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ELEMENTS OF FUNCTIONAL ISOANALYSIS

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Abstract

In this paper we outline the axiom-preserving, nonlinear, nonlocal and noncanonical isotopies of conventional mathematical structures, including units, fields, vector spaces, transformation theory, algebras, groups, geometries, Hilbert spaces, etc., which were pioneered by the theoretical physicist R. M. Santilli while at the Department of Mathematics of Harvard University in the early 80's. We then show that these studies imply a true generalization of conventional functional analysis, here submitted under the name of *functional isoanalysis*. The structural foundations of this new discipline are identified jointly with its classification into ten mathematically and physically different classes. The significance of functional isoanalysis is point out by recalling a number of aspects worked out in the physical literature, but which do not appear to have propagated in the mathematical literature, such as: the lack of unitary equivalence between conventional and isotopic formulations despite their abstract identity; the admittance by a Hermitean operator of infinitely different sets of eigenvalues depending on the infinitely possible, basic units; the capability of turning conventionally non-square integrable functions into isotopic square integrable forms, or of turning divergent series into isotopically convergent forms; and others. Further relevance of functional isoanalysis is presented in the subsequent paper on the formulation and application of the isotopies of the Fourier transforms.

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1: Introduction. The founders of analytic mechanics, such as Lagrange [1], Hamilton [2] and others, classified dynamical systems into:

1) The *exterior dynamical problem*, consisting of test particles which can be effectively approximated as being point-like, thus permitting the contemporary local-differential topology, while moving in the homogeneous and isotropic vacuum under action-at-a-distance interactions, thus resulting in potential-canonical equations of motion; and

2) The *interior dynamical problem*, consisting of particles which cannot be effectively approximated as being point-like, while moving within generally inhomogeneous and anisotropic physical media, thus resulting in the most general known, nonlinear, nonlocal-integral and nonpotential-noncanonical equations of motion.

The above distinction was kept until the early part of this century, but abandoned in more recent times (see, e.g., the care provided by Schwartzschild in separating his well known exterior solution [3] from the interior one [4] which is virtually unknown nowadays).

This was unfortunate because the lack of the above distinction prevented the identification of the limitations of available mathematical and physical theories, thus delaying possible advances.

As an example, the algebraic conceptions of Sophus Lie (see, e.g., ref. [5]) have acquired a fundamental role in physics because characterizing the brackets of the time evolution in classical and quantum formulations, as well as the basic symmetries of physical laws (see, e.g., ref.s [6] and quoted sources).

Nevertheless, the body of formulations today known as *Lie's theory* is exactly applicable *only* to the exterior dynamical problem, as necessary because of the underlying local-differential topology, and the potential-canonical character of the equations of motion.

The theoretical physicist Ruggero Maria Santilli, while at the Department of Mathematics of Harvard University under support from the U. S. Department of Energy, brought back to the attention of the mathematical and physical communities the above crucial distinction between exterior and interior problems, identified the consequential limitations of existing mathematical and physical theories, and submitted the so-called *axiom-preserving, nonlinear, nonlocal and noncanonical isotopies of Lie's theory* [7] under the name of *Lie-isotopic theory*, including the isotopies of classical and quantum formulations and basic symmetries. (It should be noted that the Lie-isotopic theory was introduced by Santilli as a particular case of the yet more general Lie-admissible theory – which is not considered in this paper for brevity –, and this explains the title of ref. [7]).

Typical examples of the applicability of Lie's methods are given by a satellite in a stable orbit around Earth or an electron in a quantized orbit of an atomic structure. Typical examples of applicability of Santilli's isotopic methods are given instead by the same satellite during re-entry in Earth's atmosphere along a monotonically decaying trajectory, or the electron when moving within the physical medium in the interior of a collapsing star.

Santilli's proposals were subsequently studied by a number of authors (see, e.g., ref.s [8–19] and papers quoted therein), they were recently presented in this Journal in memoirs [20,21], and were finalized in their classical formulation in the recent volumes [22,23] and in their operator form in ref.s [24,25] (see also the independent reviews [26,27]).

Thanks to contributions also by other physicists, such as A, Jannussis, A. K. Aringazin, A. O. E. Animalu, M. Nishioka, R. Mignani and others, these studies have now come to age with a variety of novel physical applications [28–33] and preliminary, yet clear experimental verifications [34–41]. Mathematical research on Santilli's isotopies is ongoing in ref.s [42–49], while the status of our mathematical knowledge in the isotopies of Lie's theory is presented in the forthcoming monograph by D. S. Sourlas and G. T. Tsagas [50].

In this paper we show that these studies imply a mathematically and physically nontrivial, step-by-step generalization of each structural aspect of functional analysis, resulting in a genuine new discipline, here submitted, apparently for the first time, under the name of *functional isoanalysis*. Additional aspects are treated in the subsequent paper [57] on the construction and application of the isotopies of the Fourier transforms.

Our presentation is intended to be mathematical because the isotopies studied in this paper have a mathematical significance per se, independent from any physical application. Nevertheless, at times we shall point out the physical needs that originated the isotopies because they still are a source of intriguing novel mathematical problems.

For guidance in the quoted literature, it should be noted that the isotopies of classical Hamiltonian are known under the names of *Birkhoffian mechanics* for nonlinear and noncanonical, but still local systems, and of *Hamilton-Santilli mechanics* for the most general possible nonlinear, nonlocal and noncanonical systems. The isotopies of quantum mechanics are known under the names of *hadronic mechanics* or *isotopic completion of quantum mechanics* or *isolocal realism* .

2: Elements of isotopic methods.

Let us briefly review the aspects of Santilli's isotopic methods which are essential for the definition and treatment of the isotopies of functional analysis at large, and those of the Fourier transforms, in particular.

2.A: ISOTOPIES OF THE UNIT: The fundamental isotopies from which all others can be uniquely derived, is the lifting of the trivial n-dimensional unit I = diag. (1, 1, 1, ..., 1) of the current formulation of Lie's theory (see, e.g., ref. [53]) into n-dimensional matrices denoted with the symbol $\hat{1}$ and called *isounits*, whose elements possess the most general known, nonlinear, nonlocal and noncanonical dependence on all possible variables (such as the local coordinates x and wavefunctions ψ) and their derivatives with respect to independent variables of arbitrary order [7,21]

$$I = \mathrm{diag.}\,(1, 1, ..., 1) \quad \Rightarrow \quad \hat{1} = \hat{1}(t, x, \dot{x}, \ddot{x}, \psi, \psi^{\dagger}, \partial\psi, \partial\psi^{\dagger}, ...)\,, \qquad (2.1)$$

A fundamental necessary condition on the isounits to characterize isotopies is that they are (conventionally) Hermitean,

$$\hat{1} \;=\; \hat{1}^{\dagger}. \qquad (2.2)$$

In fact, whenever such a condition is relaxed, the liftings $I \Rightarrow \hat{1}$, $\hat{1} \neq \hat{1}^{\dagger}$, imply the abandoned of the Lie algebras axioms in favor of the covering axioms of the Lie-admissible algebras (see ref.s [21,22] for brevity).

In this paper we shall introduce, apparently for the first time, the classification of Hermitean isounits into the following five classes:

CLASS I: ISOUNITS properly speaking, when the they are sufficiently smooth, bounded, nowhere singular, Hermitean and positive-definite. This class characterizes an isotopy of the conventional unit because of the preservation of the original axioms of I, and it is the class of primary use in physics for the characterization of ordinary particles in interior physical conditions [23,25];

CLASS II: ISODUAL ISOUNITS, when they are the same as those of Class I except that they are negative-definite. This class characterizes the *isodual isotopy*, according to Santilli's *isodual conjugation* $\hat{1} \Rightarrow \hat{1}^{d} = -\,\hat{1}$ [15,21], and is used in physics to characterize antiparticles via a reinterpretation of the negative-energy solutions of Dirac's equation [25,28].

<u>CLASS III: SINGULAR ISOUNITS, when considered at a limit which is divergent, $\hat{1} \Rightarrow \pm\infty$.</u> This class is used in physics to represent gravitational collapse into a singularity and other limit conditions [23,29].

<u>CLASS IV: INDEFINITE ISOUNITS, when they are sufficiently smooth, bounded, nowhere singular and Hermitean, and can smoothly interconnect positive-definite with negative-definite values.</u> This class is particularly useful in mathematics, e.g., for the classification and unification of all possible structures of Classes I and II. And

<u>CLASS V: GENERAL ISOUNITS, when they are solely Hermitean.</u> This is the most general possible class which, besides including the preceding ones, permits a large variety of additional realizations including those in terms of discrete structures (e.g., a lattice), discontinuous functions, distributions, etc.

From now on, unless otherwise specified, the term "isotopies" shall be solely referred to isounits of Class I.

<u>Comment</u> 2.A.1: The physically and mathematically most significant realizations of the isounits are those of nonlocal-integral character, i.e., defined over a given area or volume of integration. Despite that, units and their isotopic images coincide at the abstract level by conception.

<u>Comment</u> 2.A.2: Once the original unit I is lifted into the isounit $\hat{1}$, all mathematical and physical structures must be modified in such a way to admit $\hat{1}$ as the left and right unit. Nevertheless, the emerging isotopic formulations coincide with the original formulations at the abstract, realization-free level.

<u>Comment</u> 2.A.3: On physical grounds, Planck's constant $\hbar = 1$ characterizes the basic laws of quantum mechanics (e.g., Heisenberg's uncertainties for particles in vacuum $\Delta r \, \Delta p \approx \hbar$). Santilli's isotopies were conceived as an axiom-preserving integral isotopies of Planck's constant $\hbar \Rightarrow \hat{1}$ [7,8], with corresponding isotopies of conventional quantum mechanical laws (e.g., the isouncertainties for particles within physical media $\Delta r \, \Delta p \approx \hat{1}$ [25,29]). The argument is that, in the transition from the exterior problem in vacuum to the interior problem within physical media, exchanges of energy acquire an integral component depending on the local physical conditions. Recent experimental evidence on the Bose-Einstein correlation [24,39], on the behaviour of the meanlives of particles with speed [35,36] and other topics, even though preliminary, appears to confirm quite clearly the predictions of the isotopic theory, with particular reference to the presence of a nonlocal internal component in the strong interactions.

<u>Comment</u> 2.A.4: On mathematical grounds, Planck's constant $\hbar = 1$ is the fundamental unit of quantum mechanics. The isotopies $\hbar \Rightarrow \hat{1}$ then imply corresponding, compatible isotopies of all mathematical structures of quantum mechanics, including fields, Hilbert spaces, transformation theory, algebras, groups, representation theory, etc. In this paper we shall study only one aspect of these new methods, the implications of the isotopies $\hbar \Rightarrow \hat{1}$ for functional analysis, and confirm that they do indeed imply the lifting of Heisenberg's uncertainties $\Delta r \, \Delta p \approx \hat{1}$ of ref. [29].

<u>Comment</u> 2.A.5: One of the most intriguing and unexplored mathematical aspects of the isotopies is the study of the topology characterized by integral isounits. It is tentatively called in the physical literature an *isolocal topology*, in the sense that it is local-differential, except at the isounit. The physical needs for such a novel topology are the following. Classically, the new topology is needed to characterize a test particle in interior dynamical conditions, such as a satellite during re-entry in Earth's atmosphere with consequential integro-differential equations of motion, in which the conventional local coordinates describe the trajectory of the center-of-mass of the satellite, while the isounit describes the integral corrections of the trajectory caused by its shape. In operator theories, the new topology is needed for a much similar case, the characterization of a particle in interior dynamical conditions, such as a proton moving in the core of a star. In fact, in this latter case too we need local coordinates to describe the motion of the center-of-mass of the particle, while the isounit represents the integral corrections caused by the immersion of the wavepacket of the particle considered within those of the surrounding particles. When both classical and elementary particles return to move in vacuum (exterior problem), said integral contributions are identically null, in which case the isolocal topology must recover conventional local topologies for $\hat{1} \equiv I$.

2.B: ISOTOPIES OF FIELDS. Let $F = F(n,+,\times)$ represent ordinary fields with conventional elements n, sum + and multiplication ×, hereon restricted to have characteristics zero, by therefore resulting to be the fields of real numbers $\Re$, complex numbers C and quaternions Q. The first consequence of the isotopies $I \Rightarrow \hat{1}$ is the necessary lifting of F into the *isofields* [11-13] (see ref. [42] for a detailed treatment)

$$\hat{F} = \{ (n,+,*) \mid n \in F, * = \times T \times, \hat{1} = T^{-1} \}, \tag{2.3}$$

Ordinary numbers n, when belonging to an isofield $\hat{F}$, are called *isonumbers*. Their sum + is the conventional one, but their product $*$ must be lifted into the form $\hat{*}$ called *isoproduct*

$$n_1 \hat{*} n_2 = n_1 T n_2, \quad \hat{1} = T^{-1}, \qquad (2.4)$$

where T is called the *isotopic element*. Lifting (2.4) is a necessary condition for $\hat{1}$ to be the left and right unit of $\hat{F}$

$$\hat{1} \hat{*} n = n \hat{*} \hat{1} \equiv n, \qquad \forall \quad n \in \hat{F} \qquad (2.5)$$

Whenever needed for clarity, isofield will be indicated with the symbol $\hat{F}_T$ identifying the selected isotopic element T. A realization often used in physics is given by $\hat{F}(\hat{n},+,\hat{*})$ where $\hat{n} = n\hat{1}$.

It is evident that the classification of the isounits of Sect. 2.A implies the corresponding classification of isofields into:

CLASS I: Isofields properly speaking;
CLASS II: Isodual isofields;
CLASS III: Singular isofields;
CLASS IV: Indefinite isofields;
CLASS V: General isofields.

Comment 2.B.1: The above definition of isonumbers holds when $\hat{1}$ is an element of the original field F. The isounit $\hat{1}$ can also be an element outside the original field F, in which case the isonumbers must be lifted into the form $n \Rightarrow \hat{n} = n\,\hat{1}$, because necessary for closure.

Comment 2.B.2: An isofield $\hat{F}$ is still a field, i.e., $\hat{F} \approx F$, thus confirming the axiom-preserving character of the lifting.

Comment 2.B.3: There exist infinitely possible isofields $\hat{F}$ for each given original field F, and this illustrates the use of the plural.

Comment 2.B.4: Only the multiplication of the original field F has been lifted $\times \Rightarrow \hat{*} = \times T \times$, $1 \Rightarrow \hat{1} = T^{-1}$, while the addition + and related additive unit 0 remain the conventional ones. Studies on the lifting of the addition $+ \Rightarrow \hat{+} = + \hat{K}$, $0 \Rightarrow \hat{0} = -\hat{K}$, $\hat{K} = K\,\hat{1}$, $K \in F$, are in progress but, unlike the lifting of the multiplication, it implies the loss of the distributive laws [46] and, as such, it will not be used in the isotopies of functional analysis.

Comment 2.B.5: The *isodual isofields* $\hat{F}^d(\hat{n}^d,+,\hat{*}^d)$ (Class II) hold when $\hat{1}^d < 0$ [23,28]. They are connected to $\hat{F}(n,+,\hat{*})$ by an antiautomorphism called *isoduality* [15] and characterized by

$$\hat{1} \Rightarrow \hat{1}^d = -\hat{1}. \qquad (2.6)$$

Comment 2.B.6: The conventional field of real numbers $\Re$ with trivial unit 1 admits the *isodual image* $\Re^d$ characterized by the negative unit $1^d = -1$ [23,28]. This implies that the absolute value $|n|^d$ of an isonumber n^d in $\Re^d$ is negative. We shall then symbolically write

$$\Re^d \approx \Re\, 1^d, \quad 1^d = -1. \qquad (2.7)$$

The ordinary product of a (non-null) number $n \in \Re$ and its isodual image $n \in \Re^d$ is also negative-definite

$$n\,n^d = n\,n\,1^d = -nn = -n^2 = n^{2d}. \qquad (2.8)$$

Comment 2.B.7: For the case of complex numbers $C = \Re + i\,\Re$, the isodual field is given by [28]

$$C^d \approx \Re^d - i^d \Re^d \approx \Re^d - i\,\Re^d \approx -\Re + i\,\Re. \qquad (2.9)$$

The above structure emerges from the requirement that the product of a (non-null) number $c = a + i\,b \in C$ and its isodual image $a^d - i\,b^d$ be negative-definite

$$(a + i\,b)(a^d + i^d b^d) = (a + i\,b)(-a + i\,b) = -a^2 - b^2. \qquad (2.10)$$

For the construction of isoquaternions, one can inspect ref. [42].

Comment 2.B.8: The use under isotopies of old notions generally leads to inconsistencies. For instance, the proverbial statement "two × two = four" is mathematically incorrect because lacking the additional necessary statement "under the assumption of the trivial multiplicative unit 1". In fact, for $\hat{1} = 3^{-1}$, "two × two = twelve". Also, $1 \hat{*} n = Tn \neq n$. In fact, isofields have two elements "ones", the "conventional element one" 1 and the "multiplicative one" $\hat{1}$. They coincide in conventional fields as a particular case, but they are different and disjoint for the more general isofields.

Comment 2.B.9: It is evident that all operations depending on the multiplication are lifted under the isotopy $F(n,+,\times) \Rightarrow \hat{F}(n,+,\hat{*})$. To begin, one notes that the *isoinverse* of an isonumber , denoted $n^{-\hat{1}}$ is defined by

$$n * n^{-\hat{1}} = \hat{1}, \quad n = \hat{1}\, n^{-1}\, \hat{1}, \tag{2.11}$$

<u>Comment</u> 2.B.10: The isotopy of the multiplication demands a corresponding compatible lifting of the division. Let a / b = c be the ordinary division of two numbers a, b ($\neq 0$) $\in$ F. The *isodivision* of two isonumbers a, b $\in \hat{F}$ hereon denoted $\hat{/}$ is the isonumber $c' \in \hat{F}$ defined by

$$a \,\hat{/}\, b \equiv a * b^{-\hat{1}} = c' = c\,\hat{1}. \tag{2.12}$$

<u>Comment</u> 2.B.11: The classification of all possible isotopes of the field of characteristics zero include: 1) the conventional fields $\Re$, C and Q; 2) their infinitely possible isotopes $\hat{\Re} \approx \Re$, $\hat{C} \approx C$ and $\hat{Q} \approx Q$; 3) the isodual fields $\Re^d$, C^d and Q^d, and 4) their infinitely possible isotopes $\hat{\Re}^d \approx \Re^d$, $\hat{C}^d \approx C^d$ and $\hat{Q}^d \approx Q^d$ [21]. For the unification of all these fields, see ref. [42].

2.C: ISOTOPIES OF METRIC AND PSEUDOMETRIC SPACES. The liftings of the unit $I \Rightarrow \hat{1}$ and of the fields $F(n,+,*) \Rightarrow \hat{F}(n,+,\hat{*})$ demand, for evident mathematical consistency, the corresponding lifting of conventional, N-dimensional, metric or pseudometric spaces $S(x,g,\Re)$ with (say, real) local coordinates x and metric g over the reals $\Re$, into the *isospaces* (first introduced in ref. [12], see also ref.s [14,15])

$$S(x,g,F):\ \det g \neq 0,\ g = g^\dagger,\ x^2 = x^t g\, x \in \Re \quad \Rightarrow$$

$$\Rightarrow \quad \hat{S}(x,\hat{g},\hat{F}):\ \hat{g} = Tg,\ \ T = T^\dagger,\ \ \det T \neq 0,\ \ \hat{F} = F\,\hat{1},\ \ \hat{1} = T^{-1}, \tag{2.13a}$$

$$x^{\hat{2}} = x^t \hat{g} x = x^t \hat{g}(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, ...)\, x \in \hat{\Re}, \tag{2.13b}$$

which preserve the dimensionality of the original space, where $\hat{g}$ = Tg is called the *isometric* .

It is again evident that the classification of the basic isounits implies the corresponding classification of the isospaces into:

CLASS I: Isospaces properly speaking;
CLASS II: Isodual isospaces;
CLASS III: Singular isospaces;
CLASS IV: Indefinite isospaces; and
CLASS V: General isospaces.

<u>Comment</u> 2.C.1: The above definition of isospaces over the reals evidently extends to *vector isospaces* $\hat{S}(z,\hat{F})$ of arbitrary real or complex coordinates z over an arbitrary isofield $\hat{F}$.

<u>Comments</u> 2.C.2: The *isodual isospaces* of Class II are given by[15,25,28]

$$\hat{S}^d(x,\hat{g}^d,\hat{\Re}^d):\quad \hat{g}^d = T^d g = -\hat{g},\ \ \hat{\Re}^d = \Re\,\hat{1}^d,\ \ \hat{1}^d = (T^d)^{-1} = -\hat{1}, \tag{2.14}$$

they hold for sign $\hat{1}^d$ = sign $1^d < 0$, and are interconnected to the isospaces by isoduality.

<u>Comment</u> 2.C.3: It is easy to prove the following

PROPOSITION 2.1 [20] : the basis of a vector space remains unchanged under isotopies.

<u>Comment</u> 2.C.4: Owing to the functional dependence of $\hat{g}$, isospaces are bona-fide nonlinear, nonlocal and noncanonical generalizations of the original spaces.

<u>Comment</u> 2.C.5: Despite the above differences, the isospaces $\hat{S}(x,\hat{g},\hat{\Re})$ (the isodual spaces $\hat{S}^d(x,\hat{g}^d,\hat{\Re}^d)$) are locally isomorphic (anti-isomorphic) to the original spaces $S(x,g,\Re)$ whenever sig. g ≡ sign. $\hat{g}$ (sig. $\hat{g}^d$ = – sign. g).

<u>Comment</u> 2.C.6: An *isoscalar function* f(x) on $\hat{S}(x,g,\hat{F})$ is a function with values on the isofield, i.e.,

$$f = f(x) \in \hat{F}(n,+,\hat{*}), \tag{2.15}$$

where f(x) is an ordinary scalar function.

<u>Comment</u> 2.C.7: The local coordinates $x \in \hat{S}(x,\hat{g},\hat{F})$ are also isoscalars, in the sense that their values are in $\hat{F}$. Note that the assumption of the quantity $\hat{x} = x\,\hat{1}$ for local coordinates of an isospace would turn separation (2.13b), i.e., $x^t T x$, into the form $\hat{x}^t \hat{*} \hat{x} = x^t x\, \hat{1}$, in which the role of T and $\hat{1}$ are interchanged. The map $T \rightarrow \hat{1}$ is at times called *reciprocity transform* . This point will soon be important for the isotopies of functional analysis.

<u>Comment</u> 2.C.8: The *isosquare* of x is given by

$$x^{\hat{2}} = x * x = x\, T\, x \tag{2.16}$$

with a corresponding definition applying for the *$\hat{n}$-th isopower*

$$x^{\hat{n}} = x * x * \ldots, * x \quad \text{(n times).} \qquad (2.17)$$

<u>Comment</u> 2.C.9: The *isosquare root* $x^{\hat{\frac{1}{2}}}$ of x is defined by the condition $x^{\hat{\frac{1}{2}}} * x^{\hat{\frac{1}{2}}} = x$, and is given by

$$x^{\hat{\frac{1}{2}}} = x^{\frac{1}{2}} T^{-\frac{1}{2}}. \qquad (2.18)$$

Note that in an isospace: the isounit $\hat{1}$ is idempotent, $\hat{1} \hat{*} \hat{1} \equiv \hat{1}$; the isodivision of the isounit by itself is the isounit $\hat{1} \hat{/} \hat{1} \equiv \hat{1}$; and the isoquare root of the isounit is the isounit, $\hat{1}^{\hat{\frac{1}{2}}} \equiv \hat{1}$, thus confirming the existence of a full isotopy.

<u>Comment</u> 2.C.10: The physically most important isospaces are given by the *isoeuclidean spaces* characterized by the following isotopies of the conventional three-dimensional spaces [12,14]

$$E(r,\delta,\Re): \delta = \text{diag.}(1, 1, 1), \det \delta \neq 0, \delta = \delta^\dagger, r^2 = r^t \delta r \in \Re \quad \Rightarrow$$

$$\Rightarrow \hat{E}(r,\hat{\delta},\hat{\Re}): \hat{\delta} = T(t, r, \dot{r}, \ddot{r}, \ldots)\,\delta, \ \det T \neq 0, \ T = T^\dagger, \qquad (2.19a)$$

$$r^2 = r^\dagger \delta r = r^i \delta_{ij} r^j \quad \Rightarrow \quad r^{\hat{2}} = r^i \hat{\delta} r = r^i \hat{\delta}_{ij}(t, r, \dot{r}, \ddot{r}, \ldots) r^j. \qquad (2.19b)$$

the *isominkowski spaces* [loc. cit.]

$$M(x,\eta,\Re): x = (r, x^4, \ x^4 = c_0 t, \eta = \text{diag.}(1, 1,1, -1), x^2 = x^\mu \eta_{\mu\nu} x^\nu \in \Re \quad \Rightarrow$$

$$\Rightarrow \hat{M}(x,\hat{\eta},\hat{\Re}): \hat{\eta} = T\,\eta, \ \hat{\Re} = \Re\,\hat{1}, \ \hat{1} = T^{-1} > 0, \ x^{\hat{2}} = (x^t \hat{\eta}\, x)\,\hat{1} \in \hat{\Re}, \qquad (2.20)$$

and the *isoriemannian spaces* [21]

$$R(x,g,\Re), \ g = g(x), \ \det g \neq 0, g = g^\dagger, \ x^2 = x^t g(x)\, x \in \Re \quad \Rightarrow$$

$$\Rightarrow \hat{R}(x, \hat{g}, \hat{\Re}): \hat{g} = T(s, x, \dot{x}, \ddot{x}, \ldots)\, g(x), \ \hat{\Re} = \Re\hat{1}, \ \hat{1} = T^{-1} \qquad (2.21)$$

with corresponding isoduals

$$\hat{E}^d(r,\delta^d,\Re^d), \ \delta^d = -\hat{\delta}, \ \hat{\Re} \Rightarrow \hat{\Re}^d = \Re\,\hat{1}^d, \ \hat{1}^d = -\hat{1}, \qquad (2.22a)$$

$$\hat{M}^d(x,\hat{\eta}^d,\hat{\Re}^d): \hat{\eta}^d = -\hat{\eta}, \quad \hat{\Re}^d = -\Re\,\hat{1}^d, \quad \hat{1}^d = -\hat{1}, \qquad (2.22b)$$

$$\hat{R}^d(x,\hat{g}^d,\hat{\Re}^d): \hat{g}^d = T^d \hat{\eta} = -\hat{g}, \ \hat{\Re}^d = \Re\,\hat{1}^d, \ \hat{1}^d = -\hat{1}. \qquad (2.22c)$$

<u>Comment</u> 2.C.11: In the same way as the conventional spaces $E(r,\delta,\Re)$, $M(x,\eta,\Re)$ and $R(x,g,\Re)$ geometrize the homogeneous and isotropic vacuum, their isotopic coverings $\hat{E}(r,\hat{\delta},\hat{\Re})$, $\hat{M}(x,\hat{\eta},\hat{\Re})$ and $\hat{R}(x,\hat{g},\hat{\Re})$ geometrize inhomogeneous and anisotropic physical media. In particular, such a geometrization occurs via the basic isounit. Isospaces are therefore important for the characterization of interior dynamical systems, and are at the foundations of Santilli's isotopies of conventional relativities for the interior dynamical problem, called *isogalilean, isospecial and isogeneral relativities* [23,26,27].

<u>Comment</u> 2.C.12: Because of their assumed characteristics, the isounits (of Class I) can be diagonalized, resulting in expressions of the type

$$\hat{1} = \text{diag.}(\tilde{b}_1^{-2}, \tilde{b}_2^{-2}, \tilde{b}_3^{-2}) > 0, \quad \tilde{b}_\mu = \tilde{b}_\mu(t, \dot{r}, \ddot{r}, \ldots) > 0, \ \mu = 1, 2, 3, 4, \qquad (2.23)$$

where the $\tilde{b}$'s are called the *characteristic quantities of the medium* , generally vary from medium to medium, and they can be averaged into constants b_μ when total properties are needed (see ref.s [23,25] for details).

<u>Comment</u> 2.C.13: All metrics g of conventional gravitational models admit the decomposition $g = T(x)\,\eta$, where η is the Minkowski metric. As a result, Riemannian spaces are locally isomorphic to the isominkowskian space with $\hat{\eta} = g$ [20,23], i.e.,

$$R(x,g,\Re) \approx \hat{M}(x,\hat{\eta},\hat{\Re}), \ g(x) = T(x)\,\eta \equiv \hat{\eta}, \ \hat{\Re} = \Re\hat{1}, \ \hat{1} = [T(x)]^{-1}. \qquad (2.24)$$

The above characterization of gravity is at the foundation of Class III (singular isounits, isofields and isospaces) because at the limit of gravitational collapse into a singularity at x, the (space component of the) isotopic element T(x) is null, and the isounit becomes singular [29].

<u>Comment</u> 2.C.14: All N-dimensional, metric or pseudo-metric spaces over the reals are unified by one single, abstract isotope $\hat{E}(x,\hat{\delta},\hat{\Re})$ of Class IV of the N-dimensional Euclidean space $E(x,\delta,\Re)$ [12]. This property has permitted the unification of the Minkowski and Riemannian spaces with

consequential unified formulation of the special and general relativities [20]. Their isotopic lifting was then consequential [23].

2.D: ISOTOPIES OF UNIVERSAL ENVELOPING ASSOCIATIVE ALGEBRAS : Let ξ be a universal enveloping associative algebra (see, e.g., ref. [53]) with generic elements A, B, C,.. , trivial associative product AB and unit I. Their isotopes $\hat{\xi}$, introduced in ref. [7] under the name of *isoassociative envelopes* , coincide with ξ as vector spaces but are equipped with the isoproduct so as to admit $\hat{1}$ as the correct (right and left) unit

$$\hat{\xi}: \quad A*B = ATB, \quad T = \text{fixed}, \quad I*A = A*I \equiv A \quad \forall A \; \hat{\xi}, \quad \hat{1} = T^{-1}. \qquad (2.25)$$

Let $\xi = \xi(L)$ be the universal enveloping algebra of an N-dimensional Lie algebra L with ordered basis $\{ X_k \}$, k = 1, 2, ..., N, $[\xi(L)]^- \approx L$, and let the infinite-dimensional basis of $\xi(L)$ of the Poincaré-Birkhoff-Witt theorem [53] be given by

$$I, \quad X_k \qquad X_iX_j \quad (i \leq j), \qquad X_iX_jX_k \quad (i \leq j \leq k), \; \ldots\ldots \qquad (2.26)$$

where one recognizes the familiar standard monomials.

A fundamental result achieved by Santilli in the original proposal [7] (see also the detailed presentation in ref. [8], p. 154-163 and ref. [20]) is the following

THEOREM 2.1 (Poincaré-Birkhoff-Santilli-Witt Theorem): The cosets of $\hat{1}$ and the standard, isotopically mapped monomials form a basis of the universal enveloping isoassociative algebra $\hat{\xi}(L)$ of a Lie algebra L,

$$I, \quad X_k \qquad X_i*X_j \quad (i \leq j), \qquad X_i*X_j*X_k \quad (i \leq j \leq k), \; \ldots\ldots \qquad (2.27)$$

The implications of the theorem are fundamental for this paper. In fact, the Fourier transforms are centrally dependent on the conventional notion of exponentiation

$$e_{\xi}^{ikx} = 1 + (ikx)/1! + (ikx)(ikx)/2! + \ldots\ldots = e^{ikx}. \qquad (2.28)$$

This notion is however inapplicable under isotopies and must be replaced by the notion of *isoexponentiation* [loc. cit.]

$$e_{\hat{\xi}}^{ikx} = \hat{1} + (ikx)/1! + (ikx)*(ikx)/2! + \ldots\ldots = \hat{1}\, e^{ikTx}. \qquad (2.29)$$

where the last expression in term of the conventional exponential has been presented merely for illustrative purposes.

The nontrivial implications of the isotopies for the Fourier (as well as other) transforms can therefore be seen already in these introductory words. In fact, it originates from the appearance of the generally nonlinear and nonlocal isotopic element T in the exponent of Eq. (2.29).

Whenever needed for clarity, isoenvelopes will be denoted with the symbol $\hat{\xi}_T$ identifying the selected isotopic element T.

As well known [53], universal enveloping associative algebras $\xi(L)$ are at the true foundations Lie's theory inasmuch as they characterize Lie algebras via the attached algebra $[\xi(L)]^-$, Lie groups via exponentiation in $\xi(L($, the representation theory, etc. The universal enveloping isoassociative algebras $\hat{\xi}(L()$ then are at the foundation of the *Lie-Santilli theory* [7,26,27,50] because they also characterize the Lie-Santilli algebras as the attached algebras $[\hat{\xi}(L)]^-$, the Lie-Santilli groups via the isoexponentiation in $\hat{\xi}(L)$, the isorepresentation theory, etc.

In the same way as Lie's theory is defined over a conventional field, the Lie-Santilli theory is necessarily defined over an isofield. The classification of the isounits, isofields and isospaces presented earlier therefore implies the following classification:

CLASS I: Lie-Santilli theory properly speaking;
CLASS II: Isodual Lie-Santilli theory;
CLASS III: Singular Lie-Santilli theory;
CLASS IV: Indefinite Lie-Santilli theory;
CLASS V: General Lie-santilli theory.

<u>Comment</u> 2.D.1: The lifting $\xi \Rightarrow \hat{\xi}$ is *necessary* under the isotopy of the unit because, in general, $\hat{1}A \neq A\hat{1} \neq A$.

<u>Comment</u> 2.D.2: The preservation of the original basis X_k is requested by Proposition 2.1, thus explaining the symbol $\hat{\xi}(L)$.

<u>Comment</u> 2.D.3: Under the assumed conditions on the isounit, the isotopies preserve the simplicity or semisimplicity of the original algebra.

<u>Comment</u> 2.D.4: It is easy to prove that $L \approx [\hat{\xi}(L)]^-$ when $\hat{1} > 0$. In

general, however, the isotopies of the envelope of a Lie algebra L characterize a nonisomorphic algebra $\hat{L} \approx [\,\hat{\xi}(L)\,]^{-} \not\approx L$.

Comment 2.D.5: Santilli [7] introduced Theorem 2.1 to be able to represent with one single Lie algebra basis X_k, but arbitrary isotopies in the envelope $\hat{\xi}(L)$, nonisomorphic algebras of the same dimension N. In fact, as well known [53], a conventional envelope $\xi(L)$ represents only one algebra $L \approx [\,\xi(L)\,]^{-}$ up to local isomorphisms. On the contrary, one universal enveloping isoassociative algebra $\hat{\xi}(L)$ of Class IV represents a family of generally nonisomorphic Lie algebras as the attached algebras $\hat{L} \approx [\hat{\xi}(L)]^{-}$. Theorem 2.1 is therefore at the foundations of Santilli's isorelativities because it permits the reduction of infinite families of linear and nonlinear, local and nonlocal, canonical and noncanonical symmetries to one primitive algebraic notion $\hat{\xi}(L)$.

Comment 2.D.6: An illustration of the unifying power of $\hat{\xi}(L)$ was provided in the original proposal [7] by showing that, given the basis J_k, $k = 1, 2, 3$ (the familiar angular momentum components) of the rotational algebra SO(3), the classification of all possible universal enveloping isoassociative algebras $\hat{\xi}(SO(3))$ includes the envelopes of:

1) SO(3), trivially given by $\hat{1} = I = \text{diag.}\ (1, 1, 1)$;

2) SO(2.1) for $\hat{1} = \text{diag.}\ (1, 1, -1)$;

3) An infinite family of nonlinear, nonlocal and noncanonical semi-simple three-dimensional algebras $\hat{O}(3)$ locally isomorphic to O(3) for $\hat{1} = \text{diag.}\ (b_1^{-2}, b_2^{-2}, -b_3^{-2})$, $b_k > 0$, and

4) An infinite family of isotopes $\hat{O}(2.1)$ isomorphic to O(2.1) for $\hat{1} = \text{diag.}\ (b_1^{-2}, b_2^{-2}, -b_3^{-2})$, $b_k > 0$.

The classification was completed in the subsequent paper [15] with:

5) The isodual $SO^d(3)$ of SO(3) for $\hat{1} = \text{diag.}\ (-1, -1, -1)$;

6) The isodual $SO^d(2.1)$ of SO(2.1) for $\hat{1} = \text{diag.}\ (-1, -1, 1)$;

7) The infinite family of isotopes $\hat{O}^d(3) \approx SO^d(3)$ for $\hat{1} = \text{diag.}\ (-b_1^{-2}, -b_2^{-2}, -b_3^{-2})$, $b_k > 0$, and

8) The infinite family of isotopes $S\hat{O}^d(2.1) \approx SO^d(2.1)$ for $\hat{1} = \text{diag.}\ (-b_1^{-2}, -b_2^{-2}, b_3^{-2})$, $b_k > 0$.

Comment D.7: The above results permitted the construction of a dual, nonlinear, nonlocal and noncanonical generalization of the conventional rotational symmetry [15,23]. $S\hat{O}(3)$ resulted to be the symmetries of all infinitely possible ellipsoidical deformations of the sphere on isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ for the direct representation of

extended-deformable particles, while their isoduals $S\hat{O}^d(3)$ on $\hat{E}^d(r,\hat{\delta}^d,\hat{\mathfrak{R}}^d)$ permitted a fundamentally novel description of antiparticles [28].

Comment 2.D.8: The unifying power of $\hat{\xi}(L)$ was additionally illustrated in ref.s [12,21] by showing that the classification of all possible universal enveloping isoassociative algebras of the four-dimensional orthogonal algebra SO(4) include the characterization of:

1) all possible, compact and noncompact six-dimensional Lie algebras SO(4), SO(3.1), and SO(2.2) (and algebras locally isomorphic to them);

2) all infinitely possible isotopes $S\hat{O}(4) \approx SO(4)$, $S\hat{O}(3.1) \approx SO(3.1)$, $S\hat{O}(2.2) \approx SO(2.2)$; and

3) all possible isoduals $SO^d(4)$, $SO^d(3.1)$, $SO^d(2.2)$, $S\hat{O}(^d4)$, $S\hat{O}^d(3.1)$, $S\hat{O}^d(2.2)$.

Comment 2.D.9: The infinite family $S\hat{O}(3.1) \approx SO(3.1)$ permitted the construction of an infinite family of nonlinear, nonlocal and noncanonical generalizations of the Lorentz symmetry for the form invariance of interval (2.20). The isosymmetries $S\hat{O}(3.1)$ are at the foundation of the isospecial relativity for the description of extended-deformable particles under nonlinear, nonlocal and noncanonical interactions or of the propagation of electromagnetic waves within inhomogeneous and anisotropic physical media.

Comment 2.D.10: A fundamental open problem identified in ref. [20] is the study of the possible unification of all N-dimensional simple Lie algebras of Cartan classification into one simple abstract N-dimensional isotope $\hat{L}(N)$. This conjecture has been proved by Santilli for all orthogonal algebras, and it is expected to be provable for all Lie algebras, with technical difficulties emerging for the inclusion of the exceptional algebras, under a suitably generalized form of isofields.

Comment 2.D.11: Since the isounit has an arbitrary functional dependence, it permits the incorporation of conventional gravitational models via the decomposition of the Riemannian metric $g(x) = T(x)\,\eta$, $\eta \in M(x,\eta,\mathfrak{R})$, and the embedding of the part T(x) representing gravitation in the isounit, $\hat{1} = [T(x)]^{-1}$. Santilli then proved that the isotopes $\hat{O}(3.1)$ of the Lorentz symmetry O(3.1) constructed for the above identified gravitational isounit provide the form-invariance of conventional gravitational models for the exterior problem in vacuum (e.g., of the Schwartzschild's exterior [3] and interior [4] metrics). The liftings $T(x) \Rightarrow T(x, \dot{x}, \ddot{x}, ...)$ then permitted a generalization of conventional gravitational theories via the *isoriemannian geometry* [21], for a more adequate representation of the nonlinearity (in the velocities), nonlocality and noncanonical character of interior gravitation [23] (see also the review by this author [26]).

2.E: ISOTOPIES OF TRANSFORMATION THEORY . The last notion essential for the understanding of the isotopies of functional analysis and of the Fourier transforms is that of the applicable transformations.

Let S(x,F) be a conventional vector space with local coordinates x over a field F, and let $x' = A(w)x$ be a linear and local transformation on S(x,F), $w \in F$.

The lifting $S(x,F) \Rightarrow \hat{S}(x,\hat{F})$ requires a corresponding necessary isotopy of the transformation theory which is characterized by the so-called *isotransformations* [7,8]

$$x' = \hat{U}(w) * x = \hat{U}(w)\, T\, x, \; T \text{ fixed}, \quad x \in \hat{S}(x,\hat{F}), \; \hat{F} = F\,\hat{1}, \hat{1} = T^{-1}. \qquad (2.30)$$

Comment 2.E.1: The isotransformations verify the condition of linearity (and locality) in isospaces,

$$A * (\alpha * x + \beta * y) = \alpha * (A * x) + \beta * (A * y),$$
$$\forall x, y \in \hat{S}(x,\hat{F}), \quad \alpha, \beta \in \hat{F} \qquad (2.31)$$

Comment 2.E.2: It is easy to see that the projection of isotransformations on the original space S(x,F) is generally nonlinear and nonlocal (as well as noncanonical). In fact, Eq. (2.6) can be explicitly written in S(x,F)

$$x' = X\,T\,x = X\,T(x, \dot{x}, \ddot{x}, ...)\, x \qquad (2.32)$$

Comment 2.E.3: Linear transformations are canonical, as well known. Isolinear transformations are noncanonical, in the sense that they do not generally leave invariant the conventional (first-order)canonical action, i.e., the contact one-form $\phi_1 = p\, dr - H\, dt$. Isolinear transformation are however isocanonical in the sense that they leave invariant the isotopic action, which is the one form $\hat{\phi}_1 = d*dr - H\odot dt$ at the basis of the *isosymplectic geometry* and related isocontact extension (see ref. [21] in this Journal for brevity).

Comment 2.E.4: The following property is particularly important for this paper:

PROPOSITION 2.2 [20]: Given a nonlinear, nonlocal and noncanonical transformation $x' = X(x,...)\,x$ on a vector space $S(x,F)$, then there always exist an isotopy $F \Rightarrow \hat{F}_T$ and an isolinear and isolocal operator $\hat{A}$ on $\hat{S}(x,\hat{F}_T)$ under which the transformation can be identically rewritten in an isolinear, isolocal and isocanonical form

$$x' = X(x, ...)\, x \equiv A * x \qquad (2.33)$$

Comment 2.E.5: A primary role of the isotopic techniques is that of *turning conventionally nonlinear, nonlocal and noncanonical theories into identical isolinear, isolocal and isocanonical forms*, with evident simplifications of their treatment. This illustrates the capabilities indicated in the introduction for isotopies to provide axiom-preserving, nonlinear, nonlocal and noncanonical generalization of conventional linear, local and canonical theories.

Comment 2.E.6: The necessity of the isotopy $Ax \Rightarrow A*x$ should be kept in mind. In fact, the preservation of the conventional transformations Ax in isospaces $\hat{S}(x,\hat{F})$ implies the loss of linearity, transitivity, etc.

Comment 2.E.7: The "isolocal topology" indicated in Comment 2.A.5 as characterized by integral isounits is expected to apply at all subsequent levels of the analysis, including isospaces, isoalgebras and isosymmetries. It is hoped that topologists will study this novel topology in the needed mathematical details.

For brevity, we refer the reader interested in the isotopies of Lie algebras and Lie groups to ref.s [20-27]. With the understanding that the isotopies of Lie's theory are at their first infancy and so much remains to be done, the reader should be aware that all structural theorems of Lie's theory (such as Lie's celebrated First, Second, and Third theorems, the Baker-Campbell-Hausdorff theorem, etc.) admit consistent and nontrivial isotopic liftings.

3: Elements of functional isoanalysis. It is significative for this paper to recall that functional analysis (see, e.g., ref.s [54-56]) was born and developed primarily because of specific physical motivations, rather than abstract mathematical needs.

In fact, the French mathematician J. B. J. Fourier identified his celebrated series and transforms during his study on heat conduction; Freedholm worked on integral equations because of specific problems in classical electromagnetism; von Neumann conducted most of his studies on operator algebras because of specific physical needs; not to mention the fundamental physical role of Hilbert studies in quantum mechanics (see the historical notes of ref.s [54-56]).

It is intriguing to note that, much along the same lines, the new branch of functional analysis characterized by the isotopies of conventional formulations, and presented in this section under the name of *functional isoanalysis*, was also born out specific physical problems, given this time by Santilli's studies of nonlinear, nonlocal and noncanonical systems of the interior dynamical problem. In fact, the conventional functional analysis can be seen as the discipline which is and will remain fundamental for the *exterior* dynamical problem of particles in vacuum (see Sect. 1), while functional isoanalysis is a covering discipline specifically conceived for the more general *interior* dynamical problem of extended particles moving within physical media.

Despite its rather vast current dimension, contemporary functional analysis remains based on conventional notions, such as conventional fields, conventional vector spaces, conventional operations, etc. It is then inevitable that the isotopic generalizations of these structural foundations imply the existence of a consequential, corresponding generalization of the entire theory.

It is also significant to note that functional isoanalysis was born and completely developed in physical publications until now, this paper being the first appearing in the field in a mathematical Journal, to the author's best knowledge.

The foundations of functional isoanalysis are those reviewed in the preceding section, and consist of Santilli's studies on the isotopies of fields, vector spaces, transformation theory, algebras, groups, geometries, etc. [20.21]. In this section we shall review and expand the studies by Myung and Santilli [11] on the isotopies of Hilbert spaces. In the adjoining paper [57] we shall add Santilli's [25,51] studies on the isotopies of Dirac's delta-function, Fourier series and Fourier transforms.

As indicated in the Introduction, we are primarily interested in identifying the essential structural lines of functional isoanalysis. Technical studies of details in all necessary mathematical rigor must be deferred, for clarify, to subsequent contributions by the interested mathematician.

The first fundamental notion of isoanalysis is an isofield $\hat{F}(n,+,*)$ with isonumbers n, conventional sum +, isoproduct $* = \times T \times$, and isounit $\hat{1} = T^{-1}$. For simplicity, we shall restrict $\hat{F}$ to be of characteristic zero and to

represent the isofields of real isonumbers $\hat{\mathfrak{R}}(n,+,*)$ and of complex isonumbers $\hat{C}(c,+,*)$.

The second fundamental notion is a generic, finite-dimensional vector isospace $\hat{S}(x,\hat{C})$ on the isofield $\hat{C}$. The abstract identity of $\hat{C}(c,+,*)$ and $C(c,+,\times)$ and that of $\hat{S}(x,\hat{C})$ and $S(x,C)$ should be kept in mind to anticipate that *functional isoanalysis coincides with the conventional formulation at the abstract level by construction* (although only for the case of isounits of Class I - see below).

Recall that conventional complex numbers c can be reinterpreted as being complex isonumbers under the isotopy of the multiplication. Along similar lines, a conventional function f(x) on S(x,C) can be reinterpreted as being a function on $\hat{S}(x,\hat{C})$. In fact, it is not the value of the function f(x) which identifies the distinction between S(x,C) and $\hat{S}(x,\hat{C})$, but rather the operations on it.

Finally, the reader should recall that the isotopies automatically generalize a linear, local and canonical theory into an axiom-preserving, nonlinear, nonlocal and noncanonical form because of the arbitrary functional dependence of the isounit $\hat{1} = \hat{1}(x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \ldots)$, where x is the local coordinate and ψ represents elements of the Hilbert space.

Next, the first isotopic operation among functions on $\hat{S}(x,\hat{C})$ is the *isoscalar product* (or *isoproduct* for short) of two functions $f_1(x)$ and $f_2(x)$, which is given by [7]

$$f_1(x) * f_2(x) \; := \; f_1(x)\, \hat{T}(x, \ldots)\, f_2(x) \;\in\; \hat{S}(x,\hat{C}) , \qquad (3.1)$$

where the isotopic element $\hat{T} = \hat{1}^{-1}$ is fixed.

The *isoinner product* of two functions $f_1(x)$ and $f_2(x)$ on $\hat{S}(x,\hat{C})$ is the composition with elements in $\hat{C}$ introduced in ref. [11]

$$(f_1 \hat{,} f_2) \; := \int_a^b dx\, \overline{f}_1(x) * f_2(x) \;\in\; \hat{C}(c,+,*) , \qquad (3.2)$$

where $\overline{f}$ denotes ordinary complex conjugation.

The above foundations then imply the lifting of the conventional quantity $| f(x) |$ into the *isoabsolute value* $\hat{|} f(x) \hat{|}$ which is characterized by

$$\hat{|} f(x) \hat{|}^2 = \overline{f(x)} * f(x) , \qquad (3.3)$$

and given, from Eq.s (2.18), by

$$\lceil f(x) \rceil = (\bar{f}\, T\, f)^{\frac{1}{2}}\, \hat{1}^{\frac{1}{2}}. \qquad (3.4)$$

where $\hat{1}^{\frac{1}{2}}$ is a conventional square root. The *isonorm* $|\,|\hat{}f(x)\,|\hat{}|$ of a function f(x) is then defined by the element of the isoreals

$$|\lceil f(x) \rceil|^2 := (f \hat{,} f) = \int_a^b dx\, \bar{f}(x) * f(x) \quad \in \hat{R}. \qquad (3.5)$$

and given by

$$|\lceil f(x) \rceil| = (f \hat{,} f)^{\frac{1}{2}} = (f_1, f_2)^{\frac{1}{2}}\, \hat{1}^{\frac{1}{2}}. \qquad (3.6)$$

It should be indicated from the outset that the above definitions are not unique, owing to the degrees of freedom of the isotopies. In fact, one can consider the maps

$$f \rightarrow \hat{f} = f\, \hat{1} \in \hat{S}(\hat{x},\hat{C}), \quad c \rightarrow \hat{c} = c\, \hat{1} \in \hat{C}(\hat{c},+,*), \qquad (3.7)$$

in which case we have the map of the isoproduct

$$f_1 * f_2 = f_1 f_2\, T \quad \rightarrow \quad \hat{f}_1 * \hat{f}_2 = f_1 f_2 \hat{1}, \qquad (3.8)$$

with corresponding definitions for isoabsolute value

$$\lceil \hat{f}(x) \rceil := (\bar{f}\, f\, \hat{1})^{\frac{1}{2}}\, \hat{1}, \qquad (3.9)$$

isoinner product

$$(\hat{f} \hat{,} \hat{f}) := \hat{1} \int_a^b dx\, \bar{f}(x)\, f(x)\, \hat{1}(x,...) \quad \in \hat{R}(\hat{n},+,*). \qquad (3.10)$$

and isonorm

$$|\lceil f(x) \rceil| := (\hat{f} \hat{,} \hat{f})^{\frac{1}{2}} = (\hat{f}_1, \hat{f}_2)^{\frac{1}{2}}\, \hat{1}. \qquad (3.11)$$

The transition from the preceding formulation in terms of ordinary numbers and functions to the latter one is called a *reciprocity*

transformation [51] because based on the replacement

$$T \rightarrow \hat{1}, \qquad \hat{1} \rightarrow \hat{1}^{-1}. \qquad (3.12)$$

The latter formulation is that primarily used in physics [25] because it implies that the isotopic eigenvalues are the conventional ones (see below in this section), although both formulations emerge rather naturally, e.g., in the lifting of Dirac delta-function (see next paper [57]).

Needless to say, maps (3.7) are, by far, nonunique and a number of additional maps implying nontrivial alterations of the isoproduct are possible. Nevertheless the above two alternatives are sufficiently to identify the foundations of isoanalysis.

From these rudimentary notions it is sufficient to see the need for the following classification:

PRIMARY CLASSIFICATION: based on the characteristics of the isounit (Sect. 2.A):

CLASS I: *Functional isoanalysis* properly speaking;
CLASS II: *Isodual functional isoanalysis*;
CLASS III: *Singular functional isoanalysis* ;
CLASS IV: *Indefinite functional isoanalysis* ;
CLASS V: *General functional isoanalysis* .

SECONDARY CLASSIFICATION: based on the assumed realization of isofields and isovector spaces

SUBCLASS A: based on isofields $\hat{F}(n,+,*)$ whose elements are ordinary numbers, isospaces $\hat{S}(x,\hat{F})$ whose local coordinates are the conventional ones and, therefore, on conventional functions f(x).
SUBCLASS B: based on isofields $\hat{F}(\hat{n},+,*)$ with elements $\hat{n} = n\hat{1}$, isospaces $\hat{S}(\hat{x},\hat{F})$ with local coordinates $\hat{x} = x\,\hat{1}$ and isofunctions $\hat{f}(x) = f(x)\,\hat{1}$.

By no means the above classification is complete. In fact, the extension of isofields $\hat{F}(n,+,*)$ to include an isotopy also of the addition + [42] will expectedly imply further branches of isoanalysis. Nevertheless, the above classification is sufficient to identify the new discipline and initiate its systematic study.

A first purpose of the above classification is to separate the axiom-preserving liftings from the more general ones. As an example, an "inner" product remains inner for Classes I , but not necessarily for Class IV, and this confirms the need to use the term "isotopies" only for Class I.