SANTILLI'S LIE-ISOTOPIC GENERALIZATION OF GALILEI'S AND EINSTEIN'S RELATIVITIES

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DEDICATION

This monograph is dedicated to

Mr. MICHAIL S. GORBACHEV
President of the U.S.S.R.

because of his vision, courage and historical contributions to mankind

June 1, 1990

This monograph presents an enlarged version of the lectures delivered by Prof. **Ruggero Maria Santilli** at the INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS of Trieste, Italy, in the first part of December 1990, following notes taken at the lectures by one of the Authors, Prof. **A. Jannussis**, and subsequently enlarged thanks to the assistance of all the other Authors, as well as to the editorial assistance of the staff of THE INSTITUTE FOR BASIC RESEARCH of Palm Harbor, Florida, U.S.A.

An invitation by Prof. **Abdus Salam**, Director of the Centre, to Prof.s Santilli and Jannussis must be here acknowledged with sincere gratitude, because it permitted the organization of the original material presented at the lectures and stimulated its subsequent enlargement. Penetrating comments by Prof. Salam at the lectures resulted to be invaluable for the achievement of sufficient maturity of presentation, and for stimulating subsequent research.

*   *   *

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PREFACE

Throughout this century, Lie's theory has been developed in both mathematical and physical literatures with respect to the simplest conceivable unit, say $I = \text{Diag.}(1, 1, ..., 1)$, and the simplest conceivable product $AB - BA$, where AB is the trivial associative product. In a pioneering memoir written at Harvard University in 1978, Ruggero Maria Santilli identified, apparently for the first time, a generalized formulation of Lie's theory constructed with respect to the most general possible unit $\hat{I}$, in which case the Lie product assumes less trivial forms, such as $A * B - B * A$ where $A * B$ is still associative but of the more general type $A * B = AgB$, where g is fixed, sufficiently smooth and nonsingular, and $\hat{I} = g^{-1}$. The generalized theory was called the "Lie-isotopic theory" for certain historical reasons reviewed in the text. The original proposal of 1978 contains the development of the Lie-isotopic theory to a rather remarkable extent, including a generalization of: the theory of universal enveloping associative algebras (Poincaré-Birkhoff-Witt Theorem, etc.); Lie's celebrated First, Second and Third Theorems; Lie's transformation groups; and Lie's symmetries. The memoir concluded with the conjecture of a conceivable generalization of Galilei's Relativity in classical mechanics for extended particles moving within resistive media (which are not only Galilei-noninvariant, but also generally nonhamiltonian). This original proposal was subjected to a systematic study in subsequent years by Santilli as well as a number of independent authors, not only for the original classical profile, but also for a conceivable operator counterpart, as well as for relativistic, gravitational and gauge extensions.

This review is a guide through a considerable and disparate literature, devoted to: the identification of the state of the mathematical studies on the Lie-isotopic generalization of conventional formulations of Lie's theory; their primary applications, to classical mechanics, particle physics and astrophysics; and an outline of the proposed fundamental tests. Except for minor treatments, the studies on conceivable operator realizations are deferred to a possible separate review.

We begin with a review of the algebraic notion of isotopy and its application to associative and Lie algebras. We then pass to the notion of analytic isotopy in classical mechanics, that realized via the Birkhoffian generalization of Hamiltonian mechanics. We also indicate the notion of operator isotopy on Hilbert spaces, that realized via the hadronic generalization of quantum mechanics, as well as the methods of "hadronization," that is, the mapping of Birkhoffian into hadronic mechanics. The notion of isotopy in symplectic geometry concludes our introductory chapter.

The second chapter is devoted to a detailed review of the mathematical studies on the Lie-isotopic formulations of: enveloping associative algebras; Lie's Theorems; Lie algebras; Lie groups; and the application of the generalized theory to space-time symmetries. The second chapter ends with a fundamental theorem by Santilli on the reconstruction of the exact nature of space-time symmetries at the Lie-isotopic level, when broken at the conventional level.

The third chapter is devoted to the applications of the Lie-isotopic theory. We begin with a review of Santilli's isotopic generalization of the group of rotations and some of its properties such as: the capability by the rotational symmetry to remain exact at the Lie-isotopic level when conventionally broken, say, for spheres undergoing deformations, or for any physical condition implying a topology-preserving alteration of the Euclidean metric. We then pass to the review of Santilli's Lie-isotopic generalization of Galilei's Relativity for systems of extended-deformable particles which are nonhamiltonian (but Birkhoffian) because of motion within a resistive medium. We review the property that, again, under certain topological restrictions, the Galilei symmetry remains exact at the Lie-isotopic level when conventionally broken by nonhamiltonian forces. A number of intriguing implications and open problems are also considered. We then pass to the review of Santilli's Lie- isotopic generalization of Einstein's Special Relativity and related properties, such as: the capability of incorporating all available studies on Lorentz "noninvariance" (universality), e.g., the several phenomenological calculations predicting deviations from Einstein's behavior on the mean life of unstable hadrons at different speeds; the capability of reconstructing the Lorentz symmetry as isotopically exact for all the above models (in which it is conventionally broken); the capability to represent a disparate variety of physical conditions outside the applicability of the conventional relativity, such as deformation of charged distributions, motion of electromagnetic waves in inhomogeneous and anisotropic media, motion of electrons in metals, propagation of causal signals within dense hadronic matter, etc.; the generalization of the various laws of the conventional relativity with intriguing implications although in need of experimental preliminary confirmations; and a number of other aspects. The third chapter then passes to a review of the construction by Gasperini and Santilli of a Lie-isotopic generalization of Einstein's gravitation which is, locally, Lorentz-isotopic and Galilei-isotopic, as well as capable of resolving at least some of the numerous problematic aspects of the conventional theory available in the literature. The need for the conduction of certain basic tests on fundamental space-time symmetries (that have been regrettably ignored for decades) completes the third chapter.

In the Appendices we review a variety of topics that complement the main text, such as: Lie-isotopic generalization of gauge theories; computation of the maximal speed of causal signals within hadronic matter; Lie-isotopic field equations; and other aspects.

The situation emerging from this review is essentially as follows. From a mathematical viewpoint, there is little doubt that the Lie-isotopic theory is mathematically consistent and does provide a genuine covering of the conventional formulation of Lie's theory. The understanding is that the studies are at the beginning and so much remains to be done. From the viewpoint of theoretical physics, the classical formulations of the Lie- isotopic theory have clear applications in Newtonian mechanics, particularly for the physical systems of our everyday life, that is, with nonhamiltonian forces, for which the conventional formulations are simply inapplicable. In regard to relativistic settings, the isotopic theories are admittedly tentative, conjectural and in need of direct tests, although we are aware of no experimental or other information on the novel physical conditions considered capable of disproving the predictions of the new theory at this writing. As a matter of fact, all evidence currently available appears to favor the Lie-isotopic symmetries over the conventional ones, in a way, after all, predictable from the necessary compatibility with established Newtonian applications.

We are here referring to: phenomenological calculations on the behavior of the meanlife of unstable hadrons with energy conducted over the past several decades showing an apparent violation of the Einsteinian law, while they are clearly and directly interpreted by Santilli's covering law; the preliminary measures via neutron interferometry conducted by Rauch and his associates on the apparent deformation of the charge distribution of neutrons under external nuclear fields, with consequential alteration of the magnetic moments/rotational asymmetry, which are also directly and quantitatively interpreted by Santilli's exact, $SU(2)$-isotopic symmetry; and others. Not surprisingly, the astrophysical applications of Santilli's covering relativities appear to be in full agreement with their particle and classical counterparts. We are here referring, e.g., to the possibility of interpreting the quasar redshift as due to propagation of light within the hyperdense, inhomogeneous and anisotropic media surrounding the quasars, rather than to the currently unplausible quasars speeds of the order of ten time the speed of light in vacuum; and other very intriguing astrophysical applications.

As a result of all the above, a thrilling possibility of a new scientific edifice emerges from Santilli's pioneering studies, with predictable implications at every level of contemporary physics, most of which are still unexplored as of now. But, by far, the most important implications of Santilli's studies are from an experimental viewpoint. In fact, the studies focus the attention on considerably overdue, fundamental experiments which have been submitted in the technical literature for decades, but largely ignored until now. We are referring to experiments such as: final measures of the behavior of the mean life of unstable hadrons at different speeds; or to final measures of the expected deformation of the charge distributions of hadrons under sufficiently intense external fields; and others. All these experiments, in their currently available preliminary form, show clear deviations from the Einsteinian predictions, in favor of the prediction of Santilli's relativities and their exact, isotopic, Lorentz symmetry. This situation leaves the ultimate foundations of contemporary physics in a state of "suspended animation" which will evidently persist until the experiments are finally done, and the issue of conventional versus isotopic space-time symmetries resolved one way or the other.

This work will achieve one of its most important objectives if it succeeds in stimulating experimentalists to finally conduct these much overdue, fundamental tests.

June 1, 1990

1 INTRODUCTION

1.1 A Brief Survey of the Literature

Despite rather vast mathematical and physical studies, the formulation of Lie's theory has been essentially restricted until recently to that via the familiar Lie product $[A, B] = AB - BA$, where AB is the simplest possible associative product, e.g., that of matrices. The unit of the theory is then the trivial element, e.g., $I = \text{diag}(1, 1, ..., 1)$.

An inspection of the physical literature confirms this condition, which has its origin in the construction of quantum mechanics via the enveloping associative algebra of operators $A, B, ...$, their simplest possible product AB, and Heisenberg's time evolution $i\hbar\dot{A} = AH - HA$. An inspection of the mathematical literature confirms the same condition which has its origin, this time, in the representation theory of enveloping associative algebras also realized via the product AB.

In a pioneering memoir of 1978 (written while at the Lyman Laboratory of Physics of Harvard University), Ruggero Maria Santilli [1] identified, apparently for the first time, a generalized formulation of Lie's theory which he called *Lie- isotopic theory* for certain historical reasons reviewed later on. The central idea is that of building the theory around the most general possible unit, say $\hat{I} = (I_{ij})$, where the elements I_{ij} have an arbitrary functional or operator dependence subject only to certain topological restrictions. This demanded, of course, a generalization of the enveloping algebra, from the form with trivial product AB, into a covering form with product of the type $A * B = ATB$, where $\hat{I} = T^{-1}$. The Lie product then takes the more general form $A * B - B * A$.

Santilli was the first to realize the mathematical and physical nontriviality of the theory and to develop it to a considerable extent already in the original proposal [1]. In fact, in this first memoir one can see several theorems generalizing enveloping associative algebras, the celebrated Lie's first, second and third theorems, and the conventional notion of Lie group, into forms compatible with the most general possible unit $\hat{I}$. Under the condition that the old unit I is contained as a particular case of the generalized unit $\hat{I}$, Santilli's theory becomes a *covering* of the conventional one, in the sense of being formulated on structurally more general foundations, while admitting the conventional formulation as a trivial particular case.

Remarkably, the Lie-isotopic theory was proposed by Santilli as a particular case of a structurally yet more general theory based on the so-called

Lie-admissible algebras, which will not be reviewed in this monograph. Nevertheless, the point is important for this review because some of the subsequent advances made by Santilli and others on the Lie-isotopic theory can be identified only as a particular case of the more general Lie-admissible formulations. Perhaps this is the reason why the Lie-isotopic theory has not received until now the attention it deserves in both physical and mathematical literatures.

The subsequent memoir also of 1978 [2] and paper [3] were primarily devoted to Lie-admissible algebras, although containing advances important also for the simpler Lie-isotopic theory such as the foundation of a conceivable operator realization of the algebras, including the generalization of Heisenberg's equations of the type $i\hbar\dot{A} = A * B - B * A$. Santilli completed the year 1978 with the release of the two monographs [4,5], the first setting up the methodological foundations of the classical realization of the Lie-isotopic theory (the so-called conditions of variational selfadjointness), and the second initiating the application of hadronic mechanics to particle physics.

In 1979 we see the appearance of the first review [6] [again for the Lie-admissible approach] followed by paper [7] on the initiation of the representation theory of the generalized algebras on suitable bimodular vector spaces. Paper [8] presents an intriguing application to gauge theories.

Paper [9] of 1980 studies the difficulties of conventional quantization, and suggests their reinspection under a broader algebraic structure. Paper [10] of 1981 studies the expected existence of a conceivable generalization of quantum mechanical laws for the interior of hadrons, with particular reference to Heisenberg's uncertainty principle. Paper [11] enters deeper into conceivable physical implications for particle physics, this time for the notion of particle under external strong interactions realized with nonlocal and nonhamiltonian terms due to mutual wave overlappings.

In 1982 we see the appearance of paper [12] which consists of a review of the physical implications of the generalized Lie structures for nonpotential nonhamiltonian interactions in Newtonian, statistical and particle mechanics. Paper [13] studies the conceivable generalization of Heisenberg's and Schrödinger's equations that are expected from the broader realizations of Lie's theory. Paper [14] presents another courageous analysis, the possibility that causal signals can propagate within dense hadronic matter at speed higher than c_o, the speed of light in vacuum. At the end of 1982 we also see the appearance of monographs [15,16] on the classical realizations of his algebraic theories, the so-called Birkhoffian [15] and Birkhoffian admissible

[16] mechanics. In these monographs one can see Santilli's extended presentations of the conceivable generalizations of Lie-isotopic and Lie-admissible type, respectively, of the classical Galilean relativity for extended particles with action-at-a-distance, potential forces, as well as contact, nonpotential and nonhamiltonian forces due to motion within a resistive medium.

In 1983 we see the appearance of three central contributions. Paper [17] presents a model on the reversibility of strong interactions for center-of-mass conditions, with irreversible dynamics for each individual constituent when considering the rest of the system as external. Paper [18] is, in our opinion, the most important paper under consideration here after refs. [1,2]. It presents the foundations of a conceivable Lie-isotopic covering of Einstein special relativity for generalizations of the Minkowski metric caused by motion of extended particles within generally inhomogeneous and anisotropic physical media. The paper also provides the explicit method for the construction of an infinite class of covering transformations from the original Lorentz ones and the given generalized metric. Paper [19] provides a generalization of Wigner's theorem on quantum mechanical symmetries within the broader Lie-isotopic setting representing nonpotential nonhamiltonian forces caused by mutual wave-overlappings of particles. This paper also signals the achievement of mathematical maturity of the generalized operator formulation, with the clear understanding that its physical validity is still basically open at this writing.

In 1984 we see the appearance of another important contribution [20]. In the preceding paper [18] Santilli shows that, under certain topological restrictions, the continuous part of the Lorentz symmetry can be proved to be exact at the abstract, Lie-isotopic level when generally considered as "broken" at the simplistic level of the product $AB - BA$. Paper [20] complements these results, this time, for the discrete part of the Lorentz symmetry. In fact, the paper indicates how parity may well be an exact symmetry under weak interactions, provided the theory is realized within the context of the covering Lie-isotopic approach, because all P-breaking terms can be incorporated in the generalized unit $\hat{I}$ [as well as in other degrees of freedom]. The exact character of the P-[as well as other] symmetries then follows from the property that Lie algebras leave invariant their unit element.

In 1985 we see additional contributions by Santilli in the field. The year started with the inspiring "Journey in the Solar system" [21] (an invited contribution to the Calcutta conference). We then see the appearance of papers [22,23] specifically devoted to Lie-isotopic symmetries. These papers (which had been written prior to paper [18] and presented at a meeting of 1983) essentially provide a rigorous mathematical formulation of the process according to which a given Lie symmetry, when broken at the simpler level $AB - BA$, can be "reconstructed" as exact at the higher Lie-isotopic level $A * B - B * A$. The papers also identify the means of constructing the (generally infinite family of) covering, exact, Lie-isotopic transformations via the sole knowledge of the old transformations and of the new metric. Papers [22,23] then apply the theory to a case of truly central physical relevance: the breaking of the rotational symmetry, say, for the deformation of a spherical charge distribution under external fields, and the recovering of the exact rotational symmetry for the deformed distribution at the covering Lie-isotopic level.

In 1988 we see the appearance of four memoirs [25] which, jointly with the original memoir [1], constitute Santilli's most significant scientific contributions. In fact, these latter memoirs present a comprehensive isotopic generalization of contemporary algebras, geometries and mechanics for systems that are not only nonlinear and nonlocal (as those of the preceding contributions), but also nonlocal integral; the memoir then apply these broader mathematical tools for the construction of isotopic coverings of Galilei's, Einstein's Special and Einstein's General Relativities for interior dynamical problems; the memoirs finally present a detailed study of the mutual compatibility of the emerging generalized formulations and propose a number of experimental verifications.

In 1989 we see the appearance of four additional memoirs [26] this time devoted to the operator formulation of the isotopic theories, including a study of the "hadronization" of classical into operator formulations; the construction of the spinorial SU(2)-isotopic symmetry and its representations; some isotopic generalizations of the various properties of the conventional theory of angular momentum and spin (such as the isotopic Clebsch-Gordon coefficients, etc); the construction of the operator formulation of the isotopic Galilean and special relativities; the foundations of the isotopic field theory, including the isotopic generalization of the Klein-Gordon and Dirac's equations; the operator study of Rauch's experiment on the spinorial symmetry of neutrons; and other important topics.

Paper [25] of 1990 tests the possibilities of hadronic mechanics via a quantitative study of the possible representation of the original Rutherford's conception of the neutron as a generalized bound state of one ordinary proton and one electron, whose total angular momentum is represented via the isotopic SU(2)-symmetry to account for the expected nonlocal and nonhamiltonian effects due to total mutual penetration of the wavepackets of the constituents.

In 1991 we use the appearance of a series of papers written at the ICTP [27, 28, 29] which develop in more details the operator formulation of the isotopic special relativity based on the isotopies of the Poincaré symmetry; the construction of the generalized field theory invariant under the isotopic Poincaré symmetry, and some applications (Rauch's experiment on the spinorial symmetry and Rutherford's conception of the neutron).

Monograph [30], currently under preparation, is expected to complete the series of the preceding volumes [4,5] and [15,16]. This completes the review of the contributions written by Santilli alone.

Papers [31–44] were written by Santilli in collaboration with several authors on numerous topics related to the precedings research (see below).

A number of physicists have studied Santilli's proposal of 1978.

R. Mignani [45] made seminal contributions in the operator realization of Lie-isotopic theories, such as: the independent identification of the Lie-isotopic generalization of Schrödinger's equation; the proposal to construct a nonpotential scattering theory; the construction of the Lie-isotopic $SU(3)$ symmetry; and others.

M. Gasperini [46] made other equally seminal contributions, such as: the computation following hypothesis [14], that, within the context of contemporary gauge theories, the speed of causal signals within hadronic matter could indeed exceed c_o; the foundations of a possible Lie-isotopic generalization of gauge theories; and the foundations of a possible Lie-isotopic generalization of Einstein gravitation for the interior problem.

A team headed by A. Jannussis made numerous contributions [47] in both classical and operator realizations of Santilli's algebras. M. Nishioka [48] also made several contributions in the field, such as the expected generalization of the delta function. A. J. Kalnay [49] succeeded in quantizing Nambu's mechanics for triplets. The algebra emerging at the operator level is exactly that of Santilli's type [2]. (This aspect, which we regrettably cannot review in this paper, opens the possibility of a true quark confinement with an identically null probability of tunnel effects into free states, besides an infinite potential barrier, as studied in papers [44].

Animalu [50] conducted several, additional, independent research, such as the study of possible contributions to conventional quark theories of the generalized setting offered by hadronic mechanics, and others.

A. Tellez Arenas, J. Fronteau and R. M. Santilli [31,32] studied the statistical profile of a generalized class of physical systems characterized by the

Lie-isotopic algebras, the so-called closed variationally nonself-adjoint systems (these are systems submitted in memoir [2] which verify conventional total conservation laws, but the internal forces are of nonlocal, nonhamiltonian type).

The (mathematician) H. C. Myung and R. M. Santilli [36,37] achieved a consistent mathematical formulation of the operator realization of the Lie-isotopic algebras. These studies were then further extended via the addition of a suitable form of Hilbert spaces and reached their final form in ref. [38] by Mignani Myung and Santilli, which is here considered the best available presentation on the operator version of Lie-isotopic theories.

Additional contributions were made by A. K. Aringazin [51] such as: the application of Lie-isotopic Lorentz transformations to describe an anomalous energy dependence of some fundamental parameters of the $K^0 - \overline{K}^0$ system; the proof that Pauli's exclusion principle is valid for the center of mass of a composite system under a Lie-isotopic operator mechanics, in a way compatible with possible departures from the same principle for each individual constituent (a similar occurrence for Heisenberg's principle had been established in ref. [38]); the universal capability of the Lorentz-isotopic symmetry to include as particular cases all available research on Lorentz noninvariance; and others.

An in depth study of torsion in gravitational theories, and its apparent ultimate origin of Santilli's isotopic type has been conducted by D. Rapoport-Campodonic [52], with intriguing developments in stochastic and operator formulations.

S. Okubo [53] has also conducted a number of investigations in the field, most remarkably, the identification of certain inconsistencies which emerge in any attempt at generalization of the conventional associative enveloping algebra of quantum mechanics, and other mathematical studies.

One of the most intriguing applications has been provided by P.A.M. Dirac in two of his last (and little known) papers [53] presenting a certain generalization of his celebrated equation which resulted to have an essential isotopic structure, as shown by Santilli [27] (see Appendix C for a review).

The interested reader can identify a number of further contributions by various additional authors in the bibliographies of the above quoted papers, as well as in Proceedings [55,61].

The contributions by pure mathematicians specifically devoted to the Lie-isotopic formulation of Lie's theory (or their universal enveloping associative algebras) are grossly lacking at this time, to our best knowledge. In fact, as we shall see later on, the sole mathematical paper of which we are aware is ref. [62] by H. C. Myung on the isotensorial product of isorepresentations. Another mathematical paper connected with this review is that by E. B. Lin [63], devoted to the problem of "hadronization" (i.e., symplectic mapping of Birkhoffian into hadronic mechanics). The authors of this review are aware of several mathematical papers by mathematicians specifically devoted to the more general Lie-admissible algebras (see bibliography [64]) and, as such, they will be quoted and reviewed in a separate review of Santilli's Lie-admissible formulation of classical and operator mechanics. Nevertheless, these mathematical works are of difficult specialization to the Lie-isotopic context. It is hoped that this review will stimulate contributions by pure mathematicians, specifically, on Lie-isotopic algebras so as to be readily available for physical applications. An outline of this monograph written for mathematicians, with a list of intriguing, open, mathematical problems has been provided by these authors in paper [65].

1.2 The Notion of Algebraic Isotopy

As limpidly expressed in Santilli's writings, physical theories are a manifestation of an articulated body of formulations of algebraic, analytic, geometrical and other character. A generalized notion in any one of these formulations, to be consistent, must admit corresponding, compatible generalizations in the remaining branches of the theory. This is the case of the central notion of this review, that of isotopy (ref. [1], §2.13, pp. 287 and ff.).

Let U be an (associative or nonassociative) algebra with (abstract) elements $a, b, c, \ldots$ and (abstract) product ab over a field F with elements $\alpha, \beta, \gamma, \ldots$ (hereinafter assumed to have characteristic zero). The product ab, by assumption, verifies the basic axioms of U. For instance, if U is associative, ab verifies the associative law; if U is commutative, it verifies the commutative law; if U is a Lie algebra, it verifies the Lie algebras axioms; etc.

> *DEFINITION 1.1 (Algebraic Isotopy): An* isotopic mapping *(also called* image *or* lifting*) of an algebra U with product ab is any mapping $U \to \hat{U}$ of U into an algebra $\hat{U}$ which is the same vector space as U (i.e., the elements of U and $\hat{U}$ coincide), but is equipped with a new product $a * b$ which is such to verify the original axioms of U.*

Note that [15] the Greek for "isotopic" is "$\iota' \; \sigma \; o \; s \; \tau \; o' \; \pi \; o \; s$" which means "same configuration," precisely along the concept of the above definition.

The central property of the notion of algebraic isotopy is therefore that of preserving the character of the original algebra. Thus, if U is associative, a necessary condition for $\hat{U}$ to be an isotope of U is that the new product $a * b$ also verifies the associative law, and we shall write:

$$U : (ab)c = a(bc) \rightarrow \hat{U} : (a * b) * c = a * (b * c) \,. \qquad (1.1)$$

Similarly, if U is a Lie algebra, a necessary condition for $\hat{U}$ to be one of its possible isotopes is that $\hat{U}$ is also Lie, and we shall write

$$U : \begin{cases} ab + ba = 0 \\ (ab)c + (bc)a + (ca)b = 0 \end{cases}, \hat{U} : \begin{cases} a * b + b * a = 0 \\ (a * b) * c + (b * c) * a + (c * a) * b = 0. \end{cases} \qquad (1.2)$$

A similar situation occurs for other algebras, such as Jordan algebras, alternative algebras, etc.

Santilli identified three types of associative isotopy, each one with an attached Lie algebra isotopy. The first is the trivial one (ref. [1], p. 287)

$$U : ab \rightarrow \hat{U} : a * b = \alpha ab; \quad \alpha \in F; \quad \alpha \neq 0 \text{ and fixed} \,, \qquad (1.3)$$

evidently given by the multiplication of the old product ab by a constant (that remains fixed for all multiplications of the new algebra). The attached Lie algebra is then given by the trivial mapping

$$[a, b]_U = ab - ba \rightarrow [a, b]_{\hat{U}} = \alpha [a, b]_U. \qquad (1.4)$$

The second realization of associative isotopy, which plays a central role throughout Santilli's analysis, is given by (ref. [1], p. 352)

$$U : ab \rightarrow \hat{U} : a * b \stackrel{\text{def}}{=} aTb, \quad T \in U, \text{ invertible and fixed.} \qquad (1.5)$$

It is simple (but instructive) to verify that indeed

$$(a * b) * c = (aTb)Tc = aT(bTc) = a * (b * c). \qquad (1.6)$$

Thus, $\hat{U}$ is an isotope of U,

$$U : (ab)c = a(bc) \rightarrow \hat{U} : (a * b) * c = a * (b * c). \qquad (1.7)$$

Evidently, isotope (1.5) is not trivial. Equally non trivial is the attached Lie algebra isotopy

$$[a, b]_U = ab - ba \rightarrow [a, b]_{\hat{U}} = a * b - b * a = aTb - bTa. \qquad (1.8)$$

Since the element T does not necessarily commute with the generic elements $a, b, ...$, of the algebra, the nontriviality of mapping (1.7) follows. The interested reader is encouraged to verify that, if $[a,b]_U$ is Lie, $[a,b]_{\hat{U}}$ is also Lie, i.e., it verifies the laws

$$[a,b]_{\hat{U}} + [b,a]_{\hat{U}} = 0 \, ,$$
$$[[a,b]_{\hat{U}}, c]_{\hat{U}} + [[b,c]_{\hat{U}}, a]_{\hat{U}} + [[c,a]_{\hat{U}}, b]_{\hat{U}} = 0. \tag{1.9}$$

Isotopies (1.5) and (1.8) were assumed by Santilli at the basis of his formulation of Lie algebra isotopy, and we shall do the same in this review. In fact, the isotopic element T is sufficient to represent a generalized metric. Isotopies (1.5) and (1.8) are then amply sufficient to illustrate the mathematical and physical nontriviality of the generalized theory.

One additional algebraic isotopy was identified by Santilli [11]. It is given by

$$U : ab \rightarrow \hat{U} : a * b = WaWbW,$$
$$W \in U, \text{ idenpotent}(W^2 = W), \text{ and fixed.} \tag{1.10}$$

It is again an instructive exercise for the interested reader to verify that the above product $a * b$ is still associative. The attached anticommutative product then remains Lie, ie., the mapping

$$[a,b]_U = ab - ba \rightarrow [a,b]_{\hat{U}} = a * b - b * a = WaWbW - WbWaW \tag{1.11}$$

constitutes another example of Lie algebra isotopy.

The reader may be interested in knowing that no investigation on isotopies (1.10) and (1.11) has been conducted until now, to our best knowledge, in both mathematical and physical literatures. All available studies are referred to isotopies (1.5) and (1.8).

The reason for the lack of physical investigations by Santilli on isotopy (1.10) is the general loss of the unit under the lifting considered. In turn, the loss of the unit has fundamental drawbacks from a physical profile, such as the loss of the measure theory, the loss of the notion of quantum of energy, the loss of the Casimir invariants, etc. For these (and other reasons), Santilli centered his research on the fundamental condition that the generalized theory must admit a consistent, left and right unit [1,2], which condition is indeed verified by isotopy (1.5) as we shall review shortly.

Also, a private communication by Santilli indicates that, according to preliminary research, isotopies (1.3), (1.5) and (1.10) and their combinations are expected to exhaust all possible associative isotopies, but no rigorous study has been conducted on this problem until now.

The classification of all possible associative (and therefore Lie) isotopies is evidently important because different isotopies are expected to characterize different physical theories.

As one can see, the notion of algebraic isotopy essentially represents a sort of "*degree of freedom of the product*" for given algebra axioms. As Santilli recalls [1], the notion is rather old, and actually dates back to the early stages of the set theory [66]. In fact, the notion apparently originates within the context of *Latin squares* (two Latin squares were called "isotopically related" if they could be made to coincide via permutations). Appropriately, Santilli quotes Bruck statement [66] to the effect that the notion is "*so natural to creep in unnoticed.*" And in fact, the notion had not been applied to Lie algebras until Santilli's proposal [1] (even though some application to other nonassociative algebras, e.g., the Jordan algebras, can be identified in the specialized mathematical literature [64,67]).

1.3 The Notion of Analytic Isotopy in its Classical and Operator Realizations

Let us pass now to the analytic counterpart of the concept of isotopy. It was introduced, also for the first time to our best knowledge, in memoir [1] and developed in detail in monograph [15] for the nonlinear and nonhamiltonian, but local-differentize case considered in this work. The more general nonlocal-integral case of memoir [24] will not be considered for brevity.

By following Santilli, let us write the conventional *Hamilton's equations* (those *without* external terms) in the unified notation

$$\dot{a}^\mu = \omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}, \quad \mu = 1,2,3,...,2n ,$$

$$a = (r^k, p_k), \quad k = 1,2,...,n, \quad H = H(t,a) , \tag{1.12}$$

with *Poisson brackets* between functions A and B in phase space $(\vec{r},\vec{p})$

$$[A,B] \stackrel{\text{def}}{=} \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial B}{\partial a^\nu} \equiv \frac{\partial A}{\partial r^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k}\frac{\partial A}{\partial p_k} , \tag{1.13}$$

and *canonical commutation rules* characterizing the *fundamental Lie tensor*

$$\begin{aligned} ([a^\mu, a^\nu]) &= \begin{pmatrix} ([r^i, r^j])([r^i, p_j]) \\ ([p_i, r^j])([p_i, p_j]) \end{pmatrix} = (\omega^{\mu\nu}) \\ &= \begin{pmatrix} O_{n\times n} & I_{n\times n} \\ -I_{n\times n} & O_{n\times n} \end{pmatrix} . \end{aligned} \tag{1.14}$$

The *canonical action principle* can be written

$$\delta A(t,\vec{r}) = \delta \int_{to}^{t} [R^o_\mu \dot{a}^\mu - H] dt = 0 \ ,$$

$$R^o = (\vec{p}, \vec{0}), \tag{1.15}$$

yielding Hamilton's equations in their covariant form

$$\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu} = 0, \tag{1.16}$$

where $\omega_{\mu\nu}$ is the covariant (symplectic) counterpart of $\omega^{\mu\nu}$ with explicit local realization in phase space

$$\omega_{\mu\nu} = \frac{\partial R^o_\nu}{\partial a^\mu} - \frac{\partial R^o_\mu}{\partial a^\nu} \ , \tag{1.17.a}$$

$$(\omega_{\mu\nu}) = \begin{pmatrix} O_{n\times n} - I_{n\times n} \\ I_{n\times n} O_{n\times n} \end{pmatrix} = (\omega^{\alpha\beta})^{-1} \ . \tag{1.17.b}$$

Finally, the *Hamilton-Jacobi equations* can be written in the unified form

$$\frac{\partial A}{\partial t} + H = 0 \ ,$$

$$\frac{\partial A}{\partial a^\mu} = R^o_\mu \ , \tag{1.18}$$

where the second set of equations can be explicitly written in the familiar form

$$\frac{\partial A}{\partial r^k} = p_k \ , \tag{1.19.a}$$

$$\frac{\partial A}{\partial p_k} = 0 \ , \tag{1.19.b}$$

showing the lack of dependence of the canonical action functional in the linear momentum (a property with important implications for quantization).

*DEFINITION 1.2 [1], [15] (*Classical-analytic Isotopy*): An* isotopic mapping *(or* image *or* lifting*) of Hamilton's equations is given by any generalized form of the equations which preserves: a) the derivability from a (first- order) variational principle; b) the Lie character of the underlying brackets; and c) the existence of a generalized, but consistent, Hamilton-Jacobi theory.*

The generalization of Hamiltonian mechanics originating from the above definition was called by Santilli *Birhkhoffian mechanics* for certain historical reasons (see ref. [1], p. 259 for the first appearance of these terms, and monograph [15] for a comprehensive presentation).

Under the above definition, principle (1.15) is generalized into the most general possible *Pfaffian variational principle* (here restricted to the semi-autonomous case for simplicity)

$$\delta \hat{A}(t,a) = \delta \int_{to}^{t} [R_\mu(a)\dot{a}^\mu - H(t,a)]dt = 0 \ ,$$

$$R = R(a) = R(\vec{r},\vec{p}) \neq R^o \ , \tag{1.20}$$

with fundamental equations given by *Birkhoff's equations* in their covariant form

$$\Omega_{\mu\nu}(a)\dot{a}^\nu - \frac{\partial H(t,a)}{\partial a^\mu} = 0, \quad \mu = 1,2,...,2n \ , \tag{1.21.a}$$

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \ , \tag{1.21.b}$$

with contravariant version

$$\dot{a}^\mu - \Omega^{\mu\nu}(a)\frac{\partial H(t,a)}{\partial a^\nu} = 0 \ , \tag{1.22.a}$$

$$\Omega^{\alpha\beta} = |(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu})^{-1}|^{\alpha\beta} \ . \tag{1.22.b}$$

The algebraic brackets of the theory are given by the so-called *generalized Poisson brackets*

$$[A\hat{,}B] \stackrel{\text{def}}{=} \frac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}(a)\frac{\partial B}{\partial a^\nu} \ , \tag{1.23}$$

with *fundamental Birkhoffian brackets*

$$[a^\mu\hat{,}a^\nu] = \Omega^{\mu\nu}(a) \ , \tag{1.24}$$

which do verify the Lie algebra axioms (see the analytic, algebraic and geometrical proofs of ref. [15])

$$[A\hat{,}B] + [B\hat{,}A] = 0 \ ,$$

$$[[A\hat{,}B]\hat{,}C] + [[B\hat{,}C]\hat{,}A] + [[C\hat{,}A]\hat{,}B] = 0 \ . \tag{1.25}$$

Finally, Eqs. (1.18) are lifted into the *Birkhoffian form of the Hamilton-Jacobi equations*

$$\frac{\partial \hat{A}}{\partial t} + H = 0 , \qquad (1.26.a)$$

$$\frac{\partial \hat{A}}{\partial a^\mu} = R_\mu . \qquad (1.26.b)$$

Note that, unlike Eqs. (1.18), the generalized action functional does depend, in general, on the linear momentum, thus resulting in nontrivial generalizations of Eqs. (1.19b) (for simpler versions see below).

In summary, the notion of analytic isotopy gives rise, not to one particular algorithm, but to *an entire new mechanics* generalizing each and every aspect of the conventional Hamiltonian mechanics. It is hoped that, in this way, the reader begins to see the rather intriguing implications of Santilli's research.

Of course, the algebraic isotopy is a particular case of Definition 1.2, this time in its classical realization in the local coordinates $a = (\vec{r}, \vec{p})$

$$[A, B] = \frac{\partial A}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial B}{\partial a^\nu} \rightarrow [A \hat{,} B] = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu}(a) \frac{\partial B}{\partial a^\nu} . \qquad (1.27)$$

This proves the compatibility of the notion of isotopy at the algebraic and analytic levels (see the next section for the geometrical aspect).

From the above property we also see another seminal result achieved in memoir [1], that *Birkhoffian mechanics is a realization in classical mechanics of the Lie-isotopic algebras.* The reader interested in acquiring an expertise in Lie-isotopy is therefore urged to study monograph [15]. This point must be stressed here because this review can only serve as a guide to the existing literature.

Unlike the conventional Hamiltonian mechanics, *the Birkhoffian mechanics is directly universal*, in the sense of being able to represent *all* possible systems of the class admitted (essentially nonself-adjoint/nonhamiltonian systems verifying certain topological restrictions) in the frame of the experimenter. This property has nontrivial implications (particularly for quantization) because the mathematical algorithms of the theory can now be assured to have a direct physical significance, e.g.,: " $\vec{r}$ " represents the actual local coordinates of the experimenter; " $\vec{p}$ " represents the physical linear momentum $m\dot{\vec{r}}$; "$\vec{r} \wedge \vec{p}$ " therefore represents the angular momentum; "H" represents the actual physical energy $T + V$; etc. (see ref. [15], §4.5).

By comparison, the algorithm "$\vec{p}$" in Hamiltonian mechanics coincides with the physical linear momentum $m\dot{\vec{r}}$ only in very special cases; nevertheless, upon quantization, its operator image is rather universally assumed to be the physical linear momentum (with consequential results of equivocal character).

Let us also recall that *each formulation of Birkhoffian mechanics can be constructed via noncanonical transformations of the corresponding Hamiltonian counterpart.* In fact, Hamilton's equations *do not* preserve their form under noncanonical transformations, as well known. What has been identified by Santilli (ref. [15], §5.3) is that, under noncanonical transformations, all essential properties persist (derivability from a first-order principle; verification of Lie algebras axioms; existence of a Hamilton-Jacobi theory; etc.).

As a further aspect, the function H of Birkhoff's Eqs. (1.21) does not represent, in general, the total physical energy $T + V$ (although, as mentioned earlier, a representation of any given system always exists under the restriction $H = T + V$). In order to avoid confusions, Santilli introduced the name *Birkhoffian* for this function. The term Hamiltonian within the context of Birkhoff's equations is used only when the function represents the total energy. In the following, whenever referring to this function, we shall use the Hamiltonian H to denote specifically the restriction to the physical total physical energy $T + V$ (which is not necessarily conserved), and the Birkhoffian B to stress its departures from the total physical energy H.

As a final point, the classical Birkhoffian realization of the Lie-isotopic theory is fully established on physical grounds. Birkhoff [68] introduced his equations for a better study of the stability of the planetary orbits, although his use of Eqs. (1.21) was restricted to conservative systems. Santilli [1] rediscovered these equations (after some 51 years) and proved not only their applicability to a much larger class of Newtonian systems, but also their direct universality. For numerous physical applications along these latter lines, we refer the reader to the examples of Ref. [15], as well as to the quoted literature.

The restriction of this review only to classical realizations of the Lie-isotopy would however be a gross disservice to the reader, because, as well known, the abstract formulation of Lie's theory is directly interpretable via operator realizations.

This renders unavoidable a brief review of the operator realization. In the following we shall review the apparent generalization of quantum mechanics which emerges from these studies, with the clear understanding that, unlike its classical counterpart, the physical validity of the generalized operator formalism is not established as of this writing.

Let $\mathcal{H}$ be a Hilbert space (hereinafter assumed to be finite-dimensional) with elements $|a\rangle, |b\rangle, \ldots$ and norm over the field C of complex numbers

$$\mathcal{H} : \langle a|b\rangle = c \in C \ . \tag{1.28}$$

Let ξ be an enveloping associative algebra of operators $A, B, \ldots$ on $\mathcal{H}$ with trivial associative product AB and unit $I = \mathrm{diag}(1, 1, \ldots, 1)$,

$$\xi : IA = AI = A, A\forall\xi \ . \tag{1.29}$$

The Lie algebra L attached to ξ is then characterized by the familiar product

$$L : [A, B]_\xi = AB - BA \ , \tag{1.30}$$

which provides the structure of the first fundamental equation of quantum mechanics, *Heisenberg's equation*

$$i\dot{A} = [A, H]_\xi = AH - HA, \quad \hbar = 1. \tag{1.31}$$

Let the homomorphism $\xi \times \mathcal{H} \rightarrow \mathcal{H}$ be characterized by the *(right) modular action* of, say, an operator $H \in \xi$ on an element $|a\rangle \in \mathcal{H}$ according to the familiar eigenvalue equation

$$H|a\rangle = c|a\rangle, \quad c \in C. \tag{1.32}$$

This provides the structure of the second fundamental equation of quantum mechanics, the familiar *Schrödinger's equation*

$$i\frac{\partial}{\partial t}|a\rangle = H|a\rangle \ , \tag{1.33}$$

with corresponding well known additional aspects (such as unitary transformation theory, various physical laws, etc.).

DEFINITION 1.3 [2], [15], [36], [38] (Operator-analytic Isotopy)*: An* isotopic mapping *(or* image *or* lifting*) of Heisenberg's and Schrödinger's equations is given by compatible generalized forms that preserve: a) the existence of an underlying Hilbert space; b) the Lie character of the brackets of the time evolution; and c) the operations on the Hilbert space, such as transpose, hermiticity, unitarity, etc.*

A realization of the above operator isotopy was identified by Santilli in 1978 [1], [2] apparently for the first time. Let $\hat{\xi}$ be an isotope of ξ with product

$$\hat{\xi} : A * B \stackrel{\text{def}}{=} ATB \, , \tag{1.34}$$

where T is a generic, Hermitian, invertible and fixed, but otherwise arbitrary operator. The lifting $AB \rightarrow A*B$ evidently implies the underlying mapping of the unit, from the original trivial unit of $\xi, I = \text{diag}(1, 1, ..., 1)$, into the nontrivial operator unit $\hat{I} = T^{-1}$, called *isounit*, according to the rule

$$\hat{\xi} : \hat{I} * A = A * \hat{I} = A, \quad A \forall \hat{\xi} \, . \tag{1.35}$$

The antisymmetric algebra $\hat{L}$ attached to the isotope $\hat{\xi}$ is evidently a Lie-isotopic algebra with now familiar form

$$\hat{L} : [A, B]_{\hat{\xi}} = A * B - B * A \, . \tag{1.36}$$

The above generalized structures allowed Santilli to propose the following *Lie-isotopic generalization of Heisenberg's equation* (ref. [2], p. 752)

$$i\dot{A} = [A, H]_{\hat{\xi}} = A * H - H * A = ATH - HTA, \quad T = T^{+} . \tag{1.37}$$

The remaining realization of Definition 1.3 was accomplished in subsequent years. First, Santilli [7] pointed out the need for a full bimodular (left and right) generalization of the conventional (uni)modular representation theory. These studies lead to the proposal in 1982 by Myung and Santilli [36] of the following generalization of Schrödinger's representation (other attempts, see ref. [13], produced generalized equations not manifestly compatible with isotopy (1.37)).

The analysis was conducted by providing, apparently for the first time, a comprehensive isotopic generalization of conventional operations on a Hilbert space which, along Definition 1.3, were compatible with the iso-Heisenberg's equations.

In order to preserve linearity, the following *isotopic generalization of the field C* (called *isofield*) results to be needed (see ref. [36], pp. 1307-1309)

$$\hat{C} : \{\hat{c} | \hat{c} = c\hat{I}; \quad c \in C; \quad \hat{I} \in \hat{\xi}\} \, . \tag{1.38}$$

The elements $\hat{c}$ of $\hat{C}$ are then called *isonumbers.*

Note that $\hat{C}$ is still a field. Also, the sum in $\hat{C}$ is the conventional one, although the multiplication is isotopic, according to the rule

$$\hat{c}_1 * \hat{c}_2 = c_1 c_2 \hat{I}; \quad \hat{c}_1, \hat{c}_2 \in \hat{C} \, . \tag{1.39}$$

The achievement of compatibility with the iso-Heisenberg's equations requires the lifting of the conventional modular/eigenvalue action on $\mathcal{H}$ into the *isomodular/isoeigenvalue* form

$$\hat{\xi} \times \mathcal{H} \rightarrow \mathcal{H} : H * |a\rangle \stackrel{\text{def}}{=} HT|a\rangle = \hat{c} * |a\rangle = c|a\rangle. \qquad (1.40)$$

Note that the "numbers" of the theory, i.e., the last numbers in the above identities, remain the conventional ones as in Eqs. (1.32).

With these preliminaries, Myung and Santilli presented a generalization of all familiar operations on a *conventional* Hilbert space (see below for generalization of the Hilbert space itself) (*loc. cit.* §II, pp. 1281-1315).

Evidently, we can review here only some of the most relevant operations. Let $\mathcal{H}$ be a conventional Hilbert space with elements $|a\rangle, |b\rangle, \ldots$ and norm (1.28). A linear operator $H \in \hat{\xi}$ on $\mathcal{H}$ is called *isohermitean* iff it verifies the identity

$$H^{\hat{\dagger}} \stackrel{\text{def}}{=} T^{\dagger} H^{\dagger} T^{-1} \equiv H. \qquad (1.41)$$

The eigenvalues of isohermitean operators results to be *isoreal*, i.e., the number at the end of equalities (1.40) is real as in the conventional case.

A linear operator $U \in \hat{\xi}$ on $\mathcal{H}$ is isounitary when it verifies the rule

$$\langle a| * U^{\hat{\dagger}} * U * |b\rangle = \langle a|b\rangle, \qquad (1.42)$$

which holds iff

$$U^{\hat{\dagger}} * U = U * U^{\hat{\dagger}} = \hat{I}; U^{\hat{\dagger}} = U^{-\hat{1}}. \qquad (1.43)$$

Along similar lines, the following generalized properties hold, where conventional symbols denote conventional operations and symbols with a superscript "*hat*" denote generalized operations

$$\widehat{Tr} A = (Tr A)\hat{I}\,, \qquad (1.44.a)$$

$$\widehat{Tr}(A * B) = \widehat{Tr}(B * A)\,, \qquad (1.44.b)$$

$$\widehat{\det} A = \det(AT)\hat{I}\,, \qquad (1.44.c)$$

$$\widehat{\det}(A * B) = (\widehat{\det} A) * (\widehat{\det} B)\,, \qquad (1.44.d)$$

$$\widehat{\det} A^{-\hat{1}} = (\widehat{\det} A)^{-\hat{1}}\,. \qquad (1.44.e)$$

After these preliminary results, Myung and Santilli proposed the following *isotopic lifting of Schrödinger's equation* (also called *iso-Schrödinger's equation*) (ref. [36], p. 1332)

$$i\frac{\partial}{\partial t}|a\rangle = H * |a\rangle \stackrel{\text{def}}{=} HT|a\rangle\,. \qquad (1.45)$$

The equivalence with Eq. (1.37) was proved in *loc. cit* §3.7.

It should be indicated here that Eq.(1.45) was jointly but independently proposed by Mignani (ref. [69], p. 1128), although without the isotopic generalization of linear operations on Hilbert spaces worked out by Myung and Santilli (also, Mignani presented his generalized equations for the broader Lie-admissible level in which the T operator is nonhermitean, thus resulting in different, nonequivalent, left and right isomodular actions. See in this respect also paper [37] by Myung and Santilli).

The above results essentially established the mathematical consistency of the generalized operator theory, under the isotopic generalization of the enveloping associative algebra $\hat{\xi}$, the attached Lie-isotopic algebra $\hat{L}$, and the underlying isofield $\hat{\mathcal{C}}$, while keeping the conventional Hilbert space $\mathcal{H}$ unchanged.

The above operator realization of Definition 1.3 shall be symbolically referred to hereon with the isotopies

$$\begin{cases} \xi \to \hat{\xi}_T \ , \\ \mathcal{C} \to \hat{\mathcal{C}}_T, \quad T = T^\dagger \ , \\ \mathcal{H} \to \mathcal{H} \ , \end{cases} \tag{1.46}$$

where evidently the last mapping is the *identity isotopy.* We should stress that generalized formulations (1.46) are fully consistent on mathematical grounds, even though based on a conventional Hilbert space (see below for physical aspects). Also, we should stress that the Lie character of the formulation is centrally dependent on the (conventional) hermiticity of H on $\mathcal{H}$. In fact, in case T is not Hermitean we have the following pair of iso-Schrödinger's equations

$$i\frac{\partial}{\partial t}|a\rangle = H * |a\rangle = HT|a\rangle \ ,$$

$$\langle a|T^\dagger H = \langle a|\hat{*}H = \langle a|\frac{\partial}{\partial t}i \ ,$$

$$T \neq T^\dagger. \tag{1.47}$$

The generalized form of Heisenberg's equations corresponding to the above equations is then given by

$$i\dot{A} = (A, H) \stackrel{\text{def}}{=} ARH - HSA \ ,$$

$$R = T^\dagger \neq S = T \ , \tag{1.48}$$

which is precisely the yet broader *Lie-admissible generalization of Heisenberg's equation* proposed by Santilli (ref. [2], p. 746).

In summary, operator isotopy (1.46) is centered on the isotopic element T as one additional operator, besides the Hamiltonian, for the characterization of the time evolution laws (1.37) and (1.45), thus broadening substantially the arena of physical applicability of the theory.

Further studies revealed that the new "degree of freedom" characterized by T is still partial, and that an additional degree of freedom exists in the structure of the Hilbert space, with a corresponding further broadening of the representational capabilities of the theory (see §3).

In fact, subsequent studies by Mignani, Myung and Santilli [38] identified the following *isotopic generalization of the Hilbert space* itself (called *isohilbert space*), $\hat{\mathcal{H}}_G$ as the linear vector space with elements $|a\rangle, |b\rangle, \ldots$ and the *isoinner product*

$$\hat{\mathcal{H}}_G : \langle a\hat{|}b\rangle \stackrel{\text{def}}{=} \langle a|G|b\rangle \hat{I} = \hat{c} \in \hat{C} \; , \tag{1.49}$$

where the new operator G is hermitean and positive definite, but otherwise arbitrary. It represents an additional "hidden" degree of freedom of the theory besides that provided by the isotopic element T.

It is easy to check that the inner product (1.48) of the isohilbert space $\hat{\mathcal{H}}_G$ obeys all conditions which are used to define an abstract Hilbert space. So the isohilbert space $\hat{\mathcal{H}}_G$ may be thought of as an extended realization of the conventional Hilbert space $\mathcal{H}$ of quantum mechanics, with G being an integration measure. The two spaces are isometric to each other.

It is instructive also to verify that the following generalized Schwarz inequality holds $|\langle a\hat{|}b\rangle| \leq ||a||_G||b||_G$, where we have denoted the *isonorm* of a as $||a||_G = \langle a\hat{|}a\rangle^{1/2}$.

Generalization (1.48) demands a further enlargement of linear operations. For instance, the condition of isohermiticity now becomes

$$H^{\hat{\dagger}} = T^{-1}GH^{\dagger}TG^{-1} \equiv H \; . \tag{1.50}$$

The above results are intriguing. In fact, one can see that *for $T = G$ the generalized notion of isohermiticity coincides with the conventional hermiticity*

$$H^{\hat{\dagger}} = T^{-1}TH^{\dagger}TT^{-1} \equiv H^{\dagger}. \tag{1.51}$$

In turn, this has the direct consequence that *the observables of quantum mechanics (Hamiltonian, linear and angular momenta, etc.) remain ob-*

servables under a general isotopy of enveloping associative algebras, fields and Hilbert spaces characterized by the same isotopic element $T = G$.

In summary, the most general known isotopic formulation of operator algebras is characterized by the following liftings

$$\begin{cases} \xi \to \hat{\xi}_T , \\ \mathcal{C} \to \hat{\mathcal{C}}_T, \quad T = T^\dagger , \\ \mathcal{H} \to \hat{\mathcal{H}}_G, \quad G = G^\dagger, \quad G > 0 , \end{cases} \tag{1.52}$$

where, in general, $T \neq G$. In the following we shall however often refer to formulations (1.52) under the specialization $T = G$, owing to their capability to preserve the operation of Hermiticity of quantum mechanics (as well as other operations, see ref. [36]).

The above rudimentary review is sufficient for our purpose here: to show the *mathematical* consistency of the generalization of quantum mechanics characterized by isotopes (1.46) and (1.52). In turn, this implies the existence of a consistent operator realization of Santilli's Lie-isotopic theory. Still, in turn, this property results invaluable for the study of the theory because, as mentioned earlier, isotopes (1.46) or (1.52) provide the most direct possible interpretation of the generalized Lie theory.

A few words on the physical profile are in order here. The generalization of quantum mechanics characterized by isotopies (1.46) and (1.52) was called by Santilli *hadronic mechanics* (ref. [2], p.756) to emphasize the restriction of the intended applicability of the theory only to the *interior* of hadrons, or to the interior of strong interactions at large.

The physical foundations of the proposal are the experimental evidence of the existence, under strong interactions, of necessary conditions of mutual overlapping of the wavepackets of particles (which are generally ignorable under electromagnetic interactions as in the atomic structure). In turn, these interactions are known at the classical level to be:

- a) of *contact* type, in the sense of *zero range*, i.e., not being representable via action-at-a-distance interactions;

- b) of *nonlocal* type, in the sense of occurring throughout a volume, and not being reducible to a finite number of isolated points; and

- c) of *nonhamiltonian* type, in the sense of being, not only of nonpotential type, but actually of being beyond the representational capabilities of a Hamiltonian in the frame of the observer (see monograph

[15] for the violation of the integrability conditions for the existence of a Hamiltonian).

The same properties are evidently expected to remain for particle wavepackets (see Fig. 1).

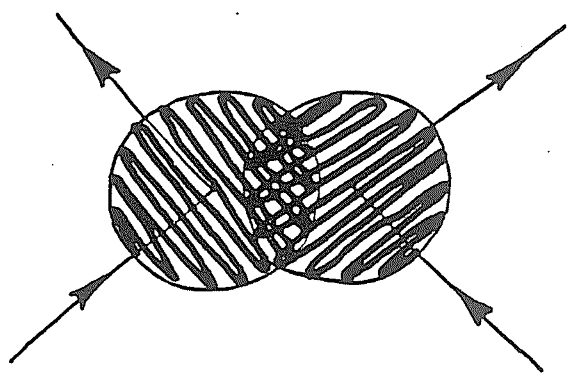


FIGURE 1. A reproduction of the slide presented by Santilli during his invited talk at the *Conference on Differential Geometric Methods in Mathematical Physics* held in Clausthal, West Germany, in 1980. The slide was intended to illustrate, for the distinguished geometers and theoreticians attending the conference, the incontrovertible experimental evidence on the nonlocal nature of the strong interactions as pointed out by the founding fathers of the theory. In fact, all hadrons are not point-like, but have a charge distribution of the order of $1F$ ($= 10^{-13} cm$) which coincides with the range of the strong interactions. Also, all known (massive) particles have a wavepacket which, again, is of the order of $1F$. Thus, a necessary condition to activate the strong interactions is that the particles enter into a state of mutual penetration of their charge distribution and wavepackets. This characterizes interactions which cannot be reduced to a finite number of isolated points, because they occur throughout the volume of mutual penetration/overlapping. Also, the interactions are of contact nature, that is, the nonlocality cannot be represented via a potential of integral type because the integrability conditions for the existence of a Hamiltonian are vi-

olated without, of course, precluding the existence of conventional potential terms. By keeping in mind that all geometries conventionally used nowadays in theoretical physics are of strictly local/differential nature, the slide was intended to stimulate the study of more general, nonlocal (e.g., integrodifferential) geometries for a more adequate representation of the interior strong problem. The Lie-isotopic theory and its various applications reviewed in this work are intended precisely as a first step toward a quantitative representation of the nonlocal/nonhamiltonian character of interior dynamical problems, in which the conventional, potential, local interactions are represented by conventional Hamiltonians, and the nonlocal, integrodifferential, and nonhamiltonian interactions are represented via the generalized unit of the theory. The symbol of overlapping spheres was subsequently assumed by Santilli as the *logo* of *The Institute for Basic Research*, at its inauguration ceremony the following August 1981.

As stressed earlier, *hadronic mechanics is not physically established as of this writing* because a large number of theoretical and experimental studies remain to be done. Nevertheless, hadronic mechanics may be applied also to account for a number of conventional applications, such as: quark confinement, hadronization processes and other cases where the perturbative techniques of QCD are known to fail to achieve a consistent description.

An apparent reason for the current resiliency toward hadronic mechanics is due to the inevitable existence of certain generalizations of basic quantum mechanics laws, such as: Heisenberg's uncertainty principle; Pauli's exclusion principle; the very notion of "particle"; etc.

The reader should however be aware that, as stressed in the literature, these *deviations from conventional quantum mechanical laws are expected only in the interior of hadrons, or in the interior of systems of strongly interacting particles, while conventional quantum mechanical laws are recovered in full for the center-of-mass motion.*

For instance, Mignani, Myung and Santilli [38] proved the validity of the *conventional* uncertainty relations for the center-of-mass motion of a composite system characterized by hadronic mechanics, in a way fully compatible with *generalized* uncertainty relations for each individual constituent. A similar situation has been proved by Santilli [17] for the time reversal, or by Aringazin [51] for Pauli's principle.

These results are important because they establish the fact that essentially *no valid experimental evidence exists at this time for disproving hadronic mechanics*, for the simple reason that all available direct tests for strong interactions are essentially center-of-mass tests. To put it differently, in order to establish experimentally the validity or invalidity of hadronic mechanics, we have to repeat the historical process that lead to the establishing of quantum mechanics. The historical experimental measures conducted for charged particles under *external* electromagnetic interactions, must be repeated, this time, for hadrons under *external* strong interaction. No direct experimental study along the latter lines evidently exists as of this writing.

In the final analysis, readers with an open mind to potentially fundamental advances should notice the evident plausibility of the occurrence: conventional quantum mechanical laws for the center-of-mass motion of hadrons, and generalized hadronic laws for their internal structure.

The physical foundations for this plausibility is provided by another seminal contribution by Santilli, the notion of *closed essentially nonselfadjoint systems*, introduced in 1978 jointly with his algebraic and classical/operational studies [1], [2]. In a few simple words, it is generally believed that the stability of a system is provided by the stability of the orbits of each individual constituent. This is essentially the case of the stability of the solar system as well as of the atomic structure.

Santilli pointed out the existence in Nature of a class of more general systems which verify all total conventional conservation laws for their center-of-mass motion, but the internal equations of motions are nonhamiltonian. (See Fig. 2.)

These broader systems are essentially provided by composite systems with each individual constituent in *unstable* conditions due to exchanges of energy, linear momentum and other physical quantities with the rest of the systems. The point is that these nonconservations are merely internal exchanges under total conserved quantities, the system being, after all, isolated.

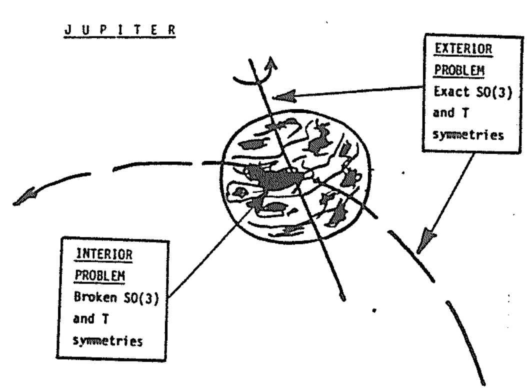


FIGURE 2. A reproduction of Figure 1, page 1208 of ref. [13], depicting a dichotomy of central relevance for the studies under review: the compatibility of the conventional symmetries and physical laws for the center-of-mass motion of celestial bodies (such as Jupiter), with manifest deviations from the same symmetries and physical laws in the interior dynamics. In fact, on one side, we have clear evidence on the stability of Jupiter's orbit in the Solar system with consequential manifest validity of the rotational symmetry for the exterior dynamics; on the other side, we have equally clear evidence for the existence in the interior motion of vortices with continuously varying angular momentum, with consequential internal violation of the rotational symmetry. Similarly, we have a manifestly reversible center-of- mass trajectory, as compared to a manifestly irreversible interior dynamics. A similar situation occurs for all other aspects at all levels of study, as we shall see, including the relativistic and the gravitational level. The dichotomy reviewed here was quantitatively interpreted by Santilli via the notion of closed- isolated systems with nonhamiltonian internal forces (see later on). The above dichotomy also provides the conceptual foundations of the fundamental experiments proposed later on in §3.5 regarding clear phenomenological predictions of apparent violation of Einstein's Special Relativity in the interior of (unstable) hadrons in flight, while the relativity is preserved for center-of-mass motions of the same hadrons, say, when moving in a particle accelerator.

The mathematical consistency of these broader systems at the classical and the operator level was also shown in the original proposals [1,2].

At the classical level, *closed nonhamiltonian* systems are characterized by the Birkhoffian equations (ref. [2], p. 624; see also monograph [15], pp. 234-237)

$$m_k \ddot{\vec{r}}_k = \vec{F}_k^{SA}(\vec{r}) + \vec{F}_k^{NSA}(t, \vec{r}, \dot{\vec{r}}, ...) , \quad (1.53.a)$$

$$\dot{H} = \frac{d}{dt}(T + V) = 0 , \quad (1.53.b)$$

$$\dot{\vec{P}}_{tot} = \frac{d}{dt}(\sum_1^{\mu}{}_k m_k \vec{P}_k) = 0 , \quad (1.53.c)$$

$$\dot{\vec{M}}_{tot} = \frac{d}{dt}(\sum_k \vec{r}_k \wedge \vec{P}_k) = 0 , \quad (1.53.d)$$

$$\dot{\vec{G}}_{tot} = \frac{d}{dt}(\sum_k m_k \vec{r}_k - t\vec{p}_k) = 0 , \quad (1.53.e)$$

where the symbols "SA" ("NSA") indicate verification (violation) of the integrability conditions for the existence of a potential, those of variational self-adjointness.

An intriguing point is that the conventional total conservation laws *are not necessarily* subsidiary constraints to the equations of motion. In fact, Eqs. (1.53.b)-(1.53.e) are verified when

$$\sum_{1}^{n}{}_{k}\vec{F}_k^{NSA} = 0\ ,$$

$$\sum_{1}^{n}{}_{k}\vec{r}_k \wedge \vec{F}_k^{NSA} = 0\ ,$$

$$\sum_{k=1}^{n} \vec{p}_k \cdot \vec{F}_k^{NSA} = 0\ , \tag{1.54}$$

which consist of seven conditions on $3n$ unknown quantities, the components of the nonhamiltonian forces $\vec{F}_k^{NSA}$. Infinite varieties of unconstrained solutions therefore exist for $n \geq 3$. The case $n = 2$ has been proved to be consistent, even though with very special features (e.g., only circular orbits are possible). The case $n = 1$ is impossible for the evident reason that an isolate particle cannot be under nonhamiltonian external forces (see Fig. 3).

CLOSED VARIATIONALLY NONSELF-ADJOINT SYSTEMS
[Isolated systems of extended particles with Hamiltonian and non-Hamiltonian internal forces]

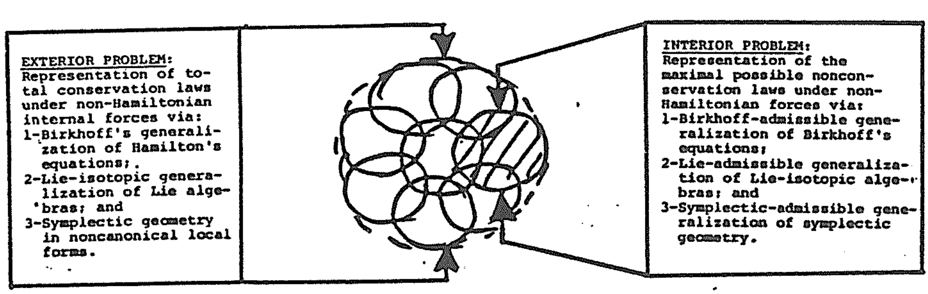


FIGURE 3. A reproduction of Figure 5.1, page 529 of ref. [16], presenting a schematic view of the notion of "closed non-self-adjoint systems" originally proposed in ref. [1], [2] and then investigated at several levels of study by a number of authors (see §1.1). Conventionally, closed-isolated systems are represented by assuming that total, conserved, quantities (such as energy H, angular momentum M, etc.) are the generators of space-time symmetries (translations, rotations, etc.). The assumption of the simplest conceivable Lie product $AB - BA$ then requires the Hamiltonian H to represent all acting internal forces. Additional technical arguments restrict all internal forces to be action-at-a-distance potential/Hamiltonian. Santilli's proposal is to assume the *same* total, conserved physical quantities H, M, etc., as the generators of isotopically lifted space-time symmetries, in which the product is less trivial, e.g., $A * B - B * A = ATB - BTA$. This yields an additional element T, besides the Hamiltonian H, to represent internal forces that are beyond the representational capability of the Hamiltonian (Fig. 1). This results into the covering notion of closed nonhamiltonian systems which are at the foundation of the studies of Lie-isotopy at all levels: Newtonian, relativistic, gravitational, statistical, etc. Remarkably, the space-time symmetries are not broken under the presence of internal nonhamiltonian forces, but merely realized in a structurally more general, but isomorphic way. This important finding was only empirically known in the early stages of the Lie-isotopic theory, and subsequently formalized in ref. [22] (see later on Theorem 2.9). The implications of these results are far reaching at all levels of study. To begin, Santilli has disproved statements such as "breaking of the Lorentz symmetry" or "Lorentz noninvariance," which are technically correct only when specifically referred to the "simplest possible realization of the Lie product $AB - BA$." In fact, Theorem 2.9 allows the reconstruction of the same symmetry as exact at the Lie-isotopic level when broken at the conventional level. Furthermore, the notion under consideration and its underlying Lie-isotopic methods, allow the possibility of constructing genuine covering of contemporary relativities, as we shall see in §3, with far reaching implications in classical as well as particle mechanics. All the above considerations refer to the "exterior problem," here intended as the description of the systems from the exterior with the emphasis on total conservation laws, along the line of monographs [4], [15]. A complementary aspect is the "interior problem" intended as the study of only one constituent of the system when all other constituents are considered as external. The emphasis is now shifted to the maximal possible nonconservation of the physical quantities of each constituent (of course in a way compatible

with total conservation laws), as the best way to maximize internal dynamical conditions. This complementary approach is along the Lie-admissible line of study of monographs [5], [16] which is not reviewed here.

The operator image of systems (1.53) was also identified by Santilli in his second memoirs of 1978. In fact, the operator H in his Eqs. (1.37) represents the total physical energy of the system and it is evidently conserved because of the Lie character of the underlying algebra. We can therefore write the following operator version of systems (1.53)

$$i\dot{H} = [H, H]_{\hat{\xi}} = H * H - H * H \equiv 0 \; ,$$

$$[\vec{P}_{tot}, H]_{\hat{\xi}} = [\vec{M}_{tot}, H]_{\hat{\xi}} = [\vec{G}_{tot}, H]_{\hat{\xi}} = 0 \; . \tag{1.55}$$

Notice that the observability of physical quantities persists because, as recalled earlier, one can select isotopes (1.52) with $T = G$, under which a total Hamiltonian H which is conventionally hermitian in quantum mechanics, remains hermitian in hadronic mechanics. Also, its eigenvalues remain real (although different!) [36].

This confirms the point touched earlier, that the center-of-mass motion of a composite system obeying hadronic mechanics, when inspected from the outside, verifies conventional physical laws. Nevertheless, the system admits in its interior a generalized integrodifferential unit $\hat{I}$ for which conventional physical laws are inapplicable, in favor of suitable covering laws.

In Santilli's words [21], the solar system is a closed Hamiltonian system whereby total stability is provided by the stability of each orbit. The planets, however, possess structures considerably more complex than that. For instance, Jupiter is an example of a closed nonhamiltonian system because, when assumed as isolated from the rest of the solar system, it verifies total conservation laws; yet its internal structure is highly nonconservative, nonhamiltonian (and irreversible).

In the transition to the particle setting, the atomic structure is analytically equivalent to that of the solar system because, again, total stability is provided by the stability of each orbit. Santilli's view is that the hadronic structure is equivalent to that of Jupiter [2], in the sense that each isolated hadron evidently verifies total conservation laws; nevertheless, the internal orbits are expected to be generally nonconservative due to the deep mutual overlapping of the wave packets of the constituents. (See Fig. 4.)

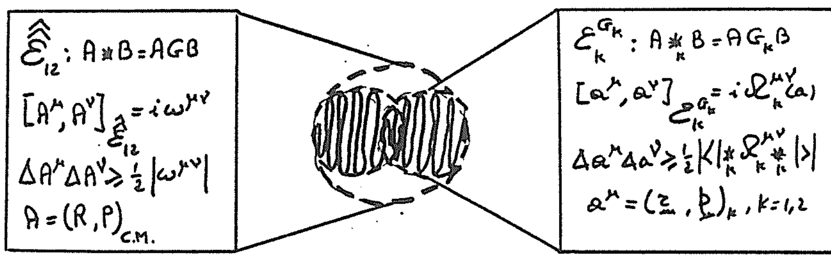


FIGURE 4. A reproduction of Figure 9.1, page 1945 of ref. [38], signaling the achievement of mathematical maturity in the operator formulation of closed nonhamiltonian systems on Hilbert spaces. Ref. [38] established the operator counterpart of the dichotomy of Figs. 2, 3, that is, the validity of conventional quantum mechanical laws for the center-of-mass motion of the state, in a way compatible with structurally more general laws for the interior dynamics. The analysis was presented for the case of Heisenberg's uncertainty principle, with guidelines for the expected extension to all other physical laws and principles of quantum mechanics. In fact, Aringazin [70] has recently proved the same occurrence for Pauli's exclusion principle. These operators results are merely indicated for the purpose of informing the reader on the existence of mathematically consistent operator counterparts of the classical models reviewed in this work, with the hope of reviewing them in detail in a future paper.

Hadronic mechanics was then applied, along the above concept of closed non-Hamiltonian systems, to the construction of a structure model of the π^0 as a generalized bound state of one electron e^- and one positron e^+, although in a mutated state expected from the condition of total mutual immersion of their wavepackets, called *eletons* and denoted with the symbols $\epsilon^\pm$ [2]. The model $\pi^0 = (\epsilon^+, \epsilon^-)$ resulted to be able to represent *all* physical characteristics of the π^0, such as: rest energy, meanlife, charge radius, total charge, spin, magnetic and electric dipole moments, as well as the primary decay. The above hadronic structure model was extended in memoir [2] to all remaining light mesons, resulting in structures of the type $\pi^\pm = (\epsilon^+, \epsilon^\pm, \epsilon^-)$, $K^0 = (\hat{\pi}^+, \hat{\pi}^-)$, $K^\pm = (\hat{\pi}^+, \hat{\pi}^\pm, \hat{\pi}^-)$, where the superscript $\hat{}$ denotes expected mutation of the characteristics of the particles caused by total immersion within the hadronic medium.

Hadronic mechanics was then applied to the quantitative interpretation of Rutherford's historical hypothesis that the neutron n is a "compressed hydrogen atom", along the representation submitted in memoir [2], $n = (p^+, \epsilon^-)$, i.e., as a generalized bound state of an ordinary (unmutated) proton p^+ and a mutated electron ϵ^-. The total angular momentum $\frac{1}{2}$ for state (p^+, ϵ^-) was first achieved in papers [24] via the construction of the $\widehat{SU}(2)$-isotopic spin symmetry and its representations, and resulted to be due to very simple constraints on the *orbital* angular momentum of the electron when "compressed" within the densiest object measured in laboratory until now. The subsequent paper [25] showed that the generalized bound state (p^+, ϵ^-) is capable of representing *all* the characteristics of the neutron, i.e., rest energy, meanlife, charge radius, total charge, spin, magnetic and electric dipole moments as well as its primary decay. A comprehensive presentation of the model is provided in paper [28]. A detailed analysis of the notion of eleton is provided in paper [27] via a generalization of conventional field equations that is invariant under the Poincaré-isotopic symmetry [26].

In this way, Santilli illustrated the possibility of achieving the primary objective for which hadronic mechanics had been suggested: the identification of the hadronic constituents with (massive) physical particles simply produced in the spontaneous decays, under the assumption of obeying a generalized mechanics when in condition of total mutual immersions, and of recovering ordinary quantum mechanics when exiting the hadronic structure.

The full compatibility of this novel structure model of hadrons with established quark models is under study by a number of authors [44,50]. Rather than being in conflict with established theories, hadronic mechanics appear to offer some genuine possibility of resolving their basic problematic aspects, such as: achieving null probability of tunnel effect for free quarks, reaching fractional charges as mutation of ordinary ones, etc.

To summarize our viewpoint, the classical analytical realization of Santilli's isotopies (Birkhoffian mechanics [15]) is nowadays established on both mathematical and physical grounds. The corresponding operator counterpart (hadronic mechanics [36]) is clearly consistent on pure mathematical grounds, but far from being established on physical grounds, although no experimental evidence can be moved against the generalized mechanics at this moment. In the final analysis, the central physical notion of the theory (that of closed nonhamiltonian system) is manifestly plausible for the representation of hadrons, as we shall see better in the final part of this analysis, and, more technically, in a possible subsequent review.

We now briefly review the process of *naive hadronization*, i.e., *the simplest possible mapping of Birkhoffian into hadronic mechanics.* This aspect is important for our analysis because it throws a deeper light in the notion of isounit of the Lie-isotopic theory (besides indicating how diversified the studies of compatibility and consistency have been conducted until now).

The conventional *naive quantization*, i.e., the mapping of classical Hamiltonian into quantum mechanics, can be characterized by the mapping of the action functional A into a *constant unit*, Planck's unit $\hbar = 1$, time $-i \log \psi$, i.e.,

$$A \to -i1 \log \psi \ , \tag{1.56}$$

under which Hamilton-Jacobi Eqs. (1.18) assume the form

$$-\frac{\partial A}{\partial t} = H \to i\frac{\partial}{\partial t}\psi = H_{op} \ ,$$

$$\frac{\partial A}{\partial \vec{r}} = \vec{p} \to -i\frac{1}{\psi}\vec{\nabla}\psi = \vec{p}_{op} \ , \tag{1.57}$$

thus becoming Schrödinger's equations

$$i\frac{\partial}{\partial t}\psi = H\psi \ ,$$

$$-i\vec{\nabla}\psi = \vec{p}\psi. \tag{1.58}$$

Animalu and Santilli [41] pointed out that mapping (1.56) is expected to be insufficient for Pfaffian action principles, because of its inability to provide a representation of the contact/nonlocal/nonhamiltonian forces of the broader systems considered. The authors proposed instead, *as naive rule of hadronization, the mapping of the Pfaffian action functional $\hat{A}$ into the operator unit of the theory*, the isounit of hadronic mechanics $\hat{I}$, time$-i \log \psi$, i.e.,

$$\hat{A} \to -i\hat{I} \log \psi. \tag{1.59}$$

For our needs we now consider the following particularized Pfaffian action

$$\hat{A} = \int_{to}^{t} [M^i_k(\vec{r}, \vec{p}) p_i \dot{r}^k - H(t, \vec{r}, \vec{p})] dt \ ,$$

$$\det(M^i_k) \neq 0 \ , \tag{1.60}$$

with Hamilton-Jacobi equations (which are still of genuine generalized nature, yet of the simpler form)

$$\frac{\partial \hat{A}}{\partial t} = H \ ,$$

$$\frac{\partial \hat{A}}{\partial \vec{r}^k} = M^i_k p_i \ ,$$

$$\frac{\partial A}{\partial \vec{p}_k} = 0 \ . \tag{1.61}$$

The application of mapping (1.59) to Eqs. (1.61) then yields the forms [41]

$$-\frac{\partial \hat{A}}{\partial t} = H \rightarrow i(\frac{\partial \hat{I}}{\partial t})\log\psi + i\hat{I}\frac{\partial}{\partial t}\psi = H^{op} \ ,$$

$$\frac{\partial \hat{A}}{\partial \vec{r}_k} = -i(\nabla_k \hat{I})\log\psi - i\hat{I}\vec{\nabla}_k\psi = M^i_k \vec{p}^{op}_i \ , \tag{1.62}$$

which can be rewritten

$$i\frac{\partial}{\partial t}\psi = [H - i\frac{\partial \hat{I}}{\partial t}\log\psi] * \psi \overset{\text{def}}{=} H^{\text{eff}} * \psi \ ,$$

$$-\, i\vec{\nabla}_k\psi = [M^i_k \vec{p}_i + i(\vec{\nabla}_k \hat{I})\log\psi] * \psi \overset{\text{def}}{=} M^i_k \vec{P}^{\text{eff}}_i * \psi \ , \tag{1.63}$$

yielding precisely the iso-Schrödinger's Eqs. (1.45), plus corresponding equations for the linear momentum. Notice the natural appearance under hadronization of a nonlinearity in the wavefunctions, besides additional nonlinearities emerging from the arbitrary functional dependence of the isotopic element (see below).

A mathematically rigorous formulation of hadronization was achieved by (the mathematician) E. B. Lin [63] via the methods of symplectic quantization. Recall that the Birkhoffian mechanics can be constructed via *noncanonical* transformations of Hamiltonian mechanics (and remains form-invariant under these general transformations). Along parallel lines, hadronic mechanics can be constructed via *nonunitary* transformations of quantum mechanics (and also remains form-invariant under the most general possible transformations) [6]. Lin essentially shows that the lifting of conventional, symplectic quantization techniques (e.g., prequantization) characterized by noncanonical (nonunitary) transformations provides precisely the desired hadronization, as expected.

This completes the objective of this section, to show that the classical and operator realizations of the notion of analytic isotopy, not only are individually consistent, but admit a consistent mapping of the former into the latter, the entire process constituting a true generalization of conventional theories.

A few comments are now in order. Evidently, the assumption of the simpler Pfaffian form (1.60) has the objective of rendering the generalized action functional independent of the linear momentum. This, in turn, allows the construction of an operator image in which the wavefunction has the familiar functional dependence $\psi(t, \vec{r})$ without a dependence on the momentum.

A personal communication by Santilli confirms the rather vast capabilities of action (1.60) to represent nonhamiltonian interactions, once the several degrees of freedom of Birkhoffian mechanics are taken into consideration (ref. [15], pp. 54-67). Nevertheless, Santilli stresses the fact that, unlike the case for general action (1.20), the direct universality of the reduced form (1.60) has not been proved as of today. In case action (1.60) does not result to be directly universal, the construction of a "wave mechanics" with "wavefunction" dependent also in the momentum, $\psi(t, \vec{r}, \vec{p})$, is inevitable.

Second, hadronization (1.62) indicates the *intrinsic nonlinearity of hadronic mechanics*, where the nonlinearity is referred also to the dependence of the equations of motion in the wavefunctions. As a matter of fact, the iso-Schrödinger's equation in its original formulation by Santilli, that in term of the *Birkhoffian operator B* [6], is the most general nonlinear as well as nonlocal equation of motion in operator form know until now. We shall write it in the explicit form

$$i\frac{\partial}{\partial t}\psi = B\hat{*}\psi = B(t, a, \psi, \psi^\dagger, ...)D(t, a, \psi, \psi^\dagger, ...)\psi \,. \qquad (1.64)$$

All known equations, nonlinear in the wavefunctions as well as in other quantities, are evidently a particular case of the above equation.

We are referring here to the *direct universality of hadronic mechanics*, i.e., the capability of representing *all* conceivable nonlinear and nonlocal equations verifying certain topological restrictions (universality) in the frame of the observer (direct universality). This is merely the operator counterpart of the classical direct universality of Birkhoffian mechanics [15].

The proof of this important property is quite easy. Recall that the universality of Birkhoff's equations ultimately results from the form-invariance of the theory under the most general possible (noncanonical) transformations. The direct universality of the iso-Heisenberg's or the iso-Schrödinger's equations then follows from their form-invariance under the most general possible (evidently nonunitary) transformations.

As an example, it is an instructive exercise for the interested reader to show that certain nonlinear wave equations currently under investigation by Weinberg [71] and others (to explore a possible nonlinearity of quantum mechanics) of the type

$$i\frac{\partial}{\partial t}\psi = \frac{\partial}{\partial \psi^\dagger}H(\psi\psi^\dagger, ...) \tag{1.65}$$

are in fact a particular case of hadronic mechanics, i.e., they can always be rewritten into an equivalent isomodular form (1.64).

But there is more. The direct universality of the theory, combined with its isotopic structure, have rather profound epistemological implications for the very notion of nonlinearity.

This is another central aspect of the Lie-isotopic theory we shall consider in more detail later on, when reviewing the isotransformation theory in the next chapter. At this point we can limit ourselves to the remark that the isotopic element D of Eq. (1.64) is arbitrary. As a result, all nonlinear terms, whether in the wavefunctions or in the other quantities, can be incorporated in the isotopic element, in which case the (nonlinear) Birkhoffian operator B is replaced by a linear Hamiltonian H, and we shall write

$$\begin{aligned} i\frac{\partial}{\partial t}\psi &= B(t,a,\psi,\psi^\dagger ...)D(t,a,\psi,\psi^\dagger, ...)\psi \\ &\equiv H(t,a)T(t,a,\psi,\psi^\dagger, ...)\psi \\ &\equiv H * \psi \,. \end{aligned} \tag{1.66}$$

The implications of the above results are rather deep. They essentially establish that, not only we have a direct universality for all possible nonlinear (and nonlocal) theories, but in addition *any possible nonlinear (and nonlocal) theory can always be rewritten in an equivalent isolinear form.* It is regrettable that the authors of studies [71] do not appear to be aware of the Lie-isotopic theory, because the intrinsic isolinear structure of Weinberg's equation (1.64) may evidently void most of their argumentations.

This is the technical reason why Santilli (private communication) does not consider *nonlinearity* a structure characterizing feature. Instead, he considers structurally fundamental the *nonlocality* and *nonhamiltonian* character caused by the deep mutual overlapping of the wave packets of strongly interacting particles.

Regrettably, we cannot enter into a detailed analysis of the implication of the isotransformation theory for Weinberg's work because this is substantially outside the scope of this review. Nevertheless, the above occurrence is important to point out the rather deep implications of the Lie-isotopic theory for a virtually endless variety of frameworks in classical, operator and other branches of physics.

In addition to the above, Weinberg's nonlinear generalization of quantum mechanics [71] is apparently afflicted by rather fundamental problematic aspects [43] essentially caused by the fact that it is based on a general, nonassociative, Lie-admissible generalization of the conventional associative envelope of quantum mechanics. These algebras are known not to possess a consistent unit [1]. As a result, all basic physical laws and quantities of quantum mechanics that are central dependent on the unit (1.29) do not possess a consistent formulation in Weinberg's theory. This is the case for the measurement theory, the notion of quantum of energy, the Casimir invariants, etc. Moreover, the nonassociative character of the underlying envelope activates the inconsistency theorems by Okubo [53] on nonassociative generalizations of Schrödinger's equations precisely of type (1.65). Finally, such a nonassociative character of the operator algebra prevents the equivalence between the Heisenberg-type and the Schrödinger-type representations in Weinberg's theory [43].

These problematic aspects have been mentioned here to point out the fact that they are all resolved by Santilli's central assumption for the construction of hadronic mechanics; the existence of the generalized unit (1.35). The occurrence is also useful to illustrate the central role of the *preservation of the associative character of the envelope*, Eq. (1.34). In fact, general Lie-admissible algebras do enter in hadronic mechanics, but for the characterization of the *brackets of the time evolution* for the exterior-open problem, while the underlying envelope remains associative. In turn, the preservation of such an underlying iso-Heisenberg and iso-Schrödinger's representations [36], and the resolution of the other problematic aspects of Weinberg's formulation.

Another aspect that is worth mentioning is the use of the iso- Schrödinger's equation for a deeper understanding of the Berry's phase [72], as studied by Mignani [73].

Next, we want to point out a fundamental feature of hadronization (1.59), according to which *the isotopic lifting of quantum mechanics is essentially centered on the replacement of Planck's constant unit* $\hbar = 1$ *with* *the operator isounit $\hat{I}$*

$$\hbar(=1) \rightarrow \hat{I}(t, a, \psi, \psi^{\dagger}, ...). \tag{1.67}$$

In turn this provides another illustration of the intriguing physical implications of the Lie-isotopic theory in general, and of Santilli's notion of generalized unit [1,2], in particular.

The epistemological implications of concept (1.67) are self-evident. They are essentially centered on the expectation that the quantum of energy, while so effective for the area of its original conception (discrete energy states of the individual electrons of the atomic structure), is expected to be insufficient for the representation of the nonlocal and nonhamiltonian conditions of wavepackets in deep mutual immersion.

This is one of the reasons why Santilli carefully avoids the use of the terms "quantization" or "quantum mechanics" when referring to the operator mechanics characterized by the Lie-isotopic theory.

We now close these analytic comments with the indication of the fact that *the Birkhoffian and hadronic mechanics constitute genuine coverings of their original counterparts, the Hamiltonian and quantum mechanics*, in the sense that:

1. the generalized theories are conceived for physical conditions intrinsically more general than those of the original theories (essentially nonhamiltonian interactions);

2. the generalized theories are constructed with mathematical methods essentially more general than those of conventional theories (Lie-isotopic methods); and

3. the generalized theories are capable of approximating the conventional ones as close as desired, e.g., for

$$\Omega \approx \omega \text{ or } \hat{I} \approx \hbar\,, \tag{1.68}$$

and they recover the conventional theories identically when all the nonhamiltonian interactions are null, e.g., for

$$\Omega \equiv \omega \text{ or } \hat{I} \equiv \hbar. \tag{1.69}$$

1.4 The Notion of Geometrical Isotopy

We now briefly touch upon another notion of isotopy, this time at the *geometrical* level.

Let M be an n-dimensional C^∞-manifold with local coordinates r_k, $k = 1, 2, ..., n$, and let T^*M be its cotangent bundle with local coordinates a^μ, $\mu = 1, 2, ..., 2n$, $a = (r, p)$. The familiar *canonical one-form* on T^*M can then be written

$$\theta_1 = p_k dr^k \equiv R^o_\mu(a) da^\mu \ , \qquad (1.70)$$

where one recognizes the same R^o as that of Eqs. (1.15).

The *fundamental symplectic two-form* on T^*M can then be written

$$\theta_2 = d\theta_1 = dp_k \wedge dr^k = \frac{1}{2}\omega_{\mu\nu} da^\mu \wedge da^\nu \ , \qquad (1.71)$$

where $\omega_{\mu\nu}$ is the covariant tensor of Eqs. (1.17).

Form (1.71) is nowhere degenerate and "closed" (in the geometrical sense that $d\theta_2 = 0$). The space T^*M, when equipped with the form θ_2, becomes a symplectic manifold in the local canonical coordinates $a = (r, p)$. All the several aspects of the symplectic geometry then follow (see, e.g., ref. [74]).

> *DEFINITION 1.4 [1],[15]* (Geometric Isotopy): *An* isotopic mapping *(or* image *or* lifting*) of a symplectic manifold with fundamental two-form (1.71) is any mapping in the same local chart that preserves the symplectic character of the two-form, i.e., its closed and nowhere degenerate character, but remains otherwise arbitrary.*

Evidently, Birkhoff's equations characterize, not only a Lie-algebra isotopy (in their contravariant form), but also a corresponding symplectic isotopy (in their covariant form).

In fact, the canonical one form (1.70) is replaced by the Pfaffian one-form

$$\hat{\theta}_1 = R_\mu(a) da^\mu. \qquad (1.72)$$

The associates two form

$$\hat{\theta}_2 = \frac{1}{2}\Omega_{\mu\nu}(a) da^\mu \wedge da^\nu \ , \qquad (1.73)$$

where the tensor $\Omega_{\mu\nu}$ is given by Eqs. (1.21b), is also closed and nondegenerate [15]. As such, the Birkhoffian two-form (1.73) characterizes the most general possible symplectic two-form in local coordinates. The direct universality of the symplectic geometry in classical mechanics then follows from that of Birkhoff's equations. This is another important result of monograph [15].

The implications of the above geometrical aspects are far reaching.

Recall that, at the abstract, coordinate-free level, all symplectic two-forms coincide. The differentiations merely emerge in local realizations, the canonical two-form being the simplest conceivable one, while the Birkhoffian two-form being the most general possible one.

Exactly the same results occur at the analytic level. In fact, Hamiltonian and Birkhoffian mechanics coincide at the abstract, coordinate-free level [15]. As a matter of fact, the latter has been constructed by Santilli precisely under the condition of coinciding with Hamiltonian mechanics at the abstract coordinate-free level.

We can therefore expect a similar occurrence at the algebraic level too. In fact, the Lie-isotopic theory has been proposed and constructed precisely in such a way to coincide with the conventional formulation at the abstract coordinate-free level. The differences merely occur in local charts: the conventional formulation of Lie's theory is the simplest conceivable one, ultimately equivalent to the canonical, analytic-geometrical counterpart. Santilli's Lie-isotopic realization is the most general possible form, which is ultimately equivalent to the Birkhoffian analytic-geometrical counterpart.

This final unity of vision is, in turn, fundamental for understanding Santilli's capability of reconstructing at the higher Lie-isotopic level, exact space-time symmetries (e.g., the rotational, Galilean and Lorentz symmetries) when conventionally broken within the context of their simplest possible realizations. The review of this occurrence is, after all, a central objective of this presentation.

1.5 Final Introductory Remarks

A few final remarks appear to be recommendable to prevent possible misrepresentations of this review.

Recall that all simple Lie algebras (over a field of characteristic zero) have been classified by Cartan a long time ago and are today well known. Thus, the reader should *not* expect new simple algebras from the Lie-isotopic lifting of the conventional Lie's theory.

Rather than looking for new algebras (or groups), the scope of the Lie-isotopic theory is that of identifying new, structurally more general realizations of known algebras (or groups).

As we shall see, the Lie-isotopic theory permits in fact the identification of a generally infinite family of physically different symmetry transformations which are all representations of the same simple, abstract, algebra.

Also, readers may tend to expect that all conventional methods currently available for Lie algebras (such as the representation theory) are directly applicable to any Lie theory, thus including the Lie-isotopic one.

This second, rather natural expectation can be readily disproved by noting that a compact Lie algebra (or group) can be turned into a noncompact form under isotopic lifting, evidently depending on the topology of the assumed isounit $\hat{I}$. Available methods, such as the representation theory for compact algebras (groups), are known not to be directly applicable for noncompact structures. A reinspection of the representation theory is then in order.

Rather than having preconceived assumptions, the reader is encouraged to enter into the study of Lie-isotopic algebras with an open mind, and the expectation that all the various methodological aspects worked out for Lie's theory must be reinspected and eventually reformulated for the covering Lie-isotopic theory.

Our final introductory remark is that Santilli's Lie-isotopic theory, despite its beauty, is far from being the ultimate Lie theory, as stressed by the author himself. This point is illustrated quite vividly by the classical Hamiltonian mechanics, because the conventional Poisson brackets have the structure [1]

$$L : [A,B]_U = \frac{\partial A}{\partial r^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k}\frac{\partial A}{\partial p_k} \stackrel{\text{def}}{=} (A,B) - (B,A) = \text{Lie}\ ,$$

$$U : (A,B) = \frac{\partial A}{\partial r^k}\frac{\partial B}{\partial p_k} = \text{ Nonassociative Lie-admissible}\ , \tag{1.74}$$

namely, the Lie algebra L of the Poisson brackets is the anticommutative algebra attached to a *nonassociative* algebra U evidently because

$$U : ((A,B),C) \neq (A,(B,C)). \tag{1.75}$$

In particular the algebra U results to be a *nonassociative Lie-admissible algebra* precisely because (as per definition of these algebras) its attached algebra $[A,B]_U$ is Lie. The same result evidently persists at the Birkhoffian level (ref. [15], p. 152).

By comparison, the algebraic structure of the conventional Heisenberg's brackets is given by

$$L : [A, B]_\varepsilon = AB - BA = \text{ Lie },$$

$$\xi : AB = \text{ Associative Lie-admissible }, \tag{1.76}$$

namely, the Lie algebra L of conventional quantum mechanics is the anti-commutative algebra attached to an *associative* algebra ξ which, as such, is also Lie-admissible.

The physical and mathematical implications of the above findings are predictably deep. On physical grounds, we have to expect problematic aspects in the quantization of conventional Hamiltonian mechanics, for the evident reason that a mapping of a nonassociative envelope U into an associative form ξ simply cannot be formulated in a consistent way (see ref. [6] for a study of this aspect).

This problematic aspect can be readily avoided in hadronic mechanics because Santilli's Lie-isotopic brackets can always be formulated according to the structure [2]

$$\hat{L} : [A, B]_U = ATB - BTA \stackrel{\text{def}}{=} (A, B) - (B, A) = \text{Lie} - \text{isotopic},$$

$$U : (A, B) = ARB - BSA = \text{ Nonassociative Lie-admissible },$$

$$T = R + S, \tag{1.77}$$

namely, a Lie-isotopic algebra, owing to its nontriviality, can always be reformulated as the antisymmetric algebra attached to a nonassociative Lie-admissible algebra. Consistency of algebraic structures with the classical counterpart (1.74) is then regained.

On mathematical grounds, the above findings establish the fact that the most *general possible formulation of Lie's theory is that via nonassociative envelopes*, along the conceptual lines so clearly expressed by the Poisson bracket, Eq. (1.74).

This is the reason why Santilli provided his primary efforts for the formulation of the theory at the nonassociative Lie-admissible level, and presented his Lie-isotopic studies only as a simpler particularization. It is remarkable that these so fundamental structures, so clearly embedded in the structure of the conventional Poisson brackets, had escaped attention in the mathematical and physical literatures until the appearance in 1978 of ref. [1,2,3].

This review is restricted to *associative Lie-admissible formulations*, although in their most general known form. The covering *nonassociative Lie-admissible formulations* shall be ignored hereon, and referred to a possible future review.

2 THE MATHEMATICAL FOUNDATIONS OF THE THEORY

2.1 Central Role of the Universal Enveloping Algebra

Let us begin by recalling the central role for Lie's theory of the universal enveloping algebra. This role is somewhat de-emphasized in the contemporary physical literature, but not in the mathematical one. We shall closely follow in this review the presentation of monograph [15], pp. 148-154.

The terms "Lie's theory" are referred today to an articulated body of sophisticated mathematical tools encompassing several disciplines. Whether in functional analysis or in the theory of linear operators, the structure of the contemporary formulation of Lie's theory can be reduced to the following three parts:

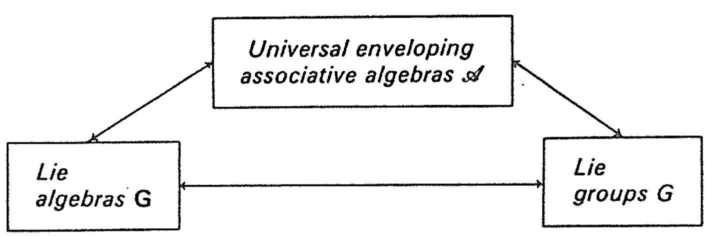


FIGURE 5. The structure of the conventional formulation of Lie's theory with the emphasis on its central mathematical structure, the universal enveloping associative algebra. The Lie-isotopic theory follows exactly the same lines, beginning with the generalization of the envelope and then following with the consequential generalization of all remaining aspects of the theory.

As duly emphasized in the mathematical literature (see, for instance, Jacobson [75], Dixmier [76], and others), a truly fundamental part of Lie's theory is the enveloping algebra ξ. In fact, the algebra ξ provides a symbiotic characterization of both the Lie algebras and the Lie groups. This is due to the fact that the basis of ξ (which is constructed via the Poincaré-Birkhoff-Witt Theorem, to be reviewed in the next section) is given by an infinite number of suitable polynomial powers of the generators X_i of $\mathbf{G}$ of the type

$$\xi : 1 \in \mathbf{F}; \quad X_i; X_i X_j (i \leq j); \quad X_i X_i X_k (i \leq j \leq k); \dots , \tag{2.1}$$

where the products $X_i\ X_j$, etc., are associative. It then follows that the Lie algebra $\mathbf{G}$

$$\mathbf{G} : [X_i, X_j] = X_i X_j - X_j X_j = C_{ij}^k X_k, \tag{2.2}$$

is (homomorphic to) the attached algebras ξ^- of ξ. The Lie group G of $\mathbf{G}$ is then the infinite power series

$$G : e^{\theta^k X_k} = 1 + \frac{\theta^k}{1!} X_k + \frac{\theta^i \theta^j}{2!} X_i X_j + \dots , \tag{2.3}$$

which, evidently, can be properly defined and treated only in the enveloping algebra (note that all terms from $X_i X_j$ on are *outside* the Lie algebra). One can then see why fundamental aspects of Lie *algebras* (such as the representation theory) are treated by mathematicians within the context of its *enveloping algebra.*

On physical grounds, the role of the enveloping algebra is equally crucial. For instance, a frequent physical problem is the computation of the magnitude of physical quantities such as the angular momentum operator M^2. While the components M_i of M are elements of the Lie algebra $\mathbf{SO}(3)$, the quantity M^2 is *outside* $\mathbf{SO}(3)$ and can only be defined in the (center of) the enveloping algebra $\xi(\mathbf{SO}(3))$. Thus, while the Lie algebra $\mathbf{SO}(3)$ essentially characterizes the components of the angular momentum and their commutation rules, the envelope $\xi(\mathbf{SO}(3))$ characterizes: 1) the components M_i; 2) their commutations relations via the attached rule $\xi^- \approx \mathbf{SO}(3)$; 3) the magnitude of the angular momentum M^2; 4) the exponentiation to the Lie group of rotations; 5) the representation theory, etc. Also, enveloping algebras play a central role in quantization at large and, specifically, in the quantization of Lie algebras and Lie groups. In short, we can state that a *truly primitive part of the contemporary formulation of Lie's theory is its universal enveloping associative algebra.*

Once the mathematical and physical origins of this occurrence are understood in full, one can easily see how any consistent generalization of the enveloping associative algebra ultimately provides a generalization of the conventional formulation of Lie's theory.

The physical motivations for this study have been pointed out in Chapter 1, and are provided by the fact that Lie algebras characterize the fundamental equations of physical theories, their time evolution. Any generalization of Lie's theory then inevitably implies the achievement of broader physical capabilities.

The mathematical motivations of the study are equally evident. In the mathematical tradition, the efforts are devoted to the formulation of theories in their most general possible form. This is typically the case for mathematical formulations such as the symplectic geometry [74], which has indeed achieved its broadest possible formulation. It is a truism to say that a similar situation within the context of Lie's theory was not in existence prior to Santilli's studies of 1978, owing to the rather general referral of the enveloping algebra, not only to its associative form, but actually to such form in its simplest possible formulation.

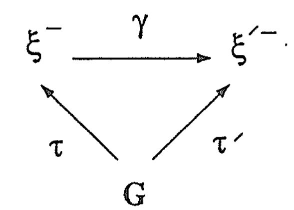
In the next section we shall review Santilli's studies toward a broader formulation of Lie's theory, beginning with the isotopic lifting of its enveloping algebra which admit a consistent, generalized, left and right unit (with the understanding that the still broader nonassociative envelopes [1] will not be considered). The reader should be aware that we shall follow Santilli's original presentation as close as possible.

2.2 Isotopic Lifting of the Universal Enveloping Associative Algebra [1], [15]

In this section we shall first review the definition of universal enveloping associative algebra and the methods for the construction of its basis according to the Poincaré-Birkhoff-Witt theorem [75]. We shall then present their *isotopic* liftings, that is, generalizations which preserve the associative character of the product. By keeping in mind the primitive character of the enveloping algebra in Lie's theory, the generalization presented in this section renders inevitable a corresponding reinspection of Lie algebras and of Lie groups.

DEFINITION 2.1 [75]: The universal enveloping associative algebra *of a Lie algebra* $\mathbf{G}$ *is the set* (ξ, τ) *where* ξ *is an associative algebra and* τ *a homomorphism of* $\mathbf{G}$ *into the attached algebra* ξ^- *of* ξ *satisfying the following properties. If* ξ' *is another associative algebra and* τ' *a homomorphism of* $\mathbf{G}$ *into* ξ', *a unique homomorphism* γ *of* ξ *into* ξ' *exists such that* $\tau' = \tau\gamma$; *i.e., the following diagram (2.4) is commutative.*

Whenever an algebra ξ belongs to the content of the definition above, we shall write $\xi(\mathbf{G})$. All Lie algebras are assumed, for simplicity, to be finite-dimensional. Also all algebras and fields are assumed to have characteristic zero, and the basis of all Lie algebras is ordered.



(2.4)

The construction of the enveloping algebra $\xi(\mathbf{G})$ is conducted as follows. Consider the algebra $\mathbf{G}$ as a (linear) vector space with basis given by the (ordered set of) generators X_i, $i = 1, 2, ..., m$. The *tensorial product* $\mathbf{G} \otimes \mathbf{G}$ is the ordinary Kronecker (or direct) product of $\mathbf{G}$ with itself as a vector space. Such a tensorial product constitutes an algebra because it satisfies the distributive and scalar laws. Also, the algebra is associative because the Kronecker product is associative. A general form of associative, *tensor* algebra which can be constructed on $\mathbf{G}$ as vector space is given by

$$\mathcal{F} = \mathbf{F}1 \oplus \mathbf{G} \oplus \mathbf{G} \otimes \mathbf{G} \oplus \mathbf{G} \otimes \mathbf{G} \otimes \mathbf{G} \oplus \ldots, \quad (2.5)$$

where F is the base field and $\oplus$ denotes the direct sum. Let $\mathcal{R}$ be the ideal generated by all elements of the form

$$[X_i, X_j] - (X_i \otimes X_j - X_j \otimes X_i), \quad (2.6)$$

where $[X_i, X_j]$ is the product of $\mathbf{G}$. Then, the universal enveloping algebra $\xi(\mathbf{G})$ of $\mathbf{G}$ is given (or, equivalently, can be defined) by the quotient

$$\xi(\mathbf{G}) = \mathcal{F}/\mathcal{R}. \quad (2.7)$$

It is possible to prove that the algebra (2.7) satisfies all the conditions of Definition 2.1 (see, for instance, Jacobson [75]).

Of utmost importance for mathematical and physical considerations is the identification of the basis of $\xi(\mathbf{G})$. The quantities

$$M_s = X_{i_1} \otimes X_{i_2} \otimes ... \otimes X_{i_s} \quad (2.8)$$

are called *standard (nonstandard) monomials* of order s depending on whether the ordering

$$i_1 \leq i_2 \leq ... \leq i_s \quad (2.9)$$

is verified (not verified). It is possible to prove that every element of $\xi(\mathbf{G})$ can be reduced to an F-linear combination of standard monomials and (cosets of) 1. This yields the following fundamental theorem on enveloping associative algebras.

Theorem 2.1 *(Poincaré-Birkhoff-Witt Theorem [75]): The cosets of* 1 *and the standard monomials form a basis of the universal enveloping associative algebra $\xi(\mathbf{G})$ of a Lie algebra* $\mathbf{G}$.

The associative envelope $\xi(\mathbf{G})$, as presented, is still abstract in the sense that the product of $\xi(\mathbf{G})$ is the tensorial product $X_i \otimes X_j$, while the product used in physical (e.g., quantum mechanical) applications is the conventional associative product $X_i X_j$. Consider then the algebra

$$A(\mathbf{G}) = F1 \oplus A^{(1)} \oplus A^{(2)} \oplus \dots ,$$

$$A^{(s)} = X_{i_1}, \quad X_{i_2} \dots X_{i_s}, \quad i_1 \leq i_2 \leq \dots \leq i_s . \tag{2.10}$$

It is possible to prove that $\xi(\mathbf{G})$ is homomorphic to $A(\mathbf{G})$, in line with Definition 2.1. Thus, the algebra $A(\mathbf{G})$ can be assumed as the universal enveloping associative algebra of $\mathbf{G}$ with basis

$$1, \quad X_i, \quad X_{i_1} X_{i_2}, \quad X_{i_1} X_{i_2} X_{i_3}, \quad \dots,$$

$$i_1 \leq i_2, \quad i_1 \leq i_2 \leq i_3, \tag{2.11}$$

and arbitrary elements

$$X_{i_1}^{k_1} X_{i_2}^{k_2} \dots X_{i_s}^{k_s} , \tag{2.12}$$

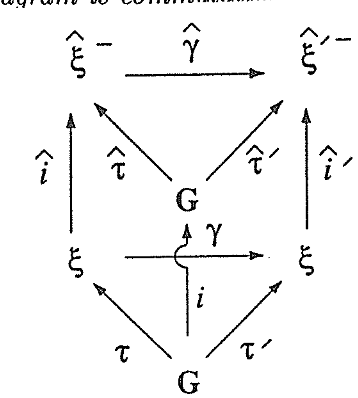
where the X's are the generators of $\mathbf{G}$. Notice that $A(\mathbf{G})$ is infinite-dimensional. The *center* of $A(\mathbf{G})$ is the set of all polynomials $P(X)$ verifying the property

$$[P(X), X_i]_A = 0 , \tag{2.13}$$

for all elements $X_i \in \mathbf{G}$. Most important elements of the center are the so-called *Casimir invariants* of $\mathbf{G}$. For additional study, we refer the interested reader to the mathematical literature on the topic [75],[76]. We move now to the identification of the desired associative-isotopic generalization

DEFINITION 2.2 [1], [15]: The isotopically mapped universal enveloping associative algebra of a Lie algebra $\mathbf{G}$ *is the set*

$((\xi,\tau),(\hat{\xi},i,\hat{\tau}))$ where: (1) (ξ,τ) is the universal enveloping associative algebra as per Definition 2.1; (2) i is an isotopic mapping of $\mathrm{G}, i\mathrm{G} = \hat{\mathrm{G}}$; (3) $\hat{\xi}$ is an associative algebra generally nonisomorphic to ξ; and (4) $\hat{\tau}$ is a homomorphism of $\hat{\mathrm{G}}$ into $\hat{\xi}^-$ such that the following properties are verified. If $\hat{\xi}'$ is still another associative algebra and $\hat{\tau}'$ a homomorphism of $\hat{\mathrm{G}}$ into $\hat{\xi}'$, a unique homomorphism $\hat{\gamma}$ of $\hat{\xi}$ into $\hat{\xi}'$ exists such that $\hat{\tau}' = \hat{\gamma}\hat{\tau}$, and two unique isotopies $\hat{i}$ and $\hat{i}'$ exist for which $\hat{i}\xi = \hat{\xi}$ and $\hat{i}'\xi' = \hat{\xi}'$, i.e., the following diagram is commutative



(2.14)

Whenever an algebra $\hat{\xi}$ verifies the conditions of the definition above, we write $\hat{\xi}(\mathrm{G})$. Again, for simplicity, we assume that all Lie algebras are finite-dimensional, all algebras and fields have characteristic zero, and all Lie algebra bases are ordered.

We are now in a position to elaborate on the insufficiency of Definition 2.1, and the need of Definition 2.2. We shall indicate first the mathematical aspect and then point out the physical profile.

The main idea of Definition 2.1 is, beginning with the basis of a Lie algebra G, to construct an enveloping algebra $\xi(\mathrm{G})$ such that $[\xi(\mathrm{G})]^- \approx \mathrm{G}$. The more general idea of Definition 2.2 is, beginning also with the basis of a Lie algebra G, to construct an enveloping algebra $\hat{\xi}(\mathrm{G})$ such that the attached algebra $[\hat{\xi}(\mathrm{G})]^-$ *is not*, in general, isomorphic to G but rather is isomorphic to an isotope $\hat{\mathrm{G}}$ of G, and we write [48]

$$[\hat{\xi}(\mathrm{G})]^- \approx \hat{\mathrm{G}} \not\approx \mathrm{G}. \tag{2.15}$$

The lack of unique association of a given basis with the envelope then ensures freedom in the realization of the associative product. Equivalently, we can say that within the context of Definition 2.1, a given basis essentially yields a single unique enveloping algebra and thus a single unique attached Lie algebra. On the contrary, within the context of Definition 2.2, a given basis yields all possible enveloping algebras and thus all possible Lie algebras of the same dimension, as we shall see. Still equivalently, we can say that, as is conventional in the contemporary formulation of Lie's theory, nonisomorphic Lie algebras are expressed via the use of *different generators* and the *same Lie product*. On the contrary, within the context of the isotopic formulation of Lie's theory, nonisomorphic Lie algebras can be obtained via the use of the *same basis* and *different Lie products*. We can therefore state that all possible enveloping associative algebras can indeed be introduced according to Definition 2.1, which is therefore suitable for the Cartan classification of Lie algebras. Definition 2.2 is more general inasmuch as, besides permitting the introduction of all possible enveloping algebras, it also permits us to construct nonisomorphic algebras via the same basis, by therefore rendering necessary the use of the most general possible realizations of the associative product.

On physical grounds, these mathematical mechanisms are at the foundation of the Lie-isotopic generalization of Hamilton's and Heisenberg's equations for closed nonself-adjoint interactions (§1.3).

As familiar, the definition of physical quantities is independent of whether or not the systems possess nonpotential interactions. When these interactions are admitted by the theory, they are represented via an alteration of the Lie algebra product. As a result, when the Hamiltonian description of a closed self-adjoint system

$$\dot{A}(a) = [A, E_{tot}] = \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial E_{tot}}{\partial a^\nu}, \tag{2.16}$$

is generalized into a Birkhoffian form (1.22) to represent the additional presence of internal, contact, nonpotential, interactions, i.e.,

$$\dot{A}(a) = [A \hat{,} E_{tot}] = \frac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}(a)\frac{\partial E_{tot}}{\partial a^\nu}, \tag{2.17}$$

the basis of the original Lie algebra remains unchanged, together with the underlying carrier space ($\mathbf{R} \times T^*M$) and the field, and only the realization of the Lie algebra product (that is, the realization of the envelope) is permitted to change. As a result, the original Lie algebra $\mathbf{G}$ with basis $X_i, (a)$ over T^*M equipped with conventional Poisson brackets is mapped into the isotope $\hat{\mathbf{G}}$, which preserves the original basis $X_i(a)$ in the same local coordinates of T^*M, although it is now equipped with the generalized Poisson brackets, i.e.,

$$\mathbf{G} : [X_i, X_j] = (X_i, X_j) - (X_j, X_i) \rightarrow \hat{\mathbf{G}} : [X_i \hat{,} X_j] = (X_i \hat{,} X_j) - (X_j \hat{,} X_i). \tag{2.18}$$

In the transition to the case of Heisenberg's equation, the situation is essentially the same and actually turns out to be more directly compatible with Definition 2.2. In fact, for consistency of the theory with its classical image, during the generalization of Heisenberg's equation (now expressed for operators),

$$i\dot{A}(a) = [A, H] = AH - HA, \tag{2.19}$$

into the Lie-isotopic form (1.37), i.e.,

$$i\dot{A}(a) = [A \hat{,} H] = ATH - HTA, \tag{2.20}$$

the nonpotential forces due to charge overlapping are expressed via the Lie-isotopic generalization of the product

$$\mathbf{G} : [X_i, X_j] = X_i X_j - X_j X_i \rightarrow \hat{\mathbf{G}} : [X_i \hat{,} X_j] = X_i T X_j - X_j T X_i. \tag{2.21}$$

Mechanism (2.21) is clearly along Definition 2.2 rather than 2.1.

The alternative approach would be that of preserving the original simplest possible product and changing the basis in order to reach direct compatibility with Definition 2.1. However, this approach has a number of problematic aspects. First of all, it is centered on the loss of the direct physical meaning of the generators (e.g., the physical linear momentum in one dimension, $p = m\dot{r}$, is replaced by abstract objects of the type $p = \alpha \exp(\beta r \dot{r})$. Secondly, the approach does not permit the achievement of the direct universality, as recalled by the preceding section. The removal of the unnecessary restrictions on the realization of the enveloping algebras is clearly preferable, both mathematically and physically.

Owing to the relevance of mechanisms (2.18) and (2.21) for this review, it is important to give an explicit example. To stress the fact that the ideas are not necessarily restricted to nonpotential interactions, we review one of the first examples of isotopy identified by Santilli, that for the harmonic oscillator in a three-dimensional Euclidean space [1], [15].

The nonisomorphic groups $\mathbf{SO}(3)$ and $\mathbf{SO}(2.1)$ are *isotopic symmetries* of the corresponding Hamiltonians

$$H(a) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}(x^2 + y^2 + z^2), \tag{2.22.a}$$

$$\hat{H}(a) = \frac{1}{2}(p_x^2 - p_y^2 + p_z^2) + \frac{1}{2}(x^2 - y^2 + z^2)\,, \qquad (2.22.b)$$

$$a = (r,p), m = k = 1\,, \qquad (2.22.c)$$

that is, they are symmetries leading to the same conservation laws of the components $M_b, b = x, y, z$, of the angular momentum via the use of Noether's theorem. Let us review the case again and reinterpret it in light of Definitions 2.1 and 2.2.

The Hamiltonian realization of the symmetry **SO**(3) of $H(a)$ is based on the Lie algebra of conserved quantities

$$\mathbf{SO}(3) : [M_x, M_y] = M_z, \quad [M_y, M_z] = M_x, \quad [M_z, M_x] = M_y\,, \qquad (2.23)$$

which is defined in terms of the conventional Poisson brackets

$$[M_b, M_c] = (M_b, M_c) - (M_c, M_b)\,, \qquad (2.24.a)$$

$$(M_b, M_c) = \frac{\partial M_b}{\partial r^i}\delta^i_j\frac{\partial M_c}{\partial p_j}; \quad (\delta^i_j) = \begin{pmatrix} +1 & 0 & \\ & +1 & \\ 0 & & +1 \end{pmatrix}. \qquad (2.24.b)$$

In the transition to the equivalent Hamiltonian $\hat{H}(a)$, the conserved quantities M_b clearly remain conserved, but the **SO**(3) symmetry is broken and is replaced by the nonisomorphic symmetry **SO**$(2.l)$. The problem now is the construction of a realization of the **SO**(2.1) algebra (the *Lorentz* algebra in $(2+1)$-dimensions) whose generators are those of the nonisomorphic **SO**(3) algebra (the *rotational* algebra in three-dimensions). This can clearly be achieved if and only if one alters the Lie algebra product. An explicit realization has been identified by Santilli [1], [15] and is given by the commutation rules

$$\mathbf{SO}(2.1) : [M_x \hat{,} M_y] = M_z, \quad [M_y \hat{,} M_z] = -M_x, \quad [M_z \hat{,} M_x] = M_y, \qquad (2.25)$$

which are now expressed in terms of the *generalized* Poisson (Birkhoffian) brackets

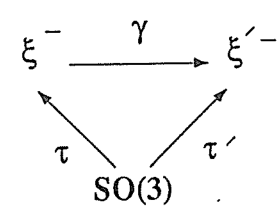
$$[M_b \hat{,} M_c] = (M_b \hat{,} M_c) - (M_c \hat{,} M_b)\,,$$

$$(M_b \hat{,} M_c) = \frac{\partial M_b}{\partial r^i}\alpha^i_j\frac{\partial M_c}{\partial p_j}, \quad (\alpha^i_j) = \begin{pmatrix} +1 & 0 & \\ & -1 & \\ 0 & & +1 \end{pmatrix}. \qquad (2.26)$$

Note that the insistence in the preservation of the same realization of the Lie algebra product, in this case, would prohibit the representation of the conservation of the angular momentum via a symmetry of the Hamiltonian $\hat{H}(a)$.

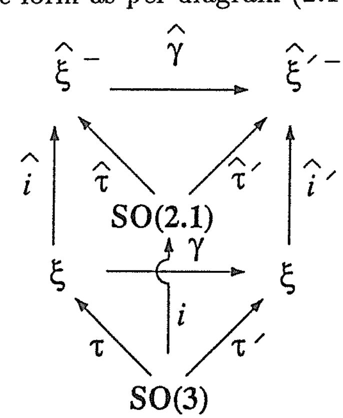
The example considered therefore establishes that one given basis (the components of the angular momentum $M = r \times p, p = m\dot{r}$) can define a hierarchy of enveloping algebras and attached Lie algebras, depending on the selected realization of the products, which is fully in line with diagram (2.4) and Definition 2.2. The example actually establishes not only the insufficiency of Definition 2.1 but also that of Definition 2.2 itself. In fact, the algebras (M_b, M_c) and $(M_b \hat{,} M_c)$ are *nonassociative*, therefore demanding a further generalization of Definition 2.1 for nonassociative enveloping algebras, even though the existence of a realization within the context of the Lie- isotopic generalization is expected to exist (§1.5).

Stated in different terms, the above example by Santilli establishes the generalization of the conventional definition of the envelope of the Lie algebra of the group of rotations as per diagram (2.4).



(2.27)

into the Lie-isotopic form as per diagram (2.14)



(2.28)

which is expected for operator-type realizations (2.21).

Note that by no means does diagram (2.28) exhaust all possible isotopies of the group of rotations. See §3.2 for details.

With a clear understanding of the new capabilities (as well as limitations) of the Lie-isotopic generalization, we pass now to the review of the generalization of Theorem 2.1 achieved by Santilli (*loc. cit.*).

The construction of an isotope $\hat{\xi}(\mathbf{G})$ of $\xi(\mathbf{G})$ can be conducted as follows. Perform an *isotopic mapping of the tensorial product* $X_i \otimes X_j$ of $\xi(\mathbf{G})$,

$$X_i \otimes X_j \to X_i * X_j \ , \tag{2.29}$$

that is, any invertible modification of the product $\otimes$ via elements of $\xi(\mathbf{G})$, of the base manifold, and of the field, which preserves: the distributive and scalar laws (to qualify as an algebra); the associativity of the product (to qualify as an isotopy), i.e.,

$$(X_i * X_j) * X_k = X_i * (X_j * X_k) \ , \tag{2.30}$$

as well as the existence of the unit $\hat{1}$. The product of two elements $X_i * X_j$ and $X_r * X_s$ is then given by

$$(X_i * X_j) * (X_r * X_s) = X_i * X_j * X_r * X_s \ , \tag{2.31}$$

and no ordering ambiguity arises because of the preservation of the associative character of the original product.

The isotope of the associative tensorial algebra (2.5) can then be written

$$\hat{\mathcal{F}} = F1 \oplus \mathbf{G} \oplus \mathbf{G} * \mathbf{G} \oplus \mathbf{G} * \mathbf{G} * \mathbf{G} \oplus \ldots . \tag{2.32}$$

Let $\hat{\mathcal{R}}$ be the ideal of $\hat{\mathcal{F}}$ generated by

$$[X_i \hat{,} X_j] - (X_i * X_j - X_j * X_i), \tag{2.33}$$

where $[X_i \hat{,} X_j]$ is the product in $\hat{\mathbf{G}}$. An *isotopically mapped universal enveloping associative algebra* $\hat{\xi}(\mathbf{G})$ of a Lie algebra $\mathbf{G}$ can then be written

$$\hat{\xi}(\mathbf{G}) = \hat{\mathcal{F}} / \hat{\mathcal{R}}. \tag{2.34}$$

Structure (2.34) is, by construction, the universal enveloping associative algebra of $\hat{\mathbf{G}}$ realized via an isotopic mapping $\mathbf{G} \to i\hat{\mathbf{G}}$.

The remaining aspects of the theory of $\hat{\xi}(\mathbf{G})$ are essentially given by an isotopic mapping of the corresponding steps for $\xi(\mathbf{G})$ outlined above.

The quantities

$$\hat{M}_s = X_{i_1} * X_{i_2} * \ldots * X_{i_s}, \tag{2.35}$$

are called *isotopically mapped standard (nonstandard) monomials* depending on whether the following ordering condition

$$i_1 \leq i_2 \leq \ldots \leq i_s \tag{2.36}$$

is verified (not verified). In the reduction of an arbitrary element of $\hat{\xi}(\mathbf{G})$

$$X_{i_1}^{k_1} * X_{i_2}^{k_2} * \ldots * X_{i_r}^{k_r}, \tag{2.37}$$

to standard monomials, a new feature arises, due to the fact that the emerging combinations of these latter monomials may occur via *functions on the base manifold.* This, in turn, occurs because the isotopy $\otimes \to *$ can be realized via functions of this type. We call these combinations $\hat{F}$- linear, where $\hat{F}$ is an isofield of type (1.38), to differentiate them from the F-linear combinations of the conventional case, that is, combinations only via elements of the field. As we shall see in the next section, these $\hat{F}$-linear combinations have a precise interpretation within the context of the isotopic Lie's theory. Despite this generalization, the construction of the basis of $\hat{\xi}(\mathbf{G})$ parallels that for $\xi(\mathbf{G})$, because $\hat{\xi}(\mathbf{G})$ is a conventional envelope for $\hat{\mathbf{G}}$. The (inverse) isotopy then simply reduces $\hat{\mathbf{G}}$ to $\mathbf{G}$.

Theorem 2.2 *(ref. [1], p. 353 and ref. [15], p. 161; Isotopic Generalization of the Poincaré-Birkhoff-Witt Theorem): The cosets of* $\hat{1}$ *and the standard isotopically mapped monomials form a basis of the isotopically mapped universal enveloping associative algebra* $\hat{\xi}(\mathbf{G})$ *of a Lie algebra* $\mathbf{G}$.

The basis is thus given by

$$\hat{1}, \quad X_i, \quad X_{i_1} * X_{i_2}, \quad X_{i_1} * X_{i_2} * X_{i_3}, \ldots$$
$$i_1 \leq i_2, \quad i_1 \leq i_2 \leq i_3 \,. \tag{2.38}$$

where $\hat{1}$ is the (abstract) unit of $\hat{\xi}$. The distinction between the tensorial realization and that used in practical applications is now lost. Indeed the mapping $X_i \otimes X_j \to X_i X_j$ can be considered, in the final analysis, a particular form of isotopy.

The explicit form of the basis depends on the assumed type of isotopy $\otimes \to *$. In turn, this depends on the realization of the basis X_i of $\mathbf{G}$, whether via matrices, quantum mechanical operators, classical functions on phase space, etc.

Suppose that the X's are realized via matrices. Then an isotopy is provided by Eq. (2.21). Let T be a polynomial on the X's (not necessarily on the center of $\hat{\xi}(\mathbf{G})$.) Then the explicit form of basis (2.38) is given by

$$1, \quad X_j, \quad X_{i_1} T X_{i_2}, \quad X_{i_1} T X_{i_2} T X_{i_3}, \ldots$$
$$i_1 \leq i_2, \quad i_1 \leq i_2 \leq i_3, \quad T = \text{fixed and invertible.} \tag{2.39}$$

Needless to say, the isotopy $X_i X_j \rightarrow X_i T X_j$ is only one example of possible associativity-preserving modifications of the product. Other associative isotopies are given by Eqs. (1.4) and (1.10).

A comment on the quantity $\hat{1}$ of Theorem 2.2 is in order here. As anticipated in §1.3, *the element* $1 \in \mathbf{F}$ *is no longer the unit element of the enveloping algebra under an isotopic mapping of the product.* In fact, for isotopic envelope (2.39) the unit element (when it exists) is given by

$$\hat{1} = T^{-1} \in \hat{F}\,, \tag{2.40}$$

because only this quantity verifies the (left and right) rules $\hat{1} * X_i = X_i * \hat{1} = X_i$ for all $X_i \in \hat{\xi}$. It should be indicated that, as we shall illustrate in §2.4, basis (2.38) can also be formulated in terms of the unit $1 \in F$ (called in this case *weak unit* [36]). This is due to the possibility of factoring out the isounit $\hat{1} \in \hat{F}$ (see, later on, Eq. (2.139)). The formulations of Theorem 2.2 in terms of the cosets of 1 (field F) or cosets of $\hat{1}$ (isofield $\hat{F}$) are, therefore, equivalent.

The restriction of the existence of the unit on all acceptable isotopies (recalled earlier) should be emphasized here. In fact, no generalization of Theorem 2.1 for isotopy (1.10) is known at this writing, precisely because of the general lack of unit $\hat{1}$ for the product $a * b = WaWbW$, $W^2 = W$, i.e., the general lack of existence of a quantity $\hat{1}$ such that $\hat{1} * a = a * \hat{1} = W\hat{1}WaW = WaW\hat{1}W = a$ for *all* $a \in \hat{\xi}$.

The restriction to the *isotopic* liftings of Theorem 2.2 is also worth a mention. In fact, Santilli presented in his original memoir [1] also a *genotopic* lifting of the theorem, i.e., a generalization of the original associative algebra ξ into a *nonassociative* Lie-admissible form. However, the nonassociativity causes problems in orderings of type (2.31) which are known to be resolvable only for a particular case of nonassociative Lie-admissible algebras called *flexible* [1]. This latter generalization was reinspected by Ktorides, Myung and Santilli [35]. We therefore defer the interested reader for details to the genotopies of Theorem 2.1 to ref.s [1,35].

An important mathematical aspect reviewed in this section is that *the knowledge of a given set of generators does not uniquely characterize a Lie algebra* because of the freedom in the selection of the enveloping algebra (product). The physical aspect treated is that *the knowledge of a Hamiltonian does not uniquely characterize the physical system* because such a characterization also depends on the explicit form of the brackets of the time evolution. As we shall see, the implications are rather intriguing. For instance, the assumption of a *Hermitian* Hamiltonian H contrary to popular belief, does not ensure that the time evolution is unitary and thus does not guarantee that $\tilde{H}$ is observable unless one specifically identifies the assumed realization of the envelope, i.e., of the assigned Lie product in Heisenberg's time evolution.

2.3 Isotopic Lifting of Lie's First, Second, and Third Theorems [1], [15]

As is well-known, an effective historical, and technical way of presenting Lie groups and Lie algebras is according to their original derivation by Sophus Lie [77] via his celebrated First, Second, and Third Theorems. In this section we shall first present these theorems, review Santilli's Lie isotopic generalization, and then show its comparability with the isotopic generalization of the enveloping algebra of the preceding section. More specifically, the objective is to show that the notion of connected Lie transformation group admits a generalization such that, when reduced in the neighborhood of the identity, admits Lie algebras with the most general possible realization of the product.

The emerging isotopic generalization of Lie's theory (that is, of the enveloping algebra, the Lie algebras, and the Lie groups) was used for the construction of the isotopic generalization of Galilei's relativity for closed non-self-adjoint systems [1], [15] with corresponding relativistic and gravitational extensions [18], [58]. Since the theory also admits operator-type realizations, its abstract formulation is expected to permit the joint treatment of closed, classical and quantum mechanical, nonpotential interactions, in much of the same way as the conventional abstract formulation of Lie's theory permits a joint treatment of closed, classical and quantum mechanical interactions of potential-Hamiltonian type. Santilli's ultimate objective is to lay the foundations for achieving, in due time, a generalization of the contemporary notion of interactions, with corresponding generalization of relativities and physical laws.

DEFINITION 2.3: Let M be a Hausdorff, second-countable, analytic, N-dimensional manifold with local coordinates $a^\mu, \mu = 1,2,...,N$ (e.g., T^*M or $R \times T^*M$). The set of transformations on M depending on r-independent parameters $\theta^i, i = 1,2,...,r$,

$$a \to a' = f(a;\theta) = \{f^\mu(a^\alpha;\theta^j)\} \quad (2.41)$$

is called a Lie transformation group *[77] when the following conditions are verified.*

1. *All functions f^μ are analytic in their variables.*
2. *For any given two transformations*

$$a' = f(a;\theta), \quad a'' = f(a';\theta') \, , \quad (2.42)$$

a set of parameters exists

$$\theta''^i = g^i(\theta,\theta') \, , \quad (2.43)$$

characterized by analytic functions g^i called group composition laws, *such that*

$$a'' = f(a;\theta'') \, . \quad (2.44)$$

3. *Transformations (2.41) recover the identity transformation at the null value of the parameters, i.e.,*

$$a = f(a;0). \quad (2.45)$$

4. *Corresponding to each transformation (2.41), there is a unique inverse transformation*

$$a = f(a';\theta^{-1}), \quad (2.46)$$

and thus the transformations are regular.

5. *The combination of any transformation (2.41) with its inverse yields the identity transformation.*

The number r of independent parameters is called the *dimension* of the Lie group.

A central property of Lie transformation groups is that they are *connected*; that is, they can be continuously connected to the identity. The primary idea of Lie's theorems is that, under the conditions indicated, the groups can be studied via their infinitesimal transformations, because a finite transformation can be recovered via infinite successions of infinitesimal transformations. Santilli [1] first reviewed these ideas by following as closely as possible their original derivation [77], as we shall do in the following. Consider transformations (2.41) with their identity

$$a' = f(a;\theta), \quad a = f(a;0), \tag{2.47}$$

and perform the infinitesimal variations

$$a' = a + da = f(a;\theta + d\theta); \quad a + \delta a = f(a;\delta\theta) \,, \tag{2.48}$$

where $d\theta$ and $\delta\theta$ represent two independent variations of the parameters. We can then write

$$da = \frac{\partial f(a;\theta)}{\partial\theta} d\theta \,, \tag{2.49.a}$$

$$\delta a = (\frac{\partial f(a;\theta)}{\partial\theta})_{\theta=0}\delta\theta \,. \tag{2.49.b}$$

The transformation $\theta + d\theta$ can be interpreted as the product of transformations relative to θ and $\delta\theta$, i.e.,

$$\theta^i + d\theta^i = \varphi^i(\theta, \delta\theta) \,, \tag{2.50}$$

for which

$$\theta^i + d\theta^i = \varphi^i(\theta, 0) + (\frac{\partial\varphi^i(\theta,\alpha)}{\partial\alpha^j})_{\alpha=0}\delta\theta^j + \dots \,. \tag{2.51}$$

Thus we can write

$$d\theta^i = \mu^i_j(\theta)\delta\theta^j \,,$$

$$\mu^i_j = (\frac{\partial\varphi^i(\theta,\alpha)}{\partial\alpha^j})_{\alpha=0} \,. \tag{2.52}$$

The formula above represents a relation between $d\theta$ and $\delta\theta$ which can also be written

$$\delta\theta^j = \lambda^j_i(\theta)d\theta^i, \quad \lambda^j_k\mu^k_i = \mu^k_i\lambda^j_k = \delta^j_i \,. \tag{2.53}$$

By putting

$$u^\mu_j(a) = (\frac{\partial f^\mu(a;\theta)}{\partial\theta^j})_{\theta=0} \,, \tag{2.54}$$

and by using Eq. (2.53), Eq. (2.49.a) can be written

$$da^\mu = u_k^\mu(a)\lambda_j^k(\theta)d\theta^j \ . \tag{2.55}$$

In this way we reach *Lie's First Theorem.*

Theorem 2.3: *When transformations (2.41) form a connected, m-dimensional, Lie group, then*

$$\frac{\partial a^\mu}{\partial \theta^j} = u_k^\mu(a)\lambda_j^k(\theta) \ , \tag{2.56}$$

where the functions u_k^μ are analytic.

Let $A(a)$ be an (analytic) function of the a variables. The infinitesimal Lie transformation $a \to a + da$ induces a variation of $A(a)$ which can be written

$$\begin{aligned} dA = \frac{\partial A}{\partial a^\mu} u_j^\mu \delta\theta^j &= \delta\theta^k u_k^\mu \frac{\partial}{\partial a^\mu} A \ . \\ &= \delta\theta^k X_k A \ . \end{aligned} \tag{2.57}$$

The m-independent quantities

$$X_k = X_k(a) = u_k^\mu(a)\frac{\partial}{\partial a^\mu} = [\frac{\partial f^\mu(a;\theta)}{\partial \theta^k}]_{\theta=0}\frac{\partial}{\partial a^\mu} \ , \tag{2.58}$$

are called the *infinitesimal generators* of the transformations (or of the group). For our later needs, we refer to the X's defined by Eqs. (2.58) as the *standard generators.*

We are now interested in the (necessary and sufficient) conditions for transformations (2.41) to constitute a Lie group. By using the converse of the Poincaré lemma, they can be written

$$\frac{\partial^2 a'^\mu}{\partial \theta^i \partial \theta^j} = \frac{\partial^2 a'^\mu}{\partial \theta^j \partial \theta^i} \ , \tag{2.59}$$

that is

$$\frac{\partial u_k^\mu}{\partial \theta^i}\lambda_j^k + u_k^\mu \frac{\partial \lambda_i^k}{\partial \theta^j} = \frac{\partial u_k^\mu}{\partial \theta^j}\lambda_i^k + u_k^\mu \frac{\partial \lambda_i^k}{\partial \theta^j} \ . \tag{2.60}$$

Thus

$$\begin{aligned} u_k^\mu\left(\frac{\partial \lambda_j^k}{\partial \theta^i} - \frac{\partial \lambda_i^k}{\partial \theta^j}\right) &= \lambda_j^k \frac{\partial u_k^\mu}{\partial \theta^i} - \lambda_i^k \frac{\partial u_k^\mu}{\partial \theta^j} = \lambda_j^k \frac{\partial u_k^\mu}{\partial a^\nu}\frac{\partial a^\nu}{\partial \theta^i} - \lambda_i^k \frac{\partial u_k^\mu}{\partial a^\nu}\frac{\partial a^\nu}{\partial \theta^j} \\ &= \lambda_j^r u_l^\nu \lambda_i^l \frac{\partial u_r^\mu}{\partial a^\nu} - \lambda_i^k u_l^\nu \lambda_j^l \frac{\partial u_k^\mu}{\partial a^\nu} \ . \end{aligned} \tag{2.61}$$

Therefore

$$u_i^\nu \frac{\partial u_j^\mu}{\partial a^\nu} - u_j^\nu \frac{\partial u_i^\mu}{\partial a^\nu} = C_{ij}^k u_k^\mu \ , \tag{2.62}$$

where

$$C_{ij}^k = \mu_i^r \mu_j^s \left(\frac{\partial \lambda_r^k}{\partial \theta^s} - \frac{\partial \lambda_s^k}{\partial \theta^r} \right) . \tag{2.63}$$

The m^3 quantities C_{ij}^k are independent from θ. This can be seen by differentiating Eq. (2.62) with respect to θ. After some simple calculations, one then sees that

$$\frac{\partial C_{ij}^k}{\partial \theta^l} = 0 \ ,$$
$$i, j, k, l = 1, 2, ..., m \ . \tag{2.64}$$

In this way we reach *Lie's Second Theorem.*

Theorem 2.4: *If $X_i, i = 1, 2, ..., m$, are the generators of an m-dimensional Lie group, they satisfy the closure relations*

$$[X_i, X_j]_\xi = X_i X_j - X_j X_i = C_{ij}^k X_k \ , \tag{2.65}$$

where the quantities C_{ij}^k are called structure constants.

The symbol ξ in Eq. (2.65) denotes an associative algebra with a conventional, associative product of operators $X_i X_j$. At closer inspection, this algebra emerges as being the *universal enveloping associative algebra* of the Lie algebra.

The *fundamental Lie's rule* (2.65) can be explicitly written

$$[X_i, X_j]_\xi = [U_i^\mu \frac{\partial}{\partial a^\mu}, U_j^\nu \frac{\partial}{\partial a^\nu}]_\xi = C_{ij}^k u_k^\alpha \frac{\partial}{\partial a^\alpha} \ , \tag{2.66}$$

where the product $[X_i, X_j]_\xi$ is Lie; that is, it satisfies the identities

$$[X_i, X_j]_\xi + [X_j, X_i]_\xi = 0 \ ,$$

$$[[X_i, X_j]_\xi, X_k]_\xi + [[X_j, X_k]_\xi, X_i]_\xi + [[X_k, X_i]_\xi, X_j]_\xi = 0 \ . \tag{2.67}$$

By substituting into these expressions the explicit form of the Lie product in terms of the structure constants, *Lie's Third Theorem* is reached.

Theorem 2.5: *The structure constants of a Lie group in standard realization obey the relations*

$$C^k_{ij} + C^k_{ji} = 0 \ ,$$
$$C^k_{ij}C^r_{kl} + C^k_{jl}C^r_{ki} + C^k_{li}C^r_{kj} = 0 \ . \tag{2.68}$$

Theorems 2.3, 2.4, and 2.5 essentially provide the correspondence between a given (connected) Lie group G and its Lie algebra **G**. In particular, they allow the characterization of a Lie group in the neighborhood of the identity via the structure constants. We have here tacitly implied that different Lie groups may exist all admitting the same Lie algebra, that is, the same structure constants. However, among all Lie groups with the same Lie algebra only one is simply connected, called the *universal covering group.*

The inverse transition from a Lie algebra to a corresponding Lie group can be characterized via the inverses of Lie's First, Second, and Third Theorems. We suggest the interested reader to study the specialized literature on this topic, such as Gilmore [78] and quoted references. We here outline one of the simplest approaches, known as the *exponential mapping* [15]. Write Eqs. (2.56) in the form

$$\frac{\partial a^\mu}{\partial \theta^i} = u^\mu_k(a)\lambda^k_i(\theta) = \lambda^k_i(\theta)X_k(a)a^\mu \ , \tag{2.69}$$

and introduce the one-dimensional parametrization

$$\theta^k = \tau\alpha^k, \quad a'^\mu = a'^\mu(\theta(\tau)) = a''^\mu(\tau) \ . \tag{2.70}$$

Then we write

$$a''^\mu(\tau) = T^\mu_\nu(\tau)a^\nu, \quad a^\nu = [a''^\nu(\tau)]_{\tau=0} \ . \tag{2.71}$$

To compute the elements $T^\mu_\nu(\tau)$, consider the equations

$$\frac{da^\mu}{d\tau} = \frac{\partial a^\mu}{\partial \theta^i}\frac{d\theta^i}{d\tau} = \alpha^k\lambda^r_k(\theta)X_r(a)a''^\mu(0) \ ,$$
$$\frac{d}{d\tau}T^\mu_\nu(\tau)a^\nu = \alpha^k\lambda^r_k(\theta)X_r(a)T^\mu_\nu(\theta)a''^\nu(0) \ . \tag{2.72}$$

However, the $a''^\nu(0)$ are arbitrary initial values. Thus the solutions of the total differential equations

$$\frac{d}{d\tau}T^\mu_\nu(\tau) = \alpha^k\lambda^r_k(\theta)X_r(a(\tau))T^\mu_\nu(\tau) \ , \tag{2.73}$$

with initial conditions

$$T^{\mu}_{\nu}(0) = \delta^{\mu}_{\nu}, \quad \frac{d}{d\tau}T^{\mu}_{\nu}(\tau)|_{\tau=0} = \alpha^k \lambda^r_k(\theta) X_r(a(0))\delta^{\mu}_{\nu} \ , \tag{2.74}$$

can be written

$$T^{\mu}_{\nu}(\tau) = \sum_{n=0}^{\infty} \frac{1}{n!}[\theta^k X_k(a(0))\delta^{\mu}_{\nu}]^n \ , \tag{2.75}$$

yielding the exponential mapping

$$a'^{\mu} = e^{\theta^k X_k}|_{\xi} a^{\mu} \ . \tag{2.76}$$

If, instead of the variables of the base manifold, we have a function of the same variables, the procedure above also applies, and we can write

$$A(a') = e^{\theta^k X_k}|_{\xi} A(a) \ . \tag{2.77}$$

In particular, the infinitesimal (standard) generators can be recovered via the rule

$$X_k = [\frac{\partial}{\partial \theta^k} e^{\theta^i X_i}|_{\xi}]_{\theta=0} \ . \tag{2.78}$$

Notice that the standard realization (2.76) of the group of transformations (2.41) is manifestly connected. The verification of the conditions to qualify as a Lie group is simple. Here we restrict ourselves to recalling that the product of two elements of group (2.76)

$$e^{X_\alpha} e^{X_\beta} = e^{X_\rho}, \tag{2.79}$$

is characterized by the so-called *Baker-Campbell-Hausdorff formula*:

$$X_\rho = X_\alpha + X_\beta + \frac{1}{2}[X_\alpha, X_\beta]_{\xi} + \frac{1}{12}[(X_\alpha - X_\beta), [X_\alpha, X_\beta]_{\xi}]_{\xi} + \ldots . \tag{2.80}$$

It is significant for our review to recall that a Lie algebra does not necessarily admit a corresponding Lie group. For specific examples of Lie algebras of this type, the reader may consult, for instance, Hurst [79]. In essence, the applicability of the exponential mapping in general, or the "integration" of a Lie algebra to a Lie group must satisfy certain (convergence) conditions of the underlying infinite series, known as *integrability conditions*. We also refer the reader in this respect to the specialized literature in the subject and, in particular, to Nelson [80].

We pass now to the review of Santilli's Lie-isotopic generalization of Lie's theorems. The prior review of the main objective may be useful here. Lie's crucial result is fundamental rule (2.65). This rule essentially characterizes Lie algebras via the conventional associative product X_iX_j of vector fields $X_i = u_i^\mu(a)\partial/\partial a^\mu$ on a manifold M. Santilli's main objective is to generalize Definition 2.3 and Lie's theorems in such a way as to characterize a Lie algebra via the most general possible associative product $X_i * X_j$ of vector fields on a manifold.

Of utmost importance is the condition that *the base manifold M with local coordinates a^μ, the parameters θ_i, and the generators X_i of the conventional formulation of Lie's theorems are not changed in their isotopic generalization.* This is due to physical requirements for the description under consideration. As we recalled earlier, the local coordinates of M customarily have a direct physical meaning such as the coordinates of the frame of the experimental setup; the parameters carry a direct physical meaning as measurable quantities such as time, angle, etc., and the generators directly represent physical quantities such as energy, angular momentum, etc. When the conventional description of self-adjoint interactions via Theorems 2.3, 2.4, and 2.5 is broadened to permit the additional presence of the nonself-adjoint interactions, the frame of the experimental observer must be preserved; measurable quantities such as time and angles must be preserved; and physical quantities such as energy and angular momentum must also be preserved unaltered.

These objectives were achieved by Santilli as follows.

DEFINITION 2.4 (ref. [1], pp. 329-368. See also ref. [15], pp. 169-173): Let

$$G : a^\nu \rightarrow a'^\nu = f^\nu(a;\theta)\,, \tag{2.81}$$

be an r-dimensional Lie transformation group G as per Definition 2.3. A Lie isotopic image *or, simply an* isotope $\hat{G}$ *of G is a set of transformations characterizable via a regular $(N \times N)$ matrix of analytic functions $(g^\mu_\nu(a;\theta))$ acting on (2.81)*

$$\hat{G} : a^\mu \rightarrow \hat{a}^\mu = g^\mu_\nu(a;\theta) f^\nu(a,\theta) = \hat{f}^\mu(a;\theta)\,,$$

$$\det(g^\mu_\nu) \neq 0, \quad g^\mu_\nu|_{\theta=0} = \delta^\mu_\nu\,, \tag{2.82}$$

which verify the following properties. (a) The transformations $\hat{a} = \hat{f}(a;\theta)$ constitute a Lie transformation group, by therefore *verifying conditions 1-5 of Definition 2.3. (b) The group $\hat{G}$ is realized via the same base manifold, the same parameters and the same generators of G. (c) When reduced in the neighborhood of the identity transformation, the group $\hat{G}$ can be characterized by a Lie algebra isotope $\hat{\mathbf{G}}$ of $\mathbf{G}$.*

Condition (c) is introduced to avoid non-Lie, Lie-admissable algebras in the neighborhood of the identity transformations [1]. As a matter of fact, it is precisely this possibility that permits the further generalization of Lie's theory of Lie-admissible type.

Since the group of transformations $\hat{f}^\mu(a;\theta)$ is a conventional, connected Lie group by assumption, it can be studied in the neighborhood of the identity as in the conventional case. The repetition of the analysis of $f(a;\theta)$ then yields the expressions

$$da^\mu = \hat{u}^\mu_k(a)\lambda^k_i(\theta)d\theta^i \, ,$$

$$\hat{u}^\mu_k(a) = |\frac{\partial}{\partial\theta^k} g^\mu_\nu(a;\theta) f^\nu(a;\theta)|_{\theta=0} \, . \tag{2.83}$$

In order to realize the isotopy, we then introduce the following reformulation in terms of the quantities of G for given $g^i_k(a)$ functions

$$\hat{u}^\mu_k(a) = g^i_k(a) u^\mu_i(a), \quad \det(g^i_k) \neq 0 \, . \tag{2.84}$$

Note that the other possibility $\hat{u}^\mu_k = g^\mu_\nu u^\nu_k$, even though conceivable (and actually more in line with Eq. (2.83)), is excluded here because it would imply the redefinition of the generators $X_k = u^\mu_k(\partial/\partial a^\mu) \to \hat{X}_k = g^\mu_\nu u^\nu_k(\partial/\partial a^\mu)$ which is *contrary* to the notion of isotopy. The analyticity of the transformations then implies the following Santilli's generalization of Lie's First Theorem.

Theorem 2.6 *[1], [15]: If transformations (2.82) characterize an isotopic image $\hat{G}$ of the Lie group G of transformations (2.81), then analytic functions $g^i_k(a)$ exist such that*

$$\frac{\partial \hat{a}^\mu}{\partial \theta^j} = g^k_i(a) u^\mu_k(a) \lambda^i_j, \quad \det g \neq 0 \, , \tag{2.85}$$

and the $u^\mu_k(a)$ functions are analytic.

This theorem, though mathematically trivial, has nontrivial implications. Indeed, it implies a modification of the structure of the group in the neighborhood of the identity, i.e.,

$$G : a'^{\mu} \approx a^{\mu} + \theta^{i} u_{i}^{\mu}(a) \rightarrow \hat{G} : \hat{a}^{\mu} \approx a^{\mu} + \theta^{i} g_{i}^{j}(a) u_{j}^{\mu}(a) \ , \qquad (2.86)$$

which is precisely the desired situation. We must now identify the integrability conditions under which such a behavior is still Lie in algebraic character, when expressed in terms of the generators and parameters of the original group. Under these conditions, we say that the quantities g_{j}^{i} of Eqs.(2.85) or (2.86) are *isotopic functions* with respect to $\hat{G}$.

The group G is Lie and thus admits the standard realization worked out earlier,

$$u_{i}^{\nu} \frac{\partial}{\partial a^{\nu}} u_{j}^{\mu} - u_{j}^{\nu} \frac{\partial}{\partial a^{\nu}} u_{i}^{\mu} = C_{ij}^{k} u_{k}^{\mu} \frac{\partial}{\partial a^{\mu}} \ , \qquad (2.87.a)$$

$$C_{ij}^{k} = \mu_{i}^{r} \mu_{j}^{s} \left(\frac{\partial \lambda_{r}^{k}}{\partial \theta^{s}} - \frac{\partial \lambda_{s}^{k}}{\partial \theta^{r}} \right) , \qquad (2.87.b)$$

$$[X_{i}, X_{j}]_{\xi} = X_{i} X_{j} - X_{j} X_{i} = C_{ij}^{k} X_{k} \ , \qquad (2.87.c)$$

$$X_{k} = u_{k}^{\mu}(a) \frac{\partial}{\partial a^{\mu}} \ . \qquad (2.87.d)$$

The group $\hat{\mathbf{G}}$ is also Lie and thus can be realized in the standard form

$$\hat{u}_{i}^{\nu} \frac{\partial}{\partial a^{\nu}} \hat{u}_{j}^{\mu} - \hat{u}_{j}^{\nu} \frac{\partial}{\partial a^{\nu}} \hat{u}_{i}^{\mu} = \hat{C}_{ij}^{k} \hat{u}_{k}^{\mu} \frac{\partial}{\partial a^{\mu}} \ , \qquad (2.88.a)$$

$$\hat{C}_{ij}^{k} = \hat{\mu}_{i}^{r} \hat{\mu}_{j}^{s} \left(\frac{\partial \hat{\lambda}_{r}^{k}}{\partial \theta^{s}} - \frac{\partial \hat{\lambda}_{s}^{k}}{\partial \theta^{r}} \right) , \qquad (2.88.b)$$

$$\left[\hat{X}_{i}, \hat{X}_{j} \right]_{\xi} = \hat{X}_{i} \hat{X}_{j} - \hat{X}_{j} \hat{X}_{i} = \tilde{C}_{ij}^{k} X_{k} \ , \qquad (2.88.c)$$

$$\hat{X}_{k} = \hat{u}_{k}^{\mu} \frac{\partial}{\partial a^{\mu}} \ . \qquad (2.88.d)$$

However, as indicated earlier, this realization generally implies a change of the generators in the transition from G to $\hat{G}$:

$$\mathbf{G} : X_{k} = u_{k}^{\mu} \frac{\partial}{\partial a^{\mu}} \rightarrow \hat{\mathbf{G}} : \hat{X}_{k} = \hat{u}_{k}^{\mu} \frac{\partial}{\partial a^{\mu}} \ , \qquad (2.89)$$

and, as such, does not verify the conditions for isotopy. To achieve the objective under consideration, Santilli introduced the following isotopy of the universal enveloping associative algebra, according to §2.2, this time realized via *functions on the base manifold* [1], [15].

$$\xi(\mathrm{G}) : X_i X_j \rightarrow \hat{\xi}(\mathrm{G}) : X_i * X_j = g_i^r X_r g_j^s X_s \, . \tag{2.90}$$

Notice that this mapping does verify the conditions of isotopy, in the sense that it is realized via the generators of the original algebra, while preserves the associativity of the product,

$$(g_i^r X_r g_j^s X_s) g_k^t X_t = g_i^r X_r (g_j^s X_s g_k^t X_t) \, . \tag{2.91}$$

The fundamental Lie rule (2.87.c) can now be rewritten

$$u_i^\nu \frac{\partial}{\partial a^\nu} * u_j^\mu - u_j^\nu \frac{\partial}{\partial a^\nu} * u_i^\mu = \hat{C}_{ij}^k u_k^\mu \, ,$$

$$\hat{C}_{ij}^k = \tilde{C}_{ij}^r g_r^k(a) \, . \tag{2.92}$$

The integrability conditions for the functions $g_k^i(a)$ to be isotopic, that is to yield rule (2.92), can then be readily computed. Thus we reach the following Santilli's generalization of Lie's Second Theorem.

Theorem 2.7[1], [15]: *Under the integrability conditions*

$$g_i^k u_k^\nu \frac{\partial}{\partial a^\nu} g_j^l - g_j^k u_k^\nu \frac{\partial}{\partial a^\nu} g_i^l = g^j g_i^s C_{rs}^l + \tilde{C}_{ij}^k g_k^l \, , \tag{2.93}$$

the generators X_i of an isotope $\hat{\mathrm{G}}$ of a Lie group G satisfy the isotopic rule of associative Lie admissibility

$$[X_i, X_j]_{\hat{\xi}} = X_i * X_j - X_j * X_i = \hat{C}_{ij}^k(a) X_k \, , \tag{2.94.a}$$

$$\hat{\xi}(\mathrm{G}) : X_i * X_j = g_i^r X_r g_j^s X_s \, , \tag{2.94.b}$$

$$X_k = u_k^\mu(a) \frac{\partial}{\partial a^\mu} \, , \tag{2.94.c}$$

where the quantities $\hat{C}_{ij}^k(a)$, called structure functions, are generally dependent on the (local) coordinates of the base manifold of the original group.

In this way Santilli reached an interpretation of the $\hat{F}$-linear combination of the isotopically mapped standard monomials of §2.2. While in the standard realization (2.87.c) the quantities $\tilde{C}_{ij}^k$ are constants (the structure *constants* of a Lie group), the corresponding quantities which emerge after the reformulation of the same group $\hat{G}$ in terms of the base manifold, the parameters, and the generators of G, acquire an explicit dependence on the local coordinates (the structure *functions* $\hat{C}^k_{ij}(a)$). This situation has numerous technical implications (e.g., from the viewpoints of the representation and classification theory) which are not reviewed here.

The use of the Lie algebra laws for the isotopically mapped product (2.94.b) yields Santilli's generalization of Lie's Third Theorem.

Theorem 2.8 *[1], [15]: The structure functions $\hat{C}^k_{ij}(a)$ of the isotopic realization of a Lie group $\hat{G}$ verify the identities*

$$\hat{C}^k_{ij} + \hat{C}^k_{ji} = 0 \ , \tag{2.95.a}$$

$$\hat{C}^k_{ij}\hat{C}^r_{kl} + \hat{C}^k_{jl}\hat{C}^r_{ki} + \hat{C}^k_{li}\hat{C}^r_{kj} + [\hat{C}^r_{ij}, X_l]_{\hat{\xi}} + [\hat{C}^r_{jl}, X_i]_{\hat{\xi}} + [\hat{C}^r_{li}, X_j]_{\hat{\xi}} \tag{2.95.b}$$

The exponentiation from the Lie algebra to the Lie group can now be formulated in terms of the *isotopic image of the exponential law* (2.77), i.e.,

$$G : e^{\theta^i X_i}|_{\xi} \rightarrow \hat{G} : e^{\theta^i X_i}|_{\hat{\xi}'} \tag{2.96}$$

which is based on the following *rule of Lie isotopy*

$$\mathbf{G} : [X_i, X_j]_{\xi} = C^k_{ij} X_k \rightarrow \hat{\mathbf{G}} : [X_i, X_j]_{\hat{\xi}} = \hat{C}^k_{ij}(a) X_k \ , \tag{2.97}$$

with consequential *isotopically mapped Baker-Campbell-Hausdorff formula* [1], [15]

$$e^{\hat{X}_\alpha} * e^{\hat{X}_\beta} = e^{\hat{X}_\rho} \quad , \hat{X} = gX \ ,$$

$$\hat{X}_\rho = \hat{X}_\alpha + \hat{X}_\beta + \frac{1}{2}[X_\alpha, X_\beta]_{\hat{\xi}} + \frac{1}{12}[(X_\alpha - X_\beta), [X_\alpha, X_\beta]_{\hat{\xi}}]_{\hat{\xi}} + \ldots , \tag{2.98}$$

whose existence is ensured by that of the standard realization. The reader can now see the emergence of the $\hat{F}$-linear combination of the basis directly in the group composition law. Clearly, the enveloping algebra underlying expressions (2.98) is the isotope $\hat{\xi}(\mathbf{G})$ of $\xi(\mathbf{G})$.

A simple example may be useful in illustrating the above analysis [1], [15]. Consider the one-parameter group of dilations

$$r' = f(r; \theta) = e^{\theta} r. \tag{2.99}$$

The standard generator for this group is given by

$$X = r\frac{\partial}{\partial r} . \qquad (2.100)$$

Indeed

$$e^{\theta r(\partial/\partial r)}r = [1 + \frac{\theta}{1!}(r\frac{\partial}{\partial r}) + \frac{\theta^2}{2!}(r\frac{\partial}{\partial r})^2 + ...]r = e^{\theta}r . \qquad (2.101)$$

The group composition law is, in this case, trivial, i.e.,

$$r'' = f(r';\theta') = e^{\theta'}r' = e^{\theta'+\theta}r . \qquad (2.102)$$

Consider now the one-parameter connected Lie group of *nonlinear* transformations

$$\hat{r} = \hat{f}(r;\theta) = \frac{r}{1-\theta r} = g(r,\theta)f(r,\theta), g = \frac{e^{\theta}}{1-\theta r} , \qquad (2.103)$$

with composition law

$$\hat{r}' = \hat{f}(\hat{r};\theta') = \frac{\hat{r}}{1-\theta'\hat{r}} = \frac{r/(1-\theta r)}{1-\theta'(1/r-\theta r)} = \frac{r}{1-(\theta'+\theta)r} . \qquad (2.104)$$

We are interested in realizing this group, as a necessary condition of isotopy, via the generator (2.100) of the different group (2.99). This implies the search for an isotopic function, that is, a function which multiplies generator (2.100) to yield the correct transformation law of $\hat{f}$ as a solution of integrability conditions (2.94). Such a solution, in the case at hand, is simple and is given by r. Indeed, the isotopically mapped exponential law (2.96) yields the correct result

$$\begin{aligned} e^{\theta r(r(\partial/\partial r))} &= [1 + \frac{\theta}{1!}(r^2\frac{\partial}{\partial r}) + \frac{\theta^2}{2!}(r^2\frac{\partial}{\partial r})^2 + ...]r \\ &= \frac{r}{1-\theta r} . \end{aligned} \qquad (2.105)$$

Thus group (2.103) can be realized as an isotopic image of group (2.99).

The case considered above is trivial in the sense that all connected one-dimensional Lie groups are (locally) isomorphic. Thus, to activate the truly nonisomorphic character of the isotope with respect to the original group, one needs more than one dimension. Such a case is already provided by the realization of **SO**(2.1) as an isotope of **SO**(3), in Eqs. (2.26). More examples will be provided in Chapter 3.

2.4 Isotopic Lifting of Space-Time Symmetry Groups on Metric Spaces [22]

After achieving the generalization of Lie's theory reviewed in the preceding sections, Santilli specialized it to metric spaces, so as to facilitate the direct application to cases of physical relevance. In this way, he achieved a result of truly important value (Theorem 2.9 below) which provides the reconstruction of an exact space-time symmetry when conventionally broken [22].

In the following we shall review Santilli's original presentation as closely as possible.

We shall use the term *metric spaces* for the n-dimensional topological spaces M over the field $\mathbf{F}$ of real numbers $\mathbf{R}$, or complex numbers $\mathbf{C}$ or quanternions $\mathbf{Q}$, equipped with a nonsingular, sesquilinear, and Hermitian composition (x, y), $x, y \in M$, characterizing the mapping

$$(x, y) : M \times M \rightarrow \mathbf{F} . \qquad (2.106)$$

Let $e = (e_1, ..., e_n)$ be a basis of M, and define the *metric tensor* via the familiar rules

$$(e_i, e_j) = g_{ij} . \qquad (2.107)$$

Then, the condition of nonsingularity is intended to ensure the existence of the inverse

$$I = g^{-1}, \quad g = (g_{ij}) , \qquad (2.108)$$

with the consequent characterization of covariant and contravariant quantities

$$x_i = g_{ij} x^j, \quad x^i = I^{ij} x_j . \qquad (2.109)$$

The conditions of sesquilinearity

$$(x, \alpha y + \beta z) = \alpha(x, y) + \beta(x, z) , \qquad (2.110)$$

or

$$(\alpha x + \beta y, z) = \overline{\alpha}(x, z) + \overline{\beta}(y, z) , \qquad (2.111)$$

where the overbar represents complex conjugation in $\mathbf{F}$, permit the realization of the composition

$$(x, y) = x^\dagger g y = x^i g_{ij} x^j , \qquad (2.112)$$

where the dagger represents Hermitian conjugation in M.

Finally, the condition of Hermiticity can be formulated via the rules

$$(x, gy) = (g^\dagger x, y) = (gx, y) , \tag{2.113}$$

and is introduced for reasons to be identified below.

Additional conditions, such as the positive-definite character of the metric, are not recommendable for a general view of the Lie- isotopic theory, and they will not be considered at this time.

Metric spaces were then indicated in Ref. [22] with the notation

$$M = M(n, g, \mathbf{F}), \quad \mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{Q} . \tag{2.114}$$

which is also adopted hereon. Some of the metric spaces admitted for $\mathbf{F} = \mathbf{R}$ are: the Euclidean space $E(3, \delta, \mathbf{R})$, $\delta = \mathrm{diag}(+1, +1, +1)$; the Minkowski space $M(3+1, \eta, \mathbf{R})$, $\eta = \mathrm{diag}(+1, +1, +1, -1)$; the Riemannian space $R(n, g(x), \mathbf{R})$, with $g(x)$, $x \in M$, being symmetric and positive definite; the Finsler space $F(n, g(x, \dot{x}), \mathbf{R})$, where $g(x, \dot{x}) = \frac{1}{2}(\partial^2 f(x, \dot{x})/\partial x^i \partial x^j)$ is positive definite (for non-null $\dot{x}$) and of rank n; and others with corresponding spaces for the fields $\mathbf{F}$ of complex numbers and quaternions. Thus, we shall assume that the metric g is nonsingular, Hermitian, and verifies the needed continuity conditions (e.g., analyticity) in all variables, and we write

$$\det g \neq 0, \quad g^\dagger = g, \quad g = g(t, x, \dot{x}, \ldots) . \tag{2.115}$$

As one can see, the above definition of a metric is as general as possible, and *does not* coincide with the more restrictive definition conventionally used in specific geometries, such as the symplectic or the Riemannian ones. This situation is permitted by the Lie-isotopic theory because it does not require restrictions on g beyond those considered here. The formalization of the metric and its restriction to specific cases would then imply particularizations (such as the removal of the dependence on the velocities) which are not warranted or recommendable for a general study in Lie isotopy.

We consider now a special case of Definition 2.3, an m-parameter, continuous Lie transformation group $G(m)$ on $M(n, g, \mathbf{F})$, i.e., a topological space $G(m)$ equipped with a binary mapping, e.g.,

$$\varphi : G(m) \times G(m) \to G(m) , \tag{2.116}$$

verifying the conditions for $G(m)$ to be a topological group, and an additional mapping

$$f : G(m) \times M \to M , \tag{2.117}$$

characterized by n analytic functions $f(w;x)$ depending on m parameters w and the local coordinates $x \in M$, which verify the conditions for $G(m)$ to be a Lie transformation group (closure, associativity, identity, and inverse).

We shall furthermore assume that the group $G(m)$ acts linearly on M, i.e.,

$$x' \stackrel{\text{def}}{=} f(w;x) = A(w)x \,, \tag{2.118}$$

under which the group conditions can be realized in the form

$$A(0) = I \,, \tag{2.119.a}$$

$$A(w)A(w') = A(w''), \quad w'' = w + w' \,, \tag{2.119.b}$$

$$A(w)A(w^{-1}) = A(w^{-1})A(w) = I, w^{-1} = -w \,, \tag{2.119.c}$$

where I is the unit matrix in n dimensions.

Among the rather large number of aspects of the theory of linear, continuous, m-parameter Lie transformation groups, we now consider for clarity the specialization of the following aspects of §2.2 and §2.3 to metric spaces:

(1) The *universal enveloping associative algebra* ξ of $G(m)$, which we shall indicate with the symbolic expression of the basis

$$\xi : I, \quad X_r, \quad X_rX_s, \quad X_rX_sX_t \,,$$

$$r \leq s, r \leq s \leq t, \quad r,s,t,... = 1,2,\ldots m \,, \tag{2.120}$$

where I is now the $m \times m$ identity of ξ,

$$IX_r = X_rI = X_r \,. \tag{2.121}$$

The X's are the generators of $G(m)$ in their fundamental $(m \times m)$ representation verifying the skew-Hermiticity property

$$X_r^\dagger = -X_r \,, \tag{2.122}$$

the product X_rX_s is the conventional associative product of matrices; and the attached Lie algebra is given by the familiar rule

$$\xi^- : [P_r, P_s]_\xi = P_rP_s - P_sP_r, \tag{2.123}$$

where the P's are polynomials in the X's.

(2) The *Lie's group* $G(m)$ of transformations on M for the case of the action to the right as in Eq. (2.118), which we shall write in the symbolic exponentiated form for continuous transformations

$$G(m) : A(w) = e^{X_1 w_1} e^{X_2 w_2} \ldots e^{X_m w_m} = \prod_{k=1}^{m} e^{X_k w_k} , \qquad (2.124)$$

and which will be reduced to the appropriate exponential form whenever we consider specific cases. The corresponding action to the left,

$$x^{\dagger\prime} = x^{\dagger} A^{\dagger}(w) , \qquad (2.125)$$

can be characterized by the operation of Hermitian conjugation, which we shall write in the symbolic form

$$G(m) : \hat{A}^{\dagger}(w) = (\prod_{k=1}^{m} e^{X_k w_k})^{\dagger} , \qquad (2.126)$$

and whose explicit form will be computed whenever the reduced form of Eq. (2.124) is known (see the case of rotations of §3.2).

(3) The *Lie algebra* $\mathbf{G}(m)$ of $G(m)$, characterized by the closure rules

$$\mathbf{G}(m) : [X_r, X_s]_{\xi} = X_r X_s - X_s X_r = C^t_{rs} X_t . \qquad (2.127)$$

The underlying methodology we shall tacitly imply is the familiar one consisting of the Poincaré-Birkhoff-Witt theorem for the characterization of the basis (2.120); the Baker-Campbell-Housdorff theorem for the composition of the exponentials (2.124) and (2.126); Lie's First, Second,and Third Theorems for the characterization of the closure rules (2.127); the representation theory; etc.

The idea of the *Lie-isotopic theory* [1] is that of generalizing the structure of the enveloping algebra ξ, of the Lie group $G(m)$, and of the Lie algebra $\mathbf{G}(m)$ in such a way to preserve the Lie character of the theory (in order to qualify for isotopy). The generalization is done via the replacement of the simplest possible, associative, Lie-admissible product $X_r X_s$ of the conventional theory into a form denoted by $X_r * X_s$ which is still associative and Lie admissible (i.e., its attached product $X_r * X_s - X_s * X_r$ is Lie); nevertheless, it is given by the structurally more general form

$$X_r * X_s = X_r g X_s . \qquad (2.128)$$

It is evident that the generalization of the product of ξ implies a step-by-step generalization of the entire formulation of Lie's theory, from basis (2.120) to groups (2.124) and (2.126), to algebra (2.127), etc.

In paper [22] Santilli investigates not the Lie-isotopic theory per se, but its action on a metric space. He therefore identified the generalization of the structure of the metric space permitting a consistent action of the Lie-isotopic theory.

For this purpose, we shall first review the notion of *metric isotopy*, that is, a generalization of a given metric space which preserves its metric character. We shall then review the corresponding Lie-isotopic theory. Finally, we shall apply the results to the case when the considered Lie and Lie-isotopic groups constitute symmetries of the metric and its isotope, respectively. This latter result will be presented via Theorem 2.9 below on the symmetry properties of isotopy which is at the foundation of the applications of Chapter 3 to rotations, Galilei and Lorentz transformations.

Consider the simplest possible metric spaces, the Euclidean space $E(n, \delta, \mathbf{F})$, $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{Q}$, with composition law

$$(x, y) = x^i \delta_{ij} x^j. \tag{2.129}$$

Suppose that the metric δ has to be modified into a form of the generic type (2.115). The emerging generalized space can be expressed via the notion of metric isotopy as follows.

Let $\hat{I} = g^{-1}$ be the inverse of the new metric according to (2.108). Introduce the isotopic lifting of the field (1.38), i.e.,

$$\hat{\mathbf{F}} = \{\hat{N} | \hat{N} = N\hat{I}, N \in \mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{Q}\}. \tag{2.130}$$

The composition of elements of the field with elements of the metric space is now done according to the redefinition of the product

$$\hat{N} * x = \hat{N} g x = N \hat{I} g x = N x. \tag{2.131}$$

Thus, the lifting $\hat{\mathbf{F}}$ of $\mathbf{F}$ essentially permits the use of a generalized composition $\hat{N} * x$ which, while being characterized by the new metric g, preserves the old values Nx.

Next, Santilli generalizes the metric space $E(n, \delta, \mathbf{F})$ into a form $\hat{E}$ that accommodates the new metric g under a mapping of the type

$$\hat{m} : \hat{E} \times \hat{E} \rightarrow \hat{\mathbf{F}}. \tag{2.132}$$

This implies that the generalized composition law must have value in $\hat{\mathbf{F}}$. A realization is given by the form patterned along the isotopic lifting of the Hilbert spaces, Eq. (1.49), i.e.,

$$\begin{aligned} (x \hat{,} y) &= \hat{I}(x, gy) = \hat{I} x^i g_{ij} x^j \\ &= (x, gy)\hat{I} = (gx, y)\hat{I}. \end{aligned} \quad (2.133)$$

Ref. [22] defines as *isotopic liftings of the Euclidean space* all possible spaces $\hat{E}(n, g, \hat{\mathbf{F}})$ over the field $\hat{\mathbf{F}} = \hat{\mathbf{R}}, \hat{\mathbf{C}}, \hat{\mathbf{Q}}$, equipped with mappings (2.132) realized via composition (2.132), where g is the new metric tensor.

It is evident that, by construction, *all possible nonsingular metrics of the same dimension are isotopes of the Euclidean metric.* This includes the Minkowskian, Riemannian, Finslerian, and other metrics.

Note that, strictly speaking, the metric spaces $\hat{E}(n, g, \mathbf{F})$ cannot be considered as isotopes of $E(n, \delta, \mathbf{F})$, owing to the lack of lifting of the field. Nevertheless, this technical point can be ignored in practical applications owing to the identity $\hat{N} * x = Nx$. We can then assume that all possible metric spaces of n dimensions over the field $\mathbf{F}$ are isotopes of the Euclidean space.

Note that, since $\hat{\mathbf{F}}$ is still a field, $\hat{E}(n, g, \hat{\mathbf{F}})$ is also a metric space in the sense indicated earlier.

It is evident that the original Lie group $G(m)$ cannot act consistently on the new spaces. In fact, to begin, the action of the group on the space cannot be formulated according to the old composition (2.118) and must be modified into the form

$$x' = \hat{A}(w) * x \stackrel{\text{def}}{=} \hat{A}(w) g x \ , \quad (2.134)$$

[where the quantities $\hat{A}(w)$ will be identified shortly]. In turn, this implies that the old composition laws (2.118) cannot be consistently preserved, and must be generalized into the form

$$\hat{A}(0) = \hat{I} \ , \quad (2.135.a)$$

$$\hat{A}(w) * \hat{A}(w') = \hat{A}(w + w') \ , \quad (2.135.b)$$

$$\hat{A}(w) * \hat{A}(-w) = \hat{A}(-w) * \hat{A}(w) = \hat{I} \ , \quad (2.135.c)$$

which are precisely the defining conditions of a *Lie-isotopic transformation group* [1], [15].

The most important property of generalized laws (2.135) is the replacement of the old unit I with the new unit $\hat{I} = g^{-1}$. Thus, the dominant feature of Santilli's isotopy under consideration is the assumption of the inverse $\hat{I}$ of the new metric g as the generalized identity of the group. Since the original identity I can be interpreted as the inverse of the metric δ of the Euclidean space, when the original group $G(m)$ is a symmetry of δ, we expect its isotopic image $\hat{G}(m)$ to constitute a symmetry of g.

To achieve this result, Santilli uses the following main lines of the Lie-isotopic theory reviewed in §2.2 and §2.3:

(1) *Isotopic lifting of the universal enveloping associative algebra.* The Poincaré-Birkhoff-Witt theorem admits a consistent isotopic generalization, resulting in the new basis

$$\hat{\xi} : \hat{I}, \quad X_r, \quad X_r * X_s, \quad X_r * X_s * X_t, \dots , \tag{2.136}$$

$$r \leq s, \quad r \leq s \leq t,$$

$$r, s, t, \dots = 1, 2, \dots, n \,.$$

expressed in terms of the isounit $\hat{I}$, which is the same as that of the group composition laws (2.135). The generators X_r are here the same as those of ξ. The attached Lie algebra is now given by the isotope

$$\begin{aligned} \hat{\xi}^- : [P_r, P_s]_{\hat{\xi}} &= P_r * P_s - P_s * P_r \\ &= P_r g P_s - P_s g P_r \stackrel{\text{def}}{=} [P_r \hat{,} P_s] \,, \end{aligned} \tag{2.137}$$

The algebra $\hat{\xi}$ is still "universal" and "enveloping" not, of course, with respect to the algebra ξ^-, but with respect to $\hat{\xi}^-$. We see in this way that the generalized metric g enters into the very structure of the Lie product, Eq. (2.137), as expected.

(2) *Isotopic lifting of the Lie group.* The new basis (2.136) permits the construction of the new group elements $\hat{A}(w)$ via the so-called *isotopic exponentiation* [1], [15]. For one-parameter actions to the right, this exponentiation is characterized by the old generator X of $G(m)$ but now expanded in the new envelope according to the rule

$$\begin{aligned} \hat{G}(1) : \hat{A}(w) &= \hat{I} + \frac{1}{1!}(Xw) + \frac{1}{2!}(Xw)^{\hat{2}} + \frac{1}{3!}(Xw)^{\hat{3}} + \dots \\ &= \hat{I} + \frac{1}{1!}(Xw) + \frac{2}{2!}(Xw)g(Xw) + \frac{1}{3!}(Xw)g(Xw)g(Xw) + \\ &= e^{Xw}|_{\hat{\xi}} \stackrel{\text{def}}{=} \hat{e}^{Xw}, \end{aligned} \tag{2.138}$$