HADRONIC JOURNAL 16, 429–457 (1993)

PROBLEMATIC ASPECTS OF q-DEFORMATIONS AND THEIR ISOTOPIC RESOLUTION

Diego F. Lopez*

State University of Campinas, 13083 Campinas, Brasil

Received July 12, 1993; Revised November 25, 1993

Abstract

By recalling that q-deformations have an impeccable mathematical structure, we outline their numerous problematic aspects of physical nature which essentially emerge whenever attempting dynamical applications, thus implying evolution in time. We outline Santilli's initiation of q-deformations back in 1967 via isotopies and genotopies of classical and quantum mechanics. We show how they permit an axiomatic reformulation of q-deformations which leaves the results unchanged while avoiding their problematic aspects. We finally point out applications and experimental verifications which would be generally precluded to q-deformations without their consistent axiomatic reformulation.

*) Currently visiting The Institute for Basic Research, Box 1577, Palm Harbor FL 34682 USA, Fax 1-813-934 9275, E-mail: ibrrms@pinet.aip.org

1: PROBLEMATIC ASPECTS OF q-DEFORMATIONS

The so-called q-deformations (see, e.g., ref.s [1] and quoted literature) include a great variety of deformations of Quantum Mechanical (QM) formalisms whose *mathematical* consistency is impeccable. Nevertheless, the q-deformations are afflicted by a number of rather serious problematic aspects of *physical* character which emerge whenever dynamical applications are attempted, thus implying the evolution in time.

The latter problematic aspects and their resolution, together with a number of applications, have been studied in great details by Santilli in the three volumes [2] on Hadronic Mechanics (HM). They were also presented by this author [3] at the *International Conference on Symmetry Methods in Physics* held this past July at the J.I.N.R. in Dubna. This paper is an extended version of note [3]. By following ref.s [2], we classify q-deformations into the following primary types:

I) Deformation of the enveloping associative algebra. Let $\xi(L)$ be the universal enveloping associative algebra of a Lie algebra L with elements A, B, ... and conventional associative product AB over a field $F(\alpha,+,\times)$ with generic elements α, conventional sum + and product $\alpha*\beta := \alpha\beta$. This first type is characterized by the following generalization of the associative product AB

$$A\,B \;\Rightarrow\; A * B = q\,A\,B, \tag{1.1}$$

where q is an element of the base field (or a function).

II) Deformation of the Lie product. Let L be a Lie algebra in quantum mechanical realization on a Hilbert space $\mathcal{H}$ over a field $F(\alpha,+,\times)$ with fundamental commutation rules $rp - pr = i$ ($\hbar = 1$). This second type of q-deformation is based on the generalized commutators

$$r\,p - p\,r \;\Rightarrow\; r\,p - q\,p\,r = i\,f(q, ...) \tag{1.2}$$

where $f(q,...)$ is a sufficiently smooth, bounded and nonsingular function.

III) Deformation of the structure constants. let L be an n-dimensional Lie algebra with ordered basis X_i, envelope $\xi(L)$ and commutation rules $[X_i, X_j] = C_{ij}^{\ k} X_k$ over a field $F(\alpha,+,\times)$. This third type of deformations, which contains the Hopf algebras and others, is based on the preservation of the original product Lie $X_iX_j - X_jX_i$, while deforming this time the structure constants

$$X_i\,X_j - X_j\,X_i = C = C_{ij}^{\ k}\,X_k \;\Rightarrow\; X_i\,X_j - X_j\,X_i = F_{ij}^{\ k}(q, ..)\,X_k, \tag{1.3}$$

where the functions $F_{ij}^{\ k}$ are also sufficiently smooth, bounded and nonsingular.

Numerous other q-deformations exist in the literature (such as the deformation of creation-annihilation operators of the above Types I, II, III) [1] which can be derived via the techniques of the above three deformations. Others can be reduced to a combination of the above three types, or are given by to a combination of one of the above types with QM structures (such as the combination of *deformed* commutators (1.2) and *conventional* Heisenberg equations for the time evolution).

Again, the mathematical consistency of all the above q-deformations is undeniable. However, when considered for physical applications they require the necessary use of the dynamical time evolution, in which case the following problematic aspects emerge as identified in detail in Vol. I of ref.s [2] (see also [3]):

A) General loss of the Hermiticity/observability of the Hamiltonian and of other physical quantities. q-deformations imply a *nonunitary time evolution,* as necessary for Types I, II, III from the lack of canonicity of the commutation rules, and demonstrable, e.g., via quantization of the corresponding, classical, noncanonical theories (see below for more details). In turn, nonunitary time evolutions imply the following generalization of the structure of the enveloping associative algebra first identified in ref. [4]

$$\xi: A\,B \;\Rightarrow\; \hat{\xi}: A' * B' = A'\,Q\,B', \quad A' = U\,A\,U^{\dagger},\; B' = U\,B\,U^{\dagger}, \qquad (1.4a)$$

$$U\,U^{\dagger} \neq I, \qquad Q = (U\,U^{\dagger})^{-1}, \qquad (1.4b)$$

which evidently also applies to the product qAB. Still in turn, the above structure implies the loss of the Hermiticity/observability of the Hamiltonian and of other physical quantities. This is due to the fact that q-deformations are defined on a *conventional Hilbert space,* while the preservation of Hermiticity under lifting (1.4) demands the joint deformation of the base field and of the Hilbert space (see later on Lemma 3.1).

B) General loss of the measurement theory and consequential lack of applicability to experiments. q-deformations are deformations of the basic associative product AB and/or of Planck's constant $\hbar$ = 1, and/or of structure constants, without a corresponding redefinition of the unit as done in Santilli's isotopic theories [2]. Therefore, *q-deformations are theories without a left and right unit which remains invariant under the time evolution.*

This occurrence is transparent in lifting (1.1) which deforms the product AB ⇒ A*B = qAB without jointly deforming the unit as done in the foundations of hadronic mechanics [2,3,4]:

$$I \;\Rightarrow\; \hat{I} = q^{-1}. \qquad (1.5)$$

The lack of basic unit can also be established for deformations of Types II and III, e.g., under time evolution with ensuing nonunitary structure, and unification of all envelopes into isotopic form (1.4). The loss of the unit then implies the

evident loss of the measurement theory, owing to the necessary existence of a well defined, left and right unit for the very concept of measurement.

It should be indicated that the problem of the unit is much deeper of what may appear at a first inspection. QM theories have in actuality *two different units,* the unit $\hbar$ = 1 of the *field* and the unit I = diag. (1, 1, ...) of the *envelope.* Under deformations (1.1) we evidently have the loss of the unit I of the envelope.

In regards to the unit of the field we have two alternatives. The first is to keep the theory with one single associative product A*B = qAB which would then apply also to numbers $\alpha * \beta = q\alpha\beta$. In this first case one has evidently the loss also of the unit of the field. The q-deformations are then theories on a Hilbert space defined *over a commutative ring without unit.* The lack of applicability to experiments is then transparent.

The second alternative, which is that followed by the current literature [1], is to define deformations (1.1) on a Hilbert space defined over a *conventional* field which, as such, does possess the unit. This evidently implies that *deformations (1.1) are theories with two different associative multiplications, one for the envelope and one for the field.* The problem is that the differentiation of these two multiplications leads to the lack of observability of the physical quantities because it prevents the needed lifting of the underlying Hilbert space *and related field.*

In summary, rather deep technical reason related to the preservation of the observability at all times demand the unification of the associative product of the envelope with that of the field, as well as the unification of their unit (see Vol. I of ref.s [2] for a detailed treatment).

C) General lack of uniqueness of mathematical structures, such as Gaussian distributions, with consequential lack of uniqueness of physical laws. One of the strengths of quantum mechanics is the *uniqueness* of its mathematical structure (such as the exponentiation and related Gaussian) which evidently implies the known uniqueness of its physical laws (such as the uniqueness of Heisenberg's uncertainties as derivable from the unique Gaussian distribution). This uniqueness can be mathematically traced to the uniqueness of the basic unit of the theory, Planck's constant, as well as to the existence of a right and left unit of the universal enveloping operator algebra ξ(L).

The mathematical implications of the general lack of the basic unit implies that *q-deformations do not possess a consistent formulation of the Poincaré-Birkhoff-Witt theorem which is applicable at all times.* In fact, a necessary condition for the very formulation of the theorem is the existence and uniqueness of a left and right unit (see Jacobson [5]).

This means *the lack of existence of a unique, infinite-dimensional basis for the envelopes of q-deformations* and, therefore, *the lack of existence of a unique form of exponentiation.* In fact, q-deformations are known for the variety of their possible "exponentiations".

Even though mathematically correct (and intriguing), the above occurrences have rather severe physical consequences identified by Santilli [2], such as *the lack of uniqueness of a Gaussian distribution with consequential lack of uniqueness of the generalized uncertainties.* A similar situation occurs for other physical laws.

It should be stressed that the above occurrences *are not* referred to different physical laws for different q-deformations, which would be physically acceptable, but to different physical laws which can be introduced in *each* q-deformation.

D) General loss of special functions under time evolution. As well known, q-deformations are formulated at a fixed value of time, and so are the related q-special functions [1]. But under time evolution the q-*number* is replaced by the *operator* Q. The inapplicability of the q-special functions under time evolution is then consequential.

Again, this occurrence is fully acceptable on mathematical grounds. However, its physical implications are rather serious, such as the impossibility of performing a partial q-wave-analysis at all times.

E) General loss of the fundamental axioms of Einstein's special and general relativities. Even though not fully identified in the literature, all q-deformations imply a structural departure from *all* basic axioms of the special and general relativities, as established by the noncanonicity of the commutation rules, the nonunitary character of the time evolution, the deformation of the structure constants of the Poincaré symmetry, etc.

Again, this occurrence can be mathematically intriguing, but it carries rather serious physical problems in the compliance with physical reality which must be addressed prior to any physical application.

The reader can derive numerous additional problematic aspects as a consequence of the above primary ones.

In the following we shall review Santilli's origination of q-deformations back in 1967 because it provides significant insights in their appropriate treatment, and then his axiomatization of q-deformations which avoids all the preceding problematic aspects. After achieving a physically consistent re-formulations, we shall then point out numerous physical applications of q-deformations which would be otherwise precluded.

2: ORIGIN OF q-DEFORMATIONS

When studying the axiomatic structure of quantum mechanics, the first and most fundamental task is the identification of the algebra characterized by the commutator $[A,B] = AB - BA$, the Lie algebra [5]. Similarly, when studying q-deformations, the identification of the algebra characterized by the "commutator" $[A,B]_q = AB - qBA$ is an evident pre-requisite for the achievement of a consistent

axiomatization.

The algebra characterized by the product $[A,B]_q$ was first introduced by the American mathematician Albert [6] back in 1948, via the following notions:

Lie-admissibility: a (generally nonassociative) algebra U with elements a, b, c, ... and (abstract) product ab over a field F is said to be *Lie-admissible* when the attached algebra U^-, which is the same vector space as U but equipped with the product $[A,b]_U = ab - ba$, is a Lie algebra. A Lie-admissible (or any other) algebra U is said to be *flexible* when it verifies the weaker form of associativity $a(ba) = (ab)a$ for all $a,b, \in U$.

Jordan-admissibility: the algebra U is said to be *Jordan-admissible* if the attached algebra U^+, which is the vector space U equipped with the product $\{a,b\}_U = ab + ba$, is a (commutative) Jordan algebra [7]. An algebra U is said to be *noncommutative Jordan algebra* when the product ab is noncommutative but verifies Jordan axiom $(ab)a^2 = a(ba^2)$.

The first introduction of q-deformations in the mathematical and physical literature was done by the physicist Santilli [8] back in 1967 as part of his Ph.D. in physics at the University of Turin, Italy. In fact, in [8], p. 573, one can see the first appearance of the product

$$(a,b) = \lambda\, ab - \mu\, ba = \rho\,[a,b] + \sigma\,\{a,b\}, \quad \lambda = \rho + \sigma, \mu = \rho - \sigma \in F, \quad (2.1)$$

which, for ab associative, was introduced as characterizing an algebra U which is Lie-admissible, Jordan-admissible, flexible as well as noncommutative Jordan. Moreover, product (2.1) was introduced as the (λ, μ)-*mutation* of a generic (not necessarily associative) algebra U with product ab, in order to distinguish it from *deformations* of an algebra as conventionally understood in mathematics.

In fact, formulation of Type III are true "deformations", but formulations of Types I and II are not thus justifying the term "mutations". Nevertheless, the term "deformation" is now entered in the literature and will be kept in this paper to avoid confusion.

It is evident that the q-deformation $[A,B]_q = AB - qBA$ is a particular case of Santilli's mutation for $\lambda = 1, \mu = q$ and ab associative.

One should note the virtually complete silence in the entire literature [1] on the above origin of q-deformations. This is rather odd because Albert's notion of Lie-admissibility, or the emergence of the still open Jordan's legacy alone, should be reason for their quotation.

To clarify the priority of product (2.1) we recall that Albert presented in [6] an abstract (and relatively short) treatment of Lie-admissibility, with more emphasis on the Jordan-admissibility because of its greater interest in the mathematics of the time. In fact. the sole explicit realization of the product in Albert's paper is given by the known realization of noncommutative Jordan algebras [6,7]

$$(a,b) = \lambda\, ab - (1-\lambda)\, ba, \qquad (2.2)$$

for ab associative. The point is that q-deformations are a particular case of Santilli's mutation (2.1) and not of Jordan's form (2.2).

Santilli is therefore the originator on both mathematical and physical grounds of theories today known as *Lie-admissible formulations*, and referred to a step-by-step generalization of Lie's theory, with realizations in classical, operator and statistical mechanics. This priority is now acknowledged in mathematical circles (see, e.g., the historical charts of ref. [10], p. 13, or the mathematical monographs of ref.s [34]). In fact, following Albert [6] and prior to paper [8], only two short mathematical notes in Lie-admissibility had appeared (see [8] and bibliography [9]), also without any specific realization.

On mathematical grounds, Lie-admissible algebras had been studied as *nonassociative* algebras, an approach still continuing in the mathematical literature [9]. On the contrary, Santilli constructed a generalization of enveloping *associative* algebras characterizing Lie-admissible algebras, groups, representation theory, etc., which subsequently resulted to be crucial for the axiomatization of q-deformations presented in below.

On physical grounds, Santilli studied already in 1968 [11] the *classical limit* of the (λ, μ)-mutations (2.1), by proving that they are a particular case of Hamilton's equations with external terms. This established that the mutations AB − BA → λAB − μBA imply the transition from closed-conservative to open-nonconservative systems, because of the loss of total antisymmetry of the product.

These initial classical studies were then complemented in 1978 [12] with the identification that *the brackets of Hamilton's equation with external terms, when properly written, characterize a general Lie-admissible algebra*. In fact, we can write for N particles in "phase space" with unified coordinates $a = (a^{\mu}) = (r_a{}^k, p_{ak})$, $\mu = 1, 2, \ldots, 6N$, $k = 1, 2, 3$, $a = 1, 2, \ldots, N$,

$$(\dot a^{\mu}) = \begin{pmatrix} \dot r_a{}^k \\ \dot p_{ak} \end{pmatrix} = \begin{Bmatrix} \partial H/\partial p_{ak} \\ -\partial H/\partial r_a{}^k + F_k \end{Bmatrix} = \left(\omega^{\mu\alpha}\, 1^{>}{}_{\alpha}{}^{\nu}(t, a, \dot a, \ldots) \frac{\partial H}{\partial a^{\nu}} \right), \qquad (2.3)$$

where $\omega^{\mu\alpha}$ is the conventional canonical Lie tensor, $1^{>}{}_{\alpha}{}^{\nu} = \delta_{\alpha}{}^{\nu} + \omega_{\alpha\rho}\, s^{\rho\nu}$, s = diag. (0, F/(∂H/∂p)) and the meaning of the symbol ">" will be identified later on. The corresponding brackets among functions in "phase space"

$$(A, B) = \frac{\partial A}{\partial a^{\mu}}\, \omega^{\mu\alpha}\, 1^{>}{}_{\alpha}{}^{\nu}(t, a, \dot a, \ldots) \frac{\partial B}{\partial a^{\nu}} \qquad (2.4)$$

are then Lie-admissible because the attached brackets are twice the conventional Poisson brackets, (A, B) − (B, A) = 2[A, B].

The need for the reformulation emerges from the fact that the brackets of the original Hamilton's equations with external terms violate the right scalar and distributive law and, as such, they do not characterize any algebra (see (see Vol. II of ref. [13] fpor details). Intriguingly, the classical brackets (2.4) *are not* Jordan-admissible, as one can verify. Only their operator counterparts (see Eq. (2.8) below) are Jordan-admissible.

These classical studies were systematically continued in monographs [13,14] via: the classical version of the Lie-admissible formulations with exponentiated group structure called *classical Lie-admissible group*

$$a' = \left(e^{w_k \partial_{\nu} X_k\, \omega^{\mu\alpha} 1^{>}{}_{\alpha}{}^{\nu} \partial_{\nu}} \right) a, \qquad (2.5)$$

admitting a non-Lie, Lie-admissible structure in the neighborhood of the identity; the Lie-admissible generalization of Lie's first, second and third theorems; the identification of the *exterior-admissible calculus*, as a generalization of the conventional exterior calculus; the introduction of the main lines of the *symplectic-admissible geometry* as the classical geometry underlying brackets (2.4); the derivation of Hamilton's equations with external term from the variational principle (despite their *variational nonselfadjointness* -NSA- [12])

$$\delta \hat A^{>} = \delta \int_{-\infty}^{+\infty} (p{>}dr - H{>}dt) = 0, \qquad (2.6)$$

where $\Phi_1{}^{>} = p{>}dr := p\, T_0{}^{>}dr$ is the *exterior-admissible one-form* characterized by a nonsymmetric matrix $T_0{}^{>}$; the Hamilton-Jacobi equations for principle (2.6), etc.

To understand the significance of these studies it is sufficient to note that they imply a generalization of Noether's theorem in which the Lie-admissible symmetry characterizes *time-rate-of-variations of physical quantities*. The conventional Noether's theorem is then an evident particular case when the time-rate-of-variation is null.

On operator grounds, Santilli was the first to introduced back in 1978 [4]: the *general Lie-admissible and Jordan admissible algebras* with brackets

$$(A, B) = APB - BQA, \qquad (2.7)$$

where P and Q are operators; the well known *Lie-admissible equations* in the infinitesimal form [4], p. 746 ($\hbar$ = 1),

$$i\, \dot A = (A, H) = A\, P\, H - H\, Q\, A, \qquad (2.8)$$

with corresponding finite form [4], p. 783,

$$A(t) = e^{iHQt} A(0) e^{-itPH}; \tag{2.9}$$

the *fundamental Lie-admissible commutation rules* [4], p. 746,

$$(a^\mu, a^\nu) = a^\mu P a^\nu - a^\nu Q a^\mu = i\,\omega^{\mu\alpha} 1_\alpha{}^\nu; \tag{2.10}$$

the first formulation of Lie-admissible operator algebras on bimodular Hilbert spaces; and other advances.

Subsequently, Fronteau, Tellez-Arenas and Santilli [15] were the first to identify in 1979 the Lie-admissible structure of the most general possible equations in statistical mechanics, those with an arbitrary collisions term C,

$$i\,\dot{\rho} = (\rho, H) = \rho P H - H Q \rho = \rho H - H \rho + C. \tag{2.11}$$

The need for the Lie-admissible reformulation stems from the fact that the brackets $\rho \times \eta = \rho H - H\rho + C$ violate the scalar and distributive laws and, therefore, do not characterize any algebra of any kind. This implies that familiar notions such as "a proton with spin ½" which are well defined for brackets $[\rho, H] = \rho H - H\rho$ have no mathematical or physical sense for brackets $\rho \times H = \rho H - H\rho + C$.

The generalized Schrödinger's counterpart of Lie-admissible equations (2.8) was identified by Myung and Santilli [16] in 1982 and, independently, Mignani [17] according to the expressions

$$i \frac{\partial}{\partial t} |\psi\rangle = H Q |\psi\rangle, \quad -i \langle\psi| \frac{\partial}{\partial t} = \langle\psi| P H. \tag{2.12}$$

The identification of the correct form of the linear momentum operator required considerable additional studies at the *classical* level [13,14], which eventually permitted Santilli [18] to reach the axiomatically correct form

$$p_k Q |\psi\rangle = -i (Q^{-1})_k{}^i \nabla_i |\psi\rangle, \quad \langle\psi| P p_k = i \langle\psi| \nabla_i (P^{-1})_k{}^i. \tag{2.13}$$

achieved via the prior identification of the Hamilton-Jacobi equations for principle (2.6). The above classical and operator formulations were then interconnected by a unique map called *isoquantization*, first identified by Animalu and Santilli (see ref.s [2]). The simplest possible case, called *naive isoquantization*, maps the Hamilton-Jacobi equations for principle (2.6) into Eq.s (2.12) via the rule

$$\hat{A}^> \Rightarrow -i\,1^> \,\mathrm{Log}\,\psi, \tag{2.14}$$

where $1^> = Q^{-1}$ for the envelope acting to the right, with corresponding conjugate quantities for the envelope acting to the left.

Note for subsequent needs the primary role of the universal enveloping associative algebras in the above Lie-admissible formulations, exactly as it is the case for the conventional Lie formulations [5].

Additional biographical data worth an indication are the following. The first deformation of the (λ,μ)-mutation of SU(2) spin was presented by Santilli at the Clausthal Conference on *Differential Geometric Methods in Physics* of 1980 [19]. The first generalizations of the rotational and Lorentz symmetries for operators P = Q was reached in [20,21]. The first identification of the underlying generalizations of symplectic, affine and Riemannian geometries was done in [22]; the first Q-operator generalization of gauge theories was reached by Gasperini [23] in 1983; the first studies of the Lie-admissible generalization of creation and annihilation operators were conducted by Jannussis *et al.* [24] beginning from 1981; Mignani [25] initiated the construction of a Lie-admissible scattering theory, subsequently completed by Santilli [2,18] via the use of special P-Q-functions; Okubo [26] identified certain "no go" theorems for operator formulations with nonassociative envelopes; Kalnay and Santilli [27] discovered the operator form of Nambu's mechanics for triplets with an essential Lie-admissible structure; Animalu [28] was the first to apply the methods to electron pairing in superconductivity; Kadeisvili [29] initiated the systematic study of special functions, distributions and transforms compatible with Lie-admissible structures; additional studies were conducted by Nishioka [30], Aringazin [31], Lopez [32], and others.

A comprehensive presentation of all these operator studies is now available in the three volumes on HM [2] (see also ref. [33] for a recent review), which is based on the main classification of HM into:

Lie-admissible formulations, applicable when the energy is not conserved, i.e., from Eq.s (2.8), $i\,\dot{H} = (H, H) = H(P - Q)H \neq 0$; and the simpler

Lie-isotopic formulations, applicable when the energy is conserved, which occur when in Eq.s (2.8) P = Q, $i\,\dot{A} = [A \,\hat{,}\, H] = AQH - HQA$, in which case the algebra is still Lie, although of a more general type.

Equivalently, the two branches can be identified via their underlying methods, which were called in ref. [12]:

Isotopies, when the original axioms are preserved, as it is the case for the Lie-isotopic branch of HM; and

Genotopies, which apply when the original axioms are replaced by covering axioms, as it is the case for the Lie-admissible branch.

As marginal comments, we should note that the scripture A*B = qAB is correct but only when q is in the center of the algebra. In fact, the "product" $A*B = qAB$ for q a fixed operator violates the left distributive and scalar laws and, as such, it does not characterize any algebra of any type. This is the reason why Santilli's writes the deformation in the form $A*B = AqB$ which now verifies the left and right scalar and distributive laws for arbitrary operator realizations of q. Similarly, the correct form of writing deformation (1.2) for arbitrary q is $rp - pqr$ because the form $rp - qpr$ for q a fixed operator does not characterize any algebra of any kind [2,8,12].

3: SANTILLI'S AXIOMATIZATION OF q-DEFORMATIONS

We now present a dual axiomatization of q-deformations worked out by Santilli [2,12,18] which avoids the problematic aspects of Sect. 1. The first is of Lie-isotopic type, and the second is of the more general Lie-admissible type. The former is sufficient for q-deformations of Type I and III, while those of Type II demand the full Lie-admissible treatment.

The emerging axiomatization is naturally applicable for operator Q with an arbitrary, nonlinear, nonlocal and noncanonical dependence $Q = Q(t, r, \dot{r}, \ddot{r}, \psi, \partial\psi, \partial\partial\psi, \ldots)$. Within this context, QM emerges as describing the *exterior particle problem*, that is, motion of point-like particles in vacuum, while HM applies for the *interior particle problems*, that is, extended-deformable particles moving within hyperdense physical media, thus resulting in the most general known equations of motions with an arbitrary nonlinearity and nonlocality (in $\dot{x}$, $\ddot{x}$, $\partial\psi, \ldots$). Also, the operator Q is restricted, by construction, to recover the identity when motion returns to be in vacuum. In this way, HM is a *covering* of QM.

Before entering in the field, the reader should be aware of its dimension. HM is first divided into the the Lie-isotopic and Lie-admissible branches, and then each of them is classified into *Kadeisvili five classes:* I (when the isotopic elements are sufficiently smooth, bounded, nowhere singular, Hermitean and positive-definite), II (when the isotopic elements are the same as in I but negative-definite), III (the union of I and II), IV (when the isotopic elements are degenerate), and IV (when the isotopic elements are arbitrary, i.e., discrete structures, distributions, lattices, etc.) [29]. For the Lie-isotopic cases these characterizations refer to the operator Q, while for the Lie-admissible case they refer to the maximal Hermitean part of P and Q. In this note we can only consider for brevity HM formulations of Class I (see ref.s [2] for the other classes).

We shall now first study the isotopic axiomatization, which can be summarized via the following basic points.

1) Recall that Lie algebras L with product $[A, B] = AB - BA$ over F are the antisymmetric algebras $[\xi(L)]^-$ attached to the universal enveloping algebra $\xi(L)$ with conventional associative product AB [5]. The first point is to focus the attention of the deformations of ξ, and construct the brackets of the time evolution only thereafter.

2) Consider the q-deformations ξ_q of ξ characterized by

$$\xi: \ AB \quad \Rightarrow \quad \xi_q : \ qAB, \tag{3.1}$$

Santilli's fundamental point is that *any deformation of the conventional product AB necessarily requires a corresponding generalization of the basic (multiplicative) unit.* In fact, it is "anathema" in number theory to change the product and keep the old unit, or viceversa, because units and products are deeply inter-related. Recall that the basic (left and right) unit of ξ is the trivial unit matrix I, $IA = AI \equiv A$, $\forall A \in \xi$.

The fundamental assumption is the interpretation of deformations (3.1) as a redefinition of the basic associative product in term of the Q-operator (of Class I) called *isotopic element* [4,12]

$$\xi_Q: \quad A * B := A\,Q\,B\,, \ Q = \text{fixed} \tag{3.2}$$

We then have the consequential generalization of the unit I into the form $\hat{1} = Q^{-1}$ called *isounit*, which is such to be the correct left and right unit of the Q-theory

$$\hat{1} = Q^{-1}, \qquad \hat{1} * A = A * \hat{1} \equiv A\,, \ \forall \ A \in \xi_Q. \tag{3.3}$$

Santilli identified other isotopies of associative algebras, such as the form $AB \Rightarrow A*B = WAWBW$ with W idempotent, $W^2 = W$, which preserves associativity. The latter isotopies were however rejected for the construction of physical theories because they do not admit a unit. This is further illustration of the emphasis throughout Santilli's studied on the preservation of the basic unit.

3) The generalization of the multiplication and related unit requires, for mathematical consistency, a generalization of the notion of "numbers". Recall that a field $F(\alpha,+,*)$ is a set of elements α, β, γ, equipped with two operations and related units, the (associative and commutative) sum + with *additive unit* 0, $\alpha + 0 = 0 + \alpha \equiv \alpha$, and the (associative but not necessarily commutative) multiplication $\times$, $\alpha\times\beta \equiv \alpha\beta$, with *multiplicative unit* I, $I\times\alpha = \alpha\times I \equiv \alpha$, which is closed under sum, multiplication and their combinations (left and right distributive laws). At the 1980 Clausthal Conference on *Differential Geometric Methods in Physics*, Santilli [22] introduced the isotopies

$$F(\alpha,+,\times) \Rightarrow \hat{F}_Q(\hat{\alpha},+,*)\,, \ \times \Rightarrow * := \times Q \times\,, \quad \alpha \Rightarrow \hat{\alpha} := \alpha\hat{1}\,, \ I \Rightarrow \hat{1} := Q^{-1}\,, \tag{3.4}$$

characterizing *isofields*. In particular, for $Q = q \in F$ the lifting $\alpha \Rightarrow \hat{\alpha} = \alpha\hat{1}$ is unnecessary because the set $\hat{F}_q(\alpha,+,*)$ is a field (see Propositions I.2.3.1 and I.2.3.2 of ref. [2]). However, the generalization of numbers $\alpha \Rightarrow \hat{\alpha} = \alpha\hat{1}$ is needed whenever Q is not an element of the original field F, as a necessary condition for isotopies, i.e., for $\hat{F}$ to preserve all the original axioms of F [22]. It is evident that this third step requires a suitable isotopic generalization of all operations on numbers, e.g., $\hat{a}^{\hat{n}} = \hat{a}*\hat{a}*...*\hat{a} = a^n \hat{1}$ (n times), $\hat{a} \hat{/} \hat{b} = a/b$, $\hat{/} = \hat{1}/$; $\hat{a}^{\hat{\frac{1}{2}}} = a^{\frac{1}{2}} \hat{1}$ (see [22]).

4) Recall that conventional carrier spaces are defined over conventional fields. The generalization of multiplication, unit and fields evidently requires, also for mathematical consistency, a compatible generalization of conventional carrier spaces, introduced for the first time by Santilli [21] in 1983. Let S(x,g,R) be a metric or pseudo-metric space with local coordinates x and (Hermitean, nowhere singular) metric g over the reals R. The isotopies necessary under Q-deformations are

$$S(x,g,R) : x^2 = x^t g\, x \in R \Rightarrow \hat{S}(x,\hat{g},\hat{R}) : x^{\hat{2}} = (x^t \hat{g}\, x)\hat{1} \in \hat{R},\ \hat{g} = Qg,\ \hat{1} = Q^{-1}. \quad (3.5)$$

Isospaces $\hat{S}(x,\hat{g},\hat{R})$ characterize fundamentally novel geometries called *isoeuclidean, isominkowskian and isoriemannian,* with intriguing mathematical and physical implications, such as the isotopic generalization of conventional angles, the geometric unification of spheres, ellipsoids and hyperboloids, etc. [2,14, 20–22].

Note that the original geometries are local-differential while Santilli's isogeometries are nonlocal-integral, as well as nonlinear in the velocities and the derivatives of the wavefunction, as needed for interior dynamical problem. This is due to the arbitrary functional dependence of the isometric $\hat{g}(t, x, \dot{x}, \ddot{x}, \psi, \partial\psi, ...)$.

A most intriguing property of the isogeometries is that they deform any given structure. However, this deformation is seen only in the projection to the original space because in isospace the original structure is preserved in its entirety. Thus, isotopies deform straight lines, circles and cones into geometric structures called *isostraight lines, isocircles* and *isocones,* which are perfect straight lines, circles and cones, respectively, in isospaces, but are deformed when projected in our space.

This remarkable occurrence is due to the joint lifting of metric $g \rightarrow \hat{g} = Qg$ *and* of the unit in the amount which is the inverse of the deformation of the metric, $I \Rightarrow \hat{1} = Q^{-1}$ and is at the foundation of the resolution of problematic aspect E (loss of Einsteinian axioms for conventional deformations). In fact, the preservation of the perfect light cone under deformations evidently permits the preservation of the basic axioms of the special relativity (see ref.s [2] for brevity).

As an example, the perfect sphere in Euclidean space E(r,δ,R) represented by the metric $g = \delta = \text{diag.}\,(1,\ 1,\ 1)$ can be deformed into the ellipsoids $\hat{g} = \hat{\delta} = Q\delta = \text{diag.}\,(b_1^2, b_2^2, b_3^2)$, $b_k \neq 0$. However, in isospace the original sphere remains perfectly spherical because of the joint lifting of the unit $I \Rightarrow \hat{1} = \text{diag.}\,(b_1^{-2}, b_2^{-2}, b_3^{-2})$, that is, for each semiaxis we have the lifting $I \Rightarrow b_k^2$ which is compensated by the opposite lifting of the relative unit $I \Rightarrow b_k^{-2}$. One of the intriguing

consequences is that the conventional rotational symmetry O(3) is evidently lost for the ellipsoidical deformations of the sphere, but it is reconstructed as an exact symmetry under Santilli's joint lifting $\delta \Rightarrow \hat{\delta} = Q\delta$ *and* $I \Rightarrow \hat{1} = Q^{-1}$.

These novel geometric properties have predictable fundamental implications, such as the reconstruction of the exact *light cone* in vacuum for electromagnetic waves propagating within physical media with a locally varying speed [2].

The reader can begin to see the horizon of novel applications which is available to q-deformations, but only when lifted into an axiomatic Q-operator/matrix form. In fact, for q-number there is no meaningful deformation of Minkowski or Riemann, trivially, because in this latter case $x^{\hat{2}} = (x^t qgx)q^{-1} = (x^t gx)qq^{-1} = x^t gx \equiv x^2$.

4) The liftings of multiplication, unit, fields and carrier spaces require a compatible lifting of the transformation theory into the so-called *isotransformations*

$$x' = U\, x \quad \Rightarrow \quad x' = \hat{U} * x = H\, Q\, x, \quad Q = \text{fixed}. \quad (3.6)$$

first introduced in ref. [12] of 1978. Note that the preservation of the old transformation $x' = Ux$ under isotopies implies the loss of linearity, transitivity, superposition principle, etc.

Note also that isotransforms are nonlinear, nonlocal and noncanonical only when projected in the original space, while they verify the axioms of linearity, locality and canonicity at the isotopic level. For this reason they are called *isolinear, isolocal and isocanonical.*

This is another property of isotopic methods permitting further applications of Q-deformations via the turning of given nonlinear-integral theories into *identical* isolinear and isolocal forms, thus being manifestly more manageable.

5) The generalization of the multiplication, unit, field, carrier spaces and transformation theory then requires a step-by-step generalization of the entire Lie theory into a form originally submitted as *Lie-isotopic theory* [12] and today known as the *Lie-Santilli theory* (see papers [23–33] and monographs [34]). We are here referring to the isotopics of all structural parts of Lie's theory, such as enveloping algebras, Lie algebras, Lie groups, representation theory, symmetries and first integrals, etc.

The fundamental isotopies, those of enveloping associative algebras, were the central topic of the original proposal [12]. Most important is the first achievement of the isotopies of the Poincaré-Birkhoff-Witt theorem on the infinite-dimensional basis of ξ, which provides the new basis of ξ_Q and the correct exponentiation under isotopies, called *isoexponentiation*

$$e_{\xi_Q}^{[illegible]} = \hat{1} + (i\hat{w}*X)/1! + (i\hat{w}*X)*(i\hat{w}*X)/2! + \ldots = (e^{[illegible]})\hat{1}. \qquad (3.7)$$

Particularly important is the *uniqueness* of the above isoexponentiation (up to isoequivalence transformations studied below), which should be compared to the various types of q–exponentiation in the literature [1].

Yet another horizon of applications for q–deformations emerge from isoexponentiation (3.7), such as the isotopic lifting of Dirac's $\delta(x)$ to spread its singularity at $x = 0$ over a finite region of space, thus removing the singularities afflicting conventional theories from the beginning [2].

6) The above isotopies imply corresponding lifting of Lie algebras into the *Lie–Santilli algebras* [12,34]

$$[X_i, X_j] = X_i X_j - X_j X_i = C_{ij}{}^k X_k \Rightarrow [X_i \hat{,} X_j] = X_i Q X_j - X_j Q X_i = \hat{C}_{ij}{}^k * X_k, \quad (3.8)$$

where the $\hat{C}$'s are called *structure isofunctions*, and depend on all needed local variables and their derivatives. Note the preservation of the Lie axioms by the isotopic product AQB – BQA (and *not* by QAB – QBA).

The existence of a unique infinite–dimensional basis for the isoexponentiations then permits the identification of the (connected) *Lie–Santilli groups*

$$x' = \hat{U}(\hat{w}) * x = (e_{\xi_Q}^{\,i X * \hat{w}}) * x = (e^{i X Q w})\, x, \qquad (3.9a)$$

$$\hat{U}(0) = \hat{U}(\hat{w}) * (\hat{U} - \hat{w}) = \hat{1} = Q^{-1},\ \hat{U}(\hat{w}) * \hat{U}(\hat{w}') = \hat{U}(\hat{w}') * \hat{U}(\hat{w}) = \hat{U}(\hat{w} + \hat{w}'), \qquad (3.9b)$$

$$(e_{\xi_Q}^{X_1}) * (e_{\xi_Q}^{X_2}) = e_{\xi_Q}^{X_3}, \qquad X_3 = X_1 + X_2 + [X_1 \hat{,} X_2] + \ldots \qquad (3.9c)$$

In turn, the above liftings imply the isotopies of the representation theory, symmetries and first integrals, etc. Note the nontriviality of the isotopies, as transparently exhibited by *the appearance of an unrestricted, nonlinear, integro–differential operator Q in the exponent of the group structure (3.9a).* In fact, the isotopic image of the conventional linear–local–canonical rotations, Lorentz and Poincaré transformations are given by highly nonlinear–nonlocal–noncanonical generalizations.

The remarkable property of Santilli's isotopies is that, despite these differences, the isotopic groups are isomorphic to the original groups for all positive–definite Q, $\hat{O}_Q(3) \approx O(3)$, $\hat{O}_Q(3.1) \approx O(3.1)$, $\hat{P}_Q(3.1) \approx P(3.1)$, $S\hat{U}_Q(2) \approx SU(2)$, $S\hat{U}_Q(3) \approx SU(3)$, etc.

In fact, the isotopies are introduced as methods for the reconstruction of exact space–time and internal symmetries when *believed* to be broken. One should expect this property from the preservation of the geodesics of the original symmetry in isospace mentioned earlier. In fact, the *isorotational group* $\hat{O}_Q(3)$ was introduced [20] to show that the *rotational* symmetry remains *exact* for all the *ellipsoidical deformations* of the sphere $\hat{\delta}$ = diag. $(b_1{}^2, b_2{}^2, b_3{}^2)$, $b_k \neq 0$. Similarly, the Lorentz and Poincaré symmetries remain *exact* for all signature preserving *nonlinear–nonlocal–noncanonical deformations* of the Minkowski metric $\hat{\eta} = Q\eta$ [21], etc.

Intriguingly, when the Q–element depends only on the local coordinates, Q = Q(x), *the isopoincaré symmetry* $\hat{P}_Q(3.1)$ *provides the universal invariance of all possible conventional Riemannian metrics* $g(x) = Q(x)\eta$.

Further physical applications of the Q–deformations then emerge in conventional gravitation, such as the characterization of the gravitational horizon as the zeros of Q, and the gravitational singularities as the zeros of the isounit $\hat{1}$ (see [2] for detail).

7) The preceding isotopies further imply a step–by–step generalization of functional analysis into a new discipline called *functional isoanalysis* [29], in which all conventional operations (say, log, derivative, integral, etc.), distributions (Dirac's delta, etc.), transforms (Fourier, Laplace and other transforms), special polynomials (Legendre polynomials, spherical harmonics, etc.), weak and strong continuity, etc. are generalized into a *unique* form compatible with the basic isounit $\hat{1} = Q^{-1}$ which is applicable at all times. See ref.s [2] for a comprehensive presentation with applications.

8) The above chain of interconnected isotopies can indeed be formulated on a *conventional* Hilbert space $\mathcal{H}$, as done in the original proposal [4]. However, this implies the general loss of Hermiticity because isohermiticity is now defined by

$$H^{\hat{\dagger}} = Q H^{\dagger} Q^{-1}. \qquad (3.10)$$

Lemma 3.1: *An operator* $H \in \xi_q$ *which is originally Hermitean under q–number–deformations at time* $t = 0$, *over a conventional Hilbert space* $\mathcal{H}$, *becomes generally nonhermitean over the same space* $\mathcal{H}$ *under nonunitary time evolutions leading to a Q–operator–deformation, unless* Q *and* H *commute.*

For this reason, Myung and Santilli [18] introduced in 1982 the *isohilbert space* $\hat{\mathcal{H}}_Q$ characterized by the lifting

$$\mathcal{H}: \langle \psi | \phi \rangle = \int d^3r\, \psi^\dagger(r)\, \phi(r) \in C \Rightarrow \hat{\mathcal{H}}_Q: \langle \psi \hat{|} \phi \rangle := \hat{1} \int d^3r\, \psi^\dagger(r)\, Q(r, \ldots) \phi(r) \in \hat{C} \quad (3.11)$$

in which case isohermiticity coincides with Hermiticity. *This is a first manifestly fundamental property of Santilli's axiomatization of Q–deformations because it permits the preservation of observability under arbitrary time evolutions* (the issue whether the observable is conserved or not is a separate one treated below). Note that for Q positive–definite the composition is still inner and $\hat{\mathcal{H}}_Q$ is still Hilbert. Note also that for Q independent of the integration variables (or constant), $\hat{\mathcal{H}}_Q \equiv \mathcal{H}$ because in this case

$$<\psi \hat{|} \phi> = <\psi | Q | \phi > \hat{1} \equiv <\psi | \phi>. \tag{3.12}$$

In this sense, *Myung-Santilli isohilbert spaces* are "hidden" in the conventional formulation of q-deformations. Note the crucial condition for consistency of lifting the field $F \Rightarrow \hat{F}_Q$. In fact, the isospace $\hat{\mathcal{H}}_Q$ on a conventional field F has no mathematical or physical sense. Interested mathematicians are encouraged to extend the results to formal aspects, such as selfadjointness.

9) The preceding isotopies imply, also for mathematical consistency, compatible and *unique* generalizations of *all* operations on $\mathcal{H}$ into forms called *isolinear operations* on $\hat{\mathcal{H}}_Q$ [2,33]. We here limit ourselves to indicate *isounitarity*

$$\hat{U} * \hat{U}^{\dagger} = \hat{U}^{\dagger} * \hat{U} = \hat{1} \quad , \text{or } \hat{U}^{-\hat{1}} = \hat{U}^{\dagger}, \tag{3.13}$$

isounitary transformations

$$A' = \hat{U} * A * \hat{U}^{\dagger}, \tag{3.14}$$

with realization in term of an isohermitean operator X,

$$\hat{U} = e_{\xi_Q}{}^{i X * \hat{w}}, \tag{3.15}$$

the notions of determinant and trace of a matrix A

$$\hat{\text{Det}}\, A = [\,\text{Det}\,(AQ)\,]\hat{1} \in \hat{F}, \quad \hat{\text{Tr}}\, A = (\text{Tr}\, A)\hat{1} \in \hat{F}, \tag{3.16}$$

the isoprojection operator

$$\hat{P} = \sum_k |\psi_k><\psi_k| Q^{-1}, \tag{3.17}$$

the isotopies of eigenvalue equations

$$H * |\hat{\psi}> = \hat{E} * |\hat{\psi}>, \quad \hat{E} \in \hat{F}_Q, \tag{3.18}$$

and similarly for all other operations.

Recall that isotopic techniques turn nonlinear, nonlocal and noncanonical theories into *identical* forms verifying the axioms of linearity, locality and canonicity at the isotopic level. This is a general property of the methods which also holds for nonunitary transforms $UU^{\dagger} \neq I$. In fact, the transformation theory of hadronic mechanics is *nonunitary*, although expressed in an identical form $\hat{U} * \hat{U}^{\dagger} = \hat{U} Q \hat{U}^{\dagger} = \hat{1} = Q^{-1}$ which verifies the axioms of unitarity at the isotopic level. This is precisely the meaning of *isounitary* transforms.

The above isooperations permit other applications of Q-deformations,

when axiomatically treated. For instance, lifting (3.18) turns Q-deformations into an *explicit realization of the theory of "hidden variables"*, with far reaching epistemological implications. In fact, we can say that HM in general, is a "completion" of QM essentially along the historical argument of Einstein, Podolsky, Rosen and others [2].

The next step requires the selection of the specific dynamical brackets constructed via the isotopies of envelopes. We now assume that the Q-operator is independent from the coordinates r to avoid gravitational profiles within physical conditions in which they are ignorable.

ISOTOPIC AXIOMATIZATION OF q-DEFORMATIONS [2]:

Fundamental assumptions: *A) representation of systems via two independent operators, the conventional Hamiltonian H = K + V, and the isotopic operator Q; B) representation of all action-at-a-distance interactions via the potential V, and all contact, nonpotential and nonlocal interactions due to mutual wave-penetration via the isotopic element Q; C) Integro-differential generalization $\hat{1} = Q^{-1}$ of Planck's unit $\hbar = 1$, with reconstruction of the entire QM formalism to admit $\hat{1}$ as the correct left and right unit, as per the preceding mathematical notions and the following physical axioms:*

Axiom I: *The states are elements of a isohilbert space $\hat{\mathcal{H}}_Q$ interpreted as (left or right) isomodule with "isoschrödinger's equation"*

$$i \frac{\hat{\partial}}{\hat{\partial} t} |\hat{\psi}> = H * |\hat{\psi}> := H Q |\hat{\psi}> = \hat{E} * |\hat{\psi}>, \quad <\hat{\psi} \hat{|} \hat{\psi}> = \hat{1}, \tag{3.19}$$

where $\hat{\partial}/\hat{\partial} t = \hat{1}_t\, \partial/\partial t$ is the isoderivative and $\hat{1}_t$ the time isounit (see ref.s [2] for brevity)

Axiom II: *Measurable quantities are represented by isocommuting isohermitean operators on $\hat{\mathcal{H}}_Q$ whose eigenvalues are conventional real numbers, e.g.,*

$$H^{\hat{\dagger}} \equiv H^{\dagger}, \quad H * |\hat{\psi}> = \hat{E} * |\hat{\psi}> \equiv E |\hat{\psi}>, \quad \hat{E} \in \hat{R}, \quad E \in R, \tag{3.20}$$

Axiom III: *The fundamental dynamical operators, coordinates r^k and momenta p_k, are characterized by isoeigenvalue equations and isocommutation rules (in momentum representation)*

$$p_k * |\hat{\psi}> = -i \hat{1}_k{}^i \nabla_i |\hat{\psi}>, \quad r^k{}_{op} * |\hat{\psi}> = \hat{r}^k{}_{scal.} * |\hat{\psi}> \equiv r^k |\hat{\psi}>, \tag{3.21a}$$

$$[a^{\mu} \hat{,} a^{\nu}] := a^{\mu} Q a^{\nu} - a^{\nu} Q a^{\mu} = i\, \omega^{\mu\alpha} \hat{1}_{\alpha}{}^{\nu}, \; a = (p, r), \; (\hat{1}_{\alpha}{}^{\nu}) = \text{diag.}(Q^{-1}, Q^{-1}). \tag{3.21b}$$

Axiom IV: *The time evolution of states is characterized by isounitary transformations with the (isohermitean) Hamiltonian as generator*

$$|\hat\psi(t)> = \hat U(t,t_o) * |\hat\psi(t_o)> = (e_{\hat\xi_Q}^{\;i H*(t_o - \hat t)}) * |\hat\psi(t_o)> \equiv e^{iHQ(t_o - t)} |\hat\psi(t_o)> . \qquad (3.22)$$

while the time evolution of operators is characterized by an equivalent, one-dimensional, Lie-Santilli group of isounitary transformations with the same Hamiltonian as generators, expressible in the finite form

$$A(t) = \hat U * A(t_o) * \hat U^{\dagger} = (e_{\hat\xi_Q}^{\;i H(t_o - t)}) * A(t_o) * (e_{\hat\xi_Q}^{\;-i(t_o - t)H}) \qquad (3.23)$$

with infinitesimal version provided by the "isoheisenberg equations"

$$i \frac{\hat\partial A}{\hat\partial t} = [A \hat{,} H] = A Q H - H Q A \qquad (3.24)$$

where $\hat\partial / \hat\partial t = \hat 1_t d/dt$.

Axiom V: *The values expected in measurements of observables are given by the isoexpectation values*

$$\hat< A \hat> := \frac{<\hat\psi| * A * |\hat\psi>}{<\psi| * |\hat\psi>} = \frac{<\hat\psi|Q A Q|\hat\psi>}{<\psi|Q|\hat\psi>}, \qquad (3.25)$$

The following comments are in order. The first fundamental result is that *Santilli's axiomatization of Q-operator-deformations of Class I coincides with conventional quantum mechanics at the abstract level.* In fact, at that level, all distinction cease to exist between $\hat F_Q$ and F, $\hat E(r,\delta,\hat R)$ and $E(r,\delta,R)$, $\hat\xi_Q$ and ξ, $\hat{\mathcal{H}}_Q$ and $\mathcal{H}$, etc. A subtle implication is that criticisms on the above axiomatization may eventually result to be criticisms on the axiomatic structure of quantum mechanics itself.

The second fundamental result is that *Santilli's axiomatization is form-invariant under its own transformation theory, the isounitary transformations.* This can be seen from the fact that the isocommutator is invariant under isounitary transformations, $\hat U*[A \hat{,} B]*\hat U^{\dagger} = [A' \hat{,} B']$, or the invariance of eigenvalues and isoexpectation values under isounitary transformation, etc. This form-invariance should be compared with the general *lack* of invariance of q-deformations under time evolution recalled in Sect. 1.

The third fundamental result is that *Santilli's isotopies achieve a true axiomatization of the quantity* Q^{-1} *assumed as the isounit of the theory.* In fact, $\hat 1 = Q^{-1}$ verifies the following properties: 1) $\hat 1$ is isoidempotent of arbitrary (finite) degree, $\hat 1^{\hat n} = \hat 1 * \hat 1 * \ldots * \hat 1 \equiv \hat 1$; 2) The isoquotient of $\hat 1$ by itself is $\hat 1$, $\hat 1 \hat / \hat 1 \equiv \hat 1$; 3) The isosquare root of $\hat 1$ is $\hat 1$, $\hat 1^{\hat{1/2}} \equiv \hat 1$; 4) $\hat 1$ isocommutes with all possible operators, $[A \hat{,} \hat 1] = A - A \equiv 0$; 5) $\hat 1$ is left invariant by isounitary transformations, $\hat U * \hat 1 * \hat U^{\dagger} \equiv U * \hat U^{\dagger} = \hat 1$; 6) $\hat 1$ is conserved in time, $i\,\hat\partial \hat 1 / \hat\partial t \equiv [\hat 1 \hat{,} H] \equiv 0$; and 7) The isoexpectation value of all possible isounits $\hat 1$ is the ordinary number 1,

$$\hat< \hat 1 \hat> = \frac{\hat\psi|Q Q^{-1} Q|\hat\psi>}{<\hat\psi|\hat Q|\hat\psi>} = 1 . \qquad (3.26)$$

The latter property has evident, far reaching, epistemological, theoretical and experimental implications studied in detail in [2]. We can here only mention the following aspects:

1) The property $\hat< \hat 1 \hat> = 1$ implies the reconstruction of Planck's constant for all measurement purposes;

2) Santilli's isotopies permit a fully *causal* treatment of *nonlocal deformations* because they are all embedded in the isounit $\hat 1$, thus resulting in an axiom-preserving isotopy of conventional causal treatments;

3) Santilli's isotopies also permit an axiomatization of discrete time theories (Kadeisvili Class V [29]) via their embedding in the isounit Q^{-1}, which therefore result to be "hidden" in, and compatibility with the *conventional* axioms of quantum mechanics, only realized in their most general possible (rather than simplest possible) form;

4) there is the impossibility for conventional experimental measures to distinguish between quantum and hadronic mechanics, unless complemented with additional tests specifically conceived for the difference [33];

5) there is the conservation of conventional total quantities of an isolated system originating from from the invariance of the basis under isotopies [22], according to which the generators of conventional and Lie-Santilli isosymmetries coincide; and others.

GENOTOPIC AXIOMATIZATION OF q-DEFORMATIONS [2]:

The above isotopic methods constitute *half* of HM. The remaining half is provided by the genotopic methods. The Lie-admissible methods were proposed by Santilli in [4] and developed in detail thereafter [2,12]. However, the largest number of applications of genotopies has been provided by Jannussis and his collaborators through several years [24].

The most salient aspect in the transition from the Lie-isotopic to the Lie-admissible formulations is the differentiation of the envelopes for the isomodular action to the right and to the left. In particular, Santilli [2,18,22] identified the origin of this distinction in the difference between the multiplication of numbers from the left and from the right, thus achieving a full axiomatization essentially based on two different sets of Axioms I-V for multiplications to the right and to the left.

Consider two real numbers α, $\beta \in R$. Their multiplication can be differentiated into two forms, $\alpha > \beta := \alpha P \beta$ and $\alpha < \beta := \alpha Q \beta$ (where P, Q are fixed nonsingulAR operators and $P \neq Q$), depending on whether "α multiplies β from the left", or "β multiplies α from the right", respectively. Note that this differentiation remains fully compatible with the commutativity of the product of ordinary numbers, i.e., we have $\alpha > \beta \equiv \beta > \alpha$ and $\alpha < \beta \equiv \beta < \alpha$, but $\alpha > \beta \neq \alpha < \beta$. The important mathematical discovery here is that *the ordering of the multiplication preserves all axioms of a field* [22].

These results permit the introduction of two different generalized fields called *genofields*, one in which only the multiplication to the right is allowed, ${}_P\hat{F}^{>}(\hat{\alpha}^{>},+,>)$ and one in which only that from the left holds ${}^{<}\hat{F}_Q({}^{<}\hat{\alpha},+,<)$, often denoted with the unified symbol ${}^{<}{}_P\hat{F}_Q{}^{>}({}^{<}\hat{\alpha}^{>},+,<>)$, with respective isounits

$$1^{>} = Q^{-1}, \quad {}^{<}1 = P^{-1}. \tag{3.27}$$

The Lie-admissible generalization of quantum mechanics therefore implies two different generalizations of Planck's constant for multiplication to the right and to the left. In turn, this implies the existence of two different chains of genotopies includinng: genospaces ${}^{<}{}_P\hat{S}_Q{}^{>}(x,{}^{<}\hat{g}^{>},{}^{<}\hat{R}^{>})$, genohilbert spaces ${}^{<}{}_P\hat{\mathcal{H}}_Q{}^{>}$, genoenvelopes ${}^{<}{}_P\hat{\xi}_Q{}^{>}$, etc.

The axiomatization is then given by two different sets of Axioms I–V, one for multiplication to the right >, and one to the left, whose identification is left to the reader for brevity [2]. We only mention that the isospace ${}^{<}{}_P\hat{\mathcal{H}}_Q{}^{>}$ acts as an isobimodule, i.e., an isomodule with different actions to the right and to the left. Eq.s (2.12) can then be rewritten in the axiomatic form

$$i \frac{\partial}{\partial t} | \hat{\psi} > = H > | \hat{\psi} > := HQ | \hat{\psi} > = E^{>} | \hat{\psi} >, \tag{3.28a}$$

$$- i < \hat{\psi} | \frac{\partial}{\partial t} = < \hat{\psi} | < H = < \hat{\psi} | P H = < \hat{\psi} | {}^{<}E, \tag{3.28b}$$

while Santilli's Lie-admissible equations (3.24) become

$$i \frac{\partial A}{\partial t} = (A , H) = A < H - H > A = A P H - H Q A. \tag{3.29}$$

with corresponding finite form of a "Lie-admissible group" [2,4,12]

$$A(t) = (e_{\xi}{}^{> \, iH(t_o - t)}) > A(t_o) < (e_{<\xi}{}^{-i(t_o - t)H}) \equiv e^{iHQ(t_o - t)} A(t_o) e^{-i(t_o - t)PH}, \tag{3.30}$$

Note that the above axiomatization has been presented so far with the sole assumptions that $P \neq Q$, P and Q nonsingular. This implies that the P and Q operators can be individually Hermitean or, in particular, can be different real numbers.

Nevertheless, Santilli conceived the Lie-admissible formulations [2,4] to provide *an axiomatization of open, nonconservative and irreversible processes*. This is achieved via the interpretation of the formulations with multiplication to the right as representing *motion forward in time* >, and those to the left for *motion backward in time* < (Eddington's arrows of time), and the assumption that the P and Q operators are interconnected by conventional Hermitean (or other) conjugation,

$$Q^{\dagger} = P \neq Q, \qquad 1^{>} = ({}^{<}1)^{\dagger} \neq {}^{<}1 . \tag{3.28}$$

Thus, isotopic equations (3.24) are *structurally reversible*, in the sense that they represent time-reversal invariant systems whenever the Hamiltonian is T-invariant, while the genotopic equations (3.29) are *structurally irreversible*, in the sense that they are irreversible even for T-symmetric Hamiltonians [21].

Note the *Hermiticity/observability* of the Hamiltonian when *nonconserved*, which is an occurrence possible for HM but not for QM (where dissipation is often represented via "imaginary potentials", $H = K + iV \neq H^{\dagger}$, in which case the brackets of the theory violate scalar and distributive laws thus losing conventional notions such as that of spin [2]).

As a final comment we recall the *direct universality* of HM [2], i.e., its capability to represent all possible systems of linear and nonlinear, local and nonlocal, Hamiltonian and nonhamiltonian, discrete and continuous type (universality), directly in the frame of the experimenter (direct universality). This universality follows form the rather vast structure of HM (recall Kadeisvili five classes per each branch) and it is based on two theorems, one for isotopic formulations valid when the total energy is conserved, $i\dot{H} = HQH - HQH \equiv 0$, and the other of genotopic type which is valid when the energy is nonconserved, $i\dot{H} = HPH - HQH \neq 0$.

4: AXIOMATIC REFORMULATION OF q-DEFORMATIONS

The first point stressed in ref.s [2] (see, e.g., App. I.7.A) is the *independence* of q-deformations from HM, as evident from the structural differences of the two theories. With this understanding, HM can provide a *reformulation* of q-deformations which leaves the results of the q-deformations unaffected, while avoiding their problematic aspects for physical applications indicated in Sect. 1.

The reformulation is centrally dependent on whether the total energy is conserved, in which case isotopic methods are applicable, or the energy is not conserved, in which case the broader genotopic methods are applicable. Equivalently, isotopies are used when the brackets of the time evolution are totally antisymmetric, otherwise genotopies are needed.

The hadronic reformulation of q-deformations is simple and can be outlined as follows [2]:

Case I: q-deformations of associative envelopes, as in the cases

$$AB \Rightarrow q\,AB\,, \qquad a\,a^{\dagger} \Rightarrow q\,a\,a^{\dagger}\,, \tag{4.1}$$

The axiomatic reformulation is then achieved by simply lifting the unit I into the isounit $\hat{1} = q^{-1}$, and constructing the chain of Santilli's isotopies of fields, spaces, Lie algebras, etc. The isotopic methods then ensure the form-invariance of the theory under arbitrary transformations.

Case II: q-deformations of eigenvalues of commutators, e.g.

$$r\,p - p\,r = i\,f(q) \neq i\,, \tag{4.2}$$

As proved by Jannussis [24], these formulations are noncanonical, thus lacking an axiomatic character when treated with *conventional* methods. However, the above q-deformations can be easily reformulated in Santilli's isotopic form. Assume f(q) as the new isounit, $\hat{1} = f(q)$. The isotopic element is then given by $Q = [f(q)]^{-1}$. The axiomatic reformulation is then given by the isoeigenvalue equation

$$p * |\hat{\psi}> := p\,Q\,|\hat{\psi}> = -\,i\,\hat{1}\,\nabla\,|\hat{\psi}>\,, \tag{4.3}$$

under which commutator (4.2) is turned into the equivalent form

$$r * p - p * r = r\,[f(q)]^{-1}\,p - p\,[f(q)]^{-1}\,r = i\,\hat{1} = i\,f(q)\,. \tag{4.4}$$

which is now form invariant under time evolutions. A similar axiomatic reformulation occurs for creation and annihilation operators with noncanonical eigenvalues. Note that Lie-admissible formulations are *inapplicable* in this case because the energy is conserved or, equivalently, the brackets are antisymmetric.

Along similar lines, the axiomatic reformulation of q-deformations of Type III is obtained by assuming the deformed structure constants $F_{ij}{}^{k}(q, ...)$ as the scalar part of the structure isofunctions of the theory, and then searching for a compatible isotopic element Q

$$X_i\,X_j - X_j\,X_i = F_{ij}{}^{k}(t, q, ...)\,X_k \Rightarrow X_i\,Q\,X_j - X_j\,Q\,X_i = \hat{F}_{ij}{}^{k}(t, q, ...)\,X_k. \tag{4.5}$$

The form-invariance of the theory under the time evolution is then ensured by the isotopic methods.

Case III: q-deformations of Lie-products, such as

$$r\,p - q\,p\,r = i\,f(q)\,. \tag{4.6}$$

The latter case requires the full use of Santilli's Lie-admissible formulations because the brackets are no longer totally antisymmetric. In fact, the multiplication to the right is isotopic, $p>r = pQr$, $Q = q$, and that to the left is conventional, $r<p = rPp$, $P = 1$, resulting in the flexible Lie-admissible, Jordan-admissible product $(p, r) = r<p - p>r$. A necessary condition of consistency of the theory is that the fundamental rules [4]

$$(a^{\mu}, a^{\nu}) = a^{\mu} < a^{\nu} - a^{\nu} > a^{\mu} = i\,\omega^{\mu\alpha}\,{}^{<}\hat{1}_{\alpha}{}^{\nu>}\,, \quad a = (r, p) \tag{4.7}$$

characterize a Lie-admissible tensor $\omega^{\mu\alpha}\,{}^{<}\hat{1}_{\alpha}{}^{\nu>}$ in a selected direction of time.

The above reformulation was first studied by Jannussis and his collaborators [24] on conventional fields. That on genofields was done by Santilli [2]. It requires the selection of one "time arrow" and then the interpretation of the function f(q,...) in rules (4.6) as the genounits for that direction. Jointly, the q-deformation of the second term in the l.h.s. is not axiomatic and must be lifted into the inverse of the selected genounit, resulting in the reformulations

$$r\,p - q\,p\,r = i\,f(q,...) \Rightarrow \begin{cases} r < p - p > q = r\,P\,p - p\,Q\,r = \hat{1}^{>}, \\ \hat{1}^{>} = f(q, ...)\,/\,q, \quad Q = q\,/\,f(q, ...), \quad P = f(q,...) \\ \text{or} \\ r < p - p > q = r\,P\,p - p\,Q\,r = {}^{<}\hat{1}\,, \\ {}^{<}\hat{1} = f(q, ...)\,/\,q, \quad Q = f(q, ...), \quad P = q\,/\,f(q,...) \end{cases} \tag{4.8}$$

The entire theory must then be reformulated on genofields, genospaces, genotrasformations, etc., for the selected direction of time. The axiomatic reformulation of other q-deformations can be done with one or the other of the above methods.

The best way to see the inevitability of Santilli axiomatic reformulation even when not desired, is by subjecting q-deformations to ***nonunitary*** transformations. As a matter of fact, nonunitary transformations themselves can be used as a method to achieve an axiomatic structure. By assuming

$$U\,U^{\dagger} = f(q) = \hat{1} \neq I, \qquad Q = (U\,U^{\dagger})^{-1}\,, \tag{4.9}$$

we have

$$U\,(r\,p - p\,r)\,U^{\dagger} = r'\,Q\,p' - p'\,Q\,r' = Q^{-1}\,, \tag{4.10a}$$

$$U\,(r\,p - q\,p\,r)\,U^{\dagger} = r'\,P\,p' - p'\,Q\,r = i\,P^{-1}\,, \quad Q = qP. \tag{4.10b}$$

In both cases, starting from a conventional formulation of a q-deformation [1] one ends up in Santilli's axiomatic form [2]. Since the latter is constructed via the most general possible nonunitary transforms, it evidently remains invariant under the same.

A few examples are now in order. To understand them, one must understand first the arena of applicability of HM, the interior problem. The search, say, for the description of *conventional* systems (such as the electron in an atomic cloud or the harmonic oscillator) within the contest of HM with a nontrivial operator Q has no physical or mathematical sense. QM is *exactly* valid for these conventional systems, which means that one must necessarily put Q = 1.

The objective of HM and related methods is that of treating the *deviations* from conventional systems caused by their immersion within a physical media or interactions with external terms. Systems which are therefore significant for HM are the electron when immersed in the core of a collapsing star, or the damping of the harmonic oscillator due to an external force, the deformation of the charge distribution of a proton or a neutron due to sufficiently intense collisions and/or external fields, and the like.

A first simple example is the damped particle represented by

$$H(t) = e^{-\gamma t} H_0, \quad H_0 = \tfrac{1}{2} p_0^2, \quad m = 1, \quad \dot{H} = -\gamma H, \qquad (4.11)$$

which is *axiomatically* represented by Santilli's Lie-admissible formulations via Eq. (3.29) with

$$P = -\tfrac{1}{2} i \gamma H_0^{-1}, \qquad Q = P^{\dagger}. \qquad (4.12)$$

Eq. (3.30) then correctly reads $i\dot{H} = (\lambda - \mu) H^2 = -i\gamma H$. The direct universality of the theory ensures the existence of axiomatic representations of other systems. Note that the Hamiltonian remains Hermitean thus observable, yet it is not conserved in time.

To our knowledge, Santilli's Lie-admissible theory is the *only* one establishing the *observability of nonconserved quantities*, as actually occurring in laboratory.

As another example, consider the Lagrangian of the harmonic oscillator $L = \tfrac{1}{2}(\dot{r}\dot{r} + krr)$ in $E(r,\phi,R)$. The lifting in $\hat{E}(r,\hat{\delta},\hat{R})$, $\hat{\delta} = Q\delta$, $Q = \exp(\gamma t)$ represents the *damped* oscillator. This case also illustrate the interplay between isotopic and genotopic formulations, in the sense that a *nonconservative* system can at times be represented with *isotopic* methods. The understanding in this case is that the algorithm "H" is just a mathematical quantity (a first integral) and does not represent the energy.

It is equally instructive for the researcher in q-deformation to see that the isotopic treatments resolve *all* the problematic aspects of Sect. 1. To begin, HM has been built under the condition of possessing a generalized, but well defined left and right unit $\hat{1}$. As now familiar, this implies a corresponding compatible isotopy of the enveloping algebra, the base fields and the Hilbert space, thus ensuring the Hermiticity/observability of the Hamiltonian and other operators at all times.

The above assumptions also imply the existence of a unique generalization of the Poincaré-Birkhoff-Witt theorem resulting in a unique exponentiation and unique structures defined on it, such as isotopic delta functions, isofourier transforms, isogaussians, etc. This implies the uniqueness of physical laws and the applicability of the isospecial functions at all times.

The preservation of the fundamental axioms of Einstein's relativities was the central reason for the very construction of the isotopics (see ref.s [2] for details).

The regaining of the applicability of the measurement theory is rather intriguing. The isoexpectation values of the isounit $\hat{1} = q^{-1}$ of q-deformations reconstruct the *conventional* unit, $\hat{<} \hat{1} \hat{>} = \hat{<} f(q) \hat{>} = 1$. Thus, the measurement theory which is applicable to Santilli's axiomatic reformulation of q-deformations is the *conventional* theory. This is necessary for physical consistency and applicability to actual experiments, because measures are conducted in our *classical* frame which, as such, cannot be modified by theoretical deformations introduced in the microworld.

5: PHYSICAL APPLICATIONS AND EXPERIMENTAL VERIFICATIONS.

Once the q-deformations are reformulated in a axiomatically consistent form, an intriguing horizon of novel physical applications and experimental verifications become within technical reach. Despite their evident tentative nature, it appears recommendable to mention these possibilities (see [2] for details).

Applications I: Direct representation of nonspherical shapes and their deformations. While QM can only provide an abstraction of hadrons as points, HM can represent their nonspherical charge distribution as well as all their infinitely possible deformations. This representation occurs at the level of first isoquantization without any need of second isoquantization, and is achieved in isominkowski space by assuming isounits of the type [20,21] $\hat{1} = Q^{-1} = \mathrm{diag.}(b_1^{-2}, b_2^{-2}, b_3^{-2}, b_4^{-2})$, $b_\mu > 0$, where: the space components represent the semiaxes, in this case, of an ellipsoid; their deformation can be represented via a dependence of the b_k from the needed external quantities (e.g., intensity of external fields); and the time component b_4^{-2} provides a novel geometrization of the medium in the interior of the particle (a form of isotopy of the conventional index of refraction). These first formulations have been applied to: the apparent deformability of the charge distribution of hadrons [19,35] with a quantitative interpretation of Rauch's interferometric experiment on the 4π symmetry; a numerical resolution of the total magnetic moments for few bodies nuclear structures; the anomalous behaviour of the meanlives of unstable hadrons with speed [37]; and other cases [2,35,36].

Applications 2: Direct representation of nonhamiltonian-nonlocal-nonlinear interior effects of strong interactions. Deep inelastic scattering of hadrons are not expected to be sole scatterings among ideal points interacting at-a-distance, because they imply mutual penetrations of the densest extended objects measured in laboratory until now. Under these conditions we expect the presence of internal interactions of the so-called "contact" type which are beyond the representational capabilities of a Hamiltonian (because NSA [12]), are nonlocal-integral, and nonlinear in the most general known form (e.g., in the derivatives of the wavefunctions). The axiomatic formulation of q-deformation permits a direct representation of these internal nonhamiltonian-nonlocal-nonlinear effects precisely via the deformation-mutation of the associative product. Then the Q-operator itself, being independent from the Hamiltonian, acquires the direct physical meaning of representing said nonhamiltonian-nonlocal-nonlinear interactions. A most representative case is the Bose-Einstein correlation because there are reasons to expect that the correlation itself is absent under only local-differential interactions. Detailed phenomenological studies [38] have shown the effectiveness of the relativistic, isotopic, nonlocal treatment of the Bose-Einstein correlation. Independent phenomenological studies [39] have shown its remarkable agreement with experimental data from the UA1 experiment.

Applications III: Chemical synthesis and artificial disintegration of hadrons. Hadronic mechanics predicts fundamentally *novel* events, that is, events beyond the predictive capacities of quantum mechanics. One of them is the prediction that the cold fusion currently observed at the molecular/atomic level in actuality originates at the level of elementary particles. The novel prediction is that massive particles have a natural tendency to form a bound state at small distances (< 1 fm) in singlet states which is enhanced at low temperature (or low energy). Santilli [4] originally formulated a quantitative representation of the cold fusion of electrons and positrons as well as of mesons at large. More recently, Animalu [28] has reached a quantitative interpretation of the electron pairing in superconductivity which is in preliminary, yet remarkable agreement with experimental data. The lifting is done via: isotopic Eq.s (3.6); the conventional QM Coulomb Hamiltonian H; and the simple isotopic element $Q = \exp(- t\ N \int d^3r\ \hat{\phi}^\dagger(r)\hat{\phi}(r))$ representing the overlapping of the electrons' wavepackets $\hat{\phi}$ and ϕ. A comprehensive theory of the cold fusion of all (massive) particles (leptons, mesons and baryons) is now available [40] with preliminary experimental verification [41].

These novel advances should not be looked lightly because they imply the possibility of producing (unstable) hadrons via chemical synthesis of lither (massive) particles. In turn, this implies the possibility of the artificial disintegration of hadrons (say, of the peripheral neutrons in a nuclear structure), with consequential emergence of a possible new technology called *hadronic technology.*

Numerous additional applications have been studied, including Q-operator-isotopies of quark theories with exact confinement, Q-isotopies of potential scattering theory, applications to discrete-time theories, etc. [2]

The significance of the axiomatization presented in this paper is that, in its absence, the above physical applications and experimental verifications are generally inapplicable to the q-deformations in their current formulation.

I want to thank the Institute for basic Research for hospitality. Thanks are in particular due to Prof. Santilli for submitting the problem, putting at my disposal the IBR computer files and providing invaluable technical assistance. I also want to thank Dr. G. F. Weiss for linguistic control, and Mrs. Fleming for the typing of the manuscript.

REFERENCES

1. T. L. Curtright, D. B. Fairlie and Z. K. Zachos Editors, *Quantum Groups,* World Scientific (1991); Mo-Lin Ge and Bao-Heng Zhao, Editors, *Introduction to Quantum Groups and Integrable Massive Models of Quantum Field Theory,* World Scientific (1991); Yu. F. Smirnov and R. M. Asherova, Editors, *Proceedings of the Fifth Workshop Symmetry Methods in Physics,* JINR, Dubna, Russia (1992)
2. R. M. Santilli, *Elements of Hadronic Mechanics,* Vol I: *Mathematical Foundations,* Vol. II: *Theoretical Foundations,* and Vol. III: *Experimental Verifications,*
3. D. F. Lopez, Origin and axiomatization of q-deformations, in *Proceedings of the 1993 Workshop Symmetry Methods in Physics* (dedicated to Ya. A. Smorodinsky), G. Pogosyan et al. Editors, JINR, Dubna, Russia, in press
4. R. M. Santilli, Hadronic J. 1, 574 (1978)
5. N. Jacobson, *Lie algebras,* Interscience (1962)
6. A. A. Albert, Trans. Amer. Math. Soc. 64, 552 (1948)
7. L. J. Paige, *Jordan Algebras*, in *Studies in Modern Algebras,* Prentice-Hall (1963)
8. R. M. Santilli, Lett. Nuovo Cimento 51, 570 (1967) (see also Suppl. Nuovo Cimento 6, 1225 (1968))
9. H. C. Baltzer et al., *Bibliography in Nonassociative Algebras,* Hadronic Press (1984)
10. Proceedings of the 1989 Conference on *Quasigroups and Nonassociative*

Algebras in *Physics*, Estonian Academy of Sciences, Tartu (1989)
11. R. M. Santilli, Meccanica 1, 1 (1968)
12. R. M. Santilli, Hadronic J. 1, 228 and 1279 (1978; (see also Phys. Rev. D20, 555 (1979); *Foundations of Theoretical Mechanics*, Vol. II, Springer-Verlag (1983))
13. R. M. Santilli, *Lie-admissible Approach to the Hadronic Structure*, Vol. I (1978), I (1982), and III (in preparation), Hadronic Press
14. R. M. Santilli, *Isotopic Generalization of Galilei's and Einstein's Relativities*, Vol.s I and II, Second Edition, Ukraine Academy of Sciences, Kiev, in press
15. J. Fronteau, A. Tellez-Arenas and R. M. Santilli, Hadronic J. 3, 130 (1979)
16. H. C. Myung and R. M. Santilli, Hadronic J. 5, 1277 and 1367 (1982)
17. R. Mignani, Hadronic J. 5, 1120 (1982)
18. R. M. Santilli, Hadronic J. Suppl. 4B, issues 1,2,3,4 (1989)
19. R. M. Santilli, Hadronic J. 4, 1166 (1981)
20. R. M. Santilli, Hadronic J. 8, 25 and 36 (1985)
21. R. M. Santilli, Lett. Nuovo Cimento 37, 337 and 545 (1983); and 38, 509 (1983)
22. R. M. Santilli, Algebras, Groups and Geometries 8, 169 and 275 (1991); and 10, 273 (1993)
23. M. Gasperini, Hadronic J. 6, 935 and 1462 (1983)
24. A. Jannussis *et al.*, Hadronic J. 5, 1923 (1982); 6, 1653 (1983); 7, 947 (1984); 14, 257 (1991); Lett. Nuovo Cimento 30, 123 (1980); 31, 177 (1981); 34, 375 (1982); 37, 119 (1983); J. Phys. A24, L775 (1991); A25, L329 (1992); A26, L233 (1993); Ann. Fond. de Broglie 18, 137 (1993) and in press; Hadronic J. 8, 387 (1985); 9, 223 (1986); 10, 75 (1987); 14, 277 (1991); and in press; Physics Essays 4, 202 (1991); Physica A187, 575 (1992)
25. R. Mignani, Lettere Nuovo Cimento 39, 413 (1984); Hadronic J. 9, 103 (1986); Physics Essays 5, 531 (1992)
26. S. Okubo, Hadronic J. 5, 1667 (1982)
27. A. J. Kalnay, Hadronic J.6, 1 (1983); A. Kalnay and R. M. Santilli, Hadronic J. 6, 1798 (1983)
28. A. O. E. Animalu, Hadronic J. 14, 459 (1990); and 16 (1993), and in press
29. J. V. KADEISVILI, Algebras, Groups and Geometries 9, 283 and 319 (1992)
30. M. Nishioka, Lettere Nuovo Cimento 39, 369 (1984); 40, 309 (1984); Nuovo Cimento 85, 331 (1985); Hadronic J. 11, 97 and 143 (1988);
31. A. K. Aringazin, Hadronic J. 12, 71 (1989); 13, 183 (1990); and in press
32 D. F. Lopez, Hadronic J. 15, 431 (1992)
33. G. Brodimas, D. S. Sourlas and A. Sotiropoulos, An introduction to Hadronic Mechanics, in *Proceedings of the International Conference on Frontiers of Fundamental Physics*, F. Selleri and M. Barone, Editors, Hadronic Press, in press
34. A.K.Aringazin, A. Jannussis, D.F.Lopez, M. Nishioka and B. Veljanoski, *Santilli's Lie-isotopic Generalization of Galilei and Einstein Relativities*, Kostarakis Publ., Athens, Greece (1990); J. V. Kadeisvili, *Santilli's Isotopies of Contemporary Algebras, Geometries and Relativities*, Second Edition, Ukraine

Academy of Sciences, Kiev, in press; D. S. Sourlas and G. T. Tsagas, *Mathematical Foundations of the Lie-Santilli Theory*, Ukraine Academy of Sciences, Kiev (1993); J. Lôhmus, E. Paal and L. Sorgsepp, *Nonassociative Algebras in Physics*, Estonian Academy of Sciences, Tartu, in press
35. R. M. Santilli, Isotopic lifting of SU(2) symmetries with applications to nuclear physics, Acta Appl. Math., in press
36. R. M. Santilli, in *Proceedings of Deuteron 1993*, V. K. Luykianov et al., Editors, JINR, Dubna, Russia, in press
37. F. Cardone, R. Mignani and R. M. Santilli, J. Phys. G18, L61 and L141 (1992)
38. R. M. Santilli, Hadronic J. 15, 1, (1993)
39. F. Cardone and R. Mignani, Univ. of Rome report No. 894 (1993)
40. R. M. Santilli, Hadronic J. 13, 513 (1990), Application of isosymmetries/Q-operator deformations to the cold fusion of elementary particles, in *Proceedings of the 1993 Workshop Symmetry Methods in Physics* (dedicated to Ya A. Smorodinsky), G. Pogosyan et al., Editors, JINR, Dubna, Russia (1993); and JINR Communication E4-93-352
41. C. Borghi, C. Giori and A. Dall'Olio, Hadronic J. 15, 239 (1992)