

## Relativistic Hadronic Mechanics: Nonunitary, Axiom-Preserving Completion of Relativistic Quantum Mechanics

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*The most majestic scientific achievement of this century in mathematical beauty, axiomatic consistency, and experimental verifications has been special relativity with its unitary structure at the operator level and canonical structure at the classical level, which has turned out to be exactly valid for point particles moving in the homogeneous and isotropic vacuum (exterior dynamical problems). In recent decades a number of authors have studied nonunitary and noncanonical theories, here generally called deformations, for the representation of broader conditions, such as extended and deformable particles moving within inhomogeneous and anisotropic physical media (interior dynamical problems). In this paper we show that nonunitary deformations, including  $q$ -,  $k$ -, quantum-, Lie-isotopic, Lie-admissible, and other deformations, even though mathematically correct, have a number of problematic aspects of physical character when formulated on conventional spaces over conventional fields, such as lack of invariance of the basic space-time units, ambiguous applicability to measurements, loss of Hermiticity-observability in time, lack of invariant numerical predictions, loss of the axioms of special relativity, and others. We then show that the classical noncanonical counterparts of the above nonunitary deformations are equally afflicted by corresponding problems of physical consistency. We also show that the contemporary formulation of gravity is afflicted by similar problematic aspects because Riemannian spaces are noncanonical deformations of Minkowskian spaces, thus having noninvariant space-time units. We then point out that new mathematical methods, called isotopies, genotopies, hyperstructures and their isoduals, offer the possibilities of constructing a nonunitary theory, known as relativistic hadronic mechanics which: (1) is as axiomatically consistent as relativistic quantum mechanics, (2) preserves the abstract axioms of special relativity, and (3) results in a completion of the conventional mechanics much along the celebrated Einstein-Podolski-Rosen argument. A number of novel applications are indicated, such as a geometric unification of the special and general relativity via the isominkowskian geometry in which the two relativities are differentiated via the invariant basic*

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unit, while preserving conventional Riemannian metrics, Einstein's field equations, and related experimental verifications; a novel operator form of gravity verifying the axioms of relativistic quantum mechanics under the universal isopoincaré symmetry; a new structure model of hadrons with conventional massive particles as physical constituents which is compatible with composite quarks and with established unitary classifications; and other novel applications in nuclear physics, astrophysics, theoretical biology, and other fields. The paper ends with the proposal of a number of new experiments, some of which may imply new practical applications, such as conceivable new forms of recycling nuclear waste. The achievement of axiomatic consistency in the study of the above physical problems has been possible for the first time in this paper thanks to mathematical advances that recently appeared in a special issue of the *Rendiconti Circolo Matematico Palermo*, and in other journals, identified in the Acknowledgments.

## 1. OUTLINE OF DEFORMATIONS

In 1948, the American mathematician A. A. Albert<sup>(1)</sup> introduced the notion of *Jordan admissible and Lie-admissible algebras* as generally nonassociative algebras  $U$  with elements  $a, b, c$ , and abstract product  $ab$  which are such that the attached algebras  $U^+$  and  $U^-$ , which are the same vector spaces as  $U$  equipped with the products  $\{a, b\}_U = ab + ba$  and  $[a, b]_U = ab - ba$ , are Jordan and Lie algebras, respectively. Albert then studied the algebra with product

$$(A, B) = p \times A \times B + (1 - p) \times B \times A \quad (1.1)$$

where  $p$  is a parameter,  $A, B$  are matrices or operators hereon assumed to be Hermitian, and  $A \times B$  is the associative product. It is easy to see that the above product is indeed jointly Jordan- and Lie-admissible because  $\{A, B\}_U = A \times B + B \times A$  and  $[A, B]_U = (1 - 2p) \times (A \times B - B \times A)$ .

As part of his Ph.D. studies in theoretical physics, Santilli<sup>(2)</sup> introduced in 1967 a stronger notion of Lie admissibility which is Albert's definition,<sup>(1)</sup> plus the condition that the algebras  $U$  admit Lie algebras in their classification. This refinement is recommendable for physical application because Albert was primarily interested in the *Jordan content* of a given algebra (for  $p=0$  product (1.1) becomes that of a *commutative Jordan algebra*), while possible physical applications are evidently enhanced by a well-defined *Lie content*. In fact, product (1.1) does not admit a (finite) value of  $p$  under which it recovers the Lie product and, therefore, it cannot be used for possible generalizations of current physical theories.

Santilli<sup>(2a)</sup> therefore introduced the realization

$$(A, B) = p \times A \times B - q \times B \times A \quad (1.2)$$

with related time evolution in the following infinitesimal and finite forms ( $\hbar = 1$ ):

$$i dA/dt = p \times A \times H - q \times H \times A \quad (1.3a)$$

$$A(t) = e^{iq \times t \times H} \times A(0) \times e^{-ip \times t \times H} \quad (1.3b)$$

where  $p$  and  $q$  are finite parameters with non-null values  $p \pm q$ ,  $A, B$  are Hermitian matrices or operators; and  $A \times B$  is also the associative product. It is easy to see that product (1.2) is Jordan-admissible, Lie-admissible, and admits Lie algebras as particular (nondegenerate) cases for  $p = q$  ( $\neq 0$ ).

Refinement<sup>(2)</sup> turns out to be insufficient in physical applications because, as we shall see shortly, the parameters  $p$  and  $q$  become *operators* under the time evolution of the theory. Santilli<sup>(3a, b)</sup> therefore introduced in 1978 the broader condition of *general Lie-admissibility* which is the notion of Ref. 1 plus the condition that the algebra  $U$  admits *Lie-isotopic* (rather than Lie) algebras in its classification.

The latter notion was realized via the *general Lie-admissible* product (first introduced in Ref. 3b, p. 719)

$$(A, B) = A \times P \times B - B \times Q \times A \quad (1.4)$$

with time evolution in infinitesimal and finite forms (Ref. 3b, pp. 741, 742)

$$i dA/dt = A \times P \times H - H \times Q \times A \quad (1.5a)$$

$$A(t) = e^{iH \times Q \times t} \times A(0) \times e^{-it \times P \times H} \quad (1.5b)$$

where  $P$  and  $Q$  are generally nonhermitian matrices or operators with non-singular and Hermitian sum  $P + Q$  admitting of parametric values  $p$  and  $q$  as particular cases. The conventional Heisenberg's equations are evidently recovered for  $P = Q = 1$ .

Note that the  $P$  and  $Q$  operators must be sandwiched in between the elements  $A$  and  $B$  to characterize an algebra as commonly understood in mathematics. In fact, the script  $P \times A \times B - Q \times B \times A$  would be acceptable for  $P$  and  $Q$  parameters, but it would violate the right distributive and scalar laws for  $P$  and  $Q$  operators (see Refs. 3a, 3b for details).

In the latter case the algebras  $U$  admit Lie algebras for  $P = Q = 1$ , and the attached antisymmetric algebra  $U^-$  is not characterized by the traditional product  $[A, B] = A \times B - B \times A$ , but rather by the product (first introduced in Ref. 3b, p. 725)

$$[A, \wedge B]_U = A \hat{\times} B - B \hat{\times} A = A \times T \times B - B \times T \times A, \quad T = P - Q = T^\dagger \quad (1.6)$$

called *Lie-isotropic*, because verifying the Lie axioms, although in a more general way, with the product  $A \hat{\times} B = A \times T \times B$  called *isoassociative* because it is more general than the conventional associative product  $A \times B$ , yet preserves associativity,  $A \hat{\times} (B \hat{\times} C) \equiv (A \hat{\times} B) \hat{\times} C$ .

According to the above results, the *nonassociative* algebra  $U$  with product  $(A, B)$ , Eq. (1.4), can be replaced by an algebra  $\hat{\xi}$  with *isoassociative* product  $A \hat{\times} B = A \times T \times B$ , in the characterization of the attached antisymmetric algebra<sup>(3a, 3b, 3d)</sup>

$$[A, \hat{\times} B]_{\sigma} = (A, B) - (B, A) \equiv [A, \hat{\times} B]_{\xi} = A \hat{\times} B - B \hat{\times} A \quad (1.7)$$

The latter property permitted a step-by-step lifting of the conventional formulation of Lie theory in terms of the isoassociative product  $A \hat{\times} B$ , including enveloping algebras, Lie algebras, Lie groups, Lie symmetries, transformation and representation theory, etc.,<sup>(4)</sup> called today *Lie-Santilli isothery* (see Ref. 5 and papers quoted therein).

As a particular case of the broader Lie-admissible formulations, Santilli<sup>(3)</sup> therefore studied the Lie-isotopic time evolution in infinitesimal and finite forms for  $T = T^{\dagger}$  (first introduced in Ref. 3b, p. 752)

$$i dA/dt = [A, \hat{\times} H]_{\xi} = A \hat{\times} H - H \hat{\times} A = A \times T \times H - H \times T \times A \quad (1.8a)$$

$$A(t) = e^{i\hbar \times T \times t} \times A(0) \times e^{-i\hbar \times T \times t} \quad (1.8b)$$

which admit conventional quantum equations for  $\hat{T} = 1$ .

The latter theory was called *isotopic*<sup>(3)</sup> in the Greek sense of being *axiom-preserving*, because the deformation is still Lie, yet of a more general nature, while the preceding theory (1.5) was called *genotopic*, in the Greek sense of being *axiom-inducing*, because the Lie axioms are replaced by the covering Lie-admissible axioms.

No operator theory has sufficient depth without well-defined classical foundations. For this reasons, Santilli conducted extensive studies on the classical counterparts of the preceding theories reported in Refs. 3d, 3e. In essence, the classical action underlying the Lie-isotopic theories resulted in the most general possible, first-order Pfaffian action in phase space

$$A = \int_{t_1}^{t_2} dt [R_{\mu}(b) db^{\mu}/dt + H(t, b)] \quad (1.9)$$

$$b = \{b^{\mu}\} = \{r^k, p_k\}$$

$$R = \{R_{\mu}\} = \{A_k(r, p), B^k(r, p)\}, \quad \mu = 1, 2, \dots, 6, \quad k = 1, 2, 3$$

whose variations yield the well-known *Birkhoff's* equations in the following covariant and contravariant forms (see Ref. 3d for historical notes and references)

$$\Omega_{\mu\nu}(b) \frac{db^{\nu}}{dt} = \frac{\partial H(t, b)}{\partial b^{\mu}} \quad (1.10a)$$

$$\frac{db^{\mu}}{dt} = \Omega^{\mu\nu}(b) \frac{\partial H(t, b)}{\partial b^{\nu}} \quad (1.10b)$$

with (nowhere degenerate) covariant and contravariant tensors

$$\Omega_{\mu\nu} = \partial R_{\nu} / \partial b^{\mu} - \partial R_{\mu} / \partial b^{\nu} \quad (1.11a)$$

$$\Omega^{\mu\nu}(b) = (|\Omega_{\alpha\beta}|^{-1})^{\mu\nu} \quad (1.11b)$$

The ensuing mechanics, called *Birkhoffian mechanics* in Ref. 3d, and *Birkhoff-Santilli isomechanics* in various references (see, e. g., Ref. 5 and papers quoted therein), was said to be isotopic because it preserves the main axioms of conventional Hamiltonian mechanics although realized in their most general possible form, i.e.: (1) derivability from the most general possible first-order action (analytic isotopy); (2) characterization by the most general possible, regular symplectic structure in local coordinates (analytic isotopy),

$$\Omega = \Omega_{\mu\nu}(b) db^{\mu} \wedge db^{\nu} \quad (1.12)$$

and (3) characterization by the most general possible regular (unconstrained) brackets verifying the Lie axioms (algebraic isotopy)

$$[A, B]^* = \frac{\partial A}{\partial b^{\mu}} \Omega^{\mu\nu}(b) \frac{\partial B}{\partial b^{\nu}} \quad (1.13)$$

Conventional classical Hamiltonian mechanics is admitted as a particular case at all levels for  $R = R^{\circ} = (p, 0)$ , as one can easily verify.

One may consult Ref. 3d for additional aspects, including: the unified treatment via the *conditions of variational self-adjointness*; the use of the *isotopies of Lie's theory*; the proof of the "direct universality" of the mechanics, i.e., its capability to represent all well-behaved local-differential nonconservative Newtonian systems (universality) in the given  $b$ -coordinates of the experimenter (direct universality); and other aspects.

Since Eqs. (1.8) and (1.10) have the most general possible (unconstrained and regular) Lie structures, the former were introduced in Ref. 3d as the operator image of the latter.

References 3a, 3e were devoted to the study of the classical counterpart of Lie-admissible equations (1.5). Conventional Newtonian forces are divided into variationally *self-adjoint* (SA) and *non-self-adjoint forces* (NSA),  $F_k(t, b) = F_k^{SA} + F_k^{NSA}$ . The SA forces are represented in terms of a conventional potential  $U(t, b)$  via the techniques of the inverse problem of Ref. 3d. The NSA forces are represented via the *algebraic tensor* of the theory, according to the equations of Refs. 3a, 3e

$$\frac{db^v}{dt} - S^{\mu\nu}(t, b) \frac{\partial H(t, b)}{\partial b^\mu} \equiv m dv_k/dt - F_k^{SA}(t, b) - F_k^{NSA}(t, b) \quad (1.14a)$$

$$(S^{\mu\nu}) = (\omega^{\mu\nu}) + (s^{\mu\nu}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (F^{NSA}/(\partial H/\partial p)) \end{pmatrix} \quad (1.14b)$$

where  $\omega^{\mu\nu}$  is the familiar canonical Lie tensor and  $S_{\mu\nu}$  is a Lie-admissible tensor because

$$S^{\mu\nu}(t, b) - S^{\nu\mu}(t, b) = 2\omega^{\mu\nu} \quad (1.15)$$

Consequently, the brackets of the time evolution

$$dA/dt = (A, H) = \frac{\partial A}{\partial b^\mu} S^{\mu\nu}(t, b) \frac{\partial H}{\partial b^\nu} \quad (1.16)$$

are Lie-admissible,

$$(A, B) - (B, A) = 2[A, B] \quad (1.17)$$

with a compatible lifting of the symplectic two-form (1.12) called *symplectic-admissible*.

The emerging mechanics was called *Birkhoff-admissible mechanics* in Ref. 3e and it is called *Birkhoff-Santilli genomechanics* in the literature.<sup>(5)</sup> Note its very simple direct universality for all possible Newtonian systems, owing to general solution (1.14b), which should be compared with the rather complex direct universality of Birkhoff's equations (1.10).

We should also recall that the Lie-admissible equations (1.14) were constructed<sup>(3a, 3e)</sup> along the original Hamilton's equations, those with *external terms* here denoted  $F_k^{NSA}$ . The important point is that the numbers of independent functions in the external terms  $F_k^{NSA}$  and in the Lie-admissible tensor  $S^{\mu\nu}$  coincide.

Reformulation (1.14) is required by the fact that the brackets of Hamilton's equations with external terms violate the condition to form any algebra, let alone the Lie algebras, thus preventing the construction of a covering mechanics. On the contrary, brackets (1.16), first of all, verify all conditions to characterize an algebra, and, second, that algebra becomes Lie-admissible, i.e., a covering of the algebraic structure of conventional Hamiltonian mechanics.

Note also that *Lie-isotopic equations* (1.10) are *structurally reversible*, that is, they are reversible for reversible Hamiltonians. On the contrary, *Lie-admissible equations* (1.14) are *structurally irreversible*, that is, they are irreversible even for reversible Hamiltonians. These main characteristics will persist throughout the analysis of this paper.

As such, the Lie-admissible equations are particularly suited for an *axiomatization of irreversibility*, that is, its representation via the structure of the theory, rather than the addition of symmetry breaking terms in a time-symmetric Lagrangian or Hamiltonian.

Since Eqs. (1.5) and (1.14) have the maximal possible (unconstrained and regular) Lie-admissible structures, the former were assumed in Refs. 3e to be the operator image of the latter. For additional aspects, the reader may inspect Ref. 3e.

Classical and operator Lie-admissible structures and their Lie-isotopic particularizations were then studied in a variety of mathematical and physical papers; see, e.g., Ref. 5 and additional papers.<sup>(6)</sup> A comprehensive bibliography up to 1984 can be found in Ref. 7, and that on subsequent works in Ref. 5d.

In 1985 Biedenharn<sup>(8a)</sup> and Macfarlane<sup>(8b)</sup> introduced the so-called *q*-deformations which were followed by a large number of papers in the field (see, e.g., Ref. 9). Still more recently, other types of deformations of relativistic quantum formulations appeared in the literature under the name of *k*-deformations (see, e.g., Ref. 10). Comprehensive studies were also conducted in the field known under the name (somewhat misleading) of *quantum groups* (see, e.g. Ref. 11).

The latter deformations are essentially reducible to the following types:

- (1) Deformations of enveloping associative algebras

$$A \times B \rightarrow A \hat{\times} B = q \times A \times B \quad (1.18)$$

- (2) Deformations of the Lie product

$$A \times B - B \times A \rightarrow A \times B - q \times A \times B \quad (1.19)$$

## (3) Deformations of the structure constants

$$X_i \times X_j - X_j \times X_i = C_{ij}^k X_k \rightarrow X_i \times X_j - X_j \times X_i = C_{ij}^k X_k \quad (1.20)$$

and numerous others studied in Sect. 2 and 3.

One can easily see that deformations (1.18) and (1.19) are particular cases of the Lie-admissible deformations (1.5), while alteration of the *structure constants*, Eq. (1.20), are true deformations as intended in mathematics. Because of this disparity, Ref. 2 suggested the name of *mutations* for *alterations of the structure of the Lie product*, with the intent of preserving the name of *deformations* for structure such as (1.20).

Nevertheless, the term "deformations" is now widely used and will be kept in this paper to avoid misinterpretations. We shall therefore call "deformation" any alteration of the structure of classical or quantum mechanics, thus including *q*-, *k*-, quantum-deformations, the deformations of Lie-isotopic and Lie-admissible type, as well as any deviation from the conventional linear, local, canonical, or unitary structure.

Ironically, by the time Biedenharn's and Macfarlane's papers<sup>(8)</sup> appeared, Santilli had already abandoned this line of inquiry because of insurmountable problematic aspects of *physical* character preliminarily reported by Lopez in Ref. 12.

Despite the appearance of the latter papers and the passing of time, the problematic aspects of deformations of classical and quantum formulations have not yet propagated in the literature, thus rendering their additional study recommendable.

The ultimate problem addressed in this paper is the following. On the one hand, the main characteristics of conventional classical and quantum formulations are those of being *canonical* and *unitary*, respectively. On the other hand, advancements in interior problems, e.g., the classical representation of the irreversibility of the structure of Jupiter, requires a *noncanonical* theory<sup>(3d, 3e)</sup> or the operator representation of a black hole structure requires a *nonunitary* theory.<sup>(13)</sup>

But, as outlined in the next section, the above classical and quantum deformations possess a number of rather serious, problematic aspects of physical nature, even though they possess an undeniable *mathematical* beauty (which perhaps accounts for the large number of papers in the field).

Above all, noncanonical-nonunitary theories violate the axioms of the special relativity, thus creating the considerable problems of identifying new axioms, proving their axiomatic consistency and, after that, establishing them experimentally.

The main problem considered in this paper is therefore the achievement of theories which are structurally noncanonical at the classical level and nonunitary at the operator level, yet formulated in such a way to be as axiomatically consistent as conventional mechanics and, above all, capable of preserving the abstract axioms of especial relativity.

As we shall see, contemporary formulations of quantum gravity are afflicted by similar problematic aspects because *Riemannian spaces are deformations of the Minkowski space which are noncanonical at the classical level and nonunitary at the operator level*. Therefore, quantum gravity suffers from essentially the same problematic aspects of the preceding nonunitary theories.

The primary objective of this paper is of *methodological* character. As such, applications and verifications will only be indicated for further studies elsewhere with the understanding that it would be unreasonable to expect their joint detailed treatment here.

## 2. PROBLEMATIC ASPECTS OF QUANTUM DEFORMATIONS, CLASSICAL DEFORMATIONS AND GRAVITY

As is well known, a necessary condition to exit the class of equivalence of quantum mechanics is that the map from quantum to deformed formulations must be *nonunitary*

$$U \times U^\dagger \neq I \quad (2.1)$$

when referred to conventional Hilbert spaces  $\mathcal{H}$  with inner product and normalization

$$\langle \psi | \phi \rangle \in C(c, +, \times), \quad \langle \psi | \psi \rangle = 1 \quad (2.2)$$

where  $C(c, +, \times)$  represents the conventional field of complex numbers  $c$  with familiar sum  $+$ , multiplication  $\times$ , and related additive unit 0 and multiplicative unit 1.

It is evident that, to be nontrivial, quantum deformations must be a nonunitary images of conventional quantum setting, otherwise they are mere equivalent quantum mechanical forms. Note that this includes not only *q*-, *k*-, and quantum-deformations, but also all Lie-admissible and Lie-isotopic formulations (1.2)–(1.8). As an example, we have the following

**Lemma 1.** The general Lie-admissible time evolution (1.5) and its Lie-isotopic particularization (1.8) are nonunitary on  $\mathcal{H}$  over  $C$ .

$t=0$  are no longer generally Hermitian at subsequent times, and the considered quantum deformations do not possess unambiguous observables.

As is well known, the numerical predictions of quantum mechanics are the result of data elaboration via special functions (and transforms). The predictions of quantum deformations are also the result of special functions although of new type specifically built per each case considered, the so-called  $q$ -,  $k$ -, quantum-special functions of Ref. 8-11. But nonunitary deformations are not form invariant under their time evolution and so are the related special functions. Problematic aspect (3) then follows because the lack of invariance of deformed special functions evidently implies the lack of invariant numerical predictions.

For instance,  $q$ -special functions at the initial time  $t=0$  no longer generally apply at a later time  $t$  because the  $q$  parameter becomes a  $Q$  operator under nonunitary transforms, according to the rule

$$q \times A \times B \rightarrow q \times U \times A \times A \times B \times U^\dagger = \hat{A} \times \hat{Q} \times \hat{B} \quad (2.6a)$$

$$U \times U^\dagger \neq I, \quad Q = q \times (U \times U^\dagger)^{-1} \\ \hat{A} = U \times A \times U^\dagger, \quad \hat{B} = U \times B \times U^{-1} \quad (2.6b)$$

and a similar situation holds in the other cases. ■

It should be stressed that Theorem 1 applies, specifically, to *nonunitary* deformations computed on a *conventional* Hilbert space over *conventional* fields. If the deformations are *unitary*, no problematic aspect evidently arises when computed over a conventional Hilbert space over  $C$ .

Similarly, if the deformations are *nonunitary* and computed over *suitably generalized* Hilbert space and fields, then consistency can be regained under certain conditions studied in Sec. 3.

The problematic aspects of the above "No-Go Theorem" are serious per se. Yet, additional problematic aspects are implied by consequences (1), (2), and (3). For instance, it is known that the probabilities of quantum mechanics are deeply linked to the invariance of the unit and its decomposition. The lack of invariance of the unit under nonunitary transforms then implies the following property (where the computation on conventional Hilbert spaces over conventional fields is assumed hereon):

**Corollary 1.A.** Nonunitary quantum deformations do not possess invariant probabilities.

Recall that the physical laws of quantum mechanics are unique in their definition and invariant under the time evolution of the theory. By recalling the several alternative possibilities of defining  $q$ -,  $k$ -, quantum-, and other

special functions (e.g., the numerous  $q$ -exponentiations existing in the literature<sup>(8-10)</sup> and their lack of invariance in time, we have the following:

**Corollary 1.B.** Nonunitary quantum deformations do not possess unique and invariant physical laws.

Recall that the causality of quantum mechanics follows from the unitarity of its time evolution. We therefore have the additional:

**Corollary 1.C.** Nonunitary quantum deformations violate causality.

But the problematic aspect considered particularly serious by this author is the following one of evident derivation from Theorem 1:

**Corollary 1.D.** Nonunitary quantum deformations violate the axioms of the special relativity.

The above occurrence can be easily illustrated by noting that, e.g., the deformed Minkowski spaces of Ref. 10 are not compatible with the Lorentz transforms, or that the corresponding deformed Poincaré symmetry is not isomorphic to the conventional symmetry. These occurrences create the sizable problems identified in Sec. 1 (which are inherent in relativistic deformations<sup>(10)</sup> of: (a) identifying new relativistic axioms which replace the Einsteinian ones; (b) proving their axiomatic consistency; and, after that, (c) establishing them experimentally.

By no means does the above analysis exhausts all physical problematic aspects of the deformations of quantum mechanics currently under study. For completeness, we mention that the rather old addition of an "imaginary potential"  $iV(r)$  to a (Hermitian) Hamiltonian  $H_0$ ,  $H = H_0 + iV(r)$ , which is frequently use in nuclear physics to represent dissipation, implies the deformation of the basic brackets from a bilinear to a triple form,

$$[A, H_0] = A \times H_0 - H_0 \times A \rightarrow [A, H, H^\dagger] = A \times H^\dagger - H \times A \quad (2.7)$$

By recalling that the brackets of the time evolution must be, for consistency, the brackets of the underlying algebras and symmetries, generalizations (2.7) imply the loss of *all* algebras as commonly understood, let alone the loss of all Lie algebras (e.g., the  $SU(2)$ -spin symmetry cannot be even defined, let alone treated, with triple systems). Under these conditions, familiar physical terms such as "protons and neutrons with spin 1/2" have no mathematical or physical meaning of any known nature (for more details on the problematic aspects of generalization (2.7), see Ref. 3b.

**Proof.** Heisenberg's time evolution in finite form has a *bimodular Lie structure*, in the sense of being characterized by an action to the right, here denoted  $U^> = \exp\{iH \times t\}$ , and an action to the left, here denoted  $<U = \exp\{-it \times H\}$ ,

$$A(t) = U^> \times A(0) \times <U = e^{iH \times t} \times A(0) \times e^{-it \times H} \quad (2.3)$$

The unitarity of the evolution follows from the familiar conjugation

$$<U = (U^>)^{\dagger} \quad (2.4)$$

under which we have the law

$$\begin{aligned} U \times U^{\dagger} &= U^{\dagger} \times U = U^> \times <U = <U \times U^> = U^> \times (U^>)^{\dagger} = (U^>)^{\dagger} \times U^> \\ &= <U \times (<U)^{\dagger} = (<U)^{\dagger} \times <U = I \end{aligned} \quad (2.5)$$

The general Lie-admissible law (1.5) violates, first, condition (2.4) and then each condition (2.5) because of the lack of commutativity of  $P$  and  $Q$  with  $H$ . The Lie-isotopic time evolution (1.8) verifies condition (2.4) but violates conditions (2.5), again because of the lack of general commutativity of  $T$  and  $H$ . Therefore, time evolutions (1.5) and (1.8) are non-unitarity. The same occurs for particular cases such as  $q$ -deformations (1.19). ■

Needless to say, the corresponding transformation theory of the classical Lie-admissible (1.14) and Lie-isotopic equations (1.10) are *noncanonical*, as studied in detail in Refs. 3d, 3e.

Even though Lie's theory is preserved, we essentially have a similar situation for deformations (1.20), in fact, to be nontrivial, the deformation of the structure constants  $C_{ij}^k \rightarrow D_{ij}^k$  must be produced at the classical level by *noncanonical* transforms with *nonunitary* image at the operator level (the reader may inspect the noncanonical deformation of the Minkowski space and of the Casimir invariants of the Poincaré symmetry of Ref. 10).

The general loss of unitarity then has the following serious problematic aspects of physical character:

**Theorem 1.** All possible nonunitary deformations of quantum mechanics computed on conventional Hilbert spaces over conventional fields, including  $q$ -,  $k$ -, quantum-, Lie-isotopic, and Lie-admissible and other deformations, have the following physical problematic aspects: (1) they lack the invariance of the unit, thus lacking unambiguous applications to measurements; (2) they lack the preservation of Hermiticity in time, thus lacking unambiguous observables; and (3) they lack invariant special functions and transforms, thus lacking invariant numerical predictions.

**Proof.** The unit of a quantum theory is the unit  $I$  of the enveloping associative operator algebra  $\xi$  with generic elements  $A, B, \dots$  and conventional associative product  $A \times B$ ,

$$I \times A = A \times I \equiv A, \quad \forall A \in \xi \quad (2.1)$$

It is well known that, by definition, the above unit is not invariant under nonunitary transforms

$$I \rightarrow I' = U \times I \times U^{\dagger} \neq I \quad (2.2)$$

and it is not generally preserved under the time evolution, e.g.,

$$i dI/dt = (I, H) = I \times P \times H - H \times Q \times I \neq 0 \quad (2.3)$$

Problematic aspect (1) then follows because the considered quantum deformations cannot be unambiguously applied to measurements, e.g., it is not possible to measure distances with a (stationary) meter of varying length.

Under a nonunitary transform, the familiar associative modular action of the Schrödinger's representation  $H \times |\psi\rangle$ , where  $H$  is an operator Hermitian at the initial time  $t=0$ , becomes

$$\begin{aligned} U \times H \times |\psi\rangle &= U \times H \times U^{\dagger} \times (U \times U^{\dagger})^{-1} \times U \times |\psi\rangle \\ &= \hat{H} \times \hat{T} \times |\hat{\psi}\rangle \end{aligned} \quad (2.4a)$$

$$U \times U^{\dagger} \neq I, \quad \hat{T} = (U \times U^{\dagger})^{-1}, \quad |\hat{\psi}\rangle = U \times |\psi\rangle, \quad \hat{H} = U \times H \times U^{\dagger} \quad (2.4b)$$

By noting that  $\hat{T}$  is Hermitian,  $\hat{T} = (U \times U^{\dagger})^{-1} = \hat{T}^{\dagger}$ , the initial condition of Hermiticity of  $H$  on  $\mathcal{H}$ ,  $\langle \psi | \times \{H \times |\psi\rangle\} \equiv \langle \psi | \times H^{\dagger} \times |\psi\rangle$ , when applied to the Hilbert space  $\mathcal{H}$  with states  $|\hat{\psi}\rangle, |\hat{\phi}\rangle$ , etc. requires the action of the transformed operator (2.4) on a *conventional* inner product, resulting in the expressions

$$\begin{aligned} \langle \hat{\psi} | \times \{\hat{H} \times \hat{T} \times |\hat{\psi}\rangle\} &\equiv \langle \hat{\psi} | \times \hat{T} \times \hat{H}^{\dagger} \times |\hat{\psi}\rangle, \\ \text{i.e., } \hat{H}^{\dagger} &= \hat{T}^{-1} \times \hat{H} \times \hat{T} \neq \hat{H} \end{aligned} \quad (2.5)$$

As such, Hermiticity is not preserved under nonunitary transforms formulated on conventional spaces  $\mathcal{H}$  over conventional fields  $C$ , because of the lack of general commutativity of  $\hat{T}$  and  $\hat{H}$ . By recalling that the time evolution of the considered class of deformations is nonunitary, problematic aspect (2) follows because operators which are Hermitian at the initial time

The same situation occurs in statistical mechanics when collisions are represented via the deformation of the Liouville equation with an external term

$$i dp/dt = [\rho, H] \rightarrow i dp/dt = [\rho, H, C] = \rho \times H - H \times \rho + C \quad (2.8)$$

In addition to the loss of all algebras, and, therefore, of all possible symmetries as currently understood, theories (2.7) and (2.8) do not have an invariant unit, thus suffering from most of the problematic aspects of Theorem 1 (see Ref. 14 for additional studies on the problematic aspects of statistical equations with external terms).

Similarly, the deformation of the *linearity* of quantum mechanics into *nonlinear theories* (hereon referred to *nonlinearity in the wavefunctions*), e.g., of the type<sup>(15)</sup>

$$H(x, p, \psi, \dots) \times \psi = E \times \psi \quad (2.9)$$

even though *mathematically* impeccable, has serious problematic aspects of *physical* nature, such as the violation of the superposition principle. As such, nonlinear theories cannot be used for consistent studies of composite systems, besides having the problematic aspects of Theorem 1 whenever the time evolution is nonunitary (see Ref. 16 for detailed studies on the problematic aspects caused by nonlinearity).

Additional deformations of quantum mechanics are based on *non-associative envelopes*, e.g., *Weinberg's theory*<sup>(17a)</sup> which can be reformulated in the methods of this paper via the general Lie-admissible structure,

$$[A, \wedge B] = (A, B) - (B, A) = \text{Lie} \quad (2.10a)$$

$$(A, B) = \frac{\partial A}{\partial \psi_k} \frac{\partial B}{\partial \psi_k^\dagger} = \text{nonassociative Lie-admissible} \quad (2.10b)$$

Even though of impeccable mathematical beauty, the latter theory violates Okubo's<sup>(6b)</sup> "*No-Go Theorem*" on deformations with *nonassociative envelopes*, under which there is the loss of the equivalence of the Heisenberg-type and Schrödinger-type representations and other problematic aspects. In addition, *Weinberg's theory possesses no unit at all in the envelope* (i.e., there is no nontrivial quantity  $E$  such that for product (2.10b)  $(E, A) \equiv (A, E) \equiv A$  for all possible generators  $A$ ), thus having physically unsettled aspects more serious than those of Theorem 1 (for detailed studies of the latter problematic aspects see Ref. 18).

Note that the attempt of Ref. 17 to reconstruct Weinberg's theory with an *associative envelope* with the brackets  $awb - bwa = (a_{ir})(w_{rs})(b_{sj}) -$

$(b_{ir})(w_{rs})(a_{sj})$  is precisely along our rule (1.7) which turns Weinberg's non-linear theory into our Lie-isotopic theory.<sup>(3b)</sup>

Yet another group of deformations of quantum mechanics affected by Theorem 1 is that of the so-called *squeezed states*<sup>(19)</sup> which are also generally nonunitary images of conventional theories. As such, they suffer the same problematic aspects of Theorem 1.

A further important type of quantum deformations is *Prigogine's non-unitary statistics*<sup>(20)</sup> introduced to attempt a reconciliation of the irreversibility of classical and quantum worlds. Being nonunitary, this theory too is affected by Theorem 1. However, unlike other deformations, Prigogine's nonunitary statistics may only require its isotopic formulation on appropriate spaces and fields to achieve invariance and axiomatic consistency, as shown in the next section.

Needless to say, the same problematic aspects exist for the more general Lie-admissible statistics in its first formulation submitted by Fron-  
teau *et al.*<sup>(61)</sup> (see Sec. 3.12 for its current mathematical formulation).

There is little doubt that *problematic aspects in deformed quantum formulations must have corresponding problematic aspects in their classical counterpart*. Recall that the Birkhoff-admissible equations (1.10) are the classical counterpart of the operator Lie-admissible equations (1.5), and the Birkhoff's equations (1.10) are the classical counterpart of the operator Lie-isotopic equations (1.8).

Recall also that the fundamental unit of classical theories is the unit  $\hat{I} = \text{Diag}(1, 1, 1)$  of the Euclidean space which represents the units of the three Cartesian coordinates (say, 1 cm) in dimensionless form. We then have the following:

**Theorem 2.** All noncanonical deformations of classical Hamiltonian mechanics formulated on conventional spaces over conventional fields, including the classical image of  $q$ -,  $k$ -, quantum-deformation, the Birkhoffian-admissible and other deformations, do not possess invariant units, with consequential problematic aspects in their application to measurements.

**Proof.** The admitted transformation theories are *noncanonical* by assumption, e.g., they leave invariant the Birkhoff's (1.11) or Birkhoff-admissible tensor (1.15). As such, they do not leave invariant the canonical tensor  $\omega_{\mu\nu}$ . In particular, a map from the Hamiltonian to the Birkhoffian mechanics is given precisely by noncanonical transforms  $b = \{r, p\} \rightarrow b'(b) = \{r', p'\}$  (see Ref. 3e for details) such that

$$\omega_{\mu\nu} \rightarrow \omega'_{\mu\nu} = \frac{\partial b^a}{\partial b^\mu} \omega_{a\beta} \frac{\partial b^\beta}{\partial b'^\nu} = \Omega_{\mu\nu}(a') \quad (2.11)$$



But the canonical tensor represents the fundamental space units of the theory,

$$(\omega_{\mu\nu}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \quad (2.12)$$

and this establishes the inability of noncanonical theories on conventional spaces over conventional fields to have invariant basic units, with ensuing problematic aspects in measurements. ■

We then have the following evident implication,

**Corollary 2.A.** The relativistic versions of all noncanonical classical theories, including the classical image of  $q$ -,  $k$ -, quantum-, and other deformations, the Birkhoffian and Birkhoffian-admissible mechanics, and other theories, violate the axioms of classical relativistic mechanics.

This illustrates the reasons why, after conducting the rather laborious classical studies of Refs. 3d, 3e, Santilli had to re-start from the beginning and identify a new form of generalized classical mechanics with invariant fundamental units. Intriguingly, a necessary condition resulted in the preservation of the abstract relativistic axioms, as we shall see in the next section.

By no means should the reader dismiss Theorem 2 following a possible impression that it has marginal implications, because the lack of invariance of the unit implies rather deep axiomatic inconsistencies which generally remain undetected by a nonexpert in the field.

As an illustration, the lack of conservation of the basic unit implies a corresponding lack of conservation of the base fields. Thus, starting from a theory defined at the initial time on conventional numbers, the same theory has to be defined at a later time on new *yet unknown numbers*. The ambiguities of noncanonical theories in their application to actual measurements are then beyond scientific doubts.

Another important class of theories with serious problematic aspects of *physical* character is given by the conventional formulation of *gravity*, i.e., that on conventional curved spaces over conventional fields (see, e.g., Ref. 21 and contributions quoted therein). In fact, we have the following:

**Theorem 3.** The basic unit of all (nowhere degenerate, real valued, and symmetric) geometries with non-null curvature over conventional fields is not invariant under the symmetries of the line element with consequential problematic aspects in applications to measurements for both classical and quantum formulations.

**Proof.** Let  $E(x, \delta, R)$  and  $\mathfrak{R}(y, g, R)$  be  $n$ -dimensional Euclidean and Riemannian spaces, respectively, with the same signature  $(+, +, \dots, +)$ , basic unit  $I = \text{diag}(1, 1, \dots, 1)$ , metrics  $\delta = \delta_{ij} = \text{Diag}(1, 1, \dots, 1)$  and  $g(y) = (g_{ij}) = g^i_j$ , and local coordinates  $x = \{x^k\}$ ,  $y = \{y^k\}$ ,  $i, j, k = 1, 2, \dots, n$ , over the reals  $R = R(n, +, \times)$ .

The transformation  $x \rightarrow y(x)$  for which the Euclidean metric is mapped into the Riemannian metric,

$$\delta_{ij} \rightarrow g_{ij}(y) = \frac{\partial y^r}{\partial x^i} \delta_{rs} \frac{\partial y^s}{\partial x^j} \quad (2.13)$$

is *noncanonical*. Therefore, the symmetries of the Riemannian line elements  $y^2 = y^i g_{ij} y^j$  are necessarily *noncanonical*. As such, these symmetries do not generally preserve the basic unit  $I$  at the classical level. The symmetries of the same line element in operator formulation are then necessarily *non-unitary* for consistency (see next section), and this proves the lack of invariance of the basic unit also for operator theories. The same proof evidently applies for indefinite signatures  $(+, +, \dots, -, -, \dots)$ . ■

To understand the implications of the above theorem, recall that for the  $(3+1)$ -dimensional Minkowskian and Riemannian geometries the basic unit is given by  $I = \text{Diag}(\{1, 1, 1\}, 1)$ , where the first three components represent the space units (say 1 cm) in dimensionless form, and the fourth component represents the time unit (say 1 sec), also in dimensionless form. The above theorem establishes that *curvature implies the lack of invariance of the fundamental space-time units*, thus activating the problematic aspects of Theorem 1.

In different terms, the adoption of the conventional Riemannian space  $\mathfrak{R}(x, g, R)$  over conventional fields  $R$  implies the adoption of a *noncanonical* theory, thus suffering from the problematic aspects of all noncanonical theories (Theorem 2). In fact, a particular case of the Birkhoffian mechanics is that in which the Euclidean metric  $\delta$  is replaced precisely by a Riemannian metric.<sup>(3d)</sup>

Therefore, *the problems in the quantization of gravity are not necessarily due to Einstein's (or other) field equations, but rather to their referral to a manifold in which the basic unit is not invariant*. This identifies a novel alternative in both classical and quantum gravity considered in the next section, that of preserving Einstein's (or other) field equations identically and searching instead for a formulation of the Riemannian spaces in which the unit is invariant (see the Sec. 3.11).

A detailed study of the lack of invariance of numerical predictions of nonunitary deformations can be found in Ref. 22b, App. 4.E, where it is shown that the numerical predictions of deviations from the conventional

uncertainties of squeezed states and other theories are not unique in their definition at the initial time and their numerical value is not preserved by the time evolution of the theory. Besides, when formulated in an axiomatically correct way, *nonunitary theories preserve Heisenberg's uncertainty*, as shown in the next section. Note that these problematic aspects also apply to quantum gravity.

This completes our study of deformations of quantum mechanics with the understanding that, by no means, do the above lines exhaust all classical and quantum deformations available in the literature. They are merely intended to identify primary classes. The author would be grateful to colleagues who care to bring to his attention additional important deformations.

In summary, the problematic aspects studied in this section confirm the majestic mathematical beauty and physical consistency of relativistic classical and quantum mechanics, and suggest caution before exiting from their *canonical and unitary* structure with *associative* enveloping algebra and *invariant* fundamental unit.

### 3. A POSSIBLE RESOLUTION OF THE PROBLEMATIC ASPECTS

#### 3.1. Foundations

According to overwhelming experimental evidence, relativistic quantum mechanics is *exactly valid* for the so-called *exterior dynamical problems*, such as the structure of atoms, the electroweak interactions at large, and others.

Despite these achievements, Prigogine *et al.*,<sup>(20)</sup> Ellis *et al.*,<sup>(13)</sup> and Santilli *et al.*<sup>(3, 22)</sup> have suggested the study of broader theories for the more general *interior dynamical problems*, such as the structure of stars, quasars, black holes, and others in which hadrons are under "contact interactions," i.e., at mutual distances equal to or smaller than their charge radius. The latter physical conditions are expected to imply novel nonlinear and non-local interactions which, being of contact type, are nonhamiltonian and therefore, nonunitary.

In short, the *nonunitary* character of new operator theories appears to be uncompromisable for possible advances, with the corresponding *non-canonical* classical counterpart. The problem considered in this section is therefore how to reach a theory as axiomatically consistent as conventional classical and quantum mechanics yet having a noncanonical and non-unitary structure.

Above all, the problem consists in reaching a noncanonical and non-unitary theory for *interior* dynamical conditions while preserving the abstract axioms of the relativistic quantum mechanics for *exterior* conditions. Moreover, to represent a smooth transition from interior to exterior conditions, all nonunitary formulations must admit conventional formulations as particular cases under a smooth limit, a condition which is assumed hereon.

The above problems were studied at length by this author. Their solution appears to be possible thanks to certain rather crucial *mathematical* advances which only recently appeared in the special issue<sup>(23)</sup> of *Rendiconti Circolo Matematico Palermo* entirely dedicated to the mathematical issues herein considered. This section is dedicated to the outline of the essential *physical* aspects.

We have attempted to render this section minimally self-sufficient. Nevertheless, its technical understanding requires a technical knowledge of at least the special issue.<sup>(23)</sup> The noninitiated reader should be aware that the studies herein presented are based on novel *mathematics*. Any appraisal based on conventional mathematics is therefore afflicted by a host of inconsistencies.

As in Secs. 1 and 2, we shall first identify the axiomatically correct operator nonunitary theory and then study the corresponding classical noncanonical counterpart.

#### 3.2. Nonunitary Image of Quantum Mechanics

The main problem of the earlier operator formulations<sup>(3)</sup> of Lie-isotopic and Lie-admissible deformations is that they are computed on *conventional* Hilbert spaces over *conventional* fields, thus suffering the problematic aspects studied in Sec. 2.

By recalling the need to preserve nonunitarity for advances, the only possible alternative for consistency is therefore their formulation on *generalized* Hilbert spaces and fields. However, in order not to exit from the axioms of special relativity, the latter generalizations have to be axiom-preserving. This is the main idea of Santilli's<sup>(3a)</sup> *isotopies*, although its realization in an axiomatically consistent form is possible only following the mathematical advances that recently appeared in Ref. 23, as illustrated below.

The best way of identifying the needed mathematical structure is by subjecting to nonunitary transforms the main aspects of conventional quantum structures (see later on for other maps). This approach yields the following nonunitary image of the algebraic structure of quantum mechanics (unit, enveloping operator algebra  $\xi$ , and attached Lie algebra  $L$ ):

$$I \rightarrow \hat{I} = U \times I \times U^\dagger = \hat{I}^\dagger \neq I, \quad \hat{T} = (U \times U^\dagger)^{-1} = \hat{T}^\dagger \quad (3.1a)$$

$$\xi: A \times B \rightarrow \hat{\xi}: U \times A \times B \times U^\dagger = \hat{A} \times \hat{T} \times \hat{B} = \hat{A} \hat{\times} \hat{B} \quad (3.1b)$$

$$\begin{aligned} L \approx \xi^-: [A, B] &= A \times B - B \times A \rightarrow \hat{L}: [\hat{A}, \hat{B}] = \hat{A} \hat{\times} \hat{B} - \hat{B} \hat{\times} \hat{A} \\ &= \hat{A} \times \hat{T} \times \hat{B} - \hat{B} \times \hat{T} \times \hat{A} \end{aligned} \quad (3.1c)$$

where  $\hat{A} = U \times A \times U^\dagger$ , etc. The nonunitary image of states and inner product of the Hilbert space is then given by

$$|\psi\rangle \rightarrow |\hat{\psi}\rangle = U \times |\psi\rangle \quad (3.2a)$$

$$\begin{aligned} \mathcal{H}: \langle \phi | \psi \rangle &\rightarrow \hat{\mathcal{H}}: \langle \phi | \times U^\dagger \times U^{\dagger-1} \times U^{-1} \times U \times |\psi\rangle \\ &= \langle \hat{\phi} | \times \hat{T} \times |\hat{\psi}\rangle = \langle \hat{\psi} | \hat{\times} |\hat{\psi}\rangle \end{aligned} \quad (3.2b)$$

where one should keep in mind that  $\hat{I}$  and  $\hat{T}$  are Hermitian. As such, they are hereon assumed to be diagonal and positive-definite (see Refs. 22 and 23 for other possibilities).

It is then easy to see that the above liftings are axiom-preserving and thus isotopic in the sense of Ref. 3a. In fact,  $\hat{\xi}$  is still associative because  $(\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{C} = \hat{A} \hat{\times} (\hat{B} \hat{\times} \hat{C})$ , and possesses the left and right unit  $\hat{I}$ ,

$$\hat{I} \hat{\times} \hat{A} = \hat{A} \hat{\times} \hat{I} \equiv \hat{A}, \quad \forall \hat{A} \in \hat{\xi} \quad (3.3)$$

Thus,  $\hat{\xi}$  is locally isomorphic to  $\xi$ , yet it is structurally broader, as desired.

Similarly, the generalized Lie product  $[\hat{A}, \hat{B}]$  (first proposed in Ref. 3b p. 725) is still Lie, as one can verify, and  $\hat{L}$  can be proved to be locally isomorphic to  $L$  (for positive-definite  $\hat{I} > 0$ ).<sup>(22)</sup> Thus the lifting  $L \rightarrow \hat{L}$  is *nonunitary yet axiom-preserving*, as desired.

Finally, the deformed composition  $\langle \hat{\phi} | \times \hat{T} \times |\hat{\psi}\rangle$  is still inner and, therefore,  $\hat{\mathcal{H}}$  is still Hilbert. The lifting  $\mathcal{H} \rightarrow \hat{\mathcal{H}}$  is therefore an isotopy, with  $\hat{\mathcal{H}}$  broader than  $\mathcal{H}$ , as desired.

Nonunitary structures (3.1) and (3.2) imply the following *Heisenberg-isotopic formulations* (first introduced by Santilli in Ref. 3b, p. 752), here considered in one dimension for simplicity:

$$i d\hat{A}/dt = [A, \hat{H}]_{\hat{\xi}} = \hat{A} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{A} = \hat{A} \times \hat{T} \times \hat{H} - \hat{H} \times \hat{T} \times \hat{A} \quad (3.4a)$$

$$[\hat{p}, \hat{r}] = \hat{p} \hat{\times} \hat{r} - \hat{r} \hat{\times} \hat{p} = \hat{p} \times \hat{T} \times \hat{r} - \hat{r} \times \hat{T} \times \hat{p} = i \times \hat{I} \quad (3.4b)$$

$$[\hat{p}, \hat{p}] = [\hat{r}, \hat{r}] = 0 \quad (3.4c)$$

and the following *Schrödinger-isotopic counterpart for the energy* (first identified by Myung and Santilli<sup>(24)</sup> and Mignani<sup>(25)</sup> and for the linear momentum (first identified by Santilli<sup>(22, 23a)</sup>

$$\hat{H} \hat{\times} |\hat{\psi}\rangle = \hat{H} \times \hat{T} \times |\hat{\psi}\rangle = E |\hat{\psi}\rangle \quad (3.5a)$$

$$\hat{p} \hat{\times} |\hat{\psi}\rangle = \hat{p} \times \hat{T} \times |\hat{\psi}\rangle = -i \hat{T} \times \hat{\nabla} |\hat{\psi}\rangle \quad (3.5b)$$

where we have assumed for simplicity that  $\hat{T}$  is independent of  $r$  so that  $U \times \nabla |\psi\rangle = U \times \nabla \times U^\dagger \times (U \times U^\dagger)^{-1} \times U \times |\psi\rangle = \hat{T} \times \nabla |\hat{\psi}\rangle$  (see later on for an arbitrary dependence).

The above structures do permit the resolution of the problem of Hermiticity of Sec. 2, because now the condition of Hermiticity reads

$$\langle \hat{\psi} | \times \hat{T} \times \{H \times \hat{T} \times |\hat{\psi}\rangle\} = (\langle \hat{\psi} | \times \hat{T} \times H^\dagger) \times \hat{T} \times |\hat{\psi}\rangle \quad (3.6a)$$

$$H^\dagger = \hat{T}^{-1} \times \hat{T} \times H \times \hat{T} \times \hat{T}^{-1} \equiv H^\dagger \quad (3.6b)$$

Thus, starting from an operator  $H$  which is Hermitian at the initial time, the nonunitarily transformed operator  $\hat{H} = U \times H \times U^\dagger$  remain Hermitian under nonunitary transforms. However, a necessary condition is that Hermiticity is *not* computed in the conventional Hilbert space  $\mathcal{H}$ , but rather in the above-defined generalized Hilbert space.

Despite this encouraging result, deformations (3.1)–(3.3) and related dynamical equations (3.4) and (3.5) are still far from physical consistency, because they are *not invariant under additional nonunitary transforms*, for which we have

$$W \times W^\dagger \neq I, \quad \hat{Z} = (W \times W^\dagger)^{-1} \quad (3.7a)$$

$$\hat{I} \rightarrow \hat{I}' = W \times \hat{I} \times W^\dagger \neq \hat{I} \quad (3.7b)$$

$$\hat{\xi}: \hat{A} \times \hat{T} \times \hat{B} \rightarrow \hat{\xi}': W \times (\hat{A} \hat{\times} \hat{B}) \times W^\dagger = \hat{A} \times \hat{Z} \times \hat{T}' \times \hat{Z} \times \hat{B}' \neq \hat{A}' \times \hat{T}' \times \hat{B}' \quad (3.7c)$$

$$\begin{aligned} \hat{L}: \hat{A} \times \hat{T} \times \hat{B} - \hat{B} \times \hat{T} \times \hat{A} &\rightarrow \hat{L}': \hat{A}' \times \hat{Z} \times \hat{T}' \times \hat{Z} \times \hat{B}' - \hat{B}' \times \hat{Z} \times \hat{T}' \times \hat{Z} \times \hat{A}' \\ &\neq \hat{A}' \times \hat{T}' \times \hat{B}' - \hat{B}' \times \hat{T}' \times \hat{A}' \end{aligned} \quad (3.7d)$$

It then follows that a theory with dynamical equations (3.4) and (3.5) is not physically consistent when formulated on generalized Hilbert space (3.2), because the "No-Go" Theorem 1 still applies.

### 3.3. Isofields and Isohilbert Spaces

Extensive studies of all possible alternatives conducted since Ref. 3 have established that the above problematic aspects are due to the fact that

a nonunitary transform cannot be consistently applied only to *part* of the quantum formalism, while the remaining formalism stays conventional. In fact, as shown in the recent works,<sup>(2,3)</sup> *the isotopic theory apparently achieves the same axiomatic consistency of conventional quantum mechanics when the entirety of the mathematical structure of quantum mechanics, without exception, is subjected to an isotopic map with the same generalized unit.*

A primary objective of this section is to indicate the problematic aspects which emerge for nonunitary theories in isotopic treatment whenever any aspect of quantum mechanics is not subject to isotopy.

To begin, transforms (3.1a) imply the generalization of the basic unit of the theory. The definition of *generalized* structures (3.1)–(3.5) on a *conventional field*  $C(c, +, \times)$  is, therefore, bound to imply axiomatic inconsistencies. This is due to the fact that the latter is still defined with respect to the *conventional* unit 1 while the former has a *generalized* unit  $\hat{I}$ .

To achieve a consistent formulation of the above nonunitary theory, the conventional fields of numbers have to be generalized into a form admitting of  $\hat{I}$ , rather than of  $I$ , as their left and right unit. This study has been conducted in detail in Ref. 26, resulting in the *isofields*  $\hat{C} = \hat{C}(\hat{c}, +, \hat{\times})$  which are rings of elements  $\hat{c} = c \times \hat{I}$ ,  $\hat{I} \notin C$ , called *isocomplex numbers* or, in general, *isonumbers*, equipped with the following *isotopic sum and multiplication*:

$$\hat{c}_1 + \hat{c}_2 = (c_1 + c_2) \times \hat{I}, \quad \hat{c}_1 \hat{\times} \hat{c}_2 = \hat{c}_1 \times \hat{T} \times \hat{c}_2 = (c_1 \times c_2) \times \hat{I} \quad (3.8)$$

under which the quantity  $\hat{I} = \hat{T}^{-1}$  is the correct left and right multiplicative unit,  $\hat{I} \hat{\times} \hat{c} = \hat{c} \hat{\times} \hat{I} \equiv \hat{c}$ ,  $\forall c \in \hat{C}$ , called *isounit*, while  $\hat{T}$  is called the *isotopic element*. The additive unit remains the conventional quantity  $\hat{0} = 0$ ,  $\hat{c} + \hat{0} = 0 + \hat{c} \equiv \hat{c}$ ,  $\forall \hat{c} \in \hat{C}$ . It is easy to see that, under these conditions,  $\hat{C}$  satisfies all axioms of a field. The lifting  $C \rightarrow \hat{C}$  is therefore an isotopy, as desired.

It is evident that all operations of fields are generalized for isofields in a simple yet unique and significant way. For instance, conventional squares  $c^2 = c \times c$  have no sense for  $\hat{C}$  and must be lifted into the *isosquare*  $\hat{c}^2 = \hat{c} \hat{\times} \hat{c}$ , with corresponding *isopower*  $\hat{c}^n = \hat{c} \hat{\times} \hat{c} \hat{\times} \dots \hat{\times} \hat{c}$ ; square roots  $c^{1/2}$  are lifted into the *isosquare roots*  $\hat{c}^{1/2} = c^{1/2} \times \hat{I}^{1/2}$ ; quotients  $a/b$  are lifted into the *isoquotient*  $\hat{a}/\hat{b} = (a/b) \times \hat{I}$ ; the norm  $|c|$  is lifted into the *isonorm*  $\hat{\uparrow} \hat{c} \hat{\uparrow} = |c| \times \hat{I}$ ; etc.

The isofield  $\hat{R} = \hat{R}(\hat{n}, +, \hat{\times})$  of *isoreal numbers*  $\hat{n} = n \times \hat{I}$ ,  $n \in R(n, +, \times)$ , is evidently a particular case of  $\hat{C}$ . For subsequent needs one should note that the isoproduct of an isonumber  $\hat{n}$  by a quantity  $Q$  is conventional,  $\hat{n} \hat{\times} Q = n \times \hat{I} \times \hat{T} \times Q = n \times Q$ . Even though the numbers are generalized, the numbers predicted by the theory are therefore conventional, as we shall see

(for detailed studies of the isoreal, isocomplex, isoquaternionic, and iso-octonionic numbers we refer the interested reader to Ref. 26).

As is well known, Hilbert spaces are defined over fields. Part of the problematic aspects of Theorem 1 is that nonunitary theories are defined on a conventional Hilbert space over *conventional fields*. It is easy to see that the use of *generalized* Hilbert space (3.2) over a *conventional* field  $C(c, +, \times)$  is bound to be axiomatically inconsistent.

In fact, the modular action  $\hat{H} \hat{\times} |\hat{\psi}\rangle = \hat{H} \times \hat{T} \times |\hat{\psi}\rangle$  and composition  $\langle \hat{\psi} | \hat{\times} |\hat{\psi}\rangle = \langle \hat{\psi} | \times \hat{T} \times |\hat{\psi}\rangle$  possess the generalized unit  $\hat{I} = \hat{T}^{-1}$ , because that is the only quantity such that  $\hat{I} \hat{\times} |\hat{\psi}\rangle \equiv |\hat{\psi}\rangle$ . Their referral to a field  $C(c, +, \times)$  with conventional unit  $I$  is then inconsistent.

To achieve axiomatic consistency, the new Hilbert space  $\hat{\mathcal{H}}$  must be referred to the isofield  $\hat{C}$  with the same basic unity  $\hat{I}$ . As a consequence, the deformed Hilbert space must have the structure of an isonumber  $c \times \hat{I}$ . This leads in a unique and unambiguous way to the *isohilbert space* characterized by the following *isoinner product and isonormalization*<sup>(22a, 23b, 24)</sup>

$$\hat{\mathcal{H}}: \langle \hat{\phi} \hat{\uparrow} \hat{\psi} \rangle = \langle \hat{\phi} | \hat{\times} |\hat{\psi}\rangle \times \hat{I} = \langle \hat{\phi} | \times \hat{T} \times |\hat{\psi}\rangle \times \hat{T}^{-1} \in \hat{C} \quad (3.9a)$$

$$\langle \hat{\psi} | \hat{\times} |\hat{\psi}\rangle = \langle \hat{\psi} | \times \hat{T} \times |\hat{\psi}\rangle = I \quad (3.9b)$$

Note that *isohermiticity coincides with conventional Hermiticity* in view of property (3.6). As a result, *all conventional quantum mechanical observables are preserved for the above isohilbert spaces over isofields.*

The conventional unitary transforms on  $\mathcal{H}$  over  $C$  are lifted under isotopies into the *isounitary transforms* on  $\hat{\mathcal{H}}$  over  $\hat{C}$

$$\hat{U} \hat{\times} \hat{U}^\dagger = \hat{U} \times \hat{T} \times \hat{U}^\dagger = \hat{U}^\dagger \hat{\times} \hat{U} = \hat{U}^\dagger \times \hat{T} \times \hat{U} = \hat{I} = \hat{T}^{-1} \quad (3.10)$$

The conventional theory of linear operators on  $\mathcal{H}$  must then be subjected to a compatible lifting on  $\hat{\mathcal{H}}$  over  $\hat{C}$  which is studied in Ref. 22a. We here merely mention the correct form of the *isoeigenvalue equations*

$$\hat{H} \hat{\times} |\hat{\psi}\rangle = \hat{H} \times \hat{T} \times |\hat{\psi}\rangle = \hat{E} \times |\hat{\psi}\rangle = (E \times \hat{I}) \times \hat{T} \times |\hat{\psi}\rangle = E \times |\hat{\psi}\rangle$$

$$\hat{H}^\dagger = \hat{H} \in \hat{E}, \quad \hat{E} \in \hat{R}, \quad E \in R \quad (3.11)$$

and of the *isoeexpectation values*

$$\langle \hat{H} \rangle = \frac{\langle \hat{\psi} | \times \hat{T} \times \hat{H} \times \hat{T} \times |\hat{\psi}\rangle}{\langle \hat{\psi} | \times \hat{T} \times |\hat{\psi}\rangle} \quad (3.12)$$

The reader can easily prove that the *isoeigenvalues of isohermitian operators are isoreal*, and that the *isoeigenvalues and isoeexpectation values*

of the same operator coincide. Note that the final numbers of isoeigenvalue equations (3.11) are conventional, and so are the isoexpectation values (because the isounits cancel in the quotient). Equations (3.11) and (3.12) therefore confirm that the final numbers of the theory are conventional. Note the necessity for each and every multiplication to require the sandwiching of the isotopic operator  $\hat{T}$  in order to have  $\hat{I}$  as the isounit (for these and all other aspects, see Ref. 22a).

The nontriviality of the isotopies here considered is illustrated by the fact that the isoeigenvalues of an operator are generally different from its conventional eigenvalues. In fact, starting from the expression  $H \times |\psi\rangle = E_0 \times |\psi\rangle$ , we have  $\hat{H} \hat{\times} |\hat{\psi}\rangle = E \times |\hat{\psi}\rangle$ , where the operator  $\hat{H}$  is the same, but the eigenvalues  $E_0$  and  $E$  are different. This result should not be surprising to the attentive reader because the theory under consideration is a nonunitary image of the quantum theory, and such transforms are known not to preserve the eigenvalues.

In actuality, Eqs. (3.11) establish that the same Hermitian operator generally possesses an infinite class of different sets of eigenvalues, one per each selected unit, thus disproving a rather popular belief that a Hermitian operator possesses a unique set of eigenvalues.

### 3.4. Isolinearity, Isolocality, Isounitariness

The current definition of (operator) isotopies, originally submitted in Ref. 3a but completed only in the recent special issue,<sup>(23)</sup> is that of maps of any given linear, local and unitary, mathematical or physical structure into the most general possible nonlinear, nonlocal, and nonunitary forms which are capable of restoring linearity, locality, and unitarity in isospaces over isofields. An understanding of these basic aspects is essential for this paper.

As we shall see better later on, the quantum mechanical representation of exterior systems (particles at large mutual distances compared to wavelengths) requires the knowledge of two quantities, the Hamiltonian  $H$  and the assumption of the trivial value  $I$  for the basic unit. Similarly, the isotopic representation of interior systems (particles at mutual distances equal to or smaller than their wavelengths) also requires the knowledge of two quantities, the Hamiltonian  $H$  representing all conventional exterior interactions and, this time, a nontrivial unit  $\hat{I}$  representing interior nonlinear, nonlocal, and nonhamiltonian effects due to overlapping of the wavepackets (which occurs also for point-like charges).

The Hamiltonian is conventional and it is only rewritten in isospace. It is therefore time to begin acquiring more knowledge on the isounit, with the understanding that, as is the case for the Hamiltonian, its explicit and unique form can be solely fixed by the physical conditions considered

regarding shape, density, and other typical interior characteristics usually ignored in the Hamiltonian.

Besides the positive-definiteness, isotopic theories leave unrestricted the functional dependence of the isounit  $\hat{I}$  and isotopic element  $\hat{T}$ , which can therefore depend on coordinates  $r$ , wavefunctions  $\psi$ , their derivative of arbitrary order, the local density  $\mu$  of the considered interior problem, its local temperature  $\tau$ , and any needed additional quantity,

$$\hat{I} = \hat{I}(r, p, \dot{p}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, \dots) > 0, \quad \hat{T} = \hat{T}(r, p, \dot{p}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, \dots) > 0 \quad (3.13)$$

Moreover, the latter dependence is unrestricted in topological character, that is, it can be arbitrarily nonlinear in the wavefunctions or in any other quantity, nonlocal, e.g., of integral type, or of other types as well as of any other admissible character, e.g., discrete in time and/or space. As an illustration, the isounit used in some of the applications (see later on Secs. 3.7 and 3.15 for more details) is of the type

$$\hat{I} = \text{Diag}(n_1^2, n_2^2, n_3^2, n_4^2) \times \exp(tN(\psi_1/\hat{\psi}_1 + \partial\psi_1/\partial\hat{\psi}_1 + \dots)) \times \int dv \psi_1^\dagger(r) \times \psi_1(r) \quad (3.14)$$

where the quantities  $n_1^2, n_2^2, n_3^2$  represent the extended, nonspherical, and deformable shapes of the hadron considered;  $n_4^2$  represents its density; the terms in the exponent  $\psi_1/\hat{\psi}_1, \partial\psi_1/\partial\hat{\psi}_1$ , etc., represent a typical nonlinearity, and the integral  $\int dv \psi_1^\dagger(r) \times \psi_1(r)$  in the exponent represents a typical nonlocality due to mutual penetration and wave-overlapping of the charge distributions of the hadrons considered. A system of particles is evidently represented by an isounit which is the tensorial product of isounits of type (3.14).

Whenever the hadrons considered are perfectly spherical and perfectly rigid,  $n_1^2 = n_2^2 = n_3^2 = 1$ , the representation of their density is ignored,  $n_4^2 = 1$ , and the mutual distances are such as to imply no appreciable overlapping of the wavepackets,  $\int dv \psi_1^\dagger(r) \times \psi_1(r) = 0$ ; then  $\hat{I} \equiv I$ , the considered nonunitary structure collapses into a unitary form, and conventional quantum mechanics is recovered identically, in accordance with our fundamental condition of Sec. 3.1.

As a result of the above occurrences, the lifting of conventional into isotopic eigenvalue equations is highly nonlinear as well as nonlocal and nonunitary:

$$\begin{aligned}
H(r, p) \times |\psi\rangle &= E \times |\psi\rangle \rightarrow H \hat{\times} |\hat{\psi}\rangle \\
&= H(r, p) \times \hat{T}(r, p, \dot{p}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, \dots) \times |\hat{\psi}\rangle \\
&= E \times |\hat{\psi}\rangle
\end{aligned} \tag{3.15}$$

As a matter of fact, the above lifting is "directly universal" for all systems with conserved Hamiltonians (see Sec. 3.12 for nonconserved Hamiltonians), namely, it can represent all possible well-behaved, nonlinear, nonlocal, and nonunitary generalizations of the conventional eigenvalue equation (universality) in the coordinates of the observer (direct universality).

The first aspect to understand here is that, despite the above generality, the mapped theory does indeed satisfy *linearity in isospace over isofields*, called *isolinearity*.<sup>(22a)</sup> In fact, the lifted theory satisfies all familiar linearity conditions,

$$\hat{A} \hat{\times} (\hat{c} \hat{\times} |\hat{\psi}\rangle) + \hat{d} \hat{\times} |\hat{\psi}\rangle = \hat{c} \hat{\times} \hat{A} \hat{\times} |\hat{\psi}\rangle + \hat{d} \hat{\times} \hat{A} \hat{\times} |\hat{\psi}\rangle \tag{3.16a}$$

$$(\hat{c} \hat{\times} \hat{A} + \hat{d} \hat{\times} \hat{B}) \hat{\times} |\hat{\psi}\rangle = \hat{c} \hat{\times} \hat{A} \hat{\times} |\hat{\psi}\rangle + \hat{d} \hat{\times} \hat{B} \hat{\times} |\hat{\psi}\rangle \tag{3.16b}$$

$$(\hat{A} \hat{\times} \hat{B}) \hat{\times} |\hat{\psi}\rangle = \hat{A} \hat{\times} (\hat{B} \hat{\times} |\hat{\psi}\rangle) \tag{3.16c}$$

As one can see, *the recovering of linearity in isospace is ensured by the embedding of all nonlinear terms in the isounit*. One can also prove that *any nonlinear theory can always be identically rewritten in an isotopic form*. In fact, all possible nonlinear theories (2.9) always admit the factorization of the nonlinear terms, which can then be assumed as the isotopic element of the theory

$$\begin{aligned}
H(r, p, \psi, \dots) \times |\hat{\psi}\rangle &= H_0(r, p) \times \hat{T}(r, p, \psi, \dots) \times |\hat{\psi}\rangle \\
&= H_0 \hat{\times} |\hat{\psi}\rangle, \quad \hat{T} = H_0^{-1} \times H
\end{aligned} \tag{3.17}$$

where  $H_0$  is the maximal (Hermitian) operator representing the total energy. A first advantage of the above identical reformulation is *the recovering of the superposition principle in isospace, thus permitting a consistent study of composite systems under nonlinear interactions*.<sup>(22b)</sup>

A similar occurrence holds for nonlocality. In fact, under the condition that all nonlocal terms are embedded in the isotopic element, isotopic theories verify *locality in isospace*, called *isolocality*.<sup>(23b)</sup> In fact, the theory is *everywhere local except at the isounit*.

A similar occurrence also holds for nonunitarity. In fact, any possible (well-behaved) nonunitary transform with the same "magnitude"  $\hat{I}$  can

always be identically reformulated in an *isounitary form on isospace over isofields*, called *isounitariness*,<sup>(23b)</sup> according to the rules

$$W \times W^\dagger = \hat{I} \neq I, \quad W = \hat{W} \times \hat{T}^{1/2} \tag{3.18a}$$

$$W \times W^\dagger = \hat{W} \times \hat{T} \times \hat{W}^\dagger = W^\dagger \times W = \hat{W}^\dagger \times \hat{T} \times W = \hat{I} \tag{3.18b}$$

Note that the actions  $H \times |\psi\rangle$  and  $H \hat{\times} |\hat{\psi}\rangle$  coincide at the abstract level. We can therefore state that *nonlinearity, nonlocality, and nonunitarity are not irreducible properties because they can be made to disappear at the abstract level under isotopies*.

The recovering of a classical canonical structure in phase space, called *isocanonicity*, is studied in Sec. 3.8.

### 3.5. Isotopic Realization of "Hidden variables" and "Completion" of Quantum Mechanics

At this intermediate stage of our analysis we can temporarily define the *isotopies of quantum mechanics* as a theory with dynamical equations (3.4) and (3.5) defined with respect to the isoenveloping operator algebras  $\hat{\xi}$  on isohilbert spaces  $\hat{\mathcal{H}}$  over isofields  $\hat{C}$  with common isounit  $\hat{I}$ .

A fundamental property is that, in view of the positive-definiteness of  $\hat{I}$  and  $\hat{T}$ , the isotopic theory coincides with quantum mechanics at the abstract realization-free level by conception<sup>(3b)</sup> and realization,<sup>(23b)</sup> because at the abstract level  $I$  and  $\hat{I}$ ,  $\xi$  and  $\hat{\xi}$ ,  $L$  and  $\hat{L}$ ,  $C$  and  $\hat{C}$ ,  $\mathcal{H}$  and  $\hat{\mathcal{H}}$ , etc., coincide (see Ref. 22b when the isounit is no longer positive-definite).

To avoid misrepresentations, we should therefore stress that by no means do the above isotopies constitute a "new theory." In fact, a new theory can only be claimed under *structurally novel axioms*. On the contrary, the above isotopic theory preserves the conventional abstract axioms by conception and construction. As such, *the isotopies merely provide new realizations of the abstract axioms of quantum mechanics, with the conventional realization recovered identically as a particular case for  $\hat{I} = I$* .

In different terms, conventional quantum mechanics holds under the (necessary) assumption that the basic unit has the trivial value  $I$ . The studies herein reported have established that such an assumption is unnecessary, and that the same axioms also hold for arbitrary positive-definite units  $\hat{I}$ . Thus, *the isotopies identify the infinite class of realizations of the same quantum axioms characterized by all infinitely possible isounits  $\hat{I}$  with the quantum unit  $\hat{I} = I$  as particular case*.

It is understood that *theories with different isounits are mathematically equivalent but physically different*, otherwise it would be like pretending

that nonunitary theories are physically equivalent to the unitary ones. Alternatively, we can say that the unit  $I$  is fixed in quantum mechanics and the same must be for *each* isounit  $\hat{I}$  of the isotopic realizations.

The abstract identity of the isotopic and conventional operator theories is illustrated by the following *new invariance law of the Hilbert space* for  $\hat{T}$  independent from the integration variables, here introduced apparently for the first time,

$$\langle \phi | \psi \rangle \equiv \langle \phi | \psi \rangle \times \hat{T} \times \hat{T}^{-1} = \langle \phi | \times \hat{T} \times | \psi \rangle \times \hat{I} = \langle \phi \uparrow \psi \rangle \quad (3.19)$$

and called *iso-self-scalarity*. The above invariance confirms the preservation of the original quantum axioms, as desired, for the preservation of Einstein's axioms and as otherwise needed for axiomatic consistency under nonunitary transforms.

Note that invariance (3.19) has remained undetected in this century. This should not be surprising because its identification required the prior discovery of *new numbers*, those with arbitrary unites.<sup>(26)</sup> In fact, invariance (3.19) cannot be defined via the conventional theory of numbers, that with the sole unit  $+1$ .

It is intriguing to note that *the isotopic theory here considered constitutes an explicit and concrete realization of the theory of "hidden variables"  $\lambda$*  (see, e.g., Ref. 27). In fact, we can rewrite Eq. (3.11) in the form

$$\hat{H} \hat{\times} |\hat{\psi}\rangle = \hat{H} \times \lambda \times |\hat{\psi}\rangle = \hat{E} \hat{\times} |\hat{\psi}\rangle = (E \times \lambda^{-1}) \times \lambda \times |\hat{\psi}\rangle = E_{\lambda} \times |\hat{\psi}\rangle$$

$$\hat{E} \in \hat{R}, \quad E \in R \quad (3.20)$$

which does evidently provide said concrete and explicit realization of the "hidden variables"  $\lambda$  actually in the more general form of "hidden operators"  $\lambda(r, p, \hat{p}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, \dots) = \hat{T} > 0$ .

The "hidden" character is an evident consequence of the preservation of the quantum mechanical axioms. Note the nontriviality of the realization. In fact, the eigenvalues of a Hermitian operator turn out to be different for different "hidden operators"  $\lambda$ .

In fact, the axiomatic structure of conventional eigenvalue expressions is given by the *modular, associative action of an operator on a state*  $H \times |\psi\rangle$ , for which  $(A \times B \times C) \times |\psi\rangle = A \times ((B \times C) \times |\psi\rangle) = (A \times B) \times (C \times |\psi\rangle)$ . These axiomatic properties are preserved for the isotopies here considered because in the latter case we have a *modular isoassociative action of an operator on an isostate*  $\hat{H} \hat{\times} |\hat{\psi}\rangle$  for which the preceding properties are preserved in isospaces in view of (3.15c). The important point from which the realization of "hidden variables" follows is that the two axioms " $H \times |\psi\rangle$ " and " $\hat{H} \hat{\times} |\hat{\psi}\rangle$ " coincide at the abstract, realization-free level on all grounds.

We can therefore state that *concrete and explicit realizations of "hidden variables" are indeed admitted by the abstract axioms of quantum mechanics, provided that they are realized in a nonunitary axiom-preserving way.*

Moreover, *the isotopies of quantum mechanics constitute a "completion" of quantum mechanics intriguingly along the celebrated argument by Einstein, Podolsky, and Rosen.<sup>(28)</sup>* In fact, the conventional unitary realization can be "completed" into an axiom-preserving nonunitary-isounitary form with evident structural broadening.

In particular, von Neumann's theorem<sup>(29)</sup> and Bell's inequalities<sup>(30)</sup> do not apply, trivially, because the considered theory is nonunitary. This reinforces the connection with the EPR argument, because *the nonunitarily transformed Bell's inequalities (the only ones applying under isotopies) can indeed admit a classical image in interior problems (only) owing to the arbitrariness in their classical limit under no unitary transforms* (see Ref. 22b, App. 4.C).

In different terms, in exterior problems in vacuum von Neumann's theorem and Bell's inequalities apply as is the case for all of quantum mechanics. In interior problems the situation is different under nonunitary-isotopic transforms, because they evidently imply a *necessary alteration of the upper boundaries* of the inequalities which now can admit a classical counterpart (see Ref. 22b, Appendix 4.C, for details and proofs, including the image of Pauli's matrices under nonunitary transforms as isorepresentations of the isotopic  $S\hat{U}(2)$  symmetry).

A reason for the still unresolved controversies in these issues is that, in order to be nontrivial, any realization of "hidden variables" or "completion" of quantum mechanics must be *outside the class of equivalence of quantum mechanics*, that is, they must have a nonunitary structure. The isotopies then emerge as the sole known methods capable of formulating them in an axiom-preserving way.

The reader should meditate a moment on the implications of the above results. For instance, the above realization of "hidden variables" and "completion" of quantum mechanics imply that *discrete time theories* (see, e.g., Ref. 31) *are compatible with the abstract axioms of quantum mechanics, provided that they are realized in their isotopic form* (i.e., via the embedding of all discrete terms in the isounit of the theory).

Virtually all applications and verifications of the isotopic theories outlined in Sec. 3.14 are, strictly speaking, realizations and verifications of the theory of "hidden variables" and of the EPR "completion" of quantum mechanics.

Almost needless to say, we have considered here only the "isotopic" realization of "hidden operators" and "completion" of quantum mechanics, without any claim that it is unique, while encouraging the identification of inequivalent realizations.

### 3.6. Isotopies of Differential Calculus, Lie's Theory and Functional Analysis

Despite all the preceding studies (conducted since the proposal<sup>(3b)</sup> of 1978 and completed by the early 1990's), the isotopic theories were still afflicted by axiomatic inconsistencies of rather subtle origin which escaped prolonged efforts at their resolutions.

We come in this way to the crucial role of the special issue of *Rendiconti*.<sup>(23)</sup> In essence, lengthy studies on all possible alternatives indicated that the inconsistencies originated where one would expect them the least, in the *ordinary differential calculus*. Even though ignored because of protracted use over centuries, the dependence of the ordinary differential calculus on the basic unit  $I$  is rather fundamental because the differential calculus acts on rings of functions *defined over conventional fields*. Such a dependence becomes nontrivial for generalized units because in this case  $d\hat{I} \neq 0$ . The use of the conventional differential calculus for theories with generalized units is then bound to be inconsistent.

One should note that this is not a mere mathematical curiosity, because the issue directly affects the basic dynamical equations which, when defined via the conventional differential calculus in the time and space derivatives as in Eqs. (3.4) and (3.5), escape all efforts to achieve invariance.

Santilli therefore introduced in Ref. 23b the *isotopies of the differential calculus*, or *isodifferential calculus* for short, which is essentially based on the following simple, yet unique and unambiguous, *isodifferentials* and *isoderivatives*

$$\hat{d}r^k = \hat{I}_i^k \times dr^i, \quad \hat{d}r_k = \hat{T}_k^i \times dr_i \quad (3.21a)$$

$$\hat{\partial}/\hat{\partial}r^k = \hat{T}_k^i \times \partial/\partial r^i, \quad \hat{\partial}/\hat{\partial}r_k = \hat{I}_i^k \times \partial/\partial r_i \quad (3.21b)$$

$$\hat{d}p_k = \hat{T}_k^i \times dp_i, \quad \hat{d}p^k = \hat{I}_i^k \times dp^i \quad (3.21c)$$

$$\hat{\partial}/\hat{\partial}p_k = \hat{I}_i^k \times \partial/\partial p_i, \quad \hat{\partial}/\hat{\partial}p^k = \hat{T}_i^k \times \partial/\partial p^i \quad (3.21d)$$

with basic properties

$$\hat{\partial}r^i/\hat{\partial}r^j = \delta_j^i, \quad \hat{\partial}r_i/\hat{\partial}r^j = \hat{T}_i^j, \quad \hat{\partial}r^i/\hat{\partial}r_j = \hat{I}_j^i, \text{ etc.} \quad (3.22)$$

and other axiom-preserving properties here omitted for brevity.<sup>(23b)</sup>

It should be noted that other definitions, such as  $\hat{d}r = d(\hat{I} \times r) = (r \times \partial\hat{I}/\partial r + \hat{I}) \times dr = \hat{I}' \times dr$ ,  $\hat{I}' = r \times \partial\hat{I}/\partial r + \hat{I}$ , lead to inconsistencies because they imply the alteration of the basic unit under the operation of differential,  $\hat{I} \rightarrow \hat{I}' \neq \hat{I}$ . This would imply the loss of the systems considered because of the lack of homomorphic map under differentiation. In fact, the original ring

of functions would be mapped into another ring with a different unit with evident problematic aspects of various nature which are in general detected only following in-depth inspection.

An important confirmation of the axiom-preserving character of the isodifferential calculus (as formulated above with an invariant isounit) will be indicated in the next section.

The next isotopies needed for axiomatic consistencies are those of Lie's theory. In fact, the use of *conventional* symmetries for *nonunitary* theories also leads to serious inconsistencies, because Lie's theory is notoriously constructed with respect to the conventional unit  $I = \text{diag}(1, 1, \dots, 1)$ , while the theories considered have the isounit  $\hat{I} \neq I$ .

This occurrence required the construction of the *step-by-step isotopies of Lie's theory*, including the isotopies of enveloping associative algebras, Lie algebras, Lie groups, Lie symmetries, transformation and representation theory, etc., which were first proposed by Santilli in,<sup>(3a)</sup> studied in detail in Refs. 3b-3d, 4, 22, 23b and numerous other contributions, and are nowadays called the *Lie-Santilli isotheory* (see Refs. 5, 23c and papers quoted therein). The latter theory is essentially the reconstruction of all aspects of the conventional formulation of Lie's theory with respect to the generalized unit  $\hat{I}$ .

Regrettably, we cannot possibly review the latter, rather vast studies and are forced to refer the interested reader to Refs. 22a, 22b, 5c, 5e, 5f, and 23c. We merely mention that by no means was the Lie-Santilli isotheory conceived to discover new Lie algebras, because all these algebras (over a field of characteristic zero) are known from Cartan's classification.

By recalling that the current formulation of Lie's theory is strictly linear, local, and unitary, the Lie-Santilli isotheory is specifically intended to provide the broadest possible *nonlinear, nonlocal, and nonunitary realizations* of known Lie algebras and groups, according to the following main lines:

(a) the *universal isoassociative enveloping algebra*  $\hat{\xi}$  with isounit  $\hat{I}$  and isotopic product  $\hat{A} \hat{\times} \hat{B}$  as characterized by the *isotopic Poincaré-Birkhoff-Witt theorem* (first formulated in the original proposal<sup>(3a, 3d)</sup> (see also Refs. 5c, 6g) with infinite-dimensional basis

$$\hat{\xi} : \hat{I}, X_k, \hat{X}_i \hat{\times} \hat{X}_j, \quad i \leq j, \quad X_i \hat{\times} \hat{X}_k, \quad i \leq j \leq k, \dots, \quad i, j, k = 1, 2, \dots, N \quad (3.23)$$

from which we have the unique and unambiguous *isoexponentiation*

$$\begin{aligned} e^{\hat{\xi}^{hw} \times X} &= \hat{e}^{hw \times X} = \hat{I} + w \times X/1! + (w \times X) \hat{\times} (w \times X)/2! + \dots \\ &= (e^{iX \times T \times w}) \times \hat{I} = \hat{I} \times (e^{hw \times T \times X}) \end{aligned} \quad (3.24)$$



(b) the *Lie-Santilli isoalgebras*, also proposed for the first time in Ref. 3a via the isotopes of Lie's theorems,

$$[\hat{X}_i, \hat{X}_j] = \hat{X}_i \hat{X}_j - \hat{X}_j \hat{X}_i = \hat{C}_{ij}^k \hat{X}_k \quad (3.25)$$

(c) the *Lie-Santilli isogroups*<sup>(3a)</sup> with rules

$$\begin{aligned} \hat{U}(\hat{w}) \hat{X} \hat{U}(\hat{w}') &= \hat{U}(\hat{w}') \hat{X} \hat{U}(\hat{w}) = \hat{U}(\hat{w} + \hat{w}') \\ \hat{U}(\hat{w}) \hat{X} \hat{U}(-\hat{w}) &= \hat{U}(0) = I, \quad \hat{w} = w \times I \in \hat{R} \end{aligned} \quad (3.26)$$

and realization in terms of isounitary operators

$$\hat{U}(\hat{w}) = \hat{e}^{i\hat{X} \hat{w}} = \hat{e}^{i\hat{X} \times w} = (e^{iX \times T \times w}) \times I, \quad \hat{X} = \hat{X}^\dagger \quad (3.27)$$

(d) the *isotransforms*

$$x' = \hat{U}(\hat{w}) \hat{X} x = (\hat{e}^{i\hat{X} \hat{w}}) \hat{X} x = (e^{iX \times T \times w}) \times x \quad (3.28)$$

(e) the *isorepresentation theory etc.*<sup>(22)</sup> and other aspects.

The *isosymmetries* are then constructed accordingly.<sup>(4, 22)</sup> The non-triviality of the Lie-Santilli isotheory emerges from the appearance of a *nonlinear integro-differential operator*  $\hat{T}(r, p, \dot{p}, \partial\psi, \partial\partial\psi, \mu, \tau, \dots)$  in the *exponent* of the group structure, as well as from the fact that it is a *nonunitary* image of the conventional theory with consequential *new weights*.

A property important for this paper is that the *Lie-Santilli isotheory preserves the generators  $X$  and parameters  $w$  of the original symmetry and merely changes the operations on them*. In fact, the generators  $X$  represent ordinary physical quantities such as coordinates, momentum, angular momentum, etc. and, as such, they are the same for all possible interactions. Similarly, the parameters  $w$  represent physical quantities such as angles, speeds, etc., and, as such, they also cannot change.

The preservation of *conventional generators* under *generalized symmetries* implies that *isosymmetries characterize new composite systems with conventional total conservation laws and generalized structure* called, for certain technical reasons, *closed-variationally non-self-adjoint system*, for which the Lie-isotopic theory was proposed in the first place<sup>(3a, 3d)</sup> (see Sec. 3.14 for applications and verifications).

The reader should be alerted that the correct formulation of the isotheory requires the use of the totality of isotopic structures at all levels, including isofields, isospaces, isodifferential calculus, etc. A number of mathematical and physical studies were initially conducted on the Lie-Santilli isotheory with only *part* of the isotopic mathematics, e.g.,

formulated over conventional fields, or via the conventional differential calculus, and some of them are continued by independent authors nowadays. The reader should be aware that these studies have turned out to be axiomatically inconsistent, suffering from essentially the same drawbacks of Theorem 1. The axiomatically consistent formulation of the Lie-Santilli isotheory, that based on the totality of the isotopic mathematics, has been achieved only recently in contributions of the special issue of *Rendiconti Circolo Matematico Palermo*,<sup>(23)</sup> and in *Mathematical Methods in Applied sciences*.<sup>(5g)</sup>

The latter occurrence is best expressed by the fact that the elaboration of the Lie-Santilli isotheory with conventional and special functions and transforms also leads to a host of inconsistencies, beginning with the use of ordinary functions, such as trigonometric, exponential, or logarithmic functions, or ordinary quantities such as matrices, determinants and traces, which have no meaning under isotopies and have to be replaced by the appropriate isotopic generalizations.<sup>(22a)</sup>

The latter aspects belong to the new field called *functional isoanalysis* whose study was initiated by Kadeisvili<sup>(32a)</sup> (who introduced the notions of *isocontinuity*), Tsagas and Sourlas<sup>(32b)</sup> (who introduced the notion of *integro-differential topology* called *Tsagas-Sourlas isotopology*), and by Santilli and others<sup>(22)</sup> (who constructed several conventional and special isofunctions, isotransforms, and isodistributions).

Needless to say, we cannot possibly review here these additional, equally vast topics and must refer the interested reader to Ref. 22. To render this section minimally self-sufficient, we mention that, unlike  $q$ -,  $k$ -, and other special functions and transforms,<sup>(8-11)</sup> those of isotopic character are *uniquely and unambiguously* defined per each given isounit, and they are *invariant* under the time evolution of the theory (see below). The reader can verify these main characteristics for isoexponentiation (3.24), and the same holds for other cases. The physical law constructed on isofunctions, isotransforms, and isodistributions are then unique and invariant, as desired.

### 3.7. Isotopies of Metric Spaces and Differential Geometries

Despite their broad character, the preceding studies are still insufficient to achieve axiomatic consistency of nonunitary theories under isotopic reformulation because of the need for one final important class of isotopies, those of metric spaces and conventional local-differential geometries. In fact, the use under isotopies of conventional carrier spaces, such as the Euclidean, Minkowskian, or Riemannian spaces and related geometries, is bound to imply axiomatic inconsistencies, because the latter

geometries are defined with respect to conventional units, while the isotopies have generalized units.

The above occurrence requires the necessary construction of the so-called *isospaces* and related *isogeometries* first presented by Santilli in Ref. 4a and then developed in numerous works (see the general presentation in Ref. 22a and the update in Ref. 23b).

The basic geometric isotopies for physical applications are those of the Euclidean geometry. Let  $E(r, \delta, R)$  be the familiar three-dimensional conventional Euclidean space with local coordinates  $r = \{r^k\}$ ,  $k = 1, 2, 3$ , basic unit  $I = \text{Diag}(1, 1, 1)$ , metric  $\delta = \text{Diag}(1, 1, 1)$ , and line element  $r^2 = r' \times \delta \times r = xx + yy + zz$  over the reals  $R = R(n, +, \times)$ . The *isoeuclidean spaces*<sup>(4a)</sup> also called *Euclidean-Santilli isospaces*, or *Santilli's isoeuclidean spaces*, are defined by<sup>(5)</sup>

$$\hat{E}(\hat{r}, \hat{\delta}, \hat{R}): \hat{r} = r \times \hat{I}, \quad \hat{\delta} \rightarrow \hat{\delta} = \hat{T} \times \delta, \quad I \rightarrow \hat{I} = \hat{T}^{-1} \quad (6.29a)$$

$$\begin{aligned} \hat{r}^2 &= \hat{r}' \hat{\times} \hat{\delta} \hat{\times} \hat{r} = (r \times \hat{I})' \times \hat{T} \times \delta \times \hat{T} \times (r \times \hat{I}) \\ &= [r' \times \delta(x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, \dots) \times r] \times \hat{I} \\ &= (x \times \hat{T}_{11} \times x + y \times \hat{T}_{22} \times y + z \times \hat{T}_{33} \times z) \times \hat{I} \in \hat{R} \end{aligned} \quad (3.29b)$$

where  $\hat{T}$  and  $\hat{I}$  are now 3-dimensional positive-definite  $3 \times 3$  matrices which, as such, can always be diagonalized. The corresponding *isotopies of the Euclidean geometry*,<sup>(4, 22, 23)</sup> also called *Santilli's isoeuclidean geometry*,<sup>(5)</sup> are the geometry of the related isospaces.

A most salient property of the isoeuclidean geometry is that it is characterized by the abstract axioms of the original Euclidean geometry, only realized in a more general way. In fact, the lifting of the metric  $\delta \rightarrow \hat{\delta} = \hat{T} \times \delta$  while the unit is lifted by the *inverse* amount,  $I \rightarrow \hat{I} = \hat{T}^{-1}$ , permits the preservation of the original geometric axioms.

More particularly, the isoeuclidean geometry satisfies the axiom of *flatness in isospace over isofields*, called *isoflatness*.<sup>(22a)</sup> Jointly, however, the isogeometry acquires an *unrestricted functional dependence of the isometric*,  $\hat{\delta} = \hat{\delta}(r, \dot{r}, \ddot{r}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, \dots)$ , with evident advantages. These features permit a novel formulation of gravity outlined in Sec. 3.11.

Another fundamental notion is the *perfect sphere in isospace*, called *isosphere*. Consider the ordinary sphere in  $E(r, \delta, R)$ ,  $r^2 = r' \times \delta \times r = xx + yy + zz \in R$ . Its image in isospace is given by the ellipsoidal deformation of each axis  $l_k \rightarrow \hat{T}_{kk}$  while each related unit is deformed by the inverse amount  $l_k \rightarrow \hat{T}_{kk}^{-1}$ , thus recovering perfect spheriodicity in isospace.

The notion of isosphere has fundamental importance for applications to hadrons which are represented precisely as isospheres (Sec. 3.14). In

fact, isospheriodicity permits the representation of the charge distribution of hadrons as they are in the physical reality, that is, *extended, nonspherical, and deformable* when represented in our space, although lifted into perfect sphericity in isospace. The latter reduction has the evident fundamental role of reconstructing the exact rotational symmetry in isospace (Sec. 3.10).

This is a most fundamental step in the preservation of the abstract Einsteinian axioms by keeping, on the one hand, the perfect spheriodicity and rigidity in isospace and, on the other hand, representing physical reality of hadrons as nonspherical and deformable via their projection in ordinary space.

For future needs the reader should keep in mind that the quantity left invariant under the lifting  $E(r, \delta, R) \rightarrow \hat{E}(\hat{r}, \hat{\delta}, \hat{R})$  is given by  $(\text{length})^2 \times (\text{unit})^2$ . In turn, such an invariance permits intriguing novel geometric studies (see, e.g., the "geometric propulsion" of Ref. 22a).

Note that identities (3.29b) imply that in practical applications the *isocoordinates*  $\hat{r} = r \times \hat{I}$  are redundant and can be reduced to the conventional coordinates  $r$ . In fact, the important property is that *the isoinvariant must be an element of the isofield*, i.e.,  $\hat{r}^2$  must have the structure  $\hat{r}^2 = n \times \hat{I} \in \hat{R}$ ,  $n \in R$ , which is possible via the use of ordinary coordinates  $r$  and the factorization of  $\hat{I}$ , resulting in the identity  $\hat{r}^2 = r^2$ .

We should also note that, as a necessary condition for consistency, *isospaces require their formulation over an isofield with the same isounit  $\hat{I}$* . This is not generally the case for conventional metric spaces where the geometric unit is  $N$ -dimensional,  $I = \text{Diag}(1, 1, \dots, 1)$ , while the unit of the base field  $R$  is the one-dimensional quantity  $+1$ . Nevertheless, the reals can be trivially redefined with respect to the unit  $I$ , thus reaching a unique unit for conventional spaces too.

For detailed studies and applications of the isoeuclidean geometry we refer the reader to Refs. 22a, 23b. We here merely mention that conventional geometries can be safely assumed to be *exactly valid for exterior dynamical problems* in vacuum. The isogeometries have been proposed to attempt a more adequate representation of *interior dynamical problems* within physical media. This is possible thanks to the arbitrary functional dependence of the isometric, which is particularly suited for the direct geometrization of interior effects which have a notorious *arbitrary* functional dependence on *velocities* and other variables (which dependence simply does not exist in current geometries).

Perhaps the most effective illustration of the isogeometries is given by the *isosymplectic geometry* originally proposed by Santilli<sup>(22a)</sup> via the use of the isotopic degrees of freedom of the product, and finalized only recently in Ref. 23b in its axiomatically correct form via the use of the isodifferential calculus.



For this reason the symplectic geometry is generally presented with the celebrated *Darboux Lemma*, establishing that, under the necessary continuity and regularity conditions, there always exists a Darboux transform  $b = \{r, p\} \rightarrow b'(b) = \{r'(r, p), p'(r, p)\}$  under which the original, local-differential and nonhamiltonian vector-field becomes (locally) Hamiltonian,

$$\omega_- | X(b) \times db \neq dH(b), \quad \omega_- | X'(b') \times db' = dH'(b') \quad (3.36)$$

Nevertheless, it is not generally explained in the literature in differential geometry (see, e.g., the comprehensive literature in Vol. I of Ref. 3d) that *the use of Darboux's lemma implies the violation of Einstein's axioms as well as the practical impossibility of conducting experiments in the transformed frame.*

This is due to the fact that Darboux's transforms  $b = \{r, p\} \rightarrow b'(b) = \{r'(r, p), p'(r, p)\}$  are highly *nonlinear* and, as such, the transformed frames are highly *noninertial*, thus violating the inertial character of the frames as requested by the special relativity. Along the same lines, the Darboux coordinates cannot be realized in actual experiments because one cannot possibly move a heavy measuring device, from its original position at rest in  $r$ , to nonlinear Darboux trajectories, e.g., of the type  $r' = N \times \exp\{Mr \times p\}$ ,  $N, M \in R$ .

In view of the above drawbacks, Santilli introduced in Ref. 23b the following *alternative of Darboux's lemma*:

**Lemma 3.** Any (well behaved) vector-field  $X(b)$  which is not (locally) Hamiltonian in a neighborhood of a point of the chart  $b = (r, p)$  of the local observer always admits an isotopy under which it becomes (locally) isohamiltonian, i.e., there always exists a Hamiltonian  $H(b)$  and an isotopy of the basic unit  $I \rightarrow \hat{I}$  with corresponding isotopy of the differential  $dr \rightarrow \hat{d}b = \hat{I} \times db$  under which we have the identity in the original neighborhood

$$\hat{\omega}_- | X(b) \times \hat{d}b = \hat{d}H(b) \quad (3.37)$$

The above result is important to prevent a predictable tendency to turn theories which are noncanonical or nonunitary in the frame of the experimenter into forms which are canonical or unitary in hypothetical reference frames. In fact, the latter have only the appearance of preserving established knowledge, while in reality they violate it.

Because of the above occurrences, all studies herein reported, beginning with Ref. 3, solely admit "direct representations" of physical systems, that is, *representations in the fixed coordinates of the experimenter.* Only after achieving such a representation may the use of the transformation theory have physical relevance.

Intriguingly, the isotopies have permitted the achievement of the "direct universality" for the representation of all well-behaved *local-differential* and nonpotential Newtonian systems in the frame of the experimenter via the Birkhoff-Santilli mechanics.<sup>(3,5)</sup> This direct universality is extended to the representation of all well-behaved *nonlocal-integral* and nonpotential Newtonian systems via the isohamiltonian mechanics<sup>(23b)</sup> (see next section). The same direct universality is extended in this paper to operator systems of nonlocal and nonpotential type (see Secs. 3.5 and 3.9).

The isotopies of the Minkowskian geometry are outlined in Sec. 3.10 and are the fundamental ones for the preservation of the relativistic axioms. The isotopies of the Riemannian geometry are studied in detail in Refs. 22a, 23b with the understanding that they have a *mathematical* character in view of Theorem 3, and *are not* recommended for applications (see Sec. 3.11 for details).

### 3.8. Isotopies of Newtonian, Lagrangian, and Hamiltonian Mechanics

We are finally equipped to identify the generalized noncanonical and nonunitary formulations with the desired axiomatic consistency and the capability to preserve relativistic axioms at the abstract level. It is expedient to begin with a brief outline of the isotopies of the truly fundamental equations of dynamics, *Newton's* equations, and of Lagrange's and Hamilton's mechanics identified in Ref. 23b, and then pass to the study of their operator counterpart. In this way the reader can see that the broadening of the representational capabilities of operator isotopies originate at the primitive *Newtonian* level. For all technical details of this section we refer the reader to Ref. 23b. Additional studies are presented in Ref. 6w.

We assume hereon the following notation: the symbol " $\times$ " represents the conventional associative product of classical and quantum mechanics; the symbol " $\hat{\times}$ " ( $-\times\hat{T}\times$ ) represents the isoassociative product; whenever no symbol of product appears, we assume the conventional associative product; all isotopies have the same Hermitian and positive-definite isounits, the  $3 \times 3$ -dimensional *space isounit*  $\hat{I}_s = \hat{T}_s^{-1} = \hat{I}^\dagger > 0$  and the one-dimensional *time isounit*  $\hat{I}_t = \hat{T}_t^{-1} = \hat{I}_t^\dagger > 0$  (which are generally different (because of different dimensionality); the *velocity isounit* is assumed to be identical to the space isounit for simplicity,  $\hat{I}_v \equiv \hat{I}_s$  (because  $v^k$  is contravariant as  $r^k$ ); the *momentum isounit* is instead given by  $\hat{I}_p = \hat{T}_p = \hat{I}_s^{-1}$  (because  $p_k$  is covariant); the study of equivalence classes characterized by more general isounits  $\hat{I}_v \neq \hat{I}_s$  and  $\hat{I}_p \neq \hat{T}_p$  is left to the interested reader; whenever no ambiguity arises, we shall denote  $\hat{I}_s$  and  $\hat{T}_s$  by the generic symbols  $\hat{I}$  and  $\hat{T}$ ; the main isodifferentials are given by  $\hat{d}r^k = \hat{I}_r^k \times dr^i$ ,  $\hat{d}p^k = \hat{I}_k^i \times dp^i$ ,  $\hat{d}t = \hat{I}_t \times dt$  with isoderivatives  $\hat{\partial}/\hat{\partial}r^k = \hat{T}_k^i \times \partial/\partial r^i$ ,

$\hat{\partial}/\hat{\partial}\theta^k = \hat{T}_k^i \times \partial/\partial\theta^i$ ,  $\hat{\partial}/\hat{\partial}\beta_k = \hat{I}_k^i \times \partial/\partial\beta_i$ ,  $\hat{\partial}/\hat{\partial}\hat{t} = \hat{I}_i \times \partial/\partial\hat{t}$  and properties  $\hat{\partial}^i/\hat{\partial}\hat{r}^j = \delta_j^i$ ,  $\hat{\partial}^i/\hat{\partial}\hat{r}^j = \delta_j^i$ ,  $\hat{\partial}^i/\hat{\partial}\hat{r}^j = \hat{T}_i^k \times \delta_{kj}$ ,  $\hat{\partial}^i/\hat{\partial}\hat{r}^j = \hat{I}_k^i \times \delta^{kj}$ , etc.; all quantities with a "hat" (such as  $\hat{r}$ ,  $\hat{p}$ ,  $\hat{H}$ ,  $\hat{T}$ , etc.) are computed in isospaces over isofields, while all quantities without a "hat" (such as  $r$ ,  $p$ ,  $H$ ,  $T$ , etc.) are the corresponding projections computed on conventional spaces over conventional fields; the isoscalar character of the coordinates  $\hat{r} = r \times \hat{I}$ ,  $\hat{t} = t \times \hat{I}$ , shall be generally ignored for simplicity in view of the identity  $\hat{r}^2 \equiv r^2$  of Sec. 3.7, although the symbol  $\hat{r}$  will be preserved to denote computation on isospaces over isofields; finally the ordinary multiplication by the isounit is generally omitted for computational simplicity (e.g., we shall omit the isotopy of the quotient  $a/\hat{b} = (a/b) \times \hat{I}$ ), with the understanding that it is mathematically necessary for consistency for any structure of isoscalar character on  $\hat{R}$ .

The configuration space of nonrelativistic isomechanics is given by the Kronecker product of isoeuclidean spaces  $\hat{S}(\hat{t}, \hat{r}, \hat{\theta}) = \hat{E}(\hat{t}, \hat{R}_t) \times \hat{E}(\hat{r}, \hat{\delta}, \hat{R}) \times \hat{E}(\hat{\theta}, \hat{\delta}, \hat{R})$  with total isounit  $\hat{I}_{\text{tot}} = \hat{I}_t \times \hat{I}_r \times \hat{I}_\theta$ . The fundamental isonewton equations, first introduced in Ref. 23b, p 31, are given by

$$\hat{m} \hat{\times}_i \frac{d\hat{v}_k}{d\hat{t}} - \frac{d}{d\hat{t}} \frac{\partial \hat{U}(\hat{t}, \hat{r}, \hat{\theta})}{\partial \hat{\theta}^k} + \frac{\partial \hat{U}(\hat{t}, \hat{r}, \hat{\theta})}{\partial \hat{r}^k} = 0 \quad (3.38a)$$

$$\hat{U}(\hat{t}, \hat{r}, \hat{\theta}) = \hat{U}_k(\hat{t}, \hat{r}) \hat{\theta}^k + \hat{U}_0(\hat{t}, \hat{r}) \quad (3.38b)$$

where  $\hat{m} = m \times \hat{I}_i$  ( $\neq 0$ ) is the isotopic mass, that is, the image of the Newtonian mass in isospace with isounit  $\hat{I}_i$ .

**Theorem 4 [23b].** All possible sufficiently smooth, regular, non-linear, nonlocal-integral and nonhamiltonian Newton's equations in conventional representations always admit in a neighborhood of a point  $(t, r, v)$  of the local variables a representation in terms of the isotopic equations (3.38).

**Proof.** The inverse isotopic problem is here defined as the computation of the isounits and potentials from given equations of motion according to the identifications

$$\begin{aligned} \hat{m} \hat{\times}_i \hat{T}_i^k \frac{d(\hat{T}_k^i v_i)}{dt} - \hat{T}_i^k \frac{d}{dt} \hat{T}_i^k \frac{\partial \hat{U}(\hat{t}, \hat{r}, \hat{\theta})}{\partial \hat{\theta}^i} + \hat{T}_i^k \frac{\partial \hat{U}(\hat{t}, \hat{r}, \hat{\theta})}{\partial \hat{r}^i} \\ = \hat{m} \hat{T}_i^k \hat{T}_i^k \frac{dv_i}{dt} - \hat{T}_i^k \hat{T}_i^k \frac{\partial \hat{U}_i(t, r)}{\partial r^s} v^s + \hat{T}_i^k \frac{\partial \hat{U}_0(t, r)}{\partial r^i} + \hat{m} \hat{T}_i^k \frac{d\hat{T}_k^i}{dt} v_i \\ = \hat{T}_i^k \{ m dv_i/dt - F^{\text{SA}}(i(t, r, v)) - F^{\text{NSA}}_i(t, r, v) \} = 0 \end{aligned} \quad (3.39)$$

where the unknowns are the isotopic elements  $\hat{T}_i$ ,  $\hat{T}_k^i$  and the potential  $\hat{U}_k$ ,  $\hat{U}_0$ ; SA(NSA) stands for *variational self-adjointness* (non-self-adjointness), i.e., the verification (violation) of the necessary and sufficient conditions for the existence of a potential (see, Ref. 3d for details); and the potentials  $\hat{U}_k$  and  $\hat{U}_0$  are computed from the SA force via the techniques of the ordinary inverse problem (see also Ref. 3d).

Sufficient conditions for the above identities are then given by

$$m \times T_i \times d\hat{v}_i/dt = m \times dv_i/dt \quad (3.40a)$$

$$\hat{T}_i^k \times \frac{\partial \hat{U}_i(t, r)}{\partial r^s} \times \hat{\theta}^s = \frac{\partial U_i(t, r)}{\partial r^s} \times v^s \quad (3.40b)$$

$$\frac{\partial \hat{U}_0(t, r)}{\partial r^i} = \frac{\partial U_0(t, r)}{\partial r^i} \quad (3.40c)$$

$$m \hat{T}_i \frac{d\hat{T}_k^i(t, r, \dots)}{dt} v_i = -\hat{T}_k^i F^{\text{NSA}}_i(t, r, v) \quad (3.40d)$$

which are overdetermined and, as such, always admit a solution in the unknown quantities  $\hat{T}_i$ ,  $\hat{T}_k^i$ ,  $\hat{U}_k$ , and  $\hat{U}_0$  for given equations of motion. In fact, the simplest possible solution exists for *diagonal* space isounit and *constant* time isounit,

$$\hat{I}_k^i = \delta_k^i e^{f_k(t, r, v)}, \quad \hat{I}_i = \text{const} > 0, \quad \hat{\theta} = \hat{T}_i v \quad (3.41)$$

for which

$$\hat{U}_k(t, r) = U_k(t, r), \quad \hat{U}_0(t, r) = U_0(t, r) \quad (3.42a)$$

$$f_k(t, x, v) = -m^{-1} \int_0^t dt F^{\text{NSA}}_k(t, r, v)/v_k \quad (3.42b)$$

where there are no repeated indices and the remaining potentials are computed from the self-adjoint forces.<sup>(3d)</sup>

The physical advantages in the transition from the conventional to isotopic Newton's equations are the following.

**Corollary 4.A.** The isonewton equations permit a representation of the actual, extended and nonspherical shape of the body considered and of its possible deformations via the generalized unit (or isotopic element) of the theory.

Recall that Newton's equations can only approximate the body considered as a *massive point*, as is well known since Newton's time. The point-like representation of particles then persists under analytic representations via Hamilton's equations as well as under symplectic map to quantum mechanics. A representation of the extended character of particles is reached in *second* quantization via the form factors. However, this representation is restricted to perfectly spherical and rigid shapes in order not to violate the fundamental rotational symmetry.

On the contrary, the isonewton's equations can represent the actual *extended, nonspherical and deformable* shape of the body considered. As a simple illustration, suppose that the body considered is a rigid spheroidal ellipsoid with semiaxes  $n_1^2, n_2^2, n_3^2 = \text{const}$ . Such a shape is directly represented by the isotopic element of the theory in the simple diagonal form

$$\hat{T} = \text{diag}(n_1^{-2}, n_2^{-2}, n_3^{-2}), \quad n_k = \text{const} > 0, \quad k = 1, 2, 3, \quad \hat{T}_i = 1 \quad (3.43)$$

The representation of the shape in isospace  $\hat{S}(\hat{t}, \hat{r}, \hat{v})$  is then embedded in the *isoderivatives* of the isotopic equations and, when projected in the conventional space  $S(t, x, v)$ , can be written

$$m \hat{T}_k^i \frac{dv_i}{dt} - \hat{T}_k^i \frac{\partial U_i(t, x)}{\partial x^i} v^s + \hat{T}_k^i \frac{\partial U_0(t, x)}{\partial x^i} = 0 \quad (3.44)$$

namely, the shape terms  $\hat{T}_k^i$  are admitted as factors.

Note that *the representation of shape occurs in isospace because, when projected in the conventional Euclidean space, the shape terms cancel out by recovering the conventional point-like character of Newton's equations*. The representation of shapes more complex than the spheroidal ellipsoids is possible with nondiagonal isounits. The representation of the *deformations* of the original shape due to motion within resistive media or other effects, can be achieved via a suitable functional dependence of the  $\hat{T}_k^i$  terms on velocities, pressure, etc.<sup>(22b)</sup>

**Corollary 4.B.** The isonewton equations permit a novel representation of variationally non-self-adjoint forces via the isotopy of the underlying geometry according to the rules

$$m dv_k/dt - F^{\text{NSA}}_k(t, r, v) = \hat{T}_k^i m d(\hat{T}_i^j \times v_j)/dt \quad (3.45)$$

while leaving unchanged the representation of conventional self-adjoint forces (except for the constant factor  $\hat{T}_i$  of  $U_k$ ).

In fact, the non-self-adjoint forces are embedded in the *covariant* coordinates in isospace  $\hat{\theta}_i = \hat{T}_i^j v_j$ , where the  $v_j$ 's are the covariant coordinates in conventional space. The novelty therefore lies in the fact that nonpotential forces are represented by the isogeometry itself.

The simplicity of representation (3.39) should be kept in mind and compared to the complexity of the conventional solution of the *inverse problem of Newtonian mechanics*,<sup>(3d)</sup> i.e., the representation of non-self-adjoint systems via a Lagrangian or a Hamiltonian. Moreover, under the assumed conditions, the latter exists in the fixed coordinates  $(t, r, v)$  of the observer only for a restricted class called *nonessentially non-self-adjoint* [*loc. cit.*], while isorepresentation (3.39) always exists in the given coordinates  $(t, r, v)$  under broader conditions.

As an example, the equation of the linearly damped particle in one dimension

$$m dv/dt + \gamma v = 0, \quad \gamma \in R(n, +, \times), \quad \gamma > 0 \quad (3.46)$$

admits isorepresentation (3.39) with values

$$\hat{T} = \hat{S}_0 \times e^{\gamma t/2m}, \quad \hat{T}_i = 1, \quad U_k = U_0 = 0 \quad (3.47)$$

where  $\hat{S}_0$  is a *shape factor*, e.g., of the spheroidal type (3.43) prolate in the direction of motion. In this way, the isotopic Newton equations represent: (1) the non-self-adjoint force  $F^{\text{NSA}} = -\gamma v$  experienced by an object moving within a physical medium; (2) its extended character (which is necessary for the existence of the resistive force); and (3) the deformation of the original shape (in the case considered a perfect sphere) caused by the medium.

The equation for the linearly damped harmonic oscillator in one dimension

$$m\ddot{r} + \gamma\dot{r} + kr = 0, \quad k \in R(n, +, \times), \quad k > 0 \quad (3.48)$$

admits isorepresentation (3.39) with the value

$$\hat{T} = \hat{S}_0 \times e^{\gamma t/2m}, \quad U_0 = -\frac{1}{2}kr^2, \quad U_k = 0, \quad \hat{T}_i = 1 \quad (3.49)$$

where  $\hat{S}_0$  represents the shape of the body oscillating within a resistive medium. The interested reader can construct a virtually endless variety of isorepresentations of non-self-adjoint forces.

**Corollary 4.C.** The isonewton equations permit the representation of nonlocal-integral forces when completely embedded in the isounit of the theory.

The strictly local-differential character of quantum mechanics originates from the corresponding character of Newton's equations which are equipped with the local-differential topology of the Euclidean space. In fact, the latter carries over at the level of Lagrangian and Hamiltonian mechanics, and then persists under quantization.

One of the most significant advances of our isonewtons equations is that they can represent *nonlocal-integral forces*. As we shall see, the latter characteristic then carries over at all subsequent levels, thus permitting quantitative studies of the historical legacy of the *nonlocality of the strong interactions*. As a matter of fact, Santilli conceived the above isotopic broadening of Newton's equations precisely as the necessary foundations for nonlocal treatment of strong interactions.

Consider as an example the integrodifferential equation

$$m \, dv/dt + \gamma v^2 \int_{\sigma} d\sigma \mathcal{F}(\sigma, \dots) = 0, \quad \gamma > 0 \tag{3.50}$$

representing an extended object (such as a spaceship during reentry in our atmosphere) with local-differential center-of-mass trajectory  $r(t)$  and corrective terms of integral type due to the shape (surface)  $\sigma$  of the body moving within a resistive medium, where  $\mathcal{F}$  is a suitable kernel depending on  $\sigma$  as well as on other variables such as pressure, temperature, density, etc. The above equation admits isorepresentation (3.39) with the values

$$\hat{T} = \hat{S}_{\sigma} e^{\gamma m^{-1} r \int_{\sigma} d\sigma \mathcal{F}(\sigma, \dots)}, \quad \hat{T}_i = 1, \quad U_k = U_0 = 0 \tag{3.51}$$

where  $\hat{S}_{\sigma}$  is the shape factor. Similar isorepresentations can be easily constructed by the interested reader.

In summary, the isonewtonian mechanics permits the identification beginning at the purely classical level of the isounit as representing the shape of the body considered, as well as its nonlocal and nonpotential interactions.

We now outline the isotopies of Lagrange's and Hamilton's mechanics first identified in Ref. 23b via the isodifferential calculus for the direct analytic representation of isonewton's equations.

**Proposition 1.** All well-behaved action functionals of first or higher order in Euclidean space  $S(t, x, v) = E(t, R_t) \times E(x, \delta, R) \times E(v, \delta, R)$  can always be identically rewritten as first-order isoaction functionals in isospace  $\hat{S}(\hat{t}, \hat{R}_t) = \hat{E}(\hat{t}, \hat{R}_t) \times \hat{E}(\hat{t}, \hat{\delta}, \hat{R}) \times \hat{E}(\hat{t}, \hat{\delta}, \hat{R})$ ,

$$\hat{A} = \int_{t_1}^{t_2} dt \mathcal{L}(t, r, v, a, \dots) \equiv \int_{\hat{t}_1}^{\hat{t}_2} d\hat{t} \hat{\mathcal{L}}(\hat{t}, \hat{r}, \hat{v}) \tag{3.52a}$$

$$\begin{aligned} \hat{\mathcal{L}} &= \frac{1}{2} m \hat{v}^i \hat{\delta}_{ij} \hat{v}^j - \hat{U}^i(\hat{t}, \hat{r}) \hat{\delta}_{ij} \hat{v}^j - U_0(\hat{t}, \hat{r}) \\ &= \frac{1}{2} m \hat{v}_k \hat{v}^k - \hat{U}_k(\hat{t}, \hat{r}) v^k - \hat{U}_0(\hat{t}, \hat{r}) \end{aligned} \tag{3.52b}$$

where we have used the *isointegral*  $\hat{\int} = \int \hat{T}^{(22a)}$ . Note that identity (3.52) is overdetermined because, for each given  $\mathcal{L}$ , it admits infinitely many choices of  $\hat{m}$ ,  $\hat{T}_i$ ,  $\hat{T}^j$ ,  $\hat{U}_k$ , and  $\hat{U}_0$ . These isotopic degrees of freedom can be eliminated via the preceding inverse isotopic problem, i.e., via the use of equations of motion, and their study is left to the interested reader.

The *isovariational calculus* is a simple extension of the isodifferential calculus and it is here omitted for brevity.<sup>(23b)</sup> Its application to isoaction (3.52a) yields the following fundamental *isolagrange equations*, first introduced in Ref. 23b, p. 44, and here presented along an actual *isopath*  $\hat{P}_0$ ,

$$\mathcal{L}_k(\hat{P}_0) = \left\{ \frac{d}{d\hat{t}} \frac{\partial \hat{\mathcal{L}}(\hat{t}, \hat{r}, \hat{v})}{\partial \hat{v}^k} - \frac{\partial \hat{\mathcal{L}}(\hat{t}, \hat{r}, \hat{v})}{\partial \hat{r}^k} \right\} (\hat{P}_0) = 0 \tag{3.53}$$

We shall say that the isonewton equations admit a *direct isoanalytic representation*, when there exists one isolagrangian  $\hat{\mathcal{L}}(\hat{t}, \hat{x}, \hat{d})$  under which all the following identities occur:

$$\begin{aligned} \hat{r}_k^i & \left\{ \frac{d}{d\hat{t}} \frac{\partial \hat{\mathcal{L}}(\hat{t}, \hat{r}, \hat{v})}{\partial \hat{v}^i} - \frac{\partial \hat{\mathcal{L}}(\hat{t}, \hat{r}, \hat{v})}{\partial \hat{r}^i} \right\} \\ &= \hat{r}_k^i \left\{ \hat{m} \frac{d\hat{v}_i}{d\hat{t}} - \frac{\partial \hat{U}_i(\hat{t}, \hat{r})}{\partial \hat{r}^s} \frac{d\hat{r}^s}{d\hat{t}} + \frac{\partial \hat{U}_0(\hat{t}, \hat{r})}{\partial \hat{r}^i} \right\} \\ &= \left\{ m \frac{dv_k}{dt} - \frac{\partial U_k(t, r)}{\partial r^s} \frac{dr^s}{dt} + \frac{\partial U_0(t, r)}{\partial r^k} - F^{\text{NSA}}_k(t, r, v) \right\}^{\text{NSA}} = 0 \end{aligned} \tag{3.54a}$$

$$\hat{\mathcal{L}}(\hat{t}, \hat{r}, \hat{v}) = \frac{1}{2} m \hat{v}^k \hat{v}_k - \hat{U}, \quad \hat{U}(\hat{t}, \hat{r}, \hat{v}) = \hat{U}_k(\hat{t}, \hat{r}) \hat{v}^k + \hat{U}_0(\hat{t}, \hat{r}) \tag{3.54b}$$

All remaining aspects of the conventional Lagrangian mechanics are then subjected to similar isotopies.<sup>(23b)</sup> Note that Lagrange originally proposed his celebrated equations *with external terms* under the assumption of representing the kinetic and potential energies with the Lagrangian and representing all remaining forces and effects with the external terms.

The isolagrange equations have been constructed along the same spirit, because all nonLagrangian forces/external terms are represented with the isounit. It should be stressed that the replacement of the external terms with the isounit is *necessary* for the preservation of Einsteinian axioms under variationally non-self-adjoint interactions.

We now study the isotopies of Hamilton's equations which, according to our notation indicated at the beginning on this section, are defined in the isospace  $\hat{S}(\hat{t}, \hat{r}, \hat{p}) = \hat{E}(\hat{t}, \hat{R}_t) \times \hat{E}(\hat{r}, \hat{\delta}, \hat{R}_r) \times \hat{E}(\hat{p}, \hat{\delta}, \hat{R}_p)$  with total isounit  $\hat{I}_{\text{tot}} = \hat{I}_t \times \hat{I}_r \times \hat{I}_p = \hat{I}_t \times \hat{I} \times \hat{I}$ . The *isocanonical momentum* is given by<sup>(23b)</sup>

$$\hat{p}_k = \frac{\hat{\partial} \hat{\mathcal{L}}(\hat{t}, \hat{r}, \hat{p})}{\hat{\partial} \hat{v}^k} = m \hat{v}_k - \hat{U}_k(\hat{t}, \hat{r}) \quad (3.55)$$

under the condition that the isolagrangian is regular in a  $(2n+1)$ -dimensional region  $\mathfrak{R}$  of points  $(\hat{t}, \hat{r}, \hat{p})$

$$\text{Det} \left( \frac{\hat{\partial}^2 \hat{\mathcal{L}}(\hat{t}, \hat{r}, \hat{p})}{\hat{\partial} \hat{v}^i \hat{\partial} \hat{v}^j} \right) (\mathfrak{R}) \neq 0 \quad (3.6)$$

thus admitting a unique set of implicit functions  $\hat{v}^k = \hat{v}^k(\hat{t}, \hat{r}, \hat{p})$ .

The *isolegendre transform* can then be defined by [*loc. cit.*]

$$\begin{aligned} \hat{\mathcal{L}}[\hat{t}, \hat{x}, \hat{v}(\hat{t}, \hat{r}, \hat{p})] &= \hat{p}_k \hat{v}^k(\hat{t}, \hat{r}, \hat{p}) - \frac{1}{2} m \hat{v}_i(\hat{t}, \hat{r}, \hat{p}) \hat{v}^j(\hat{t}, \hat{r}, \hat{p}) \\ &\quad + \hat{U}_k(\hat{t}, \hat{r}) \hat{v}^k(\hat{t}, \hat{r}, \hat{p}) + \hat{U}_0(\hat{t}, \hat{r}) \\ &= \hat{p}_k \hat{p}^k / 2m + \hat{V}^k(\hat{t}, \hat{r}) \hat{p}_k + \hat{V}^0(\hat{t}, \hat{r}) = \hat{H}(\hat{t}, \hat{r}, \hat{p}) \end{aligned} \quad (3.57)$$

The isovariation of the isoaction in momentum representation then yields the fundamental *isohamilton's equations*, first submitted in Ref. 23b, p. 47,

$$\frac{d\hat{x}^k}{d\hat{t}} = \frac{\hat{\partial} \hat{H}(\hat{t}, \hat{x}, \hat{p})}{\hat{\partial} \hat{p}_k}, \quad \frac{d\hat{p}_k}{d\hat{t}} = - \frac{\hat{\partial} \hat{H}(\hat{t}, \hat{x}, \hat{p})}{\hat{\partial} \hat{x}^k} \quad (3.58)$$

which can be written in unified notation

$$\left( \frac{\hat{\partial} \hat{R}_v^0}{\hat{\partial} \hat{b}^\mu} - \frac{\hat{\partial} \hat{R}_\mu^0}{\hat{\partial} \hat{b}^\nu} \right) \frac{d\hat{b}^\nu}{d\hat{t}} - \frac{\hat{\partial} \hat{H}(\hat{t}, \hat{b})}{\hat{\partial} \hat{b}^\mu} = 0 \quad (3.59a)$$

$$\hat{R}^0 = \{ \hat{R}_\mu^0 \} = \{ \hat{p}_k, \hat{0} \} \quad (3.59b)$$

or in the following covariant and contravariant forms:

$$\hat{\omega}_{\mu\nu} \frac{d\hat{b}^\nu}{d\hat{t}} = \frac{\hat{\partial} \hat{H}(\hat{t}, \hat{b})}{\hat{\partial} \hat{b}^\mu} \quad (3.60a)$$

$$\frac{d\hat{b}^\mu}{d\hat{t}} = \hat{\omega}^{\mu\nu} \frac{\hat{\partial} \hat{H}(\hat{t}, \hat{b})}{\hat{\partial} \hat{b}^\nu} \quad (3.60b)$$

with explicit expressions

$$(\omega_{\mu\nu}) = \left( \frac{\hat{\partial} \hat{R}_v^0}{\hat{\partial} \hat{b}^\mu} - \frac{\hat{\partial} \hat{R}_\mu^0}{\hat{\partial} \hat{b}^\nu} \right) = \begin{pmatrix} 0_{N \times N} & -I_{N \times N} \\ I_{N \times N} & 0_{N \times N} \end{pmatrix} \quad (3.61a)$$

$$(\omega^{\alpha\beta}) = \left( \frac{\hat{\partial} \hat{R}_v^0}{\hat{\partial} \hat{b}^\mu} - \frac{\hat{\partial} \hat{R}_\mu^0}{\hat{\partial} \hat{b}^\nu} \right)^{-1} = \begin{pmatrix} 0_{N \times N} & I_{N \times N} \\ -I_{N \times N} & 0_{N \times N} \end{pmatrix} \quad (3.61b)$$

holding in view of the properties of the isodifferential calculus

$$\hat{\partial} \hat{R}_v^0 / \hat{\partial} \hat{b}^\mu \equiv \partial R_v^0 / \partial b^\mu \quad (3.62)$$

The above equations confirm the fundamental invariance of the canonical tensor under isotopies first identified via the isosymplectic geometry in the preceding section. The above equations also confirm the classical realization (3.34) of the Lie-Santilli isobrackets. In fact, the time evolution under Eq. (3.60b) can be written

$$d\hat{A}/d\hat{t} = [\hat{A}, \hat{H}] \quad (3.63)$$

The equivalence of the isolagrangian and isohamiltonian equations under the assumed regularity and invertibility of the isolegendre transform can be proved as in the conventional case.

We also have the *isotopic Hamilton-Jacobi equations*, first identified in Ref. 23b, p. 48, which have an evident fundamental character for quantization,

$$\frac{\hat{\partial} \hat{A}}{\hat{\partial} \hat{t}} + \hat{H}(\hat{t}, \hat{r}, \hat{p}) = 0, \quad \frac{\hat{\partial} \hat{A}}{\hat{\partial} \hat{r}^k} - \hat{p}_k = 0 \quad (3.64)$$

plus initial conditions here ignored.

We finally quote also from Ref. 23b the following direct universality of isanalytic mechanics,



**Theorem 5.** All possible sufficiently smooth and regular, nonlinear, nonlocal, and non-self-adjoint dynamical systems always admit a direct isorepresentation in a star-shaped neighborhood of a point of their variables via the isolagrange or the isohamilton equations on isospaces over isofields.

Note the *abstract identity between the conventional and isotopic mechanics*. Since the isounits are positive-definite, at the abstract level there is no distinction between  $dt$  and  $\hat{d}t$  or  $dr$  and  $\hat{d}r$ ,  $L$  and  $\hat{L}$ ,  $H$  and  $\hat{H}$  etc. The isolagrange and isohamilton equations therefore coincide at the abstract level with the conventional equations. This illustrates the axiom-preserving character of the isotopies, this time, in analytic mechanics.

This is another basic point of our efforts to preserve Einsteinian axioms. In fact, we have established with the above results that *classical relativistic isotopic mechanics coincides at the abstract level with the conventional relativistic mechanics*.

Another important property is that *the transformation theory of the isohamilton's equations is isocanonical, that is, it preserves the conventional canonical structure*,

$$\hat{b} = (\hat{r}, \hat{p}) \rightarrow \hat{b}'(\hat{b}) = \{\hat{r}'(\hat{r}, \hat{p}), \hat{p}'(\hat{r}, \hat{p})\} \quad (3.65a)$$

$$\hat{\omega}_{\mu\nu} \rightarrow \hat{\omega}'_{\mu\nu} = \frac{\hat{\partial}\hat{b}^\alpha}{\hat{\partial}\hat{\theta}'^\mu} \hat{\times} \hat{\omega}_{\alpha\beta} \hat{\times} \frac{\hat{\partial}\hat{b}^\beta}{\hat{\partial}\hat{b}'^\nu} \equiv \hat{\omega}_{\mu\nu} \equiv \omega_{\mu\nu} \quad (3.65b)$$

This permits the main result of this subsection, according to which *the isohamiltonian mechanics preserves the basic units and the canonical symplectic tensor, thus confirming the capability to preserve Einstein's axioms at the abstract level*.

The advantages of the isoanalytic over the conventional mechanics are evident. For instance, the isohamilton's equations are directly universal in the fixed inertial frame of the observer while the conventional Hamilton's equations are not, thus forcing the use of the Darboux's maps to *noninertial* frames with the consequential loss of the axioms of the special relativity indicated in the preceding sections. Also, Hamilton's equations are strictly local-differential from the underlying conventional topology, while the isohamilton equations are integrodifferential thanks to the underlying broader Tsagas-Sourlas isotopology. Finally, Hamilton's equations cannot possibly represent the shape of the particles considered, while such a representation is possible under isotopies.

We close this section with a comparison between the isoanalytic mechanics outlined above and the Birkhoffian mechanics of Ref. 3d, so as to identify the reasons why the latter was insufficient for the studies herein

considered. Both mechanics are directly universal for well-behaved local-differential systems. As such, they must admit an interconnecting relation within the fixed frame of the experimenter.

In fact, the Pfaffian action (1.9) can be identically rewritten in the isotopic form according to the rules

$$\int_{t_1}^{t_2} [R_\mu(b) \times db^\mu - H(t, b) \times dt] \equiv \int_{\hat{t}_1}^{\hat{t}_2} [\hat{R}_\mu^\circ(b) \times \hat{d}\hat{b}^\mu - \hat{H}(\hat{t}, \hat{b}) \times dt] \\ \hat{R}^\circ = \{\hat{p}, \hat{0}\}, \quad \hat{d}b = \hat{I}_{\text{tot}} \times db = (\hat{R}/\hat{R}^\circ) \times db \\ \hat{b}^\mu \equiv b^\mu, \quad \hat{H} \equiv H, \quad \hat{d}\hat{t} = dt \quad (3.66)$$

Also, the totally antisymmetric Lie-isotopic tensor  $\Omega^{\mu\nu}(b)$  always admits the factorization into the canonical Lie tensor  $\omega^{\mu\nu}$  and a regular symmetric matrix  $\hat{T}_\mu^\nu$

$$\Omega^{\mu\nu} \equiv \omega^{\alpha\beta} \times \hat{T}_\beta^\nu \quad (3.67)$$

As a result, *Birkhoff's equations can always be rewritten in an identical isohamiltonian form* (here given for the simple case with  $\hat{I}_t = 1$ )

$$\frac{db^\mu}{dt} - \Omega^{\mu\nu}(b) \frac{\partial H(t, b)}{\partial b^\nu} \equiv \frac{db^\mu}{dt} - \omega^{\mu\nu} \frac{\partial H(t, b)}{\partial b^\nu} \quad (3.68)$$

The above reformulation is nontrivial, mathematically and physically. Mathematically, it implies the lifting of the totality of conventional methods, from numbers to topology. Physically, the above reformulation is fundamental to preserve the axioms of classical relativistic mechanics, which is not possible with Birkhoff's mechanics (Theorem 2).

In summary, the inability of the Birkhoffian mechanics, as well as of the noncanonical generalizations of Hamiltonian mechanics to preserve Einsteinian axioms, is due to their representation in *conventional metric spaces over conventional fields*, because the same theories when properly reformulated on isospaces over isofields can indeed preserve said axioms.

As a historical note, we should recall that Hamilton was fully aware of the lack of universal character of what is today called the Hamiltonian and, for this reason, he proposed his celebrated equations *with external terms*. The latter were eliminated in the literature of this century owing to the successes of analytic mechanics for the treatment of conservative systems, such as planetary and atomic systems.

The studies herein considered are based on a return to the original conception by Hamilton on the limited representational capabilities of the Hamiltonian. In fact, the isounit has essentially the same function of

Hamilton's external terms to such an extent that they both have the same number of independent elements.

As indicated in Sec. 1, the isotopic representation of the historical external terms is *necessary* because the latter imply the abandonment of Lie algebras in the brackets of the time evolution in favor of the broader Lie-admissible algebras,<sup>(3c)</sup> thus implying the inability to preserve the axioms of special relativity.

### 3.9. Isotopies of Quantization and Nonrelativistic Quantum Mechanics

We are now equipped to identify, apparently for the first time, the main elements of the *isotopies of nonrelativistic quantum mechanics*, also known as *nonrelativistic hadron mechanics*, in the version characterized by the isodifferential calculus of the recent memoir.<sup>(23b)</sup> For a comprehensive list of contributions and related references on the formulation prior to the isodifferential calculus, we suggest to consult for brevity Ref. 22b.

The systematic use of the isotopic liftings or, equivalently, nonunitary transforms, yield the following main mathematical structure:

(1) The basic *isofields* of isoreal  $\hat{R} = \hat{R}(\hat{n}, +, \hat{\times})$  or isocomplex numbers  $\hat{C} = \hat{C}(\hat{c}, +, \hat{\times})$  with isonumbers  $\hat{n} = n \times \hat{I}$ ,  $\hat{c} = c \times \hat{I}$ , and isoproduct  $\hat{a} \hat{\times} \hat{b} = \hat{a} \times \hat{T} \times \hat{b}$ ,  $\hat{I} = \hat{T}^{-1}$ ,  $\hat{a} = \hat{n}$ ,  $\hat{c}$ ;

(2) The *enveloping isoassociative operator algebra*  $\hat{\xi}$  over  $\hat{C}$  with elements  $X = \{X_k\}$ , and isoassociative product  $\hat{X}_i \hat{\times} \hat{X}_j$ , infinite-dimensional basis (3.23), and isoexponentiation (3.24);

(3) The *isohilbert spaces*  $\hat{\mathcal{H}}$  over  $\hat{C}$  with isoinner product  $\langle \hat{\phi} | \times \hat{T} \times | \hat{\psi} \rangle \times \hat{I} \in \hat{C}$  and isonormalization  $\langle \hat{\psi} | \times \hat{T} \times | \hat{\psi} \rangle = \hat{I}$ ;

(4) The *isoeuclidean spaces*  $\hat{E}(\hat{r}, \hat{\delta}, \hat{R})$  with isometric  $\hat{\delta}(r, \hat{r}, \psi, \partial\psi, \dots) = \hat{T}(r, \hat{r}, \psi, \partial\psi, \dots) \times \hat{\delta}$ ,  $\hat{\delta} = \text{diag}(1, 1, 1)$ ; and biosphere  $\hat{r}^2 = (r' \times \hat{\delta} \times r) \times \hat{I} \in \hat{R}$  (Sec. 3.7);

(5) The *Lie-Santilli isotheory* with isoalgebra (3.25), isogroup (3.27), and related isosymmetries in isounitary realization.

The operator theory herein considered can be uniquely and unambiguously derived from the isohamiltonian mechanics of the preceding section via simple isotopies of the conventional naive or symplectic quantization. Recall that the simplest possible map of Hamiltonian into quantum mechanics, called *naive quantization*, is characterized by the map of the canonical action functional

$$A \rightarrow -i \times \hbar \times \text{Ln} |\psi(t, r)\rangle \quad (3.69)$$

which maps the conventional Hamilton-Jacobi equations into the Schrödinger equations

$$i\partial/\partial t |\psi\rangle = H \times \hbar^{-1} \times |\psi\rangle \quad (3.70a)$$

$$p_k \times \hbar^{-1} \times |\psi\rangle = -i\partial/\partial r^k |\psi\rangle \quad (3.70b)$$

with corresponding Heisenberg's equation

$$i dA/dt = A \times \hbar^{-1} \times H - H \times \hbar^{-1} \times A \quad (3.71a)$$

$$p_i \times \hbar^{-1} \times r^j - f^j \times \hbar^{-1} p_i = i \times \delta_i^j \quad (3.71b)$$

$$p_i \times \hbar^{-1} \times p_j - p_j \times \hbar^{-1} \times p_i = r^i \times \hbar^{-1} \times r^j - r^j \times \hbar^{-1} \times r^i = 0 \quad (3.71c)$$

where Planck's constant has been moved from the traditional place of writing it, the l.h.s., to the r.h.s. for reasons which will appear clear momentarily.

But the isoaction (3.52) is of arbitrary order in conventional space and, therefore, the preceding map is not applicable. We therefore introduced the map called naive *isoquantization*, here considered for the simpler case when  $\hat{I}$  does not depend explicitly on the local coordinates.

$$\hat{A}(\hat{t}, \hat{r}) \rightarrow -i\hat{I}(\hat{r}) \text{Ln} \hat{\psi}(\hat{t}, \hat{r}) \quad (3.72)$$

under which the isotopic Hamilton/Jacobi equations (3.64) yield the *isoschrödinger equations*

$$\begin{aligned} i\hat{\partial}/\hat{\partial}\hat{t}|\hat{\psi}\rangle &= i\hat{I}_t \times \partial/\partial\hat{t}|\hat{\psi}\rangle = \hat{H} \hat{\times} |\hat{\psi}\rangle = \hat{H} \times \hat{T} \times |\hat{\psi}\rangle = \hat{E} \hat{\times} |\hat{\psi}\rangle \\ &= E \times \hat{\psi} \end{aligned} \quad (3.73a)$$

$$\hat{p}_k \hat{\times} |\hat{\psi}\rangle = \hat{p}_k \times \hat{T} \times |\hat{\psi}\rangle = -i\hat{\partial}/\hat{\partial}\hat{r}^k |\hat{\psi}\rangle = -i\hat{T}_k^i \times \partial/\partial r^i |\hat{\psi}\rangle \quad (3.73b)$$

with equivalent *isoheisenberg equations*

$$\begin{aligned} i \hat{d}\hat{A}/\hat{d}\hat{t} &= -i\hat{I}_t \times d\hat{A}/d\hat{t} = [\hat{A}, \hat{H}]_{\hat{\xi}} = \hat{A} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{A} \\ &= \hat{A} \times \hat{T} \times \hat{H} - \hat{H} \times \hat{T} \times \hat{A} \end{aligned} \quad (3.74a)$$

$$[\hat{p}_j, \hat{r}^j] = \hat{p}_i \hat{\times} \hat{r}^j - \hat{r}^j \hat{\times} \hat{p}_i = \hat{p}_i \times \hat{T} \times \hat{r}^j - \hat{r}^j \times \hat{T} \times \hat{p}_i = i\hat{\delta}_i^j \quad (3.74b)$$

$$[\hat{p}_i, \hat{p}_j] = [\hat{r}^i, \hat{r}^j] \equiv 0 \quad (3.74c)$$

The use of the full dependence of  $\hat{I}$  merely implies the enlargement of the isohamiltonian (for details see Schuch<sup>(16)</sup> and Ref. 22b).

The isotopies can also be applied to the symplectic quantization (first studied by Lin<sup>(61)</sup>) yielding the same fundamental equations (3.73) and (3.74), as the reader can verify.<sup>(23b)</sup>

Note the essential role of the isodifferential calculus<sup>(23b)</sup> for the identification of the axiomatically correct dynamical equations, as well as for the construction of the fundamental isocommutation rules.

The most salient aspect emerging from the comparison of the conventional and isotopic quantization is that *the classical isounit of Newton's equations assumes at the operator level the role of the generalized Planck's constant,*

$$\hat{h} = I \rightarrow \hat{h} = \hat{I} \quad (3.75)$$

Also, *the isounit is the fundamental invariant of the isotheory* because of the dynamical conservation law

$$i \hat{d}\hat{I}/\hat{d}t = \hat{I} \times \hat{H} - \hat{H} \hat{\times} \hat{I} = \hat{H} - \hat{H} \equiv 0 \quad (3.76)$$

(plus its invariance indicated below). Moreover,  $\hat{I}$  satisfies all axioms of the conventional Planck's unit,

$$\hat{I}^n = \hat{I} \hat{\times} \hat{I} \hat{\times} \dots \hat{\times} \hat{I} \hat{\times} \hat{I}, \quad \hat{I}^{\sqrt{2}} \equiv \hat{I}, \quad \hat{I} / \hat{I} \equiv \hat{I}, \text{ etc.} \quad (3.77)$$

and, finally, its isoexpectation values recover the conventional Planck's value,

$$\langle \hat{I} \rangle = \frac{\langle \hat{\psi} | \times \hat{I} \times \hat{I}^{-1} \times \hat{I} \times | \hat{\psi} \rangle}{\langle \hat{\psi} | \times \hat{I} \times | \hat{\psi} \rangle} = I = \hbar \quad (3.78)$$

In turn, the above properties imply that *the center-of-mass trajectories of the systems represented by the isotopic completion of quantum mechanics verify Heisenberg's uncertainties.* In fact, from Eq. (3.74b) we have ( $\hbar = 1$ )

$$\Delta r \Delta p \geq \frac{1}{2} \langle [\hat{r}, \hat{p}] \rangle = \frac{1}{2} \quad (3.79)$$

The latter property establishes the axiomatic character of Heisenberg's uncertainties, i.e., their invariance under isotopies, and confirms the lack of need of deviations from the same laws under nonunitary transforms indicated in Sec. 2. In fact, the anomalies of "squeezed states"<sup>(22)</sup> can be reformulated in an axiomatically consistent and invariant way in which there is no departure from Heisenberg's uncertainties.

The latter results establish the novel character of the composite systems characterized by the isotopic mechanics, as first anticipated via the isosymmetries, because: (1) the systems verify conventional total conservation laws (Sec. 3.7); (2) the center-of-mass trajectory of the systems is

conventional; nevertheless (3) the systems admit a generalized internal structure with linear and nonlinear, local and nonlocal, and potential as well as nonpotential internal effects.

A dominant feature of the isotopic equations is the conservation of the total energy due to the antisymmetry of the isoproduct,

$$i \hat{d}\hat{H}/\hat{d}t = \hat{H} \times \hat{H} - \hat{H} \hat{\times} \hat{H} \equiv 0 \quad (3.80)$$

The generalized composite systems represented by the theory can therefore be assumed to be *closed-isolated*.

Recall that conventional quantum mechanics represents systems via the sole knowledge of the Hamiltonian  $H$  under the *tacit assumption of the simplest possible space-time units*  $I$ . The isotopic theory requires instead the knowledge of the Hamiltonian *plus the assumed space-time units*. In particular, the Hamiltonian  $\hat{H}$  represents all conventional action-at-a-distance interactions mediated by particle exchanges, while the isounit  $\hat{I}$  can represent all contact-nonhamiltonian interactions which, as such, cannot possibly be mediated by particle exchanges, such as the nonlinear and nonlocal interactions of Sec. 3.8.

An objection is at times voiced that the isotheory is "too general" because it admits infinitely possible operators  $\hat{I}$ . This is equivalent to the statement that Newton's equations are too general because they admit an infinite possibility of different forces, or that quantum mechanics is too general because it admits infinitely possible Hamiltonians. In reality, a primary value of the isotheory is precisely the unrestricted character of the isounit, which has been uniquely determined for all interior systems studied until now (Sec. 3.14).

Also, the isotopic lifting of *each* exterior system admits *many different* possible isounits, because a system of particles at large mutual distances in vacuum can be brought into a large variety of interior conditions depending on density, temperature, size, chemical composition, etc.

It appears that the above isotopies of quantum mechanics do indeed preserve all axiomatic properties of quantum mechanics.<sup>(22, 23)</sup> This can be seen from the fact that the isotopic and conventional mechanics coincide at the abstract level by conception and construction. Any possible inconsistency is therefore expected to be due to the erroneous or lack of use of the isotopies. Alternatively, we can say that *any criticism on the axiomatic structure of the above isotopic theory is de facto a criticism on the axiomatic structure of quantum mechanics.*

In particular, it is possible to show that the isotopic completion of quantum mechanics resolves the problematic aspects identified in Sec. 2. To begin, the isotheory is invariant under additional nonunitary transforms,

provided that they have the same "magnitude"  $\hat{I}$  and are written in the isotopic form (3.18), which leaves numerically invariant the isounit

$$\hat{I} \rightarrow \hat{I}' = W \hat{\times} \hat{I} \hat{\times} W^\dagger = \hat{W} \times \hat{T} \times \hat{T}^{-1} \times \hat{T} \times W^\dagger = \hat{W} \hat{\times} \hat{W}^\dagger = \hat{I} \quad (3.81)$$

as well as the isoassociative and Lie-Santilli products

$$\hat{W} \hat{\times} (\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{W}^\dagger = \hat{A}' \hat{\times} \hat{B}' \quad (3.82a)$$

$$\hat{W} \hat{\times} (\hat{A} \hat{\times} \hat{B} - \hat{B} \hat{\times} \hat{A}) \hat{\times} \hat{W}^\dagger = \hat{A}' \hat{\times} \hat{B}' - \hat{B}' \hat{\times} \hat{A}' \quad (3.82b)$$

This resolves the problematic aspects of Sec. 2 due to lack of invariance of the unit, lack of conservation of probabilities, and lack of uniqueness and invariance of physical laws.

The preservation of the original Hermiticity follows from Properties (3.6) combined with the latter invariances. The invariance of the numerical predictions can be established via the invariance of the isospecial functions which is omitted here for brevity (see for details Ref. 22). The preservation of causality for the above isotopic theory is essentially equivalent to that for quantum theory, and it is ensured by the abstract identity and local isomorphism of unitary group with their isounitary formulation. The preservation of the Einsteinian axioms will be investigated in the next section.

Note that the isounitary theory is based on the assumption of one fixed isounit, e.g., nonunitary transforms such that  $U \times U^\dagger = \hat{I}' \neq \hat{I}$  are not allowed, evidently because they would imply the transition to *different physical systems*. This occurrence is fully equivalent to the corresponding one in quantum mechanics in which only transformations such that  $U \times U^\dagger = I$  are admitted and others such that  $U \times U^\dagger = \hat{I} \neq I$  are excluded.

A fundamental implication of the isotopic completion of quantum mechanics is that *the validity of quantum mechanics for the center-of-mass treatment of strongly interacting systems, by no means, can be used as an argument for the necessary validity of the same mechanics for the interior structure*. In fact, the same center-of-mass characteristics are also verified by structurally more general theories.

Stated in plain language, the above results imply that quantum mechanics can be safely stated as being exactly valid for *exterior* systems such as atomic structures and electroweak interactions at large (Sec. 3.1), but *the problem of the physical laws holding for strong interactions remains basically open at this writing on both theoretical and experimental grounds*, as pointed out in Ref. 3b.

This completes our rudimentary study of nonrelativistic isotopies. For further details we refer the interested reader to Refs. 22, 23.

### 3.10. Isotopies of Relativistic Quantum Mechanics

Consider a conventional Minkowski space  $M = M(x, \eta, R)$  with coordinates  $x = \{x^\mu\} = \{r, c_0 t\}$ , where  $c_0$  is the speed of light in vacuum, basic unit  $I = \text{Diag}(+1, +1, +1, +1)$  and metric  $\eta = \text{diag}(+, +1, +1, -1)$  over the reals  $R = R(n, +, \times)$ .

The fundamental isospaces of this paper, the *isominkowskian* spaces  $\hat{M} = \hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ , were first introduced in Ref. 4a in 1983 jointly with the isotopies of the Lorentz symmetry. They are called *Minkowski-Santilli isospaces*<sup>(5)</sup> and can be characterized by the expressions

$$\hat{M}(\hat{x}, \hat{\eta}, \hat{R}): \hat{x} = \{\hat{x}^\mu\}, \hat{\eta}(x, \hat{x}, \psi, \partial\psi, \mu, \tau, \dots) = \hat{T}(x, \hat{x}, \psi, \partial\psi, \mu, \tau, \dots) \times \eta \quad (3.83a)$$

$$\begin{aligned} (\hat{x} - \hat{y})^{\hat{2}} &= [(\hat{x} - \hat{y})^\mu \hat{x} \hat{N}_{\mu\nu}(x, \hat{x}, \psi, \partial\psi, \dots) \hat{x}(\hat{x} - \hat{y})^\nu] \\ &= [(x - y)^\mu \times \hat{\eta}_{\mu\nu}(x, \hat{x}, \psi, \partial\psi, \dots) \times (x - y)^\nu] \times \hat{I} \\ &= [(x^1 - y^1) \times T_{11}(x, \psi, \dots) \times (x^1 - y^1) \\ &\quad + (x^2 - y^2) \times T_{22}(x, \psi, \dots) \times (x^2 - y^2) \\ &\quad + (x^3 - y^3) \times T_{33}(x, \psi, \dots) \times (x^3 - y^3) \\ &\quad - (x^4 - y^4) \times T_{44}(x, \psi, \dots) \times (x^4 - y^4)] \times \hat{I} \\ &= (x - y)^{\hat{2}} \in \hat{R} \end{aligned} \quad (3.83b)$$

$$\hat{T} = \text{Diag}(T_{11}, T_{22}, T_{33}, T_{44}), \quad \hat{I} = \hat{T}^{-1} \quad (3.83c)$$

$$\hat{T}_{\mu\mu}(x, \hat{x}, \psi, \partial\psi, \partial^2\psi, \mu, \tau, \dots) > 0 \quad (3.83d)$$

$$\hat{N}_{\mu\nu} = \hat{\eta}_{\mu\nu} \times \hat{I}, \quad \hat{\eta}^{\mu\nu} = (|\hat{\eta}_{\alpha\beta}|^{-1})^{\mu\nu} \quad (3.83d)$$

The *isominkowskian geometry*,<sup>(22a)</sup> also called *Santilli's isominkowskian geometry*,<sup>(5)</sup> is the geometry of isospaces  $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ , and its outline is omitted here for brevity (see Refs. 22, 23).

The first property of particular importance for this note is that *isominkowski spaces are locally isomorphic to the conventional Minkowski space*.<sup>(4)</sup> In fact, the deformation of the metric  $\eta \rightarrow \hat{\eta} = \hat{T} \times \eta$  while the basic unit of the original space is deformed by the *inverse* amount,  $I \rightarrow \hat{I} = \hat{T}^{-1}$ , implies the preservation of all original axioms. In particular, isominkowskian spaces are *isoflat*, that is, they verify the axiom of flatness in isospace (but not so in their projection to the original space  $M$ ).

The local isomorphism  $\hat{M}(\hat{x}, \hat{\eta}, \hat{R}) \approx M(x, \eta, R)$  has evident fundamental importance for the main objective of these studies, the preservation of the axioms of the special relativity under nonunitary theories, for which

purpose the isominkowskian spaces and related geometry were proposed in the first place.<sup>(4)</sup>

The advantages of the isotopies  $M(x, \eta, R) \rightarrow \hat{M}(\hat{x}, \hat{\eta}, \hat{R})$  are transparent. In fact, *the isominkowski spaces are "directly universal," in the sense that they admit all infinitely possible signature-preserving deformations of the Minkowski space (universality), directly in the x-frame of the experimenter (direct universality)* (this property was studied in detail by Aringazin.<sup>(33)</sup>) All possible, well-behaved, real-valued and symmetric metrics are therefore particular cases of the isominkowskian metric  $\hat{\eta}$ , and this includes Riemannian, Finslerian, nondesarguesian, and any other possible (3+1)-dimensional metric. As a result, the isominkowskian geometry permits intriguing geometric unifications (see next section) which are not evidently possible with conventional geometries.

Recall that the quantity  $I = \text{Diag}\{1, 1, 1\}, 1$  is the space-time unit of the conventional Minkowski space. The  $4 \times 4$  quantity  $\hat{I}$  therefore represents the *space-time isounits*. In this interpretation the isounit is often written in the form

$$\hat{I} = \text{Diag}(\{n_1^2, n_2^2, n_3^2\}, n_4^2), n_\mu(x, \hat{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, \dots) \neq 0 \quad (3.84)$$

where  $\{n_1^2, n_2^2, n_3^2\}$  are the *space isounits* in dimensionless form and  $n_4^2$  is the *time-isounit* also in dimensionless form (the  $n$ 's are also called in the literature the *characteristic functions* of the isominkowski space). The isocoordinates  $\hat{x}$  must therefore be referred to the above isounits.

Recall that "the universal constancy of the speed of light"  $c_0$  is a philosophical abstraction because the speed of light is constant *only in vacuum* (exterior problem) and becomes locally varying within physical media (interior problem). For instance, the speed of electromagnetic waves propagating in our atmosphere varies with the density to assure yet different values in water, glass, oil, etc.

The rather frequent reduction of light in interior conditions to photons in second quantization scattering through molecules is here ignored because, e.g., the speed within our atmosphere of radio waves of 1 m wavelength must be first represented *classically* before operator theories in first, and then second, quantization can have sense.

According to our basic assumptions, the Minkowskian geometry is *exactly valid* for exterior conditions in vacuum, where it provides a "direct geometrization," that is, a *geometrization via the metric  $\eta$* , of the constant speed  $c_0$  as well as of the *homogeneity and isotropy* of the medium in which it propagates (ether).

The objective of the isominkowskian geometry is to provide a direct geometrization, this time *via the isometric  $\hat{\eta}$* , of interior conditions, e.g.,

electromagnetic waves propagating with locally varying speeds within *inhomogeneous and anisotropic physical media* (with the understanding that the background ether remains homogeneous and isotropic).

In fact, the basic isoinvariant characterized by isounit (3.43) is given by

$$x^{\hat{2}} = (x^1 x^1 / n_1^2 + x^2 x^2 / n_2^2 + x^3 x^3 / n_2^2 - t t c_0^2 / n_4^2) \times \hat{I} \quad (3.85)$$

The above invariant does indeed permit the direct geometrization of locally varying speeds of light  $c = c_0 / n_4$  where  $n_4$  is the familiar index of refraction.<sup>(4)</sup>

The same invariant (3.85) also permits a direct geometrization of the inhomogeneity and anisotropy of the medium considered. In fact, the index of refraction is complemented in isoinvariant (3.85) by the space-counterparts  $n_k^2$ , which emerge from mere space-time symmetrization, or just application of the Lorentz transforms. As one can see, the geometry based on isointerval (3.85) is *inhomogeneous*, e.g., because of the local variation of the density represented via the dependence of the isometric  $\hat{\eta}$  in the local coordinates, and *anisotropic*, e.g., because of a preferred direction in the medium caused by an intrinsic angular momentum and represented via different characteristic functions  $\eta_\mu$ .

The preservation of the Einsteinian axioms despite variable speeds of light is ensured by the isominkowskian geometry. In fact, jointly with the deformation of the speed  $c_0 \rightarrow c = c / n_4$  we have the deformation of the corresponding time unit in the *inverse* amount. This leaves  $c_0$  as the maximal causal speed in isospace  $\hat{M}$ . The same occurrence holds for the space components.

More specifically, the light cone in  $\hat{M}$ , called *isolight cone*,<sup>(22a)</sup> is a perfect cone *identical* to the conventional light cone in  $M$ , including its characteristic angle (which sets the value  $c_0$ ). Under lifting  $M \rightarrow \hat{M}$  each axis  $1_\mu$ , of the original cone is deformed by the amount  $1_\mu^\mu \rightarrow \hat{I}_\mu^\mu = n_\mu^{-2}$ , while the related units are deformed by the inverse amount  $I_\mu^\mu \rightarrow \hat{I}_\mu^\mu = n_\mu^2$ , thus preserving the perfect cone in isospace. The locally varying speed of light  $c = c_0 / n_4$  and the deformed cone appear only in the *projection* of  $\hat{M}$  on  $M$ . For a proof of these properties (which requires the use of isotrigonometric and isohyperbolic functions) we refer the interested reader for brevity to Ref. 22a.

According to an occurrence similar to the iso-self-scalarly of the Hilbert space, Eq. (3.19), and of the fundamental symplectic structure, Eq. (3.31), the *conventional* Minkowskian line element admits the following *new invariance law*, here introduced apparently for the first time,

$$\eta \rightarrow \hat{\eta} = n^{-2} \times \eta, \quad I \rightarrow \hat{I} = n^2, \quad n \in R, n \neq 0 \quad (3.86a)$$

$$\begin{aligned}
 (x-y)^2 &= [(x-y)^\mu \times \eta_{\mu\nu} \times (x-y)^\nu] \times I \\
 &\equiv [(x-y)^\mu \times (n^{-2} \times \eta_{\mu\nu}) \times (x-y)^\nu] \times \hat{I} = (x-y)^{\hat{2}} \\
 &= [(x-y)^\mu \times \hat{\eta}_{\mu\nu} \times (x-y)^\nu] \times \hat{I} = (x-y)^{\hat{2}} \quad (3.86b)
 \end{aligned}$$

also called *iso-self-scalarity*. The new invariance illustrates again the axiom-preserving character of the isotopies, this time, for relativistic theories. In fact, law (3.86) establishes that *the isotopies are admitted by the invariance of the conventional Minkowskian line element*.

Invariance (3.86) represents a direct geometrization of *homogeneous and isotropic physical media* such as water, for which  $n_1 = n_2 = n_3 = n_4 = n$ , and permits a resolution of the ambiguities of special relativity when its applicability is extended from the vacuum to physical media. In fact, isoinvariant (3.86) represents the physical speed of light in water,  $c = c_0/n < c_0$ , while the maximal causal speed remains  $c_0$ . This permits the causal interpretation of electrons traveling in water faster than the local speed of light (Cerenkov light), and provides the correct relativistic addition of speeds (which in the conventional treatment on  $M$  does not permit the recovering of the speed of light in water as the sum of two speeds of light, while jointly assuming  $c_0$  as the maximal speed to salvage causality). For these and related aspects we refer the interested reader to Ref. 22b. Independent studies can be found in Ref. 5g.

Again, the reader should not be surprised that invariance (3.86) escaped attention throughout this century, because its discovery required the prior identification of *new numbers* with arbitrary unit.<sup>(26)</sup>

The maximal possible isolinear isosymmetry of separation (3.83b) has been constructed by Santilli in Ref. 4 under the name of *isopoincaré symmetry*  $\hat{P}(3.1)$ , and it is called the *Poincaré-Santilli isosymmetry* (see Refs. 5, 23c and references quoted therein). In particular, the isotopies of rotations have been studied in Ref. 4b, those of  $SU(2)$ -spin in Ref. 4c, those of the Lorentz symmetry in Ref. 4a, those of the Poincaré symmetry in Ref. 4d, and those of the spinorial covering of Poincaré in Ref. 4e, with a general study in Ref. 22 including initial studies of isorepresentations.

The isosymmetry  $\hat{P}(3.1)$  is essentially given by the image of the conventional symmetry  $P(3.1)$  under the lifting of the unit  $I \rightarrow \hat{I}$ . Since  $\hat{I}$  is positive-definite,  $\hat{P}(3.1)$  is locally isomorphic to  $P(3.1)$  by conception and construction. Nevertheless, it provides the invariance of the most general possible line element in isospace, Eq. (3.83b). By recalling the direct universality of the latter for all possible signature-preserving nonlinear integro-differential deformations of the Minkowski space, we can anticipate from the outset a corresponding direct universality of the Poincaré-Santilli isosymmetry.

The explicit construction of the isosymmetry  $\hat{P}(3.1)$  is one of the primary applications of the Lie-Santilli isothory, and can be summarized as follows. By using the same convention as those of Sec. 3.8, we have the following isodifferential, derivatives and related properties on isospace  $\hat{M}$  (see Sec. 3.8 for conventions),

$$\hat{d}\hat{x}^\mu = \hat{I}^\mu \times d\hat{x}^\nu, \quad \hat{\partial}_\mu = \hat{\partial}/\hat{\partial}\hat{x}^\mu = \hat{T}_\mu^\nu \times \partial_\nu = \hat{T}_\mu^\nu \times \partial/\partial\hat{x}^\nu \quad (3.87a)$$

$$\hat{\partial}\hat{x}^\mu/\hat{\partial}\hat{x}^\nu = \delta_\nu^\mu, \quad \hat{\partial}\hat{x}_\mu/\hat{\partial}\hat{x}^\nu = \hat{\eta}_{\mu\nu}, \quad \hat{\partial}\hat{x}^\mu/\hat{\partial}\hat{x}_\nu = \hat{\eta}^{\mu\nu}, \quad \hat{\partial}\hat{x}_\mu/\hat{\partial}\hat{x}_\nu = \delta_\mu^\nu \quad (3.87b)$$

$$\hat{d}\hat{p}_\mu = \hat{T}_\mu^\nu \times d\hat{p}_\nu, \quad \hat{\partial}/\hat{\partial}\hat{p}_\mu = \hat{I}^\mu \times \partial/\partial\hat{p}_\nu \quad (3.87c)$$

The conventional relativistic four-momentum operator  $p_\mu \times |\psi\rangle = -i\partial_\mu |\psi\rangle$  is lifted into the *relativistic iso-four-momentum* with related *fundamental relativistic isocommutation rules*

$$\hat{p}_\mu \hat{x} |\hat{\psi}\rangle = \hat{p}_\mu \times \hat{T} \times |\hat{\psi}\rangle = -i\hat{\partial}_\mu |\hat{\psi}\rangle = -i\hat{T}_\mu^\nu \times \partial_\nu |\hat{\psi}\rangle \quad (3.88a)$$

$$\begin{aligned}
 [\hat{x}_\mu, \hat{p}_\nu] \hat{x} |\hat{\psi}\rangle &= (\hat{x}_\mu \times \hat{T} \times \hat{p}_\nu - \hat{p}_\nu \times \hat{T} \times \hat{x}_\mu) \times \hat{T} \times |\hat{\psi}\rangle \\
 &= i\hat{N}_{\mu\nu} \hat{x} |\hat{\psi}\rangle = i\hat{\eta}_{\mu\nu} \times |\hat{\psi}\rangle \quad (3.88b)
 \end{aligned}$$

$$\hat{N}_{\mu\nu} = \hat{\eta}_{\mu\nu} \times \hat{I} \in \hat{R}, \quad \hat{\eta}_{\mu\nu} \in R \quad (3.88c)$$

where the quantities  $\hat{N}_{\mu\nu}$  represent the realization of  $\hat{N} = (\hat{N}_{\mu\nu})$  as an *isomatrix*,<sup>(23b)</sup> i.e., a matrix whose elements are isofunctions in  $\hat{R}$ .

The original generators and parameters of  $\hat{P}(3.1)$  are preserved unchanged under isotopies (Sec. 3.6) and we write in standard notion

$$X = \{X_k\} = \{M_{\mu\nu}, p_\alpha\}, \quad M_{\mu\nu} = x_\mu \times p_\nu - x_\nu \times p_\mu \quad (3.89a)$$

$$w = \{w_k\} = \{(\theta, v), a\} \in R, \quad k = 1, 2, \dots, 10, \quad \mu, \nu = 1, 2, 3, 4 \quad (3.89b)$$

by keeping in mind that the  $w$ 's can be rewritten as isoscalar  $\hat{w} = w \times \hat{I}$  for mathematical completeness, yet  $\hat{w} \hat{x} X \equiv w \times X$ .

The *connected Poincaré-Santilli isogroup*  $\hat{P}_0(3.1) = S\hat{O}(3.1) \hat{x} (3.1)$ , where  $S\hat{O}(3.1)$  is the *connected Lorentz-Santilli isogroup*<sup>(4a)</sup> and  $\hat{T}(3.1)$  is the group of *isotranslations* on  $\hat{M}$ ,<sup>(4d)</sup> can be written via the *isoexponentiation* (3.24)

$$\hat{P}_0(3.1): \hat{A}(\hat{w}) = \prod_k \hat{e}^{iX \hat{x} \hat{w}} = \left( \prod_k e^{iX \times T \times w} \right) \times \hat{I} = \tilde{A}(w) \times \hat{I} \quad (3.90)$$

under which the isotransforms can be written

$$x' = \hat{A}(\hat{w}) \hat{x} x = [\tilde{A}(\hat{w}) \times \hat{I}] \times \hat{T} \times x = \tilde{A}(w) \times x \quad (3.91)$$

The preservation of the original dimension is ensured by the *isotopic Baker–Campbell–Hausdorff theorem*<sup>(3a)</sup> (see also Refs. 3d, 5c, 5d).

To identify the isoalgebra  $\hat{p}_0(3.1)$  of  $\hat{P}_0(3.1)$ , we use the isodifferential calculus on  $\hat{M}$  which yields the *isocommutation rules*<sup>(4a, 4d)</sup>

$$[\hat{M}_{\mu\nu}, \hat{M}_{\alpha\beta}] = i(\hat{\eta}_{\nu\alpha} \times \hat{M}_{\mu\beta} - \hat{\eta}_{\mu\alpha} \times \hat{M}_{\nu\beta} - \hat{\eta}_{\nu\beta} \times \hat{M}_{\mu\alpha} + \hat{\eta}_{\mu\beta} \times \hat{M}_{\nu\alpha}) \quad (3.92a)$$

$$[\hat{M}_{\mu\nu}, \hat{p}_\alpha] = i(\hat{\eta}_{\mu\alpha} \times \hat{p}_\nu - \hat{\eta}_{\nu\alpha} \times \hat{p}_\mu), \quad [\hat{p}_\alpha, \hat{p}_\beta] = 0 \quad (3.92b)$$

where  $[\hat{A}, \hat{B}] = \hat{A} \times \hat{T} \times \hat{B} - \hat{B} \times \hat{T} \times \hat{A}$ .

The *isocasimir invariants* are then given by<sup>(4d)</sup>

$$C^{(0)} = \hat{I} = \hat{T}^{-1} \quad (3.93a)$$

$$C^{(2)} = \hat{p}^2 = \hat{p}_\mu \hat{\times} \hat{p}^\mu = \hat{\eta}^{\mu\nu} \times \hat{p}_\mu \hat{\times} \hat{p}_\nu \quad (3.93b)$$

$$C^{(4)} = \hat{W}_\mu \hat{\times} \hat{W}^\mu, \quad \hat{W}_\mu = \varepsilon_{\mu\alpha\beta\rho} \hat{M}^{\alpha\beta} \hat{\times} \hat{p}^\rho \quad (3.93c)$$

The local isomorphism  $\hat{p}_0(3.1) \approx p_0(3.1)$  is ensured by the positive-definiteness of  $\hat{T}$ . In fact, the use of the generators in the form  $M_\nu^\mu = x^\mu p_\nu - x^\nu p_\mu$  would yield the *conventional* structure constants under a *generalized* Lie product, as one can verify.

The explicit form of the isotransforms can be easily computed from isoexponentiations (3.24) via the knowledge of the conventional transforms and the given deformation of the Minkowski metric, i.e., the given isotopic components  $\hat{T}_{\mu\nu}$  of iso-invariant (3.83b). Moreover, the convergence of the original exponentiation into a finite form plus the assumed topological restrictions on  $\hat{T}$  assure the convergence of the isoexponentiation.

The *isrotations* were first computed along these lines in Ref. 4b and can be written in the  $(\hat{x}, \hat{y})$ -plane

$$\hat{x}' = \hat{x} \times \cos(\hat{T}_{11}^{1/2} \times \hat{T}_{22}^{1/2} \times \theta_3) - \hat{y} \times \hat{T}_{11}^{-1/2} \times \hat{T}_{22}^{1/2} \times \sin(\hat{T}_{11}^{1/2} \times \hat{T}_{22}^{1/2} \times \theta_3) \quad (3.94a)$$

$$\hat{y}' = \hat{x} \times \hat{T}_{11}^{1/2} \times \hat{T}_{22}^{-1/2} \times \sin(\hat{T}_{11}^{1/2} \times \hat{T}_{22}^{1/2} \times \theta_3) + \hat{y} \cos(\hat{T}_{11}^{1/2} \times \hat{T}_{22}^{1/2} \times \theta_3) \quad (3.94b)$$

(see Ref. 22b for general isrotations in all three Euler angles and related isorepresentations).

As one can verify, isotransforms (3.94) leave invariant all infinitely possible ellipsoids

$$r' \times \delta \times r = x \hat{T}_{11} x + y \hat{T}_{22} y + z \hat{T}_{33} z = \text{inv.} \quad (3.95)$$

However, the above ellipsoids become perfect spheres in isospace, the isospheres  $r^2 = (r' \times \delta \times r) \times \hat{I}$  of Sec. 3.7. This isosphericity is the geometric origin of the isomorphism  $\hat{O}(3) \approx O(3)$ , as well as of the preservation of the *rotational* invariance for the *ellipsoidal deformations* of the sphere.<sup>(4b)</sup> In fact, the *isogeodesic* of  $S\hat{O}(3)$  are perfect circles, only defined in isospaces.<sup>(22a, 23b)</sup>

To understand the above occurrence, the reader should note that conventional relativistic theories have *only one interpretation*, that on  $M$ . Isorelativistic theories have instead *two different interpretations*, that in isospace  $\hat{M}$  as well as its projection in the original space  $M$ .

The connected *Lorentz–Santilli isosymmetry*  $S\hat{O}(3.1)$  is characterized by the isorotations and the *isoboosts* first introduced in Ref. 4a, which can be written in the  $(\hat{x}^3, \hat{x}^4)$ -plane

$$\hat{x}^{1'} = \hat{x}^1, \quad \hat{x}^{2'} = \hat{x}^2 \quad (3.96a)$$

$$\begin{aligned} \hat{x}^{3'} &= \hat{x}^3 \times \sinh(\hat{T}_{33}^{1/2} \times \hat{T}_{44}^{1/2} \times v) - \hat{x}^4 \times \hat{T}_{33}^{-1/2} \times \hat{T}_{44}^{1/2} \times \cosh(\hat{T}_{33}^{1/2} \times \hat{T}_{44}^{1/2} \times v) \\ &= \hat{y} \times (\hat{x}^3 - \hat{T}_{33}^{-1/2} \times \hat{T}_{44}^{1/2} \times \hat{\beta} \times \hat{x}^4) \end{aligned} \quad (3.96b)$$

$$\begin{aligned} \hat{x}^{4'} &= -\hat{x}^3 \times \hat{T}_{33}^{1/2} \times c_0^{-1} \times \hat{T}_{44}^{-1/2} \times \sinh(\hat{T}_{33}^{1/2} \times \hat{T}_{44}^{1/2} \times v) \\ &\quad + \hat{x}^4 \times \cosh(\hat{T}_{33}^{1/2} \times \hat{T}_{44}^{1/2} \times v) \\ &= \hat{y} \times (\hat{x}^4 - \hat{T}_{33}^{1/2} \times \hat{T}_{44}^{-1/2} \times \hat{\beta} \times \hat{x}^3) \end{aligned} \quad (3.96c)$$

where

$$\hat{\beta} = (v_k \times \hat{T}_{kk} \times v_k / c_0 \times \hat{T}_{44} \times c_0)^{1/2}, \quad \hat{\gamma} = (1 - \hat{\beta}^2)^{-1/2} \quad (3.97)$$

Note that the above isotransforms are generally *nonlinear nonlocal and noncanonical in their projection in M* precisely as expected. The above transforms are, however, *isolinear, isolocal, and isocanonical on  $\hat{M}$* . Moreover, the isotransforms are formally similar to the Lorentz transforms, as also expected from their isotopic character. This completion of these occurrences via the isolight cone in isospace confirms the local isomorphism  $S\hat{O}(3.1) \approx SO(3.1)$ .<sup>(4)</sup>

The *isotranslations* can be written<sup>(4d)</sup>

$$x' = (\hat{e}^{ip \times a}) \hat{\times} x = x + a \times A(x, \psi, \dots), \quad p' = (\hat{e}^{ip \times a}) \hat{\times} p = p \quad (3.98a)$$

$$A_\mu = \hat{T}_{\mu\mu}^{1/2} + a^\alpha \times [\hat{T}_{\mu\mu}^{1/2}, \hat{p}_\alpha] / 1! + \dots \quad (3.98b)$$

The *isoinversions* are given by<sup>(4a, 4b)</sup>

$$\hat{\pi} \hat{\times} x = \pi \times x = (-r, x^4), \quad \hat{\ell} \hat{\times} x = \tau \times x = (r, -x^4) \quad (3.99)$$

where  $\hat{\pi} = \pi \times \hat{I}$ ,  $\hat{\ell} = \tau \times \hat{I}$ , and  $\pi, \tau$  are the conventional inversion operators.

At this point the following novelty occurs. It has been believed throughout this century that the maximal symmetry of the Minkowskian line element is ten-dimensional. The isotopies add *two* new symmetries to the conventional setting. The first is invariance (3.86) with *iso-self-scalar isotransforms*

$$\hat{x} \rightarrow \hat{x}' = n^{-1} \times \hat{x}, \quad \hat{l} \rightarrow \hat{l}' = n^2 \times \hat{l} \quad (3.100)$$

The second new invariance, called *iso-self-duality*, is introduced in Sec. 14.

For the isotopies of the spinorial covering of the Poincaré symmetry we refer the interested reader to Ref. 4d. For initial studies on the isorepresentations one may inspect Ref. 22b. The isofield equations characterized by isocasimirs (3.93) are studied in Ref. 22b.

Note that *there is nothing to compute for the desired invariance, but merely plotting isotransforms (3.94)–(3.99) on isoinvariant (3.83b) for given deformations  $\hat{T}_{\mu\nu}$  of the, Minkowski metric elements  $\eta_{\mu\nu}$ .*

The isominkowskian geometry and related isopoincaré symmetry permit the achievement of the main objective of the studies herein considered, *the preservation of the axioms of the special relativity in isospace.*<sup>(4, 22, 23)</sup> In fact, the isotopic and conventional structures coincide by conception and construction at the abstract realization-free level. Applications are evidently open to scientific debate, but *criticisms on the axiomatic structure of relativistic isotopic theories are de facto criticisms on the structure of Einstein's axioms.*

The above *axiom-preserving* results should be compared with the *departures* from Einstein's axioms which are necessary for deformations.<sup>(10)</sup> At any rate, the deformed Minkowski isospaces such as those of Ref. 10a admit the isogroup  $\hat{P}(3.1)$  as their symmetry.

A primary function of the Poincaré–Santilli isosymmetry for which it was conceived<sup>(4)</sup> is the characterization of new composite systems with conventional center-of-mass trajectories and total conservation laws, yet generalized internal structure.

These new systems can be visualized as follows. The computer visualization of the conventional Poincaré symmetry (for  $N \geq 3$ ) is expected to yield a *Keplerian system*, i.e., a system of particles at large mutual distances without collisions with a nucleus, the *Keplerian nucleus*, occupied by the heaviest constituent. This visual computerization confirms the exact validity of the Poincaré symmetry for the atomic structure, as expected.

A computer visualization of the Poincaré–Santilli isosymmetry (also for  $N \geq 3$ ) is instead expected to yield a different system in which all constituents are in mutual contact with each other and the center can be

occupied by an arbitrary constituent<sup>(3b, 3d, 22)</sup> owing to the presence of contact interactions represented by the isounit  $\hat{I} \neq I$ . These features are precisely the main characteristics of the interior systems for which isotopic theories were conceived in the first place,<sup>(3)</sup> such as the structure of Jupiter, of a star, or, for that matter, of a nucleus or a hadron. In fact, all these systems lack a Keplerian center.

When treated with conventional methods, the loss of the Keplerian center evidently implies a necessary *breaking* of the conventional Poincaré symmetry. Our isotopic methods permit instead the reconstruction of the *exact* Poincaré symmetry in isospace over isofields, thus permitting a significant unity of thought for both exterior and interior systems.

Note that the Poincaré symmetry is reconstructed as exact under the most general possible *nonlinear, nonlocal, and nonhamiltonian internal interactions.*

The isorelativistic studies outlined above are completed by the *isotopies of the special relativity*, or *isospecial relativity*<sup>(22)</sup> for short, which include a step-by-step isotopic lifting of the various aspects of the special relativity, including the isotopies of the Doppler shift, time dilation, space contraction, etc., whose outline is here omitted for brevity.

### 3.11 Isotopic Formulation and Quantization of Gravity

As is well known, gravitation is currently formulated on Riemannian spaces  $\mathfrak{R}(x, g, R)$  with (well-behaved) and symmetric metrics  $g = g(x) = g'$  and line element  $x^2 = x' \times g \times x$  on the reals  $R(n, +, \times)$ .

In Sec. 2 we have presented a main drawback of the above formulation, the fact that *the basic space-time unit  $I = \text{Diag}(1, 1, 1, 1)$  of the Riemannian geometry is not preserved by the symmetries of the line element* (Theorem 3). The occurrence is a consequence of the fact that the Riemannian metric is a noncanonical deformation of the Minkowskian one. Therefore, the Riemannian geometry suffers of all the problematic aspects of noncanonical theories.

The proof of Theorem 3 can now verified with the preceding results. In fact, the universal symmetry of all possible line element  $x^2 = x' \times g \times x$  is the Poincaré–Santilli isosymmetry  $\hat{P}(3.1)$ , only specialized to the particular case  $\hat{\eta} = \hat{\eta}(x) = g(x)$ . But its isotransforms are nonunitary-isounitary. The lack of invariance of the basic space-time units whenever the curvature is non-null then follows as in Theorem 1.

The above occurrences are not of marginal relevance, because they have rather serious implications, such as a confirmation of the historical ambiguities in the compatibility of general and special relativities, lack of achievement of a form of quantum gravity as consistent as relativistic



quantum mechanics, inability to reach a unification of gravity with other interactions, problematic aspect in the applicability of both classical and operator theories to measurements, and others.

New possibilities for resolving in due time this impasse are offered by the isotopies. Even though the studies are at their beginning, mentioning them may be of relevance for an overall view.

The main hypothesis of the *isotopic formulation of gravity*, also called *isogravitation* (first presented at the VII M. Grossmann Meeting on General Relativity at Stanford University in July 1994;<sup>(34)</sup> see also Refs. 34b, 34c) is the representation of gravity in the isominkowskian, rather than the Riemannian space. In fact, all  $(3 + 1)$ -dimensional Riemannian metrics  $g(x)$  admit the isotopic factorization

$$g(x) = \hat{T}_{gr}(x) \times \eta \quad (3.101)$$

where  $\eta$  is the conventional Minkowski metric, and  $\hat{T}$  is necessarily positive-definite (from the locally Minkowskian character of  $(3 + 1)$ -dimensional Riemannian spaces).

*Isogravitation for exterior problems in vacuum* is then characterized by the following identifications on isominkowski space  $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$

$$\hat{\eta}(x) \equiv \hat{T}_{gr}(x) \times \eta \equiv g(x) \quad (3.102a)$$

$$\hat{I}_{gr} = [\hat{T}_{gr}(x)]^{-1} \quad (3.102b)$$

where  $\hat{I}_{gr}$  is called the *gravitational isounit* and  $\hat{T}_{gr}$  the *gravitational isotopic element*.

The above assumption has the following main implications:

(1) *Isogravitation* possesses a universal symmetry, the Poincaré–Santilli isosymmetry  $\hat{P}_{gr}(3.1)$  for isounit (3.102b). This resolves a historical difference between the special and general relativities, because the former is indeed equipped with the universal symmetry  $P(3.1)$ , while the latter is not. Note the *necessity* of the isominkowskian formulation  $\mathfrak{R}(x, g, R) \rightarrow \hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ ,  $\hat{\eta} \equiv g$ . In fact, isotopic methods cannot be even defined on Riemannian spaces.

(2) *Isogravitation permits a unified treatment of relativistic and gravitational phenomena*. In fact, the two profiles can be formulated via the *same* abstract axioms, those of the *special* (rather than the general) relativity, and can be merely differentiated by the selected realization of the unit. When the conventional unit is assumed to be  $I$ , one has the conventional special relativity, while the assumption of isounit (3.102b) implies a gravitational

formulation. Evidently, this unification is a consequence of the prior unification of the underlying carrier spaces and symmetries. The unification is not a mere mathematical curiosity, because it resolves known ambiguities in the compatibility of the general with the special relativities, e.g., in regard to the total conservation laws. In fact, the mere visual inspection of the generators of  $\hat{P}(3.1)$ , Eq. (3.89a), establishes their conserved character as well as the identity of the relativistic and gravitational conservation laws without any possible ambiguity.

(3) *Isogravitation possesses an invariant basic unit at both classical and operator levels*. This is also a consequence of the universal Poincaré–Santilli isosymmetry  $\hat{P}(3.1)$ . This permits an unambiguous comparison of the predictions of the theory with actual measurements, besides being necessary for the achievement of real compatibility between the general and special relativities.

(4) *Isogravitation permits the preservation of Einstein's field equations and related experimental verifications*. This is a peculiarity of the isominkowskian geometry which is *flat in isospace*, but its metric is assumed to coincide with the Riemannian metric. As a result, the formalism of the Riemannian geometry (covariant derivative, connection, etc.) can be preserved in isospace  $\hat{M}(x, \hat{\eta}, \hat{R})$  and merely referred to a different field (see Ref. 34d for details).

(5) *Isogravitation admits a new operator form called "operator isogravity," which is as axiomatically consistent as relativistic quantum mechanics*. In fact, the relativistic operator isothory of the preceding section can be specialized to the gravitational isotopic element and isounit (3.102), yielding an operator theory which verifies the same abstract axioms of relativistic quantum mechanics. This may resolve known problematic aspects of conventional quantum gravity. Note that the local isocommutativity of the linear momenta, Eq. (3.92b), confirms the isoflat character of isogravity.

The *interior isogravitation* is given by the above theory with isotopic elements and isounits of the general type

$$\hat{T} = \hat{T}_0(x, \hat{x}, \hat{x}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, \dots) \times \hat{T}_{gr}(x) \quad (3.103a)$$

$$\hat{I}_{gr} = [\hat{T}_{gr}(x)]^{-1} \times [\hat{T}_0(x, \hat{x}, \hat{x}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, \dots)]^{-1} \quad (3.103b)$$

where  $\hat{T}_0$  represents internal nonlinear, nonlocal, and nonlagrangian effects. The latter theory can then provide a direct gravitational geometrization of the locally varying speed of light in interior media.

As an example, a simple lifting of the Schwarzschild metric for isounits can be written

$$\hat{T}_{kk} = 1/(1 - M/r) \times n_k^2, \quad \hat{T}_{44} = (1 - M/r) \times c_0^2/n_4^2 \quad (3.104)$$

which evidently permits a direct geometric characterization of the locally varying speed  $c = c_0/n_4$  for light propagating within inhomogeneous and anisotropic interior gravitational media. No such direct geometrization is evidently possible in Riemannian spaces.

Intriguingly, all main characteristics (1)–(5) above remain valid for interior conditions. This is due to the unrestricted functional dependence of the isounit.

We can therefore conclude by saying that, the conventional Poincaré symmetry applies for systems which are: linear, local, potential, canonical, unitary, exterior, relativistic, and classical or quantum mechanical. The broader Poincaré–Santilli isosymmetry holds instead for systems which are linear or nonlinear, local or nonlocal, potential or nonpotential, canonical or noncanonical, unitary or nonunitary, exterior or interior, classical or quantum mechanical, and relativistic or gravitational.

### 3.12. Genotopic Formulations

The isotopies were proposed in Ref. 3 as a particular case of the broader genotopies. The main idea is that the isotopies are *axiom-preserving*, while the genotopies (also from the Greek meaning of the word) are *axiom-inducing*, that is, they imply the abandonment of the original axiomatic structure in favor of a broader structure under the condition of admitting the isotopies as a particular case.

Even though the genotopies are considerably less developed than the isotopies at this writing, their indication may be of value because of insufficiencies of the isotopies in certain interior problems, e.g., those of irreversible type.<sup>(13)</sup>

The basic assumption of the genotopies is the relaxation of the Hermiticity of the isounit,  $\hat{I} \neq \hat{I}^\dagger$ , and its realization via nowhere singular, real-valued, and nonsymmetric  $N \times N$  matrices (in which case the transpose  $t$  is sufficient for conjugation). Besides generalized products and related units, the genotopies require the additional ordering of the multiplication to the right and to the left<sup>(3a, 3b, 26)</sup>

$$\hat{A} \hat{\times} \hat{B} = \hat{A} \times \hat{T} \times \hat{B} \rightarrow \begin{cases} \hat{A} > \hat{B} = \hat{A} \times \hat{T} > \times \hat{B}, & \hat{I} > = (\hat{T} >)^{-1} & (3.105a) \\ \hat{A} < \hat{B} = \hat{A} \times < \hat{T} \times \hat{B}, & < \hat{I} = (< \hat{T})^{-1} & (3.105b) \end{cases}$$

with interconnecting map  $\hat{I} > = (< \hat{T})'$ , which are usually assumed to represent *motion forward and backward in time*, respectively.

This essentially implies a duplication of the isotopic methods one per each direction of time, and requires the construction of *genofields*, *genospaces*, *genoalgebras*, *genogeometries*, etc.<sup>(22)</sup>

As an illustration, the *forward genofield*<sup>(26)</sup>  $\hat{C} > (\hat{e} >, +, >)$  is the ring of *genonumbers*  $\hat{e} > = c \times \hat{I} >$  with conventional sum  $+$  and additive unit 0, and genoproduct (3.105a). A similar situation occurs for the *backward genofield*  $< \hat{C} (< \hat{e}, +, <)$ . For each genofield the genomultiplication is commutative (for complex genonumbers), although the result is different for different orderings, i.e.,  $c > d = d > c$ ,  $c < d = d < c$ , but  $c > d \neq c < d$ . The important point is that each genofield  $\hat{C} > (\hat{e} >, +, >)$  and  $< \hat{C} (< \hat{e}, +, <)$  verifies the axioms of a field<sup>(23)</sup> under a selected ordering of the multiplication.

Similarly, the *forward genospaces* are characterized by the following *forward geometrics* and *genounits*:

$$\hat{S} > (\hat{x} >, \hat{g} >, \hat{R} >): \hat{x} > = x \times \hat{I} >, \quad \hat{g} > = \hat{T} > \times g, \quad \hat{I} > = (\hat{T} >)^{-1} \quad (3.106a)$$

$$x^{\hat{2}} > = (x' \times \hat{g} > \times x) \times \hat{I} > \in \hat{R} > \quad (3.106b)$$

which also preserve the original axioms despite the loss of symmetric character of the metric. A similar situation exists for the *backward genospaces*  $< \hat{S} (< \hat{x}, < \hat{g}, < \hat{R})$ .

An intriguing consequence is that, contrary to rather popular beliefs, the Riemannian axioms admit a nonsymmetric metric  $\hat{g} > = \hat{T} > \times g \neq \hat{g} >'$ ,  $g = g'$ , provided that the underlying unit  $\hat{I} >$  is the inverse of the nonsymmetric component of the metric  $\hat{T} >$  (see Refs. 22, 23 for details). Similar occurrences hold for all remaining genostructures.

The fundamental physical theory is the genotopy of *Newtonian mechanics on the forward genospace*

$$\hat{S} > (\hat{t} >, \hat{x} >, \hat{\theta} >) = \hat{E} > (\hat{t} >, \hat{R} >_t) \times \hat{E} > (\hat{x} >, \hat{\delta} >, \hat{R} >) \times \hat{E} > (\hat{\theta} >, \hat{\delta} >, \hat{R} >) \quad (3.107)$$

with *forward genonewton's equations*, first identified in Ref. 23b,

$$\hat{m} > > \frac{d^{\hat{2}} \hat{\theta} >}{d\hat{t} >} - \frac{\hat{a} > \hat{\delta} > \hat{O}(\hat{t}, \hat{x}, \hat{\theta})}{d\hat{t} >} \frac{\hat{\delta} > \hat{O}(\hat{t}, \hat{x}, \hat{\theta})}{\hat{\delta} \hat{\theta}^k >} + \frac{\hat{\delta} > \hat{O}(\hat{t}, \hat{x}, \hat{\theta})}{\hat{\delta} \hat{x}^k >} = 0 \quad (3.108)$$

and corresponding *backward genonewton equations* here omitted for brevity.

The next important theory is the genotopy of analytic mechanics with *forward genohamilton's equations* [loc. cit.]

$$\hat{d}^> \hat{b}^\mu / \hat{d}\hat{t}^> = \hat{\omega}^{>\mu\nu} \hat{\delta}^> \hat{H} \hat{\delta}^> \hat{b}^\nu \quad (3.109a)$$

$$\hat{\omega}^{>\mu\nu} = \omega^{\mu\nu} \times \hat{I}^> \equiv S^{\mu\nu} = \omega^{\mu\nu} + s^{\mu\nu} \quad (3.109b)$$

$$\hat{I}^> = \omega_{\alpha\beta} \times s^{\alpha\beta} \quad (3.109c)$$

and related genotopies of the Hamilton–Jacobi equations, with corresponding backward version.

The reader should note reformulation (3.109) of the Lie-admissible equations (1.14) which is similar to the isotopic reformulation (3.67) of Birkhoff's equations.

The genotopies of quantization then yield the following *genoschrödinger equations*:<sup>(23b)</sup> (first submitted by Mignani and by Myung and Santilli in 1982; see Ref. 22b for literature)

$$i\hat{\delta}^> / \hat{\delta}^> |\hat{\psi}^>\rangle = \hat{H}^> |\hat{\psi}^>\rangle = \hat{H} \times \hat{T}^> \times |\hat{\psi}^>\rangle = \hat{E}^> |\hat{\psi}^>\rangle = E \times |\hat{\psi}^>\rangle \quad (3.110a)$$

$$\hat{p}_k^> |\hat{\psi}^>\rangle = \hat{p}_k \times \hat{T}^> \times |\hat{\psi}^>\rangle = -i\hat{T}_i^> \times \partial / \partial r^i |\hat{\psi}^>\rangle \quad (3.110b)$$

with corresponding *Lie-admissible genoheisenberg's equations*<sup>(1, 3b, 22, 23b)</sup>

$$i\hat{d}^> \hat{A} / \hat{d}\hat{t}^> = (A, \hat{H}) = \hat{A} < \hat{H} - \hat{H} > \hat{A} \times < \hat{T} \times \hat{H} - \hat{H} \times \hat{T} > \hat{A} \quad (3.111)$$

where each ordered product must be referred to its corresponding genofields, genospaces, etc., and exponentiated form (1.5b).

A dominant feature of the genotopic methods is that they imply the *time rate of variation of the energy*,

$$i\hat{d}^> H / \hat{d}\hat{t}^> = H \times ( < T - T > ) \times H \neq 0 \quad (3.112)$$

Genotopic methods were therefore proposed<sup>(3a, 3b)</sup> to represent *open-non-conservative systems under the most general possible external interactions*.

It can be proved that the product  $A < B | < \xi - B > A | \xi >$  verifies the *Lie-admissible axioms*<sup>(1, 2, 3)</sup> when computed in conventional spaces over conventional fields, but it verifies the *Lie axioms* when interpreted as a genobimodule in which each ordered product is computed in the related genospace over the related genofield.<sup>(22, 23)</sup> The latter property permits a further, step-by-step lifting of the Lie-isotopic isothory into a form called in the literature *Lie-Santilli genotheory*.<sup>(5)</sup>

Similarly, *nonconservation law* (3.112) holds only in its projection in conventional spaces over conventional fields because, when computed in genospaces over genofields, it recovers conservation. The *genosymmetries* therefore

permits the characterization of *time rate of variations of physical quantities* as originally proposed in Ref. 3a, with isosymmetries and conventional symmetries and conservation laws as particular cases.

The factorization of the *conventional* canonical Lie structure  $\omega^{\mu\nu}$  in the Lie-admissible tensor  $\hat{\omega}^{>\mu\nu}$ , Eq. (3.109b), the rescaling of all methods with respect to the new genounit  $\hat{I}^>$ , and the computation of the dynamical equations on genospaces over genofields permit the preservation of conventional axioms under genotopies, an occurrence holding at both classical and operator levels.<sup>(22, 23b)</sup>

Thus *genotopic theories are axiom-inducing in their projection in conventional spaces over conventional fields, but they are axiom-preserving when formulated in genospaces over genofields*.

As a result, the *genotopies constitute a still broader realization of "hidden variables" and "completion" of quantum mechanics much along the EPR argument* (Sec. 3.5).

In their projection on conventional spaces over conventional fields, the genotopies are particularly suited for an *axiomatization of irreversibility*, that is, its representation irrespective of whether the Hamiltonian is reversible or not. Stated in different terms, potential interactions are notoriously irreversible. The suggestion of Ref. 3 was therefore that of representing the irreversibility of the physical reality, not with a Hamiltonian or a Lagrangian (which are assumed as reversible), but rather with their dynamical equations. This is precisely the case for Eq. (3.110).

However, when formulated on genospaces over genofields, *genotopies are structurally reversible*. This permits an intriguing and novel reconciliation of the irreversibility of physical reality with the reversible structure of quantum mechanics.

In summary, isotopies are recommended for the study of *closed-isolated systems with conventional-reversible center-of-mass trajectories and generalized internal structure*, although isotopies can also represent irreversibility, e.g., via an irreversible isounit  $\hat{I}(t, \dots) \neq \hat{I}(-t, \dots)$ . The broader genotopies are instead recommended for *open-nonconservative systems due to interactions with an external term with irreversible and generalized internal structure*.

We finally mention that, besides irreversible interior problems such as black holes,<sup>(13)</sup> the genotopic methods have turned out to be particularly promising for theoretical biology.<sup>(35)</sup> In fact, biological systems are notoriously irreversible because they either grow or decay in time and, as such, they require formulations characterizing *time rate of variations* of given characteristics of size, weight, etc.

Within such a setting, conventional quantum mechanics is transparently insufficient, owing to its notorious characterization of conservation laws.

As indicated earlier, isotopic formulations can be constructed via a step-by-step *nonunitary* transforms  $U \times U^\dagger = \hat{I} \neq I$  of conventional quantum formulations. As we shall study in a subsequent paper,<sup>(3B)</sup> the genotopic formulations can be constructed via a step-by-step transform of quantum formulations of a more general type characterized by  $A \times B^\dagger = \hat{I}^\dagger$  and  $A^\dagger \times B = \hat{I}$ , where  $A$  and  $B$  are generally different nonunitary operators.

In fact, the latter transforms applied to numbers yield the genonumbers to the right  $\hat{n}^\dagger = A \times n \times B^\dagger = n \times (AB^\dagger) = n \times \hat{I}^\dagger$  and to the left  $A^\dagger \times n \times B = n \times (A^\dagger B) = n \times \hat{I}$ , with the correct genoproducts, and the same occurs for all subsequent aspects.

### 3.13. Hyperstructural Formulations

We should also indicate for completeness that the genotopies, in turn, have recently turned out to be particular cases of the *multivalued hyperstructures with a left and right unit* as proposed in Ref. 23b. In this case we have an ordering of the multiplication to the right or to the left as in the genotopies, but each genounit and genotopic element is given by an ordered *set* of values, e.g.,

$$\{I^\dagger\} = \{I_1^\dagger, I_2^\dagger, \dots, I_N^\dagger\} = \{\hat{T}^\dagger\}^{-1} = \{\hat{T}_1^\dagger, \hat{T}_2^\dagger, \dots, \hat{T}_N^\dagger\} \quad (3.113a)$$

$$\{<I\} = \{<I_1, <I_2, \dots, <I_N\} = \{<\hat{T}\}^{-1} = \{<\hat{T}_1, <\hat{T}_2, \dots, <\hat{T}_N\} \quad (3.113b)$$

with corresponding *multivalued hyperproduct*

$$\begin{aligned} \hat{A} \{>\} \hat{B} &= \hat{A} \times \{\hat{T}_1^\dagger, \hat{T}_2^\dagger, \dots, \hat{T}_N^\dagger\} \times \hat{B} \\ &= \{\hat{A} \times \hat{T}_1^\dagger \times \hat{B}, \hat{A} \times \hat{T}_2^\dagger \times \hat{B}, \hat{A} \times \hat{T}_2^\dagger \times \hat{B}, \dots, \hat{A} \times \hat{T}_N^\dagger \times \hat{B}\} \end{aligned} \quad (3.114a)$$

$$\begin{aligned} \hat{A} \{<\} \hat{B} &= \hat{A} \times \{<\hat{T}_1, <\hat{T}_2, \dots, <\hat{T}_N\} \times \hat{B} \\ &= \{\hat{A} \times <\hat{T}_1 \times \hat{B}, \hat{A} \times <\hat{T}_1 \times \hat{B}, \hat{A} \times <\hat{T}_2 \times \hat{B}, \dots, \hat{A} \times <\hat{T}_N \times \hat{B}\} \end{aligned} \quad (3.114b)$$

This implies the construction of a third layer of new mathematics called *hypermathematics*, recently studied in Ref. 23b (see page 18 of Web Site 3h for an outline with open problems). For instance, we have the *hyperfield to the right*  $\{\hat{C}^\dagger\}(\{\hat{n}^\dagger\}, +, \{>\})$  with *hypernumbers to the right*  $\{\hat{c}^\dagger\} = c \times \{I^\dagger\}$  and conventional sum and additive unit, and multivalued product (3.114a). The important point is that, despite the yet broader character, the above hyperfield satisfies all axioms of a field (via

strong identities, rather than the weak identities generally used in hyperstructures). Along similar lines we have the *hyperspaces*<sup>(23b)</sup>

$$\{S^\dagger\}(\{\hat{x}^\dagger\}, \{\hat{g}^\dagger\}, \{\hat{R}^\dagger\}): \{\hat{g}^\dagger\} = \{\hat{T}^\dagger\} \times g, \quad \{I^\dagger\} = \{\hat{T}^\dagger\}^{-1} \quad (3.115a)$$

$$\{x^{\hat{2}}\} = (x' \times \{\hat{g}^\dagger\} \times x) \times \{I^\dagger\} \in \{\hat{R}^\dagger\} \quad (3.115b)$$

which evidently represents a multiplicity of genospaces. We similarly have *hyperalgebras, hypergeometries, etc.*

The fundamental theory is the *hypernewtonian mechanics* defined on hyperconfiguration space with basic *forward hypernewton's equations*, first identified in Ref. 23b,

$$\{\hat{n}^\dagger\} \{>\} \frac{\{\hat{d}^\dagger\} \{\hat{o}_k^\dagger\}}{\{\hat{d}\hat{t}^\dagger\}} - \frac{\{\hat{d}^\dagger\} \{\hat{\sigma}^\dagger\} \{\hat{O}(t, x, \theta)^\dagger\}}{\{\hat{\sigma}\} \{\hat{o}^k\}} + \frac{\{\hat{\sigma}^\dagger\} \{\hat{O}(t, x, \theta)^\dagger\}}{\{\hat{\sigma}\} \{\hat{x}^k\}} = 0 \quad (3.116)$$

with corresponding *backward hypernewton equations* here omitted for brevity.

The next important theory is the *hyperanalytic mechanics with forward hyperhamilton's equations* [loc. cit.]

$$\{\hat{d}^\dagger\} \hat{b}^\mu / \{\hat{d}\hat{t}^\dagger\} = \{\hat{\omega}^{\mu\nu}\} \{\hat{\sigma}^\dagger\} \hat{H} / \{\hat{\sigma}\hat{b}^\nu\} \quad (3.117a)$$

$$\{\hat{\omega}^{\mu\nu}\} = \omega^{\mu\nu} \times \{I^\dagger\}, \quad \{I^\dagger\} = \omega_{\alpha\beta} \{s^{\alpha\beta}\} \quad (3.117b)$$

and related hyperlifting of the Hamilton-Jacobi equations, with corresponding backward versions.

The hyperlifting of quantization then yields the following *forward hyperschrödinger equations*<sup>(23b)</sup>

$$\begin{aligned} i\{\hat{\sigma}^\dagger\} / \{\hat{\sigma}\hat{t}^\dagger\} |\hat{\sigma}\rangle &= \hat{H} \{>\} |\hat{\psi}^\dagger\rangle = \hat{H} \times \{\hat{T}^\dagger\} \times |\hat{\psi}^\dagger\rangle \\ &= \{\hat{E}^\dagger\} \{>\} |\hat{\psi}^\dagger\rangle = \{\hat{E}^\dagger\} \times |\hat{\psi}^\dagger\rangle \end{aligned} \quad (3.118a)$$

$$\begin{aligned} \hat{p}_k \{>\} |\hat{\psi}^\dagger\rangle &= \hat{p}_k \times \{\hat{T}^\dagger\} \times |\hat{\psi}^\dagger\rangle \\ &= -i\{\hat{T}_k^{\dagger i}\} \times \partial / \partial r_i |\hat{\psi}^\dagger\rangle \end{aligned} \quad (3.118b)$$

and related *forward hyperheisenberg's equations*

$$\begin{aligned} i\{\hat{d}^\dagger\} \hat{A} / \{\hat{d}\hat{t}^\dagger\} &= (A, \hat{H}) = \hat{A} \{>\} \hat{H} - \hat{H} \{>\} \hat{A} \\ &= \hat{A} \times \{<\hat{T}\} \times \hat{H} - \hat{H} \times \{\hat{T}^\dagger\} \times \hat{A} \end{aligned} \quad (3.119a)$$

where each ordered product must be referred to its corresponding hyperfields, hyperspaces, etc.

A dominant feature of the above hyperformulations is that they imply the *time rate of variation*

$$i\{\hat{d}^>\} \hat{H}/\{\hat{d}t^>\} = \hat{H}(\{>\}) - \{\langle\} \hat{H} \neq 0 \quad (3.120)$$

It is evident that the above hypertheory is structurally nonconservative and irreversible when formulated in conventional spaces over conventional fields. The theory was suggested by this author for a characterization of systems for which the genotopies are insufficient, e.g., those of biological type.<sup>(35b)</sup>

It should be indicated that *hyperstructures are axiom-preserving when formulated on hyperspaces over hyperfields*, in much of the same occurrence as that for genotheories (Sec. 3.12).

We reach in this way a *third, multivalued hyperstructural realization of "hidden variables" and "completion" of quantum mechanics, in addition to the isotopic and genotopic realizations of the preceding sections.*

The reader is then encouraged to meditate a moment on the vastity of the implications of the legacy of Einstein, Podolsky, and Rosen<sup>(28)</sup> when treated with the appropriate novel mathematics.

### 3.14. Isodual Methods for Antimatter

This presentation would be incomplete and somewhat misleading without an indication that all the preceding methods are solely intended for the representation of *matter* and, if applied for the description of *antimatter* (see, e.g., the historical account by Dirac<sup>(36)</sup>), they imply a number of inconsistencies.

As an example the sole transition from classical to quantum mechanics of contemporary physics is the naive or symplectic quantization. It then follows that the operator image of current classical descriptions of antimatter *is not* the charge (or PCT) conjugate as needed for consistency, but rather a conventional *particle* with the mere change of the sign of the charge.

The origin of the inconsistency is that *any correct representation of antimatter, whether classical or quantum mechanics, must be an antiautomorphic (or, more generally, antiisomorphic) image of the description of matter, as it is the ease for charge conjugation.*

The above occurrence required an additional laborious search by this author of the *mathematics* suitable for the description of antimatter beginning at the *classical* level. It soon emerged that no available mathematics would

meet the above requirement, because the entirety of contemporary mathematics is based on the conventional unit  $+1$ , while a correct description of antimatter requires a conjugation at *all* levels.

In fact, preliminary attempts based on antiautomorphic maps of *part* but not *all* of mathematics soon emerged as possessing axiomatic inconsistencies similar to the partial use of the isotopies. A similar insufficiency emerged for the novel iso-, geno-, and hypermathematics outlined in the preceding sections.

After a comprehensive investigation of all possible alternatives, Santilli introduced in Refs. 4b of 1985 the following *antiisomorphic conjugation* of the basic isounit,

$$\hat{I} > 0 \rightarrow \hat{I}^d = -\hat{I}^\dagger = -\hat{I} < 0 \quad (3.121)$$

under the name of *isoduality*, which admits the conjugation of the conventional unit of contemporary mathematics as a particular case,  $1 > 0 \rightarrow \hat{I}^d = -1$ .

Then, *all* conventional or generalized methods have to be subjected to isoduality, for consistency. For instance, it is not enough to change the sign of the unit in number theory, because the quantity  $-1$  *is not* the unit of negative numbers,  $(-1) \times (-n) = +n \neq -n$ ,  $n \in R(n, +, \times)$ , and the same situation holds under isotopy.

Jointly with map (3.121), the isotopic product must be also subjected to isoduality, yielding the *isodual isomultiplication*

$$\hat{A} \hat{\times} \hat{B} = \hat{A} \times \hat{I} \times \hat{B} \rightarrow \hat{A} \hat{\times}^d \hat{B} = \hat{A} \times \hat{I}^d \times \hat{B} = -\hat{A} \hat{\times} \hat{B} \quad (3.122a)$$

$$\hat{I} \rightarrow \hat{I}^d = -\hat{I}, \quad \hat{I}^d = (\hat{I}^d)^{-1} \quad (3.122b)$$

under which  $\hat{I}^d$  is indeed the left and right unit,  $\hat{I}^d \hat{\times}^d \hat{A} = \hat{A} \hat{\times}^d \hat{I}^d \equiv \hat{A}$  for all possible  $\hat{A}$ , with particularization for the ordinary product  $A \times^d b = A \times (-1) \times B = -AB$ .

The above rules permit the construction of the following chain of yet novel mathematics specifically built for antimatter (see Ref. 26 for the isodualities of numbers, isonumbers, and genonumbers, Ref. 23b for isodualities of analytic and quantum mechanics, Ref. 37c for a general treatment and page 18 of Web Site 3h for an outline with open mathematical problems):

(1) *Isodual mathematics*, including new numbers, the *isodual numbers*, which are ordinary numbers with *negative unit*,  $a^d = a \times \hat{I}^d = -a^\dagger$ ,  $a = n, c, q$  and related *isodual fields*  $F^d(a^d, +, \times^d)$ ,  $F = R, C, Q$ ; *isodual metric spaces*  $S^d(x^d, g^d, R^d)$ ; *isodual geometries, algebras, mechanics, etc.*

(2) *Isodual isomathematics*, including *isodual isonumbers*  $\hat{a}^d = a \times I^d = -\hat{a}^d$ ,  $\hat{a} = \hat{n}, \hat{e}, \hat{q}$  and related *isodual isofields*  $\hat{F}^d(\hat{a}^d, +, \hat{x}^d)$ ,  $F = R, C, Q$ ; *isodual isospaces*  $\hat{S}^d(\hat{x}^d, \hat{g}^d, \hat{R}^d)$ ; *isodual isogeometries, isoalgebras, isomechanics*; etc.

(3) *Isodual genomathematics*, with isodualities of all various aspects; and

(4) *Isodual hypermathematics*, characterized by the isodualities of multivalued hyperstructures.

Each element of the above chain of isodual mathematics is an antiautomorphic image of the corresponding formulation with positive units. Moreover, all the above methods admit *negative-definite norms*,

$$|a^d|^d = |a| \times I^d = -|a|, \quad \uparrow \hat{a}^d \uparrow^d = |\hat{a}| \times \hat{I}^d = -\uparrow \hat{A} \uparrow, \text{ etc.} \quad (3.123)$$

This implies the return to the original conception of antimatter (Stueckelberg *et al.*), namely, *all characteristics which are positive for matter become negative for antimatter under its isodual representation, including negative mass, negative energy, negative (magnitude of) the angular momentum, etc., and motion backward in time.*

A novelty is that all *negative* physical characteristics are now referred to *negative* units. This removes the problem of causality for motion backward in time, because *motion backward in time referred to a negative unit is fully equivalent, although antiautomorphic to motion forward in time referred to a positive unit.*

Recall that antiparticles were predicted by Dirac<sup>(36)</sup> in the *negative-energy solutions of his equations*,<sup>(36)</sup> which however behave unphysically, thus calling for the celebrated "hole theory" in second quantization. Santilli<sup>(37a)</sup> pointed out that *negative-energy solutions behave in a fully physical way when referred to negative units and, when properly formulated in Hilbert spaces, isoduality is equivalent to charge conjugation.*

The latter advances resolved the historical inconsistency of negative-energy solutions and permitted a full treatment of antiparticles at the level of *first quantization*, which is absent in current theories. In fact, such a treatment requires the prior identification of a *new antiisomorphic image of quantum mechanics, the isodual quantum mechanics*, with corresponding antiautomorphic image of the special relativity, called *isodual special relativity*, first identified in Ref. 22 (see Ref. 37c for additional studies).

In particular, Ref. 23b achieves a complete equivalence in the treatment of matter and antimatter at all possible levels, beginning with antiautomorphic *Newton's equations*, and then passing to Lagrange's and Hamilton's equations, quantization, quantum mechanics, etc. In this way,

antimatter becomes equipped with its own isodual quantization, thus resolving the problematic aspects indicated earlier.

The fundamental physical theory is in this case the *isodual Newtonian mechanics* defined on the *isodual isospace*  $\hat{S}^d(\hat{r}^d, \hat{x}^d, \hat{v}^d) = \hat{E}^d(\hat{r}^d, \hat{R}^d) \times \hat{E}^d(\hat{x}^d, \hat{g}^d, \hat{R}^d) \times \hat{E}^d(\hat{v}^d, \hat{g}^d, \hat{R}^d)$  with *isodual isounit*  $\hat{I}_{\text{tot}}^d = \hat{I}_t^d \times \hat{I}^d \times \hat{I}^d = -\hat{I}_{\text{tot}}$  and fundamental *isodual Newton's equations for antimatter*, first submitted in Ref. 23b,

$$\hat{m}^d \times^d \frac{\hat{a}^d \hat{v}_k^d}{\hat{a}^d \hat{r}^d} - \frac{\hat{a}^d \hat{\sigma}^d \hat{U}^d(\hat{r}^d, \hat{x}^d, \hat{v}^d)}{\hat{a}^d \hat{r}^d} + \frac{\hat{\sigma}^d \hat{U}^d(\hat{r}^d, \hat{x}^d, \hat{v}^d)}{\hat{\sigma}^d \hat{x}^{kd}} = 0 \quad (3.124)$$

We then have the *isodual Hamilton equation*

$$\omega_{\mu\nu}^d \frac{d^d b^{\nu d}}{d^d t^d} = \frac{\partial^d \hat{H}^d(r^d, b^d)}{\partial^d b^{\mu d}} \quad (3.125)$$

corresponding *isodual Hamilton-Jacobi equations, isodual naive or symplectic quantization* and, finally, the *isodual quantum mechanics* with basic equations

$$i \times^d \hbar^d \times^d (\partial^d / \partial t^d) |\psi\rangle^d = H^d \times^d |\psi\rangle^d = E^d \times^d |\psi\rangle^d \quad (3.126a)$$

$$i \times^d \hbar^d \times^d d^d a^d / d^d t^d = A^d \times^d \hbar^d - H^d \times^d a^d \quad (3.126b)$$

where we have used the *iso-self-duality* of  $i$ ; i.e., its invariance under isoduality,  $i^d = -i^{\dagger} = -\bar{i} = i$ , the isodual Hilbert space  $\mathcal{H}^d$  is characterized by the isodual states  $|\psi\rangle^d = -(|\psi\rangle)^{\dagger}$  and isodual inner product  $\langle \phi | \psi \rangle^d = \langle \phi |^d \times^d |\psi\rangle^d \times I^d \in C^d$ . As one can see, the isodual eigenvalues  $E^d$  are *negative* as expected for consistency.

We also have the additional, *novel symmetry* of the *conventional* Minkowskian line element in  $M(x, \eta, R)$

$$I \rightarrow \hat{I}^d = -I, \quad \eta \rightarrow \eta^d = -\eta \quad (3.127a)$$

$$\begin{aligned} (x-y)^2 &= [(x-y)^{\mu} \times \eta_{\mu\nu} \times (x-y)^{\nu}] \times I \\ &= [(x^d - y^d)^{\mu} \times^d \eta_{\mu\nu}^d \times^d (x^d - y^d)^{\nu}] \times \hat{I}^d \\ &= (x-y)^{d2d} \in R^d \end{aligned} \quad (3.127b)$$

as well as the additional novel symmetry of the *conventional* inner product of Hilbert space  $\mathcal{H}$

$$\langle \psi | \times | \psi \rangle \times I \equiv \langle \psi |^d \times^d | \psi \rangle^d \times I^d = \langle \psi | \psi \rangle^d \quad (3.128)$$

The above invariances establish that the *isodual representation of antimatter is admitted by the very axioms of relativistic classical and quantum mechanics*. In fact, the same line element and inner product, and therefore the same physical laws, apply to both, matter and antimatter under different antiautomorphic realizations.

Rather remarkably, the *conventional* Dirac equation has a *negative unit* embedded in its gamma matrices, even though not interpreted as such during this century. Its symmetry then turns out to be the iso-self-dual structure  $\mathcal{P}(3.1) \times \mathcal{P}^d(3.1)$ , contrary to a rather popular belief that it is solely given by the spinorial covering of the Poincaré symmetry  $\mathcal{P}(3.1)$ . Note that the latter is *ten* dimensional, while the former is *twenty-two* dimensional.

The isotopy of the Dirac equations and related iso-self-dual isosymmetry  $\mathcal{P}(3.1) \times \mathcal{P}^d(3.1)$  and isogeometry  $\hat{M}(\hat{x}, \hat{\eta}, \hat{R}) \times \hat{M}^d(\hat{x}^d, \hat{\eta}^d, \hat{R}^d)$  emerge as being of such broad applications as to characterize a new cosmology called *isocosmology*,<sup>(22)</sup> with the same amount of matter and antimatter in the universe, and *null total characteristics of energy, linear momentum, time, etc.* The covering genotopic formulations imply a broader cosmology, called *genocosmology*, in which the Universe is open (or there is continuous creation). Finally, the still broader hyperformulations imply a rather complex cosmology *inclusive of biological structures* with the same characteristics of the genocosmology plus unlimited (yet ordered) multivalued structure.

Again, the new symmetries (3.127) and (3.128) have remained undetected during this century because of the prior need of new numbers, this time, *numbers with a negative left and right unit*.

In conclusion, in this paper we have established that the vision by Einstein, Podolsky, and Rosen<sup>(28)</sup> on the lack of completion of quantum mechanics is indeed correct, inasmuch as quantum mechanics can be first completed in an axiom-preserving way with isotopic, genotopic, and hyperstructural liftings, and then, still in an axiom-preserving way, can be completed with isodual conventional, isodual isotopic, isodual genotopic, and isodual hyperstructural realizations, thus admitting a total of *seven* axiom-preserving structural generalizations.

The abstract axioms of this entire, rather vast construction are those of conventional quantum mechanics, merely assumed in different realizations of progressive complexity for the characterization of systems of progressively increasing methodological needs.

### 3.15. Outline of Applications to Interior Hadronic, Nuclear, Astrophysical, and Other Systems

As indicated in Sec. 1, this paper is devoted to the identification of axiomatically consistent nonunitary formulations capable of preserving the

abstract axioms of the special relativity. As such the paper cannot enter into detailed studies on applications and verifications. Nevertheless, at least an indication of some of the available applications with related references appears to be necessary to justify the preceding study. At any rate, this author believes that the conceptual foundations of any applications are the basic ones, because technical developments are merely consequential. Needless to say, these applications are in their infancy and so much remains to be done.

(A) **Applications to Hadron Physics.** As stated earlier, relativistic quantum mechanics (and evidently quantum field theory) can be safely assumed to be exactly valid for *electroweak interactions* at large, as well as, more generally, for all conditions in which there is no appreciable overlapping of the wavepackets of particles.

The isotopic completion of relativistic quantum mechanics was conceived for the primary purpose of attempting *new models of structure, interaction, and scattering of hadrons*<sup>(3, 4, 22)</sup> (for which this author suggested the name of "hadronic mechanics,"<sup>(3b)</sup> as well as, more generally, for all conditions of particles in which there is an appreciable overlapping of *wavepackets*, irrespective of whether *charges* are point-like or not.

The conditions of the former case imply the validity of the point-like approximation of the particles and their wavepackets with consequential validity of the underlying *local-differential* geometry and topology. Moreover, point-like particles can only have *action-at-a-distance interactions* which, as such, are representable with a potential and related particle exchanges. The exact validity of quantum mechanics then follows.

By contrast, strong interactions are profoundly different from the electroweak ones. The range of strong interactions is of the same order of magnitude as the size (charge distribution) of all hadrons ( $\approx 1 \text{ fm} = 10^{-13} \text{ cm}$ ). Thus, *a necessary condition to activate strong interactions is that hadrons enter into conditions of mutual penetration and overlapping*. But hadrons are some of the densest objects measured in the laboratory until now. This implies the historical legacy by Bloch'intsev, Fermi, and others on the *nonlocal* structure of the strong interactions. But the latter interactions are of *contact* type, that is, they are due to the physical contact of the wavepackets. As such, they have no potential, they cannot be mediated by particle exchanges, and, consequently, they should be represented with anything, *except* the Hamiltonian (or the Lagrangian).

Relativistic hadronic mechanics (Sec. 3.10) represents all potential interactions with the conventional Hamiltonians  $H(r, p)$  and all contact interactions with an integrodifferential generalization of Planck's unit  $h = I \rightarrow \hat{h} = \hat{I}(x, p, \psi, \partial\psi, \dots)$  under the condition that the appropriate expectation

value reproduce the conventional,  $\langle I \rangle = \hbar$ , Eq. (3.78). This representation permits the verification of conventional total conservation laws and conventional quantum behavior of the center-of-mass trajectories (e.g., verification of the conventional uncertainties as shown in Sec. 3.9), yet admits the most general possible nonlinear, nonlocal, and nonpotential internal structure.

As such, it has been stressed in the text that, by no means, relativistic hadronic mechanics is a new theory, because it merely provides a *new realization* of exactly the same axioms of relativistic quantum mechanics, essentially differentiated by the selected basic unit. In particular, relativistic hadronic mechanics has emerged to be a "completion" of relativistic quantum mechanics much along the historical legacy by Einstein, Podolsky, and Rosen<sup>(28)</sup> (Sec. 3.5).

Therefore, the validity for strong interactions of the *abstract axioms* of relativistic quantum mechanics is outside scientific debates at this point of our knowledge, and we solely study in this paper the validity for strong interactions of alternative *realizations* of the same axioms.

As a result, there exist no *a priori* theoretical objections against the axiomatic structure of hadronic mechanics because they are *de facto* objections against the axiomatic structure of *quantum mechanics*. Also, there exist no *a priori* experimental objections against the description of strongly interacting systems provided by hadronic mechanics because it reproduces conventional center-of-mass behaviour.

The selection of the validity for the interior problem of strong interactions of the simplest possible quantum *realization*  $\hbar = I$  of the unit, or of the hadronic *realization*  $\hbar = \hat{I}(x, p, \psi, \partial\psi, \dots)$ , must therefore be left to a scientific comparison of the plausibility of the predictions of the two realizations and their comparative confrontation with experimental evidence.

In the latter respect one should avoid transparent inconsistencies, such as the confrontation of *hadronic* predictions with experimental data elaborated with *quantum* methods because, as stressed in this paper, conventional quantum methods have no meaning for hadronic mechanics of any type, beginning with fundamental notions such as numbers, angles, trigonometric and hyperbolic functions, etc., and then passing to special functions, transforms, and distributions, etc. To conduct a scientific inquiry, one must therefore elaborate with *quantum* methods the data for *quantum* mechanics and with *hadronic* methods those for *hadronic* mechanics.

This includes above all the use of *diffident scattering theories*, the conventional *potential scattering theory* for quantum mechanics and the covering *isoscattering theory* for hadronic mechanics (see Ref. 22b, Chap. 12). In fact, by keeping in mind that the latter is a *nonunitary* image of the former, the reader should expect *different numerical results*, beginning with *different definitions of differential cross sections* [*loc. cit.*].

Equivalently, the reader should expect that the *same total cross section*, here referred to the number of scattered particles in a given solid angle, has *different numerical interpretations* in quantum and hadronic mechanics. After all, the reader should keep in mind that the two scattering theories are interconnected by *nonunitary* transforms.

It is at this point that all mathematical and physical efforts reviewed in this paper acquire their true significance. In fact, the current theory of strong interactions, that based on the assumption of the exact validity of conventional quantum mechanics, has now reached a clear impasse due to well known basic problems unsolved for decades (see below). This calls for the initiation of the scientific process of trial and error for a structural revision of the current quark theory along the teaching of the history of physics, that is, by preserving unchanged results of clear value and implementing the others in a broader description.

In this latter respect, *the unitary classification of hadrons into families can be safely assumed as being of final character* owing to its now historical capability to predict new hadrons. The aspects currently debated in various physical circles are restricted to the *different problem of the structure of each individual hadron of a given unitary multiplet*.

More specifically, the latter problematic aspect, which constitute the physical foundations for the applicability of the hadronic completion of quantum mechanics, are the following:

(I) *Inability by current theories to achieve a rigorous confinement of quarks* which, alone, should be sufficient grounds for structural revisions. Note that the lack of confinement is deeply linked to the studies herein reported, because the problem is due to the assumption of the *same* mechanics for *both* the exterior problem in vacuum and the *interior* structural problem, with consequential finite transition probabilities for free quarks originating from Heisenberg's uncertainties. The assumption, instead, of the conventional mechanics for the interior behavior and a generalized mechanics for the structure does indeed permit the achievement of a rigorous confinement via the incoherence of the two Hilbert spaces, as indicated below.

(II) *Inability to formulate gravity for matter composed of quarks*. Gravity can be solely formulated in our space-time, while quarks can be solely formulated in mathematical unitary spaces, without any possible interconnection (in view of the O'Rafeartaigh theorem). At any rate, the original and primary physical meaning of unitary theories is that of *classification* of hadrons into families. It is then evident that *no gravity can be defined for a classification*. This occurrence, alone, should also be grounds for structural revisions of current quark theories. As treated in



detail in the original proposal to build hadronic mechanics,<sup>(3b)</sup> this problem is due to the use of *one single theory*, the quark theory, for *both* the *classification* of hadrons into families and the *structure* of each individual member of a given family, which is unprecedented in the history of physics. In fact, history teaches that atoms required *two different models*, the Mendeleev model of *classification* of atoms into families and a different, yet compatible, quantum model of *structure* of each atom of a given family. The same differentiation proved to be necessary for nuclei, molecules, and other structures. It was therefore an easy prediction of Ref. 3b that the same differentiation will eventually emerge in one way or another as being necessary for hadrons too. Note that *the theory valid for Mendeleev's classification, classical mechanics, proved to be insufficient for the resolution of the problem of structure, which required the advent of a new mechanics, quantum mechanics*. The current scientific scene for hadrons appears to be essentially similar, the methods valid for the classification of hadrons, quantum mechanics, this time being insufficient for the structural problem.<sup>(3b)</sup>

(III) *Inability to introduce quark masses as unambiguous physical masses in our space-time*. A necessary well-known condition for a mass to be physical, that is, to exist in our space-time, is being the eigenvalue of the second-order Casimir invariant of the Poincaré symmetry,  $m^2 = p^2/c_0^2$ . But quarks are not admitted as representations of the Poincaré symmetry in view of their fractional charges and other anomalous properties, and their masses cannot therefore be introduced as eigenvalues of said Poincaré Casimir. As a result, *on strict scientific grounds, quark "masses" have the sole meaning of parameters in mathematical unitary spaces, rather than physical masses in our space-time*. As such, quark masses cannot possibly originate gravity in any known consistent way. The lack of admission of quarks by the special relativity as physical particles in our space-time is also sufficient, alone, to warrant structural revisions of current theories because of the evident conflict in assuming *mathematical* structures in unitary spaces as *physical* constituents of hadrons in our space-time.

(IV) *Inability for quarks to be the "elementary" constituents of hadrons*. This occurrence was first pointed out by Santilli<sup>(38)</sup> back in 1981 and forgotten by everybody in the field, although now rather widely accepted by the physics community in view of the current impasse created by conventional quark theories. In fact, a rather general current trend is that *quarks are composites*. It is evident that such an assumption implies the admission that the fundamental *elementary* constituents of hadrons are not quarks but other particles, as predicted since the proposal of Ref. 3b. As also pointed out in Ref. 38 and forgotten for over a decade, it should be

noted that the assumption of quarks as composites appears to be the only line of research capable of reconciling the two different problems, the established  $SU(3)$ -color theory for the classification of hadrons into families, and a new theory for the structure of *each individual hadron* of a given unitary multiplet. In turn, this is the only possible way permitting the definition of gravity, not for quarks in their unitary spaces, but for their physical constituents in real space-time.

(V) *The historical legacy on the nonlocality of the structure of hadrons and the strong interactions at large* (indicated earlier). As is well known, current unitary theories, QCD and all that, are *strictly local theories*. Such a mathematical structure can be safely assumed to be exactly valid for the problem of *classification* of hadrons into families, thus confirming the validity of the unitary classification. However, the same local structure cannot be expected to be of "final" character for the different problem of the *structure* of hadrons which implies the mutual penetration of some of the densest media measured in laboratory by mankind until now. As stressed in the original proposal to build hadronic mechanics,<sup>(3b)</sup> the problem of the nonlocality of the hadronic structure will sooner or later force a revision of current theories, the advantage being evidently gained by those physicists who admit it first.

Above all, a revision of current theories on the hadronic structure (only, and not that of classification) is warranted by *rather clear theoretical, phenomenological, and experimental evidence*. To begin, *the Minkowski space and related Poincaré symmetry are not exact already for interior media of low density, such as our atmosphere*. As indicated in Sec. 3.10, this is due to: the locally varying character of the speed of electromagnetic waves within physical media such as our atmosphere, water, glass, oil, etc.; the inability of reducing the above *classical* setting to photons scattering through molecules *in second quantization*, e.g., for electromagnetic waves of one meter wavelength; ambiguities in the applicability of special relativity within physical media, such as speeds of electrons greater than the local speed of light or loss of the relativistic sum of speed under the causal speed in vacuum; and other problematic aspects).<sup>(22b)</sup>

Under these premises, the expectation that the Minkowski space and the Poincaré symmetry in their current simplest possible formulation are as valid for the hadronic structure as they are for the atomic structure, has little scientific credibility, owing to the hyperdense character of the hadronic structure as compared to the virtually empty character of the atomic structure. In fact, the expectation has a number of questionable consequences such as the fact that the hadronic constituents must travel free in vacuum in the same way as electron evolve in the atomic structure,

or that hadrons have a "tiny atomic structure," and the like. This provides solid grounds for the "inapplicability" (and not "violation") *atomic realization* of the Minkowski geometry and the Poincaré symmetry within hadronic media in favor of more general *hadronic realizations* of the same axioms.

All direct phenomenological calculations<sup>(39)</sup> confirm such a plausible assumption. As an example, Nielsen and Picket<sup>(39g)</sup> identified deviations from the Minkowski metric in the interior of pions and kaons via the use of conventional gauge theories in the Higgs sector, which we write in the form

$$\hat{\eta} = \text{Diag}((1 - 3\alpha), (1 - 3\alpha), (1 - 3\alpha), -(1 + \alpha)) = \hat{T} \times \eta$$

$$\hat{T} = \text{Diag}(n_1^{-2}, n_2^{-2}, n_3^{-2}, n_4^{-2}), \quad \eta = \text{Diag}(+1, +1, +1, -1) \quad (3.129)$$

where

$$\text{For pions: } n_k^{-2} = 1 + 1.2 \times 10^{-3}, \quad n_4^{-2} = 1 - 3.7 \times 10^{-3} \quad (3.130a)$$

$$\text{For kaons: } n_k^{-2} = 1 + 2.0 \times 10^{-4}, \quad n_4^{-2} = 1 + 6.00 \times 10^{-4} \quad (3.130b)$$

Geometrically and numerically similar results are reached by the other studies.<sup>(39)</sup>

The reader should be aware of the fact that, even though not widely known, the lack of exact character of the conventional realization of the Minkowskian geometry and the Poincaré symmetry in the interior of hadrons is also supported by a number of experiments.

The first supporting experiments were conducted by Aronson *et al.*<sup>(40)</sup> at Fermilab in 1983 on the measure of the behavior of the meanlife of  $K_s^0$  with energy ranging from 30 to 100 GeV, which show *deviations* from the Minkowskian geometry. Additional experiments were performed by Grossman *et al.*<sup>(41)</sup> also at Fermilab in 1987 for the same behavior of the  $K_s^0$ , but for the *different* energy range from 100 to 350 GeV, and they claim *verification* of the Minkowskian geometry. Unfortunately, *the data of the Grossman experiment<sup>(41)</sup> were elaborated in a frame in which there is no CP violation, which is known to yield no geometric anomalies, as shown by D. Y. Kim<sup>(39c)</sup> and others.* The results of the tests in Ref. 41, besides being inapplicable for the energy range 30–100 GeV, cannot be considered as final in their own range of 100–350 GeV because they are dependent on questionable theoretical assumptions in the data elaboration. It should be indicated that the data of the test in Ref. 40 too have considerable statistical errors. Therefore, both measures<sup>(40,41)</sup> have to be repeated with better accuracy and without questionable theoretical assumptions in the data elaboration to claim any scientific conclusion.

Other experiments directly relevant for the geometry inside hadrons are those on the Bose–Einstein correlation (see, e.g., Ref. 42) of  $p - \bar{p}$  annihilation at very high energy<sup>(43)</sup> and very low energy.<sup>(44)</sup> These tests are perhaps more relevant for the problem considered because, even though not widely admitted, it is scientifically established that *correlation cannot exist for strict local structures*. Thus, *measures<sup>(43,44)</sup> can be interpreted as possible direct experimental evidence on the nonlocality of the structure of hadrons*. Deviations from the Minkowskian geometry then follow.

Numerous other experiments currently exists on anomalous Minkowskian behavior some of which are somewhat hidden in "semi-phenomenological adjustment," such as the measure of photons and astrophysical matter traveling at speeds higher than the speed of light in vacuum, and they are not reviewed here for brevity (see Refs. 22c, 34d).

The isotopic theories, including Santilli's isominkowskian geometry on isospaces  $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$  and isopoincaré symmetry  $\hat{P}(3.1)$  (Sec. 3.10), appear to be particularly suited as foundations for a new generation of theories on the structure, interaction, and scattering of hadrons, in view of the following reasons:

(1) *The isotheories are directly universal, thus admitting as particular cases all possible deformations of the Minkowskian geometry.<sup>(33)</sup> In fact, deformations of the type (3.129) are precisely of isominkowskian type. Generalized structures  $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$  and  $\hat{P}(3.1)$  therefore apply under deformations even when not desired.*

(2) *The isotheories are the only known theories permitting the preservation of the abstract axioms, symmetries, and physical laws of special relativity. All other deformations considered in this paper<sup>(8-21)</sup> imply the violation of Einsteinian axioms in one form or another. As noted in Sec. 1, the use of the latter deformations implies the sizable problems of identifying new axioms, proving their consistency, and establishing them experimentally before applying them to the hadronic structure. Note the significance of reconstructing the exact rotational, Lorentz, and Poincaré symmetries in isospace (Sec. 3.10).<sup>(4)</sup> As an example, Nielsen and Picket<sup>(39g)</sup> call the  $\alpha$ -parameter in metric (3.129) the "Lorentz asymmetry parameter." In reality, it was shown by Santilli in Ref. 4a that the Lorentz symmetry remains fully exact for metric (3.129) because it can be realized with respect to the unit  $\hat{I} = \text{Diag}((1 - \alpha/3)^{-1}, (1 - \alpha/3)^{-1}, (1 - \alpha/3)^{-1}, (1 - \alpha)^{-1})$ .*

(3) *The isotheories reproduce conventional total conservation laws and center-of-mass trajectories, including Heisenberg's uncertainties (Sec. 3.9). By comparison, other theories, such as those on "squeezed states,"<sup>(19)</sup> imply*

unnecessary deviations from established center-of-mass trajectories, as indicated earlier.

(4) The isounits  $\hat{I} = \text{Diag}(n_1^{-2}, n_2^{-2}, n_3^{-2}, n_4^{-2})$  represents a geometrization of the extended, nonspherical, and deformable shape of the charge distribution of hadrons via the characteristic functions  $n_1^{-2}, n_2^{-2}, n_3^{-2}$ , as well as a representation of the density of the hadron considered via the characteristic function  $n_4^{-2}$ . By comparison, the representation of the extended size of hadrons via conventional methods requires the second quantization which can only represent perfectly spherical and perfectly rigid shapes (evidently as a necessary condition not to violate the basic rotational symmetry), contrary to evidence in hadron physics. No representation of nonspherical and deformable shapes is known for other theories, to our best knowledge. No representation of the density (which is an important characteristic varying from hadron to hadron) exists in other theories, whether conventional or generalized.

(5) The isotheories admit excellent fits on available experimental data, which include:

(5.A) The fit of the data by Aronson *et al.*<sup>(40)</sup> for the  $K_s^0$  via the isominkowskian geometry conducted by Cardone *et al.*<sup>(45a)</sup> with numerical results

$$n_k^{-2} = 0.9023 \pm 0.004, \quad n_4^{-2} = 1.003 \pm 0.0021 \quad (3.131)$$

(5.B) The fit of both seemingly discordant data by Aronson *et al.*<sup>(40)</sup> and Grossman *et al.*<sup>(41)</sup> via a unified isominkowskian representation conducted by Cardone *et al.*<sup>(45b)</sup> with values

$$n_k^{-2} = 0.909080 \pm 0.0004, \quad n_4^{-2} = 1.002 \pm 0.002 \quad (3.132)$$

(5.C) The theoretical elaboration of the Bose-Einstein correlation at high energy<sup>(43)</sup> with the isominkowskian geometry and the isopoincaré symmetry conducted by Santilli<sup>(46)</sup> and its fit to the UA1 experimental data conducted by Cardone and Mignani<sup>(47)</sup> with numerical values

$$n_1^{-1} = 0.267 \pm 0.054, \quad n_2^{-1} = 0.437 \pm 0.035 \quad (3.133a)$$

$$n_3^{-1} = 1.661 \pm 0.013, \quad n_4^{-1} = 1.653 \pm 0.015 \quad (1.133b)$$

which shows the known elongated ellipsoid of the  $p - \bar{p}$  fireball (here represented in a scale invariant form). It should be noted that the representation of the correlation requires the presence in the expectation values of states of cross terms which are absent in the conventional expectation value

of quantum mechanics.<sup>(42)</sup> Therefore, on strict scientific grounds, two-point correlation functions are outside the arena of exact applicability of quantum mechanics. In fact, correlation functions are nowadays obtained by throwing in "semiphenomenological" parameters of unknown origin, such as the "chaoticity,"<sup>(42)</sup> which in actuality represents precisely the deviation from quantum mechanics realizations of the axioms.<sup>(46)</sup> By comparison, the isoexpectation values of the hadronic realization, Eq. (3.78), do indeed admit the needed cross terms for nondiagonal isotopic elements  $\hat{T}$ , and they produce a final form of the two-point isocorrelation function without any *ad hoc* parameter, and with the sole assumption of null longitudinal momentum transfer (which is experimentally verified).<sup>(43, 44)</sup>

Other experimental data on Minkowskian anomalies, such as those of speeds higher than that of light in vacuum, are directly represented by the isominkowskian geometry, e.g., via the new invariance (3.85) for  $n < 1$  (see Refs. 22c, 34d for brevity).

In summary, we can state that all available theoretical, phenomenological, and experimental information appears to support the validity of Santilli's isominkowskian geometry and of isopoincaré symmetry inside hadrons, pending final experimental resolutions (see below).

In regard to the construction of the new theory of hadronic structure based on isotopic methods, the main contributions have been the following: Mignani<sup>(6d)</sup> studied the isotopies of  $SU(3)$  and proved its local isomorphism to the conventional symmetry; Kalnay<sup>(66j)</sup> constructed an operator image of Nambu's mechanics for triplets which was then proved by Kalnay and Santilli<sup>(6k)</sup> to be a realization of hadronic mechanics with Eq. (3.74a) and isotopic element  $\hat{T} = H_1^{-1} + H_2^{-1} - 2$ , where  $H_1$  and  $H_2$  are Nambu's Hamiltonians; Santilli<sup>(48)</sup> then presented preliminary studies on the explicit construction of the isoquark theory (that is, the theory of quarks defined as isorepresentations of  $S\hat{U}(3)$  and obeying the isotopic completion of relativistic quantum mechanics). Comprehensive studies are available in monographs<sup>(22)</sup> whose primary objective is the hadronic structure.

The main characteristics of the isoquark theory are the following<sup>(48, 4e, 49, 22)</sup>

(i) Isoquarks have the same quantum numbers of quarks. This property is technically achieved via the construction of isorepresentations of  $S\hat{U}(3)$  obeying Klimyk's rule.<sup>(5g, 22a, 23c)</sup> The two theories are not therefore distinguishable on grounds of current theoretical and experimental knowledge.

(ii) Isoquarks have an exact confinement. This is due to the incoherence of the internal isohilbert space  $\hat{\mathcal{H}}$  with the conventional external Hilbert space  $\mathcal{H}$ , which imply an identically null transition probability

for free isoquarks (but *not* for free quarks) even in the absence of a potential barrier (asymptotic freedom at all energies);

(iii) *Isoquarks have convergent perturbative expansions.* This is due to the fact that conventionally divergent series are *isorenormalized* into a convergent form for isotopic elements sufficiently smaller than 1 in absolute value,  $|\hat{T}| \ll 1$ , condition which is intriguingly verified by Kalnay's<sup>(6i)</sup> quantum version of Nambu's mechanics for triplets and other versions. It is evident that the above features are not readily possible for conventional quark theories, thus confirming the plausibility of the isotopic formulation.

(iv) *Isoquarks are reducible isounitary isorepresentations of Santilli's isopoincaré symmetry,* e.g., isoquarks are composites.<sup>(48)</sup>

(v) *The constituents of unstable hadrons are assumed to be ordinary massive particles produced free in the spontaneous decays, generally those with the lowest decay mode (tunnel effect of the constituents).* It is in this final aspect where all studies reported in this paper acquire their full significance and for which they were proposed in the first place.<sup>(3b)</sup> In fact, the above assumption can be readily proved to be impossible under conventional quantum mechanics. On the contrary, the assumption of the isotopic completion of quantum mechanics inside hadrons does indeed permit the identification of the hadronic constituents as ordinary massive particles, according to the following main lines:

(va) *Ordinary massive particles experience "mutations"<sup>(2)</sup> in the transition from conventional conditions in vacuum to the interior of hyperdense hadronic media.* These mutations are technically represented via the transition from the conventional to the isotopic symmetry. The new states are called "isoparticles" and are essential to achieve consistency for the reasons indicated above.

(vb) *Nonlinear, nonlocal, and nonpotential internal effects imply new renormalizations of conventional characteristics.* Recall that all interactions imply renormalizations. Those of potential-Lagrangian type are well known and are here ignored. The *new* character of said normalizations originates from the fact that the interactions considered are not representable with a Lagrangian. Thus, isoparticles have different rest energy, different charge, different magnetic moments, etc. than the corresponding values in vacuum. These occurrences are confirmed by a mere visual inspection of the isocasimirs (3.93) and are evidently essential to "build" quarks inside hadronic media (see below).

(vc) *Half-odd-integer angular momenta are prohibited for quantum mechanics but are admitted in hadronic mechanics.* Recall that half-odd-integer angular momenta violate Hermiticity and unitarity on a conventional

Hilbert space and, as such, all structure models based on quantum mechanics must exclude them. On the contrary, half-odd-integer angular momenta are fully admitted for the isohilbert space, under a full preservation, this time, of isohermiticity and isounitariness, as established by a mere visual inspection of the new iso-self-scalar invariance (3.19) [see Ref. 22b or a study via the isorepresentation theory of  $SU(2)$ ]. As a result, when and only when applicable, hadronic mechanics allows the construction of structure models which are unthinkable with conventional settings (see below).

(vd) *Nonlinear, nonlocal, and nonpotential interactions of particles in singlet coupling are attractive even under repulsive Coulomb barriers, while triplet coupling are repulsive.* This property was established in Refs. 6f and 49 where the lifting of conventional Coulomb equations characterized by an isounit of type (3.14) produced an *explicitly attractive force between the two identical electrons* of the Cooper pair in superconductivity, in remarkable agreement with experimental data. The property is evidently general, and can be used for a deeper understanding of known facts, such as how identical protons can be bound together in nuclei (charge independence of the nuclear force), or how the two identical electrons of the helium atom can generally orbit together. The property can also be used to attempt new knowledge, e.g., the understanding of how an electron can "exclude" another electron when there is no interaction carrying energy, exactly as permitted by the *contact* interactions represented by isounits of type (3.14).

(ve) *Nonlinear, nonlocal, and nonpotential interactions do not carry appreciable binding energy.* This latter property was established in Ref. 4e where it was shown that the conventional (negative) binding energy is essentially due to conventional long-range interactions (evidently adjusted by the nonpotential effects).

The above main lines have permitted the construction of new models of the structure of various *unstable* hadrons which are studied in detail elsewhere (see the forthcoming Ref. 22c). We here mention that the original proposal of 1978 to build hadronic mechanics,<sup>(3b)</sup> Sec. 5, contained the hadronic model on the structure of the  $\pi^0$  as a "compressed positronium," that is, the transition from the quantum mechanical (QM) representation of the positronium in singlet coupling to the corresponding state described by hadronic mechanics (HM) yields a representation of the totality of the characteristics of the  $\pi^0$ , including rest energy, charge, mean life, charge radius, electric and magnetic moments, parity, etc.,

$$\text{Positronium} = (e_1^-, e_1^+)_{\text{QM}} \rightarrow \pi^0 = (\hat{e}_1^-, \hat{e}_1^+)_{\text{HM}} \quad (3.134)$$

where  $e^\pm \rightarrow \hat{e}^\pm$  represents mutation, the total energy of the isoelectrons is of about 67 MeV, and the binding is only of nonlinear, nonlocal and non-potential type (because of its absorption of Coulomb interactions,<sup>(49)</sup> with small binding energy.<sup>(3b)</sup>

Note that the  $\pi^\circ$  constituents are freely emitted and are the massive particle emitted in the decay with the lowest mode,  $\pi^\circ \rightarrow e^- + e^+$  (hadronic tunnel effects of the constituents), in which we evidently have the inverse transition  $\hat{e}^\pm \rightarrow e^\pm$ .

Intriguingly, the indicial hadronic equation admits one and only one energy level, that of the  $\pi^\circ$ ; see Ref. 3b, pages 837–840 (this is called the *hadronic suppression of atomic spectra*.<sup>(3b, 22)</sup> In fact, any excitation of the hadronic state would imply distances bigger than 1 fm, for which all non-potential effects are no longer appreciable, i.e.,  $\hat{I} \equiv I$ , and hadronic mechanics recovers quantum mechanics identically, thus recovering the infinite *atomic spectrum*. It was stressed in Ref. 3b that, as a necessary condition of consistency, the *hadronic bound state must admit only one energy level*. Equivalently, we can say that the positronium can admit only one *additional state* at distances  $\leq 1$  fm,<sup>(3b)</sup> because the existence of a spectrum in the latter conditions would imply the assumption, again, of a “tiny atomic structure” inside hadrons.

Similarly, relativistic hadronic mechanics in its isospinorial form has permitted the construction in Ref. 4e of a structure model of the neutron as synthesized in new stars, *from protons and electrons only*, essentially along Rutherford's historical conception of the neutron as a “compressed hydrogen atom.” In fact, the isotopic completion

$$\text{Hydrogen atom} = (p_{\uparrow}^-, p_{\downarrow}^+)_{\text{QM}} \rightarrow n = (p_{\uparrow}^+, p_{\downarrow}^-)_{\text{HM}} \quad (3.135)$$

permits the representation, again, of the totality of the intrinsic characteristics of the neutron. The well-known historical objections were based on the exact validity of quantum mechanics for the hyperdense medium inside the proton, and, therefore, have no final character. In fact, they are easily resolved by hadronic mechanics. For instance, the need in compression (3.135) for an unacceptable “positive” binding energy (because the sum of the rest energies of the constituents is *smaller* than that of the neutron) is easily resolved by the isorenormalization of the rest energy of the constituents; the quantum inability to reach the magnetic moment of the neutron from those of the protons and electron is easily resolved by the mutations of the latter due to their mutual immersion; the impossibility of reaching a total spin 1/2 from a proton and an electron each of spin 1/2 is easily resolved by the admission of half-odd-integer angular momenta (by recalling that the proton is about 2,000 times heavier than the electron,

an inspection soon reveals that the *orbital* angular momentum of the electron must coincide with the *spin* of the proton as a condition for stability; see Ref. 4e for details); and the same occurs for the other characteristics of charge radius, mean life, etc.

Again, the hadronic indicial equations admits only one energy level in compression (3.135), that of the neutron, as necessary for consistency. Again, excited states imply distances bigger than 1 fm for which  $\hat{I} \equiv I$  and the atomic spectra is recovered identically. Again, the neutron constituents are the massive particles emitted free in the spontaneous decay  $n \rightarrow p^+ + e^- + \bar{\nu}$  in which  $\hat{p}^+ \approx p^+$  and  $\hat{e}^- \rightarrow e^- + \bar{\nu}$ .

It should be stressed that *quark theories cannot represent the synthesis of the neutron as occurring in early stars, from protons and electrons only*. In fact, their use requires the presence of the baryonic octet which *does not exist as yet in early stars* (which are notoriously composed solely of hydrogen). This provides a clear illustration of the limitations which are inherent to a theory for *classification* when jointly used to provide the *structure* of each element of a given multiplet.

Numerous additional hadronic structure models are possible along the above lines in a *quantitative and axiomatically consistent way*, such as

$$\text{Muonic atom} = (\mu_{\uparrow}^+, \mu_{\downarrow}^+)_{\text{QM}} \rightarrow \eta = (\hat{\mu}_{\uparrow}^+, \hat{\mu}_{\downarrow}^+)_{\text{HM}} \quad (3.136a)$$

$$\text{Pionic atom} = (\pi^+, \pi^-)_{\text{QM}} \rightarrow K_s^\circ = (\hat{\pi}^+, \hat{\pi}_{\downarrow}^-)_{\text{HM}} \quad (3.136b)$$

$$A = (\hat{\eta}, \hat{\pi}^\circ)_{\text{HM}}, \quad \sum^\pm = (\hat{p}^\pm, \hat{\pi}^\circ)_{\text{HM}}, \text{ etc.} \quad (3.136c)$$

where the reader should keep in mind that, under mutation, conventional differences, e.g., between  $\mu$  and  $\pi$ , are lost, i.e.,  $\hat{\mu}^\pm \approx \hat{\pi}^\pm$ . The emerging structure then resembles a number of early proposals in strong interactions, such as “bootstrapping,” with the understanding that their realization is now made consistently possible because of the advent of relativistic hadronic mechanics (see Ref. 22c for technical details and historical references).

In summary, *relativistic hadronic mechanics permits the consistent reduction of all hadrons to the protons and electrons only*, as predicted in Ref. 3b. Note that, besides representing all characteristics usually treated in quark theories, the model based on hadronic mechanics permits the additional representation of the mean life and charge radius (which are notoriously *not* represented by quark theories), as well as the reason why the latter is essentially the same for all hadrons.

The *difference* and *complementarity* of the new structure model with quark theories should be indicated for completeness. The *difference* was first pointed out in Ref. 3b, Sec. 5.2, via the fact that *the number of constituents of  $\pi^\circ$  and  $\pi^\pm$  is the same for quark models while it must increase*

by one unit for the new models. In fact, quark models assume that the constituents are two for both  $\pi^0$  and  $\pi^\pm$ . On the contrary, for the hadronic model we have  $\pi^0 = (\hat{e}_\uparrow^-, \hat{e}_\uparrow^+)_{\text{HM}}$  and the additional model proposed in Ref. 3b, Sec. 5.2,

$$\pi^\pm = (\hat{\pi}^0, \hat{e}^\pm)_{\text{HM}} = (\hat{e}_\uparrow^-, \hat{e}^\pm, \hat{e}_\uparrow^+)_{\text{HM}} \quad (3.137)$$

Again, the latter model is absolutely impossible for quantum mechanics (because of the inability to represent the total null spin of the  $\pi^\pm$ , and this may be the reason for the resiliency in its consideration by a part of the scientific community since its proposal in 1978). However, the model is readily possible for hadronic mechanics. In fact, for consistency, the total angular momentum of the central isoelectrons  $\hat{e}^\pm$  must be the same as that of the  $\pi^0$ , an occurrence identical to that of model (1.35).

A similar disparity between quark and hadronic models occurs in the transition from the model  $K_s^\pm = (\hat{\pi}_\uparrow^+, \hat{\pi}_\uparrow^-)_{\text{HM}}$  to

$$K^\pm = (\hat{K}^0)_{\text{HM}} = (\hat{\pi}^+, \hat{\pi}^\pm, \hat{\pi}^-)_{\text{HM}} \quad (3.138)$$

where one should again keep in mind the equivalence under mutation  $\hat{\mu}^\pm \approx \hat{\pi}^\pm$  and the axiomatic constrain of relativistic hadronic mechanics of recovering conventional total characteristics, including the behavior of the center-of-mass.

In summary, as one can verify, the above new structure model of hadrons is essentially that proposed in 1978, Ref. 3b, Sec. 5, with the primary subsequent advance given by the isotopies of the  $SU(2)$  spin symmetry (achieved in Ref. 4c of 1993) and the availability of the mechanics in an axiomatically consistent form (which can be claimed only in this paper following the mathematical advances of Ref. 23).

The complementarity of the hadronic and quark model has been indicated in Ref. 48 and it is studied in detail in Ref. 22c. In short, fractional charges are impossible in conventional space-time as is well known, but they are readily possible under isotopies (because they verify the condition of conventional total quantities, thus being compatible with the isorelativistic axioms). As such, relativistic hadronic mechanics permits the construction of quarks with conventional fractional charged and other standard characteristics as granules of real and virtual physical constituents. In other words, the "count of physical constituents," while fully correct in classical and, to some extent, in atomic structures, is questionable already in conventional quantum field formulations, and it becomes definitely untenable under yet more general hadronic settings. The dominant physical reality for hadrons is their hyperdense hadronic medium. Such a medium

cannot granulate into quarks for conventional relativistic quantum realization, but it can for the covering isotopic completion.

The compatibility between the novel isoquark theory and the established unitary classification can be achieved as follows for the case of the octet of mesons. On one side, mesons are assumed to be constituted by two composite isoquarks, i.e., isostates with reducible, thus multidimensional isounits. This assures the preservation of all conventional results in the classification via the techniques of this paper.

On the other side, the novel structure model of mesons is indeed reducible to a two-body isothory. For instance, we have  $\pi^\pm = (\hat{e}^+, \hat{e}^\pm, \hat{e}^-)_{\text{HM}} \equiv (\hat{\pi}^0, \hat{e}^\pm)_{\text{HM}}$ , and the same happens for all other mesons with actual, real, physical constituents produced free in the spontaneous decays. But isoparticles are characterized by the isounits. Thus, we can define the two isoquarks for the octet of mesons as reducible isorepresentations of  $SU(3)$  in the mathematical unitary space characterized by reducible isounits each of which is the tensorial product of eight isounits representing the eight pairs of actual, real, physical constituents of the mesons in our space-time. This permits, for the first time to our knowledge, a consistent definition of gravity under unitary theories, the achievement of a rigorous exact confinement, and the other advances mentioned earlier. The compatibility for baryons is essentially the same.

The reader should note the emergence of the most advanced formulations submitted in this paper, those of multivalued hyperstructural type. For additional details, one may consult Sect. V, page 19 of Web Site 3h.

It should be stressed that, after comprehensive studies of all possible alternatives, the above complementarity is the only one known to this author which permits the resolution of the existing problematic aspects (lack of confinement, lack of gravity, etc.), as well as the reconciliation of the unitary model of classification with the new model of structure with physical constituents defined in our space time.

Also, as in the preceding nuclear and atomic structures, the admission of constituents which can be produced free permits for the first time the predictions of novel practical applications, including conceivable new forms of subnuclear energy inherent in the inverse of the artificial synthesis of the neutron<sup>(4e)</sup> (the latter requiring, while the former releasing, 0.80 MeV), which are currently under study.<sup>(22c)</sup>

**(B) Applications in Nuclear Physics.** It is easy to predict that the above advances at the level of the structure of hadrons, and those in the structure of the neutron in particular, permit the study of basically novel aspects in nuclear physics. To begin, the representation of nucleons as extended, nonspherical, and deformable has permitted the achievement of

the first exact-numerical representation in scientific record of the total magnetic moment of the deuteron and of few-body nuclei,<sup>(50)</sup> which has eluded relativistic quantum mechanics for some three quarters of a century despite all possible corrections, including the inability of the latest attempts via the polarizability of quark orbits (because they produce a correction too small as compared to the missing 1% of the experimental value).

We recalled earlier the regaining of the exact  $SU(2)$ -isospin symmetry in nuclear physics via the embedding of all symmetry-breaking terms in the isounit.<sup>(3c)</sup> Numerous other aspects are currently under study, including: the representation of nuclear oscillations and deformations via an *exact* rotational symmetry; the treatment of nuclear dissipation via an axiomatically correct theory (rather than the axiomatically inconsistent triple systems of type (2.7)); the prediction under certain conditions of nuclear reactions caused by nonlinear, nonlocal, and nonpotential effects *against* the Coulomb barrier<sup>(49)</sup> which are rather lightly dismissed via conventional linear, local, and potential theories; and others.<sup>(22c)</sup>

Relativistic hadronic mechanics, however, permits deeper novel insights into nuclear forces. For instance, it implies the termination of the historical process whereby one keeps adding potentials to the Hamiltonian, admitting instead contact, nonlinear, nonlocal, and nonpotential interactions represented with generalized units, and reinterpreting some of the terms believed to be of "potential" type as being in reality of "contact" type without any potential energy. This is the case of the isotopic reinterpretation of the charge independence of nuclear interactions via the nonpotential effects of Ref. 49 indicated earlier. One should keep in mind that, even though quantitatively smaller than those in the hadronic structure, nonlinear, nonlocal, and nonpotential effects are indeed present in nuclear physics and may eventually be the origin of the lack of exact character of quantum mechanics in the field as compared to the majestic validity of the same discipline for the atomic phenomenology.

Among possible new applications, we mention ongoing theoretical and experimental studies on *conceivable new recyclings of nuclear waste based on the inverse of the synthesis of the neutron*<sup>(4c)</sup> (its stimulated decay) and other means, which can be used by nuclear power plants in house, thus avoiding altogether the need for its transportation to, and storage in, a yet unidentified dumping site (which is projected by the U.S. Department of Energy to cost some 230 billion dollars during the first five years). After all, mean lives are perennial and immutable for the Poincaré symmetry, but *not* for its isotopic completion.

(C) **Applications to Astrophysics and Cosmology.** Additional intriguing applications to astrophysics should also be indicated, such as the

exact numerical representation of the difference in cosmological redshift between *physically connected* quasars and galaxies, originally proposed in Ref. 51a, studied in Ref. 51b, and extended in Ref. 51c to the representation of the internal quasar blueshift and redshift.

Other applications exist on cosmology based on Santilli's isominkowskian geometry and its isodual (Sec. 3.14), with suggestive properties, such as equal distribution of matter and antimatter in the Universe and null total physical quantities of energy, time, etc., lack of need of the missing mass, and others.<sup>(22b, 34d)</sup>

(D) **Applications to Antimatter.** Additional independent applications of the methods studied in this paper exist in the "new physics of antimatter" via the use of the isodual isominkowskian geometry and isodual isopoincaré symmetry.<sup>(37)</sup> The latter theories reproduce known electroweak data, represent gravitational *attraction* for particle-antiparticles bound states in both fields of matter and antimatter, and predict *antigravity* for *elementary* antiparticles and their bound states in the field of Earth.

The latter prediction becomes mandatory following a forgotten *identification* (rather than "unification") of the gravitational and electromagnetic fields achieved by Santilli a couple of decades ago<sup>(37e)</sup> from the primary electromagnetic origin of mass. Such an identification then imposes the equivalence of the two phenomenologies, including the capability to reverse gravity. The above findings are so strong that the absence of antigravity would imply the lack of identification of gravitational and electromagnetic fields, with the consequential need of rewriting the foundations of electromagnetism and particle physics into a form avoiding the primary electromagnetic origin of mass.

The isodual theory of antimatter also predicts the existence of a *new photon*, called *isodual photon*,<sup>(37d)</sup> which is predicted to be emitted by the antihydrogen atom and be solely distinguishable from the ordinary photon via gravitational interactions.

If confirmed, the latter studies may permit, in due time, the first possibility on record to ascertain whether a far-away galaxy or quasar is made up of matter or of antimatter.

(E) **Applications in Theoretical Biology.** Particularly suggestive and novel are the applications of our new methods in theoretical biology.<sup>(35)</sup> The insufficiency of quantum mechanics in hadronic and nuclear physics is still reason for scientific debate in physical circles, while the same insufficiency must be widely admitted in theoretical biology for scientific credibility. In fact, physical systems such as a nucleus or a hadron are *stable*, thus being *reversible in time* and requiring *conservation laws* for their

quantitative treatment. On the contrary, biological systems *grow or decay*, thus being *structurally irreversible* and requiring *time rate of variations* of given characteristics of size, weight, etc. The insufficiency of quantum mechanics for biological systems is then beyond *credible* doubt.

Note that the above occurrence renders the *genotopic* methods (Sec. 3.12) better than the isotopic ones for applications to theoretical biology, with the multivalued hyperstructural methods (Sec. 3.13) being preferable owing to the complexities of the biological systems.

As a first elementary illustration, computer visualization conducted by Illert in Ref. 35a has shown that the *shape* of sea shells can indeed be represented in Euclidean three-dimensional space, but not their *growth in time*. In fact computer visualizations have shown that, under the strict implementation of the Euclidean axioms, sea shells grow in a deformed way and then crack. Santilli in Refs. 35a, 35b has then shown that the imposition instead of the axioms of the isoeuclidean geometry (with a time asymmetric isounit) permits a fully regular growth under computer visualization, with the representation via the geno-euclidean geometry being more axiomatically correct, and that under the hypereuclidean geometry being the most appropriate.

The above studies have established that sea shells and other biological structures are *perceived* by our sensorial capacities as belonging to our three-dimensional Euclidean space in view of our three Eustachian tubes, but the same structures may actually exist in a much more complex world.

As an example, quantitative studies on bifurcations of biological structures have indicated the need of a *bona-fide* "space-time machine" (closed loop in the forward time cone), which violates causality when formulated in conventional Minkowski space-time, but it is readily admitted by our new iso-self-scalar invariance (3.86) and iso-self-dual invariance, Eq. (3.127).

Also applicable to these biological aspects appears to be the "geometric propulsion" indicated earlier,<sup>(22a)</sup> according to which locomotion occurs *without the applicability of Newtonian forces and via the alteration instead of the geometry*. In fact, this new conception of locomotion is based on the isotopic invariant  $\text{length}^2 \times \text{unit}^2$  of Sec. 3.7 in which the alteration of the space unit implies a necessary inverse alteration of the distance, thus causing locomotion in space, while its application in space-time, including isoduality, implies the most general possible formulation of the "space-time machine."

The reader should keep in mind that other models of "space-time machine" currently under study in the physical literature require the complexification of the Minkowski metric, quantum fluctuations and the like. On the contrary the "space-time machine" in the above most general

possible realization based on the "geometric propulsion" in space and time is fully admitted by the *conventional* Minkowski space, in view of our new iso-self-scalar and iso-self-dual invariances recalled above.

Intriguingly, the possible realization in physics of these advanced geometric notions appears to be far in the future, because the alteration of the isounit in isogravitational representation (Sec. 3.11) requires the availability in small regions of space of extremely large amounts of energy which can be only conjectured at this time as existing in the ether. By comparison, the same "geometric locomotion" and "space-time machine" appear to be *already existent* in biological structures, and in actuality they appear to be *necessary* for quantitative representations of events such as bifurcations, the transportation of water in trees up to very high levels, certain manifestly non-Newtonian osmotic motions in cells, and others.<sup>(35)</sup>

The main aspect is that, despite its limitations, our sensory perception is expected to detect the existence of geometries axiomatically *inequivalent* to the Euclidean geometry. In fact, our senses do indeed detect the transition from a *flat* to a *curved* geometry. The *axiom-preserving character* of the isotopies, genotopies, and hyperstructures and their isoduals has then fundamental relevance for their application to biological structure, because it renders them compatible with our sensory perception.<sup>(35)</sup>

This implies the compatibility with our sensory perception of our *isotopic, genotopic or multidimensional-hyperstructural times and their isoduals*,<sup>(22b, 23b, 35)</sup> i.e., we *perceive* time as one-dimensional and one-directional while the methods of this paper render compatible with such a perception substantially more complex notions of times, and the same occurrence exists for space.

In closing, we suggest the following experiments:

**Proposed experiment 1:** *The finalization of the deformability of the intrinsic magnetic moment of neutrons under sufficiently intense external fields, as theoretically studied by Eder<sup>(6v)</sup> and preliminarily measured by Rauch<sup>(52)</sup> via neutron interferometric techniques.* This test is of evidently fundamental relevance because: it can provide independent verification of the old hypothesis that the alteration of conventional intrinsic magnetic moments of protons and neutrons when bound in a nuclear structure is necessary for a representation of total nuclear magnetic moments; it can establish the notion of isoparticle which is at the foundation of the new structure model of hadrons with physical constituents indicated earlier; it may permit new technological advances, such as the design of new equipment for the recycling of nuclear waste by the utilities themselves, thus avoiding their transportation to and storage in an as yet unidentified site; and other advances.



**Proposed experiment 2:** *Measure the difference of the redshift "component" of the tendency toward the red of sun light at sunset and sunrise.* This difference is visible to the naked eye and is predicted to have an isotopic origin.<sup>(22b, 34d)</sup> The measures can evidently confirm or deny the validity of Santilli's isominkowskian geometry within physical media such as our atmosphere, resolve the vexing problem of the origin of the large difference in redshifts of physically connected quasars and galaxies, as well as identify the geometry most effective in astrophysics and cosmology.

**Proposed experiment 3:** *Finalize the direct and indirect measures on deviations from the Minkowskian geometry inside hadrons, such as the measures on the behavior of the mean lives of unstable hadrons with energy,<sup>(41, 42)</sup> the Bose-Einstein correlation,<sup>(51)</sup> and others.* These measures are of such a fundamental character that they render conjectural any theory on the structure of hadrons, whether quarks of isoparticles, prior to their scientific finalization in a form without theoretically questionable assumptions in the data elaborations.

**Proposed experiment 4:** *Measure the total cross section of the reaction  $\gamma + n \rightarrow p + e + \bar{\nu}$  to confirm or deny the prediction of a peak for photons with 1.294 MeV.<sup>(4e)</sup>* This test can confirm or deny the new model of the hadronic structure with physical constituents in its most fundamental case, the synthesis of the neutron from protons and electrons only as occurring in stars, as well as confirm or deny the new subnuclear hadronic energy. An additional possible test is recommended for the reaction  $\gamma + \pi^0 \rightarrow$  photons which is predicted to have a peak for incident photons of 67 MeV.<sup>(22c)</sup>

**Proposed experiment 5:** *Measure the gravity of positrons in horizontal flight in Earth's field inside a suitably designed, sufficiently long and shielded vacuum tube, which has been theoretically predicted to experience antigravity<sup>(37b)</sup> and experimentally studied as being within technical feasibility.<sup>(53)</sup>* The latter experiment too, if successful, would have far-reaching implications for all of science. As an indication, the detection of antigravity would establish that the totality of contemporary mathematics, let alone of theoretical physics, that based on the trivial unit +1, is inapplicable for the treatment of antimatter (Sec. 3.14).

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