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REPRESENTATION OF ANTIPARTICLES VIA ISODUAL NUMBERS, SPACES AND GEOMETRIES

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As well known, antiparticles originate from the negative-energy solutions of relativistic equations. However, such solutions behave unphysically when treated with our conventional space-time. This has forced the construction of various models to avoid inconsistencies, such as the celebrated "hole theory". In this paper we use recent mathematical advances in number theory which have identified new numbers n^d whose unit is -1, and therefore with norm $|n^d|^d < 0$, called *isodual numbers*. These new numbers have then induced for the generalization of conventional space-time, algebras, geometries and mechanics which are the image of conventional structures under an antihomomorphic conjugation called *isoduality*. In this paper we show that isoduality to a different space-time is equivalent to charge conjugation in our own space-time, thus achieving a novel, rather intriguing interpretation of antiparticles as being characterizable via isodual numbers, spaces, algebras, geometries and mechanics. One of the implications of the analysis is the emergence of a hitherto unknown *isodual universe*, geometrically separate from our own universe, possessing not only negative energies $|E^d|^d < 0$, but also evolving backward in time $|t^d|^d < 0$.

1. Introduction

The entire physical reality is represented in contemporary theoretical physics as existing in one single space, the Euclidean, Minkowski or Riemannian space (see e.g. [1]). This includes not only elementary particles of which ordinary matter is composed of, but also antiparticles and corresponding antimatter.

In fact, given a particle (such as an electron e^- or a proton p^+) in Minkowski space, the corresponding antiparticles (the positron e^+ or antiproton p^- [2-4] is constructed via charge conjugation and other means, but with the tacit understanding that the antiparticle belongs to the same Minkowski space of the ordinary particle.

This implies that *in contemporary theoretical physics both particles and antiparticles, as well as more generally matter and antimatter, are assumed to exist in the same physical space.*[1-4].

This view has been structurally altered by recent advances known as the *isotopies of contemporary algebras, geometries and mechanics*[5-18] because they have identified the existence of fundamentally new spaces, submitted under the name of *isodual spaces*, which apparently permit the characterization of antiparticles and antimatter in a way considerably deeper than the contemporary one.

Following the rudimentary proposal of **[9b]** (see also **[11b]** Chapters III, IV and V), in this paper we shall study the characterization of antiparticles and antimatter in the *isodual Minkowski space*, with the understanding that the corresponding characterization in the *isodual Euclidean space* is a simple particular case. The *isodual Riemannian spaces* [9h, 10b, 11a, b] are quite intriguing indeed because they permit the identification of *isodual universes* fundamentally unknown to this day, which will be investigated in a subsequent paper.

The isodual spaces have remained unknown until recently because the entirety of contemporary theoretical physics has been constructed on the basis of, and it is crucially dependent on the assumption of the simplest conceivable *unit*, such as the number 1 or the 4-dimensional unit matrix $1 = \text{diag.}(1, 1, 1, 1)$. The isotopies of contemporary algebras, geometries and mechanics are instead based on the most general possible units $\hat{I}$, called *isounits*, which have the most general conceivable functional dependence of the type $\hat{I} = \hat{I}(s, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \ldots)$. The lifting $I \Rightarrow \hat{I}$ then requires corresponding compatible generalizations of all theoretical formulations of contemporary physics, without any exclusion known to this author at this writing, and including: fields; metric and pseudo-metric spaces;

enveloping algebras; Lie algebras; Lie group; transformation theory; representation theory; symplectic, affine and Riemannian geometries; conventional, classical and quantum Hamiltonian mechanics; conventional space-time and internal symmetries; etc.

The generalizations are said to be *isotopic* if they preserve the original axioms, although realized in their most general nonlinear, nonlocal and noncanonical form. Thus, $\hat{I}$ is an isotope of I if and only if it preserves the original axioms of I, i.e., nowhere degeneracy, Hermiticity and positive-definiteness, $\hat{I} > 0$. Similarly, according to the original proposal[5a], a generalization of a Lie algebra is called a ***Lie-isotopic algebra*** if it is constructed in such a way to admit the quantity $\hat{I}$ for isounit, while preserving the basic axioms of Lie's theory (antisymmetry of the product and Jacobi's identity), with similar results for Lie-isotopic groups, the isotopic symmetries, the isotopic representation theory, etc.[10-18].

The identification of the isodual spaces has been permitted precisely by the above new techniques, and it essentially consists of the lifting of the basic unit $I \Rightarrow \hat{I}$ for the particular quantity $\hat{I} = (-I)$ which is called ***isodual unit*** and denoted with the symbol $I^d = (-I)$, while the terms ***isodual isounit*** and the symbol $\hat{I}^d$ are reserved for the isodual expression $\hat{I}^d = (-\hat{I})$ of the most general possible iso unit $\hat{I}$).

The reason why isodual spaces (and isospaces) have remained completely unidentified until recently is now clear. In fact, their treatment requires a simple but nontrivial generalization of the basic unit, with corresponding generalizations of fields, metric spaces, Lie algebras, etc. while contemporary physics is entirely restricted to the use of the simplest conceivable unit I.

The reader should be aware that, strictly speaking, the lifting $I \Rightarrow I^d$ is not an isotopy because $I > 0$ while $I^d < 0$, thus implying that the original axioms of I are not preserved. But the absolute values of the original and new unit coincide. This illustrates the meaning of the term *isoduality* (or other terms such as *anti-isotopy, anti-isomorphisms*, etc.).

The fundamental notion of this paper, that of isospace on an isofield, was introduced for the first time in ([8a], Section II) in regard to the formulation and proof of a general theorem on isosymmetries. The notion of isoduality was introduced for the first time in ([8b], Definition 1, p. 42) as part of the classification of all possible isotopes $\hat{O}(3)$ of the rotational symmetry $O(3)$. The first application of these notions to the lifting of the Minkowski space and related Lorentz symmetry was submitted in[7a], and then extended to operator formulations in[7b] via an isotopy of Wigner's theory on unitary symmetries, later treated in more details in memoirs[9]. The interested reader may note that ref.s[8a, b] were written considerably earlier than[7a,b], but they appeared in print considerably later than the latter because of unusual editorial difficulties reported in ([8a], p. 26).

The studies on the isotopies of contemporary algebras, geometries and mechanics are now well under way (see the mathematical monograph [15] and quoted contributions). However, except [7-11], the complementary notion of isoduality (or anti-isotopy) has remained unexplored, thus motivating the analysis of this paper.

2. Isotopies of Contemporary Formulations

As recalled in Section 1, the *isotopies* are given by axiom-preserving, nonlinear, nonlocal and noncanonical realizations of any given linear, local and canonical structure, and were constructed for the purpose of enlarging the representational capabilities of contemporary formulations.

More specifically, contemporary formulations essentially permit the treatment of the so-called *exterior dynamical problem*, consisting of point-like particles moving in the homogeneous and isotropic vacuum, thus resulting in the familiar local-differential and canonical-Hamiltonian equations of motions verifying linear symmetries. On the contrary, the isotopic formulations have been constructed for a quantitative treatment of the more general *interior dynamical problem*, which consists of extended-deformable particles moving within inhomogeneous and anisotropic physical media, thus resulting in the most general known nonlocal-integral and non-canonical-nonhamiltonian equations of motion verifying expectedly nonlinear symmetries.

In this paper we shall study the isodual image of the *conventional* formulations, thus restricting our analysis to the conventional exterior problem in vacuum. Nevertheless, a knowledge of the isotopic formulations is important for this paper, first because a necessary pre-requisite for its technical understanding, and also because the exterior, local-canonical isoduality of this paper can be extended by the interested reader to the covering notion of nonlocal-noncanonical isoduality of the interior dynamical problem.

The fundamental isotopy is that of the unit. Consider a relativistic quantum field theory on Minkowski space $M(x, \eta, \mathcal{R})$, with invariant line element s, local coordinates x, and metric η over the reals $\mathcal{R}$. The basic unit of such space is then the four-dimensional familiar form $I = \text{diag.}(1, 1, 1, 1)$. The isotopes of I are given by the infinitely possible 4×4 matrices $\hat{I}$ with the most general admissible nonlinear and nonlocal dependence on: s, the local coordinates x, their derivative with respect to s of arbitrary order, the wavefunctions ψ and $\psi^\dagger$ and their derivatives, and any needed additional quantity, under the condition of preserving the original axioms of I (in order to qualify as an isotopy)[5a, b, 8a, 7a]

$$I = \text{diag.}(1, 1, 1, \ldots, 1) \Rightarrow \hat{I} = \hat{I}(s, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \ldots), \quad (2.1a)$$

$$\det I \neq 0, I = I^\dagger, I > 0 \Rightarrow \det \hat{I} \neq 0, \hat{I} = \hat{I}^\dagger, \hat{I} > 0. \quad (2.1b)$$

The isotopic lifting of the unit $I \Rightarrow \hat{I}$ then implies corresponding compatible liftings of all the mathematical tools used in contemporary theoretical physics, which can be outlined as follows.

First, conventional fields F (restricted in this paper to the fields of real numbers $\mathcal{R}$ or complex numbers C or quarternions Q) must be generalized into the forms, called *isofields*[8a,10a,11a]

$$F = \{n : n = n1\} \Rightarrow \hat{F} = \{\hat{n} : \hat{n} = n\hat{I}, n \in F\}, \quad (2.2a)$$

$$n_1 + n_2 \Rightarrow \hat{n}_1 + \hat{n}_2 = (n_1 + n_2)\,\hat{I}, \tag{2.2b}$$

$$n_1 n_2 \Rightarrow \hat{n}_1 * \hat{n}_2 = \hat{n}_1\, T\, \hat{n}_2 = (n_1\, n_2)\,\hat{I}, \tag{2.2c}$$

$$\hat{I} = T^{-1} > 0, \tag{2.2d}$$

namely, isofields preserve the conventional additive unit 0 and related sum, but require a generalization of the conventional associative product ab of arbitrary quantities a,b into the *isoassociative product* $a * b = a\,T\,b$, where T is fixed and called the *isotopic element*, Then, under the condition $\hat{I} = T^{-1}$, $\hat{I}$ is the correct left and right unit of the theory

$$\hat{I} * a \equiv a * \hat{I} \equiv a, \tag{2.3}$$

in which case (only) it is called the *isounit*.

The generalization of the basic unit and fields then implies, for evident compatibility, a corresponding isotopy of metric or pseudo-metric spaces $S(x, g, \mathcal{R})$ with local coordinates and metric g over the reals $\mathcal{R}$ into the *isospaces*[8a, 10a, 11a]

$$S(x, g, \mathcal{R}) \Rightarrow \hat{S}(x, \hat{g}, \mathcal{R}), \hat{g} = Tg, \ \mathcal{R} \approx \mathcal{R}\hat{I} = T^{-1}, \tag{2.4}$$

namely, the original, local-differential; metric g and separation $x^2 = x^t g x$ are lifted into the most general possible nonlocal (integral) *isometric* $\hat{g} = T(s, x, \dot{x}, \ldots)\, g$ and *isoseparation* $\hat{x}^2 = x^t \hat{g}\, x$. The structural novelty of the isospaces is that if, jointly with the lifting $g \Rightarrow \hat{g} = Tg$, we have the lifting of the underlying field $\mathcal{R} \Rightarrow \mathcal{R} \approx \mathcal{R}\hat{I}$ with $\hat{I} = T^{-1}$, then the original local topology can be preserved because it is insensitive to the functional dependence of its own unit, once positive-definite.

The close connection between conventional spaces and their isotopic images is made more precise by the following

LEMMA 1 *Under conditions* $\hat{g} = Tg$, $\mathcal{R} = \mathcal{R}\hat{I}$, $\hat{I} = T^{-1} > 0$ *(and sufficient smoothness), all infinitely possible isospaces* $\hat{S}(x, \hat{g}, \hat{I})$ *are locally isomorphic to the original space* $S(x, g, \mathcal{R})$, $\hat{S}(x, \hat{g}, \hat{\mathcal{R}}) \approx S(x, g, \mathcal{R})$.

We should recall that the basis of a vector, metric or pseudo-metric space is unchanged by isotopies (Proposition 3.1, **[10a]**, p. 181). Note also that each given conventional space admits an infinite number of different, although geometrically equivalent isotopic images, evidently because of the infinitely possible different isounits $\hat{I}$.

The most important isospaces are given by the *isoeuclidean spaces*[8,7,10]

$$\hat{E}(r, \hat{\delta}, \mathcal{R}) : \hat{\delta} = T\,\delta,\ \delta = \text{diag.}\,(1, 1, 1)\ \mathcal{R} = \mathcal{R}\hat{I}, \hat{I} = T^{-1},$$

$$\hat{r}^2 = (r^t\, \hat{\delta}\, r)\,\hat{I}, \tag{2.5}$$

the *isominkowskian spaces*[7,10,11]

$$\hat{M}(x, \hat{\eta}, \hat{\mathcal{R}}), \hat{\eta} = T\,\eta,\ \eta = \text{diag.}\,(1, 1, 1, -1) \in M(x, \eta, \mathcal{R}),$$

$$\hat{R} = R\,\hat{I}, \hat{I} = T^{-1}, \hat{x}^2 = (x^t\,\hat{\eta}\,x)\,\hat{I}, \tag{2.6}$$

and the *isoriemannian spaces*[10,11]

$$\hat{R}(x, \hat{g}, \hat{\mathcal{A}}), \hat{g} = Tg, g \in R(x, g\,\hat{\mathcal{A}}), \mathcal{R} = \hat{\mathcal{R}}\hat{I}, \hat{I} = T^{-1}, \hat{x}^2 = (x^t\,\hat{g}\,x)\,\hat{I}, \tag{2.7}$$

where $E(r, \partial, \mathcal{R})$, $M(x, \eta, \mathcal{R})$ and $R(x, g, \hat{\mathcal{A}})$ are the conventional Euclidean, Minkowski and Riemannian spaces, respectively.

As one can see, isospaces (2.5), (2.6) and (2.7) are naturally set for a direct geometrization of the interior physical characteristics, that is, their direct representation via the isometrics. In fact, isospaces can directly represent the inhomogenuity of interior physical media, e.g., via a dependence of the isometric on the locally varying density;

similarly, the anisotropy of the interior problem, e.g., due to an intrinsic angular momentum can be directly represented via a factorization of the spinning direction in the isometric; the integral structure of the interior problem can be directly represented via integral realizations of the isometrics; etc. (see [11] for a detailed study and examples).

The generalization of the basic unit, fields and carrier spaces then implies a corresponding, step-by-step generalization of Lie's theory into the *Lie-isotopic theory* of the original proposal[5a]. Recall that the entire Lie theory is constructed with respect to, and crucially dependent on the conventional unit I. The Lie-isotopic theory is instead an axiom-preserving reformulation of the coventional Lie's theory, this time, constructed with respect to the most general possible nonlinear and integral isounits $\hat{I}$, and consists of the following main branches[5a,6b,10a,11b,15]:

1) The isotopies of the universal enveloping associative algebras ξ with elements A, B, ...(say, vector fields or operators) and trivial associative product AB over F into the following structure called *universal enveloping isoassociative algebras*

$$\hat{\xi} : A * B \stackrel{\text{def}}{=} ATB, \hat{I} * A = A * \hat{I} = A, \forall A \in \xi, \tag{2.8}$$

with underlying isotopy of the Poincaré-Birkhoff-Witt theorem and related infinite-dimensional basis for an (ordered) set of generators X_i, $i = 1, 2, ..., n$ first submitted in [5a]

$$\hat{I}, X_i, X_i * X_j \ (i \leq j), X_i * X_j * X_k, \ (i \leq j \leq k), \dots \tag{2.9}$$

2) The isotopies of Lie algebras $L \approx \xi^-$ and of the celebrated Lie's First, Second and Third Theorem with familiar product $[X_i, X_j]_\xi = X_i X_j - X_j X_i = C^k_{ij} X_k$, where the C's are the familiar structure constants, which are given by the *Lie-isotopic algebras and theorems* first submitted in [5a].

$$\hat{L} \approx \xi^- : [X_i, X_j]_\xi = [\hat{X_i X_j}] \stackrel{\text{def}}{=} X_i * X_j - X_j * X_i$$

$$= X_i T X_j - X_j T X_i = \hat{C}^k_{ij} (s, x, \dots) X_k, \tag{2.10}$$

where, as one can verify, the isotopic product $X_i T X_j - X_j T X_i$ satisfies the Lie algebra axioms, and the $\hat{C}$'s are called the structure functions;

3) The isotopies of the conventional (connected) Lie groups G (also necessary under lifting (2.1) because of the evident lack of mathematical meaning of the conventional $\exp(iwX) = \exp_\xi(iwX)$, $w \in F$). The isotopies are expressible in terms of the exponentiation in ξ (permitted by the existence of infinite basis (2.10) and called *isoexponentiations*), characterize the *Lie-isotopic groups* also submitted for the first time in [5a], and can be written for the one-dimensional case

$$\hat{G}_1 : U(w) = \hat{I} + iwX/1! + (iwX) * (iwX)/2! + (iwX) * (iwX)$$

$$* (iwX)/3! + \dots \stackrel{\text{def}}{=} e_{\hat{\xi}}^{iwX} = \hat{I}(e_\xi^{iwTX}) = (e_\xi^{XTwi}) \hat{I}, \tag{2.11}$$

with ready generalization to the case of more than one dimension (via the isotopic Baker-Campbell-Hausdorff theorem[5a]). The Lie- isotopic groups evidently violate the conventional group laws, but verify instead the *isotopic group laws*[5a]

$$U(w) * U(w') = U(w') * U(w) = U(w + w'), \ U(0) = \hat{I},$$

$$U(w) * U(-w) = \hat{I}, \tag{2.12}$$

where the product $U(w) * U(w')$ is evidently isoassociative. This permits the preservation of the Lie group axioms at the abstract, realization-free level where the conventional associative product AB and its isotopic image $A * B$ coincide.

4) The isotopy of the conventional linear transformations $X' = AX$

(requested because they would violate linearity, transitivity and other basic properties if defined on an isospace) which are given by the so-called *isotransformations*

$$x' = A * x = ATx,\ T = \text{fixed},\ \hat{I} = T^{-1}, \qquad (2.13)$$

Note that transformations (2.13) are *isolinear and isolocal* in the considered isospace (in the sense that they verify the isotopic axioms of linearity and locality), but when projected in the original space they are generally nonlinear and nonlocal

$$x' = A * x = A\,T(s, x, \dot{x}, \ldots)\,x = B(x, \ldots)\,x, \qquad (2.14)$$

owing to the arbitrary functional dependence of T. This ensures that the isotopies of conventional formulations yield nonlinear and nonlocal theories, as desired;

5) The isotopy of the conventional representations, called *isorepresentations*, which essentially permit the construction of an infinite class of nonlinear and nonlocal representations of given groups; etc.

The isotopies of the basic unit, fields, carrier spaces and Lie's theory evidently demand the construction of corresponding compatible isotopies of the conventional symplectic, affine and Riemannian geometries. This study has been conducted in **[10, 11a]** and it cannot be reviewed here for brevity.

Finally, all the above isotopies have been predictably constructed for their physically most important applications, the isotopies of classical and quantum mechanics, and interconnecting map, called *isoquantization*. For the isotopies of classical mechanics we refer the reader to the recent volumes[11]. In order to render this minimally understandable and self-sufficient, it is important to outline the isotopies of quantum mechanics, also known under the name of *hadronic mechanics*, which were originally proposed in **[5b]** (see **[12-14]** for recent presentations).

Hadronic mechanics is essentially an operator realization of the

Lie- isotopic theory on a suitable Hilbert space characterizable via the following basic structures:

I) The ***universal enveloping isoassociative operator algebra*** $\hat{\xi}$ of equations (2.8) and (2.9);

II) The *isofields* of real $\mathcal{A}$ and complex numbers $\hat{C}$ of equations (2.2); and

III) The *isohilbert spaces* $\hat{\mathcal{H}}$ with conventional states $|\psi>, |\phi>, \ldots$ and *isoinner product* or $\hat{C}$

$$\hat{\mathcal{H}}: \langle \phi | \psi \rangle = \langle \phi | T | \psi > \hat{I} \in \hat{C}. \qquad (2.15)$$

A step-by-step isotopic generalization of the entire formulation of quantum mechanics then follows. Here, we merely recall for subsequent need that the conventional notion of Hermiticity (observability) remains invariant under the isotopies characterized by structures I, II and III above, while the notion of unitarity is lifted into the *isounitarity*

$$U * \hat{U}^{\dagger} = \hat{U}^{\dagger} * U = \hat{I}. \qquad (2.16)$$

Similarly, the conventional expectation values are lifted into the *isoexpectation values*

$$\langle A \rangle = \langle \psi | T A T | \psi \rangle \hat{I} \in \hat{C}; \qquad (2.17)$$

the conventional expression $H|\psi\rangle = E^0|\psi\rangle$ is lifted into the *isoeigenvalue equation*.

$$H * |\psi\rangle = HT|\psi\rangle = \hat{E} * |\psi. = E\hat{I}T|\psi\rangle = E|\psi\rangle; \qquad (2.18)$$

while the basic dynamical equations of hadronic mechanics are given by the *isoschrödinger's equation* (for $h = 1$ hereinafter assumed)

$$i\frac{\partial}{\partial t}|\psi\rangle = H * |\psi\rangle, \qquad (2.19a)$$

$$p_\mu * | \psi \rangle + p_\mu T | \psi \rangle = - i \hat{I}_\mu^\nu \frac{\partial}{\partial x^\nu} | \psi \rangle, \tag{2.19b}$$

(where H and p are certain effective Hamiltonian and linear momentum operators, respectively), with corresponding, isounitarily equivalent *isoheisenberg's equations* in their infinitesimal form (originally proposed in **[5b]** as an operator realization of the Lie-isotopic theory of the preceding **[5a]**)

$$i \dot{A} = [A, H]_\xi = A * H - H * A = A\, T H - H T A = A\, T H - H T A$$

$$= A\, \dot{T}(s, x, \dot{x}, \ddot{x}\, \psi\, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \ldots)\, H$$

$$- H\, \dot{T}(s, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \ldots)\, A, \tag{2.20}$$

and finite form characterized by the following Lie-isotopic group of isounitary transformations

$$A(t) = e^{-iHTt} A(0)\, e^{-tTH} = \hat{I} e_\xi^{iHt} * A(0) * e_\xi^{iH} \hat{I}. \tag{2.21}$$

The interested reader is referred to reviews[12-14] for remaining aspects of hadronic mechanics, including the isotopies of conventional operations, quantities and physical laws on a Hilbert space (determinant, trace, Dirac's delta distribution, superposition, casuality, etc.), as well as a number of applications to the Bose-Einstein correlation, exact confinement of quarks, etc.

The fundamental concept the reader should keep in mind is that hadronic and quantum mechanics coincide, by construction, at the abstract, realization-free level. The mathematical consistency of hadronic mechanics is therefore established at this writing. Only its physical effectiveness is under scrutiny[14].

3. Isodual Minkowski Spaces and Related Formulations

The *isodual theory* is essentially given by the reformulation of the isotopic theory of the preceding section for the *isodual unit*

$$\hat{I} = - I = \text{diag.}(-1, -1, -1, \ldots, -1) \stackrel{\text{def}}{=} I^d < 0, \tag{3.1}$$

and the realization of the isotopic element T via the *isodual element*

$$\dot{T} = (-I^d)^{-1} = \text{diag.}(-1, -1, -1, \ldots, -1) = T^d < 0, \tag{3.2}$$

which, in the particular case considered, trivially coincide, $I^d \equiv T^d$

Thus, isoduality is not an isotopy because of the lack of preservation of the original axioms. The isodual and isotopic theories therefore admit nontrivial differences illustrated below.

First, liftings (3.1) and (3.2) imply the following *isodual field of real numbers*

$$\mathcal{R}^d = \{ n^d : n^d = n\, I^d = -n, \in \mathcal{R} \}\ \mathcal{R}^d \approx \mathcal{R} I^d \approx - \mathcal{R}, \tag{3.3}$$

which are the image of the original fields $\mathcal{R}$ under reversal of the sign of all numbers. Thus, the identity of two numbers belonging to $\mathcal{R}$ and $\mathcal{R}^d$ is characterized by the following property

$$n\,|\,\mathcal{R} \equiv - n\,|\,\mathcal{R}^d, \tag{3.4}$$

which the reader should keep in mind to understand the realization of antiparticles in isodual spaces.

Note also the sum of isodual numbers is the conventional *operation*, but its value is the opposite of the conventional one,

$$n_1^d + n_2^d = (n_1 + n_2)\, I^d = - (n_1 + n_2)\, I, \tag{3.5}$$

while for the multiplication we have

$$n_1^d * n_2^d = (n_1 I^d)\, T^d\, (n_2 I^d) = (n_1 n_2)\, I^d = - (n_1 n_2)\, I. \tag{3.6}$$

Moreover, the modulus of a number in $\mathcal{R}$ is given by $|n|_{\mathcal{R}} = |nn|^{1/2}\ I = + |nn|^{1/2}$, while the corresponding *isomodulus* in $\mathcal{R}^d$ is given by

$$|n|^d_{\mathcal{R}} = |n * n|^{1/2d} I^d = -|n\, n^d|^{1/2} = -|nn|^{1/2} = -|n|_{\mathcal{R}}. \qquad (3.7)$$

In the following, whenever no confusion arises, we shall ignore the multiplication by the trivial quantity I.

Next, we introduce the notion of *isodual complex field* which can be defined by

$$C^d = \{c^d \mid c^d = \bar{c}\, I^d, c \in C\}\ , C^d \approx \bar{C}\, I^d \approx -\bar{C}, \qquad (3.8)$$

namely, via the multiplication by the isodual unit I^d, plus the operation of complex conjugation. A complex number $a + ib$ in C is therefore identical to the complex conjugate number $-a + i\,b$ in C^d, i.e.,

$$(a + i\,b)|_C \equiv (-a + i\,b)\ C^d. \qquad (3.9)$$

Structure (3.8) can be proved in a number of ways. First, recall that complex numbers have the structure $C \approx \mathcal{R} + i\,\mathcal{R}$. Thus, their isoduals are given by

$$C^d \approx \mathcal{R}^d + i^d x^d \mathcal{R}^d = (\mathcal{R} + i^d x^d \mathcal{R})\, I^d = (-\mathcal{R}) + (i)\,(-1)\ (-\mathcal{R}) = -\mathcal{R} + i\,\mathcal{R} \qquad (3.10)$$

namely, by the isoduality of both the real unit 1 and the product i. This begins to illustrate the lack of equivalence between the isotopic and isodual theories. Note that $i^d = -\bar{i} = i$.

Alternatively, and equivalently, definition (3.8) is needed to reach a consistent (that is, negative) *isomodulus of a complex number* $c = a + ib$

$$|c^d|_{C^d} = (c\, c^d)^{1/2d} = -|a^2 + b^2|^{1/2} = -|c|_C. \qquad (3.11)$$

The reader can then easily construct all remaining operations of C^d. Equally instructive (and intriguing) is the construction of the *isodual quaternion field* Q^d, here left to the interested reader for brevity.

As recalled in Section 2, the isodualities of the basic unit and fields require, for consistency, a necessary corresponding isoduality of the conventional (Euclidean and) Minkowski space. We therefore have the following

DEFINITION 1 [8, 7, 11] *Let* $M(X, \eta, \mathcal{R})$ *be the conventional* (3 + 1)*-dimensional Minkowski space with local coordinates* $x = (x^\mu)$, $\mu = 1, 2, 3, 4$ *metric* $\eta = diag.\,(1, 1, 1, -1)$, *unit* $I = \text{diag.}\,(1, 1, 1, 1)$ *and invariant separation over the reals* $\mathcal{R}$ *with unit* I.

$$M(x, \eta, \mathcal{R}) : x^2 = (x^t\, \eta\, x)\ 1 = x^\mu\, \eta_{\mu\nu}\, x^\nu = x^1 x^1 + x^2 x^2 + x^3 x^3$$
$$- x^4 x^4 \in \mathcal{R}, x^4 = ct, \qquad (3.13)$$

Then, the "isodual Minkowski space" $M^d(x, \eta^d, \mathcal{R}^d)$ *is the image of* $M(x, \eta, \mathcal{R})$ *characterized by the isodual unit* $\hat{I} = I^d = (T^d)^{-1} = T^d = -I = -$ diag. $(1, 1, 1, 1)$, *over the isodual field with isodual unit* $1^d = -1$

$$M^d(x, \eta^d, \mathcal{R}^d) : \eta^d = T^d\, \eta = -\eta, \mathcal{R}^d \approx \mathcal{R}\, 1^d = -\mathcal{R} \qquad (3.14)$$

with "isodual invariant" and "isodual distance" given respectively by

$$x^{2d} = (x^t\, \eta^d\, x)\ 1^d = (x^\mu\, \eta^d_{\mu\nu}\, x^\nu)\ 1^d = = (-x^1 x^1 - x^2 x^2 - x^3 x^3 + x^4 x^4)\ 1^d \equiv x^2 \qquad (3.15a)$$

$$|x - y|_{M^d(x, \eta^d, \mathcal{R}^d)} = -|x - y|_{M(x, \eta, \mathcal{R})} \qquad (3.15b)$$

Lemma 1 of Section 2 shows that, for positive-definite isounits $\hat{I}$, isospaces are isomorphic to the original spaces. Then, by defining an *anti-isomorphism* as one to-minus-one and onto, it is easy to prove the following

PROPOSITION 1. *The conventional Minkowski space* $M(x, \eta, \mathcal{R})$ *and its isodual* $M^d(x, \eta^d, \mathcal{R}^d)$ *are anti-isomorphic.*

It is important to understand that *isoduality is a new operation independent from all conventional operations, such as space and time inversions, trivially, because all conventional operations cannot alter their own basic unit.*

To see this important point, consider the *isoinversions*[8,10,11]

$$x' = \hat{P} * x = Px = (-x, x^4), x' = \hat{T} * x = Tx = (x, -x^4), \qquad (3.16)$$

where $\hat{P} = P\hat{I}$, $\hat{T} = T\hat{I}$, and P and T are the conversional space and time inversions, respectively. Then, it is easy to see that the lifting $M(x, \eta, \mathcal{R}) \Rightarrow M^d(x, \eta^d, \mathcal{R}^d)$ cannot be reached via the inversions, trivially, because x^2 is invariant under P and T, as well known, and the same occurs for all other operations.

Another important point to understand is the difference between the space $M(x, \eta^d, \mathcal{R})$ characterized by the sole change of the sign of the metric $\eta \Rightarrow \eta^d = -\eta$ (which is studied in details in **[19]** and $M^d(x, \eta^d \mathcal{R}^d)$. The former is a *conventional* space, i.e., a space based on the conventional unit I, while the latter is an *isodual* space, i.e. space whose definition necessarily requires a generalized unit.

Also, note that the mapping $M(X, \eta, \mathcal{R}) \Rightarrow M(X, \eta^d, \mathcal{R})$ is not an isomorphism (or an antimorphis) because, for a fixed coordinate x, the mapping changes space-like into time-like vectors and viceversa, evidently because of the inversion of the sign of the separation while preserving the underlying field, i.e.

$$M(x, \eta, \mathcal{R}) : x^2 \in \mathcal{R} \Rightarrow Mx, \eta^d, \mathcal{R}) : x^{2d} = -x^2 \in \mathcal{R} \qquad (3.17)$$

On the contrary, the isodual, mapping $M(x, \eta, \mathcal{R}) \Rightarrow M^d(x, \eta^d, \mathcal{R}^d)$ preserves the original space-like or time-like character because, jointly with mapping (3.17), we have a corresponding mapping of the field,

$$M(x, \eta, \mathcal{R}) : x^2 \in \mathcal{R} \Rightarrow M^d(x, \eta^d \mathcal{R}^d) : x^{2d} = -x^2 \otimes \mathcal{R}^d = -\mathcal{R} \qquad (3.18)$$

This occurrence is an indication of how misleading can be a direct

interpretation of a given sign, unless treated in a mathematically correct way. We have recalled earlier that a numerical value, say, $x = 3$, has no mathematical meaning unless referred to a specifically selected unit and related field. If the conventional field $\mathcal{R}$ is selected, then (and only then) the expression $x = 3$ has the conventional meaning. However, if the isodual field $\mathcal{R}^d$ is selected, then the mathematically correct expression is $3^d = 3I^d = -3$.

Moreover, the isoproduct of any isodual number n^d with a given quantity Q coincides with the conventional product.

$$n^d * Q = n I^d T^d Q \equiv nQ. \qquad (3.19)$$

Thus, when dealing with isodual fields, for practical purposes one may consider ordinary products such as xQ, where x is, say a local coordinate. However the correct expression in $M^d(x, \eta^d, \mathcal{R}^d)$ is $x^d Q$, in which case the correct coordinate is $x^d = x I^d = -x$.

In short, all numbers of the conventional Minkowski space, including space and time coordinates, change sign under isoduality.

Next, the *isodual Lorentz symmetry* O^d (3.1) of $\hat{M}^d(x, \eta^d, \hat{\mathcal{R}}^d)$ is a simple particular case of the *isolorentz symmetries* $\hat{O}$ (3.1) for $\hat{I} = I^d = -I$. Both the isolorentz and isodual lorentz symmetries have been originally proposed in [7] and then studied in detail in **[11b]**, jointly with the *isodual isospecial relativity*, that is, the image of the special relativity under isoduality.

Here, we limit ourselves to the review that, for an arbitrary isotopic element $T = \text{diag.}(b_1^2, b_2^2, b_3^2, b_4^2)$, where the b's are arbitrary, positive-definite functions of all needed local variables, the *isolorentz transformations* can be written

$$x'^1 = x^1, x'^2 = x^2, x'^3 = \hat{\gamma}(x^3 - \beta x^4), x'^4 = \hat{\gamma}(x^4 - \hat{\beta} x^3), \qquad (3.20)$$

where

$$\beta = v/c_0, \hat{\beta} = \frac{vb_3}{cb_4}, \qquad \hat{\gamma}^2 = 1 - \frac{v^k b_k^2 v^k}{cb_4^2 c}, \tag{3.21}$$

and they leave invariant the general separation on isominkowski spaces

$$\hat{M}(x, \hat{\eta}, \hat{R}) : x^{\hat{2}} = x^i b_1^2 x^i + x^2 b_2^2 + x^3 b_3^2 x^3 - x^4 b_4^2 x^4. \tag{3.22}$$

Now, the conventional Lorentz transformations are given by $b_\mu = 1$, while the isodual transformations occur for $b_\mu^d = i, \mu = 1, 2, 3, 4$. Moreover, the isodual image of isoexponentiation (2.11) implies that the parameters are now defined in the isodual field, i.e., we have the parameters $w^d = -w$. Moreover in the exponent of isoexponentiation (2.11) we have the isoduality $iwx \to iw^d x^d x \equiv iwx$, where $\times^d$ is the multiplication in $\mathcal{R}^d$, yielding the final result $I^d \{\exp(iwT^d X\} I^d = -\{\exp(-iXw\}$.

Thus, the isodual Lorentz and Poincaré transformations are given by an overall change of sign joint with the change of the sign of the parameters

$$O^d(3.1) : \Lambda^d(w) = -\Lambda(-w) \tag{3.23a}$$

$$T^d(3.1) : x'^d = x^d + x^{0d} = -(x - x^0) \tag{3.23b}$$

We have the following

PROPOSITION 2 *The isodual Poincaré symmetry $P^d(3.1)$ is anti-isomorphic to the conventional symmetry $P(3.1)$.*

We also have the following

COROLLARY 2A *The separation in Minkowski space, equatoin (3.15a) and the Casimir invariants remain unchanged under isoduality, i.e.*

$$x_M^{2d} \equiv x_M^2. \tag{3.24a}$$

$$(p^2 + m^2)_{P(3.1)} \equiv (p^2 + m^2)_{P^d(3.1)}. \tag{3.24b}$$

The proper understanding and use of the above properties requires care in the interpretation of the results and the knowledge of identity (3.4) for the isodual field $\mathcal{R}^d$ underlying $P^d(3.1)$.

The above results can then be lifted to full isotopic conditions with nontrivial isounits and $\hat{I}^d$, yielding two additional structures, the isopoincaré symmetry $\hat{P}(3.1) \approx P(3.1)$ and its isodual $\hat{P}^d(3.1) \approx P^d(3.1)$ holding for interior physical conditions, for which two corresponding generalizations of separation, Casimir invariant and transformations hold, also interconnected by isoduality.

Similar results hold in particular for the *isodual isorotational symmetry* $O^d(3) \in O^d(3.1)$ studied in **[8b, 11b]**.

The properties identified in this section permit the achievement of the most important result of the isodual theories, which can be expressed as follows

Universal Invariance Law Under Isoduality **[11B]**: *All physical laws in Minkowski space which are invariant under the Poincaré symmetry $P(3.1)$ are also invariant under the isodual symmetry $P^d(3.1)$.*

To clarify this new principle, recall that the Casimir invariants of the Poincaré symmetry remain unchanged under isoduality (and this will soon result to be the case for all laws of the special relativity). But then each law can be considered as jointly holding for both spaces, our physical Minkowski space. and its isodual.

Recall that the magnitude (norm) of all quantities in isodual fields is negative definite. Then, the most salient features of the isodual symmetries is that they characterize particles with negative-definite energy

$$|E^d|^d < 0, \tag{3.25}$$

as well as evolving backward in time, because

$$| t^d |^d < 0. \tag{3.26}$$

As we shall see in the next sections, the above principle appears to permit a novel interpretation of antimatter, as well as stimulate rather intriguing epistemological possibilities.

Finally, the reader should be aware that the introductory lines of this section represent only part of the isodual theory, that of Lie-isotopic type for the exterior dynamical problem. Another part is given by the isodual theory of isominkowski spaces for interior dynamical problems. Additional parts are given by the still more general, exterior and interior, isodual theories of Lie-admissible type (see **[5a,b]** or the Appendixes of **[11a]**).

4. Antiparticles in Isodual Spaces

As indicated in Section 1, it is currently believed that antiparticles such as the positron e^+ or the antiproton p^- exist in the same Minkowski space $M(X, \eta, \mathcal{R})$ of the ordinary particles e^- and p^+.

The purpose of this paper is to study the following hypothesis rudimentary submitted in [9b]:

HYPOTHESIS *Consider particles existing in our Physical Minkowski space $M(X, \eta, \mathcal{R})$. Then their antiparticles exist in the isodual Minkowski space $M^d(X, \eta^d, \mathcal{R}^d)$ and are characterized by the isodual mapping*

$$I = \text{diag.}(1, 1, 1, 1) \Rightarrow I^d = -I = \text{diag.}(-1, -1, -1, -1), \tag{4.1}$$

that is, by the isotopies of conventional fields, pseudo-metric spaces, Lie's theory, symplectic, affine and Riemannian geometries, and quantum mechanics characterized by the isodual unit I^d, with the understanding that there exist particles, such as the neutral pion, which are "self-isodual", that is, common to both $M(x, \eta, \mathcal{R})$ and $M^d(x, \eta^d, \mathcal{R}^d)$.

The purpose of this paper is to study this hypothesis in more details. For this, we need the operator theory on $M^d(x, \eta^d \mathcal{R}^d)$ here introduced under the name of *isodual relativistic quantum mechanics* which is a realization of hadronic mechanics of Section 2 for the particular isounit $\hat{I} = I^d$ and, as such, it is characterized by the following main structures:

I) The *enveloping isodual associative operator algebra* ξ^d of with elements $A, B, \ldots$, and isodual product

$$\xi^d. A * B = A\, T^d B = -AB,\ T^d = (I^d)^{-1} = \text{diag.}(-1, -1, -1, -1), \tag{4.2a}$$

$$I^d * A = A * I^d = A,\ \forall A \in \xi^d, \tag{4.2b}$$

II) The *isodual fields* $\mathcal{R}^d \approx \mathcal{R} I^d$ and $C^d \approx \bar{c}\, I^d$, and

III) The *isodual Hilbert Spaces* $\mathcal{H}^d$ with states $| \Psi \rangle^\delta = -\langle \overline{\Psi} |$ and composition

$$\mathcal{H}^d : \langle \phi | \psi \rangle^d = \langle \bar{\phi} | T^d | \bar{\psi} \rangle I^d \in C^d. \tag{4.3}$$

The above structures imply a simple but rather subtle change of the entire formulation of relativistic mechanics. First, the notion of Hermiticity remains unchanged, i.e. $H^{\dagger d} \equiv H^\dagger$. Nevertheless, the right modular action of the Hermitean operator H on a state $| \psi \rangle$ is chanced into the form

$$H | \psi \rangle \Rightarrow -\langle \bar{\psi} | * \dot{H} = -\langle \bar{\psi} | * E^d = -\langle \overline{\Psi} | E \tag{4.4}$$

which confirms the inversion of the sign of ordinary numbers under isoduality plus conjugation, i.e. if $\langle x | \psi \rangle = \psi(x)$ is a wavepacket in $\mathcal{H}$, its image in $\mathcal{H}^d$ is given by $-(\psi(x))^\dagger$.

The notion of *isodual unitarity* is also equivalent to the conventional one owing to the properties

$$U * U^{\dagger d} = U^{\dagger d} * U = I^d \equiv -UU^\dagger = -U^\dagger U = -1. \tag{4.5}$$

The notion of *isodual expectation value* and *isodual probabilities* however change sign, being elements of C^d. In fact, we have

$$\langle H \rangle^d = \langle \overline{\psi} | * H * | \overline{\psi} \rangle I^d = -\langle \overline{\psi} | H | \overline{\psi} \rangle \in C^d. \tag{4.6}$$

In particular, it is possible to prove that *the isodual eigenvalues and expectation values of a Hermitean operator are isodual reals.*

Next, as expected, the fundamental dynamical equations are invariant under isoduality, that is, they are given by the adjoint of the conventional ones. In fact, the *isodual Schrödinger's equation* is given by

$$-i^d x^d \langle \overline{\psi} | \frac{\partial}{\partial t} = \langle \overline{\psi} | * H = +i \langle \overline{\psi} | \frac{\partial}{\partial t} = -\langle \overline{\psi} | H, \tag{4.7}$$

while the *isodual Heisenberg's equation* is given by

$$\left(i \frac{dA}{dt} \right)^d = (-i) \frac{dA}{dt} = [A, H]_{\xi^d} = A * H = H * A = -(AH - HA). \tag{4.8}$$

We leave for brevity as an instructive exercise for the interested reader the construction of the explicit isoduality of all other operations of, such as trace, determinant, projection operator, etc.

Based on the above isodual relativistic mechanics, one can readily see the following mapping of plane waves under isoduality

$$\psi(x) - N \exp(i k^\mu \eta_{\mu\nu} x^\nu) \Rightarrow \psi^d(x) = -N \exp(i^d k^\mu \eta^d_{\mu\nu} x^\nu) = -\psi(x). \tag{4.9}$$

Now, for the free particle in $M(x, \eta, \mathcal{R})$ we have from Corollary 2B (for $c = 1$ and by ignoring 1)

$$p^2 + m^2 = p_\mu \eta^{\mu\nu} p_\nu + m^2 \Rightarrow p_\mu \eta^{d\mu\nu} p_\nu + m^{2d} = -(p^2 + m^2). \tag{4.10}$$

Similarly, in view of rule (3.4), a charge e in $M(x, \mathcal{R})$ is seen in $M^d(x, \eta^d, \mathcal{R}^d)$ as $-e$. As a result,

$$eA_\mu(x) = e^d x^d A^d_\mu(x) = -eA_\mu \tag{4.11}$$

and the following property holds.

PROPOSITION 3 *A particle of charge e under an external electromagnetic field with potentials $A_\mu(x)$ is invariant under isoduality, i.e.,*

$$[p_\mu - eA_\mu(x)] \eta^{\mu\nu} [p_\nu - eA_\nu(x)] + m^2\}_{M(x,\eta,\mathcal{R})}$$

$$\equiv \{[p^d_\mu - e^d x^d A^d_\mu(x)] x^d \eta^{d\mu\nu} x^d [p^d_\nu - e^d x^d A^d_\nu(x)] + m^{2d}\}_{M^d(x,\eta^d,\mathcal{R}^d)}, \tag{4.12}$$

The invariance of the corresponding Klein-Gordon equation is consequential and its proof is left to the interested reader. The proof of the isodual invariance of the Lagrangian and Hamiltonian densities is then straightforward (with the understanding that $L^d = -L$ and $H^d = -H$). ...

The behavior of Dirac's equations under isoduality is also intringuing. First, recall that the sign of the γ-matrices is changed under charge conjugation (see, e.g.[20], p. 176). On the contrary, it is easy to prove that under isoduality we have $\gamma^d_\mu = -\gamma_\mu$, because the decomposition of the isodual Casimir $-(p^2 + m^2)$ leads to the properties

$$-(p^2 + m^2) = (\gamma^{d\mu} p_\mu + im) * (\gamma^{d\nu} p_\nu - im), \tag{4.13a}$$

$$\{\gamma^d, \gamma^d_\nu\}_{\xi^d} = -\{\gamma^d, \gamma^d_\nu\}_\xi = \{\gamma_\mu, \gamma_\nu\}_\xi = 2\eta^d_{\mu\nu} = -2\eta_{\mu\nu}. \tag{4.13b}$$

Next, the linear momentum operator remains unchanged under charge conjugation. On the contrary, by recalling (2.19b), the behavior of the linear momentum under isoduality is given by

$$-\langle \overline{\psi} | * p^d_\mu = i^d x^d \langle \overline{\psi} | \frac{\partial}{\partial x^\nu} I^{d\nu}_\mu = -i \langle \overline{\psi} | \frac{\partial}{\partial x^\mu}, \tag{4.14}$$

while it is easy to see that the helicity is iso-self-dual.

We therefore have the following

PROPOSITION 4 *The conventional Dirac equation for a charged particle under an external electromagnetic field on Minkowski space $M(x, \eta, \mathcal{R})$ is mapped under isoduality into the adjoint equation for the antiparticle in isodual Minkowski space $M^d(x, \eta^d \mathcal{R}^d)$. i.e.,*

$$\{\gamma_\mu \eta^{\mu\nu}[p_\nu - e A_\nu(x)] + im\} \mid \psi\rangle = 0 \Rightarrow \langle \overline{\psi} \mid * \{\gamma^d_\mu \eta^{d\mu\nu}[p^d_\nu - e^d A^d_\nu(x)]$$

$$- i^d x^d m^d\} = - \langle \overline{\psi} \mid \{\gamma_\mu \eta^{\mu\nu}[p_\nu + eA_\nu(x)] - i m\} = 0. \quad (4.15)$$

while the four-current remains unchanged, i.e.,

$$J_\mu = i \langle \overline{\psi} \mid \gamma_\mu \mid \psi\rangle \Rightarrow J^d = i x^d x^d \langle \overline{\psi} \mid \gamma^d_\mu \mid \psi\rangle \equiv J_\mu. \quad (4.16)$$

We have reached in this way the following

PROPOSITION 5 *The conventional charge conjugation for particles on Minkowski space $M(x, \eta, \mathcal{R})$ is equivalent to the "isodual conjugation", i.e., the mapping of the equations of motion from $M(x, \eta, \mathcal{R})$ to its isodual image $\mathcal{M}^d(\eta^d, \mathcal{R}^d)$*

Next, it is important to clarify that the conventional solutions of relativistic equations, including both the Klein-Gordon and Dirac's equations admit solutions in both spaces $M(x, \eta, \mathcal{R})$ and $M^d(x, \eta^d, \mathcal{R}^d)$. This property should be expected from the fact that the fundamental Casimir invariant $p^2 + m^2$ is common to both spaces.

Consider in this respect the explicit form of the plane-wave solution of the Klein-"Gordon equation. As well known, (see e.g.[20], p. 29), both positive and negative energies (or frequencies) are possible according to the expression

$$\psi(x) = \exp(ik^\mu \eta_{\mu\nu} x^\nu) = \exp(ik \times x - i\,\omega\, t),\ \omega = \pm (k^2 + m^2)^{1/2} \quad (4.17)$$

But, by recalling rule (3.4), any positive energy (or frequency) on $M(x, \eta, \mathcal{R})$ coincides with the corresponding negative value in $M^d(x, \eta^d, \mathcal{R}^d)$. In fact, the isodual conjugation

$$\psi(x)_{M(x,\eta,\mathcal{R})} = \exp(i\,k \times x - i\,\omega \times t) \Rightarrow \overline{\psi}(x)_{M^d(x,\eta^d,\mathcal{R}^d)}$$

$$= \exp(-i^d k^d \times^d x + i^d \omega^d \times^d t) = \exp(-i\,k\,x + i\,\omega\,t), \quad (4.18)$$

where we have kept the space-time coordinates unchanged because they are the same as for the positive-energy solutions.

We leave it to the interested reader for brevity to verify that a fully similar situation occurs for Dirac's equation, which admits solutions in both $M(x, \eta, \mathcal{R})$ and $M^d(x, \eta^d, \mathcal{R}^d)$.

The interested reader may also easily prove the existence of particles which are iso-self-duals, such as the photons γ and the π^0 meson.

Finally, note that the problematic aspects for the Lorentz invaraince of the solution with positive energy in $M(x, \eta, \mathcal{R})$, $\exp i(K \times k + |\omega| t)$ are resolved in isodual spaces via the O^d (3.1) invariance of the expressions $\exp i(-k \times k + |\omega| t)$ on $M^d(x, \eta^d, \mathcal{R}^d)$.

5. Concluding Remarks

The analysis of this paper confirms the possibility of representing antiparticles and antimatter in general as belonging to a space different than our own physical Minkowski space $M(x, \eta, \mathcal{R})$, called isodual Minkowski space $M(x, \eta^d, \mathcal{R}^d)$, which is characterized by a negative unit $I^d = -I$ called isodual unit, with $\eta^d = \eta I^d = -\eta$, $\mathcal{R}^d \approx \mathcal{R} I^d \approx -\mathcal{R}$.

In such isodual space all quantities are conjugate to the corresponding quantities in our own space, in the sense that: the numerical unit 1 becomes -1; the imaginary unit i becomes $-i$; all positive numbers become negative and viceversa; right is turned into left; up is turned in down; forward motion in time is turned in backward motion in time; conventionally positive distances and magnitudes are turned into negative values: conventional Hilbert states

$|\psi\rangle$ are turned into their antiadjoint $-\langle\bar{\psi}|$; etc.

The above structure provides a fundamentally new resolution of the historical problem of the negative-energy solutions of conventional relativistic equations, such as the Klein-Gordon and Dirac equations, as merely being fully acceptable solutions on the isodual space $M^d(X, \eta^d, \mathcal{R}^d)$.

Recall that the negative-energy solutions caused great difficulties for several decades and, even as of today, the various solutions existing in the literature[1-4] do not appear to be immune from criticisms or uneasiness because of the underlying assumption of the existence of infinite seas of states.

The roots of the problem can be identified in the assumption of only one space-time for both positive and negative energy solutions, in which case the latter have unphysical behavior, such as a particle accelerates in a direction opposite to the applied force, etc. Moreover, the problem had to be considered seriously because the two solutions are not disjoint, but admit a finite transition probability from one to the other.

The assumption of two separate, but complementary space-times interconnected by isoduality resolves the problem of the negative energy solutions without any need for the assumption of infinite seas of states and related "hole theory", and prior to any second quantization, as desired. In fact, negative energy solutions behave unphysical if one insists in assuming that they have the same identity I of the conventional solution, but they behave in a fully physical way if one assumes that their unit is the isodual one I^d.

At any rate, positive and negative energy solutions emerge in a rather natural way as being fully symmetrical and they are found in the solutions for both particles and antiparticles. As such, they can be naturally associated to different spaces interconnected by isoduality, thus resulting in the elimination of the need for infinite seas of states of either type.

As a historical note, Dirac[21] came rather close to the notion of

isodual space, because he encouraged the admission of negative energies and probabilities, by indicating that they are as real "negative sums of money". However, to achieve a full physical acceptability, i.e., a full equivalence with our physical world he lacked at the time of his paper[21] the isotopies of conventional mathematical structures.

Besides these new perspectives in the problem of the negative-energy, negative-probability solutions, the primary results of our analysis are the following:

1) The spaces $M(X, \eta^d, \mathcal{R}^d)$ are characterized by the isodual Poincaré symmetry P^d (3.1) and the isodual special relativity in a way fully similar, although different, than the corresponding characterizations in $M(x, \eta, \mathcal{R})$;

2) All physical laws that are invariant under the conventional Poincaré symmetry P (3.1) on $M(x, \eta, \mathcal{R})$ are jointly invariant under the isodual symmetry P^d (3.1) on $M^d(X, \eta^d, \mathcal{R}^d)$;

3) Particles existing in our space $M(X, \eta, \mathcal{R})$ become antiparticles when interpreted as existing in $M(X, \eta^d, \mathcal{R}^d)$.

The above results have rather intriguing epistemological (as well as theoretical and mathematical) implications, such as:

a) the isodual space-time coexists with our own, because every space-time point $X = (r, ct)$ can be interpreted as either belonging to our physical Minkowski space or to its isodual;

b) Particles and matter at large can be thought as being unique, in the sense that their differentiation into matter or antimatter depends on the assumed space.

c) The two space-times are not disjoint, but appear to be in direct communication because conventional relativistic equations, such as Dirac's equation, have a finite transition probability from states to isostates and viceversa.

These results may permit the elimination of the very notion of antiparticle. In fact, one may assumes that each space has the same particle. In fact, as elaborated in the preceding section, a given physical electron in Minkowski space becomes a positron in the isodual space, with the net result that each space has only one particle, one being the isodual image of the other.

But perhaps the most intriguing possibilities occur when the results of this paper are extended to the Riemannian spaces and their isoduals. In this case, as we shall see in a subsequent paper, we have the apparent emergence of an entire new universe of isodual type hitherto unknown, with rather intriguing possibilities regarding basic advances in gravitation.

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