an isospace $\hat{S}(x,\hat{g},\hat{R}(n, +, \hat{\times})$, then a *pseudoisospace* is given by the image $\hat{S}(x,\hat{g},\hat{R}(n,\hat{+},\hat{\times})$ of the original space characterized by the further lifting $+ \rightarrow \hat{+} = + K +$, $0 \rightarrow \hat{0} = - K$. The *isodual pseudoisospace* is then defined accordingly.

6: ISOTRANSFORMATIONS

Another notion needed for this paper is the applicable transformation theory. Consider an isospace $\hat{S}(x,\hat{\delta},\hat{R}(n,+,\hat{\times}))$. Conventional linear, local and canonical transformations $x' = A \times x$ are now afflicted by a host of mathematical inconsistencies (such as the violation of linearity, transitivity and others) whenever applied to $\hat{S}(x,\hat{g},\hat{R}(n,+,\hat{\times}))$. For this reason this author introduced in ref. [14] the *isotransformations* as the right isomodular actions on $\hat{S}(x,\hat{\delta},\hat{R}(n,+,\hat{\times}))$

$$x' = A \hat{\times} x = A \times T \times x, \tag{6.1}$$

where the isotopic element T is fixed for all $x \in \hat{S}$, which now do verify all needed conditions, although expressed in their isotopic form. The left isotransformations are defined by

$$x^{t\,\prime} = x^t \hat{\times} A^t = x^t \times T \times A^t, \tag{6.2}$$

because $T = T^t$ for the considered Class I of isonumbers and isospaces.

The *isodual isotransformations* are given by

$$x'^{\,d} = A^d \hat{\times}^d x^d. \tag{6.3}$$

Transformations (6.1) are called *isolinear* because they coincide with the conventional linear transformations at the abstract level. Note that all nonlinear transformations $x' = B(x)$ can be always cast into an identical isolinear form [29]

$$x' = B(x) \equiv A \hat{\times} x, \qquad T = A^{-1} \times B \times x^{-1}. \tag{6.4}$$

We can then say that linearity is a true axiomatic structure, but nonlinearity is not because it can be made to disappear under isotopies.

Transformation (6.1) are also *isolocal*, in the sense that they coincide with the conventional local transformations $x' = A \times x$ at the abstract level. Again, all nonlocal integral transformations $x' = I(x)$ verifying the needed continuity conditions can always be identically written in an isotopic form [29]

$$x' = I(x) \equiv A \hat{\times} x, \qquad T = A^{-1} \times I \times x^{-1}. \tag{6.5}$$

In this way one can see that locality is a true axiomatic structure, but nonlocality is not because it can be made to disappear at the abstract level under isotopies.

Transformations (6.1) are finally called *isocanonical*, in the sense that they generally violate the conditions to constitute canonical transformations, but they nevertheless coincide with conventional canonical transformations at the abstract level. Thus, the canonical structure is a true axiomatic structure, but its absence (violation of the integrability conditions for the existence of a Hamiltonian, the *conditions of variational selfadjointness* [15]) is also not a true axiomatic structure because it can be made to disappear at the abstract level under isotopies. In fact, in ref. [15] one can see the construction of the *Birkhoffian generalization of Hamiltonian mechanics* for the representation of all possible, sufficiently smooth and regular, local but nonlinear and nonhamiltonian systems, under the condition that Birkhoffian and Hamiltonian mechanics coincide at the abstract level.

The *pseudoisotransformations* are then isotransformations (4.1) on a pseudoisospace $\hat{S}(x,\hat{g},\hat{R}(n,\hat{+},\hat{\times})$. However, while the original transformations (4.1) are distributive, the latter ones are not,

$$A \hat{\times} (x \hat{+} x') \neq A \hat{\times} x \hat{+} A \hat{\times} x', \tag{6.6}$$

in view of Proposition 4.3. The *isodual pseudoisotransformations* are then defined accordingly.

7: ISOALGEBRAS, PSEUDOISOALGEBRAS AND THEIR ISODUALS

A further notion needed for our analysis is the applicable definition of algebra and of the representation theory. An isovector space $\hat{U}$ with elements A, B, C, ... and isomultiplication $\hat{o}$ over an isofield $\hat{F}(a,+,\hat{\times})$ of elements a, b, c, and isomultiplication $a \hat{\times} b$ with multiplicative isounit $\hat{1} = T^{-1}$ is called an (associative or nonassociative) *isoalgebra* [14,29], when it satisfies the left and right scalar and distributive laws

$$(a \hat{\times} A) \hat{o} B = A \hat{o} (a \hat{\times} B) = a \hat{\times} (A \hat{\times} B),$$

$$(A \hat{\times} a) \hat{o} B = A \hat{o} (B \hat{\times} a) = (A \hat{o} B) \hat{\times} a,$$

$$A \hat{o} (B + C) = A \hat{o} B + A \hat{o} C, \quad (B + C) \hat{o} A = B \hat{o} A + C \hat{o} A. \quad (7.1)$$

for all elements A, B, C $\in \hat{A}$ and a, b, c $\in \hat{F}$.

Note the differentiation between the isomultiplication $A\hat{o}B$ of the elements of the algebras, which are, say, matrices, from the isomultiplication of the elements of the isofields $a\hat{\times}b$, which can be ordinary numbers.

The isoalgebra $\hat{U}$ is called an *isodivision algebra* when the equation $A \hat{\times} x = B$ always admit a solution for $A \neq 0$.

Recall that a basis e_k, k = 1, 2, ..., m of a conventional algebra U (i.e., one verifying the conventional form of the scalar and distributive laws) remains unchanged under isotopies, except for possible renormalization $e_k \to \hat{e}_k$ (ref. [29], Proposition 3.1, p. 181). Thus a generic element $A \in \hat{U}$ can be written

$$A = \sum_{k=1,...,m} \hat{n}_k \hat{\times} \hat{e}_k, \qquad \hat{n}_k \in \hat{F}(\hat{a},+,\hat{\times}). \quad (7.2)$$

The *isonorm* of $\hat{U}$ in the basis considered is then given by

$$\hat{|} A \hat{|} = (\sum_{k=1,...,m} n_k \times n_k)^{\frac{1}{2}} \times \hat{1} \in \hat{F}. \quad (7.3)$$

An isoalgebra $\hat{U}$ is called *isonormed*, when the isonorm verifies the axiom

$$\hat{|} A \hat{o} B \hat{|} = \hat{|} A \hat{|} \hat{\times} \hat{|} B \hat{|} \in \hat{F}, \ \hat{|} \hat{n} \hat{\times} A \hat{|} = \hat{|} \hat{n} \hat{|} \hat{\times} \hat{|} A \hat{|}, \quad (7.4)$$

where we have differentiated the product $A\hat{o}B$ of the elements A and B of the algebras from the product $\hat{|} A \hat{|} \hat{\times} \hat{|} B \hat{|}$ of the elements of the isofield $\hat{|} A \hat{|}$ and $\hat{|} B \hat{|}$.

The isoalgebra $\hat{U}$ is said to be *isoassociative* when it verifies the isoassociative law

$$A \hat{o} (B \hat{o} C) = (A \hat{o} B) \hat{o} C, \quad \forall A, B, C \in \hat{U}; \quad (7.5)$$

it is said to be *isoalternative* when it verifies the isoalternative laws

$$A^{\hat{2}} \hat{o} B = A \hat{o} (A \hat{o} B), \quad A \hat{o} B^{\hat{2}} = (A \hat{o} B) \hat{o} B, \quad (7.6)$$

it is said to be *Lie-isotopic* when the product $A\hat{o}B$ verifies the Lie algebra axioms in isotopic form (anticommutativity and Jacobi law) as in the realization [14,29]

$$A \hat{o} B = A T B - B T A, \ A T, B T, \text{etc.} = \text{assoc.} \quad (7.7)$$

and it is said to be *Lie-admissible* when the antisymmetric product attached to $A\hat{o}B$

$$[A \hat{,} B] := A \hat{o} B - B \hat{o} A, \quad (7.8)$$

is Lie-isotopic as in the realization

$$A \hat{o} B = A R B - B S A. \quad (7.9)$$

The *isodual isoalgebras* $\hat{U}^d$ are then those characterized over an isodual isofield $\hat{F}^d(n^d,+,\hat{\times}^d)$.

Suppose that $\hat{U}$ is isoassociative and let $\hat{R}_A$ and $\hat{L}_A$ represent the right and left isomultiplication of the element $A \in \hat{U}$. It is possible to prove that the map $A \to \hat{R}_A$ $(A \to \hat{L}_A)$ constitutes a homomorphism (antihomomorphism) of $\hat{U}$ into the algebra of all isolinear transformations of $\hat{U}$ as a vector isospace called *right (left) isorepresentations* (see the forthcoming paper [35] for details).

A dominant aspect of the transition from conventional representations to the covering isorepresentations, for which the isonumbers were conceived, is the transition from the conventional linear, local and canonical representations currently used in physics to their most general possible nonlinear, nonlocal and noncanonical form.

As well known, the distributive laws are basic axioms for any structure to characterize an "algebra" as commonly understood [7–10]. The image of an isoalgebra $\hat{U}$ under the transition an isofield $\hat{F}(a,+,\hat{\times})$ to a pseudoisofield $\hat{F}(a,\hat{+},\hat{\times})$ then implies the loss of the basic distributive laws and, for this reason it will be called *pseudoisoalgebra*.

8: REALIZATION OF ISOREAL NUMBERS AND THEIR ISODUALS

8.A: Realization of ordinary real numbers. Recall (see, e.g., ref. [7]) that conventional real numbers $n \in R(n,+,\times)$ are realized on the one-dimensional real Euclidean space $E_1(x,\delta,R(n,+,\times))$, which essentially represents a straight line with origin at 0, local coordinates x, metric $\delta = 1$, additive unit 0 and multiplicative unit 1. In fact, the *dilations*

$$x' = n \times x, \qquad n \in R(n,+,\times), \qquad x,\, x' \in E_1(x,\delta,R) \,, \tag{8.1}$$

characterize an isomorphism of the reals $R(n,+,\times)$ into the commutative one-dimensional group of dilations G(1).

The trivial basis is

$$e = 1, \tag{8.2}$$

with the familiar norm

$$|\, n \,| = (\, n \times n\,)^{\frac{1}{2}} > 0 \,, \tag{8.3}$$

verifying axioms (7.4),

$$|\, n \times n' \,| = |\, n \,| \times |\, n' \,| \,. \tag{8.4}$$

This show that *real numbers constitute a one-dimensional normed associative and commutative algebra* U(1).

8.B: Realization of isodual real numbers. Isodual real numbers $n^d \in R^d(n^d,+,\times^d)$ are conventional numbers n, although defined with respect to the isodual unit $1^d = -1$. The isodual conjugation for real numbers can then be written

$$n = n \times 1 \quad \rightarrow \quad n^d = n \times 1^d = -n \,. \tag{8.5}$$

All numerical values therefore change sign under isoduality. One should however keep in mind that such a sign inversion occurs only when the isodual real numbers are projected in the field of conventional real numbers.

As an example, the negative integer number −3 referred to the negative unit − 1 is fully equivalent to the positive integer number +3 referred to the positive unit +1, and this illustrates that the change of sign under isoduality occurs only in the projection of the isodual numbers in the original conventional field.

The representation of $R^d(n^d,+,\times^d)$ requires the use of the one-dimensional, real isodual Euclidean space $E^d(x,\delta^d,R^d(n^d,+,\times^d))$, which is also a straight line, this time with conventional additive unit 0, and isodual multiplicative unit $1^d = -1$. The *isodual dilations* are then given by

$$x' = n^d \times^d x = n \times x \,. \tag{8.6}$$

They establish an isomorphism between $R^d\ (n^d,+,\times^d\ )$ and the isodual group of dilations $G^d(1)$, i.e., the conventional group G(1) reformulated with respect to the multiplicative unit 1^d.

Note that $E_1(x,\delta,R)$ and $E_1{}^d(x,\delta^d,R^d)$ are anti-isomorphic and the same property holds for G(1) and $G^d(1)$. Note that isodual dilations (8.2) coincide with the conventional ones (8.1), and this could be a reason for the lack of detection of isodual numbers until ref.s [18–20].

The *isodual basis* is now

$$e^d = 1^d \tag{8.7}$$

with *isodual norm*

$$|\, n \,|^d := (n \times n)^{\frac{1}{2}} \times 1^d = |\, n \,| \times 1^d = -|\, n \,| < 0 \tag{8.8}$$

verifying axioms (7.4)

$$|\, n^d \times n'^d \,|^d = |\, n^d \,|^d \times^d |\, n'^d \,|^d \,. \tag{8.9}$$

This shows that *isodual real numbers constitute a one-dimensional isodual, associative and commutative normed algebra* $U^d(1)$ *which is anti-isomorphic to* U(1).

8.C: Realization of isoreal numbers. We consider now the isoreal numbers $\hat{n} = n \times \hat{1}$ as elements of an isofield of Class I [32], $\hat{R}_I(\hat{n},\hat{+},\hat{\times})$ with isomultiplication $\hat{\times} = \times T \times$, and multiplicative isounit $\hat{1} = T^{-1} > 0$ generally outside the original set $R(n,+,\times)$. Their representation requires the isoeuclidean spaces of

Class I, $\hat{E}_{I,I}(x,\hat{\delta},\hat{R}(\hat{n},+,\hat{\times}))$, $\hat{\delta} = T\delta$, over $\hat{R}(\hat{n},+,\hat{\times})$, which are the isotopes of the conventional one-dimensional Euclidean spaces $E_I(x,\delta,R)$.

One should keep in mind that:

A) $\hat{E}_{I,I}(x,\hat{\delta},\hat{R})$ is a simple, yet bona-fide *isoriemannian space* [30], because $\hat{\delta} = T \times \delta = \hat{\delta}(t, x, \dot{x}, \ddot{x}, ...)$, where the local dependence is generally nonlinear, nonlocal and noncanonical in all variables;

B) $\hat{E}_{I,I}(x,\hat{\delta},\hat{R})$ is not a Riemannian space because of the intrinsic dependence of the isometric $\hat{\delta}$ on the derivatives $\dot{x}$, $\ddot{x}$, ... as well as the fact that the basic unit is not the conventional quantity I; and

C) Despite their differences, the conventional Euclidean space $E_I(x,\delta,R)$ and its isotopic covering $\hat{E}_{I,I}(x,\hat{\delta},\hat{R})$ are locally isomorphic due to the joint liftings $\delta \rightarrow \hat{\delta} = T\times\delta$ *and* $I \rightarrow \hat{1} = T^{-1}$.

Thus, the one-dimensional isospace $\hat{E}_{I,I}(x,\hat{\delta},\hat{R})$ represents a generalization of the conventional straight line, here called an *isoline*, because of its intrinsically nonlinear, nonlocal and noncanonical metric $\hat{\delta}(t, x, \dot{x}, \ddot{x}, ...)$ with multiplicative isounit $\hat{1} = \hat{1}(t, x, \dot{x}, \ddot{x}, ...)$.

$\hat{R}_I(\hat{n},\hat{+},\hat{\times})$ can then be realized via the *isodilations* on $\hat{E}_{I,I}(x,\hat{\delta},\hat{R})$

$$x' = \hat{n} \hat{\times} x = n \times x, \tag{8.10}$$

which formally coincide the original dilations (8.1), as it is the case for the isodual dilations, thus providing a reason for the lack of detection of the isoreal numbers until recently.

Isodilations (8.10) characterize an isomorphism of the isoreal numbers with the one-dimensional isogroup of isodilations $\hat{G}(1)$, i.e., the group G(1) realized with respect to the isounit $\hat{1}$. The isomorphism $E(x,\delta,R(n,+,\times)) \approx \hat{E}_{I,I}(x,\hat{\delta},\hat{R}(\hat{n},+,\hat{\times}))$ then readily implies $\hat{G}(1) \approx G(1)$.

The *isobasis* is given by

$$\hat{e} = \hat{1}, \tag{8.11}$$

while the *isonorm* can be defined by

$$\hat{|}\hat{n}\hat{|} := (n \times n)^{\frac{1}{2}} \times \hat{1} = |n| \times \hat{1}, \tag{8.12}$$

namely, by the conventional norm, only rescaled to the new unit $\hat{1}$, which is the essence of the transition from real number n to their isotopes $\hat{n} = n\times\hat{1}$.

In particular, axioms (7.4) trivially hold,

$$\hat{|}\hat{n} \hat{\times} \hat{n}'\hat{|} = \hat{|}\hat{n}\hat{|} \hat{\times} \hat{|}\hat{n}'\hat{|}. \tag{8.13}$$

with the same product inside and out because referred to the same elements. One can see that the *isoreal numbers constitute a one-dimensional, isonormed, isoassociative and isocommutative isoalgebra* $\hat{U}(1) \approx U(1)$.

8.D: Realization of isodual isoreal numbers. We consider now the isodual numbers $\hat{n}^d = n\times\hat{1}^d$, $\hat{1}^d = -\hat{1}$ belonging to an isodual isofield $\hat{R}_{II}{}^d(\hat{n}^d,+,\hat{\times}^d)$. In this case we need the one-dimensional, isodual isoeuclidean space of Class II, $\hat{E}_{II}{}^d(x,\hat{\delta}^d,\hat{R}^d)$, and the *isodual isodilations*

$$x' = \hat{n}^d \hat{\times}^d x, \tag{8.14}$$

which also coincide with the conventional dilations (8.1), by characterizing an isomorphism of the isodual isoreal numbers with the one-dimensional isodual isogroup $\hat{G}^d(1)$, i.e., the image of G(1) under the isodual isounit $\hat{1}^d = -\hat{1}$. the underlying isomorphism $E_I{}^d(x,\delta^d,R^d(n^d,+,\times^d)) \approx \hat{E}_{II,I}{}^d(x,\hat{\delta}^d,\hat{R}^d(\hat{n}^d,+,\hat{\times}^d))$ then implies $\hat{G}^d(1) \approx G^d(1)$.

The *isodual isobasis* is given by

$$\hat{e}^d = \hat{1}^d, \tag{8.15}$$

with *isodual isonorm*

$$\hat{|}\hat{n}^d\hat{|}^d := (n \times n)^{\frac{1}{2}} \times \hat{1}^d = -\hat{|}\hat{n}\hat{|}, \tag{8.16}$$

verifying axioms (7.4),

$$\hat{|}\hat{n}^d \hat{\times}^d \hat{n}'\hat{|}^d = \hat{|}\hat{n}^d\hat{|}^d \hat{\times}^d \hat{|}\hat{n}'^d\hat{|}^d. \tag{8.17}$$

Thus, *the isodual isoreal numbers are a realization of the one-dimensional isodual, isonormed, isoassociative and isocommutative isoalgebra* $\hat{U}^d(1) \approx U^d(1)$.

The pseudoisoreal numbers $\hat{n} \in \hat{R}(\hat{n},\hat{+},\hat{\times})$ and their isoduals $\hat{n}^d \in \hat{R}^d(\hat{n}^d,\hat{+}^d,\hat{\times}^d)$ can be readily constructed from the above lines although, as now familiar, they are no longer distributive.

9: REALIZATION OF ISOCOMPLEX NUMBERS AND THEIR ISODUALS

9.A: Realization of ordinary complex numbers. Recall (see, e.g., ref. [7]) that conventional complex numbers $c = n_0 + n_1 \times i \in C(c,+,\times)$, where n_0, $n_1 \in$

$R(n,+,\times)$, and i is the imaginary unit, are represented in a *Gauss plane* [1] which is essentially a realization of the two-dimensional Euclidean space $E_2(x,\delta,R(n,+,\times))$ with basic separation

$$x^2 = x^t \delta x = x_i \delta_{ij} x_j = x_1^2 + x_2^2 \in R(n,+,\times) \qquad (9.1)$$

whose group of isometries, the one-dimensional Lie group O(2), is the invariance of the circle, as well known. We then expect complex number to be representable via the fundamental representation of O(2) (see below).

The correspondence between complex numbers $c = n_0 + n_1 \times i$ and the Gauss plane with points $P = (x_1 , x_2)$ is then made one-to-one by the *dilative rotations*

$$z' = (x_1 + x_2 \times i)' = c \odot z = (n_0 + n_1 \times i) \odot (x_1 + x_2 \times i), \qquad (9.2)$$

with multiplication rule

$$c \odot z = (n_0 , n_1) \odot (x_1 , x_2) =$$

$$= (n_0 \times x_1 - n_1 \times x_2 , n_0 \times x_2 + n_1 \times x_1) . \qquad (9.3)$$

which is known to preserve all properties to characterize a field, thus establishing a one-to-one correspondence between complex numbers and points in the Gauss plane. Transformations (9.3) also forms a two-dimensional group G(2) in one to one correspondence with $C(c,+,\times)$.

Complex numbers also admit the matrix representation

$$c := n_0 \times 1_0 + n_1 \times i_1 = \begin{pmatrix} n_0 & n_1 \times i \\ n_1 \times i & n_0 \end{pmatrix}, \qquad (9.4a)$$

$$1_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad i_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (9.4b)$$

which are the identity and fundamental representation of O(2), respectively, as expected.

The *norm* is then given by the familiar expression

$$|c| = |n_0 + n_1 \times i| := (\mathrm{Det}\, c)^{\frac{1}{2}} = (\bar{c} \times c)^{\frac{1}{2}} = (n_0^2 + n_1^2)^{\frac{1}{2}}. \qquad (9.5)$$

and readily verifies axiom (7.4)

$$|c \odot c'| = |c| \times |c'| \in R, \quad c, c' \in C. \qquad (9.6)$$

where now we have different products because referred to different elements.

Finally, the identification of the basis in terms of matrices (9.4b)

$$e_1 = 1_0, \quad e_2 = i_1, \qquad (9.7)$$

implies the equally well known result that *complex numbers constitute a two-dimensional, normed, associative and commutative algebra* U(2).

9.B: Realization of isodual complex numbers. We now consider the isodual complex numbers

$$C^d = \{ (c^d,+,\times^d) \mid \times^d = -\times;\ 1^d = -1;\ c^d = \bar{c} \times 1^d = -\bar{c},\ \bar{c} \in \bar{C} \}, \qquad (9.8)$$

where $\bar{c}$ is the usual complex conjugation. Thus, given a complex number $c = n_0 + n_1 \times i$, its isodual is given by

$$c^d = -\bar{c} = n_0{}^d + n_1{}^d \times \bar{i} = -n_0 - n_1 \times \bar{i} = -n_0 + n_1 \times i \in C^d. \qquad (9.9)$$

In this case we need the two-dimensional isodual Euclidean space $E_2{}^d(x,\delta^d,R^d(n^d,+,\times^d))$ with basic invariant

$$x^{2d} = x^t \delta^d x = x_i \delta^d{}_{ij} x_j = x_1{}^{2d} + x_2{}^{2d} =$$

$$= x_1 \times^d x_1 + x_2 \times^d x_2 = -x_1{}^2 - x_2{}^2 \in R^d(n^d,+,\times^d) \qquad (9.10)$$

whose group of isometries is the one-dimensional isodual Lie group $O^d(2)$, i.e., the image of O(2) under the lifting $I = \mathrm{diag.}\,(1,1) \rightarrow I^d = \mathrm{diag.}\,(-1,-1)$ [20]. We then expect isodual complex numbers to be characterized by the isorepresentation of O(2).

We can then introduce the *isodual Gauss plane* as the image of the conventional plane under isoduality. The correspondence between isodual complex numbers and the isodual Gauss plane with points $P = (x_1 , x_2)$ is then made one-to-one by the *isodual dilative rotations*

$$z' = (x_1 + x_2 \times i)' = c^d \odot^d z = (-n_0 + n_1 \times i) \odot^d (x_1 + x_2 \times i), \qquad (9.11)$$

with multiplication rules

$$c^d \odot^d z = (-n_0 , n_1) \odot^d (x_1 , x_2) =$$

$$= (-n_0 \times x_1 \times n_1 \times x_2 , -n_0 \times x_2 + n_1 \times x_1) , \qquad (9.12)$$

which can be easily shown to preserve all properties to characterize a field. Also isodual transformations (9.12) form an isodual group $G^d(2)$ antiisomorphic to G(2).We therefore see that, as expected, the one-to-one correspondence between complex numbers and the Gauss plane persists under isoduality.

Isodual complex numbers also admit the matrix representation

$$c^d := n_0^d \times I_0^d + n_1^d \times i_1^d = \begin{pmatrix} -n_0 & n_1 \times i \\ n_1 \times i & -n_0 \end{pmatrix}, \quad (9.13a)$$

$$I_0^d = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad i_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad (9.13b)$$

which are the isodual unit and isodual representations of $O^d(2)$, respectively.

The *isodual norm* , from rule (3.6), is now given by

$$|c^d|^d = |-n_0 + n_1 \times i\,|^d := [\det_R (c^d \times T^d)]^{\frac{1}{2}} \times I_0^d = (\bar{c}^d \times^d c^d)^{\frac{1}{2}} \times I_0^d, \quad (9.14)$$

can be written

$$|c^d|^d = (c \times \bar{c}) \times I_0^d = (n_0^2 + n_1^2) \times I_0^d. \quad (9.15)$$

and also verifies axioms (7.4),

$$|c^d \odot^d c'^d|^d = |c^d|^d \times^d |c'^d|^d \in R^d, \quad c^d, c'^d \in C^d. \quad (9.16)$$

Finally, the identification of the *isodual basis* in terms of matrices (9.13b)

$$e_1^d = I_0^d, \quad e_2^d = i_1^d, \quad (9.17)$$

implies that *isodual complex numbers constitute a two-dimensional, isodual, normed, associative and commutative algebra* $U^d(2)$ *which is anti-isomorphic to* U(2).

9.C: Realization of isocomplex numbers. We consider now the isofield of isocomplex numbers

$$\hat{C} = \{ (\hat{c}, \hat{+}, \hat{\times}) \,|\, \hat{\times} = \times T \times, \hat{1} = T^{-1}, \hat{c} = c \times \hat{1}, c \in C(c,+,\times) \}, \quad (9.18)$$

with generic element $\hat{c} = \hat{n}_0 + \hat{n}_1 \times i$. In this case we need the two-dimensional isoeuclidean space of Class I, $\hat{E}_{I,2}(x,\hat{\delta},\hat{R}(\hat{n},+,\hat{\times}))$. Their realization most used in the

physical literature is that with a diagonalized and positive-definite isotopic element and isounit

$$T = \mathrm{diag.}(b_1^2, b_2^2), \quad \hat{1} = \mathrm{diag.}(b_1^{-2}, b_2^{-2}), \; b_k > 0, \; k = 1, 2, \quad (9.19)$$

with basic isoseparation

$$x^2 = (x^t \hat{\delta} x) \times \hat{1} = (x_i \hat{\delta}_{ij} x_j) = (x_1 b_1^2 x_1 + x_2 b_2^2 b_2) \times \hat{1} \in \hat{R}(\hat{n},+,\hat{\times}), \quad (9.20)$$

whose group of isometries is the Lie-isotopic group $\hat{O}(2) \sim O(2)$ [20], i.e., the group O(2) constructed with respect to the multiplicative isounit $\hat{1} = \mathrm{diag.}(b_1^{-2}, b_2^{-2})$, which provides the invariance of all possible ellipses with semiaxes $a = b_1^{-2}$, $b = b_2^{-2}$ as the infinitely possible deformation of the circle (9.1). We then expect that isocomplex numbers are characterizable via the fundamental isorepresentation of $\hat{O}(2)$.

We now introduce the *isogauss plane* which is the set of points $P = (\hat{x}_1, x_2)$ on $\hat{E}_{I,2}(x,\hat{\delta},\hat{R}(\hat{n},+,\hat{\times}))$ for the characterization of isocomplex numbers $\hat{c} = (\hat{n}_0, \hat{n}_1)$.

The correspondence between the isocomplex numbers $\hat{C}(\hat{c},+,\hat{\times})$ and the isogauss plane can be made one-to-one by the *isodilative isorotations*

$$z' = (x_1 + x_2 \times i)' = \hat{c} \,\hat{\odot}\, z \quad (9.21)$$

with isomultiplication rule

$$\hat{c} \,\hat{\odot}\, z = (\hat{n}_0, \hat{n}_1) \,\hat{\odot}\, (x_1, x_2) =$$

$$= ([(n_0 \times x_0) \times \hat{1} - \Delta^{\frac{1}{2}} \times (n_1 \times x_2) \times \hat{1}], [(n_0 \times x_2) \times \hat{1} + (n_1 \times x_1) \times \hat{1}]),$$

$$\Delta = \mathrm{Det}\, T = b_1^2 \times b_2^2, \quad (9.22)$$

where the appearance of the $\Delta^{\frac{1}{2}}$ factor will be justified shortly, and confirmed later on in this section for the case of isoquaternions and isooctonions.

Isocomplex numbers also admit the following two-by-two matrix representation

$$\hat{c} = \hat{n}_0 \times \hat{1}_0 \;\hat{+}\; n_1 \hat{i}_1 = \begin{pmatrix} n_0 \times b_1^{-2} & i \times n_1 \times b_1^2 \times \Delta^{-\frac{1}{2}} \\ i \times n_1 \times b_2^2 \times \Delta^{-\frac{1}{2}} & n_0 \times b_1^{-2} \end{pmatrix}, \quad (9.23a)$$

$$\hat{1} = \hat{1}_0 = \begin{pmatrix} b_1^{-2} & 0 \\ 0 & b_2^{-2} \end{pmatrix}, \quad \hat{i}_1 = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & i \times b_1^2 \\ i \times b_2^2 & 0 \end{pmatrix}, \quad (9.23b)$$

$$\Delta = \text{Det}\, T = b_1{}^2\, b_2{}^2\,, \tag{9.23c}$$

which verify rule (9.22) and characterize the isounit and the fundamental (adjoint) representation of $\hat{O}(2)$ respectively (see ref. [23,38] and the following article [40] in this Journal on the fundamental isorepresentation of the isotopic $\hat{su}(2)$ algebras, as well as the application to isoquaternions provided below).

Then, the set $\hat{S}(\hat{c},\hat{+},\hat{\times})$ of matrices (9.23a) is closed under addition and isomultiplication, each element possesses the isoinverse

$$\hat{c}^{-1} = \hat{c}^{-1} \times \hat{1} \tag{9.24}$$

where $\hat{c}^{-1}$ is the ordinary inverse. Thus, $\hat{S}(\hat{c},+,\hat{\times})$ is an isofield and the local isomorphism $\hat{S}(\hat{c},+,\hat{\times}) \sim \hat{C}(\hat{c},+,\hat{\times})$ follows.

It is easy to see that the isogauss plane possesses all axioms to characterize an isofield. In particular, isotransformations (9.22) form a two-dimensional isodilation isogroup $\hat{G}(2) \sim G(2)$. As expected, the one-to-one correspondence between complex numbers and Gauss plane is preserved under isotopy.

The implications are however nontrivial, as illustrated by a number of properties, such as the lack of existence of a unitary transformations $c' = U \hat{\circ} c \hat{\circ} \hat{U}$, $U \hat{\circ} U^\dagger = U^\dagger \hat{\circ} U = I = \text{diag.}\,(1,\, 1)$, mapping representations (9.4) into their isotopic form (9.23). The understanding that a transformation does indeed exist, but it is isounitary $\hat{c} = \hat{U} \hat{\circ} c \hat{\circ} \hat{U}^\dagger$, $\hat{U} \hat{\circ} \hat{U}^\dagger = \hat{U}^\dagger \hat{\circ} \hat{U} = \hat{1}$.

Another way to see the nontriviality of the isotopy is by noting that the conventional trigonometry is inapplicable to the isogauss plane. In fact, conventional functions such as cos α, sin α, etc. which are well defined in the Gauss plane, have no mathematical meaning in our isogauss plane because it is isocurved. A study of the generalization of trigonometric functions needed for the isogauss plane shall be presented elsewhere [27].

The reader should be aware that, by no means, realization (9.23) is unique, owing to the intriguing "degrees of freedom" of the isotopic formulations which are not studied here for brevity (see, ref.s [15,20,21])..

The *isonorm* is defined, from Eq.s (3.4) by

$$\hat{|}\, \hat{c}\, \hat{|} = [\, \text{Det}_R(\, \hat{c} \times T)\,]^{\frac{1}{2}} \times \hat{1}_0 = (\, n_0{}^2 + \Delta\, n_1{}^2\,)^{\frac{1}{2}} \times \hat{1}_0\,, \tag{9.25}$$

and readily verifies axiom (7.4),

$$\hat{|}\, \hat{c} \hat{\circ} \hat{c}'\, \hat{|} = \hat{|}\, \hat{c}\, \hat{|}\; \hat{\times}\; \hat{|}\, \hat{c}'\, \hat{|} \in \hat{R}, \quad \hat{c}, \hat{c}' \in \hat{C} \tag{9.26}$$

Finally, the *isobasis*

$$\hat{e}_1 = \hat{1}_0\,, \quad \hat{e}_2 = \hat{1}\,, \tag{9.27}$$

show that *isocomplex numbers constitute a two-dimensional, isonormed, isoassociative and isocommutative isoalgebras over the isoreals* $\hat{O}(2) \sim U(2)$.

9.D: Representation of isodual isocomplex numbers. We consider now the isodual isocomplex numbers

$$\hat{C}^d = \{\, (\hat{c}^d,+,\hat{\times}^d) \mid \hat{c}^d = -\, \bar{c}\, \hat{1}^d,\;\; \hat{\times}^d = \times\, T^d\, \times,\; T^d = -\, T,\; \hat{1}^d = T^{d-1}\,,\; c \in C(c,+,\times)\,\}\,, \tag{9.28}$$

with generic element $\hat{c}^d = \hat{n}^d + \hat{n}_1{}^d \times i^d = -\,\hat{n}_0 + \hat{n}_1 \times i$. In this case we need the two-dimensional isodual isoeuclidean space of Class II, $\hat{E}_{II,2}{}^d(x,\delta^d,\hat{R}^d(\hat{n}^d,+,\hat{\times}^d))$ with realization

$$T^d = \text{diag.}\,(\,-b_1{}^2\,, -b_2{}^2\,)\,, \quad \hat{1}^d = \text{diag.}\,(\,-b_1{}^{-2}\,, -b_2{}^{-2}\,)\,, \quad b_k > 0,\;\; k = 1,\, 2, \tag{9.29}$$

and basic isodual isoseparation

$$x^{2d} = (\, x^t\, \delta^d\, x\,) \times \hat{1}^d = (x_i\, \delta^d{}_{ij}\, x_j\,) \times \hat{1}^d =$$

$$= (-\, x_1\, b_1{}^2\, x_1 \,-\, x_2\, b_2{}^2\, b_2\,) \times \hat{1}^d \in \hat{R}^d\,(\hat{n}^d,+,\hat{\times}^d\,), \tag{9.30}$$

whose group of isometries is the isodual isoorthogonal group $\hat{O}^d(2) \sim O^d(2)$ [20].

The *isodual isogauss plane* is then the set of points $P = (\hat{x}_1,\, x_2)$ on $\hat{E}_{1,2}{}^d(x,\delta^d,\hat{R}^d(\hat{n}^d,+,\hat{\times}^d))$ for the characterization of isocomplex numbers $\hat{c} = (-\hat{n}_0\,,\, \hat{n}_1\,)$.

The correspondence between the isodual isocomplex numbers $\hat{C}^d(\hat{c}^d,+,\hat{\times}^d)$ and the isodual isogauss plane can be made one-to-one by the *isodual isodilative isorotations*

$$z' = (\, x_1 + x_2 \times i\,)' = \hat{c}^d\, \hat{\circ}^d\, z \tag{9.31}$$

with multiplication rule

$$\hat{c}\, \hat{\circ}^d\, z = (\, \hat{n}_0\,, \hat{n}_1\,)\, \hat{\circ}^d\, (\, x_1\,, x_2\,) = \tag{9.32}$$

$$= (\,[\,(\,-\,n_0 \times x_0\,) \times \hat{1} + \Delta^{\frac{1}{2}} \times (\, n_1 \times x_2\,) \times \hat{1}\,],\, [\,(-\,n_0 \times\, x_2\,) \times \hat{1} + (\, n_1 \times x_1\,) \times \hat{1}\,]\,)$$

It is easy to see that the isodual isogauss plane preserves all axioms to characterize an isodual isofield. Also, isodual isotransformations (9.32) forms an isodual isogroup $\hat{G}^d(2) \sim G^d(2)$. As expected, the one-to-one correspondence between complex numbers and Gauss plane is also preserved under isodual isotopy.

Isodual isocomplex numbers also admit the following two-by-two matrix representation

$$\hat{c}^d = \hat{n}_0{}^d \times \hat{1}_0{}^d + n_1{}^d \times \hat{1}^d = \begin{pmatrix} -n_0 \times b_1{}^{-2} & i \times n_1 \times b_1{}^2 \times \Delta^{-\frac{1}{2}} \\ i \times n_1 \times b_2{}^2 \times \Delta^{-\frac{1}{2}} & -n_0 \times b_2{}^{-2} \end{pmatrix}, \quad (9.33a)$$

$$\hat{1}^d = \hat{1}_0{}^d = \begin{pmatrix} -b_1{}^{-2} & 0 \\ 0 & -b_2{}^{-2} \end{pmatrix}, \quad \hat{1}^d = \begin{pmatrix} 0 & -i \times b_1{}^2 \times \Delta^{-\frac{1}{2}} \\ -i \times b_2{}^2 \times \Delta^{-\frac{1}{2}} & 0 \end{pmatrix}, \quad (9.33b)$$

which satisfies isomultiplication rule (9.32), which characterize the isodual isounit and fundamental representation of $\hat{O}^d(2)$, respectively.

Then, the set $\hat{S}^d(\hat{c}^d,+,\hat{\times}^d)$ of matrices (9.33a) is closed under addition and isomultiplication, each element possesses the isodual isoinverse

$$\hat{c}^{-\hat{1}d} = (\hat{c}^d)^{-1} \times \hat{1}^d \quad (9.34)$$

Thus $\hat{S}^d(\hat{c}^d,+\hat{\times}^d)$ is an isofield. The local isomorphism $\hat{S}^d(\hat{c}^d,+,\hat{\times}^d) \sim \hat{C}^d(\hat{c}^d,+,\hat{\times}^d)$ is then consequential.

The *isodual isonorm* is defined, from Eq.s (3.6), by

$$\uparrow \hat{c}^d \uparrow^d = [\, \mathrm{Det}_R (\hat{c}^d \times T^d)\,]^{\frac{1}{2}} \times \hat{1}_0{}^d = (n_0{}^2 + \Delta \times n_1{}^2)^{\frac{1}{2}} \times \hat{1}^d, \quad (9.35)$$

and readily verifies axioms (7.4),

$$\uparrow \hat{c}^d \hat{o}^d \hat{c}'^d \uparrow^d = \uparrow \hat{c}^d \uparrow^d \hat{\times}^d \uparrow \hat{c}'^d \uparrow^d \in \hat{R}^d, \quad \hat{c}^d, \hat{c}'^d \in \hat{C}^d. \quad (9.36)$$

Finally, the *isodual isobasis*

$$\hat{e}_1{}^d = \hat{1}_0{}^d, \quad \hat{e}_2{}^d = \hat{1}_1{}^d, \quad (9.37)$$

shows that *isodual isocomplex numbers constitute a two-dimensional, isodual, isonormed, isoassociative and isocommutative isoalgebras over the isodual isoreals isoreals* $\hat{O}^d(2) \sim \hat{U}^d(2)$.

The extension of the above results to the pseudoisocomplex numbers $\hat{C}(\hat{c},\hat{+},\hat{\times})$, $\hat{+} = +K \times \hat{1}$, $\hat{0} = -K \times \hat{1}$ and their isoduals $\hat{C}^d(\hat{c}^d,\hat{+}^d,\hat{\times}^d)$ is straighforward. Note that in this case a generic pseudoisocomplex number is given by

$$\hat{c} = n_0 \hat{+} n_1 \times \hat{1} = n'_0 + n'_1 \times \hat{1}, \; n'_0 = n_0 + K, \; n'_1 = n_1 + K, \quad (9.38)$$

thus showing an intrinsic rescaling of both the real and imaginary parts.

10: REALIZATION OF ISOQUATERNIONS AND THEIR ISODUALS

10.A: Realization of quaternions. Among their various realizations (see ref. [39]), we consider now the conventional form of quaternions $q \in Q(q,+,\times)$ [see also ref.s [7,8]) with realization in the complex Hermitean Euclidean plane $E_2(z,\delta,C)$ with separation

$$E_2(z,\delta,C): \quad z^\dagger z = \bar{z}_i \delta_{ij} z_j = \bar{z}_1 z_1 + \bar{z}_2 z_2 \quad \delta^\dagger \equiv \delta. \quad (10.1)$$

whose basic (unimodular) invariant is SU(2). We therefore expect quaternions to be characterizable via the fundamental (adjoint) representation of SU(2), i.e., by Pauli's matrices, as reviewed below.

Quaternions can be realized via pairs of complex numbers, $q = (c_1, c_2)$. A *Hermitean dilative rotation* on $E_2(z,\delta,C)$, i.e., one leaving invariant $z^\dagger z$, is given by

$$z'_1 = c_1 \odot z_1 + c_2 \odot z_2, \qquad z'_2 = -\bar{c}_2 \odot z_1 + \bar{c}_1 \odot z_2, \quad (10.2)$$

where the dilation is represented by the value $\bar{c}_1 \odot c_1 + \bar{c}_2 \odot c_2 \neq 1$. Again, transformations (10.2) form a group G(4), this time associative but non-commutative, which is in one-to-one correspondence with quaternions.

Rule (10.2) characterizes the following matrix representation of quaternions $Q(q,+,\times)$ over the field of complex numbers $C(c,+,\times)$

$$q = \begin{pmatrix} c_1 & c_2 \\ -\bar{c}_2 & \bar{c}_1 \end{pmatrix}, \quad (10.3)$$

which is also one-to-one. By assuming

$$c_1 = n_0 + n_3 \times i, \quad c_2 = n_1 + n_2 \times i, \quad (10.4)$$

matrix (10.3) admits the representation

$$q = n_0 \times I_0 + n_1 \times i_1 + n_2 \times i_2 + n_3 \times i_3, \quad (10.5)$$

where the i's are the celebrated two-dimensional *Pauli's matrices* plus the two-dimensional identity,

$$1_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; i_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; i_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; i_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad (10.6)$$

with fundamental properties

$$i_n \times i_m = -\epsilon_{nmk}\, i_k, \quad n \neq m, \quad n, m = 1, 2, 3, \quad (10.7)$$

where ϵ_{nmk} is the conventional totally antisymmetric tensor of rank three. The algebra A of Pauli's matrices is closed under commutators, and characterize the fundamental representation of the Lie algebra su(2)

$$[\, i_n, i_m\,] = i_n \times i_m - i_m \times i_n = -2\,\epsilon_{nmk}\, i_k, \quad (10.8)$$

with Casimir invariants 1_o and $i^2 = \sum_{k=1,2,3} i_k^{\,2}$,

$$[1_o, i_k] = [\, i^2, i_k\,] = 0, \quad k = 1, 2, 3, \quad (10.9)$$

and eigenvalues on a two-dimensional basis ψ with normalization $\psi\dagger \times \psi = 1$

$$\sum_{k=1,2,3} i_k^{\,2} \times \psi = \sum_{k=1,2,3} i_k \times i_k \times \psi = -3 \times \psi, \quad (10.10)$$

By noting that

$$q\dagger = n_o\, 1_o - n_1\, i_1 - n_2\, i_2 - n_3\, i_3, \quad (10.11)$$

the *norm* of q can be written

$$|\, q\,| = (\, q\dagger q\,)^{\frac{1}{2}} = (\, \textstyle\sum_{k=0,1,2,3} n_k^{\,2}\,)^{\frac{1}{2}}, \quad (10.12)$$

and also satisfies axioms (7.4),

$$|\, q \circ q'\,| = |\, q\,| \times |\, q\,| \in R, \quad q, q' \in Q. \quad (10.13)$$

The *basis*

$$e_1 = 1_0, \quad e_{k+1} = i_k, \quad k = 1, 2, 3, \quad (10.14)$$

then establishes that *quaternions constitute a normed, associative, noncommutative algebra of dimensions 4 over the reals* U(4) [7,8].

10.B: Realization of the isodual quaternions. We consider now the isodual quaternions $q^d \in Q^d(q^d,+,\times^d)$ which can be represented via the isodual complex Hermitean Euclidean space $E_2^{\,d}(z,\delta^d,C^d(c^d,+,\times^d))$ with separation

$$(\,\bar z_i\, \delta^d_{\;ij}\, z_j\,) \times 1^d = (\, -\bar z_1 z_1 - \bar z_2 z_2\,) \times 1^d \in R^d. \quad (10.15)$$

Isodual quaternions can be realized via pairs of isodual complex numbers, $q^d = (c_1^{\,d}, c_2^{\,d})$. An *isodual Hermitean dilative rotation* on $E_2^{\,d}(z,\delta^d,C^d(c^d,+,\times^d))$, i.e., one leaving invariant $z\dagger\delta^d z$, is given by

$$z'_1 = c_1^{\,d} \odot^d z_1 - \bar c_2^{\,d} \odot^d z_2, \quad z'_2 = c_2^{\,d} \odot^d z_1 + \bar c_1^{\,d} \odot^d z_2, \quad (10.16)$$

where the dilation is represented by the value $\bar c_1^{\,d} \odot^d c_1^{\,d} + \bar c_2^{\,d} \odot^d c_2^{\,d} \neq -1$. Again, transformations (10.16) form an associative but noncommutative isodual group $G^d(4)$, which is in one-to-one correspondence with isodual quaternions $Q^d(q^d,+,\times^d)$.

Rule (10.16) characterizes the following matrix representation of isodual quaternions over the field of isodual complex numbers $C^d(c^d,+,\times^d)$

$$q^d = \begin{pmatrix} c_1^{\,d} & -\bar c_2^{\,d} \\ c_2^{\,d} & \bar c^d \end{pmatrix}, \quad (10.17)$$

By assuming

$$c_1^{\,d} = -n_0 + n_3 \times i, \quad c_2^{\,d} = -n_1 + n_2 \times i, \quad (10.18)$$

and by recalling that $-\bar c^d = c$, we have the representation

$$q^d = n_o^{\,d} + n_1^{\,d} \times i_1^{\,d} + n_2^{\,d} \times i_2^{\,d} + n_3 \times i_3^{\,d} =$$
$$= -n_o + n_1 \times i_1 + n_2 \times i_2 + n_3 \times i_3, \quad (10.19)$$

where the i's are the Pauli's matrices reviewed above.

The *isodual norm* is then defined by

$$|\, q^d\,|^d = [\, (q^d)^\dagger \times^d q\,]^{\frac{1}{2}} \times 1^d = [\, \mathrm{Det}_C (q^d \times T^d)\,]^{\frac{1}{2}} \times 1^d =$$
$$= (\, \textstyle\sum_{k=0,1,2,3} n_k^{\,2}\,)^{\frac{1}{2}} \times 1^d, \quad (10.20)$$

with property

$$|\, q^d \odot^d q'^d\,|^d = |\, q^d\,|^d \times^d |\, q'^d\,|^d \in R^d, \quad q^d, q'^d \in Q^d. \quad (10.21)$$

The use of the isodual basis

$$e^d_{\;1} = 1^d_{\;0}, \quad e^d_{\;k+1} = i_k, \; k = 1, 2, 3, \quad (10.22)$$

then shows that *isodual quaternions constitute an isodual four-dimensional, normed, assocoiative and noncommutative algebra over the isodual reals* $U^d(4)$, *which is antiisomorphic to* U(4).

10.C: Realization of isoquaternions. To study the isoquaternions $\hat{q} \in \hat{Q}(\hat{q},+,\hat{\times})$, we need the *two-dimensional, complex Hermitean isoeuclidean space* of Class I, $\hat{E}_{I,2}(z,\hat{\delta},\hat{C})$ on the isofield $\hat{C}(\hat{c},+,\hat{\times})$ with separation

$$z\dagger\,\hat{\delta}\,z = \bar{z}_i\,\hat{\delta}_{ij}\,z_j = \bar{z}_1\,b_1^2 z_1 \;\hat{+}\; \bar{z}_2\,b_2^2\,z_2 \quad \hat{\delta}\dagger \equiv \hat{\delta} > 0, \tag{10.23}$$

basic isotopic element and isounit

$$T = \mathrm{Diag.}\,(b_1^2, b_2^2), \qquad \hat{1} = \mathrm{Diag.}\,(b_1^{-2}, b_2^{-2}), \tag{10.24}$$

whose (unimodular) invariance group is the Lie-isotopic group $\hat{SU}(2)$ [21,23,27].

A *Hermitean isodilative isorotation* on $\hat{E}_{I,2}(z,\hat{\delta},\hat{C}(\hat{c},+,\hat{\times}))$, i.e., one leaving invariant $z\dagger\hat{\delta}z$, is given by

$$z'_1 = \hat{c}_1\,\hat{\circ}\,z_1 + \hat{c}_2\,\hat{\circ}\,z_2, \qquad z'_2 = -\hat{c}_2\,\hat{\circ}\,z_1 + \bar{c}_1\,\hat{\circ}\,z_2, \tag{10.25}$$

where the dilation is represented by the value $\bar{c}_1\hat{\circ}\hat{c}_1 + \bar{c}_2\hat{\circ}\hat{c}_2 \neq \hat{1}$.

The map of isoquaternions into two-by-two matrices on $\hat{C}(\hat{c},\hat{+},\hat{\times})$ must now be characterized by the fundamental (adjoint) isorepresentations of the Lie-isotopic algebra $\hat{su}(2)$ studied in ref.s [21,23,38] (see several alternatives in [38] and the review in the subsequent article [40] in this Journal) which can be written

$$\hat{1}_0 = \begin{pmatrix} b_1^{-2} & 0 \\ 0 & b_2^{-2} \end{pmatrix}, \tag{10..26}$$

$$\hat{1}_1 = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & i \times b_1^2 \\ i \times b_2^2 & 0 \end{pmatrix}, \hat{1}_2 = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & b_1^2 \\ -b_2^2 & 0 \end{pmatrix}, \hat{1}_3 = \Delta^{-\frac{1}{2}} \begin{pmatrix} i \times b_2^2 & 0 \\ 0 & -i \times b_1^2 \end{pmatrix},$$

$$\Delta = \mathrm{Det}\,T = b_1^2\,b_2^2, \tag{10.26c}$$

and are called *isopauli matrices* [loc. cit.]. As expected, the $\hat{1}$-matrices verify the isotopic image of properties (10.7), i.e.,

$$\hat{1}_n \hat{\times} \hat{1}_m = -\Delta^{\frac{1}{2}}\,\epsilon_{nmk}\,\hat{1}_k, \qquad n \neq m, \quad n, m = 1, 2, 3. \tag{10.28}$$

and are therefore closed under isocommutators (as a necessary condition to have an isotopy), resulting the Lie-isotopic algebra

$$[\hat{1}_n \hat{,} \hat{1}_m] := \hat{1}_n \hat{\times} \hat{1}_m - \hat{1}_m \hat{\times} \hat{1}_n = -2\,\Delta^{\frac{1}{2}}\,\epsilon_{nmk}\,\hat{1}_k. \tag{10.29}$$

with *isocasimir invariants* $\hat{1}_0$ and $\hat{1}^2 = \sum_{k=1,2,3} \hat{1}_k^2 = \sum_{k=1,2,3} \hat{1}_k \times \hat{1}_k$,

$$[\hat{1}_0 \hat{,} \hat{1}_k] = [\hat{1}^2 \hat{,} \hat{1}_k] = 0, \quad k = 1, 2, 3, \tag{10.30}$$

and generalized eigenvalues on a two-dimensional basis $\hat{\psi}$ with isonormalization $\hat{\psi}^\dagger\hat{\times}\hat{\psi} = \hat{1}$

$$\hat{1}_3 * \psi = \pm\,\Delta^{\frac{1}{2}} \times \psi, \;\; \sum_{k=1,2,3} \hat{1}_k^2 \hat{\times} \hat{\psi} = \sum_{k=1,2,3} \hat{1}_k \hat{\times} \hat{1}_k \hat{\times} \hat{\psi} = -3 \times \Delta \times \hat{\psi}, \tag{10.31}$$

Note the complete abstract identity of the isotopic $\hat{su}(2)$ with the conventional su(2) algebra [38]. Nevertheless, it is easy to prove that Pauli's matrices and their isotopic covering are not unitarily equivalent, and this establishes the nontriviality of the isotopies here considered.

Note that the isoinvariance $\hat{O}(2)$ of the isocomplex numbers is a subgroup of $\hat{SU}(2)$ characterizable by $\hat{1}_1$. Note also that there exist isopauli matrices with $\Delta = 1$ (see [38] and the following review [40]).

Isoquaternions can therefore be written in the form (apparently presented here for the first time)

$$\hat{q} = n_0\,\hat{1}_0 + n_1\,\hat{1}_1 + n_2\,\hat{1}_2 + n_3\,\hat{1}_3 =$$

$$= \begin{pmatrix} (n_0 \times b_1^{-2} + i \times n_3 \times b_2^2 \times \Delta^{-\frac{1}{2}}) & (i \times n_1 - n_2) \times b_1^2 \times \Delta^{-\frac{1}{2}} \\ (i \times n_1 + n_2) \times b_2^2 \times \Delta^{-\frac{1}{2}} & (n_0 \times b_2^{-2} + i \times n_3 \times b_1^2 \times \Delta^{-\frac{1}{2}}) \end{pmatrix}. \tag{10.32}$$

It is straightforward to show that the set $\hat{S}(\hat{q},+,\hat{\times})$ of all possible expression (10.32) preserves the axioms of the original set $S(q,+,\times)$. In fact, the set $\hat{S}(\hat{q},\hat{+},\hat{\times})$ is a four-dimensional vector space over the isoreals $\hat{R}(\hat{n},+,\hat{\times})$ which is closed under the operation of conventional addition and isomultiplication, thus being an isofield. The isomorphism $\hat{S}(\hat{q},+,\hat{\times}) \approx \hat{Q}(\hat{q},+,\hat{\times})$ then follows.

The *isonorm* of the isoquaternions is given by

$$\hat{|}\hat{q}\hat{|} = [\,\mathrm{Det}_R(\hat{q} \times T)\,]^{\frac{1}{2}} \times \hat{1}_0 = (\,\hat{q}\dagger \hat{\times} \hat{q}\,) \times \hat{1}_0 =$$

$$= [\,n_0^2 + \Delta(\,n_1^2 + n_2^2 + n_3^2)\,]^{\frac{1}{2}} \times \hat{1}_0, \tag{10.33}$$

which should be compared with expression (10.12) for the ordinary quaternions, with basic rule

$$\hat{|}\hat{q}\,\hat{\circ}\,\hat{q}'\hat{|} = \hat{|}\,\hat{q}\,\hat{|} \hat{\times} \hat{|}\hat{q}'\hat{|} \in \hat{R}, \quad \hat{q}, \hat{q}' \in \hat{Q}. \tag{10.34}$$

The *isobasis*

$$\hat{e}_1 = \hat{1}_0, \quad \hat{e}_{k+1} = \hat{1}_k, \quad k = 1, 2, 3, \tag{10.35}$$

then establishes that *isoquaternions constitute a four-dimensional, isonormed, isoassociative, non-isocommutative isoalgebras over the isoreals* $\hat{U}(4) \approx U(4)$.

10.D: Realization of isodual isoquaternions. The *isodual isoquaternions* can be characterized via the two-dimensional isodual complex Hermitean isoeuclidean space of Class II over the isodual isocomplex field, $\hat{E}^d_{II,2}(z, \hat{\delta}^d, \hat{C}^d(\hat{c}^d, +, \hat{\times}^d))$, with separation

$$z^\dagger \hat{\delta}^d z = \bar{z}_1 \hat{\times}^d z_1 + \bar{z}_2 \hat{\times}^d z_2 = -\bar{z}_1 b_1^2 z_1 - \bar{z}_2 b_2^2 z_2, \tag{10.36}$$

with basic isodual isotopic element and isodual isounit

$$T^d = \text{Diag.}(-b_1^2, -b_2^2), \quad \hat{1}^d = \text{Diag.}(-b_1^{-2}, -b_2^{-2}), \tag{10.37}$$

whose (unimodular) invariance is now that of the isodual Lie-isotopic group $\hat{SO}^d(2)$ [21,23,27]. An *isodual Hermitean isodilative isorotation* on $\hat{E}^d_{II,2}(z, \hat{\delta}^d, \hat{C}^d(\hat{c}^d, +, \hat{\times}^d))$, i.e., one leaving invariant $z^\dagger \hat{\delta} z$, is given by

$$z'_1 = \hat{c}^d_1 \hat{\odot}^d z_1 - \bar{c}^d_2 \hat{\odot}^d z_2, \qquad z'_2 = \hat{c}^d_2 \hat{\odot}^d z_1 + \bar{c}^d_1 \hat{\odot}^d z_2, \tag{10.38}$$

where the dilation is represented by the value $\bar{c}^d_1 \hat{\odot}^d \hat{c}^d_1 + \bar{c}^d \hat{\odot}^d \hat{c}^d_2 \neq \hat{1}^d$.

Isoquaternions then admit a realization in terms of the isodual isorepresentation of $\hat{SO}^d(2)$ which can be written

$$\hat{q}^d = \hat{n}_0{}^d \hat{+} \hat{n}_1{}^d \times \hat{1}_1{}^d \hat{+} \hat{n}_2{}^d \times \hat{1}_2{}^d + \hat{n}_3{}^d \times \hat{1}_3{}^d =$$

$$= -\hat{n}_0 + \hat{n}_1 \times \hat{1}_1 + \hat{n}_2 \times \hat{1}_2 + \hat{n}_3 \times \hat{1}_3 =$$

$$= \begin{pmatrix} (-n_0 \times b_1^{-2} + i \times n_3 \times b_2^2 \times \Delta^{-\frac{1}{2}}) & (i \times n_1 - n_2) \times b_1^2 \times \Delta^{-\frac{1}{2}} \\ (i \times n_1 + n_2) \times b_2^2 \times \Delta^{-\frac{1}{2}} & (-n_0 \times b_2^{-2} + i \times n_3 \times b_1^2 \times \Delta^{-\frac{1}{2}}) \end{pmatrix} \cdot \tag{10.39}$$

It is again easy to show that the set $\hat{S}^d(\hat{q}^d, +, \times^d)$ of all possible matrices (10.39) is an isofield. The isomorphism $\hat{S}^d(\hat{q}^d, +, \times^d) \approx \hat{Q}^d(\hat{q}^d, +, \hat{\times}^d)$ then follows.

The *isodual isonorm* is now given by

$$\hat{|}\hat{q}^d\hat{|}^d = [\,\text{Det}_R(\hat{q}^d \times T^d)\,]^{\frac{1}{2}} \times \hat{1}_0{}^d = (\hat{q}^{\dagger d} \times^d \hat{q}^d)^{\frac{1}{2}} \times \hat{1}_0{}^d =$$

$$= [\, n_0^2 + \Delta(n_1^2 + n_2^2 + n_3^2)]^{\frac{1}{2}} \times \hat{1}_0{}^d, \tag{10.40}$$

and also verified the basic rule

$$\hat{|}\hat{q}^d \hat{\odot}^d \hat{q}'^d\hat{|}^d = \hat{|}\hat{q}^d\hat{|}^d \hat{\times}^d \hat{|}\hat{q}'^d\hat{|}^d \in \hat{R}^d, \quad \hat{q}^d, \hat{q}'^d \in \hat{Q}^d. \tag{10.41}$$

The *isodual isobasis*

$$\hat{e}^d_1 = \hat{1}^d_0, \quad \hat{e}^d_{k+1} = \hat{1}^d_k, \quad k = 1, 2, 3, \tag{10.42}$$

then shows that *isodual isoquaternions constitute a four-dimensional, isodual, isonormed, isoassociative, non-isocommutative isoalgebra over the isodual isoreals* $\hat{O}^d(4) \approx U^d(4)$.

We shall leave to the interested reader the study of the isotopies of other forms of quaternions, the *split quaternions, antiquaternions and semiquaternions* [39], as well as the study of *pseudoisoquaternions* and their isoduals.

11: REALIZATION OF ISOOCTONIONS AND THEIR ISODUALS

The realizations of octonion, isodual octonions, isooctonions and isodual isooctonions follow very closely the corresponding realizations at the quaternionic level. In particular, the realizations of the isooctonions and their isoduals follows very closely the construction of isoquaternions and their isoduals from isocomplex numbers and their isoduals.

11.A: Realization of octonions. Recall (see, e.g., ref.s [7,9,39] and contributions quoted therein), that the octonions $o \in O(o, +, \times)$ can be realized via two quaternions, $o = (q_1, q_2)$, with composition rules

$$o \odot o' = (q_1, q_2) \odot (q'_1, q'_2) = (q_1 \odot q'_1 + q_1 \odot q'_2, -\bar{q}_1 \odot q'_2 + \bar{q}_1 \odot q_2), \tag{11.1}$$

The antiautomorphic conjugation of an octonion is given by

$$\bar{o} = (\bar{q}_1, -q_2). \tag{11.2}$$

It is then possible to introduce the *norm*

$$|o|^2 := (\bar{o} \odot o) = |q_1| + |q_2|, \tag{11.3}$$

which also verified the basic axiom

$$|o \odot o'| = |o| \times |o'| \in R, \quad o, o' \in O. \tag{11.4}$$

We finally recall that *the octonions form an eight dimensional normed, nonassociative and noncommutative, alternative algebra* U(8) *over the field of reals* R(n,+,×) [loc. cit.].

11.A: Realization of isodual octonions. The *isodual octonions* are defined via the isoconjugation

$$o^d = (q_1^d, q_2^d) \tag{11.5}$$

this time, over the isodual reals $R^d(n^d,+,\times^d)$, and are therefore different than the conventional conjugate octonions $\bar{o}$, Eq.(11.2). Their isodual multiplication is

$$o^d \odot^d o'^d = (q_1^d, q_2^d) \odot^d (q_1'^d, q_2'^d) =$$

$$= (q_1^d \odot^d q_1'^d - \bar{q}_1^d \odot q_2'^d,\ q_1^d \odot^d q_2'^d + \bar{q}_1^d \odot^d q_2^d), \tag{11.6}$$

the isodual antiautomorphism is the given by

$$\bar{o}^d = (\bar{q}_1^d, -q_2^d). \tag{11.7}$$

It is then possible to introduce the *isodual norm*

$$|o^d|^{d\,2} := (\bar{o}^d \odot^d o^d) \times I^d = |q_1^d|^d + |q_2^d|^d \tag{11.8}$$

which also verifies the basic axiom

$$|o^d \odot^d o'^d|^d = |o^d|^d \times^d |o'^d|^d \in R^d, \quad o^d, o'^d \in O^d. \tag{11.9}$$

Thus, *the isodual octonions form an eight dimensional isodual, normed, nonassociative, alternative and noncommutative algebra* $U^d(8)$ *over the isodual real numbers* $R^d(n^d,+,\times^d)$.

11.C: Realization of the isooctonions. Isooctonions $\hat{o} \in \hat{O}(\hat{o},+,\hat{\times})$ can be defined as the pair of isoquaternions , $\hat{o} = (\hat{q}_1, \hat{q}_2)$ over the isoreals $\hat{R}(\hat{n},+,\hat{\times})$ with multiplication rules

$$\hat{o} \hat{\odot} \hat{o}' = (\hat{q}_1, \hat{q}_2) \hat{\odot} (\hat{q}'_1, \hat{q}'_2) = (\hat{q}_1 \hat{\odot} \hat{q}'_1 + \hat{q}_1 \hat{\odot} \hat{q}'_2, -\bar{q}_1 \hat{\odot} \hat{q}'_2 + \bar{q}_1 \hat{\odot} \hat{q}_2), \tag{11.10}$$

It is then easy to see that the lifting $o \rightarrow \hat{o}$ is an isotopy, thus preserving all original axioms of o. In fact, we have the antiautomorphic conjugation

$$\bar{o} = (\bar{q}_1, -\hat{q}_2), \tag{11.11}$$

and the *isonorm*

$$|\hat{o}|^2 := (\bar{o} \hat{\odot} \hat{o}) \times \hat{I} = |\hat{q}_1| + |\hat{q}_2| \tag{11.12}$$

with property

$$|\hat{o} \hat{\odot} \hat{o}'| = |\hat{o}| \hat{\times} |\hat{o}'| \in \hat{R}, \quad \hat{o}, \hat{o}' \in \hat{O}. \tag{11.13}$$

It is then easy to see that *isooctonions form an eight dimensional isonormed, non-isoassociative, non-isocommutative, isoalternative isoalgebra* $\hat{U}(8) \sim U(8)$ *over the isoreals* $\hat{R}(\hat{n},+,\hat{\times})$.

11.D: Realization of the isodual isooctonions. The notion of isoduality applies also to the isooctonions yielding the isodual isooctonions $\hat{o}^d = (\hat{q}_1^d, \hat{q}_2^d)$ with composition rule

$$\hat{o}^d \hat{\odot}^d \hat{o}'^d = (\hat{q}_1^d, \hat{q}_2^d) \hat{\odot}^d (\hat{q}_1'^d, \hat{q}_2'^d) =$$

$$= (\hat{q}_1^d \hat{\odot}^d \hat{q}_1'^d - \bar{q}_1^d \hat{\odot}^d \hat{q}_2'^d,\ q_1^d \hat{\odot}^d q_2'^d + \bar{q}_1^d \hat{\odot}^d q_2^d), \tag{11.14}$$

Then we have the isodual isoantiautomorphism

$$\bar{o}^d = (\bar{q}_1^d, -\hat{q}_2^d). \tag{11.15}$$

the *isodual isonorm*

$$|\hat{o}^d|^{d\,2} := (\bar{o}^d \hat{\odot}^d \hat{o}^d) \times \hat{I}^d = |\hat{q}_1^d|^d + |\hat{q}_2^d|^d \tag{11.16}$$

which also verifies the basic axiom

$$|\hat{o}^d \hat{\odot}^d \hat{o}'^d|^d = |\hat{o}^d|^d \hat{\times}^d |\hat{o}'^d|^d \in \hat{R}^d, \quad \hat{o}^d, \hat{o}'^d \in \hat{O}^d. \tag{11.17}$$

It is then possible to prove that *isodual isooctonions form an eight dimensional isodual, isonormed, non-isoassociative, non-isocommutative, but isoalternative isoalgebra* $\hat{U}^d(8) \sim U^d(8)$ *over the isodual isofield* $\hat{R}^d(\hat{n}^d,+,\times^d)$.

The extension of the results to the pseudoisooctonions and their isoduals is left to interested readers.

12: CLASSIFICATION OF ISCNORMED ISOALGEBRAS WITH IDENTITY AND THE CONJECTURE OF NEW "HIDDEN NUMBERS"

Historically, the "numbers" studied in this paper are those permitting a solution of the following problem (see, e.g., ref. [8])

$$(a_1^2 + a_2^2 + \dots + a_n^2) \times (b_1^2 + b_2^2 + \dots + b_n^2) = A_1^2 + A_2^2 + \dots + A_n^2, \quad (12.1a)$$

$$A_k = \sum_{r,s} c_{krs} \times a_r \times b_s . \quad (12.1b)$$

where all the a's, b's and c's are elements of a field $F(a,+,\times)$ with conventional operations $+$ and $\times$. As well known, the only possible solutions of problem (12.1) are of dimension 1, 2, 4, 8 (Theorem 1.1).

The isotopies and pseudoisotopies of the theory of numbers evidently creates the problem of the possible existence of *"hidden numbers"*, that is, new solutions of dimension different than 1, 2, 4, 8 which are hidden in the operations $\times$ and/or $+$. This problem essentially asks whether the classification of Theorem 1.1 persists under isotopies, pseudoisotopies and their isodualities, or it is incomplete.

It is easy to see that the reformulation of problem (12.1) under the isotopies of the multiplication

$$\times \rightarrow \hat{\times} = \times T \times, \quad 1 \rightarrow \hat{1} = T^{-1}, \quad (12.2)$$

does not lead to new solutions. In fact, Problem (12.1) under lifting (12.2) is given by

$$(a_1^2 + a_2^2 + \dots + a_n^2) \hat{\times} (b_1^2 + b_2^2 + \dots + b_n^2) = A_1^2 + A_2^2 + \dots + A_n^2, \quad (12.3a)$$

$$A_k = \sum_{r,s} c_{krs} \hat{\times} a_r \hat{\times} b_s , \quad (12.3b)$$

where now all the a's, b's and c's belong to an isofield of the type $\hat{F}(a,+,\hat{\times})$, in which case $\hat{1}$ is an element of the original field F (Proposition 4.1). Problem (12.3) can be written in conventional operations

$$(a_1^2 + a_2^2 + \dots + a_n^2) \times (b_1^2 + b_2^2 + \dots + b_n^2) = T^{-2} \times (A_1^2 + A_2^2 + \dots + A_n^2), \quad (12.4a)$$

$$A_k = T^2 \times \sum_{r,s} c_{krs} \times a_r \times b_s , \quad (12.4)$$

The substitution of of the latter expression into the former, then recovers Problem (12.1) identically. The reformulation in the isofield $\hat{F}(\hat{a},+,\hat{\times})$ is also equivalent to the original one. We can therefore summarize the studies of this paper with the following isotopies and isodualities of Theorem 1.1:

THEOREM 12.1: All possible isonormed isoalgebras with multiplicative isounit over the isoreals are the isoalgebras of dimension 1 (isoreals), 2 (isocomplex), 4 (isoquaternions) and 8 (isooctonions), and the classification persists under isoduality.

Nevertheless, there exists a third formulation of pseudoisotopic type (Proposition 4.3 and Definition 4.2) characterized by the further lifting of the addition

$$+ \rightarrow \hat{+} = + \hat{K}, \quad 0 \rightarrow \hat{0} = - \hat{K}, \quad \hat{K} = K \times \hat{1}, \quad (12.5)$$

A more general formulation of Problem (12.1) can be written over the pseudoisofield $\hat{F}(\hat{a},\hat{+},\hat{\times})$, where the elements $\hat{a}$, the addition $\hat{+}$ and the multiplication $\hat{\times}$ are lifted

$$(\hat{a}_1^2 \hat{+} \hat{a}_2^2 \hat{+} \dots \hat{+} \hat{a}_n^2) * (\hat{b}_1^2 \hat{+} \hat{b}_2^2 \hat{+} \dots \hat{+} \hat{b}_n^2) = \hat{A}_1^2 \hat{+} \hat{A}_2^2 \hat{+} \dots \hat{+} \hat{A}_n^2, \quad (12.6a)$$

$$\hat{A}_k = \sum_{r,s} \hat{c}_{krs} * \hat{a}_r * \hat{b}_s = (\sum_{r,s} c_{krs} \, a_r \, b_s) \hat{1} = A_k \hat{1}, \quad (12.6b)$$

and can be rewritten in conventional operations

$$[(a_1^2+a_2^2+\dots+a_n^2)\hat{1} + (n-1) K \hat{1}] T [(b_1^2+b_2^2+\dots+b_n^2)\hat{1} + (n-1) K \hat{1}] =$$

$$= (A_1^2 + A_2^2 + \dots + A_n^2)\hat{1} + (n-1) K \hat{1}, \quad (12.7a)$$

$$A_k = (\sum_{r,s} c_{krs} \, a_r \, b_s)\hat{1}, \quad (12.7)$$

where we have the cancellation of the isounit as in preceding cases, but the appearance of the "hidden" degree of freedom K.

The existence of the "hidden numbers", that is, of solutions of problem (12.7) of dimension other than 1, 2, 4, 8, is here submitted, apparently for the first time, *as a conjecture under the pseudoisofield* $\hat{F}(\hat{a},\hat{+},\hat{\times})$*, i.e., under the loss of the needed axioms of a field, such as the distributive laws* (Proposition 2.3.3), although without technical study at this time.

We merely limit ourselves to indicate the existence of the following example of "hidden numbers" of dimension 3

$$(\hat{1}^{\hat{2}} \hat{+} \hat{2}^{\hat{2}} \hat{+} \hat{3}^{\hat{2}}) * (\hat{5}^{\hat{2}} + \hat{6}^{\hat{2}} + \hat{7}^{\hat{2}}) = \hat{12}^{\hat{2}} \hat{+} \hat{24}^{\hat{2}} \hat{+} \hat{30}^{\hat{2}}. \qquad (12.8)$$

Note the original combinations for the numbers on the r.h.s $12 = 2\times 6$, $24 = 2\times 5 + 2\times 7$, $30 = 3\times 3 + 3\times 7$, although a solution in three dimension does not exist, i.e.,

$$(1^2 + 2^2 + 3^2)(5^2 + 6^2 + 7^2) \neq 12^2 + 24^2 + 30^2. \qquad (12.9)$$

However, the more general problem (12.8) can be written

$$[\,(1^2 + 2^2 + 3^2)\hat{1} + 2K\hat{1}\,]\,T\,[\,(5^2 + 6^2 + 7^2)\,\hat{1} + 2K\hat{1}\,] =$$

$$= (12^2 + 24^2 + 30^2)\hat{1} + 2K\hat{1}, \qquad (12.10)$$

and reduces to the equation in K

$$4K^2 + 246K - 80 = 0, \qquad (2.A.11)$$

with solution

$$K = 0.325.......... \qquad (2.A.11)$$

Thus, a solution exists under the relaxations of a sufficient number of axioms of the original fields, in addition to the loss of distributivity. In fact, in the case considered we start from the set of integers which is a field. However, the emerging solution for K is not an integer. This implies the loss of closure under the isoaddition (see the comments after Proposition 4.3) for the case of integers. However, closure is regained if the field is enlarged to include all real numbers. The issue whether such solutions of problem (12.9) do indeed form a pseudoisofield is left to the interested mathematician.

Note that Problems (12.3) and (12.6) are restricted to dimensions $n \leq 8$. This is due to the fact that algebras of dimensions higher than 8 are no longer alternative [8], and such a property is expected to persist under isotopies and pseudoisotopies.

Among endless novel (and intriguing) problems identified by the isofields which are still open at this writing, we indicate:

> The novel notion of "number with a singular unit", i.e., the isofields of Class IV which are at the foundations of the isotopic studies of gravitational collapse and are vastly unknown at this writing;

> The study of isofields of isocharacteristic $p \neq 0$, to see whether new fields, and therefore new Lie algebras, are permitted by the isotopies;

> The study of the integro-differential topology characterized by isofields with local-differential structure and integral isounits; and others.

APPENDIX A: GENONUMBERS AND THEIR ISODUALS

In the main text of this paper we have studied the degrees of freedom in the characterization of numbers originating from the *addition* and *multiplication*. The emerging generalized field are at the foundations of the Lie-isotopic theory [14,15,37]. In this appendix we shall indicate the existence of a third degree of freedom originating in the *ordering* of the above operations, which results in a further generalization of fields, this time, at the foundation of the Lie-admissible algebras [14,15,27]

Let $F(a,+,\times)$ be a field of ordinary numbers with generic elements a, b, c, ..., addition $a + b = b + a$ and multiplication $a\times b$. Each of these operations can be defined with respect to the following:

Ordering of the multiplication: multiplication of a time b from the left, $a \times^{>} b$, and multiplication of b time a from the right, $a\, {}^{<}\!\times b$, here called *genomultiplications.*

Ordering of the addition: addition of a to b from the left, $a +^{>} b$, and addition of b to a from the right, $a\, {}^{<}\!+ b$, here called *genoaddition*;

Let us study genomultiplications. The first property to note is that the ordering of the multiplication is fully compatible with its basic axioms, such as alternativity (for octonions), associativity (for quaternions) and commutativity (for complex and real numbers). In fact in the latter case we have

$$a \times^{>} b \equiv b \times^{>} a, \qquad a\, {}^{<}\!\times b \equiv b\, {}^{<}\!\times a. \qquad (A.1)$$

However, the identity of the two ordered multiplications is an un-necessary condition of the current theory of numbers, because the two genomultiplications can be assumed to be different

$$a \times^{>} b \neq a\, {}^{<}\!\times b, \qquad (A.2)$$

with realization

$$a \times^{>} b := a R b, \qquad a\, {}^{<}\!\times b := a S b, \quad R \neq S, \qquad (A.3)$$

where R and S are fixed, sufficiently smooth, bounded and nowhere singular (but not necessarily Hermitean) elements outside the original field, here called *genotopic elements*.

In the multiplication of two integers, say, two and three, we then have the following cases:

1) The conventional multiplications "two multiplied by three and three multiplied by two equal six", which hold under the (generally tacit) assumption of the number one as the unit with consequential conventional multiplication;

2) The isotopic multiplications "two multiplied by three and three multiplied by two equal twelve" which hold for the isotopic element T = 2 and isounit $\hat{1} = 1/4$; and

3) The genotopic multiplications "two multiplied by three from the right and three multiplied by two from the right equal twelve", and "two multiplied by three and three multiplied by two from the left equal eighteen" which hold for the isotopic element for the right ordering R = 2 and that for the left ordering S = 3.

We can then introduce the following two generalized (left and right) units, here called *genounits*

$$\hat{1}^> = R^{-1}: \qquad \hat{1}^> \hat{\times}^> a = a \hat{\times}^> \hat{1}^> \equiv a\,, \tag{A.4a}$$

$$^<\hat{1} = S^{-1}: \qquad ^<\hat{1} \, ^<\hat{\times} a = a \, ^<\hat{\times} \, ^<\hat{1} \equiv a\,, \tag{A.4b}$$

It is then easy to see that all axioms and properties of Definition 4.1 are preserved under the restriction of the multiplication to one of the above two orderings for all dimensions 1, 2, 4, 8. This yields a new type of fields here called *genofield* and denoted with the symbols $\hat{F}^>(\hat{a}^>, +, \hat{\times}^>)$, $^<\hat{F}(^<\hat{a},+,^<\hat{\times})$, or with the unified symbol $^<\hat{F}^>(^<\hat{a}^>,+,^<\hat{\times}^>)$ where the need to select one ordering at the time is understood.

All properties of isofields also extend to genofields, as the reader is encouraged to verify. In particular, we have the *isodual genofields* characterized by the antiautomorphic conjugations

$$R \rightarrow R^d = -R, \qquad S \rightarrow S^d = -S, \tag{A.5}$$

denoted $^<\hat{F}^{>d}(^<\hat{a}^{>d},+,^<\hat{\times}^{>d})$.

It is evident that isofields are a particular case of genofield when the genotopic elements coincide

$$^<\hat{F}^>(^<\hat{a}^>,+,^<\hat{\times}^>)\big|_{R = S = T} = \hat{F}(\hat{a},+,\hat{\times}). \tag{A.6}$$

We now show that genofields are the correct fields underlying the Lie-admissible algebras. Let [A,B] = AB – BA be the conventional Lie product among generic quantities A, B (such as vector fields on a cotangent bundle or operators on a Hilbert space), where AB is the conventional associative product.

The general realization of the Lie-admissible algebras [14] can be constructed via the following R-S-mutation of the above Lie product

$$(A, B) := A\,R\,B - B\,S\,A, \tag{A.7}$$

and results to be Lie-admissible because the attached antisymmetric product

$$[A \hat{,} B] := (A, B) - (B, A) = A\,T\,B - B\,T\,A, \quad T = R - S, \tag{A.8}$$

is Lie-isotopic.

The lifting $[A, B] \rightarrow [A \hat{,} B]$ was called an *isotopy* in ref. [14], while the lifting $[A, B] \rightarrow (A, B)$ was called a *genotopy* (Sect. 1), and this motivates the corresponding names of "isofields" and "genofields".

Now, the Lie-isotopic algebras are characterized by one single isotopy of the enveloping associative algebra and related unit

$$AB = A \times B \rightarrow A \hat{\times} B = A\,T\,B, \qquad I \rightarrow \hat{1} = T^{-1}. \tag{A.9}$$

As such, to be consistently formulated, Lie-isotopic algebras must be defined over an isofield $\hat{F}(\hat{a},+,\hat{\times})$ with isounit $\hat{1} = T^{-1}$.

Note that, strictly speaking, the conventional multiplication × admits no ordering because $I^> \equiv {}^<I = I$. The above orderings exist for the isomultiplication $\hat{\times} = \times T \times$ because in this case we can have different isounits $\hat{1}^> \neq {}^<\hat{1}$.

It is then evident that the Lie-admissible algebras are generated by two different isotopies of the original associative enveloping algebras with corresponding isotopies of the units

$$AB \rightarrow ARB := A \hat{\times}^> B, \qquad I \rightarrow \hat{1}^> = R^{-1}, \tag{A.10a}$$

$$BA \rightarrow BSA := B \, ^<\hat{\times} A\,, \qquad I \rightarrow {}^<\hat{1} = S^{-1}. \tag{A.10b}$$

and, as such, they must be defined over the genofields $^<\hat{F}^>(^<\hat{a}^>,+,^<\hat{\times}^>)$ with isounits $^<\hat{1}^>$.

In Eq.s (A.10) we have presented the right and left isomultiplications and related isounits as disjoint, in which case the isounits can indeed be Hermitean and real-valued, thus admitting of Kadeisvili classification into Classes I, II, III, IV, V.

Nevertheless, the realizations used in physics are those when the forward and back genounits are inter-related by a conjugation, such as the Hermitean conjugation

$$\hat{1}^{>} = ({}^{<}\hat{1})^{\dagger}. \qquad \text{(A.11)}$$

In this case the genofields assume particular physical significance because they provide an axiomatization of irreversibility (see [27] for details).

The preceding results on the ordering of the multiplication extend to the ordering of the addition. The understanding is that, as it was the case for the conventional multiplication, the conventional addition admits no meaningful ordering because $\hat{0}^{>} = {}^{<}\hat{0} \equiv 0$. The ordering exists for the isoaddition $\hat{+} = + K +$, because in this case $\hat{+}^{>} \neq {}^{<}\hat{+}$, $K^{>} \neq {}^{<}K$. The understanding is that genofield are closed under distributive law, while this is no longer the case under the genoadditions ${}^{<}\hat{+}^{>}$.

We reach in this way the broadest possible generalization of the conventional theory of numbers permitted by the isotopies and genotopies, that characterized by:

1) *pseudogenofields* ${}^{<}\hat{F}^{>}({}^{<}\hat{a}^{>}, {}^{<}\hat{+}^{>}, {}^{<}\hat{\times}^{>})$, here defined via the genotopies of all aspects of conventional fields $F(a,+,\times)$, including elements $a \to {}^{<}\hat{a}^{>}$, addition $+ \to {}^{<}\hat{+}^{>}$ and related unit $0 \to {}^{<}\hat{0}^{>}$, and multiplication $\times \to {}^{<}\hat{\times}^{>}$ and related unit $1 \to {}^{<}\hat{1}^{>}$.

2) *isodual pseudogenofields* ${}^{<}\hat{F}^{>d}({}^{<}\hat{a}^{>d}, {}^{<}\hat{+}^{>d}, {}^{<}\hat{\times}^{>d})$ here defined via the isoduality of pseudogenofields.

The emerging broadening of the theory of numbers is then considerable because we now have:

A) Conventional numbers of dimension 1, 2, 4, 8 and their isoduals;
B) Isonumbers of the same dimension and their isoduals;
C) Genonumbers of the same dimensions and their isodual;
D) Pseudoisonumbers of the same dimension and their isoduals;
E) Pseudogenonumbers of the same dimension and their isoduals;
F) "Hidden pseudoisonumbers" of dimension 3, 4, 5, 7 and their isoduals;
G) "Hidden pseudogenonumbers" of dimension 3, 4, 5, 7, and their isoduals;

each of which can be defined over a field of characteristic 0 of $p \neq 0$, as well as in Kadeisvili topologically different classes whenever applicable.

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