SANTILLI'S LIE-ISOTOPIC GENERALIZATION OF GALILEI'S AND EINSTEIN'S RELATIVITIES

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DEDICATION

This monograph is dedicated to

Mr. MICHAIL S. GORBACHEV
President of the U.S.S.R.

because of his vision, courage and
historical contributions to mankind

June 1, 1990

This monograph presents an enlarged version of the lectures delivered by Prof. **Ruggero Maria Santilli** at the INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS of Trieste, Italy, in the first part of December 1990, following notes taken at the lectures by one of the Authors, Prof. **A. Jannussis**, and subsequently enlarged thanks to the assistance of all the other Authors, as well as to the editorial assistance of the staff of THE INSTITUTE FOR BASIC RESEARCH of Palm Harbor, Florida, U.S.A.

An invitation by Prof. **Abdus Salam**, Director of the Centre, to Prof.s Santilli and Jannussis must be here acknowledged with sincere gratitude, because it permitted the organization of the original material presented at the lectures and stimulated its subsequent enlargement. Penetrating comments by Prof. Salam at the lectures resulted to be invaluable for the achievement of sufficient maturity of presentation, and for stimulating subsequent research.

* * *

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PREFACE

Throughout this century, Lie's theory has been developed in both mathematical and physical literatures with respect to the simplest conceivable unit, say $I = \text{Diag.}(1, 1, ..., 1)$, and the simplest conceivable product $AB - BA$, where AB is the trivial associative product. In a pioneering memoir written at Harvard University in 1978, Ruggero Maria Santilli identified, apparently for the first time, a generalized formulation of Lie's theory constructed with respect to the most general possible unit $\hat{I}$, in which case the Lie product assumes less trivial forms, such as $A * B - B * A$ where $A * B$ is still associative but of the more general type $A * B = AgB$, where g is fixed, sufficiently smooth and nonsingular, and $\hat{I} = g^{-1}$. The generalized theory was called the "Lie-isotopic theory" for certain historical reasons reviewed in the text. The original proposal of 1978 contains the development of the Lie-isotopic theory to a rather remarkable extent, including a generalization of: the theory of universal enveloping associative algebras (Poincaré-Birkhoff-Witt Theorem, etc.); Lie's celebrated First, Second and Third Theorems; Lie's transformation groups; and Lie's symmetries. The memoir concluded with the conjecture of a conceivable generalization of Galilei's Relativity in classical mechanics for extended particles moving within resistive media (which are not only Galilei-noninvariant, but also generally nonhamiltonian). This original proposal was subjected to a systematic study in subsequent years by Santilli as well as a number of independent authors, not only for the original classical profile, but also for a conceivable operator counterpart, as well as for relativistic, gravitational and gauge extensions.

This review is a guide through a considerable and disparate literature, devoted to: the identification of the state of the mathematical studies on the Lie-isotopic generalization of conventional formulations of Lie's theory; their primary applications, to classical mechanics, particle physics and astrophysics; and an outline of the proposed fundamental tests. Except for minor treatments, the studies on conceivable operator realizations are deferred to a possible separate review.

We begin with a review of the algebraic notion of isotopy and its application to associative and Lie algebras. We then pass to the notion of analytic isotopy in classical mechanics, that realized via the Birkhoffian generalization of Hamiltonian mechanics. We also indicate the notion of operator isotopy on Hilbert spaces, that realized via the hadronic generalization of quantum mechanics, as well as the methods of "hadronization," that is, the mapping of Birkhoffian into hadronic mechanics. The notion of isotopy in symplectic geometry concludes our introductory chapter.

The second chapter is devoted to a detailed review of the mathematical studies on the Lie-isotopic formulations of: enveloping associative algebras; Lie's Theorems; Lie algebras; Lie groups; and the application of the generalized theory to space-time symmetries. The second chapter ends with a fundamental theorem by Santilli on the reconstruction of the exact nature of space-time symmetries at the Lie-isotopic level, when broken at the conventional level.

The third chapter is devoted to the applications of the Lie-isotopic theory. We begin with a review of Santilli's isotopic generalization of the group of rotations and some of its properties such as: the capability by the rotational symmetry to remain exact at the Lie-isotopic level when conventionally broken, say, for spheres undergoing deformations, or for any physical condition implying a topology-preserving alteration of the Euclidean metric. We then pass to the review of Santilli's Lie-isotopic generalization of Galilei's Relativity for systems of extended-deformable particles which are nonhamiltonian (but Birkhoffian) because of motion within a resistive medium. We review the property that, again, under certain topological restrictions, the Galilei symmetry remains exact at the Lie-isotopic level when conventionally broken by nonhamiltonian forces. A number of intriguing implications and open problems are also considered. We then pass to the review of Santilli's Lie- isotopic generalization of Einstein's Special Relativity and related properties, such as: the capability of incorporating all available studies on Lorentz "noninvariance" (universality), e.g., the several phenomenological calculations predicting deviations from Einstein's behavior on the mean life of unstable hadrons at different speeds; the capability of reconstructing the Lorentz symmetry as isotopically exact for all the above models (in which it is conventionally broken); the capability to represent a disparate variety of physical conditions outside the applicability of the conventional relativity, such as deformation of charged distributions, motion of electromagnetic waves in inhomogeneous and anisotropic media, motion of electrons in metals, propagation of causal signals within dense hadronic matter, etc.; the generalization of the various laws of the conventional relativity with intriguing implications although in need of experimental preliminary confirmations; and a number of other aspects. The third chapter then passes to a review of the construction by Gasperini and Santilli of a Lie-isotopic generalization of Einstein's gravitation which is, locally, Lorentz-isotopic and Galilei-isotopic, as well as capable of resolving at least some of the numerous problematic aspects of the conventional theory available in the literature. The need for the conduction of certain basic tests on fundamental space-time symmetries (that have been regrettably ignored for decades) completes the third chapter.

In the Appendices we review a variety of topics that complement the main text, such as: Lie-isotopic generalization of gauge theories; computation of the maximal speed of causal signals within hadronic matter; Lie-isotopic field equations; and other aspects.

The situation emerging from this review is essentially as follows. From a mathematical viewpoint, there is little doubt that the Lie-isotopic theory is mathematically consistent and does provide a genuine covering of the conventional formulation of Lie's theory. The understanding is that the studies are at the beginning and so much remains to be done. From the viewpoint of theoretical physics, the classical formulations of the Lie- isotopic theory have clear applications in Newtonian mechanics, particularly for the physical systems of our everyday life, that is, with nonhamiltonian forces, for which the conventional formulations are simply inapplicable. In regard to relativistic settings, the isotopic theories are admittedly tentative, conjectural and in need of direct tests, although we are aware of no experimental or other information on the novel physical conditions considered capable of disproving the predictions of the new theory at this writing. As a matter of fact, all evidence currently available appears to favor the Lie-isotopic symmetries over the conventional ones, in a way, after all, predictable from the necessary compatibility with established Newtonian applications.

We are here referring to: phenomenological calculations on the behavior of the meanlife of unstable hadrons with energy conducted over the past several decades showing an apparent violation of the Einsteinian law, while they are clearly and directly interpreted by Santilli's covering law; the preliminary measures via neutron interferometry conducted by Rauch and his associates on the apparent deformation of the charge distribution of neutrons under external nuclear fields, with consequential alteration of the magnetic moments/rotational asymmetry, which are also directly and quantitatively interpreted by Santilli's exact, $SU(2)$-isotopic symmetry; and others. Not surprisingly, the astrophysical applications of Santilli's covering relativities appear to be in full agreement with their particle and classical counterparts. We are here referring, e.g., to the possibility of interpreting the quasar redshift as due to propagation of light within the hyperdense, inhomogeneous and anisotropic media surrounding the quasars, rather than to the currently unplausible quasars speeds of the order of ten time the speed of light in vacuum; and other very intriguing astrophysical applications.

As a result of all the above, a thrilling possibility of a new scientific edifice emerges from Santilli's pioneering studies, with predictable implications at every level of contemporary physics, most of which are still unexplored as of now. But, by far, the most important implications of Santilli's studies are from an experimental viewpoint. In fact, the studies focus the attention on considerably overdue, fundamental experiments which have been submitted in the technical literature for decades, but largely ignored until now. We are referring to experiments such as: final measures of the behavior of the mean life of unstable hadrons at different speeds; or to final measures of the expected deformation of the charge distributions of hadrons under sufficiently intense external fields; and others. All these experiments, in their currently available preliminary form, show clear deviations from the Einsteinian predictions, in favor of the prediction of Santilli's relativities and their exact, isotopic, Lorentz symmetry. This situation leaves the ultimate foundations of contemporary physics in a state of "suspended animation" which will evidently persist until the experiments are finally done, and the issue of conventional versus isotopic space-time symmetries resolved one way or the other.

This work will achieve one of its most important objectives if it succeeds in stimulating experimentalists to finally conduct these much overdue, fundamental tests.

June 1, 1990

1 INTRODUCTION

1.1 A Brief Survey of the Literature

Despite rather vast mathematical and physical studies, the formulation of Lie's theory has been essentially restricted until recently to that via the familiar Lie product $[A, B] = AB - BA$, where AB is the simplest possible associative product, e.g., that of matrices. The unit of the theory is then the trivial element, e.g., $I = \text{diag}(1, 1, ..., 1)$.

An inspection of the physical literature confirms this condition, which has its origin in the construction of quantum mechanics via the enveloping associative algebra of operators $A, B, ...$, their simplest possible product AB, and Heisenberg's time evolution $i\hbar\dot{A} = AH - HA$. An inspection of the mathematical literature confirms the same condition which has its origin, this time, in the representation theory of enveloping associative algebras also realized via the product AB.

In a pioneering memoir of 1978 (written while at the Lyman Laboratory of Physics of Harvard University), Ruggero Maria Santilli [1] identified, apparently for the first time, a generalized formulation of Lie's theory which he called *Lie- isotopic theory* for certain historical reasons reviewed later on. The central idea is that of building the theory around the most general possible unit, say $\hat{I} = (I_{ij})$, where the elements I_{ij} have an arbitrary functional or operator dependence subject only to certain topological restrictions. This demanded, of course, a generalization of the enveloping algebra, from the form with trivial product AB, into a covering form with product of the type $A * B = ATB$, where $\hat{I} = T^{-1}$. The Lie product then takes the more general form $A * B - B * A$.

Santilli was the first to realize the mathematical and physical nontriviality of the theory and to develop it to a considerable extent already in the original proposal [1]. In fact, in this first memoir one can see several theorems generalizing enveloping associative algebras, the celebrated Lie's first, second and third theorems, and the conventional notion of Lie group, into forms compatible with the most general possible unit $\hat{I}$. Under the condition that the old unit I is contained as a particular case of the generalized unit $\hat{I}$, Santilli's theory becomes a *covering* of the conventional one, in the sense of being formulated on structurally more general foundations, while admitting the conventional formulation as a trivial particular case.

Remarkably, the Lie-isotopic theory was proposed by Santilli as a particular case of a structurally yet more general theory based on the so-called

Lie-admissible algebras, which will not be reviewed in this monograph. Nevertheless, the point is important for this review because some of the subsequent advances made by Santilli and others on the Lie-isotopic theory can be identified only as a particular case of the more general Lie-admissible formulations. Perhaps this is the reason why the Lie-isotopic theory has not received until now the attention it deserves in both physical and mathematical literatures.

The subsequent memoir also of 1978 [2] and paper [3] were primarily devoted to Lie-admissible algebras, although containing advances important also for the simpler Lie-isotopic theory such as the foundation of a conceivable operator realization of the algebras, including the generalization of Heisenberg's equations of the type $i\hbar\dot{A} = A * B - B * A$. Santilli completed the year 1978 with the release of the two monographs [4,5], the first setting up the methodological foundations of the classical realization of the Lie-isotopic theory (the so-called conditions of variational selfadjointness), and the second initiating the application of hadronic mechanics to particle physics.

In 1979 we see the appearance of the first review [6] [again for the Lie-admissible approach] followed by paper [7] on the initiation of the representation theory of the generalized algebras on suitable bimodular vector spaces. Paper [8] presents an intriguing application to gauge theories.

Paper [9] of 1980 studies the difficulties of conventional quantization, and suggests their reinspection under a broader algebraic structure. Paper [10] of 1981 studies the expected existence of a conceivable generalization of quantum mechanical laws for the interior of hadrons, with particular reference to Heisenberg's uncertainty principle. Paper [11] enters deeper into conceivable physical implications for particle physics, this time for the notion of particle under external strong interactions realized with nonlocal and nonhamiltonian terms due to mutual wave overlappings.

In 1982 we see the appearance of paper [12] which consists of a review of the physical implications of the generalized Lie structures for nonpotential nonhamiltonian interactions in Newtonian, statistical and particle mechanics. Paper [13] studies the conceivable generalization of Heisenberg's and Schrödinger's equations that are expected from the broader realizations of Lie's theory. Paper [14] presents another courageous analysis, the possibility that causal signals can propagate within dense hadronic matter at speed higher than c_o, the speed of light in vacuum. At the end of 1982 we also see the appearance of monographs [15,16] on the classical realizations of his algebraic theories, the so-called Birkhoffian [15] and Birkhoffian admissible

[16] mechanics. In these monographs one can see Santilli's extended presentations of the conceivable generalizations of Lie-isotopic and Lie-admissible type, respectively, of the classical Galilean relativity for extended particles with action-at-a-distance, potential forces, as well as contact, nonpotential and nonhamiltonian forces due to motion within a resistive medium.

In 1983 we see the appearance of three central contributions. Paper [17] presents a model on the reversibility of strong interactions for center-of-mass conditions, with irreversible dynamics for each individual constituent when considering the rest of the system as external. Paper [18] is, in our opinion, the most important paper under consideration here after refs. [1,2]. It presents the foundations of a conceivable Lie-isotopic covering of Einstein special relativity for generalizations of the Minkowski metric caused by motion of extended particles within generally inhomogeneous and anisotropic physical media. The paper also provides the explicit method for the construction of an infinite class of covering transformations from the original Lorentz ones and the given generalized metric. Paper [19] provides a generalization of Wigner's theorem on quantum mechanical symmetries within the broader Lie-isotopic setting representing nonpotential nonhamiltonian forces caused by mutual wave-overlappings of particles. This paper also signals the achievement of mathematical maturity of the generalized operator formulation, with the clear understanding that its physical validity is still basically open at this writing.

In 1984 we see the appearance of another important contribution [20]. In the preceding paper [18] Santilli shows that, under certain topological restrictions, the continuous part of the Lorentz symmetry can be proved to be exact at the abstract, Lie-isotopic level when generally considered as "broken" at the simplistic level of the product $AB - BA$. Paper [20] complements these results, this time, for the discrete part of the Lorentz symmetry. In fact, the paper indicates how parity may well be an exact symmetry under weak interactions, provided the theory is realized within the context of the covering Lie-isotopic approach, because all P-breaking terms can be incorporated in the generalized unit $\hat{I}$ [as well as in other degrees of freedom]. The exact character of the P-[as well as other] symmetries then follows from the property that Lie algebras leave invariant their unit element.

In 1985 we see additional contributions by Santilli in the field. The year started with the inspiring "Journey in the Solar system" [21] (an invited contribution to the Calcutta conference). We then see the appearance of papers [22,23] specifically devoted to Lie-isotopic symmetries. These papers (which had been written prior to paper [18] and presented at a meeting of 1983) essentially provide a rigorous mathematical formulation of the process according to which a given Lie symmetry, when broken at the simpler level $AB - BA$, can be "reconstructed" as exact at the higher Lie-isotopic level $A * B - B * A$. The papers also identify the means of constructing the (generally infinite family of) covering, exact, Lie-isotopic transformations via the sole knowledge of the old transformations and of the new metric. Papers [22,23] then apply the theory to a case of truly central physical relevance: the breaking of the rotational symmetry, say, for the deformation of a spherical charge distribution under external fields, and the recovering of the exact rotational symmetry for the deformed distribution at the covering Lie-isotopic level.

In 1988 we see the appearance of four memoirs [25] which, jointly with the original memoir [1], constitute Santilli's most significant scientific contributions. In fact, these latter memoirs present a comprehensive isotopic generalization of contemporary algebras, geometries and mechanics for systems that are not only nonlinear and nonlocal (as those of the preceding contributions), but also nonlocal integral; the memoir then apply these broader mathematical tools for the construction of isotopic coverings of Galilei's, Einstein's Special and Einstein's General Relativities for interior dynamical problems; the memoirs finally present a detailed study of the mutual compatibility of the emerging generalized formulations and propose a number of experimental verifications.

In 1989 we see the appearance of four additional memoirs [26] this time devoted to the operator formulation of the isotopic theories, including a study of the "hadronization" of classical into operator formulations; the construction of the spinorial SU(2)-isotopic symmetry and its representations; some isotopic generalizations of the various properties of the conventional theory of angular momentum and spin (such as the isotopic Clebsch-Gordon coefficients, etc); the construction of the operator formulation of the isotopic Galilean and special relativities; the foundations of the isotopic field theory, including the isotopic generalization of the Klein-Gordon and Dirac's equations; the operator study of Rauch's experiment on the spinorial symmetry of neutrons; and other important topics.

Paper [25] of 1990 tests the possibilities of hadronic mechanics via a quantitative study of the possible representation of the original Rutherford's conception of the neutron as a generalized bound state of one ordinary proton and one electron, whose total angular momentum is represented via the isotopic SU(2)-symmetry to account for the expected nonlocal and nonhamiltonian effects due to total mutual penetration of the wavepackets of the constituents.

In 1991 we use the appearance of a series of papers written at the ICTP [27, 28, 29] which develop in more details the operator formulation of the isotopic special relativity based on the isotopies of the Poincaré symmetry; the construction of the generalized field theory invariant under the isotopic Poincaré symmetry, and some applications (Rauch's experiment on the spinorial symmetry and Rutherford's conception of the neutron).

Monograph [30], currently under preparation, is expected to complete the series of the preceding volumes [4,5] and [15,16]. This completes the review of the contributions written by Santilli alone.

Papers [31–44] were written by Santilli in collaboration with several authors on numerous topics related to the precedings research (see below).

A number of physicists have studied Santilli's proposal of 1978.

R. Mignani [45] made seminal contributions in the operator realization of Lie-isotopic theories, such as: the independent identification of the Lie-isotopic generalization of Schrödinger's equation; the proposal to construct a nonpotential scattering theory; the construction of the Lie-isotopic $SU(3)$ symmetry; and others.

M. Gasperini [46] made other equally seminal contributions, such as: the computation following hypothesis [14], that, within the context of contemporary gauge theories, the speed of causal signals within hadronic matter could indeed exceed c_o; the foundations of a possible Lie-isotopic generalization of gauge theories; and the foundations of a possible Lie-isotopic generalization of Einstein gravitation for the interior problem.

A team headed by A. Jannussis made numerous contributions [47] in both classical and operator realizations of Santilli's algebras. M. Nishioka [48] also made several contributions in the field, such as the expected generalization of the delta function. A. J. Kalnay [49] succeeded in quantizing Nambu's mechanics for triplets. The algebra emerging at the operator level is exactly that of Santilli's type [2]. (This aspect, which we regrettably cannot review in this paper, opens the possibility of a true quark confinement with an identically null probability of tunnel effects into free states, besides an infinite potential barrier, as studied in papers [44].

Animalu [50] conducted several, additional, independent research, such as the study of possible contributions to conventional quark theories of the generalized setting offered by hadronic mechanics, and others.

A. Tellez Arenas, J. Fronteau and R. M. Santilli [31,32] studied the statistical profile of a generalized class of physical systems characterized by the

Lie-isotopic algebras, the so-called closed variationally nonself-adjoint systems (these are systems submitted in memoir [2] which verify conventional total conservation laws, but the internal forces are of nonlocal, nonhamiltonian type).

The (mathematician) H. C. Myung and R. M. Santilli [36,37] achieved a consistent mathematical formulation of the operator realization of the Lie-isotopic algebras. These studies were then further extended via the addition of a suitable form of Hilbert spaces and reached their final form in ref. [38] by Mignani Myung and Santilli, which is here considered the best available presentation on the operator version of Lie-isotopic theories.

Additional contributions were made by A. K. Aringazin [51] such as: the application of Lie-isotopic Lorentz transformations to describe an anomalous energy dependence of some fundamental parameters of the $K^0 - \overline{K}^0$ system; the proof that Pauli's exclusion principle is valid for the center of mass of a composite system under a Lie-isotopic operator mechanics, in a way compatible with possible departures from the same principle for each individual constituent (a similar occurrence for Heisenberg's principle had been established in ref. [38]); the universal capability of the Lorentz-isotopic symmetry to include as particular cases all available research on Lorentz noninvariance; and others.

An in depth study of torsion in gravitational theories, and its apparent ultimate origin of Santilli's isotopic type has been conducted by D. Rapoport-Campodonic [52], with intriguing developments in stochastic and operator formulations.

S. Okubo [53] has also conducted a number of investigations in the field, most remarkably, the identification of certain inconsistencies which emerge in any attempt at generalization of the conventional associative enveloping algebra of quantum mechanics, and other mathematical studies.

One of the most intriguing applications has been provided by P.A.M. Dirac in two of his last (and little known) papers [53] presenting a certain generalization of his celebrated equation which resulted to have an essential isotopic structure, as shown by Santilli [27] (see Appendix C for a review).

The interested reader can identify a number of further contributions by various additional authors in the bibliographies of the above quoted papers, as well as in Proceedings [55,61].

The contributions by pure mathematicians specifically devoted to the Lie-isotopic formulation of Lie's theory (or their universal enveloping associative algebras) are grossly lacking at this time, to our best knowledge. In fact, as we shall see later on, the sole mathematical paper of which we are aware is ref. [62] by H. C. Myung on the isotensorial product of isorepresentations. Another mathematical paper connected with this review is that by E. B. Lin [63], devoted to the problem of "hadronization" (i.e., symplectic mapping of Birkhoffian into hadronic mechanics). The authors of this review are aware of several mathematical papers by mathematicians specifically devoted to the more general Lie-admissible algebras (see bibliography [64]) and, as such, they will be quoted and reviewed in a separate review of Santilli's Lie-admissible formulation of classical and operator mechanics. Nevertheless, these mathematical works are of difficult specialization to the Lie-isotopic context. It is hoped that this review will stimulate contributions by pure mathematicians, specifically, on Lie-isotopic algebras so as to be readily available for physical applications. An outline of this monograph written for mathematicians, with a list of intriguing, open, mathematical problems has been provided by these authors in paper [65].

1.2 The Notion of Algebraic Isotopy

As limpidly expressed in Santilli's writings, physical theories are a manifestation of an articulated body of formulations of algebraic, analytic, geometrical and other character. A generalized notion in any one of these formulations, to be consistent, must admit corresponding, compatible generalizations in the remaining branches of the theory. This is the case of the central notion of this review, that of isotopy (ref. [1], §2.13, pp. 287 and ff.).

Let U be an (associative or nonassociative) algebra with (abstract) elements $a, b, c, ...$ and (abstract) product ab over a field F with elements $\alpha, \beta, \gamma, ...$ (hereinafter assumed to have characteristic zero). The product ab, by assumption, verifies the basic axioms of U. For instance, if U is associative, ab verifies the associative law; if U is commutative, it verifies the commutative law; if U is a Lie algebra, it verifies the Lie algebras axioms; etc.

> *DEFINITION 1.1 (Algebraic Isotopy): An* isotopic mapping *(also called* image *or* lifting*) of an algebra U with product ab is any mapping $U \rightarrow \hat{U}$ of U into an algebra $\hat{U}$ which is the same vector space as U (i.e., the elements of U and $\hat{U}$ coincide), but is equipped with a new product $a * b$ which is such to verify the original axioms of U.*

Note that [15] the Greek for "isotopic" is "$\iota' \ \sigma \ o \ s \ \tau \ o' \ \pi \ o \ s$" which means "same configuration," precisely along the concept of the above definition.

The central property of the notion of algebraic isotopy is therefore that of preserving the character of the original algebra. Thus, if U is associative, a necessary condition for $\hat{U}$ to be an isotope of U is that the new product $a * b$ also verifies the associative law, and we shall write:

$$U : (ab)c = a(bc) \to \hat{U} : (a * b) * c = a * (b * c) \,. \tag{1.1}$$

Similarly, if U is a Lie algebra, a necessary condition for $\hat{U}$ to be one of its possible isotopes is that $\hat{U}$ is also Lie, and we shall write

$$U : \begin{cases} ab + ba = 0 \\ (ab)c + (bc)a + (ca)b = 0 \end{cases}, \hat{U} : \begin{cases} a * b + b * a = 0 \\ (a * b) * c + (b * c) * a + (c * a) * b = 0. \end{cases} \tag{1.2}$$

A similar situation occurs for other algebras, such as Jordan algebras, alternative algebras, etc.

Santilli identified three types of associative isotopy, each one with an attached Lie algebra isotopy. The first is the trivial one (ref. [1], p. 287)

$$U : ab \to \hat{U} : a * b = \alpha ab; \quad \alpha \in F; \quad \alpha \neq 0 \text{ and fixed} \,, \tag{1.3}$$

evidently given by the multiplication of the old product ab by a constant (that remains fixed for all multiplications of the new algebra). The attached Lie algebra is then given by the trivial mapping

$$[a, b]_U = ab - ba \to [a, b]_{\hat{U}} = \alpha [a, b]_U. \tag{1.4}$$

The second realization of associative isotopy, which plays a central role throughout Santilli's analysis, is given by (ref. [1], p. 352)

$$U : ab \to \hat{U} : a * b \stackrel{\text{def}}{=} aTb, \quad T \in U, \text{ invertible and fixed.} \tag{1.5}$$

It is simple (but instructive) to verify that indeed

$$(a * b) * c = (aTb)Tc = aT(bTc) = a * (b * c). \tag{1.6}$$

Thus, $\hat{U}$ is an isotope of U,

$$U : (ab)c = a(bc) \to \hat{U} : (a * b) * c = a * (b * c). \tag{1.7}$$

Evidently, isotope (1.5) is not trivial. Equally non trivial is the attached Lie algebra isotopy

$$[a, b]_U = ab - ba \to [a, b]_{\hat{U}} = a * b - b * a = aTb - bTa. \tag{1.8}$$

Since the element T does not necessarily commute with the generic elements $a, b, ...,$ of the algebra, the nontriviality of mapping (1.7) follows. The interested reader is encouraged to verify that, if $[a,b]_U$ is Lie, $[a,b]_{\hat{U}}$ is also Lie, i.e., it verifies the laws

$$[a,b]_{\hat{U}} + [b,a]_{\hat{U}} = 0 \; ,$$
$$[[a,b]_{\hat{U}}, c]_{\hat{U}} + [[b,c]_{\hat{U}}, a]_{\hat{U}} + [[c,a]_{\hat{U}}, b]_{\hat{U}} = 0. \qquad (1.9)$$

Isotopies (1.5) and (1.8) were assumed by Santilli at the basis of his formulation of Lie algebra isotopy, and we shall do the same in this review. In fact, the isotopic element T is sufficient to represent a generalized metric. Isotopies (1.5) and (1.8) are then amply sufficient to illustrate the mathematical and physical nontriviality of the generalized theory.

One additional algebraic isotopy was identified by Santilli [11]. It is given by

$$U : ab \rightarrow \hat{U} : a * b = WaWbW,$$
$$W \in U, \text{ idenpotent}(W^2 = W), \text{ and fixed.} \qquad (1.10)$$

It is again an instructive exercise for the interested reader to verify that the above product $a * b$ is still associative. The attached anticommutative product then remains Lie, ie., the mapping

$$[a,b]_U = ab - ba \rightarrow [a,b]_{\hat{U}} = a * b - b * a = WaWbW - WbWaW \quad (1.11)$$

constitutes another example of Lie algebra isotopy.

The reader may be interested in knowing that no investigation on isotopies (1.10) and (1.11) has been conducted until now, to our best knowledge, in both mathematical and physical literatures. All available studies are referred to isotopies (1.5) and (1.8).

The reason for the lack of physical investigations by Santilli on isotopy (1.10) is the general loss of the unit under the lifting considered. In turn, the loss of the unit has fundamental drawbacks from a physical profile, such as the loss of the measure theory, the loss of the notion of quantum of energy, the loss of the Casimir invariants, etc. For these (and other reasons), Santilli centered his research on the fundamental condition that the generalized theory must admit a consistent, left and right unit [1,2], which condition is indeed verified by isotopy (1.5) as we shall review shortly.

Also, a private communication by Santilli indicates that, according to preliminary research, isotopies (1.3), (1.5) and (1.10) and their combinations are expected to exhaust all possible associative isotopies, but no rigorous study has been conducted on this problem until now.

The classification of all possible associative (and therefore Lie) isotopies is evidently important because different isotopies are expected to characterize different physical theories.

As one can see, the notion of algebraic isotopy essentially represents a sort of "*degree of freedom of the product*" for given algebra axioms. As Santilli recalls [1], the notion is rather old, and actually dates back to the early stages of the set theory [66]. In fact, the notion apparently originates within the context of *Latin squares* (two Latin squares were called "isotopically related" if they could be made to coincide via permutations). Appropriately, Santilli quotes Bruck statement [66] to the effect that the notion is "*so natural to creep in unnoticed.*" And in fact, the notion had not been applied to Lie algebras until Santilli's proposal [1] (even though some application to other nonassociative algebras, e.g., the Jordan algebras, can be identified in the specialized mathematical literature [64,67]).

1.3 The Notion of Analytic Isotopy in its Classical and Operator Realizations

Let us pass now to the analytic counterpart of the concept of isotopy. It was introduced, also for the first time to our best knowledge, in memoir [1] and developed in detail in monograph [15] for the nonlinear and nonhamiltonian, but local-differentize case considered in this work. The more general nonlocal-integral case of memoir [24] will not be considered for brevity.

By following Santilli, let us write the conventional *Hamilton's equations* (those *without* external terms) in the unified notation

$$\dot{a}^\mu = \omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}, \quad \mu = 1,2,3,...,2n \ ,$$

$$a = (r^k, p_k), \quad k = 1,2,...,n, \quad H = H(t,a) \ , \tag{1.12}$$

with *Poisson brackets* between functions A and B in phase space $(\vec{r},\vec{p})$

$$[A,B] \stackrel{\text{def}}{=} \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial B}{\partial a^\nu} \equiv \frac{\partial A}{\partial r^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k}\frac{\partial A}{\partial p_k} \ , \tag{1.13}$$

and *canonical commutation rules* characterizing the *fundamental Lie tensor*

$$\begin{aligned} ([a^\mu,a^\nu]) &= \begin{pmatrix} ([r^i,r^j])([r^i,p_j]) \\ ([p_i,r^j])([p_i,p_j]) \end{pmatrix} = (\omega^{\mu\nu}) \\ &= \begin{pmatrix} O_{n\times n} & I_{n\times n} \\ -I_{n\times n} & O_{n\times n} \end{pmatrix} . \end{aligned} \tag{1.14}$$

The *canonical action principle* can be written

$$\delta A(t,\vec{r}) = \delta \int_{to}^{t} [R^o_\mu \dot{a}^\mu - H] dt = 0 \ ,$$

$$R^o = (\vec{p}, \vec{0}), \tag{1.15}$$

yielding Hamilton's equations in their covariant form

$$\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu} = 0, \tag{1.16}$$

where $\omega_{\mu\nu}$ is the covariant (symplectic) counterpart of $\omega^{\mu\nu}$ with explicit local realization in phase space

$$\omega_{\mu\nu} = \frac{\partial R^o_\nu}{\partial a^\mu} - \frac{\partial R^o_\mu}{\partial a^\nu} \ , \tag{1.17.a}$$

$$(\omega_{\mu\nu}) = \begin{pmatrix} O_{n\times n} - I_{n\times n} \\ I_{n\times n} O_{n\times n} \end{pmatrix} = (\omega^{\alpha\beta})^{-1} \ . \tag{1.17.b}$$

Finally, the *Hamilton-Jacobi equations* can be written in the unified form

$$\frac{\partial A}{\partial t} + H = 0 \ ,$$

$$\frac{\partial A}{\partial a^\mu} = R^o_\mu \ , \tag{1.18}$$

where the second set of equations can be explicitly written in the familiar form

$$\frac{\partial A}{\partial r^k} = p_k \ , \tag{1.19.a}$$

$$\frac{\partial A}{\partial p_k} = 0 \ , \tag{1.19.b}$$

showing the lack of dependence of the canonical action functional in the linear momentum (a property with important implications for quantization).

*DEFINITION 1.2 [1], [15] (*Classical-analytic Isotopy*): An* isotopic mapping *(or* image *or* lifting*) of Hamilton's equations is given by any generalized form of the equations which preserves: a) the derivability from a (first- order) variational principle; b) the Lie character of the underlying brackets; and c) the existence of a generalized, but consistent, Hamilton-Jacobi theory.*

The generalization of Hamiltonian mechanics originating from the above definition was called by Santilli *Birhkhoffian mechanics* for certain historical reasons (see ref. [1], p. 259 for the first appearance of these terms, and monograph [15] for a comprehensive presentation).

Under the above definition, principle (1.15) is generalized into the most general possible *Pfaffian variational principle* (here restricted to the semi-autonomous case for simplicity)

$$\delta \hat{A}(t,a) = \delta \int_{to}^{t} [R_\mu(a)\dot{a}^\mu - H(t,a)]dt = 0 \ ,$$

$$R = R(a) = R(\vec{r},\vec{p}) \neq R^o \ , \tag{1.20}$$

with fundamental equations given by *Birkhoff's equations* in their covariant form

$$\Omega_{\mu\nu}(a)\dot{a}^\nu - \frac{\partial H(t,a)}{\partial a^\mu} = 0, \quad \mu = 1, 2, ..., 2n \ , \tag{1.21.a}$$

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \ , \tag{1.21.b}$$

with contravariant version

$$\dot{a}^\mu - \Omega^{\mu\nu}(a)\frac{\partial H(t,a)}{\partial a^\nu} \ = \ 0 \ , \tag{1.22.a}$$

$$\Omega^{\alpha\beta} \ = \ |(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu})^{-1}|^{\alpha\beta} \ . \tag{1.22.b}$$

The algebraic brackets of the theory are given by the so-called *generalized Poisson brackets*

$$[A\hat{,}B] \stackrel{\text{def}}{=} \frac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}(a)\frac{\partial B}{\partial a^\nu} \ , \tag{1.23}$$

with *fundamental Birkhoffian brackets*

$$[a^\mu\hat{,}a^\nu] = \Omega^{\mu\nu}(a) \ , \tag{1.24}$$

which do verify the Lie algebra axioms (see the analytic, algebraic and geometrical proofs of ref. [15])

$$[A\hat{,}B] + [B\hat{,}A] = 0 \ ,$$

$$[[A\hat{,}B]\hat{,}C] + [[B\hat{,}C]\hat{,}A] + [[C\hat{,}A]\hat{,}B] = 0 \ . \tag{1.25}$$

Finally, Eqs. (1.18) are lifted into the *Birkhoffian form of the Hamilton-Jacobi equations*

$$\frac{\partial \hat{A}}{\partial t} + H = 0 , \qquad (1.26.a)$$

$$\frac{\partial \hat{A}}{\partial a^\mu} = R_\mu . \qquad (1.26.b)$$

Note that, unlike Eqs. (1.18), the generalized action functional does depend, in general, on the linear momentum, thus resulting in nontrivial generalizations of Eqs. (1.19b) (for simpler versions see below).

In summary, the notion of analytic isotopy gives rise, not to one particular algorithm, but to *an entire new mechanics* generalizing each and every aspect of the conventional Hamiltonian mechanics. It is hoped that, in this way, the reader begins to see the rather intriguing implications of Santilli's research.

Of course, the algebraic isotopy is a particular case of Definition 1.2, this time in its classical realization in the local coordinates $a = (\vec{r}, \vec{p})$

$$[A, B] = \frac{\partial A}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial B}{\partial a^\nu} \rightarrow [A \hat{,} B] = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu}(a) \frac{\partial B}{\partial a^\nu} . \qquad (1.27)$$

This proves the compatibility of the notion of isotopy at the algebraic and analytic levels (see the next section for the geometrical aspect).

From the above property we also see another seminal result achieved in memoir [1], that *Birkhoffian mechanics is a realization in classical mechanics of the Lie-isotopic algebras.* The reader interested in acquiring an expertise in Lie-isotopy is therefore urged to study monograph [15]. This point must be stressed here because this review can only serve as a guide to the existing literature.

Unlike the conventional Hamiltonian mechanics, *the Birkhoffian mechanics is directly universal*, in the sense of being able to represent *all* possible systems of the class admitted (essentially nonself-adjoint/nonhamiltonian systems verifying certain topological restrictions) in the frame of the experimenter. This property has nontrivial implications (particularly for quantization) because the mathematical algorithms of the theory can now be assured to have a direct physical significance, e.g.,: " $\vec{r}$ " represents the actual local coordinates of the experimenter; " $\vec{p}$ " represents the physical linear momentum $m\dot{\vec{r}}$; "$\vec{r} \wedge \vec{p}$ " therefore represents the angular momentum; "H" represents the actual physical energy $T + V$; etc. (see ref. [15], §4.5).

By comparison, the algorithm "$\vec{p}$" in Hamiltonian mechanics coincides with the physical linear momentum $m\dot{\vec{r}}$ only in very special cases; nevertheless, upon quantization, its operator image is rather universally assumed to be the physical linear momentum (with consequential results of equivocal character).

Let us also recall that *each formulation of Birkhoffian mechanics can be constructed via noncanonical transformations of the corresponding Hamiltonian counterpart.* In fact, Hamilton's equations *do not* preserve their form under noncanonical transformations, as well known. What has been identified by Santilli (ref. [15], §5.3) is that, under noncanonical transformations, all essential properties persist (derivability from a first-order principle; verification of Lie algebras axioms; existence of a Hamilton-Jacobi theory; etc.).

As a further aspect, the function H of Birkhoff's Eqs. (1.21) does not represent, in general, the total physical energy $T + V$ (although, as mentioned earlier, a representation of any given system always exists under the restriction $H = T + V$). In order to avoid confusions, Santilli introduced the name *Birkhoffian* for this function. The term Hamiltonian within the context of Birkhoff's equations is used only when the function represents the total energy. In the following, whenever referring to this function, we shall use the Hamiltonian H to denote specifically the restriction to the physical total physical energy $T + V$ (which is not necessarily conserved), and the Birkhoffian B to stress its departures from the total physical energy H.

As a final point, the classical Birkhoffian realization of the Lie-isotopic theory is fully established on physical grounds. Birkhoff [68] introduced his equations for a better study of the stability of the planetary orbits, although his use of Eqs. (1.21) was restricted to conservative systems. Santilli [1] rediscovered these equations (after some 51 years) and proved not only their applicability to a much larger class of Newtonian systems, but also their direct universality. For numerous physical applications along these latter lines, we refer the reader to the examples of Ref. [15], as well as to the quoted literature.

The restriction of this review only to classical realizations of the Lie-isotopy would however be a gross disservice to the reader, because, as well known, the abstract formulation of Lie's theory is directly interpretable via operator realizations.

This renders unavoidable a brief review of the operator realization. In the following we shall review the apparent generalization of quantum mechanics which emerges from these studies, with the clear understanding that, unlike its classical counterpart, the physical validity of the generalized operator formalism is not established as of this writing.

Let $\mathcal{H}$ be a Hilbert space (hereinafter assumed to be finite-dimensional) with elements $|a\rangle, |b\rangle, \ldots$ and norm over the field $\mathcal{C}$ of complex numbers

$$\mathcal{H} : \langle a|b\rangle = c \in \mathcal{C} \ . \tag{1.28}$$

Let ξ be an enveloping associative algebra of operators $A, B, \ldots$ on $\mathcal{H}$ with trivial associative product AB and unit $I = \mathrm{diag}(1, 1, \ldots, 1)$,

$$\xi : IA = AI = A, A \forall \xi \ . \tag{1.29}$$

The Lie algebra L attached to ξ is then characterized by the familiar product

$$L : [A, B]_\xi = AB - BA \ , \tag{1.30}$$

which provides the structure of the first fundamental equation of quantum mechanics, *Heisenberg's equation*

$$i\dot{A} = [A, H]_\xi = AH - HA, \quad \hbar = 1. \tag{1.31}$$

Let the homomorphism $\xi \times \mathcal{H} \rightarrow \mathcal{H}$ be characterized by the *(right) modular action* of, say, an operator $H \in \xi$ on an element $|a\rangle \in \mathcal{H}$ according to the familiar eigenvalue equation

$$H|a\rangle = c|a\rangle, \quad c \in \mathcal{C}. \tag{1.32}$$

This provides the structure of the second fundamental equation of quantum mechanics, the familiar *Schrödinger's equation*

$$i\frac{\partial}{\partial t}|a\rangle = H|a\rangle \ , \tag{1.33}$$

with corresponding well known additional aspects (such as unitary transformation theory, various physical laws, etc.).

DEFINITION 1.3 [2], [15], [36], [38] (Operator-analytic Isotopy)*: An* isotopic mapping *(or* image *or* lifting*) of Heisenberg's and Schrödinger's equations is given by compatible generalized forms that preserve: a) the existence of an underlying Hilbert space; b) the Lie character of the brackets of the time evolution; and c) the operations on the Hilbert space, such as transpose, hermiticity, unitarity, etc.*

A realization of the above operator isotopy was identified by Santilli in 1978 [1], [2] apparently for the first time. Let $\hat{\xi}$ be an isotope of ξ with product

$$\hat{\xi} : A * B \stackrel{\text{def}}{=} ATB \, , \tag{1.34}$$

where T is a generic, Hermitian, invertible and fixed, but otherwise arbitrary operator. The lifting $AB \rightarrow A*B$ evidently implies the underlying mapping of the unit, from the original trivial unit of $\xi, I = \text{diag}(1, 1, ..., 1)$, into the nontrivial operator unit $\hat{I} = T^{-1}$, called *isounit*, according to the rule

$$\hat{\xi} : \hat{I} * A = A * \hat{I} = A, \quad A \forall \hat{\xi} \, . \tag{1.35}$$

The antisymmetric algebra $\hat{L}$ attached to the isotope $\hat{\xi}$ is evidently a Lie-isotopic algebra with now familiar form

$$\hat{L} : [A, B]_{\hat{\xi}} = A * B - B * A \, . \tag{1.36}$$

The above generalized structures allowed Santilli to propose the following *Lie-isotopic generalization of Heisenberg's equation* (ref. [2], p. 752)

$$i\dot{A} = [A, H]_{\hat{\xi}} = A * H - H * A = ATH - HTA, \quad T = T^{+}. \tag{1.37}$$

The remaining realization of Definition 1.3 was accomplished in subsequent years. First, Santilli [7] pointed out the need for a full bimodular (left and right) generalization of the conventional (uni)modular representation theory. These studies lead to the proposal in 1982 by Myung and Santilli [36] of the following generalization of Schrödinger's representation (other attempts, see ref. [13], produced generalized equations not manifestly compatible with isotopy (1.37)).

The analysis was conducted by providing, apparently for the first time, a comprehensive isotopic generalization of conventional operations on a Hilbert space which, along Definition 1.3, were compatible with the iso-Heisenberg's equations.

In order to preserve linearity, the following *isotopic generalization of the field C* (called *isofield*) results to be needed (see ref. [36], pp. 1307-1309)

$$\hat{C} : \{\hat{c} | \hat{c} = c\hat{I}; \quad c \in C; \quad \hat{I} \in \hat{\xi}\} \, . \tag{1.38}$$

The elements $\hat{c}$ of $\hat{C}$ are then called *isonumbers.*

Note that $\hat{C}$ is still a field. Also, the sum in $\hat{C}$ is the conventional one, although the multiplication is isotopic, according to the rule

$$\hat{c}_1 * \hat{c}_2 = c_1 c_2 \hat{I}; \quad \hat{c}_1, \hat{c}_2 \in \hat{C} \, . \tag{1.39}$$

The achievement of compatibility with the iso-Heisenberg's equations requires the lifting of the conventional modular/eigenvalue action on $\mathcal{H}$ into the *isomodular/isoeigenvalue* form

$$\hat{\xi} \times \mathcal{H} \to \mathcal{H} : H * |a\rangle \stackrel{\text{def}}{=} HT|a\rangle = \hat{c} * |a\rangle = c|a\rangle. \tag{1.40}$$

Note that the "numbers" of the theory, i.e., the last numbers in the above identities, remain the conventional ones as in Eqs. (1.32).

With these preliminaries, Myung and Santilli presented a generalization of all familiar operations on a *conventional* Hilbert space (see below for generalization of the Hilbert space itself) (*loc. cit.* §II, pp. 1281-1315).

Evidently, we can review here only some of the most relevant operations. Let $\mathcal{H}$ be a conventional Hilbert space with elements $|a\rangle, |b\rangle, \ldots$ and norm (1.28). A linear operator $H \in \hat{\xi}$ on $\mathcal{H}$ is called *isohermitean* iff it verifies the identity

$$H^{\hat{\dagger}} \stackrel{\text{def}}{=} T^{\dagger} H^{\dagger} T^{-1} \equiv H. \tag{1.41}$$

The eigenvalues of isohermitean operators results to be *isoreal*, i.e., the number at the end of equalities (1.40) is real as in the conventional case.

A linear operator $U \in \hat{\xi}$ on $\mathcal{H}$ is isounitary when it verifies the rule

$$\langle a| * U^{\hat{\dagger}} * U * |b\rangle = \langle a|b\rangle, \tag{1.42}$$

which holds iff

$$U^{\hat{\dagger}} * U = U * U^{\hat{\dagger}} = \hat{I}; U^{\hat{\dagger}} = U^{-\hat{1}}. \tag{1.43}$$

Along similar lines, the following generalized properties hold, where conventional symbols denote conventional operations and symbols with a superscript "*hat*" denote generalized operations

$$\begin{aligned}
\widehat{Tr} A &= (Tr A)\hat{I}\,, && (1.44.a)\\
\widehat{Tr}(A * B) &= \widehat{Tr}(B * A)\,, && (1.44.b)\\
\widehat{\det} A &= \det(AT)\hat{I}\,, && (1.44.c)\\
\widehat{\det}(A * B) &= (\widehat{\det} A) * (\widehat{\det} B)\,, && (1.44.d)\\
\widehat{\det} A^{-\hat{1}} &= (\widehat{\det} A)^{-\hat{1}}\,. && (1.44.e)
\end{aligned}$$

After these preliminary results, Myung and Santilli proposed the following *isotopic lifting of Schrödinger's equation* (also called *iso-Schrödinger's equation*) (ref. [36], p. 1332)

$$i\frac{\partial}{\partial t}|a\rangle = H * |a\rangle \stackrel{\text{def}}{=} HT|a\rangle\,. \tag{1.45}$$

The equivalence with Eq. (1.37) was proved in *loc. cit* §3.7.

It should be indicated here that Eq.(1.45) was jointly but independently proposed by Mignani (ref. [69], p. 1128), although without the isotopic generalization of linear operations on Hilbert spaces worked out by Myung and Santilli (also, Mignani presented his generalized equations for the broader Lie-admissible level in which the T operator is nonhermitean, thus resulting in different, nonequivalent, left and right isomodular actions. See in this respect also paper [37] by Myung and Santilli).

The above results essentially established the mathematical consistency of the generalized operator theory, under the isotopic generalization of the enveloping associative algebra $\hat{\xi}$, the attached Lie-isotopic algebra $\hat{L}$, and the underlying isofield $\hat{\mathcal{C}}$, while keeping the conventional Hilbert space $\mathcal{H}$ unchanged.

The above operator realization of Definition 1.3 shall be symbolically referred to hereon with the isotopies

$$\begin{cases} \xi \to \hat{\xi}_T \ , \\ \mathcal{C} \to \hat{\mathcal{C}}_T, \quad T = T^\dagger \ , \\ \mathcal{H} \to \mathcal{H} \ , \end{cases} \tag{1.46}$$

where evidently the last mapping is the *identity isotopy.* We should stress that generalized formulations (1.46) are fully consistent on mathematical grounds, even though based on a conventional Hilbert space (see below for physical aspects). Also, we should stress that the Lie character of the formulation is centrally dependent on the (conventional) hermiticity of H on $\mathcal{H}$. In fact, in case T is not Hermitean we have the following pair of iso-Schrödinger's equations

$$i\frac{\partial}{\partial t}|a\rangle = H * |a\rangle = HT|a\rangle \ ,$$

$$\langle a|T^\dagger H = \langle a|\hat{*}H = \langle a|\frac{\partial}{\partial t}i \ ,$$

$$T \neq T^\dagger. \tag{1.47}$$

The generalized form of Heisenberg's equations corresponding to the above equations is then given by

$$i\dot{A} = (A, H) \stackrel{\text{def}}{=} ARH - HSA \ ,$$

$$R = T^\dagger \neq S = T \ , \tag{1.48}$$

which is precisely the yet broader *Lie-admissible generalization of Heisenberg's equation* proposed by Santilli (ref. [2], p. 746).

In summary, operator isotopy (1.46) is centered on the isotopic element T as one additional operator, besides the Hamiltonian, for the characterization of the time evolution laws (1.37) and (1.45), thus broadening substantially the arena of physical applicability of the theory.

Further studies revealed that the new "degree of freedom" characterized by T is still partial, and that an additional degree of freedom exists in the structure of the Hilbert space, with a corresponding further broadening of the representational capabilities of the theory (see §3).

In fact, subsequent studies by Mignani, Myung and Santilli [38] identified the following *isotopic generalization of the Hilbert space* itself (called *isohilbert space*), $\hat{\mathcal{H}}_G$ as the linear vector space with elements $|a\rangle, |b\rangle, ...$ and the *isoinner product*

$$\hat{\mathcal{H}}_G : \langle a\hat{|}b\rangle \stackrel{\text{def}}{=} \langle a|G|b\rangle \hat{I} = \hat{c} \in \hat{C} \ , \tag{1.49}$$

where the new operator G is hermitean and positive definite, but otherwise arbitrary. It represents an additional "hidden" degree of freedom of the theory besides that provided by the isotopic element T.

It is easy to check that the inner product (1.48) of the isohilbert space $\hat{\mathcal{H}}_G$ obeys all conditions which are used to define an abstract Hilbert space. So the isohilbert space $\hat{\mathcal{H}}_G$ may be thought of as an extended realization of the conventional Hilbert space $\mathcal{H}$ of quantum mechanics, with G being an integration measure. The two spaces are isometric to each other.

It is instructive also to verify that the following generalized Schwarz inequality holds $|\langle a\hat{|}b\rangle| \leq ||a||_G ||b||_G$, where we have denoted the *isonorm* of a as $||a||_G = \langle a\hat{|}a\rangle^{1/2}$.

Generalization (1.48) demands a further enlargement of linear operations. For instance, the condition of isohermiticity now becomes

$$H^{\hat{\dagger}} = T^{-1} G H^{\dagger} T G^{-1} \equiv H \ . \tag{1.50}$$

The above results are intriguing. In fact, one can see that *for $T = G$ the generalized notion of isohermiticity coincides with the conventional hermiticity*

$$H^{\hat{\dagger}} = T^{-1} T H^{\dagger} T T^{-1} \equiv H^{\dagger}. \tag{1.51}$$

In turn, this has the direct consequence that *the observables of quantum mechanics (Hamiltonian, linear and angular momenta, etc.) remain ob-*

servables under a general isotopy of enveloping associative algebras, fields and Hilbert spaces characterized by the same isotopic element $T = G$.

In summary, the most general known isotopic formulation of operator algebras is characterized by the following liftings

$$\begin{cases} \xi \to \hat{\xi}_T \,, \\ \mathcal{C} \to \hat{\mathcal{C}}_T, \quad T = T^\dagger \,, \\ \mathcal{H} \to \hat{\mathcal{H}}_G, \quad G = G^\dagger, \quad G > 0 \,, \end{cases} \tag{1.52}$$

where, in general, $T \neq G$. In the following we shall however often refer to formulations (1.52) under the specialization $T = G$, owing to their capability to preserve the operation of Hermiticity of quantum mechanics (as well as other operations, see ref. [36]).

The above rudimentary review is sufficient for our purpose here: to show the *mathematical* consistency of the generalization of quantum mechanics characterized by isotopes (1.46) and (1.52). In turn, this implies the existence of a consistent operator realization of Santilli's Lie-isotopic theory. Still, in turn, this property results invaluable for the study of the theory because, as mentioned earlier, isotopes (1.46) or (1.52) provide the most direct possible interpretation of the generalized Lie theory.

A few words on the physical profile are in order here. The generalization of quantum mechanics characterized by isotopies (1.46) and (1.52) was called by Santilli *hadronic mechanics* (ref. [2], p.756) to emphasize the restriction of the intended applicability of the theory only to the *interior* of hadrons, or to the interior of strong interactions at large.

The physical foundations of the proposal are the experimental evidence of the existence, under strong interactions, of necessary conditions of mutual overlapping of the wavepackets of particles (which are generally ignorable under electromagnetic interactions as in the atomic structure). In turn, these interactions are known at the classical level to be:

- a) of *contact* type, in the sense of *zero range*, i.e., not being representable via action-at-a-distance interactions;

- b) of *nonlocal* type, in the sense of occurring throughout a volume, and not being reducible to a finite number of isolated points; and

- c) of *nonhamiltonian* type, in the sense of being, not only of nonpotential type, but actually of being beyond the representational capabilities of a Hamiltonian in the frame of the observer (see monograph

[15] for the violation of the integrability conditions for the existence of a Hamiltonian).

The same properties are evidently expected to remain for particle wavepackets (see Fig. 1).

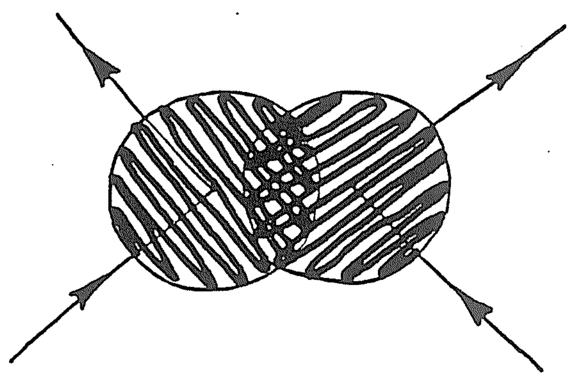


FIGURE 1. A reproduction of the slide presented by Santilli during his invited talk at the *Conference on Differential Geometric Methods in Mathematical Physics* held in Clausthal, West Germany, in 1980. The slide was intended to illustrate, for the distinguished geometers and theoreticians attending the conference, the incontrovertible experimental evidence on the nonlocal nature of the strong interactions as pointed out by the founding fathers of the theory. In fact, all hadrons are not point-like, but have a charge distribution of the order of $1F$ ($= 10^{-13}cm$) which coincides with the range of the strong interactions. Also, all known (massive) particles have a wavepacket which, again, is of the order of $1F$. Thus, a necessary condition to activate the strong interactions is that the particles enter into a state of mutual penetration of their charge distribution and wavepackets. This characterizes interactions which cannot be reduced to a finite number of isolated points, because they occur throughout the volume of mutual penetration/overlapping. Also, the interactions are of contact nature, that is, the nonlocality cannot be represented via a potential of integral type because the integrability conditions for the existence of a Hamiltonian are vi-

olated without, of course, precluding the existence of conventional potential terms. By keeping in mind that all geometries conventionally used nowadays in theoretical physics are of strictly local/differential nature, the slide was intended to stimulate the study of more general, nonlocal (e.g., integrodifferential) geometries for a more adequate representation of the interior strong problem. The Lie-isotopic theory and its various applications reviewed in this work are intended precisely as a first step toward a quantitative representation of the nonlocal/nonhamiltonian character of interior dynamical problems, in which the conventional, potential, local interactions are represented by conventional Hamiltonians, and the nonlocal, integrodifferential, and nonhamiltonian interactions are represented via the generalized unit of the theory. The symbol of overlapping spheres was subsequently assumed by Santilli as the *logo* of *The Institute for Basic Research*, at its inauguration ceremony the following August 1981.

As stressed earlier, *hadronic mechanics is not physically established as of this writing* because a large number of theoretical and experimental studies remain to be done. Nevertheless, hadronic mechanics may be applied also to account for a number of conventional applications, such as: quark confinement, hadronization processes and other cases where the perturbative techniques of QCD are known to fail to achieve a consistent description.

An apparent reason for the current resiliency toward hadronic mechanics is due to the inevitable existence of certain generalizations of basic quantum mechanics laws, such as: Heisenberg's uncertainty principle; Pauli's exclusion principle; the very notion of "particle"; etc.

The reader should however be aware that, as stressed in the literature, these *deviations from conventional quantum mechanical laws are expected only in the interior of hadrons, or in the interior of systems of strongly interacting particles, while conventional quantum mechanical laws are recovered in full for the center-of-mass motion.*

For instance, Mignani, Myung and Santilli [38] proved the validity of the *conventional* uncertainty relations for the center-of-mass motion of a composite system characterized by hadronic mechanics, in a way fully compatible with *generalized* uncertainty relations for each individual constituent. A similar situation has been proved by Santilli [17] for the time reversal, or by Aringazin [51] for Pauli's principle.

These results are important because they establish the fact that essentially *no valid experimental evidence exists at this time for disproving hadronic mechanics*, for the simple reason that all available direct tests for strong interactions are essentially center-of-mass tests. To put it differently, in order to establish experimentally the validity or invalidity of hadronic mechanics, we have to repeat the historical process that lead to the establishing of quantum mechanics. The historical experimental measures conducted for charged particles under *external* electromagnetic interactions, must be repeated, this time, for hadrons under *external* strong interaction. No direct experimental study along the latter lines evidently exists as of this writing.

In the final analysis, readers with an open mind to potentially fundamental advances should notice the evident plausibility of the occurrence: conventional quantum mechanical laws for the center-of-mass motion of hadrons, and generalized hadronic laws for their internal structure.

The physical foundations for this plausibility is provided by another seminal contribution by Santilli, the notion of *closed essentially nonselfadjoint systems*, introduced in 1978 jointly with his algebraic and classical/operational studies [1], [2]. In a few simple words, it is generally believed that the stability of a system is provided by the stability of the orbits of each individual constituent. This is essentially the case of the stability of the solar system as well as of the atomic structure.

Santilli pointed out the existence in Nature of a class of more general systems which verify all total conventional conservation laws for their center-of-mass motion, but the internal equations of motions are nonhamiltonian. (See Fig. 2.)

These broader systems are essentially provided by composite systems with each individual constituent in *unstable* conditions due to exchanges of energy, linear momentum and other physical quantities with the rest of the systems. The point is that these nonconservations are merely internal exchanges under total conserved quantities, the system being, after all, isolated.

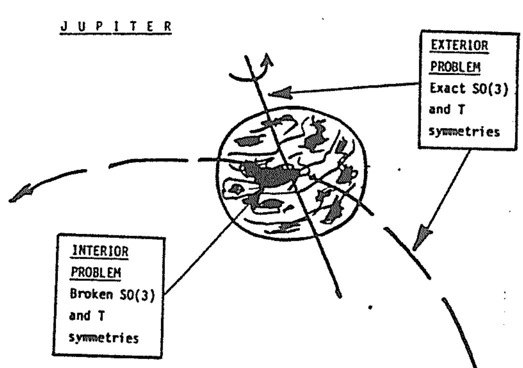


FIGURE 2. A reproduction of Figure 1, page 1208 of ref. [13], depicting a dichotomy of central relevance for the studies under review: the compatibility of the conventional symmetries and physical laws for the center-of-mass motion of celestial bodies (such as Jupiter), with manifest deviations from the same symmetries and physical laws in the interior dynamics. In fact, on one side, we have clear evidence on the stability of Jupiter's orbit in the Solar system with consequential manifest validity of the rotational symmetry for the exterior dynamics; on the other side, we have equally clear evidence for the existence in the interior motion of vortices with continuously varying angular momentum, with consequential internal violation of the rotational symmetry. Similarly, we have a manifestly reversible center-of- mass trajectory, as compared to a manifestly irreversible interior dynamics. A similar situation occurs for all other aspects at all levels of study, as we shall see, including the relativistic and the gravitational level. The dichotomy reviewed here was quantitatively interpreted by Santilli via the notion of closed- isolated systems with nonhamiltonian internal forces (see later on). The above dichotomy also provides the conceptual foundations of the fundamental experiments proposed later on in §3.5 regarding clear phenomenological predictions of apparent violation of Einstein's Special Relativity in the interior of (unstable) hadrons in flight, while the relativity is preserved for center-of-mass motions of the same hadrons, say, when moving in a particle accelerator.

The mathematical consistency of these broader systems at the classical and the operator level was also shown in the original proposals [1,2].

At the classical level, *closed nonhamiltonian* systems are characterized by the Birkhoffian equations (ref. [2], p. 624; see also monograph [15], pp. 234-237)

$$m_k \ddot{\vec{r}}_k = \vec{F}_k^{SA}(\vec{r}) + \vec{F}_k^{NSA}(t, \vec{r}, \dot{\vec{r}}, ...) , \tag{1.53.a}$$

$$\dot{H} = \frac{d}{dt}(T + V) = 0 , \tag{1.53.b}$$

$$\dot{\vec{P}}_{tot} = \frac{d}{dt}(\sum_1^{\mu}{}_k m_k \vec{P}_k) = 0 , \tag{1.53.c}$$

$$\dot{\vec{M}}_{tot} = \frac{d}{dt}(\sum_k \vec{r}_k \wedge \vec{P}_k) = 0 , \tag{1.53.d}$$

$$\dot{\vec{G}}_{tot} = \frac{d}{dt}(\sum_k m_k \vec{r}_k - t\vec{p}_k) = 0 , \tag{1.53.e}$$

where the symbols "SA" ("NSA") indicate verification (violation) of the integrability conditions for the existence of a potential, those of variational self-adjointness.

An intriguing point is that the conventional total conservation laws *are not necessarily* subsidiary constraints to the equations of motion. In fact, Eqs. (1.53.b)-(1.53.e) are verified when

$$\sum_{1}^{n} {}_k\vec{F}_k^{NSA} = 0 \, ,$$

$$\sum_{1}^{n} {}_k\vec{r}_k \wedge \vec{F}_k^{NSA} = 0 \, ,$$

$$\sum_{k=1}^{n} \vec{p}_k \cdot \vec{F}_k^{NSA} = 0 \, , \tag{1.54}$$

which consist of seven conditions on $3n$ unknown quantities, the components of the nonhamiltonian forces $\vec{F}_k^{NSA}$. Infinite varieties of unconstrained solutions therefore exist for $n \geq 3$. The case $n = 2$ has been proved to be consistent, even though with very special features (e.g., only circular orbits are possible). The case $n = 1$ is impossible for the evident reason that an isolate particle cannot be under nonhamiltonian external forces (see Fig. 3).

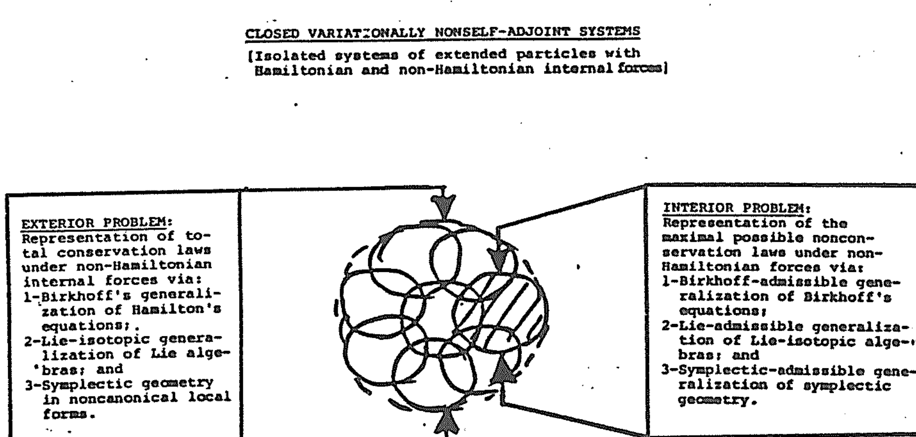


FIGURE 3. A reproduction of Figure 5.1, page 529 of ref. [16], presenting a schematic view of the notion of "closed non-self-adjoint systems" originally proposed in ref. [1], [2] and then investigated at several levels of study by a number of authors (see §1.1). Conventionally, closed-isolated systems are represented by assuming that total, conserved, quantities (such as energy H, angular momentum M, etc.) are the generators of space-time symmetries (translations, rotations, etc.). The assumption of the simplest conceivable Lie product $AB - BA$ then requires the Hamiltonian H to represent all acting internal forces. Additional technical arguments restrict all internal forces to be action-at-a-distance potential/Hamiltonian. Santilli's proposal is to assume the *same* total, conserved physical quantities H, M, etc., as the generators of isotopically lifted space-time symmetries, in which the product is less trivial, e.g., $A * B - B * A = ATB - BTA$. This yields an additional element T, besides the Hamiltonian H, to represent internal forces that are beyond the representational capability of the Hamiltonian (Fig. 1). This results into the covering notion of closed nonhamiltonian systems which are at the foundation of the studies of Lie-isotopy at all levels: Newtonian, relativistic, gravitational, statistical, etc. Remarkably, the space-time symmetries are not broken under the presence of internal nonhamiltonian forces, but merely realized in a structurally more general, but isomorphic way. This important finding was only empirically known in the early stages of the Lie-isotopic theory, and subsequently formalized in ref. [22] (see later on Theorem 2.9). The implications of these results are far reaching at all levels of study. To begin, Santilli has disproved statements such as "breaking of the Lorentz symmetry" or "Lorentz noninvariance," which are technically correct only when specifically referred to the "simplest possible realization of the Lie product $AB - BA$." In fact, Theorem 2.9 allows the reconstruction of the same symmetry as exact at the Lie-isotopic level when broken at the conventional level. Furthermore, the notion under consideration and its underlying Lie-isotopic methods, allow the possibility of constructing genuine covering of contemporary relativities, as we shall see in §3, with far reaching implications in classical as well as particle mechanics. All the above considerations refer to the "exterior problem," here intended as the description of the systems from the exterior with the emphasis on total conservation laws, along the line of monographs [4], [15]. A complementary aspect is the "interior problem" intended as the study of only one constituent of the system when all other constituents are considered as external. The emphasis is now shifted to the maximal possible nonconservation of the physical quantities of each constituent (of course in a way compatible

with total conservation laws), as the best way to maximize internal dynamical conditions. This complementary approach is along the Lie-admissible line of study of monographs [5], [16] which is not reviewed here.

The operator image of systems (1.53) was also identified by Santilli in his second memoirs of 1978. In fact, the operator H in his Eqs. (1.37) represents the total physical energy of the system and it is evidently conserved because of the Lie character of the underlying algebra. We can therefore write the following operator version of systems (1.53)

$$i\dot{H} = [H, H]_{\hat{\xi}} = H * H - H * H \equiv 0 \ ,$$

$$[\vec{P}_{tot}, H]_{\hat{\xi}} = [\vec{M}_{tot}, H]_{\hat{\xi}} = [\vec{G}_{tot}, H]_{\hat{\xi}} = 0 \ . \tag{1.55}$$

Notice that the observability of physical quantities persists because, as recalled earlier, one can select isotopes (1.52) with $T = G$, under which a total Hamiltonian H which is conventionally hermitian in quantum mechanics, remains hermitian in hadronic mechanics. Also, its eigenvalues remain real (although different!) [36].

This confirms the point touched earlier, that the center-of-mass motion of a composite system obeying hadronic mechanics, when inspected from the outside, verifies conventional physical laws. Nevertheless, the system admits in its interior a generalized integrodifferential unit $\hat{I}$ for which conventional physical laws are inapplicable, in favor of suitable covering laws.

In Santilli's words [21], the solar system is a closed Hamiltonian system whereby total stability is provided by the stability of each orbit. The planets, however, possess structures considerably more complex than that. For instance, Jupiter is an example of a closed nonhamiltonian system because, when assumed as isolated from the rest of the solar system, it verifies total conservation laws; yet its internal structure is highly nonconservative, nonhamiltonian (and irreversible).

In the transition to the particle setting, the atomic structure is analytically equivalent to that of the solar system because, again, total stability is provided by the stability of each orbit. Santilli's view is that the hadronic structure is equivalent to that of Jupiter [2], in the sense that each isolated hadron evidently verifies total conservation laws; nevertheless, the internal orbits are expected to be generally nonconservative due to the deep mutual overlapping of the wave packets of the constituents. (See Fig. 4.)

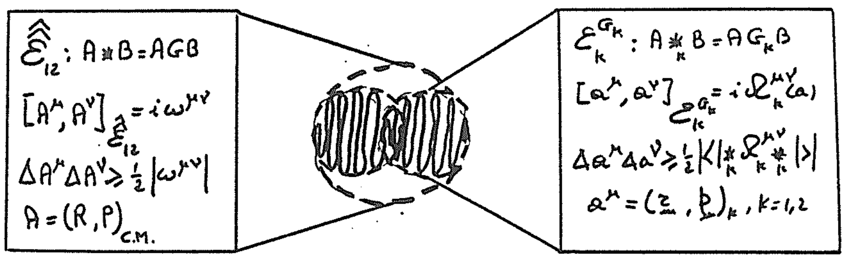


FIGURE 4. A reproduction of Figure 9.1, page 1945 of ref. [38], signaling the achievement of mathematical maturity in the operator formulation of closed nonhamiltonian systems on Hilbert spaces. Ref. [38] established the operator counterpart of the dichotomy of Figs. 2, 3, that is, the validity of conventional quantum mechanical laws for the center-of-mass motion of the state, in a way compatible with structurally more general laws for the interior dynamics. The analysis was presented for the case of Heisenberg's uncertainty principle, with guidelines for the expected extension to all other physical laws and principles of quantum mechanics. In fact, Aringazin [70] has recently proved the same occurrence for Pauli's exclusion principle. These operators results are merely indicated for the purpose of informing the reader on the existence of mathematically consistent operator counterparts of the classical models reviewed in this work, with the hope of reviewing them in detail in a future paper.

Hadronic mechanics was then applied, along the above concept of closed non-Hamiltonian systems, to the construction of a structure model of the π^0 as a generalized bound state of one electron e^- and one positron e^+, although in a mutated state expected from the condition of total mutual immersion of their wavepackets, called *eletons* and denoted with the symbols $\epsilon^\pm$ [2]. The model $\pi^0 = (\epsilon^+, \epsilon^-)$ resulted to be able to represent *all* physical characteristics of the π^0, such as: rest energy, meanlife, charge radius, total charge, spin, magnetic and electric dipole moments, as well as the primary decay. The above hadronic structure model was extended in memoir [2] to all remaining light mesons, resulting in structures of the type $\pi^\pm = (\epsilon^+, \epsilon^\pm, \epsilon^-)$, $K^0 = (\hat{\pi}^+, \hat{\pi}^-)$, $K^\pm = (\hat{\pi}^+, \hat{\pi}^\pm, \hat{\pi}^-)$, where the superscript ^ denotes expected mutation of the characteristics of the particles caused by total immersion within the hadronic medium.

Hadronic mechanics was then applied to the quantitative interpretation of Rutherford's historical hypothesis that the neutron n is a "compressed hydrogen atom", along the representation submitted in memoir [2], $n = (p^+, \epsilon^-)$, i.e., as a generalized bound state of an ordinary (unmutated) proton p^+ and a mutated electron ϵ^-. The total angular momentum $\frac{1}{2}$ for state (p^+, ϵ^-) was first achieved in papers [24] via the construction of the $\widehat{SU}(2)$-isotopic spin symmetry and its representations, and resulted to be due to very simple constraints on the *orbital* angular momentum of the electron when "compressed" within the densiest object measured in laboratory until now. The subsequent paper [25] showed that the generalized bound state (p^+, ϵ^-) is capable of representing *all* the characteristics of the neutron, i.e., rest energy, meanlife, charge radius, total charge, spin, magnetic and electric dipole moments as well as its primary decay. A comprehensive presentation of the model is provided in paper [28]. A detailed analysis of the notion of eleton is provided in paper [27] via a generalization of conventional field equations that is invariant under the Poincaré-isotopic symmetry [26].

In this way, Santilli illustrated the possibility of achieving the primary objective for which hadronic mechanics had been suggested: the identification of the hadronic constituents with (massive) physical particles simply produced in the spontaneous decays, under the assumption of obeying a generalized mechanics when in condition of total mutual immersions, and of recovering ordinary quantum mechanics when exiting the hadronic structure.

The full compatibility of this novel structure model of hadrons with established quark models is under study by a number of authors [44,50]. Rather than being in conflict with established theories, hadronic mechanics appear to offer some genuine possibility of resolving their basic problematic aspects, such as: achieving null probability of tunnel effect for free quarks, reaching fractional charges as mutation of ordinary ones, etc.

To summarize our viewpoint, the classical analytical realization of Santilli's isotopies (Birkhoffian mechanics [15]) is nowadays established on both mathematical and physical grounds. The corresponding operator counterpart (hadronic mechanics [36]) is clearly consistent on pure mathematical grounds, but far from being established on physical grounds, although no experimental evidence can be moved against the generalized mechanics at this moment. In the final analysis, the central physical notion of the theory (that of closed nonhamiltonian system) is manifestly plausible for the representation of hadrons, as we shall see better in the final part of this analysis, and, more technically, in a possible subsequent review.

We now briefly review the process of *naive hadronization*, i.e., *the simplest possible mapping of Birkhoffian into hadronic mechanics.* This aspect is important for our analysis because it throws a deeper light in the notion of isounit of the Lie-isotopic theory (besides indicating how diversified the studies of compatibility and consistency have been conducted until now).

The conventional *naive quantization*, i.e., the mapping of classical Hamiltonian into quantum mechanics, can be characterized by the mapping of the action functional A into a *constant unit*, Planck's unit $\hbar = 1$, time $-i\log\psi$, i.e.,

$$A \to -i 1 \log\psi \ , \tag{1.56}$$

under which Hamilton-Jacobi Eqs. (1.18) assume the form

$$-\frac{\partial A}{\partial t} = H \to i\frac{\partial}{\partial t}\psi = H_{op} \ ,$$
$$\frac{\partial A}{\partial \vec{r}} = \vec{p} \to -i\frac{1}{\psi}\vec{\nabla}\psi = \vec{p}_{op} \ , \tag{1.57}$$

thus becoming Schrödinger's equations

$$i\frac{\partial}{\partial t}\psi = H\psi \ ,$$
$$-i\vec{\nabla}\psi = \vec{p}\psi. \tag{1.58}$$

Animalu and Santilli [41] pointed out that mapping (1.56) is expected to be insufficient for Pfaffian action principles, because of its inability to provide a representation of the contact/nonlocal/nonhamiltonian forces of the broader systems considered. The authors proposed instead, *as naive rule of hadronization, the mapping of the Pfaffian action functional $\hat{A}$ into the operator unit of the theory*, the isounit of hadronic mechanics $\hat{I}$, time$-i\log\psi$, i.e.,

$$\hat{A} \to -i\hat{I}\log\psi. \tag{1.59}$$

For our needs we now consider the following particularized Pfaffian action

$$\hat{A} = \int_{to}^{t} [M_k^i(\vec{r},\vec{p})p_i\dot{r}^k - H(t,\vec{r},\vec{p})]dt \ ,$$
$$\det(M_k^i) \neq 0 \ , \tag{1.60}$$

with Hamilton-Jacobi equations (which are still of genuine generalized nature, yet of the simpler form)

$$\frac{\partial \hat{A}}{\partial t} = H \ ,$$

$$\frac{\partial \hat{A}}{\partial \vec{r}^k} = M^i_k p_i \ ,$$

$$\frac{\partial A}{\partial \vec{p}_k} = 0 \ . \tag{1.61}$$

The application of mapping (1.59) to Eqs. (1.61) then yields the forms [41]

$$-\frac{\partial \hat{A}}{\partial t} = H \rightarrow i(\frac{\partial \hat{I}}{\partial t})\log\psi + i\hat{I}\frac{\partial}{\partial t}\psi = H^{op} \ ,$$
$$\frac{\partial \hat{A}}{\partial \vec{r}_k} = -i(\nabla_k \hat{I})\log\psi - i\hat{I}\vec{\nabla}_k\psi = M^i_k \vec{p}^{op}_i \ , \tag{1.62}$$

which can be rewritten

$$i\frac{\partial}{\partial t}\psi = [H - i\frac{\partial \hat{I}}{\partial t}\log\psi] * \psi \stackrel{\text{def}}{=} H^{\text{eff}} * \psi \ ,$$

$$-i\vec{\nabla}_k\psi = [M^i_k \vec{p}_i + i(\vec{\nabla}_k \hat{I})\log\psi] * \psi \stackrel{\text{def}}{=} M^i_k \vec{P}^{\text{eff}}_i * \psi \ , \tag{1.63}$$

yielding precisely the iso-Schrödinger's Eqs. (1.45), plus corresponding equations for the linear momentum. Notice the natural appearance under hadronization of a nonlinearity in the wavefunctions, besides additional nonlinearities emerging from the arbitrary functional dependence of the isotopic element (see below).

A mathematically rigorous formulation of hadronization was achieved by (the mathematician) E. B. Lin [63] via the methods of symplectic quantization. Recall that the Birkhoffian mechanics can be constructed via *noncanonical* transformations of Hamiltonian mechanics (and remains form-invariant under these general transformations). Along parallel lines, hadronic mechanics can be constructed via *nonunitary* transformations of quantum mechanics (and also remains form-invariant under the most general possible transformations) [6]. Lin essentially shows that the lifting of conventional, symplectic quantization techniques (e.g., prequantization) characterized by noncanonical (nonunitary) transformations provides precisely the desired hadronization, as expected.

This completes the objective of this section, to show that the classical and operator realizations of the notion of analytic isotopy, not only are individually consistent, but admit a consistent mapping of the former into the latter, the entire process constituting a true generalization of conventional theories.

A few comments are now in order. Evidently, the assumption of the simpler Pfaffian form (1.60) has the objective of rendering the generalized action functional independent of the linear momentum. This, in turn, allows the construction of an operator image in which the wavefunction has the familiar functional dependence $\psi(t,\vec{r})$ without a dependence on the momentum.

A personal communication by Santilli confirms the rather vast capabilities of action (1.60) to represent nonhamiltonian interactions, once the several degrees of freedom of Birkhoffian mechanics are taken into consideration (ref. [15], pp. 54-67). Nevertheless, Santilli stresses the fact that, unlike the case for general action (1.20), the direct universality of the reduced form (1.60) has not been proved as of today. In case action (1.60) does not result to be directly universal, the construction of a "wave mechanics" with "wavefunction" dependent also in the momentum, $\psi(t,\vec{r},\vec{p})$, is inevitable.

Second, hadronization (1.62) indicates the *intrinsic nonlinearity of hadronic mechanics*, where the nonlinearity is referred also to the dependence of the equations of motion in the wavefunctions. As a matter of fact, the iso-Schrödinger's equation in its original formulation by Santilli, that in term of the *Birkhoffian operator* B [6], is the most general nonlinear as well as nonlocal equation of motion in operator form know until now. We shall write it in the explicit form

$$i\frac{\partial}{\partial t}\psi = B\hat{*}\psi = B(t,a,\psi,\psi^{\dagger},...)D(t,a,\psi,\psi^{\dagger},...)\psi\,. \qquad (1.64)$$

All known equations, nonlinear in the wavefunctions as well as in other quantities, are evidently a particular case of the above equation.

We are referring here to the *direct universality of hadronic mechanics*, i.e., the capability of representing *all* conceivable nonlinear and nonlocal equations verifying certain topological restrictions (universality) in the frame of the observer (direct universality). This is merely the operator counterpart of the classical direct universality of Birkhoffian mechanics [15].

The proof of this important property is quite easy. Recall that the universality of Birkhoff's equations ultimately results from the form-invariance of the theory under the most general possible (noncanonical) transformations. The direct universality of the iso-Heisenberg's or the iso-Schrödinger's equations then follows from their form-invariance under the most general possible (evidently nonunitary) transformations.

As an example, it is an instructive exercise for the interested reader to show that certain nonlinear wave equations currently under investigation by Weinberg [71] and others (to explore a possible nonlinearity of quantum mechanics) of the type

$$i\frac{\partial}{\partial t}\psi = \frac{\partial}{\partial \psi^\dagger} H(\psi\psi^\dagger, ...) \tag{1.65}$$

are in fact a particular case of hadronic mechanics, i.e., they can always be rewritten into an equivalent isomodular form (1.64).

But there is more. The direct universality of the theory, combined with its isotopic structure, have rather profound epistemological implications for the very notion of nonlinearity.

This is another central aspect of the Lie-isotopic theory we shall consider in more detail later on, when reviewing the isotransformation theory in the next chapter. At this point we can limit ourselves to the remark that the isotopic element D of Eq. (1.64) is arbitrary. As a result, all nonlinear terms, whether in the wavefunctions or in the other quantities, can be incorporated in the isotopic element, in which case the (nonlinear) Birkhoffian operator B is replaced by a linear Hamiltonian H, and we shall write

$$\begin{aligned} i\frac{\partial}{\partial t}\psi &= B(t, a, \psi, \psi^\dagger ...)D(t, a, \psi, \psi^\dagger, ...)\psi \\ &\equiv H(t, a)T(t, a, \psi, \psi^\dagger, ...)\psi \\ &\equiv H * \psi \,. \end{aligned} \tag{1.66}$$

The implications of the above results are rather deep. They essentially establish that, not only we have a direct universality for all possible nonlinear (and nonlocal) theories, but in addition *any possible nonlinear (and nonlocal) theory can always be rewritten in an equivalent isolinear form.* It is regrettable that the authors of studies [71] do not appear to be aware of the Lie-isotopic theory, because the intrinsic isolinear structure of Weinberg's equation (1.64) may evidently void most of their argumentations.

This is the technical reason why Santilli (private communication) does not consider *nonlinearity* a structure characterizing feature. Instead, he considers structurally fundamental the *nonlocality* and *nonhamiltonian* character caused by the deep mutual overlapping of the wave packets of strongly interacting particles.

Regrettably, we cannot enter into a detailed analysis of the implication of the isotransformation theory for Weinberg's work because this is substantially outside the scope of this review. Nevertheless, the above occurrence is important to point out the rather deep implications of the Lie-isotopic theory for a virtually endless variety of frameworks in classical, operator and other branches of physics.

In addition to the above, Weinberg's nonlinear generalization of quantum mechanics [71] is apparently afflicted by rather fundamental problematic aspects [43] essentially caused by the fact that it is based on a general, nonassociative, Lie-admissible generalization of the conventional associative envelope of quantum mechanics. These algebras are known not to possess a consistent unit [1]. As a result, all basic physical laws and quantities of quantum mechanics that are central dependent on the unit (1.29) do not possess a consistent formulation in Weinberg's theory. This is the case for the measurement theory, the notion of quantum of energy, the Casimir invariants, etc. Moreover, the nonassociative character of the underlying envelope activates the inconsistency theorems by Okubo [53] on nonassociative generalizations of Schrödinger's equations precisely of type (1.65). Finally, such a nonassociative character of the operator algebra prevents the equivalence between the Heisenberg-type and the Schrödinger-type representations in Weinberg's theory [43].

These problematic aspects have been mentioned here to point out the fact that they are all resolved by Santilli's central assumption for the construction of hadronic mechanics; the existence of the generalized unit (1.35). The occurrence is also useful to illustrate the central role of the *preservation of the associative character of the envelope*, Eq. (1.34). In fact, general Lie-admissible algebras do enter in hadronic mechanics, but for the characterization of the *brackets of the time evolution* for the exterior-open problem, while the underlying envelope remains associative. In turn, the preservation of such an underlying iso-Heisenberg and iso-Schrödinger's representations [36], and the resolution of the other problematic aspects of Weinberg's formulation.

Another aspect that is worth mentioning is the use of the iso- Schrödinger's equation for a deeper understanding of the Berry's phase [72], as studied by Mignani [73].

Next, we want to point out a fundamental feature of hadronization (1.59), according to which *the isotopic lifting of quantum mechanics is essentially centered on the replacement of Planck's constant unit* $\hbar = 1$ *with* *the operator isounit* $\hat{I}$

$$\hbar(=1) \rightarrow \hat{I}(t, a, \psi, \psi^\dagger, ...). \tag{1.67}$$

In turn this provides another illustration of the intriguing physical implications of the Lie-isotopic theory in general, and of Santilli's notion of generalized unit [1,2], in particular.

The epistemological implications of concept (1.67) are self-evident. They are essentially centered on the expectation that the quantum of energy, while so effective for the area of its original conception (discrete energy states of the individual electrons of the atomic structure), is expected to be insufficient for the representation of the nonlocal and nonhamiltonian conditions of wavepackets in deep mutual immersion.

This is one of the reasons why Santilli carefully avoids the use of the terms "quantization" or "quantum mechanics" when referring to the operator mechanics characterized by the Lie-isotopic theory.

We now close these analytic comments with the indication of the fact that *the Birkhoffian and hadronic mechanics constitute genuine coverings of their original counterparts, the Hamiltonian and quantum mechanics*, in the sense that:

1. the generalized theories are conceived for physical conditions intrinsically more general than those of the original theories (essentially nonhamiltonian interactions);

2. the generalized theories are constructed with mathematical methods essentially more general than those of conventional theories (Lie-isotopic methods); and

3. the generalized theories are capable of approximating the conventional ones as close as desired, e.g., for

$$\Omega \approx \omega \text{ or } \hat{I} \approx \hbar\,, \tag{1.68}$$

and they recover the conventional theories identically when all the nonhamiltonian interactions are null, e.g., for

$$\Omega \equiv \omega \text{ or } \hat{I} \equiv \hbar. \tag{1.69}$$

1.4 The Notion of Geometrical Isotopy

We now briefly touch upon another notion of isotopy, this time at the *geometrical* level.

Let M be an n-dimensional C^∞-manifold with local coordinates r_k, $k = 1, 2, ..., n$, and let T^*M be its cotangent bundle with local coordinates a^μ, $\mu = 1, 2, ..., 2n$, $a = (r, p)$. The familiar *canonical one-form* on T^*M can then be written

$$\theta_1 = p_k dr^k \equiv R^o_\mu(a) da^\mu \ , \tag{1.70}$$

where one recognizes the same R^o as that of Eqs. (1.15).

The *fundamental symplectic two-form* on T^*M can then be written

$$\theta_2 = d\theta_1 = dp_k \wedge dr^k = \frac{1}{2}\omega_{\mu\nu} da^\mu \wedge da^\nu \ , \tag{1.71}$$

where $\omega_{\mu\nu}$ is the covariant tensor of Eqs. (1.17).

Form (1.71) is nowhere degenerate and "closed" (in the geometrical sense that $d\theta_2 = 0$). The space T^*M, when equipped with the form θ_2, becomes a symplectic manifold in the local canonical coordinates $a = (r, p)$. All the several aspects of the symplectic geometry then follow (see, e.g., ref. [74]).

> *DEFINITION 1.4 [1],[15]* (Geometric Isotopy)*: An* isotopic mapping *(or* image *or* lifting*) of a symplectic manifold with fundamental two-form (1.71) is any mapping in the same local chart that preserves the symplectic character of the two-form, i.e., its closed and nowhere degenerate character, but remains otherwise arbitrary.*

Evidently, Birkhoff's equations characterize, not only a Lie-algebra isotopy (in their contravariant form), but also a corresponding symplectic isotopy (in their covariant form).

In fact, the canonical one form (1.70) is replaced by the Pfaffian one-form

$$\hat{\theta}_1 = R_\mu(a) da^\mu. \tag{1.72}$$

The associates two form

$$\hat{\theta}_2 = \frac{1}{2}\Omega_{\mu\nu}(a) da^\mu \wedge da^\nu \ , \tag{1.73}$$

where the tensor $\Omega_{\mu\nu}$ is given by Eqs. (1.21b), is also closed and nondegenerate [15]. As such, the Birkhoffian two-form (1.73) characterizes the most general possible symplectic two-form in local coordinates. The direct universality of the symplectic geometry in classical mechanics then follows from that of Birkhoff's equations. This is another important result of monograph [15].

The implications of the above geometrical aspects are far reaching.

Recall that, at the abstract, coordinate-free level, all symplectic two-forms coincide. The differentiations merely emerge in local realizations, the canonical two-form being the simplest conceivable one, while the Birkhoffian two-form being the most general possible one.

Exactly the same results occur at the analytic level. In fact, Hamiltonian and Birkhoffian mechanics coincide at the abstract, coordinate-free level [15]. As a matter of fact, the latter has been constructed by Santilli precisely under the condition of coinciding with Hamiltonian mechanics at the abstract coordinate-free level.

We can therefore expect a similar occurrence at the algebraic level too. In fact, the Lie-isotopic theory has been proposed and constructed precisely in such a way to coincide with the conventional formulation at the abstract coordinate-free level. The differences merely occur in local charts: the conventional formulation of Lie's theory is the simplest conceivable one, ultimately equivalent to the canonical, analytic-geometrical counterpart. Santilli's Lie-isotopic realization is the most general possible form, which is ultimately equivalent to the Birkhoffian analytic-geometrical counterpart.

This final unity of vision is, in turn, fundamental for understanding Santilli's capability of reconstructing at the higher Lie-isotopic level, exact space-time symmetries (e.g., the rotational, Galilean and Lorentz symmetries) when conventionally broken within the context of their simplest possible realizations. The review of this occurrence is, after all, a central objective of this presentation.

1.5 Final Introductory Remarks

A few final remarks appear to be recommendable to prevent possible misrepresentations of this review.

Recall that all simple Lie algebras (over a field of characteristic zero) have been classified by Cartan a long time ago and are today well known. Thus, the reader should *not* expect new simple algebras from the Lie-isotopic lifting of the conventional Lie's theory.

Rather than looking for new algebras (or groups), the scope of the Lie-isotopic theory is that of identifying new, structurally more general realizations of known algebras (or groups).

As we shall see, the Lie-isotopic theory permits in fact the identification of a generally infinite family of physically different symmetry transformations which are all representations of the same simple, abstract, algebra.

Also, readers may tend to expect that all conventional methods currently available for Lie algebras (such as the representation theory) are directly applicable to any Lie theory, thus including the Lie-isotopic one.

This second, rather natural expectation can be readily disproved by noting that a compact Lie algebra (or group) can be turned into a noncompact form under isotopic lifting, evidently depending on the topology of the assumed isounit $\hat{I}$. Available methods, such as the representation theory for compact algebras (groups), are known not to be directly applicable for noncompact structures. A reinspection of the representation theory is then in order.

Rather than having preconceived assumptions, the reader is encouraged to enter into the study of Lie-isotopic algebras with an open mind, and the expectation that all the various methodological aspects worked out for Lie's theory must be reinspected and eventually reformulated for the covering Lie-isotopic theory.

Our final introductory remark is that Santilli's Lie-isotopic theory, despite its beauty, is far from being the ultimate Lie theory, as stressed by the author himself. This point is illustrated quite vividly by the classical Hamiltonian mechanics, because the conventional Poisson brackets have the structure [1]

$$L : [A, B]_U = \frac{\partial A}{\partial r^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k}\frac{\partial A}{\partial p_k} \stackrel{\text{def}}{=} (A, B) - (B, A) = \text{Lie}\ ,$$

$$U : (A, B) = \frac{\partial A}{\partial r^k}\frac{\partial B}{\partial p_k} = \text{ Nonassociative Lie-admissible}\ , \tag{1.74}$$

namely, the Lie algebra L of the Poisson brackets is the anticommutative algebra attached to a *nonassociative* algebra U evidently because

$$U : ((A, B), C) \neq (A, (B, C)). \tag{1.75}$$

In particular the algebra U results to be a *nonassociative Lie-admissible algebra* precisely because (as per definition of these algebras) its attached algebra $[A, B]_U$ is Lie. The same result evidently persists at the Birkhoffian level (ref. [15], p. 152).

By comparison, the algebraic structure of the conventional Heisenberg's brackets is given by

$$L : [A, B]_\varepsilon = AB - BA = \text{ Lie },$$

$$\xi : AB = \text{ Associative Lie-admissible }, \tag{1.76}$$

namely, the Lie algebra L of conventional quantum mechanics is the anti-commutative algebra attached to an *associative* algebra ξ which, as such, is also Lie-admissible.

The physical and mathematical implications of the above findings are predictably deep. On physical grounds, we have to expect problematic aspects in the quantization of conventional Hamiltonian mechanics, for the evident reason that a mapping of a nonassociative envelope U into an associative form ξ simply cannot be formulated in a consistent way (see ref. [6] for a study of this aspect).

This problematic aspect can be readily avoided in hadronic mechanics because Santilli's Lie-isotopic brackets can always be formulated according to the structure [2]

$$\hat{L} : [A, B]_U = ATB - BTA \stackrel{\text{def}}{=} (A, B) - (B, A) = \text{Lie} - \text{isotopic },$$

$$U : (A, B) = ARB - BSA = \text{ Nonassociative Lie-admissible },$$

$$T = R + S , \tag{1.77}$$

namely, a Lie-isotopic algebra, owing to its nontriviality, can always be reformulated as the antisymmetric algebra attached to a nonassociative Lie-admissible algebra. Consistency of algebraic structures with the classical counterpart (1.74) is then regained.

On mathematical grounds, the above findings establish the fact that the most *general possible formulation of Lie's theory is that via nonassociative envelopes*, along the conceptual lines so clearly expressed by the Poisson bracket, Eq. (1.74).

This is the reason why Santilli provided his primary efforts for the formulation of the theory at the nonassociative Lie-admissible level, and presented his Lie-isotopic studies only as a simpler particularization. It is remarkable that these so fundamental structures, so clearly embedded in the structure of the conventional Poisson brackets, had escaped attention in the mathematical and physical literatures until the appearance in 1978 of ref. [1,2,3].

This review is restricted to *associative Lie-admissible formulations*, although in their most general known form. The covering *nonassociative Lie-admissible formulations* shall be ignored hereon, and referred to a possible future review.

2 THE MATHEMATICAL FOUNDATIONS OF THE THEORY

2.1 Central Role of the Universal Enveloping Algebra

Let us begin by recalling the central role for Lie's theory of the universal enveloping algebra. This role is somewhat de-emphasized in the contemporary physical literature, but not in the mathematical one. We shall closely follow in this review the presentation of monograph [15], pp. 148-154.

The terms "Lie's theory" are referred today to an articulated body of sophisticated mathematical tools encompassing several disciplines. Whether in functional analysis or in the theory of linear operators, the structure of the contemporary formulation of Lie's theory can be reduced to the following three parts:

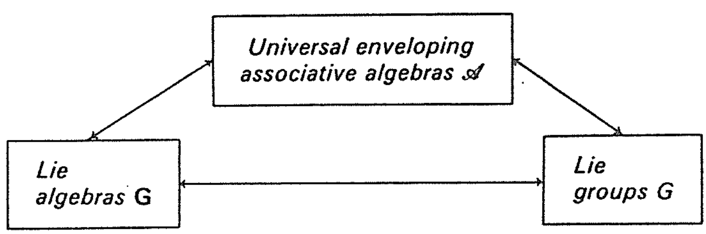


FIGURE 5. The structure of the conventional formulation of Lie's theory with the emphasis on its central mathematical structure, the universal enveloping associative algebra. The Lie-isotopic theory follows exactly the same lines, beginning with the generalization of the envelope and then following with the consequential generalization of all remaining aspects of the theory.

As duly emphasized in the mathematical literature (see, for instance, Jacobson [75], Dixmier [76], and others), a truly fundamental part of Lie's theory is the enveloping algebra ξ. In fact, the algebra ξ provides a symbiotic characterization of both the Lie algebras and the Lie groups. This is due to the fact that the basis of ξ (which is constructed via the Poincaré-Birkhoff-Witt Theorem, to be reviewed in the next section) is given by an infinite number of suitable polynomial powers of the generators X_i of $\mathbf{G}$ of the type

$$\xi : 1 \in \mathbf{F}; \quad X_i; X_iX_j(i \leq j); \quad X_iX_iX_k(i \leq j \leq k); \dots , \tag{2.1}$$

where the products $X_i\ X_j$, etc., are associative. It then follows that the Lie algebra $\mathbf{G}$

$$\mathbf{G} : [X_i, X_j] = X_iX_j - X_jX_j = C_{ij}^k X_k, \tag{2.2}$$

is (homomorphic to) the attached algebras ξ^- of ξ. The Lie group G of $\mathbf{G}$ is then the infinite power series

$$G : e^{\theta^k X_k} = 1 + \frac{\theta^k}{1!} X_k + \frac{\theta^i \theta^j}{2!} X_i X_j + \dots , \tag{2.3}$$

which, evidently, can be properly defined and treated only in the enveloping algebra (note that all terms from X_iX_j on are *outside* the Lie algebra). One can then see why fundamental aspects of Lie *algebras* (such as the representation theory) are treated by mathematicians within the context of its *enveloping algebra.*

On physical grounds, the role of the enveloping algebra is equally crucial. For instance, a frequent physical problem is the computation of the magnitude of physical quantities such as the angular momentum operator M^2. While the components M_i of M are elements of the Lie algebra $\mathbf{SO}(3)$, the quantity M^2 is *outside* $\mathbf{SO}(3)$ and can only be defined in the (center of) the enveloping algebra $\xi(\mathbf{SO}(3))$. Thus, while the Lie algebra $\mathbf{SO}(3)$ essentially characterizes the components of the angular momentum and their commutation rules, the envelope $\xi(\mathbf{SO}(3))$ characterizes: 1) the components M_i; 2) their commutations relations via the attached rule $\xi^- \approx \mathbf{SO}(3)$; 3) the magnitude of the angular momentum M^2; 4) the exponentiation to the Lie group of rotations; 5) the representation theory, etc. Also, enveloping algebras play a central role in quantization at large and, specifically, in the quantization of Lie algebras and Lie groups. In short, we can state that a *truly primitive part of the contemporary formulation of Lie's theory is its universal enveloping associative algebra.*

Once the mathematical and physical origins of this occurrence are understood in full, one can easily see how any consistent generalization of the enveloping associative algebra ultimately provides a generalization of the conventional formulation of Lie's theory.

The physical motivations for this study have been pointed out in Chapter 1, and are provided by the fact that Lie algebras characterize the fundamental equations of physical theories, their time evolution. Any generalization of Lie's theory then inevitably implies the achievement of broader physical capabilities.

The mathematical motivations of the study are equally evident. In the mathematical tradition, the efforts are devoted to the formulation of theories in their most general possible form. This is typically the case for mathematical formulations such as the symplectic geometry [74], which has indeed achieved its broadest possible formulation. It is a truism to say that a similar situation within the context of Lie's theory was not in existence prior to Santilli's studies of 1978, owing to the rather general referral of the enveloping algebra, not only to its associative form, but actually to such form in its simplest possible formulation.

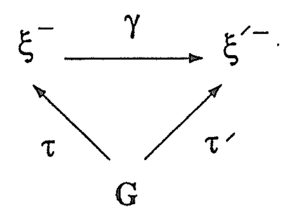
In the next section we shall review Santilli's studies toward a broader formulation of Lie's theory, beginning with the isotopic lifting of its enveloping algebra which admit a consistent, generalized, left and right unit (with the understanding that the still broader nonassociative envelopes [1] will not be considered). The reader should be aware that we shall follow Santilli's original presentation as close as possible.

2.2 Isotopic Lifting of the Universal Enveloping Associative Algebra [1], [15]

In this section we shall first review the definition of universal enveloping associative algebra and the methods for the construction of its basis according to the Poincaré-Birkhoff-Witt theorem [75]. We shall then present their *isotopic* liftings, that is, generalizations which preserve the associative character of the product. By keeping in mind the primitive character of the enveloping algebra in Lie's theory, the generalization presented in this section renders inevitable a corresponding reinspection of Lie algebras and of Lie groups.

> *DEFINITION 2.1 [75]: The* universal enveloping associative algebra *of a Lie algebra* $\mathbf{G}$ *is the set* (ξ, τ) *where* ξ *is an associative algebra and* τ *a homomorphism of* $\mathbf{G}$ *into the attached algebra* ξ^- *of* ξ *satisfying the following properties. If* ξ' *is another associative algebra and* τ' *a homomorphism of* $\mathbf{G}$ *into* ξ'*, a unique homomorphism* γ *of* ξ *into* ξ' *exists such that* $\tau' = \tau\gamma$*; i.e., the following diagram (2.4) is commutative.*

Whenever an algebra ξ belongs to the content of the definition above, we shall write $\xi(\mathbf{G})$. All Lie algebras are assumed, for simplicity, to be finite-dimensional. Also all algebras and fields are assumed to have characteristic zero, and the basis of all Lie algebras is ordered.



(2.4)

The construction of the enveloping algebra $\xi(\mathbf{G})$ is conducted as follows. Consider the algebra $\mathbf{G}$ as a (linear) vector space with basis given by the (ordered set of) generators X_i, $i = 1, 2, ..., m$. The *tensorial product* $\mathbf{G} \otimes \mathbf{G}$ is the ordinary Kronecker (or direct) product of $\mathbf{G}$ with itself as a vector space. Such a tensorial product constitutes an algebra because it satisfies the distributive and scalar laws. Also, the algebra is associative because the Kronecker product is associative. A general form of associative, *tensor* algebra which can be constructed on $\mathbf{G}$ as vector space is given by

$$\mathcal{F} = \mathbf{F}1 \oplus \mathbf{G} \oplus \mathbf{G} \otimes \mathbf{G} \oplus \mathbf{G} \otimes \mathbf{G} \otimes \mathbf{G} \oplus \ldots, \tag{2.5}$$

where F is the base field and $\oplus$ denotes the direct sum. Let $\mathcal{R}$ be the ideal generated by all elements of the form

$$[X_i, X_j] - (X_i \otimes X_j - X_j \otimes X_i)\,, \tag{2.6}$$

where $[X_i, X_j]$ is the product of $\mathbf{G}$. Then, the universal enveloping algebra $\xi(\mathbf{G})$ of $\mathbf{G}$ is given (or, equivalently, can be defined) by the quotient

$$\xi(\mathbf{G}) = \mathcal{F}/\mathcal{R}. \tag{2.7}$$

It is possible to prove that the algebra (2.7) satisfies all the conditions of Definition 2.1 (see, for instance, Jacobson [75]).

Of utmost importance for mathematical and physical considerations is the identification of the basis of $\xi(\mathbf{G})$. The quantities

$$M_s = X_{i_1} \otimes X_{i_2} \otimes ... \otimes X_{i_s} \tag{2.8}$$

are called *standard (nonstandard) monomials* of order s depending on whether the ordering

$$i_1 \le i_2 \le ... \le i_s \tag{2.9}$$

is verified (not verified). It is possible to prove that every element of $\xi(\mathbf{G})$ can be reduced to an F-linear combination of standard monomials and (cosets of) 1. This yields the following fundamental theorem on enveloping associative algebras.

Theorem 2.1 *(Poincaré-Birkhoff-Witt Theorem [75]): The cosets of* 1 *and the standard monomials form a basis of the universal enveloping associative algebra* $\xi(\mathbf{G})$ *of a Lie algebra* $\mathbf{G}$.

The associative envelope $\xi(\mathbf{G})$, as presented, is still abstract in the sense that the product of $\xi(\mathbf{G})$ is the tensorial product $X_i \otimes X_j$, while the product used in physical (e.g., quantum mechanical) applications is the conventional associative product $X_i X_j$. Consider then the algebra

$$A(\mathbf{G}) = F1 \oplus A^{(1)} \oplus A^{(2)} \oplus \ldots ,$$

$$A^{(s)} = X_{i_1}, \quad X_{i_2} \ldots X_{i_s}, \quad i_1 \leq i_2 \leq \ldots \leq i_s . \tag{2.10}$$

It is possible to prove that $\xi(\mathbf{G})$ is homomorphic to $A(\mathbf{G})$, in line with Definition 2.1. Thus, the algebra $A(\mathbf{G})$ can be assumed as the universal enveloping associative algebra of $\mathbf{G}$ with basis

$$\mathbf{1}, \quad X_i, \quad X_{i_1} X_{i_2}, \quad X_{i_1} X_{i_2} X_{i_3}, \quad \ldots,$$

$$i_1 \leq i_2, \quad i_1 \leq i_2 \leq i_3, \tag{2.11}$$

and arbitrary elements

$$X_{i_1}^{k_1} X_{i_2}^{k_2} \ldots X_{i_s}^{k_s} , \tag{2.12}$$

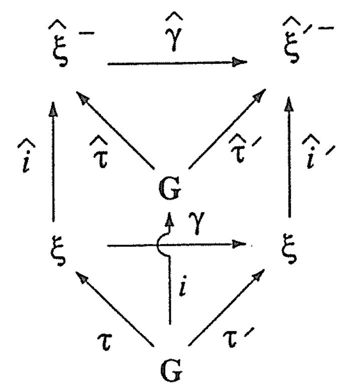
where the X's are the generators of $\mathbf{G}$. Notice that $A(\mathbf{G})$ is infinite-dimensional. The *center* of $A(\mathbf{G})$ is the set of all polynomials $P(X)$ verifying the property

$$[P(X), X_i]_A = 0 , \tag{2.13}$$

for all elements $X_i \in \mathbf{G}$. Most important elements of the center are the so-called *Casimir invariants* of $\mathbf{G}$. For additional study, we refer the interested reader to the mathematical literature on the topic [75],[76]. We move now to the identification of the desired associative-isotopic generalization

DEFINITION 2.2 [1], [15]: The isotopically mapped universal enveloping associative algebra of a Lie algebra $\mathbf{G}$ *is the set*

$((\xi,\tau),(\hat{\xi},i,\hat{\tau}))$ where: (1) (ξ,τ) is the universal enveloping associative algebra as per Definition 2.1; (2) i is an isotopic mapping of $\mathrm{G}, i\mathrm{G} = \hat{\mathrm{G}}$; (3) $\hat{\xi}$ is an associative algebra generally nonisomorphic to ξ; and (4) $\hat{\tau}$ is a homomorphism of $\hat{\mathrm{G}}$ into $\hat{\xi}^-$ such that the following properties are verified. If $\hat{\xi}'$ is still another associative algebra and $\hat{\tau}'$ a homomorphism of $\hat{\mathrm{G}}$ into $\hat{\xi}'$, a unique homomorphism $\hat{\gamma}$ of $\hat{\xi}$ into $\hat{\xi}'$ exists such that $\hat{\tau}' = \hat{\gamma}\hat{\tau}$, and two unique isotopies $\hat{i}$ and $\hat{i}'$ exist for which $\hat{i}\xi = \hat{\xi}$ and $\hat{i}'\xi' = \hat{\xi}'$, i.e., the following diagram is commutative



(2.14)

Whenever an algebra $\hat{\xi}$ verifies the conditions of the definition above, we write $\hat{\xi}(\mathrm{G})$. Again, for simplicity, we assume that all Lie algebras are finite-dimensional, all algebras and fields have characteristic zero, and all Lie algebra bases are ordered.

We are now in a position to elaborate on the insufficiency of Definition 2.1, and the need of Definition 2.2. We shall indicate first the mathematical aspect and then point out the physical profile.

The main idea of Definition 2.1 is, beginning with the basis of a Lie algebra G, to construct an enveloping algebra $\xi(\mathrm{G})$ such that $[\xi(\mathrm{G})]^- \approx \mathrm{G}$. The more general idea of Definition 2.2 is, beginning also with the basis of a Lie algebra G, to construct an enveloping algebra $\hat{\xi}(\mathrm{G})$ such that the attached algebra $[\hat{\xi}(\mathrm{G})]^-$ *is not*, in general, isomorphic to G but rather is isomorphic to an isotope $\hat{\mathrm{G}}$ of G, and we write [48]

$$[\hat{\xi}(\mathrm{G})]^- \approx \hat{\mathrm{G}} \not\approx \mathrm{G}. \qquad (2.15)$$

The lack of unique association of a given basis with the envelope then ensures freedom in the realization of the associative product. Equivalently, we can say that within the context of Definition 2.1, a given basis essentially yields a single unique enveloping algebra and thus a single unique attached Lie algebra. On the contrary, within the context of Definition 2.2, a given basis yields all possible enveloping algebras and thus all possible Lie algebras of the same dimension, as we shall see. Still equivalently, we can say that, as is conventional in the contemporary formulation of Lie's theory, nonisomorphic Lie algebras are expressed via the use of *different generators* and the *same Lie product.* On the contrary, within the context of the isotopic formulation of Lie's theory, nonisomorphic Lie algebras can be obtained via the use of the *same basis* and *different Lie products.* We can therefore state that all possible enveloping associative algebras can indeed be introduced according to Definition 2.1, which is therefore suitable for the Cartan classification of Lie algebras. Definition 2.2 is more general inasmuch as, besides permitting the introduction of all possible enveloping algebras, it also permits us to construct nonisomorphic algebras via the same basis, by therefore rendering necessary the use of the most general possible realizations of the associative product.

On physical grounds, these mathematical mechanisms are at the foundation of the Lie-isotopic generalization of Hamilton's and Heisenberg's equations for closed nonself-adjoint interactions (§1.3).

As familiar, the definition of physical quantities is independent of whether or not the systems possess nonpotential interactions. When these interactions are admitted by the theory, they are represented via an alteration of the Lie algebra product. As a result, when the Hamiltonian description of a closed self-adjoint system

$$\dot{A}(a) = [A, E_{tot}] = \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial E_{tot}}{\partial a^\nu}, \tag{2.16}$$

is generalized into a Birkhoffian form (1.22) to represent the additional presence of internal, contact, nonpotential, interactions, i.e.,

$$\dot{A}(a) = [A \hat{,} E_{tot}] = \frac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}(a)\frac{\partial E_{tot}}{\partial a^\nu}, \tag{2.17}$$

the basis of the original Lie algebra remains unchanged, together with the underlying carrier space ($\mathbf{R} \times T^*M$) and the field, and only the realization of the Lie algebra product (that is, the realization of the envelope) is permitted to change. As a result, the original Lie algebra $\mathbf{G}$ with basis $X_i,(a)$ over T^*M equipped with conventional Poisson brackets is mapped into the isotope $\hat{\mathbf{G}}$, which preserves the original basis $X_i(a)$ in the same local coordinates of T^*M, although it is now equipped with the generalized Poisson brackets, i.e.,

$$\mathbf{G} : [X_i, X_j] = (X_i, X_j) - (X_j, X_i) \rightarrow \hat{\mathbf{G}} : [X_i \hat{,} X_j] = (X_i \hat{,} X_j) - (X_j \hat{,} X_i). \tag{2.18}$$

In the transition to the case of Heisenberg's equation, the situation is essentially the same and actually turns out to be more directly compatible with Definition 2.2. In fact, for consistency of the theory with its classical image, during the generalization of Heisenberg's equation (now expressed for operators),

$$i\dot{A}(a) = [A, H] = AH - HA, \tag{2.19}$$

into the Lie-isotopic form (1.37), i.e.,

$$i\dot{A}(a) = [A \hat{,} H] = ATH - HTA, \tag{2.20}$$

the nonpotential forces due to charge overlapping are expressed via the Lie-isotopic generalization of the product

$$\mathbf{G} : [X_i, X_j] = X_i X_j - X_j X_i \rightarrow \hat{\mathbf{G}} : [X_i \hat{,} X_j] = X_i T X_j - X_j T X_i. \tag{2.21}$$

Mechanism (2.21) is clearly along Definition 2.2 rather than 2.1.

The alternative approach would be that of preserving the original simplest possible product and changing the basis in order to reach direct compatibility with Definition 2.1. However, this approach has a number of problematic aspects. First of all, it is centered on the loss of the direct physical meaning of the generators (e.g., the physical linear momentum in one dimension, $p = m\dot{r}$, is replaced by abstract objects of the type $p = \alpha \exp(\beta r \dot{r})$. Secondly, the approach does not permit the achievement of the direct universality, as recalled by the preceding section. The removal of the unnecessary restrictions on the realization of the enveloping algebras is clearly preferable, both mathematically and physically.

Owing to the relevance of mechanisms (2.18) and (2.21) for this review, it is important to give an explicit example. To stress the fact that the ideas are not necessarily restricted to nonpotential interactions, we review one of the first examples of isotopy identified by Santilli, that for the harmonic oscillator in a three-dimensional Euclidean space [1], [15].

The nonisomorphic groups $\mathbf{SO}(3)$ and $\mathbf{SO}(2.1)$ are *isotopic symmetries* of the corresponding Hamiltonians

$$H(a) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}(x^2 + y^2 + z^2) , \tag{2.22.a}$$

$$\hat{H}(a) = \frac{1}{2}(p_x^2 - p_y^2 + p_z^2) + \frac{1}{2}(x^2 - y^2 + z^2) , \tag{2.22.b}$$
$$a = (r,p), m = k = 1 , \tag{2.22.c}$$

that is, they are symmetries leading to the same conservation laws of the components $M_b, b = x, y, z$, of the angular momentum via the use of Noether's theorem. Let us review the case again and reinterpret it in light of Definitions 2.1 and 2.2.

The Hamiltonian realization of the symmetry $\mathbf{SO}(3)$ of $H(a)$ is based on the Lie algebra of conserved quantities

$$\mathbf{SO}(3) : [M_x, M_y] = M_z, \quad [M_y, M_z] = M_x, \quad [M_z, M_x] = M_y , \tag{2.23}$$

which is defined in terms of the conventional Poisson brackets

$$[M_b, M_c] = (M_b, M_c) - (M_c, M_b) , \tag{2.24.a}$$
$$(M_b, M_c) = \frac{\partial M_b}{\partial r^i}\delta^i_j\frac{\partial M_c}{\partial p_j}; \quad (\delta^i_j) = \begin{pmatrix} +1 & 0 & \\ & +1 & \\ 0 & & +1 \end{pmatrix} . \tag{2.24.b}$$

In the transition to the equivalent Hamiltonian $\hat{H}(a)$, the conserved quantities M_b clearly remain conserved, but the $\mathbf{SO}(3)$ symmetry is broken and is replaced by the nonisomorphic symmetry $\mathbf{SO}(2.l)$. The problem now is the construction of a realization of the $\mathbf{SO}(2.1)$ algebra (the *Lorentz* algebra in $(2+1)$-dimensions) whose generators are those of the nonisomorphic $\mathbf{SO}(3)$ algebra (the *rotational* algebra in three-dimensions). This can clearly be achieved if and only if one alters the Lie algebra product. An explicit realization has been identified by Santilli [1], [15] and is given by the commutation rules

$$\mathbf{SO}(2.1) : [M_x \hat{,} M_y] = M_z, \quad [M_y \hat{,} M_z] = -M_x, \quad [M_z \hat{,} M_x] = M_y, \tag{2.25}$$

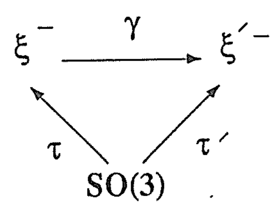
which are now expressed in terms of the *generalized* Poisson (Birkhoffian) brackets

$$[M_b \hat{,} M_c] = (M_b \hat{,} M_c) - (M_c \hat{,} M_b) ,$$
$$(M_b \hat{,} M_c) = \frac{\partial M_b}{\partial r^i}\alpha^i_j\frac{\partial M_c}{\partial p_j}, \quad (\alpha^i_j) = \begin{pmatrix} +1 & 0 & \\ & -1 & \\ 0 & & +1 \end{pmatrix} . \tag{2.26}$$

Note that the insistence in the preservation of the same realization of the Lie algebra product, in this case, would prohibit the representation of the conservation of the angular momentum via a symmetry of the Hamiltonian $\hat{H}(a)$.

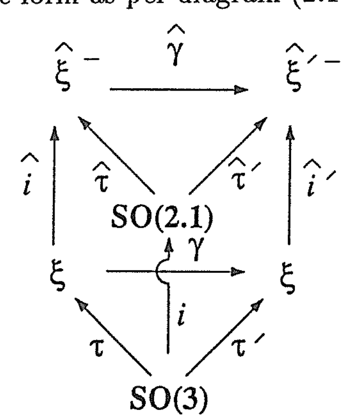
The example considered therefore establishes that one given basis (the components of the angular momentum $\mathrm{M} = r \times p, p = m\dot{r}$) can define a hierarchy of enveloping algebras and attached Lie algebras, depending on the selected realization of the products, which is fully in line with diagram (2.4) and Definition 2.2. The example actually establishes not only the insufficiency of Definition 2.1 but also that of Definition 2.2 itself. In fact, the algebras (M_b, M_c) and $(M_b \hat{,} M_c)$ are *nonassociative*, therefore demanding a further generalization of Definition 2.1 for nonassociative enveloping algebras, even though the existence of a realization within the context of the Lie- isotopic generalization is expected to exist (§1.5).

Stated in different terms, the above example by Santilli establishes the generalization of the conventional definition of the envelope of the Lie algebra of the group of rotations as per diagram (2.4).



(2.27)

into the Lie-isotopic form as per diagram (2.14)



(2.28)

which is expected for operator-type realizations (2.21).

Note that by no means does diagram (2.28) exhaust all possible isotopies of the group of rotations. See §3.2 for details.

With a clear understanding of the new capabilities (as well as limitations) of the Lie-isotopic generalization, we pass now to the review of the generalization of Theorem 2.1 achieved by Santilli (*loc. cit.*).

The construction of an isotope $\hat{\xi}(\mathbf{G})$ of $\xi(\mathbf{G})$ can be conducted as follows. Perform an *isotopic mapping of the tensorial product* $X_i \otimes X_j$ of $\xi(\mathbf{G})$,

$$X_i \otimes X_j \rightarrow X_i * X_j \ , \tag{2.29}$$

that is, any invertible modification of the product $\otimes$ via elements of $\xi(\mathbf{G})$, of the base manifold, and of the field, which preserves: the distributive and scalar laws (to qualify as an algebra); the associativity of the product (to qualify as an isotopy), i.e.,

$$(X_i * X_j) * X_k = X_i * (X_j * X_k) \ , \tag{2.30}$$

as well as the existence of the unit $\hat{1}$. The product of two elements $X_i * X_j$ and $X_r * X_s$ is then given by

$$(X_i * X_j) * (X_r * X_s) = X_i * X_j * X_r * X_s \ , \tag{2.31}$$

and no ordering ambiguity arises because of the preservation of the associative character of the original product.

The isotope of the associative tensorial algebra (2.5) can then be written

$$\hat{\mathcal{F}} = F1 \oplus \mathbf{G} \oplus \mathbf{G} * \mathbf{G} \oplus \mathbf{G} * \mathbf{G} * \mathbf{G} \oplus \ldots . \tag{2.32}$$

Let $\hat{\mathcal{R}}$ be the ideal of $\hat{\mathcal{F}}$ generated by

$$[X_i \hat{,} X_j] - (X_i * X_j - X_j * X_i), \tag{2.33}$$

where $[X_i \hat{,} X_j]$ is the product in $\hat{\mathbf{G}}$. An *isotopically mapped universal enveloping associative algebra* $\hat{\xi}(\mathbf{G})$ of a Lie algebra $\mathbf{G}$ can then be written

$$\hat{\xi}(\mathbf{G}) = \hat{\mathcal{F}} / \hat{\mathcal{R}}. \tag{2.34}$$

Structure (2.34) is, by construction, the universal enveloping associative algebra of $\hat{\mathbf{G}}$ realized via an isotopic mapping $\mathbf{G} \rightarrow i\hat{\mathbf{G}}$.

The remaining aspects of the theory of $\hat{\xi}(\mathbf{G})$ are essentially given by an isotopic mapping of the corresponding steps for $\xi(\mathbf{G})$ outlined above.

The quantities

$$\hat{M}_s = X_{i_1} * X_{i_2} * \ldots * X_{i_s}, \tag{2.35}$$

are called *isotopically mapped standard (nonstandard) monomials* depending on whether the following ordering condition

$$i_1 \leq i_2 \leq \ldots \leq i_s \tag{2.36}$$

is verified (not verified). In the reduction of an arbitrary element of $\hat{\xi}(\mathbf{G})$

$$X_{i_1}^{k_1} * X_{i_2}^{k_2} * \ldots * X_{i_r}^{k_r}, \tag{2.37}$$

to standard monomials, a new feature arises, due to the fact that the emerging combinations of these latter monomials may occur via *functions on the base manifold.* This, in turn, occurs because the isotopy $\otimes \rightarrow *$ can be realized via functions of this type. We call these combinations $\hat{F}$- linear, where $\hat{F}$ is an isofield of type (1.38), to differentiate them from the F-linear combinations of the conventional case, that is, combinations only via elements of the field. As we shall see in the next section, these $\hat{F}$-linear combinations have a precise interpretation within the context of the isotopic Lie's theory. Despite this generalization, the construction of the basis of $\hat{\xi}(\mathbf{G})$ parallels that for $\xi(\mathbf{G})$, because $\hat{\xi}(\mathbf{G})$ is a conventional envelope for $\hat{\mathbf{G}}$. The (inverse) isotopy then simply reduces $\hat{\mathbf{G}}$ to $\mathbf{G}$.

Theorem 2.2*(ref. [1], p. 353 and ref. [15], p. 161; Isotopic Generalization of the Poincaré-Birkhoff-Witt Theorem): The cosets of* $\hat{1}$ *and the standard isotopically mapped monomials form a basis of the isotopically mapped universal enveloping associative algebra* $\hat{\xi}(\mathbf{G})$ *of a Lie algebra* $\mathbf{G}$.

The basis is thus given by

$$\hat{1}, \quad X_i, \quad X_{i_1} * X_{i_2}, \quad X_{i_1} * X_{i_2} * X_{i_3}, \ldots$$
$$i_1 \leq i_2, \quad i_1 \leq i_2 \leq i_3 \,. \tag{2.38}$$

where $\hat{1}$ is the (abstract) unit of $\hat{\xi}$. The distinction between the tensorial realization and that used in practical applications is now lost. Indeed the mapping $X_i \otimes X_j \rightarrow X_i X_j$ can be considered, in the final analysis, a particular form of isotopy.

The explicit form of the basis depends on the assumed type of isotopy $\otimes \rightarrow *$. In turn, this depends on the realization of the basis X_i of $\mathbf{G}$, whether via matrices, quantum mechanical operators, classical functions on phase space, etc.

Suppose that the X's are realized via matrices. Then an isotopy is provided by Eq. (2.21). Let T be a polynomial on the X's (not necessarily on the center of $\hat{\xi}(\mathbf{G})$.) Then the explicit form of basis (2.38) is given by

$$1, \quad X_j, \quad X_{i_1}TX_{i_2}, \quad X_{i_1}TX_{i_2}TX_{i_3}, \ldots$$

$$i_1 \leq i_2, \quad i_1 \leq i_2 \leq i_3, \quad T = \text{fixed and invertible.} \tag{2.39}$$

Needless to say, the isotopy $X_iX_j \rightarrow X_iTX_j$ is only one example of possible associativity-preserving modifications of the product. Other associative isotopies are given by Eqs. (1.4) and (1.10).

A comment on the quantity $\hat{1}$ of Theorem 2.2 is in order here. As anticipated in §1.3, *the element* $1 \in \mathbf{F}$ *is no longer the unit element of the enveloping algebra under an isotopic mapping of the product.* In fact, for isotopic envelope (2.39) the unit element (when it exists) is given by

$$\hat{1} = T^{-1} \in \hat{F}\,, \tag{2.40}$$

because only this quantity verifies the (left and right) rules $\hat{1} * X_i = X_i * \hat{1} = X_i$ for all $X_i \in \hat{\xi}$. It should be indicated that, as we shall illustrate in §2.4, basis (2.38) can also be formulated in terms of the unit $1 \in F$ (called in this case *weak unit* [36]). This is due to the possibility of factoring out the isounit $\hat{1} \in \hat{F}$ (see, later on, Eq. (2.139)). The formulations of Theorem 2.2 in terms of the cosets of 1 (field F) or cosets of $\hat{1}$ (isofield $\hat{F}$) are, therefore, equivalent.

The restriction of the existence of the unit on all acceptable isotopies (recalled earlier) should be emphasized here. In fact, no generalization of Theorem 2.1 for isotopy (1.10) is known at this writing, precisely because of the general lack of unit $\hat{1}$ for the product $a * b = WaWbW$, $W^2 = W$, i.e., the general lack of existence of a quantity $\hat{1}$ such that $\hat{1} * a = a * \hat{1} = W\hat{1}WaW = WaW\hat{1}W = a$ for *all* $a \in \hat{\xi}$.

The restriction to the *isotopic* liftings of Theorem 2.2 is also worth a mention. In fact, Santilli presented in his original memoir [1] also a *genotopic* lifting of the theorem, i.e., a generalization of the original associative algebra ξ into a *nonassociative* Lie-admissible form. However, the nonassociativity causes problems in orderings of type (2.31) which are known to be resolvable only for a particular case of nonassociative Lie-admissible algebras called *flexible* [1]. This latter generalization was reinspected by Ktorides, Myung and Santilli [35]. We therefore defer the interested reader for details to the genotopies of Theorem 2.1 to ref.s [1,35].

An important mathematical aspect reviewed in this section is that *the knowledge of a given set of generators does not uniquely characterize a Lie algebra* because of the freedom in the selection of the enveloping algebra (product). The physical aspect treated is that *the knowledge of a Hamiltonian does not uniquely characterize the physical system* because such a characterization also depends on the explicit form of the brackets of the time evolution. As we shall see, the implications are rather intriguing. For instance, the assumption of a *Hermitian* Hamiltonian H contrary to popular belief, does not ensure that the time evolution is unitary and thus does not guarantee that $\tilde{H}$ is observable unless one specifically identifies the assumed realization of the envelope, i.e., of the assigned Lie product in Heisenberg's time evolution.

2.3 Isotopic Lifting of Lie's First, Second, and Third Theorems [1], [15]

As is well-known, an effective historical, and technical way of presenting Lie groups and Lie algebras is according to their original derivation by Sophus Lie [77] via his celebrated First, Second, and Third Theorems. In this section we shall first present these theorems, review Santilli's Lie isotopic generalization, and then show its comparability with the isotopic generalization of the enveloping algebra of the preceding section. More specifically, the objective is to show that the notion of connected Lie transformation group admits a generalization such that, when reduced in the neighborhood of the identity, admits Lie algebras with the most general possible realization of the product.

The emerging isotopic generalization of Lie's theory (that is, of the enveloping algebra, the Lie algebras, and the Lie groups) was used for the construction of the isotopic generalization of Galilei's relativity for closed non-self-adjoint systems [1], [15] with corresponding relativistic and gravitational extensions [18], [58]. Since the theory also admits operator-type realizations, its abstract formulation is expected to permit the joint treatment of closed, classical and quantum mechanical, nonpotential interactions, in much of the same way as the conventional abstract formulation of Lie's theory permits a joint treatment of closed, classical and quantum mechanical interactions of potential-Hamiltonian type. Santilli's ultimate objective is to lay the foundations for achieving, in due time, a generalization of the contemporary notion of interactions, with corresponding generalization of relativities and physical laws.

DEFINITION 2.3: Let M be a Hausdorff, second-countable, analytic, N-dimensional manifold with local coordinates $a^\mu, \mu = 1, 2, ..., N$ (e.g., T^*M or $R \times T^*M$). The set of transformations on M depending on r-independent parameters $\theta^i, i = 1, 2, ..., r$,

$$a \to a' = f(a; \theta) = \{f^\mu(a^\alpha; \theta^j)\} \tag{2.41}$$

is called a Lie transformation group *[77] when the following conditions are verified.*

1. *All functions f^μ are analytic in their variables.*
2. *For any given two transformations*

   $$a' = f(a; \theta), \quad a'' = f(a'; \theta')\ , \tag{2.42}$$

   a set of parameters exists

   $$\theta''^i = g^i(\theta, \theta')\ , \tag{2.43}$$

   characterized by analytic functions g^i called group composition laws, *such that*

   $$a'' = f(a; \theta'')\ . \tag{2.44}$$

3. *Transformations (2.41) recover the identity transformation at the null value of the parameters, i.e.,*

   $$a = f(a; 0). \tag{2.45}$$

4. *Corresponding to each transformation (2.41), there is a unique inverse transformation*

   $$a = f(a'; \theta^{-1}), \tag{2.46}$$

   and thus the transformations are regular.
5. *The combination of any transformation (2.41) with its inverse yields the identity transformation.*

The number r of independent parameters is called the *dimension* of the Lie group.

A central property of Lie transformation groups is that they are *connected*; that is, they can be continuously connected to the identity. The primary idea of Lie's theorems is that, under the conditions indicated, the groups can be studied via their infinitesimal transformations, because a finite transformation can be recovered via infinite successions of infinitesimal transformations. Santilli [1] first reviewed these ideas by following as closely as possible their original derivation [77], as we shall do in the following. Consider transformations (2.41) with their identity

$$a' = f(a;\theta), \quad a = f(a;0), \tag{2.47}$$

and perform the infinitesimal variations

$$a' = a + da = f(a;\theta + d\theta); \quad a + \delta a = f(a;\delta\theta)\ , \tag{2.48}$$

where $d\theta$ and $\delta\theta$ represent two independent variations of the parameters. We can then write

$$da = \frac{\partial f(a;\theta)}{\partial\theta} d\theta\ , \qquad (2.49.a)$$

$$\delta a = (\frac{\partial f(a;\theta)}{\partial\theta})_{\theta=0}\delta\theta\ . \qquad (2.49.b)$$

The transformation $\theta + d\theta$ can be interpreted as the product of transformations relative to θ and $\delta\theta$, i.e.,

$$\theta^i + d\theta^i = \varphi^i(\theta, \delta\theta)\ , \tag{2.50}$$

for which

$$\theta^i + d\theta^i = \varphi^i(\theta, 0) + (\frac{\partial\varphi^i(\theta,\alpha)}{\partial\alpha^j})_{\alpha=0}\delta\theta^j + \dots\ . \tag{2.51}$$

Thus we can write

$$d\theta^i = \mu^i_j(\theta)\delta\theta^j\ ,$$

$$\mu^i_j = (\frac{\partial\varphi^i(\theta,\alpha)}{\partial\alpha^j})_{\alpha=0}\ . \tag{2.52}$$

The formula above represents a relation between $d\theta$ and $\delta\theta$ which can also be written

$$\delta\theta^j = \lambda^j_i(\theta)d\theta^i, \quad \lambda^j_k\mu^k_i = \mu^k_i\lambda^j_k = \delta^j_i\ . \tag{2.53}$$

By putting

$$u^\mu_j(a) = (\frac{\partial f^\mu(a;\theta)}{\partial\theta^j})_{\theta=0}\ , \tag{2.54}$$

and by using Eq. (2.53), Eq. (2.49.a) can be written

$$da^\mu = u^\mu_k(a)\lambda^k_j(\theta)d\theta^j \ . \tag{2.55}$$

In this way we reach *Lie's First Theorem.*

Theorem 2.3: *When transformations (2.41) form a connected, m-dimensional, Lie group, then*

$$\frac{\partial a^\mu}{\partial \theta^j} = u^\mu_k(a)\lambda^k_j(\theta) \ , \tag{2.56}$$

where the functions u^μ_k are analytic.

Let $A(a)$ be an (analytic) function of the a variables. The infinitesimal Lie transformation $a \rightarrow a + da$ induces a variation of $A(a)$ which can be written

$$\begin{aligned} dA = \frac{\partial A}{\partial a^\mu}u^\mu_j\delta\theta^j &= \delta\theta^k u^\mu_k \frac{\partial}{\partial a^\mu}A \ . \\ &= \delta\theta^k X_k A \ . \end{aligned} \tag{2.57}$$

The m-independent quantities

$$X_k = X_k(a) = u^\mu_k(a)\frac{\partial}{\partial a^\mu} = [\frac{\partial f^\mu(a;\theta)}{\partial \theta^k}]_{\theta=0}\frac{\partial}{\partial a^\mu} \ , \tag{2.58}$$

are called the *infinitesimal generators* of the transformations (or of the group). For our later needs, we refer to the X's defined by Eqs. (2.58) as the *standard generators.*

We are now interested in the (necessary and sufficient) conditions for transformations (2.41) to constitute a Lie group. By using the converse of the Poincaré lemma, they can be written

$$\frac{\partial^2 a'^\mu}{\partial\theta^i\partial\theta^j} = \frac{\partial^2 a'^\mu}{\partial\theta^j\partial\theta^i} \ , \tag{2.59}$$

that is

$$\frac{\partial u^\mu_k}{\partial\theta^i}\lambda^k_j + u^\mu_k\frac{\partial\lambda^k_i}{\partial\theta^j} = \frac{\partial u^\mu_k}{\partial\theta^j}\lambda^k_i + u^\mu_k\frac{\partial\lambda^k_i}{\partial\theta^j} \ . \tag{2.60}$$

Thus

$$\begin{aligned} u^\mu_k(\frac{\partial\lambda^k_j}{\partial\theta^i} - \frac{\partial\lambda^k_i}{\partial\theta^j}) &= \lambda^k_j\frac{\partial u^\mu_k}{\partial\theta^i} - \lambda^k_i\frac{\partial u^\mu_k}{\partial\theta^j} = \lambda^k_j\frac{\partial u^\mu_k}{\partial a^\nu}\frac{\partial a^\nu}{\partial\theta^i} - \lambda^k_i\frac{\partial u^\mu_k}{\partial a^\nu}\frac{\partial a^\nu}{\partial\theta^j} \\ &= \lambda^r_j u^\nu_l\lambda^l_i\frac{\partial u^\mu_r}{\partial a^\nu} - \lambda^k_i u^\nu_l\lambda^l_j\frac{\partial u^\mu_k}{\partial a^\nu} \ . \end{aligned} \tag{2.61}$$

Therefore

$$u_i^\nu \frac{\partial u_j^\mu}{\partial a^\nu} - u_j^\nu \frac{\partial u_i^\mu}{\partial a^\nu} = C_{ij}^k u_k^\mu \ , \tag{2.62}$$

where

$$C_{ij}^k = \mu_i^r \mu_j^s (\frac{\partial \lambda_r^k}{\partial \theta^s} - \frac{\partial \lambda_s^k}{\partial \theta^r}) \ . \tag{2.63}$$

The m^3 quantities C_{ij}^k are independent from θ. This can be seen by differentiating Eq. (2.62) with respect to θ. After some simple calculations, one then sees that

$$\frac{\partial C_{ij}^k}{\partial \theta^l} = 0 \ ,$$

$$i, j, k, l = 1, 2, ..., m \ . \tag{2.64}$$

In this way we reach *Lie's Second Theorem.*

Theorem 2.4: *If $X_i, i = 1, 2, ..., m$, are the generators of an m-dimensional Lie group, they satisfy the closure relations*

$$[X_i, X_j]_\xi = X_i X_j - X_j X_i = C_{ij}^k X_k \ , \tag{2.65}$$

where the quantities C_{ij}^k are called structure constants.

The symbol ξ in Eq. (2.65) denotes an associative algebra with a conventional, associative product of operators $X_i X_j$. At closer inspection, this algebra emerges as being the *universal enveloping associative algebra* of the Lie algebra.

The *fundamental Lie's rule* (2.65) can be explicitly written

$$[X_i, X_j]_\xi = [U_i^\mu \frac{\partial}{\partial a^\mu}, U_j^\nu \frac{\partial}{\partial a^\nu}]_\xi = C_{ij}^k u_k^\alpha \frac{\partial}{\partial a^\alpha} \ , \tag{2.66}$$

where the product $[X_i, X_j]_\xi$ is Lie; that is, it satisfies the identities

$$[X_i, X_j]_\xi + [X_j, X_i]_\xi = 0 \ ,$$

$$[[X_i, X_j]_\xi, X_k]_\xi + [[X_j, X_k]_\xi, X_i]_\xi + [[X_k, X_i]_\xi, X_j]_\xi = 0 \ . \tag{2.67}$$

By substituting into these expressions the explicit form of the Lie product in terms of the structure constants, *Lie's Third Theorem* is reached.

Theorem 2.5: *The structure constants of a Lie group in standard realization obey the relations*

$$C^k_{ij} + C^k_{ji} = 0 \ ,$$
$$C^k_{ij}C^r_{kl} + C^k_{jl}C^r_{ki} + C^k_{li}C^r_{kj} = 0 \ . \tag{2.68}$$

Theorems 2.3, 2.4, and 2.5 essentially provide the correspondence between a given (connected) Lie group G and its Lie algebra **G**. In particular, they allow the characterization of a Lie group in the neighborhood of the identity via the structure constants. We have here tacitly implied that different Lie groups may exist all admitting the same Lie algebra, that is, the same structure constants. However, among all Lie groups with the same Lie algebra only one is simply connected, called the *universal covering group.*

The inverse transition from a Lie algebra to a corresponding Lie group can be characterized via the inverses of Lie's First, Second, and Third Theorems. We suggest the interested reader to study the specialized literature on this topic, such as Gilmore [78] and quoted references. We here outline one of the simplest approaches, known as the *exponential mapping* [15]. Write Eqs. (2.56) in the form

$$\frac{\partial a^\mu}{\partial \theta^i} = u^\mu_k(a)\lambda^k_i(\theta) = \lambda^k_i(\theta)X_k(a)a^\mu \ , \tag{2.69}$$

and introduce the one-dimensional parametrization

$$\theta^k = \tau\alpha^k, \quad a'^\mu = a'^\mu(\theta(\tau)) = a''^\mu(\tau) \ . \tag{2.70}$$

Then we write

$$a''^\mu(\tau) = T^\mu_\nu(\tau)a^\nu, \quad a^\nu = [a''^\nu(\tau)]_{\tau=0} \ . \tag{2.71}$$

To compute the elements $T^\mu_\nu(\tau)$, consider the equations

$$\frac{da^\mu}{d\tau} = \frac{\partial a^\mu}{\partial \theta^i}\frac{d\theta^i}{d\tau} = \alpha^k\lambda^r_k(\theta)X_r(a)a''^\mu(0) \ ,$$
$$\frac{d}{d\tau}T^\mu_\nu(\tau)a^\nu = \alpha^k\lambda^r_k(\theta)X_r(a)T^\mu_\nu(\theta)a''^\nu(0) \ . \tag{2.72}$$

However, the $a''^\nu(0)$ are arbitrary initial values. Thus the solutions of the total differential equations

$$\frac{d}{d\tau}T^\mu_\nu(\tau) = \alpha^k\lambda^r_k(\theta)X_r(a(\tau))T^\mu_\nu(\tau) \ , \tag{2.73}$$

with initial conditions

$$T^{\mu}_{\nu}(0) = \delta^{\mu}_{\nu}, \quad \frac{d}{d\tau}T^{\mu}_{\nu}(\tau)|_{\tau=0} = \alpha^k \lambda^r_k(\theta) X_r(a(0))\delta^{\mu}_{\nu} \ , \tag{2.74}$$

can be written

$$T^{\mu}_{\nu}(\tau) = \sum_{n=0}^{\infty} \frac{1}{n!}[\theta^k X_k(a(0))\delta^{\mu}_{\nu}]^n \ , \tag{2.75}$$

yielding the exponential mapping

$$a'^{\mu} = e^{\theta^k X_k}|_{\xi} a^{\mu} \ . \tag{2.76}$$

If, instead of the variables of the base manifold, we have a function of the same variables, the procedure above also applies, and we can write

$$A(a') = e^{\theta^k X_k}|_{\xi} A(a) \ . \tag{2.77}$$

In particular, the infinitesimal (standard) generators can be recovered via the rule

$$X_k = [\frac{\partial}{\partial \theta^k} e^{\theta^i X_i}|_{\xi}]_{\theta=0} \ . \tag{2.78}$$

Notice that the standard realization (2.76) of the group of transformations (2.41) is manifestly connected. The verification of the conditions to qualify as a Lie group is simple. Here we restrict ourselves to recalling that the product of two elements of group (2.76)

$$e^{X_\alpha} e^{X_\beta} = e^{X_\rho}, \tag{2.79}$$

is characterized by the so-called *Baker-Campbell-Hausdorff formula*:

$$X_\rho = X_\alpha + X_\beta + \frac{1}{2}[X_\alpha, X_\beta]_{\xi} + \frac{1}{12}[(X_\alpha - X_\beta), [X_\alpha, X_\beta]_{\xi}]_{\xi} + \ldots . \tag{2.80}$$

It is significant for our review to recall that a Lie algebra does not necessarily admit a corresponding Lie group. For specific examples of Lie algebras of this type, the reader may consult, for instance, Hurst [79]. In essence, the applicability of the exponential mapping in general, or the "integration" of a Lie algebra to a Lie group must satisfy certain (convergence) conditions of the underlying infinite series, known as *integrability conditions*. We also refer the reader in this respect to the specialized literature in the subject and, in particular, to Nelson [80].

We pass now to the review of Santilli's Lie-isotopic generalization of Lie's theorems. The prior review of the main objective may be useful here. Lie's crucial result is fundamental rule (2.65). This rule essentially characterizes Lie algebras via the conventional associative product $X_i X_j$ of vector fields $X_i = u_i^\mu(a)\partial/\partial a^\mu$ on a manifold M. Santilli's main objective is to generalize Definition 2.3 and Lie's theorems in such a way as to characterize a Lie algebra via the most general possible associative product $X_i * X_j$ of vector fields on a manifold.

Of utmost importance is the condition that *the base manifold M with local coordinates a^μ, the parameters θ_i, and the generators X_i of the conventional formulation of Lie's theorems are not changed in their isotopic generalization.* This is due to physical requirements for the description under consideration. As we recalled earlier, the local coordinates of M customarily have a direct physical meaning such as the coordinates of the frame of the experimental setup; the parameters carry a direct physical meaning as measurable quantities such as time, angle, etc., and the generators directly represent physical quantities such as energy, angular momentum, etc. When the conventional description of self-adjoint interactions via Theorems 2.3, 2.4, and 2.5 is broadened to permit the additional presence of the nonself-adjoint interactions, the frame of the experimental observer must be preserved; measurable quantities such as time and angles must be preserved; and physical quantities such as energy and angular momentum must also be preserved unaltered.

These objectives were achieved by Santilli as follows.

DEFINITION 2.4 (ref. [1], pp. 329-368. See also ref. [15], pp. 169-173): Let

$$G : a^\nu \to a'^\nu = f^\nu(a;\theta) , \tag{2.81}$$

be an r-dimensional Lie transformation group G as per Definition 2.3. A Lie isotopic image *or, simply an* isotope $\hat{G}$ *of G is a set of transformations characterizable via a regular $(N \times N)$ matrix of analytic functions $(g^\mu_\nu(a;\theta))$ acting on (2.81)*

$$\hat{G} : a^\mu \to \hat{a}^\mu = g^\mu_\nu(a;\theta) f^\nu(a,\theta) = \hat{f}^\mu(a;\theta) ,$$

$$\det(g^\mu_\nu) \neq 0, \quad g^\mu_\nu|_{\theta=0} = \delta^\mu_\nu , \tag{2.82}$$

which verify the following properties. (a) The transformations $\hat{a} = \hat{f}(a;\theta)$ constitute a Lie transformation group, by therefore *verifying conditions 1-5 of Definition 2.3. (b) The group $\hat{G}$ is realized via the same base manifold, the same parameters and the same generators of G. (c) When reduced in the neighborhood of the identity transformation, the group $\hat{G}$ can be characterized by a Lie algebra isotope $\hat{\mathbf{G}}$ of $\mathbf{G}$.*

Condition (c) is introduced to avoid non-Lie, Lie-admissable algebras in the neighborhood of the identity transformations [1]. As a matter of fact, it is precisely this possibility that permits the further generalization of Lie's theory of Lie-admissible type.

Since the group of transformations $\hat{f}^\mu(a;\theta)$ is a conventional, connected Lie group by assumption, it can be studied in the neighborhood of the identity as in the conventional case. The repetition of the analysis of $f(a;\theta)$ then yields the expressions

$$da^\mu = \hat{u}^\mu_k(a)\lambda^k_i(\theta)d\theta^i ,$$

$$\hat{u}^\mu_k(a) = |\frac{\partial}{\partial\theta^k} g^\mu_\nu(a;\theta)f^\nu(a;\theta)|_{\theta=0} . \tag{2.83}$$

In order to realize the isotopy, we then introduce the following reformulation in terms of the quantities of G for given $g^i_k(a)$ functions

$$\hat{u}^\mu_k(a) = g^i_k(a)u^\mu_i(a), \quad \det(g^i_k) \neq 0 . \tag{2.84}$$

Note that the other possibility $\hat{u}^\mu_k = g^\mu_\nu u^\nu_k$, even though conceivable (and actually more in line with Eq. (2.83)), is excluded here because it would imply the redefinition of the generators $X_k = u^\mu_k(\partial/\partial a^\mu) \to \hat{X}_k = g^\mu_\nu u^\nu_k(\partial/\partial a^\mu)$ which is *contrary* to the notion of isotopy. The analyticity of the transformations then implies the following Santilli's generalization of Lie's First Theorem.

Theorem 2.6 *[1], [15]: If transformations (2.82) characterize an isotopic image $\hat{G}$ of the Lie group G of transformations (2.81), then analytic functions $g^i_k(a)$ exist such that*

$$\frac{\partial\hat{a}^\mu}{\partial\theta^j} = g^k_i(a)u^\mu_k(a)\lambda^i_j, \quad \det g \neq 0 , \tag{2.85}$$

and the $u^\mu_k(a)$ functions are analytic.

This theorem, though mathematically trivial, has nontrivial implications. Indeed, it implies a modification of the structure of the group in the neighborhood of the identity, i.e.,

$$G : a'^{\mu} \approx a^{\mu} + \theta^i u_i^{\mu}(a) \rightarrow \hat{G} : \hat{a}^{\mu} \approx a^{\mu} + \theta^i g_i^j(a) u_j^{\mu}(a) \ , \tag{2.86}$$

which is precisely the desired situation. We must now identify the integrability conditions under which such a behavior is still Lie in algebraic character, when expressed in terms of the generators and parameters of the original group. Under these conditions, we say that the quantities g_j^i of Eqs.(2.85) or (2.86) are *isotopic functions* with respect to $\hat{G}$.

The group G is Lie and thus admits the standard realization worked out earlier,

$$u_i^{\nu} \frac{\partial}{\partial a^{\nu}} u_j^{\mu} - u_j^{\nu} \frac{\partial}{\partial a^{\nu}} u_i^{\mu} = C_{ij}^k u_k^{\mu} \frac{\partial}{\partial a^{\mu}} \ , \tag{2.87.a}$$

$$C_{ij}^k = \mu_i^r \mu_j^s \left(\frac{\partial \lambda_r^k}{\partial \theta^s} - \frac{\partial \lambda_s^k}{\partial \theta^r} \right) , \tag{2.87.b}$$

$$[X_i, X_j]_{\xi} = X_i X_j - X_j X_i = C_{ij}^k X_k \ , \tag{2.87.c}$$

$$X_k = u_k^{\mu}(a) \frac{\partial}{\partial a^{\mu}} \ . \tag{2.87.d}$$

The group $\hat{\mathbf{G}}$ is also Lie and thus can be realized in the standard form

$$\hat{u}_i^{\nu} \frac{\partial}{\partial a^{\nu}} \hat{u}_j^{\mu} - \hat{u}_j^{\nu} \frac{\partial}{\partial a^{\nu}} \hat{u}_i^{\mu} = \hat{C}_{ij}^k \hat{u}_k^{\mu} \frac{\partial}{\partial a^{\mu}} \ , \tag{2.88.a}$$

$$\hat{C}_{ij}^k = \hat{\mu}_i^r \hat{\mu}_j^s \left(\frac{\partial \hat{\lambda}_r^k}{\partial \theta^s} - \frac{\partial \hat{\lambda}_s^k}{\partial \theta^r} \right) , \tag{2.88.b}$$

$$\left[\hat{X}_i, \hat{X}_j\right]_{\xi} = \hat{X}_i \hat{X}_j - \hat{X}_j \hat{X}_i = \tilde{C}_{ij}^k X_k \ , \tag{2.88.c}$$

$$\hat{X}_k = \hat{u}_k^{\mu} \frac{\partial}{\partial a^{\mu}} \ . \tag{2.88.d}$$

However, as indicated earlier, this realization generally implies a change of the generators in the transition from G to $\hat{G}$:

$$\mathbf{G} : X_k = u_k^{\mu} \frac{\partial}{\partial a^{\mu}} \rightarrow \hat{\mathbf{G}} : \hat{X}_k = \hat{u}_k^{\mu} \frac{\partial}{\partial a^{\mu}} \ , \tag{2.89}$$

and, as such, does not verify the conditions for isotopy. To achieve the objective under consideration, Santilli introduced the following isotopy of the universal enveloping associative algebra, according to §2.2, this time realized via *functions on the base manifold* [1], [15].

$$\xi(\mathrm{G}) : X_i X_j \to \hat{\xi}(\mathrm{G}) : X_i * X_j = g_i^r X_r g_j^s X_s \,. \tag{2.90}$$

Notice that this mapping does verify the conditions of isotopy, in the sense that it is realized via the generators of the original algebra, while preserves the associativity of the product,

$$(g_i^r X_r g_j^s X_s) g_k^t X_t = g_i^r X_r (g_j^s X_s g_k^t X_t) \,. \tag{2.91}$$

The fundamental Lie rule (2.87.c) can now be rewritten

$$u_i^\nu \frac{\partial}{\partial a^\nu} * u_j^\mu - u_j^\nu \frac{\partial}{\partial a^\nu} * u_i^\mu = \hat{C}_{ij}^k u_k^\mu \,,$$

$$\hat{C}_{ij}^k = \tilde{C}_{ij}^r g_r^k(a) \,. \tag{2.92}$$

The integrability conditions for the functions $g_k^i(a)$ to be isotopic, that is to yield rule (2.92), can then be readily computed. Thus we reach the following Santilli's generalization of Lie's Second Theorem.

Theorem 2.7*[1], [15]: Under the integrability conditions*

$$g_i^k u_k^\nu \frac{\partial}{\partial a^\nu} g_j^l - g_j^k u_k^\nu \frac{\partial}{\partial a^\nu} g_i^l = g^j g_i^s C_{rs}^l + \tilde{C}_{ij}^k g_k^l \,, \tag{2.93}$$

the generators X_i of an isotope $\hat{\mathrm{G}}$ of a Lie group G satisfy the isotopic rule of associative Lie admissibility

$$[X_i, X_j]_{\hat{\xi}} = X_i * X_j - X_j * X_i = \hat{C}_{ij}^k(a) X_k \,, \tag{2.94.a}$$

$$\hat{\xi}(\mathrm{G}) : X_i * X_j = g_i^r X_r g_j^s X_s \,, \tag{2.94.b}$$

$$X_k = u_k^\mu(a) \frac{\partial}{\partial a^\mu} \,, \tag{2.94.c}$$

where the quantities $\hat{C}_{ij}^k(a)$, called structure functions, are generally dependent on the (local) coordinates of the base manifold of the original group.

In this way Santilli reached an interpretation of the $\hat{F}$-linear combination of the isotopically mapped standard monomials of §2.2. While in the standard realization (2.87.c) the quantities $\tilde{C}_{ij}^k$ are constants (the structure *constants* of a Lie group), the corresponding quantities which emerge after the reformulation of the same group $\hat{G}$ in terms of the base manifold, the parameters, and the generators of G, acquire an explicit dependence on the local coordinates (the structure *functions* $\hat{C}^k_{ij}(a)$). This situation has numerous technical implications (e.g., from the viewpoints of the representation and classification theory) which are not reviewed here.

The use of the Lie algebra laws for the isotopically mapped product (2.94.b) yields Santilli's generalization of Lie's Third Theorem.

Theorem 2.8 *[1], [15]: The structure functions $\hat{C}^k_{ij}(a)$ of the isotopic realization of a Lie group $\hat{G}$ verify the identities*

$$\hat{C}^k_{ij} + \hat{C}^k_{ji} = 0 \,, \tag{2.95.a}$$

$$\hat{C}^k_{ij}\hat{C}^r_{kl} + \hat{C}^k_{jl}\hat{C}^r_{ki} + \hat{C}^k_{li}\hat{C}^r_{kj} + [\hat{C}^r_{ij}, X_l]_{\hat{\xi}} + [\hat{C}^r_{jl}, X_i]_{\hat{\xi}} + [\hat{C}^r_{li}, X_j]_{\hat{\xi}} \tag{2.95.b}$$

The exponentiation from the Lie algebra to the Lie group can now be formulated in terms of the *isotopic image of the exponential law* (2.77), i.e.,

$$G : e^{\theta^i X_i}|_{\xi} \to \hat{G} : e^{\theta^i X_i}|_{\hat{\xi}'} \tag{2.96}$$

which is based on the following *rule of Lie isotopy*

$$\mathbf{G} : [X_i, X_j]_{\xi} = C^k_{ij} X_k \to \hat{\mathbf{G}} : [X_i, X_j]_{\hat{\xi}} = \hat{C}^k_{ij}(a) X_k \,, \tag{2.97}$$

with consequential *isotopically mapped Baker-Campbell-Hausdorff formula* [1], [15]

$$e^{\hat{X}_\alpha} * e^{\hat{X}_\beta} = e^{\hat{X}_\rho} \quad , \hat{X} = gX \,,$$

$$\hat{X}_\rho = \hat{X}_\alpha + \hat{X}_\beta + \frac{1}{2}[X_\alpha, X_\beta]_{\hat{\xi}} + \frac{1}{12}[(X_\alpha - X_\beta), [X_\alpha, X_\beta]_{\hat{\xi}}]_{\hat{\xi}} + \ldots, \tag{2.98}$$

whose existence is ensured by that of the standard realization. The reader can now see the emergence of the $\hat{F}$-linear combination of the basis directly in the group composition law. Clearly, the enveloping algebra underlying expressions (2.98) is the isotope $\hat{\xi}(\mathbf{G})$ of $\xi(\mathbf{G})$.

A simple example may be useful in illustrating the above analysis [1], [15]. Consider the one-parameter group of dilations

$$r' = f(r; \theta) = e^{\theta} r. \tag{2.99}$$

The standard generator for this group is given by

$$X = r\frac{\partial}{\partial r} \,. \tag{2.100}$$

Indeed

$$e^{\theta r(\partial/\partial r)}r = [1 + \frac{\theta}{1!}(r\frac{\partial}{\partial r}) + \frac{\theta^2}{2!}(r\frac{\partial}{\partial r})^2 + ...]r = e^{\theta}r \,. \tag{2.101}$$

The group composition law is, in this case, trivial, i.e.,

$$r'' = f(r'; \theta') = e^{\theta'}r' = e^{\theta'+\theta}r \,. \tag{2.102}$$

Consider now the one-parameter connected Lie group of *nonlinear* transformations

$$\hat{r} = \hat{f}(r;\theta) = \frac{r}{1-\theta r} = g(r,\theta)f(r,\theta), g = \frac{e^{\theta}}{1-\theta r} \,, \tag{2.103}$$

with composition law

$$\hat{r}' = \hat{f}(\hat{r};\theta') = \frac{\hat{r}}{1-\theta'\hat{r}} = \frac{r/(1-\theta r)}{1-\theta'(1/r-\theta r)} = \frac{r}{1-(\theta'+\theta)r} \,. \tag{2.104}$$

We are interested in realizing this group, as a necessary condition of isotopy, via the generator (2.100) of the different group (2.99). This implies the search for an isotopic function, that is, a function which multiplies generator (2.100) to yield the correct transformation law of $\hat{f}$ as a solution of integrability conditions (2.94). Such a solution, in the case at hand, is simple and is given by r. Indeed, the isotopically mapped exponential law (2.96) yields the correct result

$$\begin{aligned} e^{\theta r(r(\partial/\partial r))} &= [1 + \frac{\theta}{1!}(r^2\frac{\partial}{\partial r}) + \frac{\theta^2}{2!}(r^2\frac{\partial}{\partial r})^2 + ...]r \\ &= \frac{r}{1-\theta r} \,. \end{aligned} \tag{2.105}$$

Thus group (2.103) can be realized as an isotopic image of group (2.99).

The case considered above is trivial in the sense that all connected one-dimensional Lie groups are (locally) isomorphic. Thus, to activate the truly nonisomorphic character of the isotope with respect to the original group, one needs more than one dimension. Such a case is already provided by the realization of **SO**(2.1) as an isotope of **SO**(3), in Eqs. (2.26). More examples will be provided in Chapter 3.

2.4 Isotopic Lifting of Space-Time Symmetry Groups on Metric Spaces [22]

After achieving the generalization of Lie's theory reviewed in the preceding sections, Santilli specialized it to metric spaces, so as to facilitate the direct application to cases of physical relevance. In this way, he achieved a result of truly important value (Theorem 2.9 below) which provides the reconstruction of an exact space-time symmetry when conventionally broken [22].

In the following we shall review Santilli's original presentation as closely as possible.

We shall use the term *metric spaces* for the n-dimensional topological spaces M over the field $\mathbf{F}$ of real numbers $\mathbf{R}$, or complex numbers $\mathbf{C}$ or quanternions $\mathbf{Q}$, equipped with a nonsingular, sesquilinear, and Hermitian composition (x, y), $x, y \in M$, characterizing the mapping

$$(x, y) : M \times M \to \mathbf{F} . \tag{2.106}$$

Let $e = (e_1, ..., e_n)$ be a basis of M, and define the *metric tensor* via the familiar rules

$$(e_i, e_j) = g_{ij} . \tag{2.107}$$

Then, the condition of nonsingularity is intended to ensure the existence of the inverse

$$I = g^{-1}, \quad g = (g_{ij}) , \tag{2.108}$$

with the consequent characterization of covariant and contravariant quantities

$$x_i = g_{ij} x^j, \quad x^i = I^{ij} x_j . \tag{2.109}$$

The conditions of sesquilinearity

$$(x, \alpha y + \beta z) = \alpha(x, y) + \beta(x, z) , \tag{2.110}$$

or

$$(\alpha x + \beta y, z) = \overline{\alpha}(x, z) + \overline{\beta}(y, z) , \tag{2.111}$$

where the overbar represents complex conjugation in $\mathbf{F}$, permit the realization of the composition

$$(x, y) = x^{\dagger} g y = x^i g_{ij} x^j , \tag{2.112}$$

where the dagger represents Hermitian conjugation in M.

Finally, the condition of Hermiticity can be formulated via the rules

$$(x, gy) = (g^{\dagger}x, y) = (gx, y) , \tag{2.113}$$

and is introduced for reasons to be identified below.

Additional conditions, such as the positive-definite character of the metric, are not recommendable for a general view of the Lie- isotopic theory, and they will not be considered at this time.

Metric spaces were then indicated in Ref. [22] with the notation

$$M = M(n, g, \mathbf{F}), \quad \mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{Q} . \tag{2.114}$$

which is also adopted hereon. Some of the metric spaces admitted for $\mathbf{F} = \mathbf{R}$ are: the Euclidean space $E(3, \delta, \mathbf{R})$, $\delta = \text{diag}(+1, +1, +1)$; the Minkowski space $M(3+1, \eta, \mathbf{R})$, $\eta = \text{diag}(+1, +1, +1, -1)$; the Riemannian space $R(n, g(x), \mathbf{R})$, with $g(x)$, $x \in M$, being symmetric and positive definite; the Finsler space $F(n, g(x, \dot{x}), \mathbf{R})$, where $g(x, \dot{x}) = \frac{1}{2}(\partial^2 f(x, \dot{x})/\partial x^i \partial x^j)$ is positive definite (for non-null $\dot{x}$) and of rank n; and others with corresponding spaces for the fields $\mathbf{F}$ of complex numbers and quaternions. Thus, we shall assume that the metric g is nonsingular, Hermitian, and verifies the needed continuity conditions (e.g., analyticity) in all variables, and we write

$$\det g \neq 0, \quad g^{\dagger} = g, \quad g = g(t, x, \dot{x}, ...) . \tag{2.115}$$

As one can see, the above definition of a metric is as general as possible, and *does not* coincide with the more restrictive definition conventionally used in specific geometries, such as the symplectic or the Riemannian ones. This situation is permitted by the Lie-isotopic theory because it does not require restrictions on g beyond those considered here. The formalization of the metric and its restriction to specific cases would then imply particularizations (such as the removal of the dependence on the velocities) which are not warranted or recommendable for a general study in Lie isotopy.

We consider now a special case of Definition 2.3, an m-parameter, continuous Lie transformation group $G(m)$ on $M(n, g, \mathbf{F})$, i.e., a topological space $G(m)$ equipped with a binary mapping, e.g.,

$$\varphi : G(m) \times G(m) \to G(m) , \tag{2.116}$$

verifying the conditions for $G(m)$ to be a topological group, and an additional mapping

$$f : G(m) \times M \to M , \tag{2.117}$$

characterized by n analytic functions $f(w;x)$ depending on m parameters w and the local coordinates $x \in M$, which verify the conditions for $G(m)$ to be a Lie transformation group (closure, associativity, identity, and inverse).

We shall furthermore assume that the group $G(m)$ acts linearly on M, i.e.,

$$x' \stackrel{\text{def}}{=} f(w;x) = A(w)x\ , \tag{2.118}$$

under which the group conditions can be realized in the form

$$\begin{aligned} A(0) &= I\ , & (2.119.\text{a}) \\ A(w)A(w') &= A(w''), \quad w'' = w + w'\ , & (2.119.\text{b}) \\ A(w)A(w^{-1}) &= A(w^{-1})A(w) = I, w^{-1} = -w\ , & (2.119.\text{c}) \end{aligned}$$

where I is the unit matrix in n dimensions.

Among the rather large number of aspects of the theory of linear, continuous, m-parameter Lie transformation groups, we now consider for clarity the specialization of the following aspects of §2.2 and §2.3 to metric spaces:

(1) The *universal enveloping associative algebra* ξ of $G(m)$, which we shall indicate with the symbolic expression of the basis

$$\xi : I, \quad X_r, \quad X_rX_s, \quad X_rX_sX_t\ ,$$

$$r \leq s, r \leq s \leq t, \quad r,s,t,... = 1,2,\dots m\ , \tag{2.120}$$

where I is now the $m \times m$ identity of ξ,

$$IX_r = X_rI = X_r\ . \tag{2.121}$$

The X's are the generators of $G(m)$ in their fundamental ($m \times m$) representation verifying the skew-Hermiticity property

$$X_r^\dagger = -X_r\ , \tag{2.122}$$

the product X_rX_s is the conventional associative product of matrices; and the attached Lie algebra is given by the familiar rule

$$\xi^- : [P_r, P_s]_\xi = P_rP_s - P_sP_r, \tag{2.123}$$

where the P's are polynomials in the X's.

(2) The *Lie's group* $G(m)$ of transformations on M for the case of the action to the right as in Eq. (2.118), which we shall write in the symbolic exponentiated form for continuous transformations

$$\begin{aligned} G(m) : A(w) &= e^{X_1 w_1} e^{X_2 w_2} ... e^{X_m w_m} \\ &= \prod_{k=1}^{m} e^{X_k w_k} , \end{aligned} \tag{2.124}$$

and which will be reduced to the appropriate exponential form whenever we consider specific cases. The corresponding action to the left,

$$x^{\dagger\prime} = x^{\dagger} A^{\dagger}(w) , \tag{2.125}$$

can be characterized by the operation of Hermitian conjugation, which we shall write in the symbolic form

$$G(m) : \hat{A}^{\dagger}(w) = (\prod_{k=1}^{m} e^{X_k w_k})^{\dagger} , \tag{2.126}$$

and whose explicit form will be computed whenever the reduced form of Eq. (2.124) is known (see the case of rotations of §3.2).

(3) The *Lie algebra* $\mathbf{G}(m)$ of $G(m)$, characterized by the closure rules

$$\mathbf{G}(m) : [X_r, X_s]_{\xi} = X_r X_s - X_s X_r = C^t_{rs} X_t . \tag{2.127}$$

The underlying methodology we shall tacitly imply is the familiar one consisting of the Poincaré-Birkhoff-Witt theorem for the characterization of the basis (2.120); the Baker-Campbell-Housdorff theorem for the composition of the exponentials (2.124) and (2.126); Lie's First, Second,and Third Theorems for the characterization of the closure rules (2.127); the representation theory; etc.

The idea of the *Lie-isotopic theory* [1] is that of generalizing the structure of the enveloping algebra ξ, of the Lie group $G(m)$, and of the Lie algebra $\mathbf{G}(m)$ in such a way to preserve the Lie character of the theory (in order to qualify for isotopy). The generalization is done via the replacement of the simplest possible, associative, Lie-admissible product $X_r X_s$ of the conventional theory into a form denoted by $X_r * X_s$ which is still associative and Lie admissible (i.e., its attached product $X_r * X_s - X_s * X_r$ is Lie); nevertheless, it is given by the structurally more general form

$$X_r * X_s = X_r g X_s . \tag{2.128}$$

It is evident that the generalization of the product of ξ implies a step-by-step generalization of the entire formulation of Lie's theory, from basis (2.120) to groups (2.124) and (2.126), to algebra (2.127), etc.

In paper [22] Santilli investigates not the Lie-isotopic theory per se, but its action on a metric space. He therefore identified the generalization of the structure of the metric space permitting a consistent action of the Lie-isotopic theory.

For this purpose, we shall first review the notion of *metric isotopy*, that is, a generalization of a given metric space which preserves its metric character. We shall then review the corresponding Lie-isotopic theory. Finally, we shall apply the results to the case when the considered Lie and Lie-isotopic groups constitute symmetries of the metric and its isotope, respectively. This latter result will be presented via Theorem 2.9 below on the symmetry properties of isotopy which is at the foundation of the applications of Chapter 3 to rotations, Galilei and Lorentz transformations.

Consider the simplest possible metric spaces, the Euclidean space $E(n, \delta, \mathbf{F})$, $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{Q}$, with composition law

$$(x, y) = x^i \delta_{ij} x^j. \tag{2.129}$$

Suppose that the metric δ has to be modified into a form of the generic type (2.115). The emerging generalized space can be expressed via the notion of metric isotopy as follows.

Let $\hat{I} = g^{-1}$ be the inverse of the new metric according to (2.108). Introduce the isotopic lifting of the field (1.38), i.e.,

$$\hat{\mathbf{F}} = \{\hat{N} | \hat{N} = N\hat{I}, N \in \mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{Q}\}. \tag{2.130}$$

The composition of elements of the field with elements of the metric space is now done according to the redefinition of the product

$$\hat{N} * x = \hat{N} g x = N \hat{I} g x = N x. \tag{2.131}$$

Thus, the lifting $\hat{\mathbf{F}}$ of $\mathbf{F}$ essentially permits the use of a generalized composition $\hat{N} * x$ which, while being characterized by the new metric g, preserves the old values Nx.

Next, Santilli generalizes the metric space $E(n, \delta, \mathbf{F})$ into a form $\hat{E}$ that accommodates the new metric g under a mapping of the type

$$\hat{m} : \hat{E} \times \hat{E} \rightarrow \hat{\mathbf{F}}. \tag{2.132}$$

This implies that the generalized composition law must have value in $\hat{\mathbf{F}}$. A realization is given by the form patterned along the isotopic lifting of the Hilbert spaces, Eq. (1.49), i.e.,

$$\begin{aligned}(x\hat{,}y) &= \hat{I}(x,gy) = \hat{I}x^i g_{ij} x^j \\ &= (x,gy)\hat{I} = (gx,y)\hat{I}. \end{aligned} \tag{2.133}$$

Ref. [22] defines as *isotopic liftings of the Euclidean space* all possible spaces $\hat{E}(n, g, \hat{\mathbf{F}})$ over the field $\hat{\mathbf{F}} = \hat{\mathbf{R}}, \hat{\mathbf{C}}, \hat{\mathbf{Q}}$, equipped with mappings (2.132) realized via composition (2.132), where g is the new metric tensor.

It is evident that, by construction, *all possible nonsingular metrics of the same dimension are isotopes of the Euclidean metric.* This includes the Minkowskian, Riemannian, Finslerian, and other metrics.

Note that, strictly speaking, the metric spaces $\hat{E}(n, g, \mathbf{F})$ cannot be considered as isotopes of $E(n, \delta, \mathbf{F})$, owing to the lack of lifting of the field. Nevertheless, this technical point can be ignored in practical applications owing to the identity $\hat{N} * x = Nx$. We can then assume that all possible metric spaces of n dimensions over the field $\mathbf{F}$ are isotopes of the Euclidean space.

Note that, since $\hat{\mathbf{F}}$ is still a field, $\hat{E}(n, g, \hat{\mathbf{F}})$ is also a metric space in the sense indicated earlier.

It is evident that the original Lie group $G(m)$ cannot act consistently on the new spaces. In fact, to begin, the action of the group on the space cannot be formulated according to the old composition (2.118) and must be modified into the form

$$x' = \hat{A}(w) * x \stackrel{\text{def}}{=} \hat{A}(w) g x \ , \tag{2.134}$$

[where the quantities $\hat{A}(w)$ will be identified shortly]. In turn, this implies that the old composition laws (2.118) cannot be consistently preserved, and must be generalized into the form

$$\hat{A}(0) = \hat{I} \ , \tag{2.135.a}$$

$$\hat{A}(w) * \hat{A}(w') = \hat{A}(w + w') \ , \tag{2.135.b}$$

$$\hat{A}(w) * \hat{A}(-w) = \hat{A}(-w) * \hat{A}(w) = \hat{I} \ , \tag{2.135.c}$$

which are precisely the defining conditions of a *Lie-isotopic transformation group* [1], [15].

The most important property of generalized laws (2.135) is the replacement of the old unit I with the new unit $\hat{I} = g^{-1}$. Thus, the dominant feature of Santilli's isotopy under consideration is the assumption of the inverse $\hat{I}$ of the new metric g as the generalized identity of the group. Since the original identity I can be interpreted as the inverse of the metric δ of the Euclidean space, when the original group $G(m)$ is a symmetry of δ, we expect its isotopic image $\hat{G}(m)$ to constitute a symmetry of g.

To achieve this result, Santilli uses the following main lines of the Lie-isotopic theory reviewed in §2.2 and §2.3:

(1) *Isotopic lifting of the universal enveloping associative algebra.* The Poincaré-Birkhoff-Witt theorem admits a consistent isotopic generalization, resulting in the new basis

$$\hat{\xi} : \hat{I}, \quad X_r, \quad X_r * X_s, \quad X_r * X_s * X_t, \dots , \tag{2.136}$$

$$r \leq s, \quad r \leq s \leq t,$$

$$r, s, t, \dots = 1, 2, \dots, n \ .$$

expressed in terms of the isounit $\hat{I}$, which is the same as that of the group composition laws (2.135). The generators X_r are here the same as those of ξ. The attached Lie algebra is now given by the isotope

$$\begin{aligned} \hat{\xi}^- : [P_r, P_s]_{\hat{\xi}} &= P_r * P_s - P_s * P_r \\ &= P_r g P_s - P_s g P_r \stackrel{\text{def}}{=} [P_r \hat{,} P_s] \ , \end{aligned} \tag{2.137}$$

The algebra $\hat{\xi}$ is still "universal" and "enveloping" not, of course, with respect to the algebra ξ^-, but with respect to $\hat{\xi}^-$. We see in this way that the generalized metric g enters into the very structure of the Lie product, Eq. (2.137), as expected.

(2) *Isotopic lifting of the Lie group.* The new basis (2.136) permits the construction of the new group elements $\hat{A}(w)$ via the so-called *isotopic exponentiation* [1], [15]. For one-parameter actions to the right, this exponentiation is characterized by the old generator X of $G(m)$ but now expanded in the new envelope according to the rule

$$\begin{aligned} \hat{G}(1) : \hat{A}(w) &= \hat{I} + \frac{1}{1!}(Xw) + \frac{1}{2!}(Xw)^{\hat{2}} + \frac{1}{3!}(Xw)^{\hat{3}} + \dots \\ &= \hat{I} + \frac{1}{1!}(Xw) + \frac{2}{2!}(Xw)g(Xw) + \frac{1}{3!}(Xw)g(Xw)g(Xw) + \\ &= e^{Xw}|_{\hat{\xi}} \stackrel{\text{def}}{=} \hat{e}^{Xw}, \end{aligned} \tag{2.138}$$

which, for clarity of practical computation, can be reexpressed via the following expansion in the old envelope

$$\begin{aligned}\hat{G}(1):\hat{A}(w) &= [1+\frac{1}{1!}(Xgw)+\frac{1}{2!}(Xgw)(Xgw)+\frac{1}{3!}(Xgw)\\ &\qquad (Xgw)(Xgw)+\ldots]\hat{I}\\ &= (e^{Xgw}|_{\xi})\hat{I} = e^{X*w}\hat{I}\\ &= \hat{I}(e^{wgX}|_{\xi}) = \hat{I}e^{w*X}. \end{aligned} \tag{2.139}$$

It is evident that the elements $\hat{A}(w)$ so constructed verify all the rules (2.135), and thus they constitute the desired Lie- isotopic lifting of $G(1)$. The generalization to more than one dimension is permitted by the *Lie-isotopic generalization of the Campbell-Baker-Hausdorff theorem*

$$\hat{e}^{\alpha} * \hat{e}^{\beta} = \hat{e}^{\gamma},$$

$$\gamma = \alpha+\beta+\frac{1}{2}[\alpha\hat{,}\beta]+\frac{1}{12}[(\alpha-\beta)\hat{,}[\alpha\hat{,}\beta]]+\ldots\ , \tag{2.140}$$

under which we have the desired Lie-isotopic lifting of the Lie transformation group (2.124), here written, again, in the symbolic form

$$\begin{aligned}\hat{G}(m):\hat{A}(w) &= \hat{e}^{X_1w_1} * \hat{e}^{X_2w_2} * \ldots * \hat{e}^{X_mw_m}\\ &= \prod_{k=1}^{m} * \hat{e}^{X_kw_k}\\ &= (e^{X_1*w_1}e^{X_2*w_2}\ldots e^{X_m*w_m})\hat{I}\\ &= (\prod_{k=1}^{m} e^{X_k*w_k})\hat{I}\ . \end{aligned} \tag{2.141}$$

The action of the Lie-isotopic group to the left,

$$x^{\dagger'} = x^{\dagger} * \hat{A}^{\dagger}(w)\ , \tag{2.142}$$

is given, for the one-parameter case, by the expansion of the old generator $X^{\dagger}$ in the new envelope ξ, according to the rule

$$\begin{aligned}\hat{G}(1):\hat{A}^{\dagger}(w) &= \hat{I}+\frac{1}{1!}(wX^{\dagger})+\frac{1}{2!}(wX^{\dagger})^{\hat{2}}+\frac{1}{3!}(wX^{\dagger})^{\hat{3}}\\ &= \hat{I}+\frac{1}{1!}(wX^{\dagger})+\frac{1}{2!}(wX^{\dagger})g(wX^{\dagger})+\ldots\\ &= e^{wX^{\dagger}}|_{\hat{\xi}} = \hat{e}^{wX^{\dagger}} = \hat{e}^{-wX}\ , \end{aligned} \tag{2.143}$$

with reformulation in ξ for practical calculations

$$\begin{aligned}\hat{G}(1):\hat{A}^{\dagger}(w) &= \hat{I}[1+\frac{1}{1!}(wgX^{\dagger})+\frac{1}{2!}(wgX^{\dagger})(wgX^{\dagger})+\ldots] \\ &= \hat{I}(e^{wgX^{\dagger}}|_{\xi})=\hat{I}e^{wgX^{\dagger}} \\ &= e^{X^{\dagger}gw}\hat{I}=\hat{I}e^{-w*X}\ , \end{aligned} \tag{2.144}$$

and m-parameter expression here symbolically written

$$\hat{G}(m):\hat{A}^{\dagger}(w)=\hat{I}(\prod_{k=1}^{m}e^{X_k*w_k})^{\dagger}\ , \tag{2.145}$$

whose explicit form will be computed in specific cases (see, e.g., the case of the isotopic rotations in §3.2). It remains to prove that the operation of Hermitian conjugation, as conventionally defined, also acts consistently under isotopy in $\hat{E}(n,g,\hat{\mathbf{F}})$. The fact that this is not the case in general is known [36]. Nevertheless, as for case (1.51), the operation of Hermiticity persists for the particular case under consideration here, that for which the isotopic element of the envelope coincides with that of the composition [38], as is readily seen by using the property (2.115) and definition (2.133)

$$\begin{aligned}(x\hat{,}\hat{A}*y) &= \hat{I}(x,g\hat{A}gy) \\ &= \hat{I}((g\hat{A})^{\dagger}x,gy)=\hat{I}(\hat{A}^{\dagger}gx,gy) \\ &= \hat{I}(\hat{A}^{\dagger}*x,y)\ , \end{aligned} \tag{2.146}$$

for which

$$(e^{Xgw})^{\dagger}=e^{wg^{\dagger}X^{\dagger}}=e^{-wgX}\ . \tag{2.147}$$

(3) *Isotopic lifting of the Lie algebra.* This is characterized by the isotopic generalization of Lie's first, second, and third theorems here expressed according to the rules

$$\begin{aligned}\hat{\mathrm{G}}(m):[X_r\hat{,}X_s] &= X_r*X_s-X_s*X_r \\ &= X_rgX_s-X_sgX_r \\ &= \hat{C}^{i}_{rs}(x)*X_i\ , \end{aligned} \tag{2.148}$$

$$\hat{C}^{i}_{rs}=C^{i}_{rs}\hat{I},$$

where the $\hat{C}$'s are the *structure functions.* As is the case for the expansion (2.138), rules (2.148) can also be reformulated in ξ according to either one of the following expressions, useful for practical calculations

$$\begin{aligned}[X_r \hat{,} X_s] &= X_r g X_s - X_s g X_r \\ &= [X_r g, X_s g]\hat{I} \\ &= [X_r, X_s]g + X_r[g, X_s] + X_s[X_r, g] \\ &= \hat{I}[gX_r, gX_s] \\ &= g[X_r, X_s] + [X_r, g]X_s + [g, X_s]X_r \ , \end{aligned} \tag{2.149}$$

each one derivable from the other via the Jacobi law.

The primary lines of the Lie-isotopic theory as outlined above are sufficient for the main task of this section, that dealing with symmetries of arbitrary metrics g.

Suppose that the original (conventional) Lie transformation group $G(m)$ is a symmetry group of the composition (x, y) in $E(n, \delta, \mathbf{F})$, or, equivalently, of the metric δ, according to the familiar conditions

$$x^{\dagger\prime} x' \equiv x^{\dagger\prime} \delta x' = x^\dagger A^\dagger \delta A x = x^\dagger \delta x \equiv x^\dagger x \ , \tag{2.150}$$

which can hold identically iff

$$A^\dagger \delta A \equiv A^\dagger A = A A^\dagger \equiv A \delta A^\dagger = I = \delta^{-1} \ , \tag{2.151}$$

i.e.,

$$\begin{aligned} A^\dagger &= A^{-1} \ , \\ (\det A)^2 &= (\det I)^2 = 1 \ . \end{aligned} \tag{2.152}$$

As is well known, when conditions (2.152) are verified, we have the *orthogonal groups* $O(n, \mathbf{R})$, the *unitary groups* $\mathbf{U}(n, \mathbf{C})$, and others. When realizations of the continuous type are considered, we have the *special orthogonal groups* $\mathbf{SO}(n, \mathbf{R})$ or the *special unitary groups* $\mathbf{SU}(n, \mathbf{C})$. In this latter case, the determinant of the transformation is 1, and the discrete transformations (e.g., inversions) are excluded.

Santilli [22] investigated the behavior of the symmetry (2.150) under an isotopic lifting of the Euclidean space $E(n, \delta, \mathbf{F})$ and of the group $G(m)$ to a form characterized by an arbitrary metric (2.115). For this purpose, we recall that the composition law of $\hat{E}(n, g, \hat{\mathbf{F}})$ is based on the term

$$x^\dagger * x = x^\dagger g x \ . \tag{2.153}$$

We therefore have a symmetry when the following conditions are identically verified

$$x^{\dagger\prime} * x' = x^{\dagger} * \hat{A}^{\dagger} * \hat{A} * x = x^{\dagger} * x \,, \tag{2.154}$$

which can hold iff

$$\hat{A}^{\dagger} g \hat{A} = \hat{A} g \hat{A}^{\dagger} = \hat{I} \,, \tag{2.155}$$

i.e., iff

$$\hat{A}^{\dagger} = \hat{A}^{-\hat{1}} \,,$$
$$(\det\hat{A})^2 = (\det\hat{I})^2 \,, \tag{2.156}$$

where the inverse is computed, of course, with respect to $\hat{I}$.

It is easy to see that, when the original transformations verify conditions (2.150), their images under lifting necessarily verify the new conditions (2.154). In fact, for the case of continuous transformations, we have, from Eqs. (2.142) and (2.143),

$$\hat{A}^{\dagger}(w) = \hat{A}(-w) \,. \tag{2.157}$$

Therefore, conditions (2.154) are reduced to one of the conditions for the very existence of a Lie-isotopic group, Eq. (2.135).

The rules (2.155) can be expressed in a form particularly suitable for practical applications. Redefine the elements of $\hat{G}(m)$ according to the forms

$$\hat{A}(w) = B(w)\hat{I}, \quad B(w) = \prod_{k=1}^{m} e^{X_k * w_k} \,,$$

$$\hat{A}^{\dagger}(w) = \hat{I} B^{\dagger}(w), \quad B^{\dagger}(w) = (\prod_{k=1}^{m} e^{X_k * w_k})^{\dagger} \,. \tag{2.158}$$

Then, conditions (2.145) can be equivalently expressed as

$$B^{\dagger} g B = g \,, \tag{2.159.a}$$
$$(\det B)^2 = 1 \,, \tag{2.159.b}$$

which hold identically under the Lie-isotopic liftings of continuous transformations owing to the identity

$$\begin{aligned} e^{-wgX} g e^{Xgw} &= g - w(gXg - gXg) + \frac{1}{2} w^2 (gXgXg - gXgXg) + \ldots \\ &= g \,. \end{aligned} \tag{2.160}$$

For the case of discrete transformations, Santilli introduces the following *Lie-isotopic lifting of inversions*

$$\hat{\mathcal{P}} * x = (\mathcal{P}\hat{I})gx = \mathcal{P}x = -x \ , \tag{2.161}$$

where $\mathcal{P}$ is the conventional total inversion. The preservation of the symmetry then results from known expressions of the type

$$\mathcal{P}g\mathcal{P} = g \ , \tag{2.162}$$

whose validity is trivial.

We reach in this way Santilli's main result, which can be formulated as follows.

Theorem 2.9 *[22]: Let $G(m)$ be an m-parameter Lie symmetry group of the composition $x^\dagger \delta x$ of an n- dimensional Euclidean space $E(n, \delta, \mathbf{F})$ over the field $\mathbf{F}$ of real numbers $\mathbf{R}$, of complex numbers $\mathbf{C}$, or of quarternions $\mathbf{Q}$. Then the isotopic lifting $\hat{G}(m)$ of $G(m)$ characterized by a nonsingular, Hermitian, and sufficiently smooth metric g in the local variables leaves invariant the generalized composition $x^\dagger gx$ of the isotopic space $\hat{E}(n, g, \hat{\mathbf{F}}), \hat{\mathbf{F}} = \mathbf{F}\hat{I}, \hat{I} = g^{-1}$.*

All physical applications of Chapter 3 can be considered as applications of the above theorem to specific cases of physical relevance.

Note that the explicit construction of the Lie-isotopic transformations (as well as of the entire theory) can be done following the knowledge only of the original symmetry and of the new metric.

Note also that all Lie algebras admit the following *trivial Lie isotopy*

$$\begin{aligned} \hat{\mathbf{G}}(m) : [\hat{X}_r \hat{,} \hat{X}_s] &= \hat{X}_r * \hat{X}_s - \hat{X}_s * \hat{X}_r && (2.163.a) \\ &= (X_r X_s - X_s X_r)\hat{I} = C^k_{rs} \hat{X}_k \ , \\ \hat{X} &= X\hat{I}, \quad X \in \mathbf{G}(m), && (2.163.b) \end{aligned}$$

with a self-evident isomorphism $\hat{\mathbf{G}}(m) \approx \mathbf{G}(m)$. The above trivial isotopy should be excluded from the content of Theorem 2.9 because it does not provide the invariance of the generalized composition law. This can be readily seen from the fact that the exponentials (2.141) and (2.144), when realized for the generators $\hat{X}_k$, coincide with the original exponentials (except for the factorization of the new unit), and no genuine lifting has actually occurred.

Theorem 2.9 has clearly far reaching mathematical and physical implications, which can be only partially reviewed here. To begin, Theorem 2.9 provides a new concept of *covering Lie-isotopic symmetry* under the sole condition that the original metric δ is contained as a particular case of the new metric g.

But Theorem 2.9 applies for an infinite variety of possible new metrics. As a result, *a given, conventional, Lie symmetry $G(m)$ admits an infinite class of covering Lie-isotopic symmetries $\hat{G}(m)$*. The implications of these findings will become transparent in the next section when we shall show that *the explicit form of the Lie-isotopic symmetry transformations evidently varies with the varying of g*.

Furthermore, under certain topological conditions on the new metric (identified in the next chapter), *the original Lie symmetry $G(m)$ and its infinite class of Lie-isotopic coverings $\hat{G}(m)$, not only become locally isomorphic, but they actually coincide at the abstract realization-free level.*

This is evidently permitted by the abstract formulation of the symmetry, that in terms of an *abstract enveloping algebra* with abstract product, say, ab, and its *realization*, first in terms of the trivial associative product AB, resulting into the familiar notion of symmetry $G(m)$ as commonly available in the mathematical and physical literature, and then its isotopic liftings $A * B = \alpha AB$ or AgB, or $WAWBW$ ($W^2 = W$) resulting in Santilli's notion of infinite covering symmetries $\hat{G}(m)$.

Yet in turn, the above properties of Theorem 2.9 are at the foundation of the capabilities by Santilli to "reconstruct" an exact Lie symmetry when conventionally broken (see the next chapter for specific cases).

Still another property of Theorem 2.9 of considerable mathematical and physical importance is *the intrinsic nonlinear character of Santilli's Lie-isotopic theory, even though expressed in a formally linear form, the isolinear form.*

In fact, the transformations underlying Theorem 2.9, Eqs. (2.134), have an intrinsically nonlinear structure in the coordinates x, their derivatives $\dot{x}$ with respect to independent parameters, etc., and we shall write

$$x' = B(w; x, \dot{x}, \ldots)x \tag{2.164}$$

where the nonlinearity evidently emerges from the arbitrary dependence of the metric in expansion (2.139), i.e.,

$$B = \exp(Xg(x, \dot{x}, \ldots))w|_{\xi}. \tag{2.165}$$

Nevertheless, *nonlinear transformations (2.164) can always be written in the equivalent isolinear form*

$$x' = \hat{A}(w) * x. \tag{2.166}$$

The mathematical implications of the above results are evident, and linked to the possibility (not yet explored so far) of turning complex nonlinear problems into more manageable, equivalent, isolinear forms.

The physical implications are equally far reaching. In fact, the intrinsic isolinear character of the Lie-isotopic theory is the technical reason underlying Santilli's view that the expected nonlinearity of the strong interactions is not a structure characterizing feature; only their expected contact/nonlocal/nonhamiltonian character is.

In particular, the capability of turning all possible nonlinear models, such as Weinberg's attempt (1.65), into an equivalent isolinear form, is expected to void most of the experimental argumentations currently presented on nonlinearity.

As a further comment, the isotopic liftings of Euclidean spaces reviewed here are expected to be extendable to accommodate antisymmetric metrics and their symplectic symmetry groups. In fact, liftings (2.138) and (2.144) are possible also for antisymmetric metrics. The restriction to Hermitian metrics was done by Santilli because of compatibility condition (2.147), having in mind operator-type applications based on the completion of the Euclidean spaces into Hilbert spaces.

This completes our review of Santilli's mathematical studies on his Lie-isotopic theory which, with the sole exception of paper [62] known to us, constitute all mathematical studies on the topic available at this time.

2.5 Some Open Mathematical Problems

It is clearly remarkable for one single individual to work out the generalized formulation of Lie's theory to the extent reviewed in the preceding sections (as well as its applications reviewed in the final part of this presentation). Nevertheless, the mathematical research on the Lie-isotopic theory is only at a beginning, and so much remains to be done. The number of open mathematical problems is so large to prevent their comprehensive identification. We merely limit ourselves here to identify open mathematical problems that are relevant for the physical applications considered in the next section. A presentation of the open mathematical problems for mathematicians has been done by these authors in ref. [65].

To begin, the virtual entirety of basic definitions of Lie's theory need a suitable reinspection and reformulation into corresponding covering notions that are directly applicable to the Lie-isotopic theory. This is the case for the notions of: *compact and noncompact algebras; simple and semisimple algebras; Cartan's decomposition; Killing form;* etc.

All these notions in their familiar presentation have an unequivocal meaning because referred to one specific realization of the Lie product, the simplest possible one $AB - BA$. The same notions, unless properly redefined, become ambiguous when referred to Santilli's product $A*B-B*A = AgB - BgA$ because of the infinite family of possible isotopic elements g all with potentially different topologies.

Once these fundamental notions of Lie's theory have been properly reviewed, one can pass to the study of basic methodological aspects which have remained untouched as of now.

A central open mathematical problem is the *representation theory of Lie-isotopic algebras and groups.* Santilli's studies reviewed here, e.g., §2.4, essentially provide the fundamental representation, as we shall see in Chapter 3. The case of the general representation theory has been studied by Santilli only for the $\widehat{SU}(2)$-isotopic group [24] and that of the $\widehat{SU}(3)$-isotopic group is under study in ref. [44]. But, again, a general study of the representation theory is lacking as of now (Spring 1990) to our best knowledge.

The mathematical relevance of the problem is expressed by the fact that the exclusion of the trivial isotopy (2.163) prevents a simplistic lifting of the conventional theory. Also, the infinite variety of isotopic transformations (§2.4) demands a reinspection of the representation theory from its foundations.

The physical relevance of the representation theory is also self-evident. It can be best expressed as essential to characterize the notion of "particle" within the arena of physical applicability of the Lie-isotopic theory, i.e., the notion of "hadron" under contact/nonlocal/nonhamiltonian strong interactions (§1.3).

Of particular relevance are studies of the *representation theory of Santilli's isotopic group of rotations* (§3.2), *and of Lorentz transformations* (§3.4), which are evidently essential for possible basic advances, e.g., on the notion of intrinsic angular momentum (spin) of one hadron under *external* strong interactions of the considered type.

Another mathematical aspect in need of a comprehensive study is that of the product of the above representation, i.e., *the isotensorial product of Lie-isotopic representations.* Studies on this aspect were initiated in the only pure mathematical contribution to Lie-isotopy known to these authors, Ref. [62], but so much remains to be done.

The physical relevance of the isotensorial products of isorepresentations is evidently provided by the need, reviewed in §1.3, of recovering conventional, total, quantum mechanical quantities for an isolated bound system of strongly interaction particles, while admitting generalized internal laws.

A further mathematical problem deserving specific studies is the *contraction and expansion of Santilli's Lie-isotopic groups.* Recall from §1.3 that the "hadronization" of the classical Birkhoffian mechanics into the operator form, hadronic mechanics, proved to be particularly valuable for the understanding of both new mechanics. A quite similar situation occurs for Lie-isotopic groups. As we shall review in the next sections, Santilli applied his theory to the isotopic lifting of the Galilei and Lorentz symmetries. While the contraction of the Lorentz symmetry into the Galilean one (and the inverse expansion) is well known, no study has been conducted until now on its covering Lie-isotopic setting. Its value for a deeper understanding of the Galilei- isotopic and Lorentz-isotopic symmetries (see the next section) is evident.

The educated reader can easily identify numerous, additional, mathematical problems of fundamental, yet open character.

It is hoped in this way the reader can see the need, anticipated earlier, for a re-inspection of the entire Lie's theory and its reformulation into a covering form directly applicable to Lie-isotopic algebras and groups.

This review would have achieved a primary objective, if it succeeds in stimulating this much needed, independent mathematical research.

The authors of this review would be grateful to all mathematicians who can send to their attention (at the address of The Institute for Basic Research, P.O. Box 1577, Palm Harbor, FL 34682-1577, U.S.A., Fax 813-934-9275) any mathematical research directly or indirectly related to associative-isotopic and Lie-isotopic algebras or groups.

3 THE PHYSICAL FOUNDATIONS OF THE THEORY

3.1 Introductory Aspects

The Lie-isotopic theory was conceived by Santilli for the specific purpose of attempting a generalization of conventional space-time symmetries and related relativities [1].

In this section we shall review the state of the research in the isotopic lifting of:

a) the rotation group [23],[24],[25];

b) the Galilei group and related relativity [1],[15],[24],[25]; and

c) the Poincaré group and related special relativity [18],[24],[25],[26],[27].

For completeness, we shall also review Santilli's [18],[16], [25],[26] and Gasperini's [81], [82], [83] research on a conceivable isotopic generalization of Einstein's gravitation. Gasperini's lifting of gauge theories [84], [85] shall be reviewed in the Appendices, jointly with a number of other aspects. All known applications shall be either reviewed or indicated to the interested reader. This section shall end with a review of much overdue experiments.

Regrettably, we are unable to review numerous intriguing applications of the Lie-isotopic theory because of their intrinsic operator character, such as: Kalnay's [40] hadronization of Nambu's mechanics; Santilli's [30] true confinement of quarks with null probability of tunnel effects; Mignani's [45] nonpotential scattering theory; Nishioka's [48] studies; Animalu's [50] research; the studies by Jannussis and collaborators [47]; and others.

A few introductory comments appear to be recommendable, not only because of the manifestly delicate nature of the review, but also in order to prevent unnecessary misrepresentations.

The best way to present the material is that along the spirit of the original proposals:

1. The Lie-isotopic theory provides true, mathematically consistent generalizations of conventional space-time symmetries. As such, they are intriguing on pure mathematical grounds alone [1].

2. The nonrelativistic, isotopic, space-time symmetries have clear applications in classical mechanics [15].

3. The relativistic [18] and gravitational [5], [16], [26] isotopic, space-time symmetries are conjectural at this time because of the lack of certain fundamental tests recommended since quite some time.

As reviewed in Chapter 2, the single most dominant *mathematical* concept in Santilli's Lie-isotopic theory is the *generalized notion of unit*, the isounit $\hat{I} = g^{-1}$. The single, most dominant *physical* concept in the applications of the Lie-isotopic theory is the *notion of extended particles moving within a physical medium*, such as: propagation of light in gaseous or liquid media; motion of a satellite in Earth's atmosphere; motion of the wave-packet of a hadronic constituent within the "hadronic medium" [2] (the medium composed by the wavepackets of the remaining constituents); and other cases.

If the particles considered are assumed as being point-like, the Lie-isotopic theory has no relevance known at this time and none of the structures reviewed below has a known physical meaning.

In fact, systems of point-like particles can only admit action-at-a-distance interactions of potential-Hamiltonian type without collisions. The conventional Lie's theory then applies in full without need for any generalization. This is the case irrespective of whether the particles move in empty space or in a physical medium, for that medium too becomes composed of isolated, point-like constituents. A similar situation occurs also for an extended particle moving in vacuum under long range, external, potential forces. In fact, under these conditions, the size of the particle can be effectively ignored. (This is the case, e.g., for the wave-packet of an electron when a member of an atomic cloud.)

The physical arena changes significantly when the size of the particles must be specifically taken into account, e.g., when the particles move within a physical medium and/or experience a deformation of their shape. In these latter conditions the particles experience interactions which are generally of nohamiltonian, and therefore non-Lie character (§1.3). It is at this point that Santilli's Lie-isotopic theory offers intriguing possibilities for a quantitative treatment. In fact, as now familiar, all nonhamiltonian forces can be incorporated in the generalized unit of the theory, while the Hamiltonian can represent conventional interactions.

A second fundamental physical concept in Santilli's studies is that *empty space (vacuum) remains conventionally homogeneous and isotropic. It is the physical medium in which motion of extended objects occurs which is, in general, inhomogeneous and anisotropic.* To put it differently, the Lie-isotopic symmetries were not conceived for treating the conventional space. After all, by ignoring certain galactic indications, there is no available evidence disproving the homogeneity and isotropy of space in our Earthly environment. Conventional space-time symmetries are then the *only* ones applicable, as stressed by Santilli himself [1].

The inhomogeneity and anisotropy of physical media (whether classical or operational), leads to the inevitable breaking of conventional space-time symmetries beginning with the rotational symmetry (§3.2); and then passing to the Galilei symmetry (§3.3); the Lorentz symmetry (§3.4) and, inevitably, Einstein's gravitation (§3.5).

Admittedly, Santilli's Lie-isotopic theory is only tentative at this time, and recommended as a conceivable first step for a future more adequate treatment. But the breaking of conventional space-time symmetries under the physical conditions considered is simply out of any question. The interested reader is urged to study the classification of the various forms of breaking of conventional space-time symmetries provided by the variational self-adjointness, as originally presented in ref. [1], and subsequently reviewed in detail in monographs [4], [5].

Another important concept in Santilli's studies is *the experimental evidence of the deformability of extended particles.* Again, conventional space-time theories are strictly referred to *rigid bodies.* This is typically the case of the theory of rotations, as well known. But absolutely rigid objects do not exist in Nature. When the deformability of objects is admitted, conventional space-time symmetries are inapplicable, as stressed again by Santilli [1], [18], [20].

As an example, the conventional rotational symmetry is manifestly broken to a sphere which is jointly experiencing a rotation and a deformation. The inapplicability of the Galilei and Lorentz symmetries is then consequential, owing to the central role of the rotational symmetry (as well as for additional reasons).

The deformation of extended particles moving in vacuum but under sufficiently intense external forces is therefore another arena of possible physical applications for which the Lie-isotopic theory was conceived. Again, the effectiveness of Santilli's approach is unknown at this time, for it was merely proposed as a first quantitative step. Nevertheless, the breaking of conventional space-time symmetries under this second class of physical conditions is simply out of any question.

Note the independence of the breakings caused by deformation from those caused by the inhomogeneous and anisotropic character of the physical medium.

A third class of breakings is given by the ultimate essence of *contact interactions*, that of being *instantaneous* no matter whether in "nonrelativistic" or "relativistic" mechanics. After all, these interactions have a *null range* by their essential nature. We are here referring to the evidence that motion of a particle within a physical medium results in interactions between the particle and the medium which, by nature, are of "contact," that is, "instantaneous" and "null range" character, as well known in classical (but not yet in particle) mechanics.

This leads to a third physical origin of the breakings of conventional space-time symmetries, which is independent from the preceding two (regarding the inhomogeneous/anisotropic character of physical media, and the deformability of extended objects).

A fourth class of breakings of the conventional space-time symmetries is given by the strictly non-Hamiltonian character of the interactions considered. In fact, the notion of "potential" has no physical meaning for contact interaction, as a direct consequence of their zero range nature. More specifically, contact interactions have no potential "energy" and, as such, are conceptually and structurally outside the capabilities of Hamiltonian mechanics.

This leads to a fourth class of breakings of conventional space-time symmetries, that caused by the inapplicability of the canonical realization of Lie's theory, thus establishing the need for its structural generalization.

A fifth and final class of breakings of conventional space-time symmetries is given by the non-local/integral nature of contact interactions. In fact, when an extended object moves within a physical medium, it experiences a contact interaction on all its surface, thus requiring an integral representation of the same. In the transition to particle settings, we have a fully analogous situation, as stressed in the preceding chapter whenever we have deep, mutual, wave overlappings.

This leads to a fifth class of breakings of conventional space-time symmetries caused, this time, by the inapplicability of their ultimate mathematical structure, their local differential topology. In fact, a proper representation of interior dynamical systems necessarily requires a non-local/integral generalization of the conventional local/differential topology of contemporary relativities.

The reader should be aware of the existence of three "No- Reduction Theorems" [21] which prevent the simplistic reduction of the systems considered to an ideal ensemble of point-like particles in stable orbits under only potential forces. The first theorem establishes that a classical object in an interior continuously decaying trajectory due to contact, non-Hamiltonian and non-local forces, simply cannot be reduced to a finite collection of ideal point-like constituents each one in a stable orbit under potential, local/differential interactions. Vice versa, a finite collection of elementary particles all in stable orbits under potential forces simply cannot produce a classical system whose center of mass is in a decaying orbit. The second "No-Reduction Theorem" establishes that a classical body which violates conventional space-time symmetries according to any of the mechanisms here considered, simply cannot be reduced to a collection of elementary particles each one verifying said symmetry. Vice versa, a finite collection of elementary particles each one verifying the Galilei or Lorentz symmetry simply cannot result in a classical system which violates said symmetry. The third "No-Reduction Theory" establishes that a classical irreversible system cannot be consistently reduced to a finite collection of elementary particles each of which is reversible and, vice versa, a finite collection of elementary particles all in reversible conditions simply cannot produce a classical irreversible object.

These "No-Reduction Theorems" provide final proof of the open character of interior dynamical systems within inhomogeneous and anisotropic physical media, and the need for suitable covering relativities.

A final concept should be re-called here, the rather remarkable capability offered by Theorem 2.9 of *reconstructing as exact the symmetries that are broken at the conventional Lie level.* This is the case in general, not only for the rotational symmetry, but also for the Galilei symmetry, the Lorentz symmetries, or any other continuous or discrete symmetry, as we shall see.

In summary, this section shall deal with three established classes of breakings of conventional space-time symmetries, those characterized by:

A: the inhomogeneous and anisotropic character of physical media;

B: the deformability of extended physical objects;

C: the instantaneous/null range character of the (classical) contact interactions;

D: the strictly non-potential/non-Hamiltonian character of the interactions considered; and

E: the non-local/integral nature of the dynamics of extended bodies within physical media.

Santilli conceived and developed his Lie-isotopic theory for the specific purpose of attempting a generalization of Galilei's and Einstein's relativities capable of providing a first quantitative treatment of conventional symmetries, when broken according to classes A, B and C above. One of the aspects of the studies is that the broken symmetries are not left mathematically undefined, as in the conventional literature, but they are replaced with covering, exact, Lie-isotopic symmetries. In this way, Santilli put the mathematical and physical foundations for the construction of a conceivable new generation of true covering relativities, as we shall see.

In closing these introductory words, we should point out that Santilli's preparatory studies for his generalized relativies are rather vast and cannot possibly be reviewed on one single monograph. As an example, Santilli constructed in memoir [24] the isotopic generalizations of the symplectic, affine and Riemannian geometries for the direct representation of nonlinear, nonlocal and non-Hamiltonian vector-fields, and then constructed his generalized relativities.

Evidently, we cannot review these new geometries here to avoid a prohibitive length. We shall therefore content ourselves with presenting Santilli's generalized relativities for nonlinear and non-Hamiltonian systems, but in local-differential approximation based on the use of the Lie-isotopic theory and conventional geometries in their most general possible realization.

As a result, this monograph should be considered as merely preparatory for the study of Santilli's Relativities in their most general possible nonlinear, non-Hamiltonian and non-local forms [24].

3.2 Lie-isotopic Generalization of the Group of Rotations [1], [23],[24]

3.2.1 Introduction

As is well known, when absolute rigidity is relaxed to admit the deformations of the real world [86], [87], perfectly spherical objects in Euclidean space,

$$r^t \delta r = xx + yy + zz = 1, \tag{3.1}$$

can be deformed in ellipsoids

$$r^t g r = x b_1^2 x + y b_2^2 y + z b_3^2 z = 1, \tag{3.2}$$

with the consequent manifest loss of the symmetry under rotations.

Similarly, when the motion of extended objects occurs within inhomogeneous and anisotropic material media, the Euclidean invariant (3.1) is generalized to a form of the type

$$r^t g r = r^i g_{ij}(t, r, \dot{r}, ...) r^j, \tag{3.3}$$

where, in general, the metric tensor has a dependence on time, coordinates, velocities, and a number of additional physical quantities (such as temperature, density, pressure, etc.).

In this section we shall review Santilli's generalization [1], [23] of the special orthogonal rotation group $SO(3)$ which provides the invariance of all possible deformations of the sphere, Eq. (3.2), while recovering the conventional theory identically whenever the original structure (3.1) is resumed. We shall then show that the generalized theory also provides the invariance of the generalized metric $g(t, r, \dot{r}, \ldots)$. The generalization of the covering, special unitary group $SU(2)$ [24] will be reviewed in Appendix C.

These objectives are achieved via the use of the Lie-isotopic lifting of Lie symmetries presented in Chapter 2, with particular reference to Theorem 2.9.

3.2.2 Foundations of the Conventional Rotational Symmetry

The basic space is the conventional Euclidean space in three dimensions, $E(r, \delta, \mathbf{R})$, with local coordinates

$$r = r^k = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad k = 1, 2, 3, \tag{3.4}$$

and composition

$$r^t r = r^i \delta_{ij} r^j = xx + yy + zz. \tag{3.5}$$

The continuous component $SO(3)$ of the metric-preserving group $O(3)$ is given by the familiar form

$$R(\theta) = e^{J_1 \theta_1}|_\xi e^{J_2 \theta_2}|_\xi e^{J_3 \theta_3}|_\xi, \tag{3.6}$$

verifying the conditions

$$R^t R = R R^t = I,$$
$$R^t = R^{-I},$$
$$\det R = +1, \tag{3.7}$$

where the θ's are Euler's angles; the skew-Hermitian generators are given by

$$J_1 = J_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$J_2 = J_{31} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$J_3 = J_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$J_k^t = -J_k\ , \tag{3.8}$$

and the infinite series leading to the exponentiations (3.6) are computed in the universal enveloping algebra ξ with conventional associative product of matrices and unit

$$\xi : J_i J_j = \text{ associative product},$$

$$I J_i = J_i I = J_i,$$

$$I = \text{diag}(+1, +1, +1). \tag{3.9}$$

The attached Lie algebra is characterized by the familiar commutation rules

$$\mathbf{SO}(3) : [J_i, J_j]_\xi = J_i J_j - J_j J_i = -\epsilon_{ijk} J_k,$$

$$i, j, k = 1, 2, 3, \tag{3.10}$$

while the second-order Casimir invariant is given by

$$J^2 = \sum_{k=1}^{3} J_k J_k = -2I. \tag{3.11}$$

The discrete part of $O(3)$ is characterized by the inversion

$$Pr = -r,$$

$$P = \text{diag}(-1, -1, -1),$$

$$\det P = -1, \tag{3.12}$$

which, as well known, commutes with all elements of $SO(3)$. We shall keep in mind that $O(3)$ is not connected and that, since the reflections do not contain the identity, they constitute a group only when combined with $SO(3)$.

3.2.3 Lie-Isotopic Generalization of the Group of Rotations

We now introduce arbitrary, nonsingular, symmetric, and sufficiently smooth metrics over $\mathbf{R}$:

$$g = (g_{ij}) = (g_{ij}(t, r, \dot{r}, ...)) , \tag{3.13}$$

with composition law

$$r^t * r = r^t g r = r^i g_{ij} r^j , \tag{3.14}$$

characterizing the isotopic liftings $\hat{E}(\vec{r}, g, \hat{\mathbf{R}})$ of $E(\vec{r}, \delta, \mathbf{R})$, according to the specifications of §2.4.

We are interested in identifying the *Lie-isotopic liftings* $\hat{O}(3)$ *of the group of rotations* $O(3)$, that is, the set of transformations

$$r' = \hat{R}(\theta) * r \stackrel{\text{def}}{=} \hat{R}(\theta) g r , \tag{3.15}$$

which verify the conditions for constituting a Lie-isotopic group, including the isotopic rules

$$\begin{aligned} \hat{R}(0) &= \hat{I} = g^{-1} , \\ \hat{R}(\theta_1) * \hat{R}(\theta_2) &= \hat{R}(\theta_1 + \theta_2) , \\ \hat{R}(\theta)\hat{R}(-\theta) &= \hat{I} , \end{aligned} \tag{3.16}$$

while leaving invariant composition (3.14), i.e.,

$$r^{t\prime} * r' = r^t * \hat{R}^t * \hat{R} * r = r^t * r . \tag{3.17}$$

As indicated in §2.4, the latter property holds if the elements $\hat{R}(\theta) \in \hat{O}(3)$ verify the isotopic- orthogonality conditions

$$\hat{R}^t * \hat{R} = \hat{R} * \hat{R}^t = \hat{I} , \tag{3.18}$$

which can be expressed in terms of the inverse operation with respect to the new unit $\hat{I}$ (§1.3)

$$\hat{R}^t = \hat{R}^{-\hat{I}} , \tag{3.19}$$

and imply the following generalization of condition (3.7.c):

$$(\det\hat{R})^2 = (\det\hat{I})^2 , \tag{3.20}$$

or equivalently (because of the symmetric character of g),

$$[\det(\hat{R}g)]^2 = 1 . \tag{3.21}$$

The desired liftings $\hat{O}(3)$ of $O(3)$ can be explicitly constructed for each given metric g via the methods of §2.4. The first recommendable step is the isotopic lifting $\hat{\xi}$ of the envelope ξ. This is essentially achieved via the associativity-preserving generalization of the product J_iJ_j of ξ (associative isotopy),

$$\hat{\xi} : J_i * J_j \stackrel{\text{def}}{=} J_i g J_j, \text{ g fixed }, \tag{3.22}$$

with consequent generalization of the unit

$$\hat{I} = g^{-1}, \quad \hat{I} * J_i = J_i * \hat{I} = J_i, \quad i = 1, 2, 3 , \tag{3.23}$$

and of the methodology of enveloping algebras (Poincaré-Birkhoff- Witt theorem, etc.) now familiar from Chapter 2.

The Lie-isotopic groups $\hat{O}(3)$ are then constructed in such a way to admit the inverse of the metric as the *new*, generalized unit, that is, as the Casimir invariant of order zero. The invariance of the new separation (3.14) is then ensured by construction (Theorem 2.9).

The continuous component of $\hat{O}(3)$, say, $\widehat{SO}(3)$, is characterized by the reformulation of the expansion (2.124) in the new envelope $\hat{\xi}$ acccording to Eq. (2.138)

$$\widehat{SO}(3) : \hat{R}(\theta) = e^{J_1\theta_1}|_{\hat{\xi}} * e^{J_2\theta_2}|_{\hat{\xi}} * e^{J_3\theta_3}|_{\hat{\xi}} \tag{3.24}$$

and can be equivalently formulated in the old envelope ξ for computational convenience, resulting in the factorization of the isotopic unit

$$\begin{aligned} \widehat{SO}(3) : \hat{R}(\theta) &= (e^{J_1 g\theta_1}|_{\xi} e^{J_2 g\theta_2}|_{\xi} e^{J_3 g\theta_3}|_{\xi})\hat{I} \\ &= (\prod_{k=1}^{3} e^{J_k g\theta_k})\hat{I} \stackrel{\text{def}}{=} S_g(\theta)\hat{I}, \end{aligned} \tag{3.25}$$

where all possible reduced elements S_g verify rules (2.159) by construction,

$$S^t g S = g. \tag{3.26}$$

Note that, from rules $R^tR = I$ and $[R^t, R] = 0$, we have the isotopic rules $\hat{R}^t * \hat{R} = \hat{I}$ and $[\hat{R}^t, \hat{R}] = 0$, from which Eqs. (3.18) follow. For the case of factorization $\hat{R} = S_g\hat{I}$ as in Eq. (3.25), we have property (3.26) as a consequence of (3.18). However, in general, $[S^t, S] \neq 0$ and $S^tgS \neq SgS^t$. Also, $\det S = 1$, but $S^tS \neq I$. The reader interested in learning about Lie isotopy is encouraged to verify these (and other) properties [23].

The discrete component of $\hat{O}(3)$ can be characterized by the isotopic inversions (2.161), i.e.,

$$\hat{P} * r = -r, \quad \hat{P} = P\hat{I}, \tag{3.27}$$

where P characterizes the conventional inversion (3.12).

One readily verifies that the isotopic inversions alone do not constitute a Lie-isotopic group. However, the set of all possible combinations of isotopic rotations (3.24) and inversions (3.27) does form a Lie-isotopic group, as the reader is encouraged to verify.

Note that the isotopes $\hat{O}(3)$ can be explicitly constructed for each given metric g, as indicated earlier. In fact, the only unknown of Eqs. (3.24) and (3.27) is precisely the assumed metric g. Note also that the invariance of the generalized separation (3.14) is achieved for all possible metrics of the admitted class, including generally nondiagonal metrics.

To simplify the review we restrict ourselves from here on to the *canonical reference frame*, that is, the frame for which the metric is diagonal, and we shall write

$$r^t * r = r^t g r = x g_{11} x + y g_{22} y + z g_{33} z. \tag{3.28}$$

It should be noted that the reduction to a diagonal form can always be achieved for all metrics of the class admitted via a similarity transformation, as is well known in the theory of metric spaces. The reader should however keep in mind that the diagonal character of the metric holding in the canonical frame, is not generally preserved in other frames, irrespective of whether they are inertial or not.

Despite these physical limitations, the canonical frame provides a great simplification of the computations. In particular, it permits the speedy identification of the Lie- isotopic algebra via rule (2.149), i.e.,

$$\begin{aligned} [X_i, X_j]_{\hat{\xi}} &\stackrel{\text{def}}{=} [X_i \hat{,} X_j] \\ &= X_i g X_j - X_j g X_i \\ &= [X_i, X_j] g + X_i [g, X_j] + X_j [X_i, g] \\ &= g[X_i, X_j] + [X_i, g] X_j + [g, X_j] X_i \, , \end{aligned} \tag{3.29}$$

which yields the desired commutation rules

$$\widehat{\mathrm{SO}}(3) : [J_i \hat{,} J_j] = J_i g J_j - J_j g J_i = \hat{C}_{ij}^k * J_k \, ,$$

$$i, j, k = 1, 2, 3 \, , \tag{3.30}$$

where the structure functions are

$$\hat{C}^k_{ij} = -\epsilon_{ijk} g_{kk}(t, r, \dot{r}, ...)\hat{I} \ . \tag{3.31}$$

One can see in this way the generalization of the "structure constants" of the standard formulation of Lie's theory into "structure functions," as correctly predicted by the isotopic generalization of Lie's second theorem of §2.3.

The commutativity between isotopic inversions and rotations holds in the canonical frame owing to the identities

$$[J_k \hat{,} \hat{P}] = [J_k, P] = 0, \quad k = 1, 2, 3. \tag{3.32}$$

Under the conditions specified above, the isotopic inversions therefore constitute a discrete, invariant, subgroup of $\hat{O}(3)$. The decomposition of $\hat{O}(3)$ into a continuous and a discrete component can then be done essentially along the conventional lines.

The corresponding decomposition for the case of nondiagonal metrics, demands additional, specific, studies not available at this time. This is due to the fact that the topological structure of $\hat{O}(3)$ is expected to be considerably broader than that of $O(3)$. The relationship between discrete and continuous transformations for arbitrary, generally nondiagonal metrics is therefore expected to depend on delicate, yet unexplored properties (e.g., of cohomological nature).

3.2.4 Classification of Isotopic Rotational Symmetries

Our next objective is to review the classification of all possible Lie algebras $\widehat{\mathbf{O}}(3)$ characterized by all possible metrics (3.28) of the class admitted (regular, diagonal, and sufficiently smooth, but not necessarily positive or negative definite).

First, it is evident from rules (3.30) that the isotopes have no proper invariant subalgebra. The algebras $\widehat{\mathbf{SO}}(3)$ are therefore simple in the conventional (abstract) sense.

Second, to identify the compact or noncompact character of the isotopes, we consider an arbitrary element $X = a_1 J_1 + a_2 J_2 + a_3 J_3$. The Killing form of $\widehat{\mathbf{SO}}(3)$ can be written in $E(\vec{r}, \delta, \mathbf{R})$

$$\begin{aligned} K(X,X) &= tr(adX)^2 \\ &= \begin{pmatrix} 0 & -a_3 g_{22} & a_2 g_{33} \\ 3g_{11} & 0 & -a_1 g_{33} \\ -a_2 g_{11} & a_1 g_{22} & 0 \end{pmatrix}^2 \end{aligned}$$

$$= \quad -2(a_1^2 g_{22} g_{33} + a_2^2 g_{11} g_{33} + a_3^2 g_{11} g_{22}), \qquad (3.33)$$

with a trivial reformulation in $\hat{E}(\vec{r}, g, \hat{\mathbf{R}})$. One can readily see that the above form is negative definite, and the isotopes $\hat{\mathbf{O}}(3)$ are compact for all elements g_{kk} possessing the same sign (whether positive or negative). The isotopes are noncompact whenever two of the elements g_{11}, g_{22}, and g_{33} have different signs.

Since the metric elements are functions of the local variables, $g_{kk} = g_{kk}(t, r, \dot{r}, ...)$, their sign cannot, in general, be globally defined. As a consequence, we must assume an additional local restriction for the achievement of a first classification of $\widehat{\mathbf{SO}}(3)$. More particularly, we shall assume sufficient topological restrictions on the functions g_{kk} to preserve the sign of their value in the neighborhood of the considered point $(t, r, \dot{r}, ...)$ of their variables.

Under these restrictions, all possible isotopes $\widehat{\mathbf{SO}}(3)$ are characterized by all possible invariants

$$r^t g r = \pm x b_1^2 x \pm y b_2^2 y \pm z b_3^2 z \ . \qquad (3.34)$$

It is then easy to see that the only compact Lie-isotopic algebras are the following two:

$$\widehat{\mathbf{SO}_1}(3) : \text{sign} g = (+,+,+) \ ,$$
$$\widehat{\mathbf{SO}_2}(3) : \text{sign} g = (-,-,-) \ , \qquad (3.35)$$

while all the remaining six algebras are noncompact, according to the classification

$$\widehat{\mathbf{SO}_3}(3) : \text{sign} g = (+,+,-) \ ,$$
$$\widehat{\mathbf{SO}_4}(3) : \text{sign} g = (+,-,+) \ ,$$
$$\widehat{\mathbf{SO}_5}(3) : \text{sign} g = (-,+,+) \ ,$$
$$\widehat{\mathbf{SO}_6}(3) : \text{sign} g = (-,-,+) \ ,$$
$$\widehat{\mathbf{SO}_7}(3) : \text{sign} g = (-,+,-) \ ,$$
$$\widehat{\mathbf{SO}_8}(3) : \text{sign} g = (+,-,-) \ . \qquad (3.36)$$

To identify the type of algebras, we introduce the following redefinition of the generators

$$\hat{J}_1 = b_2^{-1} b_3^{-1} J_1, \quad \hat{J}_2 = b_1^{-1} b_3^{-1} J_2, \quad \hat{J}_3 = b_1^{-1} b_2^{-1} J_3. \qquad (3.37)$$

The Lie-isotopic commutation rules for the compact algebras (3.35) then become

$$\widehat{\mathbf{SO}}_1 : [\hat{J}_1 \hat{,} \hat{J}_2] = \hat{J}_3, \quad [\hat{J}_2 \hat{,} \hat{J}_3] = \hat{J}_1, \quad [\hat{J}_3 \hat{,} \hat{J}_1] = \hat{J}_2 \,,$$
$$\widehat{\mathbf{SO}}_2(3) : [\hat{J}_1 \hat{,} \hat{J}_2] = -\hat{J}_3, \quad [\hat{J}_2 \hat{,} \hat{J}_3] = -\hat{J}_1, \quad [\hat{J}_3 \hat{,} \hat{J}_1] = -\hat{J}_2 \,. \tag{3.38}$$

The second-order isotopic Casimir invariants are then given by

$$\hat{J}^{\hat{2}}_{(\alpha)} = \sum_{k=1}^{3} \hat{J}_k g_{(\alpha)} \hat{J}_k = -2\hat{I}_{(\alpha)}, \quad \alpha = 1, 2 \,. \tag{3.39}$$

Comparison of Eqs. (3.38) and (3.39) with (3.10) and (3.11), respectively, then leads to the following result.

Proposition 3.1 *[23]: All compact isotopes* $\widehat{\mathbf{SO}}(3)$ *are locally isomorphic to the* $\mathbf{SO}(3)$ *algebra, and they occur for positive or negative definite metrics.*

Under the assumed topological restrictions on the metric, the Lie-isotopic algebras are integrable to their corresponding groups. The exponentials (3.24) therefore exist and characterize well- defined, finite isotopic rotations.

Numerous examples can be explicitly computed. As an illustration, we consider a compact isotopic rotation around the third axis for the case of the isotope $\widehat{\mathbf{O}}_1(3)$. Trivial calculations then yield the group element [23]

$$\begin{aligned} \hat{R}(\theta_3) &= S_g(\theta_3)\hat{I} \\ &= \begin{pmatrix} cos\theta_3 & \frac{b_2}{b_1} sin\theta_3 & 0 \\ -\frac{b_1}{b_2} sin\theta_3 & cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{I} \,, \end{aligned} \tag{3.40}$$

with underlying transformations

$$\begin{aligned} r' &= \hat{R}(\theta_3) * r = S_g(\theta_3) r \\ &= \begin{pmatrix} x cos\theta_3 + y \frac{b_1}{b_1} sin\theta_3 \\ -x \frac{b_1}{b_2} sin\theta_3 + y cos\theta_3 \\ z \end{pmatrix} \,, \end{aligned} \tag{3.41}$$

which leave invariant the ellipsoids

$$\begin{aligned} r^{t\prime} g_{(1)} r' &= x' b_1^2 x' + y' b_2^2 y' + z' b_3^2 z' \\ &= x b_1^2 x + y b_2^2 y + z b_3^2 z \\ &= r^t g_{(1)} r \,, \end{aligned} \tag{3.42}$$

as the reader is encouraged to verify.

Note that the isotopic commutation rules of $\widehat{\mathbf{SO}_1}(3)$ and those of the conventional algebra $\mathbf{SO}(3)$ coincide at the abstract level of realization-free treatment of rotations. The same situation occurs for all other aspects of the theory, such as enveloping algebra, Casimir invariants, etc. A similar, formal unification can also be reached between the full orthogonal-isotopic group $\widehat{\mathbf{O}_1}(3)$ and the conventional one $\mathbf{O}(3)$.

A main result of ref. [23] is then expressed as follows.

Theorem 3.1 *The groups of isometries of all possible ellipsoidical deformations of the sphere,*

$$r^t g_{(1)} r = x b_1^2 x + y b_2^2 y + z b_3^2 z = 1,$$

$$b_k = b_k(t, r, \dot{r}, ...) \neq 0 \ , \tag{3.43}$$

here denoted $\widehat{O_1}(3)$, verify the following properties:

(a) *The groups $\widehat{O_1}(3)$ are all locally isomorphic to $O(3)$ when isotopically realized in such a way that their isounits $\hat{I}_{(1)}$ are the inverse of the metrics $g_{(1)}$ of ellipsoids (3.43).*

(b) *The groups $\widehat{O_1}(3)$ consist of infinitely many different (but isomorphic) realizations, corresponding to the infinite possibilities of explicit, local forms of the isounits $\hat{I}_{(1)}$ (or, equivalently, of the metrics $g_{(1)}$).*

(c) *The groups $\widehat{O_1}(3)$ constitute "isotopic coverings" of $O(3)$, in the sense that*

(c-1) the groups $\widehat{O_1}(3)$ are constructed via methods structurally more general than those of $O(3)$;

(c-2) the groups $\widehat{O_1}(3)$ apply for physical conditions broader than those of $O(3)$; and

(c-3) all groups $\widehat{O_1}(3)$ recover $O(3)$ identically whenever ellipsoids (3.43) reduce to the sphere.

The nontriviality of the notion of isotopic covering can be illustrated via the following important property.

Corollary 3.1.1 *[23]: While the action of $O(3)$ on local coordinates is linear, i.e.,*

$$r' = R(\theta)r, \tag{3.44}$$

that of its isotopic coverings $\widehat{O_1}(3)$ is generally nonlinear, i.e.,

$$\begin{aligned} r' &= \hat{R}(\theta) * r = S_g(\theta)r \\ &= S(t, r, \dot{r}, ...; \theta)r. \end{aligned} \tag{3.45}$$

An illustration of this occurrence is given by transformations (3.41). In fact, the nonlinearity occurs because the elements b_k entering into the transformations are generally dependent on the local coordinates (see Fig. 6 for some of the implications).

We pass now to the review of the noncompact forms, which, besides being useful for achieving a classification of all possible isotopic images of rotations, constitute the foundations of the Lie-isotopic lifting of special relativity (§3.4).

For the case of the noncompact algebras (3.35), isotopic rules (3.30) become

$$\begin{aligned}
\widehat{\mathbf{SO}_2}(3) &: [\hat{J}_1 \hat{,} \hat{J}_2] = -\hat{J}_3, \quad [\hat{J}_2 \hat{,} \hat{J}_3] = \hat{J}_1, \quad [\hat{J}_3 \hat{,} \hat{J}_1] = \hat{J}_2, \\
\widehat{\mathbf{SO}_4}(3) &: [\hat{J}_1 \hat{,} \hat{J}_2] = \hat{J}_3, \quad [\hat{J}_2 \hat{,} \hat{J}_3] = \hat{J}_1, \quad [\hat{J}_3 \hat{,} \hat{J}_1] = -\hat{J}_2, \\
\widehat{\mathbf{SO}_5}(3) &: [\hat{J}_1 \hat{,} \hat{J}_2] = \hat{J}_3, \quad [\hat{J}_2 \hat{,} \hat{J}_3] = -\hat{J}_1, \quad [\hat{J}_3 \hat{,} \hat{J}_1] = \hat{J}_2, \\
\widehat{\mathbf{SO}_6}(3) &: [\hat{J}_1 \hat{,} \hat{J}_2] = \hat{J}_3, \quad [\hat{J}_2 \hat{,} \hat{J}_3] = -\hat{J}_1, \quad [\hat{J}_3 \hat{,} \hat{J}_1] = -\hat{J}_2, \\
\widehat{\mathbf{SO}_7}(3) &: [\hat{J}_1 \hat{,} \hat{J}_2] = -\hat{J}_3, \quad [\hat{J}_2 \hat{,} \hat{J}_3] = -\hat{J}_1, \quad [\hat{J}_3 \hat{,} \hat{J}_1] = \hat{J}_2, \\
\widehat{\mathbf{SO}_8}(3) &: [\hat{J}_1 \hat{,} \hat{J}_2] = -\hat{J}_3, \quad [\hat{J}_2 \hat{,} \hat{J}_3] = \hat{J}_1, \quad [\hat{J}_3 \hat{,} \hat{J}_1] = -\hat{J}_2,
\end{aligned} \tag{3.46}$$

while the second-order Casimir invariants preserve form (3.39), i.e.,

$$\hat{J}^{\hat{2}}_{(\alpha)} = \sum_k \hat{J}_k g_{(\alpha)} \hat{J}_k = -2\hat{I}_{(\alpha)}, \alpha = 3, 4, ..., 8 \ . \tag{3.47}$$

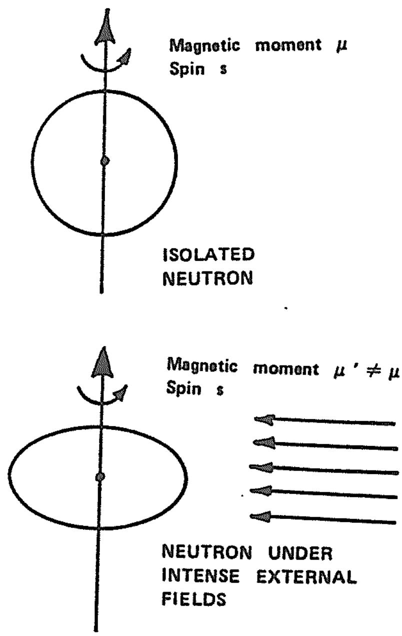


FIGURE 6. A fundamental application of the Lie-isotopic theory: the deformation/rotational-asymmetry of neutrons under intense external fields, and the exact character of isotopic rotations [23], [27]. As well known, neutrons are not point particles, but extended charge distributions with a radius of about $1F$ $(= 10^{-13}cm)$. Suppose that such distributions are perfectly spherical (an assumption already questionable [42]). Then, under sufficiently intense external fields, the particles are expected to experience a deformation of their shape precisely along the fundamental invariant (3.2) of the isotopic rotations. This deformation of shape has a number of truly fundamental, theoretical and experimental implications. On theoretical grounds, it implies the breaking of the *conventional* rotational symmetry, as manifest in the deformation of invariant (3.1) into ellipsoids (3.2). But the abstract rotational symmetry is not broken. In fact, Santilli's isotopic group $\hat{O}(3)$ provides an exact symmetry for the deformed neutron while being isomorphic to $O(3)$. Furthermore, the deformation of the charge distribution implies an alteration ("mutation" in the language of ref. [2]) of the magnetic moment of the particle, as clearly established already at the classical

level. One recovers in this way an hypothesis formulated since the early stages of nuclear physics (but oddly ignored in more recent treatises in the field), that protons and neutrons experience a deformation of their magnetic moments when members of a nuclear structure, e.g., under intense, short range, nuclear interactions. Finally, and still on theoretical grounds, the rotational-asymmetry of the figure implies a necessary breaking of the *conventional* Galilei's and Einstein's Relativities, thus creating the need for suitable generalizations of Lie-isotopic type as we shall review in the next sections. On experimental grounds, the physical occurrence depicted in the figure has a number of fundamental implications, for instance, in the controlled fusion. In fact, protons and neutrons are expected to experience an alteration of their intrinsic magnetic moments exactly at the time of initiation of the fusion process, with evident implications for confinement. The deformation/rotational asymmetry/magnetic- moment-mutation depicted in the figure has already been the subject of fundamental experiments by H. Rauch and collaborators (see ref. [88] and quoted papers) via neutron interferometric techniques. The experimenters tested the spinorial character of the neutron's $SU(2)$ symmetry via the symmetry of the wavefunction under two complete spin-flips caused by an external magnetic field. The calculations are evidently based on the *conventional* value of the magnetic moment of the neutron. As a result, deviations caused by the deformations under consideration evidently result in deviations from the $SU(2)$ symmetry. The last experimental numbers (dating back to 1978) are 715.87 ± 3.8deg. Thus, the 720 deg needed to verify the exact, conventional, rotational symmetry ARE NOT contained within the minimal value 712.97 and the maximal value 719.67 of the experimental error. A quantitative representation of Rauch's data [88] has been provided by Santilli [27] via his isotopic lifting of Dirac's equation and it will be reviewed in Appendix C. Rauch's fundamental experiment will be considered in detail in the separate review we hope to complete on the operator version of Lie-isotopic techniques on Hilbert spaces. Santilli has been a strong proponent (for over a decade now) of the repetition of Rauch's experiment by other independent experimentalists, owing to its manifestly fundamental character (see, e.g., ref. [11]). It is regrettable that the experiment has continued to be ignored by experimentalists in the field. In fact, the last available experimental numbers date back to 1978 and, most unreassuring, show a *violation*, thus rendering even more compelling the need for an experimental resolution of the issue. The clear, unquestionable plausibility of the deformation; the ready availability of all needed equipment at numerous (low energy) nuclear laboratories throughout the world; the quite moderate cost of the experiments as compared to other lesser relevant, yet much more expensive experiments preferred until now by experimenters in the field; the manifestly fundamental character of the experiment for all of theoretical physics; the equally sizable financial-administrative implications, e.g., for the investments in attempting controlled fusion via magnetic confinement; and several other aspects, have forced Santilli to raise serious issues of scientific ethics in regard to the lack of independent repetition of Rauch's experiment which are not addressed in this review.

The following result then holds:

Proposition 3.2 *[23]: All noncompact isotopes* $\widehat{\mathbf{SO}}(3)$ *are locally isomorphic to the* $\widehat{\mathbf{SO}}(2.1)$ *algebra, and they occur for (diagonal) metrics whose elements have different signs.*

Under the assumed restrictions, the noncompact isotopic algebras are also integrable to their corresponding groups. The exponentials (3.25) therefore exist, although the range of the parameters now becomes infinite.

Again, numerous examples of "noncompact isotopic rotations" can be explicitly constructed for all cases (3.36). As an illustration, we consider a "rotation" around the third axis for the case of the isotope $\widehat{SO}_4(3)$. Then, trivial calculations yield the group element [23]

$$\hat{R}(\theta_3) = \begin{pmatrix} cosh\theta_3 & -\frac{b_2}{b_1}sinh\theta_3 & 0 \\ -\frac{b_1}{b_2}sinh\theta_3 & cosh\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{I} \tag{3.48}$$

with underlying isotopic transformations

$$r' = \hat{R}(\theta_3) * r = S_g(\theta_3) r = \begin{pmatrix} xcosh\theta_3 - y\frac{b_2}{b_1}sinh\theta_3 \\ -x\frac{b_1}{b_2}sinh\theta_3 + ycosh\theta_3 \\ z \end{pmatrix}, \tag{3.49}$$

which, this time, leave invariant the hyperbolic form

$$\begin{aligned} r^{t\prime} g_{(3)} r' &= x' b_1^2 x' - y' b_2^2 y' + z' b_3^2 z' \\ &= x b_1^2 x - y b_2^2 y + z b_3^2 z \\ &= r^t g_{(3)} r. \end{aligned} \tag{3.50}$$

Again, the noncompact isotopes are indistinguishable from $SO(2.1)$ at the level of abstract, realization-free formulations.

In summary, the isotopic lifting of Lie algebras does not produce new Lie algebras, because (as stressed in §1.5) all Lie algebras over a field of characteristic zero are already known. Santilli's Lie-isotopic theory merely provides infinitely new covering realizations of known algebras. The results of Propositions 3.1 and 3.2 are therefore predictable from the simplicity of algebra (3.30). In fact, all simple, three- dimensional Lie algebras over the reals are known and are given either by **SO**(3) or by **SO**(2.1) (or by algebras isomorphic to them.)

To complete our classification, we need additional information on Lie-isotopic algebras whose metrics have opposite signs.

> *DEFINITION 3.1 [23]: Let* $\hat{\mathrm{G}}$ *be an isotopic algebra characterized by (diagonal) metrics with elements* g_{kk}. *The* isotopic dual $\hat{\mathrm{G}}^d$ *of* $\hat{\mathrm{G}}$ *is the algebra characterized by the (diagonal) metric with elements* $g^d_{kk} = -g_{kk}$, $k = 1, 2, ..., n$.

It is then easy to prove the following result.

Proposition 3.3 *[23]: Isotopically dual Lie algebras are locally isomorphic.*

Note that the proposition above includes the case when one of the algebras is conventional. We discover in this way that **SO**(3) has an image that cannot be identified via the simplest possible Lie product $AB - BA$ of current use, but demands instead the use of a more general product, such as $AgB - BgA$.

In fact, besides its conventional form, **SO**(3) can be realized via the isotopic dual, according to the expressions

$$\begin{aligned} \mathbf{SO}(3) : [J_i \hat{,} J_j] &= J_i g J_j - J_j g J_i \\ &= -\epsilon_{ijk} J_k, \\ g &= \mathrm{diag}(+1, +1, +1), \\ J^t_k &= -J_k. \end{aligned}$$

$$\begin{aligned} \mathbf{SO}^d(3) : [J_i \hat{,} J_j] &= J_i g J_j - J_j g J_i \\ &= +\epsilon_{ijk} J_k, \end{aligned}$$

$$g = \mathrm{diag}(-1,-1,-1),$$
$$J_k^t = -J_k. \qquad (3.51)$$

At the level of the full orthogonal group $O(3)$, this essentially implies the interchange of the identity I with the total inversion $\hat{I} = -I$, the latter becoming the identity of the isotopic dual. It is then easy to see that the basic algebras (3.51) and the eight isotopes (3.38) and (3.46) can be divided into two sets interconnected by isotopic duality. Until now we have considered isotopes characterized by metrics with locally definite topological characters, resulting in locally definite compact or noncompact groups. To complete his classification, Santilli indicated the existence of isotopes that can smoothly transform compact algebras into noncompact ones, and vice versa. Evidently, this topic is technically involved (and yet unexplored); it therefore demands specific, detailed investigations. We shall thus content ourselves with the mere indication of the existence of this additional class of isotopies.

For this purpose it is more effective to return to the original basis J_k of Eqs. (3.8), to the original isotopic rules (3.30), and to the generic separation (3.28), with diagonal metric elements g_{kk}. An isotopic rotation around the third axis can be readily computed from the exponentiations (3.25), resulting in the expression

$$S_g(\theta) = \begin{pmatrix} cos(\theta_3 g_{11}^{1/2} g_{22}^{1/2}) & g_{22}(g_{11}g_{22})^{1/2} sin(\theta_3 g_{11}^{1/2} g_{22}^{1/2}) & 0 \\ -g_{11}(g_{11}g_{22})^{1/2} sin(\theta_3 g_{11}^{1/2} g_{22}^{1/2}) & cos(\theta_3 g_{11}^{1/2} g_{22}^{1/2}) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (3.52)$$

It is easy to see that the above transformations do not have, in general, a globally defined compact or noncompact character. In particular, they can be isomorphic to $SO(3)$ for given values of the local variables and to $SO(2.1)$ for others. Thus, they can continuously interconnect compact and noncompact structures. Evidently, this is the most general possible isotopic lifting of rotations, which includes as particular cases all other forms considered so far.

To illustrate this occurrence, assume that the elements g_{11} and g_{33} have the value $+1$, while the element g_{22} is given by a function of the local variables $t, r, \dot{r}, \ldots$ that interconnects smoothly the values $+1$ and -1. It is then easy to see that, for the case $g_{11} = g_{22} = g_{33} = +1$, transformations

(3.52) reduce to the familiar, compact rotations

$$S_g(\theta_3) = \begin{pmatrix} cos\theta_3 & sin\theta_3 & 0 \\ -sin\theta_3 & cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{3.53}$$

while, for $g_{11} = -g_{22} = g_{33} = 1$, transformations (3.52) reduce to the equally familiar, but this time noncompact, Lorentz transformations

$$S_g(\theta_3) = \begin{pmatrix} cosh\theta_3 & -sinh\theta_3 & 0 \\ -sinh\theta_3 & cosh\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.54}$$

The generalization to metrics (3.34) is self-evident. Note the *lack* of consideration in this review of the *trivial isotopy*

$$\widehat{\mathrm{SO}}(3) : [\hat{J}_i \hat{,} \hat{J}_j] = -\epsilon_{ijk}\hat{J}_k, \quad \hat{J}_k = J_k g^{-1}, \quad J_k \in \mathrm{SO}(3), \tag{3.55}$$

$\widehat{O_0}(3)$	$g = (+1, +1, +1)$	$g = (-1, -1, -1)$	$\widehat{O_{0_d}}(3)$
$\widehat{O_1}(3)$	$g = (+b_1^2, +b_2^2, +b_3^3)$	$g = (-b_1^2, -b_2^2, -b_3^2)$	$\widehat{O_{1_d}}(3)$
$\widehat{\hat{O}_3}(3)$	$g = (+b_1^2, +b_2^2, -b_3^2)$	$g = (-b_1^2, -b_2^2, +b_3^2)$	$\widehat{O_{3_2}}(3)$
$\widehat{O_4}(3)$	$g = (+b_1^2, -b_2^2, +b_3^2)$	$g = (-b_1^2, +b_2^2, -b_3^2)$	$\widehat{O_{4_d}}(3)$
$\widehat{\hat{O}_5}(3)$	$g = (-b_1^2, +b_2^2, +b_3^2)$	$g = (+b_1^2, -b_2^2, -b_3^2)$	$\widehat{O_5}\ (3)$
$\hat{O}(3)$: $g = (g_{11}, g_{22}, g_{33})$			

FIGURE 7. A reproduction of Table I of ref.[23] presenting a preliminary classification of all possible isotopes of the conventional group of rotations denoted with $\hat{O}_o(3)$. On the left hand side we have the most notable isotopes characterized by different topologies of the metric, while on the right hand side we have their images under the notion of duality of Definition 3.1. The isotope $\hat{O}(3)$ at the bottom of the diagram symbolizes Santilli's conception of one single Lie-isotopic group which unifies all possible Lie groups of the same dimension via a metric of varying topological structure.

which does not provide the invariance of the ellipsoidical deformations of the sphere, as indicated in the closing remarks of §2.4. On the contrary, the realization

$$
\begin{aligned}
\mathbf{SO}(3) : [K_i, K_j] &= -\epsilon_{ijk} K_k, \\
K_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{b_3}{b_2} \\ -\frac{b_2}{b_3} & 0 & 0 \end{pmatrix}, \\
K_2 &= \begin{pmatrix} 0 & 0 & -\frac{b_3}{b_1} \\ 0 & 0 & 0 \\ \frac{b_1}{b_3} & 0 & 0 \end{pmatrix}, \\
K_3 &= \begin{pmatrix} 0 & \frac{b_2}{b_1} & 0 \\ -\frac{b_1}{b_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned}
\tag{3.56}
$$

even though conventional in structure (that is, realized via the conventional associative envelope without any isotopy), verifies the basic invariance property (3.26), as the reader is encouraged to verify.

Here it is important to understand that, by no means, can the results under consideration be uniquely derived via the Lie- isotopic theory. In fact, structure (3.56) indicates the possibility or recovering the form invarance of the ellipsoidical deformations of the sphere via the *conventionally realized* $O(3)$ or other ways. The Lie-isotopic liftings of Lie symmetries have been merely submitted by Santilli on grounds of their pragmatic effectiveness in constructing the covering symmetry when a given, conventionally realized Lie symmetry is broken, while admitting the latter as a particular case whenever the original physical conditions are regained.

3.2.5 Physical Applications

In order to identify physical applications, it is desirable to identify first its dynamical foundations. This, in turn, can be done more effectively in the arena of our best intuitions, Newtonian mechanics. Applications to particle physics shall be considered in Appendix C.

The Birkhoffian generalization of (classical) Hamiltonian mechanics (§1.3) evidently provides the desired dynamical setting. A knowledge of monograph [15] is therefore essential for a deep understanding of the physical applications of isotopic rotations.

For simplicity but without loss of generality, Santilli [23] first considered the case of one, free, extended particle in Euclidean space $E(\vec{r}, \delta, \mathbf{R})$, and the trivial canonical action

$$\begin{aligned} A(t,r) &= \int_{t_1}^{t_2} dt[\mathbf{p}\cdot\dot{\mathbf{r}} - \frac{1}{2}\mathbf{p}\cdot\mathbf{p}] \\ &= \int_{t_1}^{t_2} dt[p_k\dot{r}_k - H], \quad m = 1. \end{aligned} \tag{3.57}$$

Suppose that, at a given value of time, the particle experiences only contact nonhamiltonian forces due to its extended character (e.g., because of penetration within a resistive, generally anisotropic and inhomogeneous, material medium). Suppose that these physical conditions can be represented via the isotopic lifting $\hat{E}(\vec{r}, g, \hat{\mathbf{R}})$ of the Euclidean space, i.e., via the generalization of the action into the form

$$\begin{aligned} A^{\text{gen}}(t,r) &= \int_{t_1}^{t_2} dt(\mathbf{p}*\dot{\mathbf{r}} - \frac{1}{2}\mathbf{p}*\mathbf{p}) \\ &= \int_{t_1}^{t_2} dt[p_i g_{ij}\dot{r}_j - \frac{1}{2}p_i g_{ij} p_j], \end{aligned} \tag{3.58}$$

$$g = g(t, r, \dot{r}, ...),$$

which is manifestly of Birkhoffian non-Hamiltonian type with identifications

$$P_k(t,r,p) = p_i g_{ik}, \quad B = \frac{1}{2}p_i g_{ij} p_j. \tag{3.59}$$

The nonhamiltonian character of the theory can be technically established via the property that the equations of motion underlying action (3.58) generally violate the integrability conditions for the existence of a Hamiltonian in the r-frame considered [4]. The inapplicability of Hamiltonian mechanics implies, in particular, the inapplicability of the Poisson brackets for the Lie characterization of both the time evolution as well as, of course, the theory of rotations.

The direct applicability of Birkhoffian mechanics has the immediate advantage of permitting the identification of the generalized Lie product for both the time evolution and the applicable theory of rotations. It is sufficient, for illustrative purposes, to restrict ourselves to the case of a diagonal metric g with constant elements

$$g = \text{diag}(b_1^2, b_2^2, b_3^2), \quad b = \text{const.}. \tag{3.60}$$

Use of Eqs. (1.21.b) and (1.22.b) then readily yields the Lie- isotopic tensor

$$
\begin{aligned}
(\Omega^{\mu\nu}) &= \begin{pmatrix} 0 & -\frac{\partial P_i}{\partial p_j} \\ \frac{\partial P_i}{\partial p_j} & 0 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 0 & g^{-1} \\ -g^{-1} & 0 \end{pmatrix}, \qquad (3.61)
\end{aligned}
$$

with generalized brackets

$$
[A\hat{,}B] = \frac{\partial A}{\partial r_i} g_{ij}^{-1} \frac{\partial B}{\partial p_j} - \frac{\partial B}{\partial r_i} g_{ij}^{-1} \frac{\partial A}{\partial p_j}. \qquad (3.62)
$$

Simple calculations then establish the following *Newtonian realization of the isotope* $\widehat{\mathbf{SO}}_1(3)$ of *rotations* [23]

$$
\widehat{\mathbf{SO}}_1(3) : [J_i\hat{,}J_j] = \epsilon_{ijk} b_k^{-2} J_k, \qquad (3.63)
$$

with redefinition according to Eq. (3.38)

$$
\widehat{\mathbf{SO}}_1(3) : [\hat{J}_i\hat{,}\hat{J}_j] = \epsilon_{ijk} \hat{J}_k,
$$

$$
\hat{J}_1 = b_2 b_3 J_1, \quad \hat{J}_2 = b_1 b_3 J_2, \quad \hat{J}_3 = b_1 b_2 J_3, \qquad (3.64)
$$

and group form of the symbolic type

$$
\widehat{SO}_1(3) : a' = [\prod_{k=1}^{3} \exp(\theta_k \Omega^{\mu\nu} \frac{\partial J_k}{\partial a^\nu} \frac{\partial}{\partial a^\mu})] a, \qquad (3.65)
$$

with a corresponding reduction to a form of type (3.25).

The achievement of the desired objective is then confirmed by illustrative examples. For instance, an isotopic rotation around the third axis with generator J_3 can be computed via exponentials (3.65), yielding the transformations

$$
r' = \begin{pmatrix} x cos(\theta_3 b_1 b_2) - y \frac{b_2}{b_1} sin(\theta_3 b_1 b_2) \\ x \frac{b_1}{b_2} sin(\theta_3 b_1 b_2) + y cos(\theta_3 b_1 b_2) \\ z \end{pmatrix}, \qquad (3.66)
$$

with additional transformations of the type (3.40) for the generator $\hat{J}_3$.

The achievement of the form invariance of the Pfaffian action (3.58) is then consequential. Action-at-a-distance forces can be trivially incorporated in the theory via additive potentials in the Birkhoffian $B = \frac{1}{2} p^{\hat{2}}$, provided that they are properly written in $\hat{E}(\vec{r}, g, \hat{\mathbf{R}})$, e.g., with "squares" of the type (3.2).

As a further application, Santilli [23] presented a generalization of Euler's theorem (on the displacement of rigid bodies) to the case of elastic bodies. As the reader recalls [89], Euler's theorem essentially states that *the general displacement of a rigid body with one point fixed is a continuous rotation around some axis.*

Suppose that the object is an elastic sphere of radius 1, and that the fixed point is the origin of the reference frame. In the absence of deformation, the displacements of the object are given by time-dependent transformations $R = R(t) \in SO(3)$. At time $t = 0$ one can assume

$$R(0) = I = \text{diag}(+1, +1, +1). \tag{3.67}$$

At subsequent times t, the rotations are such that their eigenvalues are the elements of the conventional 3×3 unit I, i.e., there exists an eigenvector a of $R(t)$ which preserves its components in the rotated system:

$$a' = R(t)a = a, \tag{3.68}$$

or, equivalently, rotations verify the eigenvalue equations

$$[R(t) - I]a = 0 \ , \tag{3.69}$$

with secular determinant

$$\det(R - I) = 0 \ . \tag{3.70}$$

Suppose now that at time $t = t_o$ the sphere experiences a small deformation into the ellipsoid

$$r'gr = x(1+\epsilon_1)x + y(1+\epsilon_2)y + z(1+\epsilon_3)z = 1. \tag{3.71}$$

It is easy to see that the displacement can now be described via a compact isotopic rotation $\hat{R}(t) \in \widehat{SO}_1(3)$, beginning with the identification

$$\hat{R}(\epsilon) = \hat{I} = g^{-1}. \tag{3.72}$$

It is also easy to prove that the eigenvalue equation for the rigid motion, Eq. (3.69), admits the isotopic generalization

$$[\hat{R}(t) - \hat{I}] * a = [S_t(t) - I]a = 0 \ , \tag{3.73}$$

with isotopic-secular determinant (§1.3)

$$\widehat{\det}(\hat{R} - \hat{I}) = \widehat{\det}(S - I) = 0 \; , \tag{3.74}$$

where we have used the decomposition of Eq. (3.25), $\hat{R} = S_g \hat{I}$, and Theorem 2.19 of ref. [36], p. 1310.

In fact, from Eq. (3.20), $\det \hat{R}(t) = \det \hat{I}$. A step-by-step generalization of the conventional proof (see, e.g., ref. [89], pp. 119-123) then leads to the following result.

Lemma 3.1 *[23]: The isotopic eigenvalues of the compact-isotopic rotations of type* 1 *are the elements of the (diagonal) generalized unit* $\hat{I} = g^{-1}$.

Thus, much as in the conventional case, the compact-isotopic rotations admit an eigenvector that preserves its components in the transformed system. By recalling that the transformations considered here can only be continuous, the extensions to the case of finite deformations and to non-spherical objects are straightforward, yielding the following result.

Theorem 3.2 *[23]: (Isotopic Lifting of Euler's Theorem) The general displacement of an elastic body with one point fixed is a compact isotopic rotation of type* 1 *around some fixed axis.*

Numerous additional applications to the dynamics of extended, elastic, and deformable bodies are possible. Here, we limit ourselves to the indication that the isotopes of $O(3)$ seem to be naturally set for the description of deformations, with the understanding that the theory generally demands the use of nondiagonal metrics. In fact, all metrics of the theory of elasticity are permitted by the isotopic theory of rotations.

An additional class of physical applications is the motion of extended objects within generally inhomogeneous and anisotropic material media. In effect, the description of the displacement of elastic bodies (Theorem 3.2) and that of the motion within material media are complementary to each other, in the sense that they can both be reduced to suitable isotopic liftings of the Euclidean space.

To illustrate this possibility [23], consider a (classical) particle moving in a region of empty space for which the Euclidean geometry applies. Suppose now that the region considered is filled with intense radiation originating from a distant and constant source, assumed to be at infinity. It is evident that, under these novel physical conditions, the particle cannot be considered as moving in empty space. The new medium of propagation is space filled with radiation. Depending on the physical characteristics of the particle (size, charge, electric and magnetic moments, etc.), the new medium will directly affect the trajectory of the particle, that is, its dynamical evolution. In particular, the new medium is homogeneous but manifestly anisotropic, in the sense that the distribution of radiative energy is uniform, but the medium has a preferred orientation in space given by the direction of propagation of the background radiation.

Clearly, the Euclidean geometry is merely approximated for these broader physical conditions. The selection of an appropriate isotopic lifting is then relevant. We select the Finsler space with composition [23]

$$r' g r = r^i f(r, u) \delta_{ij} r^j,$$

$$f(r, u) = \frac{(r \cdot u)^2}{(r \cdot r)^2}, \tag{3.75}$$

where u is a unit vector ($u^2 = u_k u_k = 1$), here assumed along the direction of the radiation.

The Finsler space with composition (3.75) characterizes an isotope $\hat{E}(\vec{r}, g, \hat{\mathbf{R}})$ of $E(\vec{r}, \delta, \mathbf{R})$. As a result, the symmetry $\hat{O}(3)$ applies (including isotopic reflections). The reader should be aware that the symmetry $O(3)$ is broken for composition (3.74) because of its inability to preserve the preferred direction in space. The achievement of this preservation via the covering symmetry $\hat{O}(3)$ is instead ensured by the invariance of the metric under isotopic rotations, i.e.,

$$e^{-\theta g J} g e^{J g \theta} = g. \tag{3.76}$$

It is also clear that, in the transition from the Euclidean to the Finsler space, we have the transition from a flat, homogeneous, and isotropic geometry to a curved, homogeneous, and anisotropic one. Numerous intriguing properties then follow. Owing to the particular metric of Eq. (3.74), the conventional Casimir $J^2 = J_k J_k$ is preserved by the isotopic rotations,

$$\mathbf{J}^{2\prime} = \hat{R} * (\sum_{k=1}^{3} J_k J_k) * \hat{R}^t = \mathbf{J}^2, \tag{3.77}$$

as the interested reader is encouraged to verify. This result indicates that the angular momentum can be conserved also for motion within anisotropic media in which the conventional rotational symmetry is broken.

We recover in this way a result already known in analytic mechanics [1]. We are referring to the fact that the conservation of the angular momentum, by no means, necessarily implies the symmetry under the conventional rotation group. In fact, angular momentum conservation can be also characterized by isotopic symmetries.

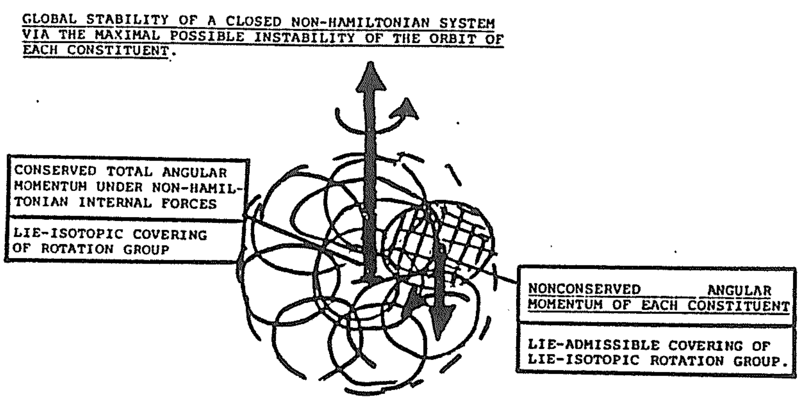


FIGURE 8. A reproduction of Fig. 5.4, p. 560 of ref. [16] representing the physical characterization of the complementary Lie-isotopic and Lie-admissible generalizations of the conventional group of rotations. Conventionally, global stability of a system is achieved via the stability of the orbits of each constituent, e.g., as in the Solar system. Closed nonhamiltonian systems, however, have identified a structurally more general global stability under maximal possible nonconservation and instability of the orbits of each constituent, e.g., as in Jupiter (see Figs. 2, 3). The Lie-isotopic lifting of rotations reviewed in this section represents the first part of Santilli's program, the characterization of total, conserved, angular momentum via the collection of the angular momenta of the constituents each, individu-

ally, nonconserved. The completion of the study requires the representation of one individual nonconserved angular momentum when all other particles are considered as external. This requires an algebra with non-antisymmetric product. The algebras selected by Santilli are the covering of the Lie-isotopic algebras known as Lie-admissible algebras. This leads to the possibility of constructing a second class of generalized relativities, specifically conceived for nonconservative conditions, which constitute coverings of the Lie-isotopic ones [16].

The generalization of the model to an inhomogeneous form is possible, and occurs, for instance, when the energy distribution of the background sea of radiation is not homogeneous. This is the case when the intensity of the radiation varies in space and time, in which case the metric (3.75) is generalized to forms of the type

$$g_{ij} = f(r,u)d_{ij}(t,r,\dot{r},...), \tag{3.78}$$

where the inhomogeneity and anisotropy are differentiated and represented by the respective terms $f(u,r)$ and $d_{ij}(t,r,\dot{r},...)$. Note that, with sufficient care, the applications of model (3.78) can also be extended to treat the motion of systems within a resistive medium with density varying in space and time, and witha preferred direction in space. Numerous additional applications are conceivable, as the reader can easily see.

As a concluding remark, we would like to indicate that, by no means, the Lie-isotopic theory of rotations is the only possibility of representing extended particles. In fact, a number of additional possibilities have been identified in the literature, most notably, Kálnay's approach via the use of intervals [90] and Prugovecki's studies via stochastic techniques [91]. Each of these approaches has its own preferred features. For instance, the Lie-isotopic approach has been conceived to achieve a covering unity of thought; Kálnay's approach is particularly tailored for certain quantum-mechanical measures; Prugovecki's approach is particularly suited for extended (perfectly spherical) particles under electromagnetic interactions.

Despite these differences, a central property of Santilli's Lie-isotopy is its "direct universality" which, for the case of classical mechanics, can be inferred from the theorem of Direct Universality of Birkhoffian mechanics (Ref. [15], Theorem 4.5.1). As a result, all possible approaches to rotations are expected to be a particular case of Santilli's isotopic group $\hat{O}(3)$.

3.3 Lie-isotopic Generalization of Galilei's Relativity [1], [16], [23]

3.3.1 Introduction

The Lie-isotopic generalization of the classical Galilei's Relativity was a first central objective of Santilli's studies, evidently conceived as a necessary step toward a compatible generalization of Einstein's Special Relativity (reviewed in the next section).

The mathematical foundations of the generalized relativity, hereinafter referred to as *Santilli's Galilean Relativity*, were achieved in the first memoirs of 1978 [1], [2], as reviewed in §2.2 and 2.3. These foundations were then complemented with studies [22] reviewed in §2.4.

The physical foundations of the generalized relativity were identified also in the original memoir of 1978 which contains the proposal of a still more general covering of Galilei's Relativity of Lie-admissible type. Studies specifically devoted to the Lie- isotopic subcase under consideration here were continued during the period 1979-1981. The covering relativity was formally submitted in 1982 in Chapter 6 of ref.[15] entitled precisely: "Generalization of Galilei's Relativity." The central part of the covering relativity, the isotopic theory of rotations, was also developed in the subsequent paper [23], reviewed in the preceding section. The Lie-isotopic spin covering was presented in papers [25,27], and reviewed in Appendix C. Finally, the formulation in its most general possible non-linear, non-Hamiltonian and nonlocal forms was reached in memoir [24a]. From here on we shall tacitly assume a sufficient knowledge of the isotope $\hat{O}(3)$ and $\widehat{SU}(2)$, as well as, particularly, knowledge of their applications.

Evidently, we cannot review here in details such rather vast research. We shall therefore review only the central aspects of the covering relativity.

For notational convenience, we shall first review the rudiments of Galilei's Relativity in classical Hamiltonian mechanics, and then pass to a review of Santilli's covering.

3.3.2 Foundations of Galilei's Relativity

Galilei's Relativity is a body of methodological tools for the form-invariant characterization of closed-isolated systems of

1. particles which can be effectively aproximated as being point-like;

2. when moving in vacuum (empty space) assumed as homogeneous and isotropic;

3. under the conditions that possible speeds are much smaller than that of light (i.e., $v \ll c_o$), quantum mechanical aspects are ignorable (i.e., $A \gg \hbar$), and gravitational effects are absent (i.e., all spaces have null curvature).

The mathematical formulation of the relativity can be summarized as follows.

Let $E(3)$ be the Euclidean space in three dimension. Let a system of N particles in $E(3)$ have the local coordinates $\vec{r}_k, k = 1, 2, ..., N$, which are the physical cordinates with respect to the observer. Let the phase space be represented via the cotangent bundle $T^*E(3)$ with local coordinates $\vec{r}_k, \vec{p}_k$, where $\vec{p}_k = m_k\dot{\vec{r}}_k$ are the physical linear momenta of the particles considered. Let $\mathbf{R}$ represent the physical time t of the observer. The basic manifold of Galilei's Relativity is then given by the $(6N + 1)$ dimensional space $\mathbf{R} \times T^*E(3)$. Its local coordinates shall be written in the unified notation

$$\mathbf{R} \times T^*E(3) : (t; \vec{r}, \vec{p}) \stackrel{\text{def}}{=} (t; a) ,$$
$$a = (a^\mu) = (\vec{r}_i, \vec{p}_j), \mu = 1, 2, ..., 6N , \tag{3.79}$$

when emphasis is needed on the symplectic geometry on $T^*E(3)$, and in the still more general notation

$$\mathbf{R} \times T^*E(3) : (t; \vec{r}, \vec{p}) \stackrel{\text{def}}{=} (b) ,$$
$$b = (b^\mu) = (t; \vec{r}, \vec{p}), \quad \mu = 0, 1, 2, \ldots, 6N , \tag{3.80}$$

when emphasis is needed on the contact geometry of the entire space $\mathbf{R} \times T^*E(3)$.

The celebrated *Galilei's transformations* can be written

$$G(3.1) : \begin{cases} t \to t' = t + t_o, \\ \vec{r}_k \to \vec{r}'_k = R(\theta)\vec{r}_k + \vec{v}_{ok}t + \vec{r}_{ok}, \\ \vec{p}_k \to \vec{p}'_k = R(\theta)\vec{p}_k + m_k\vec{v}_{ok}, \end{cases} \tag{3.81}$$

and they characterize the *Galilei group* $G(3.1)$, with ten parameters $(\vec{\theta}; \vec{v}_o; \vec{r}_o; t_o)$ and related subgroups: rotations $O_{\vec{\theta}}(3)$; Galilei's boosts $T_{\vec{v}_o}(3)$; translations in space $T_{\vec{r}_o}(3)$; and translations in time $T_{t_o}(1)$. The Lie algebra $\mathbf{G}(3.1)$ of $G(3.1)$ is then given by

$$\mathbf{G}(3.1) = [O_{\vec{\theta}}(3) \oplus T_{\vec{v}_o}(3)] \oplus [T_{\vec{r}_o}(3) + T_{t_o}(1)] , \tag{3.82}$$

where $+(\oplus)$ represents the direct sum (semidirect sum).

The following Definition is presented in ref. [15] to focus attention on some of the central methodological tools of Galilei's Relativity. For a comprehensive list of references, including some of Galilei's historical work, see ref. [15], §6.3.

*DEFINITION 3.2 (*Galilei's Relativity*): Consider a local, analytic, regular, unconstrained, conservative, Newtonian system of N particles in the unique, normal, first-order (vector field) form expressed in the local variables of its experimental observation*

$$(\dot{a}^\mu) = \begin{pmatrix} \dot{r}^{ka} \\ \dot{p}_{ka} \end{pmatrix} = (\Xi^\mu(a)) = \begin{pmatrix} p_{ka}/m_k \\ f^{SA}_{ka}(\mathbf{r}) \end{pmatrix} ,$$

$$\mu = 1, 2, \ldots, 2n = 6N; k = 1, 2, \ldots, N; a = x, y, z; \mathbf{p} = m\dot{\mathbf{r}} , \tag{3.83}$$

(where SA stands for varational self-adjointness), with the ten total conserved quantities

$$E_{tot} = T(\mathbf{p}) + V(\mathbf{r}) = X_1,$$

$$\mathbf{P}_{tot} = \sum_{k=1}^{N} \mathbf{p}_k = \sum_{k=1}^{N} m_k \mathbf{p}_k = \{X_2, X_3, X_4\},$$

$$\mathbf{M}_{tot} = \sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{p}_k = \{X_5, X_6, X_7\},$$

$$\mathbf{G}_{tot} = \sum_{k=1}^{N} (m_k \mathbf{r}_k - t\mathbf{p}_k) = \{X_8, X_9, X_{10}\}. \tag{3.84}$$

Then, Galilei'sRelativity *can be defined as a form-invariant description of closed self-adjoint systems, that is, as the symmetry of the equations of motion under the ten-parameter Lie transformation group G*(3.1) *(form-invariance):*

$$G(3.1) : b^\mu \to b'^\mu(b), \quad b = (t, a) ,$$

$$\begin{aligned} \hat{\Xi}(b) &= \hat{\Xi}^\mu(b)\frac{\partial}{\partial b^\mu} = \Xi^\mu(a)\frac{\partial}{\partial a^\mu} + \frac{\partial}{\partial t} \\ &= \hat{\Xi}^\mu(b(b'))\frac{\partial b'^a}{\partial b^\mu}\frac{\partial}{\partial b'^a} = \hat{\Xi}'^a(b')\frac{\partial}{\partial b'^a} \\ &\equiv \Xi^\alpha(b')\frac{\partial}{\partial b'^a} = \Xi^\alpha(a')\frac{\partial}{\partial a'^a} + \frac{\partial}{\partial t'} = \Xi(b') , \end{aligned} \tag{3.85}$$

whose ten generators X_k represent conservation laws (3.84) (closed self-adjoint character):

$$\dot{X}_k(b) = \frac{\partial X_k}{\partial b^\mu}\hat{\Xi}^\mu(b) \equiv 0, \quad k = 1, 2, ..., 10. \tag{3.86}$$

The relativity is characterized by the following formulations.

I. Analytic formulations. *They essentially consist of the representation of the equations of motion via the conventional Hamilton's equations*

$$[\frac{\partial R^o_\nu(a)}{\partial a^\mu} - \frac{\partial R^o_\mu(a)}{\partial a^\nu}]\dot{a}^\nu - \frac{\partial H(a)}{\partial a^\mu} = 0,$$

$$(\omega_{\mu\nu}) = (\frac{\partial R^o_\nu}{\partial a^\mu} - \frac{\partial R^o_\mu}{\partial a^\nu}) = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ 1_{n\times n} & 0_{n\times n} \end{pmatrix},$$

$$R^o = (\mathbf{p}, \mathbf{0}) \tag{3.87}$$

and related canonical formulations (canonical transformation theory; canonical perturbation theory; Hamilton-Jacobi equations; Noether's theorem; etc.);

II. Algebraic formulations. *They essentially consist of the universal enveloping associative algebra $\xi(\mathbf{G}(3.1))$ of Galilei's algebra*

$$\xi(\mathbf{G}(3.1)) = \frac{\mathcal{F}}{\mathcal{R}},$$

$$\mathcal{F} = \mathbf{F} \oplus \mathbf{G} \oplus \mathbf{G} \otimes \mathbf{G} \oplus \ldots,$$

$$\mathcal{R} : [X_i, X_j] - (X_i \otimes X_j - X_j \otimes X_i),$$

$$\mathbf{G}(3.1) \approx [\xi(\mathbf{G}(3.1))]^- : [X_i, X_j] = C^k_{ij} X_k, \tag{3.88}$$

the canonical realization of Galilei's group (here expressed in symbolic form prior to a scalar extension)

$$G(3.1) : a^\mu \to a'^\mu = \exp(\theta^k \omega^{\alpha\beta} \frac{\partial X_k}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}) a^\mu, \tag{3.89.a}$$

$$\theta^k = \{t_o; \vec{\mathbf{r}}_o; \vec{\theta}_o; \vec{\mathbf{v}}_o\}, \tag{3.89.b}$$

and related Lie's theory (representation theory, etc.).

III. Geometric formulations. *They essentially consist of the characterization of the (autonomous) equations of motion as a Hamiltonian vector field*

$$\Xi \perp \omega_2 = -dH \ , \tag{3.90}$$

with respect to the fundamental symplectic structure

$$\omega_2 = \frac{1}{2}\left(\frac{\partial R^o_\nu}{\partial a^\mu} - \frac{\partial R^o_\mu}{\partial a^\nu}\right) da^\mu \wedge da^\nu = dp_{ka} \wedge dr^{ka} \ , \tag{3.91}$$

and related symplectic and contact geometric formulations (Lie's derivatives, etc.).

Note that the "time component" of canonical realization (3.89.a) of Galilei's relativity, $a' = \exp(t\omega^{\alpha\beta}\frac{\partial H}{\partial a^\beta}\frac{\partial}{\partial a^\alpha})a$, characterizes the time evolution of the system and should not be confused with the time translation. In particular, the latter acts on time, $t \to t' = t + t_o$, while the former acts on the a variables, $a(t) \to a(t + t_o)$. Also, the latter is unique, while the former depends explicitly on the Hamiltonian, and therefore its explicit form is different for different systems.

A few comments are in order. First, we should stress the restriction of the applicability of Galilei's relativity *only* to closed self-adjoint systems. This restriction is based on the notion of (physically) *exact symmetry* applied to the case at hand. In fact, we have the combination of the *mathematical* condition of Hamiltonian form-invariance and related first integrals, with the *physical* condition that the first integrals directly represent laws of nature. The conservative character of the forces is then a consequence [16].

We can say in different terms that Definition 3.2 applies only for systems of Newtonian particles verifying the following conditions.

1. *Closure condition*: The system can be considered as isolated from the rest of the universe in order to permit the conservation laws of the total mechanical energy, the total physical linear momentum, the total physical angular momentum, and the uniform motion of the center of mass.

2. *Self-adjointness condition*: The particles can be well approximated as massive points moving in vacuum along stable orbits without collisions, in order to restrict all possible forces to those of action-at-a-distance, potential type.

3. *Form-invariance*: The ten conservation laws follow from the Galilean symmetry of the system.

The existence of physical systems obeying these conditions is unequivocal. For instance, our solar system in Newtonian approximation is indeed a system of this type, and, as such, obeys *all* conditions for the applicability of Galilei's relativity.

3.3.3 Arena of Applicability of Santilli's Covering Relativity

The applicability of Galilei's relativity is the exception, and its violation is the rule in Newtonian mechanics for several reasons. The most important is that Newtonian "particles" can be well approximated as "massive points" only under very special conditions. In fact, Newtonian systems generally imply motions of extended objects (e.g., a satellite) in a resistive medium (e.g., Earth's atmosphere), in which case their reduction to massive points would imply excessive approximations (e.g., the approximation of the satellite orbiting in our atmosphere with a conserved angular momentum). When the extended character of the objects is represented together with their motion within physical media, the dynamical conditions become unrestricted. As a result, the equations of motion break the Galilei's symmetry according to one of the mechanisms of the classification of ref. [15], §A.12 (*isotopic, self-adjoint, semicanonical, canonical, and essentially self-adjoint breakings*).

Equivalently, we can say that, if Galilei's Relativity is imposed in its exact meaning, it generally implies an excessive restriction of the acting forces, with consequentially excessive approximations of the perpetual-motion type.

In view of these and other considerations, Santilli constructed a generalization of the analytic, algebraic and geometrical foundation of Galilei's Relativity to attempt a covering relativity for the form-invariant description of closed- isolated systems of:

1'. extended-deformable particles which cannot be effectively approximated as being point-like;

2'. when moving in physical media which are generally inhomogeneous and anisotropic;

3'. under the condition that the dynamical evolution is still "nonrelativistic" (i.e., $v \ll c_o$), "classical" (i.e., $A \geq \hbar$), and "nongravitational" (i.e., null curvature).

It should be stressed that the above arena is specifically restricted to closed-isolated systems in which case the medium is evidently a part of the system (see Fig. 9).

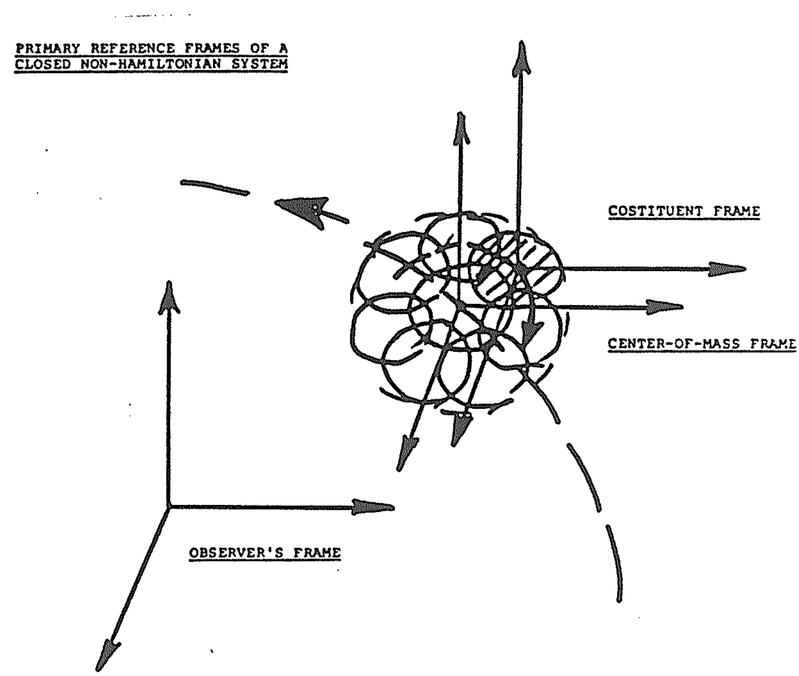


FIGURE 9. A reproduction of Fig. 5.2, p. 532 of ref. [16] depicting the three most important reference frames for closed nonhamiltonian systems: the frame of the observer, the center-of- mass frame of the system as a whole, and the center-of-mass frame of each individual constituent. In conventional dynamical systems (with action-at-a-distance interactions of point-like constituents) these frames represent all stable orbits, and result to be equivalent. The corresponding situation for closed nonhamiltonian systems is different. To begin, the orbit of one individual constituent is unstable and, therefore, it generally requires nonlinear symmetries. Secondly, the center-of-mass frame of the system as a whole generally represent stable conditions. As such, it cannot be linked to the center-of-mass frame of each constituent via linear transformations, such as the conventional Galilean (or Lorentz) transformations, but requires suitable generalizations. The Lie-isotopic generalization of Galilei's Relativity [1], [15] deals only with the observer's and center-of-

mass frame of the system as a whole. The inclusion of the center-of-mass frame of each constituent requires a still more general Lie-admissible generalization of Galilei's relativity [1], [16] which is not considered here.

The covering relativity is also applicable to other systems, e.g., when the medium is considered as external. In this case, however, the emerging "conserved quantites" are only first integral, without in general direct physical significance. In fact, the total energy, the total linear momentum and other physical quantities are generally nonconserved for open systems.

Also, the reader should be aware that conditions 1'), 2') and 3') were conceived as a sort of classical image of the structure model of hadrons [2] whose constituents have extended wave-packets moving in the hadronic medium made up of other constituents. However, classical mechanics offers numerous systems verifyng the above generalized conditions (e.g., Jupiter) in a way independent from possible operator-counterparts.

Our review shall therefore be purely classical. Particle aspects will be treated in Appendix C. In order to have the appropriate perspective, the recommended research attitude is the opposite of the conventional one. Customarily, one first assumes an established relativity, and then restricts the dynamics to that compatible with the assumed relativity. On the contrary, Santilli advocates first the assumption of dynamical conditions as identifiable in Nature, and then the search for a compatible relativity. This research attitude can be implemented according to the following three steps: the identification of the largest possible class of systems with unrestricted dynamics; the identification of the methods for their treatment; and the identification of the covering relativity.

3.3.4 Closed Non-Self-Adjoint Systems

When a system of particles is isolated from the rest of the universe, it must necessarily obey the ten conservation laws (3.84); that is, it must be closed. However, this does not necessarily imply that all internal forces are of the potential, action-at-a-distance type. In fact, closure conditions (3.84) are compatible with internal forces of contact, nonpotential, non-self- adjoint type due to internal collisions and/or motion within resistive media. This leads in a natural way to the notion of *closed non-self-adjoint systems* [2] reviewed in §1.3 in their second-order form. Their formulation for first-order systems can be presented as follows.

Implement closed self-adjoint systems (3.83) with an unrestricted collection of local and analytic forces. These additive forces can be classified into self-adjoint (SA) and non- self-adjoint (NSA), resulting in the following systems

$$\begin{aligned}(\dot{a}^{\mu}) &= \begin{pmatrix}\dot{r}^{ka}\\ \dot{p}_{ka}\end{pmatrix} = (\Gamma^{\mu}(t,a)) = (\Xi^{\mu}(a)) + (F^{\mu}(t,a)) \\ &= \begin{pmatrix}p_{ka}/m_k\\ f^{SA}_{ka}(\mathbf{r})\end{pmatrix} + \begin{pmatrix}0\\ F^{SA}_{ka}(t,\mathbf{r},\mathbf{p}) + F^{NSA}_{ka}(t,\mathbf{r},\mathbf{p},\dot{\mathbf{p}},\dots)\end{pmatrix}, \end{aligned} \quad (3.92)$$

where one can recognize: the conservative forces $f^{SA}_{ka}(\vec{r})$ verifying Galilei's Relativity; plus additional forces $F^{SA}_{ka}(t,\vec{r},\vec{p})$ that are also self-adjoint and Newtonian, but not necessarily Galilei-form-invariant; plus additional forces $F^{NSA}_{ka}(t,\vec{\mathbf{r}},\vec{\mathbf{p}},\dot{\vec{\mathbf{p}}},\dots)$ that are, in general, Galilei-form-noninvariant, non- self-adjoint, as well as non-Newtonian (that is, they can also depend on the acceleration and other non-Newtonian terms).

It should be indicated here that the original presentation [1], [15] put the emphasis on Newtonian forces. Nevertheless, following a private communication by Santilli, we have added here non-Newtonian forces, not only because the results of refs. [1], [15] are readily applicable to these forces without any modification, but also because the inclusion of accleration- dependent forces has truly intriguing implications in the operator- images of the theory for particle physics, e.g., the capability of achieving consistent nonrelativistic bound state models in which the total energy is higher than the sum of the rest energies of the constituents (a possibility which is precluded in conventional quantum mechanics). Also, explicit examples of the generalized relativity have indicated the existence of these acceleration- dependent forces, as we shall review below. Finally, acceleration- dependent forces appear, quite independently, in recent studies by A.K.T. Assis [92] and others in ordinary (nongravitational) mechanics via the postulate that the total acting forces on an individual body is null and the use of Mach's principle.

The total enegy is modified in the above implementation, trivially, because of the additional presence of potential forces,

$$E_{tot} = T(\mathbf{p}) + V(\mathbf{r}) + U(t,\mathbf{r},\mathbf{p}),$$

$$T(\mathbf{p}) = \sum_{k=1}^{N} \frac{1}{2m_k}\mathbf{p}_k \cdot \mathbf{p}_k,$$

$$F^{SA}_{k}(t,\mathbf{r},\frac{\mathbf{p}}{m}) = -\frac{\partial U}{\partial \mathbf{r}^k} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\mathbf{r}}^k}. \quad (3.93)$$

All the other total quantities (3.84.b)-(3.84.d) remain unchanged. In fact, physical quantities such as the total linear momentum $\mathbf{P}_{tot}$ are defined in a way independent from the acting forces which, clearly, can only affect their behavior in time.

DEFINITION 3.3 [2],[15]: The most general possible class of local, analytic, closed, discrete, and non-self-adjoint systems *is given by the class of all possible, consistent, generally overdetermined and constrained systems*

$$(\dot{a}^{\mu}) = \begin{pmatrix} \dot{r}^{ka} \\ \dot{p}_{ka} \end{pmatrix} = (\Gamma^{\mu}(t,a)) = \tag{3.94.a}$$

$$\begin{pmatrix} p_{ka}/m_k \\ f_{ka}^{SA}(\mathbf{r}) + F_{ka}^{SA}(t,\mathbf{r},\mathbf{p}) + F_{ka}^{NSA}(t,\mathbf{r},\mathbf{p},\dot{\mathbf{p}},...) \end{pmatrix},$$

$$\dot{X}_i(t,a) = \frac{\partial X_i}{\partial a^{\mu}}\dot{a}^{\mu} - \frac{\partial X_i}{\partial t} = 0, \tag{3.94.b}$$

$$X_1 = E_{tot} = T(\mathbf{p}) + V(\mathbf{r}) + U(t,\mathbf{r},\mathbf{p}), \tag{3.94.c}$$

$$\{X_2, X_3, X_4\} = \mathbf{P}_{tot} = \sum_{k=1}^{N} m_k \mathbf{p}_k,$$

$$\{X_5, X_6, X_7\} = \mathbf{M}_{tot} = \sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{p}_k, \tag{3.94.d}$$

$$\{X_8, X_9, X_{10}\} = \mathbf{G}_{tot} = \sum_{k=1}^{N} (m_k \mathbf{r}_k - t\mathbf{p}_k), \tag{3.94.e}$$

$$\mu = 2,1,...,6N, \quad k = 1,2,...,N, \quad a = x,y,z, \quad i = 1,2,...,10. \tag{3.94.f}$$

The primary difference between closed self-adjoint and non-self-adjoint systems is that the conservation laws of total quantities are first integrals of the equations of motion for the former, while they are, *in general*, subsidiary constraints for the latter.

The *physical existence* of closed non-self-adjoint systems is established by a simple observation of Nature. For instance, the Earth, when considered as isolated from the rest of the universe and inclusive of its atmosphere, is precisely a closed system with unrestricted internal forces, rudimentary approximated by Eqs. (3.94).

The *mathematical existence* of the systems is established by the existence theory of overdetermined systems. In fact, the following hierarchy exists of classes of consistent systems (3.94) with a dynamics of increasing complexity and methodological needs [15]:

Class α: when the conserved total physical quantities are first integrals of the vector field;

Class β: when the conserved total physical quantities constitute invariant relations of the vector field;

Class γ: when the conserved total quantities constitute bona fide subsidiary constraints of the vector field.

For brevity, we limit ourselves to the illustration of class α. The existence of the more general classes β and γ will be only indicated.

Assume for simplicity that the additive self-adjoint forces in Eqs. (3.93) are null. This implies that the original total energy (3.84) persists during the implementation of the systems with internal contact forces. We now impose the conservation laws to be the first integrals of the new systems according to the (strong) equality

$$\begin{aligned} \dot{X}_i(t,a) &= \frac{\partial X_i}{\partial a^\mu}\Gamma^\mu + \frac{\partial X_i}{\partial t} \\ &= (\frac{\partial X_i}{\partial a^\mu}\Xi^\mu + \frac{\partial X_i}{\partial t}) + \frac{\partial X_i}{\partial a^\mu}F^\mu \equiv 0. \end{aligned} \tag{3.95}$$

But the original Eqs. (3.84) are verified by assumption. Thus, conditions (3.95) reduce to

$$\frac{\partial X_i}{\partial a^\mu}F^\mu = \frac{\partial X_i}{\partial p_{ka}}F^{NSA}_{ka} \equiv 0 \ , \tag{3.96}$$

that is, the non-self-adjoint forces must be null eigenvectors of the matrix $(\partial X_i/\partial p_{ka})$. When all ten conservation laws are worked out in detail, they imply the following conditions

$$\sum_{k=1}^{N} \mathbf{p}_k \cdot \mathbf{F}_k^{NSA} \equiv 0, \tag{3.97.a}$$

$$\sum_{k=1}^{N} \mathbf{F}_k^{NSA} \equiv 0, \tag{3.97.b}$$

$$\sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{F}_k^{NSA} \equiv 0. \tag{3.97.c}$$

Note that these are conditions on non-self-adjoint forces for total physical quantities to be first integrals. As a result, conditions (3.97) are *only sufficient* for the consistency of systems (3.93) and *not necessary.*

It is now trivial to see that consistent systems of class α do indeed exist. In fact, the consistency of systems (3.94) has been reduced to that of systems (3.97). These are *functional* systems of seven equations in $3N$ unknown functions $\mathbf{F}_{ka}^{NSA}$. Solutions in the functions $\mathbf{F}_{ka}^{NSA}$ exist beginning with $N = 3$. The case $N = 2$ is a special one, inasmuch as the closure forces the orbit to be in a plane. The number of Eqs. (3.97) therefore reduces to five, while the number of functions $\mathbf{F}_{ka}^{NSA}$ is four. Despite the lack of sufficient degrees of freedom, a solution still exists, and it is reviewed later on.

The *N-body, closed, non-self-adjoint systems of class α* ($N \geq 3$) are also instructive at all levels of study. For instance, conditions (3.97) might conceivably be derived via arguments of *global stability of the system achieved via unstable orbits of the constituents.*

In fact, condition (3.97.a) (which ensures the conservation of the total energy) is clearly a first condition for global stability via unrestricted internal exchanges of energy; conditions (3.97.b) (which ensure the conservation of the total linear momentum and the uniform motion of the center of mass) are a clear expression of additional conditions of global stability via unrestricted action and reaction effects with null total value; and conditions (3.97.c) (which ensure the conservation of the total angular momentum) are clearly the last expected condition for global stability. (A first statistical study of closed non-self-adjoint systems has been conducted by Tellez-Arenas, Fronteau, and Santilli [32].)

However, as indicated earlier, conditions (3.97) are only sufficient for the systems considered. When the broader class β is admitted, Eqs. (3.95) are generalized into the weak equality for invariant relations

$$\dot{X}_i(t, a_o) = \lambda_i^j(t, a_o) V_j(t, a_o) = 0 \ , \tag{3.98}$$

that is, they hold along the solutions of the systems. In turn, conditions (3.98) themselves are only sufficient, inasmuch as the most general class of the systems (class γ) is that for which the conservation laws are bona fide subsidiary constraints of the equations of motion. The study of these latter systems is left here to the interested researcher.

3.3.5 Symmetries, First Integrals, and Conservation Laws in Birkhoffian Mechanics

As is well-known, Galilei's relativity in its contemporary interpretation is an expression of some of the most advanced analytic, algebraic, and geometric techniques of Hamiltonian Mechanics. But a necessary condition for a closed system to be non-self-adjoint is that the vector field is not Hamiltonian in the variables $(t, \mathbf{r}, \mathbf{p}), \mathbf{p} = m\dot{\mathbf{r}}$, of its experimental observation. This implies that, for systems (3.94), not only do we have the general lack of Galilei form-invariance, but we actually have the lack of applicability of the methodological foundations of the relativity. In turn, this creates the need to identify covering methods before any attempt at the construction of a covering relativity can acquire scientific value.

The direct universality of Birkhoff's equations for the representation of *all* closed non-self-adjoint systems was established in Chapter 4 of ref. [15], together with the methods for the construction of the Birkhoffian representation from the equations of motion, as well as the identification of the underlying degrees of freedom. The representation can be constructed according to the equations

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\Gamma^\nu(t,a) = \frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t},$$

$$\mu = 1, 2, ..., 6N, \tag{3.99}$$

where the Birkhoffian can be the Hamiltonian H, i.e., the total energy,

$$B = H = T(\mathbf{p}) + V(\mathbf{r}) + U(t, \mathbf{r}, \mathbf{p}), \tag{3.100}$$

and the R-functions are obtained via one of the three methods of Corollary 4.5.1d, *loc. cit.* In this way, while all self- adjoint forces are represented by the Hamiltonian, all non-self- adjoint forces are represented via the generalization of the canonical tensor $\omega_{\mu\nu}$ into the Birkhoffian form $\Omega_{\mu\nu}$ (which is not possible in Hamiltonian formulations).

The transformation theory of Birkhoff's equations is worked out in detail in Chapter 5 of ref. [15]. Regrettably, we cannot possibly review it here for brevity. The theory emerges as being a true covering of the transformation theory of Hamilton's equations. This allows the use of the Birkhoffian mechanics and its Lie-isotopic/symplectic-isotopic structure for the construction of the desired generalization of Galilei's relativity.

In the following, we shall review, for brevity, only the most essential aspects. To begin, Santilli formulated his covering theory in its broadest possible form, that of the contact geometry in unified local coordinates (3.80). For this purpose, Birkhoff's Eqs. (1.21) should be written in the unified notation

$$\tilde{\Omega}_{\mu\nu}(b)db^{\nu} = 0, \quad \mu = 0,1,2,...,6N \ ,$$

$$(\tilde{\Omega}_{\mu\nu}(b)db^{\nu}) = \begin{cases} (\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t})da^{\nu} = 0 \ , \\ (\frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}})da^{\nu} - (\frac{\partial B}{\partial a^{\mu}} + \frac{\partial R_{\mu}}{\partial t})dt = 0 \ , \end{cases} \tag{3.101}$$

where the first equation holds in view of the trivial identity

$$(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t})da^{\nu} = (\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t})\Omega^{\nu\alpha}(a)(\frac{\partial B}{\partial a^{\alpha}} + \frac{\partial R_{\alpha}}{\partial t})dt \equiv 0 \ , \tag{3.102}$$

and where, as assumed in §1.3, we use the Birkhoffian B for the case of its generic functional dependence, and the Hamiltonian H when specifically restricted to be the total energy.

The reader should be aware that (Chapter 5, ref. [15]) all possible smoothness and regularity preserving, but otherwise arbitrary transformations

$$b = \{b^{\mu}\} = \{t; \vec{r}, \vec{p}\} \to b'(b) = \{b'^{\mu}(b)\} = \{t'(t,\vec{r},\vec{p}); \vec{r}'(t,\vec{r},\vec{p}), \vec{p}'(t,\vec{r},\vec{p})\} \ , \tag{3.103}$$

are *contact-isotopic*, i.e., they preserve the contact nature of the underlying two-form

$$\tilde{\Omega}_2(b) = \frac{1}{2}\tilde{\Omega}_{\mu\nu}(b)db^{\mu} \wedge db^{\nu} = \frac{1}{2}\tilde{\Omega}'_{\mu\nu}(b')db'^{\mu} \wedge db'^{\nu} = \tilde{\Omega}'_2(b') \ ,$$

$$\tilde{\Omega}'_{\mu\nu}(b') = \frac{\partial b^{\alpha}}{\partial b'^{\mu}}\tilde{\Omega}_{\alpha\beta}(b(b'))\frac{\partial b^{\beta}}{\partial b'^{\nu}} \ . \tag{3.104}$$

To understand this property in more explicit terms, recall that Hamilton's equations preserve their canonical form only under a special class of transformations, the canonical ones. As indicated in §1.2, when Hamilton's equations are submitted to a general, noncanonical transformation, they are transformed precisely into Birkhoff's equations. Unlike the simpler case of Hamilton's equations, the covering Birkhoff's equations preserve their form under the most general possible transformations.

This point is important for the covering relativity. In fact, in the conventional Hamiltonian case, symmetries must be first canonical, and then form-invariant transformations. In the covering Birkhoffian setting the first condition is unnecessary.

We reach in this way the following definition (ref. [15], p. 238):

DEFINITION 3.4 (Symmetries in Birkhoffian Mechanics): The most general possible smoothness and regularity preserving transformations (3.103) on $\mathbf{R} \times T^*E(3)$ *are said to be* symmetries of Birkhoff's Eqs. *(3.101) when they are* identity contact-isotopic, *i.e., they leave form-invariant the contact tensor*

$$\begin{aligned} \tilde{\Omega}_{\mu\nu}(b)db^\nu &= \frac{\partial b'^\alpha}{\partial b^\mu}\tilde{\Omega}'_{\alpha\beta}(b')db'^\beta \\ &\equiv \frac{\partial b'^\alpha}{\partial b^\mu}\tilde{\Omega}_{\alpha\beta}(b')db'^\beta = 0, \end{aligned} \tag{3.105}$$

or, more explicitly, when the following particularization of transformation rules (3.104) holds

$$\begin{aligned} (\tilde{\Omega}_{\mu\nu}(b)db^\nu) &= \begin{pmatrix} (\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t})da^\nu \\ (\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu})da^\nu - (\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t})dt \end{pmatrix} \\ &= (\frac{\partial b'^\alpha}{\partial b^\mu}\tilde{\Omega}_{\alpha\beta}(b')db'^\beta) \\ &= (\frac{\partial b'^\alpha}{\partial b^\mu}) \begin{pmatrix} (\frac{\partial B}{\partial a'^\rho} + \frac{\partial R_\rho}{\partial t'})da'^\rho \\ (\frac{\partial R_\beta}{\partial a'^\alpha} - \frac{\partial R_\alpha}{\partial a'^\beta})da'^\beta - (\frac{\partial B}{\partial a'^\alpha} + \frac{\partial R_\alpha}{\partial t'})dt' \end{pmatrix} \\ &= 0, \end{aligned} \tag{3.106}$$

or

$$R_\alpha(t', a') = (R_\mu \frac{\partial a^\mu}{\partial a'^\alpha} - B\frac{\partial t}{\partial a'^\alpha})(t', a'),$$

$$B(t', a') = (B\frac{\partial t}{\partial t'} - R_\mu \frac{\partial a^\mu}{\partial t})(t', a').$$

Equivalently, we have a symmetry when the primitive one-form of Birkhoff's equations (the integrand of the Pfaff's action) is form- invariant up to Birkhoffian gauges,

$$\begin{aligned} R_\mu(a)da^\mu - B(t,a)dt \stackrel{\text{def}}{=} \tilde{R}_\mu(b)db^\mu &= \tilde{R}'_\alpha(b')db'^\alpha \\ &\equiv \left[\tilde{R}_\alpha(b') + \frac{\partial G(b')}{\partial b'^\alpha}\right] db'^\alpha, \end{aligned}$$

$$\tilde{R}'_\alpha(b') = (R_\mu \frac{\partial b^\mu}{\partial b'^\alpha})(b'). \tag{3.107}$$

Clearly, the symmetries of Hamilton's equations are a particular case of the symmetries of Birkhoff's equations, in exactly the same way as the transformation theory of Hamilton's equations is a particular case of that of Birkhoff's equations.

Most important is the property that the new time t', in general, can be not only a function of all old variables $t'(t, \mathbf{r}, \mathbf{p})$, but also the image of any old variable (Corollaries 5.3.3a and 5.3.3c, *loc. cit.*).

We move now to the review of the generalized methods for the construction of first integrals from known symmetries of Birkhoff's equations. For this purpose we suppose that given Birkhoff's equations possess the following Lie symmetry group of infinitesimal transformations

$$\begin{aligned} \hat{G}_r : (b^\mu) = \begin{pmatrix} t \\ a^\mu \end{pmatrix} \rightarrow (b'^\mu) = \begin{pmatrix} t' \\ a'^\mu \end{pmatrix} &= (b^\mu + \hat{\delta} b^\mu) = (b^\mu + w^i \hat{\alpha}_i^\mu(b)) \\ &= \begin{pmatrix} t + w^i \hat{\rho}_i(t,a) \\ a^\mu + w^i \hat{\eta}_i^\mu(t,a) \end{pmatrix}, \qquad (3.108) \end{aligned}$$

where the w's are the infinitesimal parameters.

Then, via the direct use of variational techniques, the Pfaffian action transforms according to

$$\tilde{\delta}\hat{A} = \int_{D_1} \tilde{R}_\mu(b) db^\mu - \int_{D_{t'}} \tilde{R}_\mu(b') db'^\mu = - \int_{D_{t'}} d[\tilde{\delta} G(b)], \qquad (3.109)$$

where D_t is the original (closed) interval of time, and $D_{t'}$ is its image under the transformations.

By recalling the Pfaffian variational principle (equations (5.3.50), *loc. cit.*), we can write along a possible or actual path E^o

$$\begin{aligned} \tilde{\delta} \int_{D_t} dt \tilde{R}_\mu(b) \dot{b}^\mu &= \int_{D_t} dt \tilde{\Omega}_{\mu\nu}(b) b^\nu \hat{\delta} b^\mu \\ &= - \int_{D_t} dt \frac{d}{dt} [\tilde{R}_\mu(b) \tilde{\delta} b^\mu + \tilde{\delta} G(b)](E^o) \\ &= -w^i \int_{D_t} dt \frac{d}{dt} [\tilde{R}_\mu(b) \hat{\alpha}_i^\mu(b) + G_i(b)](E^o) \\ &= -w^i \int_{D_t} dt \frac{d}{dt} [R_\mu(t,a) \hat{\eta}_i^\mu(t,a) - B(t,a) \hat{\rho}_i(t,a) \\ &- G_i(t,a)](E^o). \qquad (3.110) \end{aligned}$$

In this way we reach the following important result of refs. [1], [15]:

Theorem 3.3 (Noether's Theorem for Birkhoff's Equations) *If Birkhoff's equations admit a symmetry under an r-dimensional connected Lie Group $\hat{G}_r$ of infinitesimal transformations, then r linear combination of Birkhoff's equations exist along an admissible path which are exact differentials, i.e.,*

$$\frac{d}{dt}I_i(b) = \tilde{\Omega}_{\mu\nu}(b)b^\nu\hat{\alpha}_i^\mu,$$

$$\begin{aligned} I_i(b) &= \tilde{R}_\mu(b)\hat{\alpha}_i^\mu(b) + G_i(b) \\ &= R_\mu(t,a)\hat{\eta}_i^\mu(t,a) - B(t,a)\hat{\rho}_i(t,a) + G_i(t,a), \end{aligned}$$

$$i = 1, 2, ..., r. \tag{3.111}$$

A quite simple, alternative proof can be formulated via (a) the property that Noether's theorem also applies to first-order totally degenerate Lagrangians $L(t,a,\dot{a})$; (b) the property that Birkhoff's equations coincide with Lagrange's equations in $L(t,a,\dot{a})$; and (c) the specialization of the theory to the case at hand. This alternative approach gives rise to the quantities

$$L(t,a,\dot{a}) = R_\mu(t,a)\dot{a}^\mu - B(t,a),$$

$$\begin{aligned} I &= \frac{\partial L}{\partial \dot{a}^\mu}\hat{\delta}a^\mu - (\frac{\partial L}{\partial \dot{a}^\mu}\dot{a}^\mu - L)\hat{\delta}t + \hat{\delta}G(t,a) \\ &= R_\mu\hat{\delta}a^\mu - (R_\mu\dot{a}^\mu - R_\mu\dot{a}^\mu + B)\hat{\delta}t + \hat{\delta}G \\ &= w^i[R_\mu(t,a)\hat{\eta}_i^\mu(t,a) - B(t,a)\hat{\rho}_i(t,a) + G_i(t,a)], \end{aligned} \tag{3.112}$$

Corollary 3.3.1. *The quantities (3.111) are first integrals of Birkhoff's equations*

$$\frac{d}{dt}I_i(b)|_{E^\circ} = \tilde{\Omega}_{\mu\nu}(b)b^\nu\hat{\alpha}_i^\mu(b)|_{E^\circ} \equiv 0. \tag{3.113}$$

The covering character of Theorem 3.3 over the corresponding Hamiltonian formulations is expressed by the fact that, when the Pfaffian form becomes the canonical one (i.e., for $R = R^o = (\mathbf{p}, \mathbf{0})$ and $B = H$), we have

$$\begin{aligned} I_i &= p_{ka}\hat{\eta}_i^{ka} - H\hat{\rho}_i + G_i \\ &= \frac{\partial L}{\partial \dot{r}^{ka}}\hat{\eta}_i^{ka} - (\frac{\partial L}{\partial \dot{r}^{ka}}\dot{r}^{ka} - L)\hat{\rho}_i + G_i \end{aligned} \tag{3.114}$$

which is the formulation of the conventional Noether's theorem in Hamiltonian mechanics. Additional properties (such as the lack of necessary independence of the r first integrals, the lack of their necessary direct physical meaning, etc.) can be obtained via the extension to a Birkhoffian context of the analysis of Chart A.9, ref. [15].

We now pass to the review of the Lie algebra structure of an r-dimensional symmetry $\hat{G}_r$ of Birkhoff's equations. By recalling the lack of algebraic structure of the general nonautonomous case (Chart 4.1, *loc. cit.*), we must restrict ourselves for this purpose to semi-autonomous equations (1.21). (The capability of reducing all nonautonomous equations to this form is proved in §4.5, *loc. cit.*) Also, we assume the reader is familiar with the problematic aspects related to the physical meaning of the Birkhoffian under the reduction considered. Finally, we shall assume that Theorem 3.3 is applied to the reduced semi-autonomous form (rather than the original nonautonomous form), because symmetries are not necessarily preserved under the reduction considered.

An inspection of the notion of symmetries of Birkhoff's equations soon reveals that they *are not* canonical transformations. The necessary and sufficient condition for infinitesimal transformations to be contact-isotopic transformations is that they have the form

$$a'^{\mu} = a^{\mu} + w^i \Omega^{\mu\nu}(a) \frac{\partial X_i}{\partial a^{\nu}}(t, a),$$

$$\Omega^{\mu\nu} = (\|\frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}}\|^{-1})^{\mu\nu}, \tag{3.115}$$

where the w's are, again, the infinitesimal parameters and the X's the generators of $\hat{G}_r$.

The necessary and sufficient condition for a transformation of this type to be a symmetry is therefore that it leaves the Birkhoffian invariant, i.e.,

$$\begin{aligned} B'(a') = B(a) + \frac{\partial B}{\partial a^{\mu}} w^i \Omega^{\mu\nu} \frac{\partial X_i}{\partial a^{\nu}} &= B(a) + w^i [B \hat{,} X_i] \\ &\equiv B(a). \end{aligned} \tag{3.116}$$

Thus we reach the following additional result of refs. [1], [20]:

Theorem 3.4. (Integrability Conditions for Birkhoffian Symmetries) *Necessary and sufficient conditions for infinitesimal, contact- isotopic transformations to be symmetries of the autonomous Birkhoff's equations are that the generalized Poisson brackets of the Birkhoffian with all the* *generators* $X_i(a)$ *of the transformations are identically null, i.e.,*

$$[B\hat{,}X_i] \equiv 0, \quad i = 1, 2, ..., r. \tag{3.117}$$

The use of the isotopic generalization of Lie's theory reviewed in §2 then yields the following Corollary (see, in particular, the generalization of Lie's structure *constants* C_{ij}^k into the structure *functions* $\hat{C}_{ij}^k(a)$ of §1.3).

Corollary 3.4.1. *The Lie algebra* $\hat{\mathbf{G}}_r$ *of an r-dimensional Lie symmetry group* $\hat{G}_r$ *of Birkhoff's equations is given by the vector space (over the field* $\mathbf{F}$ *of real numbers) of the generators* X_i *verifying Eqs. (3.117) equipped with the generalized Poisson brackets as the applicable realization of the Lie product, and verifying the following closure rules expressed in terms of the structure functions* $\hat{C}_{ij}^k(a)$

$$[X_i\hat{,}X_j] = \hat{C}_{ij}^k(a) X_k. \tag{3.118}$$

In closing this topic, we can therefore say that each and every aspect of the Hamiltonian formulation of symmetries, first integrals, and conservation laws has been consistently generalized into a Birkhoffian form.

3.3.6 Construction of the Covering Relativity

At this point we review the definition of the covering relativity and then identify methods useful for its construction. We shall then review a few examples.

> *DEFINITION 3.5 [1], [15]:* Santilli's Galilean Relativity *is a description of physical systems verifying the following primary conditions:*
>
> 1. *the relativity provides a form-invariant description of closed systems of extended particles under action-at-a-distance self-adjoint interactions as well as contact non-self-adjoint interactions;*
> 2. *the relativity is based on the isotopic generalization of the methodological formulations of Galilei's Relativity, that is, on the Birkhoffian generalization of Hamiltonian mechanics, on the isotopic generalization of Lie theory, and on the symplectic and contact geometries in their most general possible local and exact realizations; and*

3. the generalized relativity recovers the conventional one identically when the systems are reduced to pointlike constituents with consequential lack of contact non-self-adjoint interactions.

By keeping in mind the conditions for a new theory to qualify as the covering of an existing one (§1.3), property 1 ensures that the new relativity applies to a physical arena broader than that of the conventional one; property 2 ensures that the new relativity is based on a generalization of the methods of the conventional one; and property 3 ensures the compatibility of the new relativity with the conventional one.

On more specific grounds, property 1 is classically realized via the construction of a ten-parameter Lie-isotopic transformation group $\hat{G}(3.1)$ *which verifies the form invariance of systems (3.94)*

$$\hat{G}(3.1) : b \to b'(b), \quad b = (t, a) ,$$

$$\begin{aligned} \tilde{\Gamma}(b) &= \tilde{\Gamma}^{\mu}(b)\frac{\partial}{\partial b^{\mu}} = \tilde{\Gamma}^{\mu}(b(b'))\frac{\partial b'^{\alpha}}{\partial b^{\mu}}\frac{\partial}{\partial b'^{\alpha}} \\ &= \tilde{\Gamma}'^{\alpha}(b')\frac{\partial}{\partial b'^{\alpha}} \equiv \tilde{\Gamma}^{\alpha}(b')\frac{\partial}{\partial b'^{\alpha}}, \end{aligned}$$

$$\tilde{\Gamma} = (1, \Gamma^{\mu}(t, a)), \tag{3.119}$$

and whose generators $X_i(b)$ *represent directly the conservation laws of total quantities (3.94.c)-(5.94.f), i.e.,*

$$\dot{X}_i(b) = 0, \quad i = 1, 2, ..., 10. \tag{3.120}$$

Property 2 is classically realized via the following formulations.

I. Isotopic generalization of Hamiltonian formulations; *which essentially consists of the representation of the equation of motion via the semiautonomous Birkhoff's equations*

$$\{[\frac{\partial R_{\nu}(a)}{\partial a^{\mu}} - \frac{\partial R_{\mu}(a)}{\partial a^{\nu}}]\dot{a}^{\nu} - \frac{\partial B(t, a)}{\partial a^{\mu}}\}_{SA} = 0 \, , \tag{3.121}$$

and related Birkhoffian covering of Hamiltonian formulations (generalized canonical transformations, generalized Hamilton-Jacobi equations, etc.).

II. Isotopic generalization of Lie's theory; *which essentially consists of the isotopic lifting of the universal enveloping associative algebra* $\hat{\xi}(\mathbf{G}(3.1))$ *of Galilei's algebra* $\mathbf{G}(3.1)$ *and attached isotopic algebra* $\hat{\mathbf{G}}(3.1)$

$$\hat{\xi}(\mathbf{G}(3.1)) = \frac{\hat{\mathcal{F}}}{\hat{\mathcal{R}}},$$

$$\hat{\mathcal{F}} = \hat{\mathbf{F}} \oplus \mathbf{G} \oplus \mathbf{G} * \mathbf{G} \oplus \cdots,$$

$$\hat{\mathcal{R}} = [X_i \hat{,} X_j] - (X_i * X_j - X_j * X_i),$$

$$\hat{\mathbf{G}}(3.1) \approx [\hat{\xi}(\mathbf{G}(3.1))]^- : [X_i \hat{,} X_j] = \hat{C}_{ij}^k(a) X_k \ , \qquad (3.122)$$

the Lie isotopic realization of the symmetry group $\hat{G}(3.1)$ *(here symbolically written prior to iso-scalar extensions)*

$$\hat{G}(3.1) : a^\mu \to a'^\mu = \exp(\theta^k \Omega^{\alpha\beta}(a) \frac{\partial X_k}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}) a^\mu,$$

$$\Omega^{\alpha\beta} = (\|\frac{\partial R_\beta}{\partial a^\alpha} - \frac{\partial R_\alpha}{\partial a^\beta}\|^{-1})^{\alpha\beta},$$

$$\{\theta^k\} = \{t_o; \vec{\mathbf{r}}_o; \vec{\theta}_o; \vec{\mathbf{v}}_o\}, \qquad (3.123)$$

and related theory (generalized representation theory, etc.).

III. Isotopic generalization of canonical geometries; *which essentially consists of the characterization of the (autonomous) equations of motion as a Birkhoffian vector field*

$$\Gamma \perp \Omega_2 = -dB \ , \qquad (3.124)$$

with respect to the exact but otherwise unrestricted symplectic structure

$$\Omega_2 = \frac{1}{2}(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}) da^\mu \wedge da^\nu, \qquad (3.125)$$

and related symplectic as well as contact geometric formulations (Birkhoffian realization of Lie's derivatives, etc.).

Finally, property 3 is classically realized via the additional condition that, together with the reduction of systems (3.94.a) to the self-adjoint and Galilei form-invariant form

$$(\Gamma^\mu)|_{F^{NSA}=0} = \begin{pmatrix} p_{ka}/m_k \\ f^{SA} + F^{NSA}_{ka} \end{pmatrix}_{F^{NSA}=0} = \begin{pmatrix} p_{ka}/m_k \\ f^{SA}_{ka} \end{pmatrix} = (\Xi^\mu), \tag{3.126}$$

we have the reduction of the group $\hat{G}(3.1)$ *to Galilei's group* $G(3.1)$, *i.e.,*

$$\hat{G}(3.1)|_{F^{NSA}=0} \equiv G(3.1),$$
$$\exp(\theta^k \Omega^{\alpha\beta}(a) \frac{\partial X_k}{\partial a^\beta} \frac{\partial}{\partial a^\alpha})|_{F^{NSA}=0} \equiv \exp(\theta^k \omega^{\alpha\beta} \frac{\partial X_k}{\partial x^\beta} \frac{\partial}{\partial a^\alpha}). \tag{3.127}$$

When all *these conditions are met, group* $\hat{G}(3.1)$ *is the* isotopic covering of Galilei's group, *herein after called the* Galilei-isotopic group.

A rather direct way of arriving at the covering relativity is the following [15]. When confronted with equations of motion violating Galilei's form-invariance, a frequent attitude is that of transforming the equations in a new coordinate system in which the applicability of familiar notions is recovered. It is intriguing to know that this is *always* possible. In fact, Theorem 6.2.1 of ref. [15] on the Indirect Universality of Hamilton's equations has the following consequence (which can be proved via the superposition of a Daurboux's and a canonical transformation).

Lemma 3.2. *Consider a non-self-adjoint and Galilei form-non-invariant system (3.94). Then a transformation always exists under which the transformed system is Galilei form- invariant.*

In particular, a transformation

$$a^\mu \to a^{*\mu}(a) \tag{3.128}$$

always exists under which the new system acquires the "free" structure

$$(\Gamma^{*\mu}(a^*)) = \begin{pmatrix} \mathbf{p}^*/m \\ 0 \end{pmatrix},$$

$$\Gamma'^{*\mu} = (\Gamma^\alpha \frac{\partial a^{*\mu}}{\partial a^\alpha})(a^*), \tag{3.129}$$

with consequential form-invariance under Galilei's group in the new coordinates

$$G(3.1): a^{*\mu} \rightarrow a^{*\prime\mu} = \exp(\theta^k \omega^{\alpha\beta} \frac{\partial X_k^*}{\partial a^{*\beta}} \frac{\partial}{\partial a^{*\alpha}}) a^{*\mu}. \quad (3.130)$$

However, this way of recovering Galilei's relativity is mathematically consistent but physically illusory. In fact, one of the uncompromisable conditions for the physical meaning of abstract mathematical algorithms is that they admit a realization in the frame of the experimental observation. It is easy to see that *the variables* $\mathbf{r}^*(\mathbf{r}, \mathbf{p})$ *and* $\mathbf{p}^*(\mathbf{r}, \mathbf{p})$ *in which symmetry (3.130) holds are generally nonrealizable experimentally.* In fact, the functional dependence of the new variables in the old is generally nonlinear, therefore implying the inability of setting measuring apparata along trajectories of the type $\mathbf{r}^* = \alpha \exp \beta \mathbf{r} \cdot \mathbf{p}$, etc.

This deficiency can be bypassed by transforming symmetry (3.130) from the mathematical coordinates $\mathbf{r}^*, \mathbf{p}^*$ to the original ones $\mathbf{r}, \mathbf{p}$ via the inverse $a^* \rightarrow a(a^*)$ of transformations (3.128). However, these transformations must be necessarily *noncanonical*, trivially, because the original vector field is non-Hamiltonian by assumption. One can then prove that, under such an inverse transformation, the conventional relativity (3.130) in mathematical coordinates transforms into the isotopic covering relativity in physical coordinates. In fact, under noncanonical transformations, Hamilton's equations transform into Birkhoff's equations; the conventional Poisson brackets transform into the generalized ones; and the conventional canonical realization of Galilei's group transforms exactly into form (3.123) according to the rules

$$\theta^k \omega^{\alpha\beta} \frac{\partial X_k^*}{\partial a^{*\beta}} \frac{\partial}{\partial a^{*\alpha}} \equiv \theta^k \Omega^{\alpha\beta}(a) \frac{\partial X_k}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}$$

$$\Omega^{\alpha\beta}(a) = \frac{\partial a^\alpha}{\partial a^{*\mu}} \omega^{\mu\nu} \frac{\partial a^\beta}{\partial a^{*\nu}},$$

$$X_k = X_k^*(t, a^*(a)). \quad (3.131)$$

We can therefore conclude by saying that Santilli's covering relativity emerges rather naturally, provided that excessive approximations of perpetual-motion-type are avoided, and the local variables permitted are restricted to be those of the experimenter.

3.3.7 Examples

We would like to review now a few specific examples.

The intriguing classical case of two particles was first identified in the original proposal for closed non-self-adjoint systems (ref. [2], pp. 622 ff), and submitted as a Newtonian limit of conceivable structure model of the neutral pion under deep mutual penetrations of the wavepackets of the constituents. The case was then studied again in additional papers (see, e.g., ref. [6]). The two-particle case was however put in a Birkhoffian/ Lie-isotopic form only recently by A. Jannussis, M. Mijatovic' and B. Veljanoski [93] who worked out also a constrained version of the three-body case. In the following we shall therefore review the main results of ref. [93].

It should be indicated here that, after "hadronization" (1.59) into their corresponding operator forms, the examples reviewed below constitute the classical foundations of the hadronic structure models of the light mesons [2] and of the neutron [25,28] with ordinary particles as physical constituents (§1.3). As a matter of fact, one of the objectives of the examples is that of identifying the generalized action (and its degrees of freedom) for which generalized quantization (1.59) is applicable, resulting in Schrödinger-isotopic equations of type (1.63). The classical examples under consideration here are therefore particularly important for applications to particle physics.

It should be indicated that Relativity 3.5 was called in ref. [93] the "Galilei-Santilli Relativity." We have submitted here the terms "Santilli's Galilean Relativity" to stress the truly profound mathematical, physical and epistemological (see below) differences between Galilei's and Santilli's relativities.

Consider the case of a closed non-self-adjoint two-particle system for which Eqs. (3.94.a) become

$$M\ddot{\vec{R}} = 0,$$
$$m\ddot{\vec{r}} = \vec{f}^{SA}(\vec{r}) + \vec{F}^{NSA}(\vec{r}, \dot{\vec{r}}, \ddot{\vec{r}}), \tag{3.132}$$

where

$$M = m_1 + m_2,$$
$$m = \frac{m_1 m_2}{m_1 + m_2},$$
$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \text{ and } \vec{r} = \vec{r}_1 - \vec{r}_2\ , \tag{3.133}$$

with the closedness conditions (3.97)

$$\vec{F}_1^{NSA} = -\vec{F}_2^{NSA} \stackrel{\text{def}}{=} \vec{F}^{NSA},$$

$$\dot{\vec{r}} \cdot \vec{F}^{NSA} = 0,$$
$$\vec{r} \times \vec{F}^{NSA} = 0. \tag{3.134}$$

The general solution of the above conditions is

$$\vec{F}^{NSA} = g[\vec{r}_1^{(2n)} - \vec{r}_2^{(2n)}],$$
$$n = 1, 2, \ldots, \tag{3.135}$$

where $g =$ const. and $\vec{r}^{(2n)}$ is 2n-th derivative of the relative coordinate. We shall take $n = 1$, i.e.,

$$\vec{F}^{NSA} = g(\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2). \tag{3.136}$$

Note that force (3.136) is non-Newtonian and that the only admissible orbit is the circle (think of gears turning one inside the other in which case the mutual orbit is circular indeed and no elliptic orbit is possible) [2]. The motion is then contained in a plane, as in the conventional Kepler case.

Further, we shall work in the Laboratory Frame. If we choose the Coulomb force

$$\vec{f}^{SA} = -\frac{k}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2), \tag{3.137}$$

where k is a constant of proportionality, the generalized two- body Kepler problem can be written for $n = 1$

$$m_1\ddot{\vec{r}}_1 = -\frac{m}{m-g}\frac{k}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2),$$
$$m_2\ddot{\vec{r}}_2 = \frac{m}{m-g}\frac{k}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2). \tag{3.138}$$

We can reduce equations (3.138) to the normal first-order form [2]

$$\begin{pmatrix} \dot{\vec{r}}_1 \\ \dot{\vec{r}}_2 \\ \dot{\vec{p}}_1 \\ \dot{\vec{p}}_2 \end{pmatrix} - \begin{pmatrix} \frac{\vec{p}_1}{m_1} \\ \frac{\vec{p}_2}{m_2} \\ -\frac{m}{m-g}\frac{k}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2) \\ \frac{m}{m-g}\frac{k}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2) \end{pmatrix} = 0\,,$$
$$\vec{r}_k = (x_k, y_k), \quad \vec{p}_k = (p_{xk}, p_{yk}), \quad k = 1, 2\,, \tag{3.139}$$

where the last expression represents the planarity of the motion.

We can identify the Birkhoffian representation if we assume [93]

$$B = \frac{m-g}{m}\frac{\vec{p}_1^{\,2}}{2m_1} + \frac{m-g}{m}\frac{\vec{p}_2^{\,2}}{2m_2} - \frac{k}{|\vec{r}_1 - \vec{r}_2|},$$

$$\{R_\mu\} = \{\frac{m-g}{m}\vec{p}_1, \frac{m-g}{m}\vec{p}_2, \vec{0}, \vec{0}\}, \tag{3.140}$$

with

$$a = (\vec{r}_1, \vec{r}_2, \vec{p}_1, \vec{p}_2). \tag{3.141}$$

The Lie-isotopic tensor is then given by the simple scalar isotopy

$$(\Omega_{\mu\nu}) = \frac{m-g}{m}(\omega_{\mu\nu}). \tag{3.142}$$

By using the Birkhoffian gauge transformation (ref. [15], p. 62)

$$R_\mu \to R'_\mu(t,a) = R_\mu(t,a) + \frac{\partial G(t,a)}{\partial a^\mu},$$

$$B \to B' = E_{tot} = B(t,a) - \frac{\partial G(t,a)}{\partial t}, \tag{3.143}$$

we have

$$B' = E_{tot} = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} - \frac{k}{|\vec{r}_1 - \vec{r}_2|},$$

$$G = -\frac{gt}{m}(\frac{\vec{p}_1}{2m_1} + \frac{\vec{p}_2}{2m_2}), \tag{3.144}$$

and

$$\{R'_\mu\} = \{\frac{m-g}{m}\vec{p}_1, \frac{m-g}{m}\vec{p}_2, -\frac{gt}{m}\frac{\vec{p}_1}{m_1}, -\frac{gt}{m}\frac{\vec{p}_2}{m_2}\}, \tag{3.145}$$

i.e.,

$$\Omega_{\mu\nu}\dot{a}^\nu = \frac{\partial B'}{\partial a^\mu} + \frac{\partial R'_\mu}{\partial t}. \tag{3.146}$$

We can now obtain the contravariant Lie-isotopic tensor, which results to be, in this case, of the simple form

$$(\Omega^{\mu\nu}) = (\Omega_{\mu\nu})^{-1} = \frac{m}{m-g}(\omega^{\mu\nu}). \tag{3.147}$$

The time component of Santilli's Galilean Relativity is then given by [93]

$$(a'^\mu) = \begin{pmatrix} \vec{r'}_1 \\ \vec{r'}_2 \\ \vec{p'}_1 \\ \vec{p'}_2 \end{pmatrix} = \begin{pmatrix} \vec{r}_1 + t'\frac{\vec{p}_1}{m_1} + \frac{t'^2\frac{m}{m-g}}{2!}\frac{A}{m_1} + \frac{t'^3\frac{m}{m-g}}{3!}\frac{c}{m_1} + \cdots \\ \vec{r}_2 + t'\frac{\vec{p}_2}{m_2} - \frac{t'^2\frac{m}{m-g}}{2!}\frac{A}{m_2} - \frac{t'^3\frac{m}{m-g}}{3!}\frac{c}{m_2} - \cdots \\ \vec{p}_1 + t'\frac{m}{m-g}A + \frac{t'^2}{2!}\frac{m}{m-g}C + \cdots \\ \vec{p}_2 - t'\frac{m}{m-g}A - \frac{t'^2}{2!}\frac{m}{m-g}C - \cdots \end{pmatrix} \tag{3.148}$$

where

$$A = -\frac{k}{|\vec{r}_1 - \vec{r}_2|^5}(\vec{r}_1 - \vec{r}_2)\ ,$$

$$C = -\frac{k}{|\vec{r}_1 - \vec{r}_2|^5}(\frac{\vec{p}_1}{m_1} - \frac{\vec{p}_1}{m_2}) + \frac{3k}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2)[(\vec{r}_1 - \vec{r}_2)(\frac{\vec{p}_1}{m_1} - \frac{\vec{p}_1}{m_2})]. \quad (3.149)$$

The form-invariance of the equation of motion can then be easily verified.

The Lie-isotopic rotation group $\widehat{SO}(2)$ can be constructed via the techniques of §3.2. Note that, since the term $m/(m-g)$ in generalized tensor (3.147) has a definite signature, $\widehat{SO}(2)$ is isomorphic or anti-isomorphic to $SO(2)$.

We should recall for the reader's convenience that, while the conventional rotational symmetry $SO(2)$ in a plane leaves invariant the familiar form $xx + yy = \text{inv.}$, the isotopic symmetry $SO(2)$ leaves invariant the form

$$xqx + yqy = \text{inv.}, \quad q = \frac{m}{m-g}. \quad (3.150)$$

The isomorphism between the rotational symmetry and its isotopic covering is therefore trivial in the two-body case (but not so from the three-body case on).

The construction of the remaining "components" of the Galilei-isotopic relativity is trivial for the two-body case and shall be left to the interested reader, jointly with the proof of its local isomorphism with the conventional components.

We consider now the three-body Kepler problem in the presence of non-self-adjoint internal forces which has been treated for the first time in ref. [93]. The equations of motion are:

$$m_1\ddot{\vec{r}}_1 = -\frac{m_1 m_2}{r_{12}^3}(\vec{r}_1 - \vec{r}_2) - \frac{m_1 m_3}{r_{13}^3}(\vec{r}_1 - \vec{r}_3) + \vec{F}_1^{NSA},$$

$$m_2\ddot{\vec{r}}_2 = -\frac{m_2 m_1}{r_{12}^3}(\vec{r}_1 - \vec{r}_2) - \frac{m_2 m_3}{r_{23}^3}(\vec{r}_2 - \vec{r}_3) + \vec{F}_2^{NSA},$$

$$m_3\ddot{\vec{r}}_3 = -\frac{m_3 m_1}{r_{13}^3}(\vec{r}_3 - \vec{r}_1) - \frac{m_3 m_2}{r_{23}^3}(\vec{r}_3 - \vec{r}_2) + \vec{F}_3^{NSA}, \quad (3.151)$$

where

$$r_{12} = |\vec{r}_1 - \vec{r}_2|, r_{13} = |\vec{r}_1 - \vec{r}_3|, \text{ and } r_{23} = |\vec{r}_2 - \vec{r}_3|. \quad (3.152)$$

From closedness conditions (3.97) we obtain for the components of the non-self-adjoint forces:

$$F_{2x} = D\frac{x_3 - x_2}{x_1 - x_3}F_{1x} ,$$

$$F_{3x} = (D\frac{x_2 - x_3}{x_1 - x_3} - 1)F_{1x} ,$$

$$F_{1y} = \frac{y_3 - y_1}{x_3 - x_1}F_{1x} ,$$

$$F_{2y} = D\frac{y_3 - y_2}{x_1 - x_3}F_{1x} ,$$

$$F_{3y} = (D\frac{y_2 - y_3}{x_1 - x_3} + \frac{y_3 - y_1}{x_1 - x_3})F_{1x} ,$$

$$F_{1z} = \frac{z_3 - z_1}{x_3 - x_1}F_{1x} ,$$

$$F_{2x} = D\frac{z_3 - z_2}{x_1 - x_3}F_{1x} ,$$

$$F_{3x} = (D\frac{z_2 - z_3}{x_1 - x_3} + \frac{z_3 - z_1}{x_1 - x_3})F_{1x} , \tag{3.153}$$

where

$$D = \frac{\frac{d}{dt}[(\vec{r}_1 - \vec{r}_3)^2]}{\frac{d}{dt}[(\vec{r}_2 - \vec{r}_3)^2]} , \tag{3.154}$$

and F_{1x} is arbitrary. Under the assumption that

$$F_{1x} = \frac{1}{2}(x_1 - x_3)(\frac{d}{dt}(r_2 - r_3)^2)F , \tag{3.155}$$

where F is an arbitrary function, formulae (3.153) take the following symmetrical expressions:

$$F_{1a} = (r^{1a} - r^{3a})(\dot{\vec{r}}_2 - \dot{\vec{r}}_3)(\vec{r}_2 - \vec{r}_3)F ,$$

$$F_{2a} = (r^{3a} - r^{2a})(\dot{\vec{r}}_1 - \dot{\vec{r}}_3)(\vec{r}_1 - \vec{r}_3)F ,$$

$$F_{3a} = -F_{1a} - F_{2a} ,$$

$$r^{ka} = (x_k, y_k, z_k), \quad a = x, y, z . \tag{3.156}$$

However, it is computationally quite elaborate to obtain the components of the Lie-isotopic tensor from this general form of non-self-adjointness. For this reason Jannussis, Mijatović and Veljanoski choose the following special case [93]:

$$\vec{F}_1^{NSA} = \gamma(\dot{\vec{r}}_2 - \dot{\vec{r}}_3) ,$$
$$\vec{F}_2^{NSA} = \gamma(\dot{\vec{r}}_3 - \dot{\vec{r}}_1) ,$$
$$\vec{F}_3^{NSA} = \gamma(\dot{\vec{r}}_1 - \dot{\vec{r}}_2) , \tag{3.157}$$

where γ is a resistive coefficient.

We can see that the above forces are Newtonian and non-self-adjoint and obey the closedness conditions (3.97.a) and (3.97.b). However, condition (3.97.c) leads to the *subsidiary constraint*

$$\vec{r}_1 \times \vec{r}_2 + \vec{r}_2 \times \vec{r}_3 + \vec{r}_3 \times \vec{r}_1 = \vec{c}, \tag{3.158}$$

where $\vec{c}$ is a constant vector. According to a definition given above, we are therefore dealing with a closed non-self- adjoint system of class γ.

The normal first-order form is [93]

$$\begin{pmatrix} \dot{\vec{r}}_1 \\ \dot{\vec{r}}_2 \\ \dot{\vec{r}}_3 \\ \dot{\vec{p}}_1 \\ \dot{\vec{p}}_2 \\ \dot{\vec{p}}_3 \end{pmatrix} - \begin{pmatrix} \frac{\vec{p}_1}{m_1} \\ \frac{\vec{p}_2}{m_2} \\ \frac{\vec{p}_3}{m_3} \\ -\frac{m_1 m_2}{r_{12}^3}(\vec{r}_1 - \vec{r}_2) - \frac{m_1 m_3}{r_{13}^3}(\vec{r}_1 - \vec{r}_3) + \gamma(\frac{\vec{p}_2}{m_2} - \frac{\vec{p}_3}{m_3}) \\ -\frac{m_2 m_1}{r_{12}^3}(\vec{r}_2 - \vec{r}_1) - \frac{m_2 m_3}{r_{23}^3}(\vec{r}_2 - \vec{r}_3) + \gamma(\frac{\vec{p}_3}{m_3} - \frac{\vec{p}_1}{m_1}) \\ -\frac{m_3 m_1}{r_{13}^3}(\vec{r}_3 - \vec{r}_1) - \frac{m_3 m_2}{r_{23}^3}(\vec{r}_3 - \vec{r}_2) + \gamma(\frac{\vec{p}_1}{m_1} - \frac{\vec{p}_2}{m_2}) \end{pmatrix} = 0. \tag{3.159}$$

If we choose

$$B = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + \frac{\vec{p}_3^2}{2m_3} - \frac{m_1 m_2}{r_{12}} - \frac{m_1 m_3}{r_{13}} - \frac{m_2 m_3}{r_{23}} = E_{tot} ,$$
$$\{R_\mu\} = \{\vec{p}_1 + \gamma\vec{r}_3, \vec{p}_2 + \gamma\vec{r}_1, \vec{p}_3 + \gamma\vec{r}_2, \vec{0}, \vec{0}, \vec{0}\} ,$$
$$a = (\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{p}_1, \vec{p}_2, \vec{p}_3) , \tag{3.160}$$

the Lie-isotopic tensor is given by

$$(\Omega^{\mu\nu}) = \begin{pmatrix} (0)_{9\times9} & & (1)_{9\times9} & \\ & 0_{3\times3} & \gamma(1)_{3\times3} & -\gamma(1)_{3\times3} \\ (-1)_{9\times9} & -\gamma(1)_{3\times3} & 0_{3\times3} & \gamma(1)_{3\times3} \\ & \gamma(1)_{3\times3} & -\gamma(1)_{3\times3} & 0_{3\times3} \end{pmatrix} . \tag{3.161}$$

The various components of Santilli's Galilean Relativity can then be computed explicitly (see ref. [93] for details).

The (local) isomorphism between the "time components" of the conventional and generalized relativities is, again, trivial. The isotope $\widehat{SO}(3)$ of the full rotation group can again be constructed from the knowledge of the generalized tensor, Eq. (3.161), and the techniques of §3.2.

Since the elements of the tensor have a topologically defined character (constants for each fixed γ), one can prove again the (local) isomorphism between $\widehat{SO}(3)$ and $SO(3)$. The nontriviality of the generalization, however, is now more transparent than in the simpler two-particle case. Unlike $SO(3)$, its covering $\widehat{SO}(3)$ leaves form-invariant an infinite- family of ellipsoids characterized by all possible values of γ. The form-invariance of the equation of motion under $\widehat{SO}(3)$ also holds, and its proof is left to the interested reader. The (local) isomorphism between the remaining components [those of "acceleration type"] is then expected from similar arguments.

These results illustrate the nontriviality of Santilli's covering relativity over the conventional one. In fact, the generators and, therefore, the physical conserved quantities remain the same. Nevertheless, in the transition from the conventional to the covering relativity we see the emergence of internal, non-self- adjoint nonhamiltonian forces which are rendered representable by structurally more general analytic, algebraic and geometrical formulations.

3.3.8 The Covering Lie-Admissible Formulations

The reader should be aware that the covering relativity reviewed here is only a *particular case* of that proposed in memoir [1], and worked out in more details in monograph [16], which is of the more general *Lie-admissible type*. The primary conceptual difference between these two relativities is the following. While the Lie-isotopic relativity is specifically constructed to represent *total conservation laws of closed/isolated systems*, the still broader relativity of Lie- admissible character is conceived to represent *time rates of variation of physical quantities for open systems* of extended particles under external contact, nonlocal and nonhamiltonian forces. In particular, while Lie-isotopic symmetries are used to represent conservation laws, the still more general Lie-admissible symmetries are used to represent time-rate-of-variations of physical quantities. The (mathematical and physical) covering nature of the latter over the former notions is evident. Regrettably, we cannot possibly review this still broader approach here (although we hope to do so in a separate review at some future time). The existence of the broader Lie-admissible relativity was used by Santilli to illustrate a central point: the lack of terminal character of physical theories, beginning with his own theories, no matter how broad they appear to be (see the hierarchy of conceivable relativities depicted in Fig. 15 of §3.5).

3.3.9 Epistemological Comments

We now pass to the review of certain epistemological aspects. At this point it becomes essential to avoid preconceived ideas, merely established because of their extended use rather than on true technical grounds.

The epistemological differences between Galilei's and Santilli's relativities are several and quite deep. We can consider here only a few. To begin, we must stress again the differences in physical attitudes. *When dealing with Galilei's Relativity, one customarily assumes first a basic symmetry, and then searches for physical systems that are compatible with that particular symmetry. In Santilli's Relativity this attitude must be reversed: one must first select a system of equations of motion as established by experimental or other information, and only then construct a relativity that is compatible with it.* The insistence in the former approach is so questionable, to have implications of scientific ethics, as it is the case when excessive approximations of physical reality are involved. In fact, the insistence on Galilei's Relativity as the sole possible relativity literally implies the acceptance of the perpetual motion in our environment. Santilli's position is quite firm on this [1]: any proposed generalization of Galilei's Relativity is evidently debatable as part of the essential scientific process of trial and error, but the *need* for a suitable generalization of Galilei's Relativity in Newtonian Mechanics must be simply out of the question.

A second aspect deserving a specific comment is the contemporary attitude of associating *only one symmetry* with each given relativity. This is certainly correct for the arena of applicability of conventional relativities (closed self-adjoint systems), but it is definitely erroneous for structurally more general systems (closed non-self-adjoint systems). In fact, the nonhamiltonian forces result in a generalization of the Lie product, and, in particular, of the basic tensor $\Omega^{\mu\nu}$ which characterizes the structure of the Lie-isotopic transformation group. Different nonhamiltonian forces then result into different tensors $\Omega^{\mu\nu}$ and, thus, different Lie-isotopic transformations.

It follows that, *while Galilei's relativity 3.2 characterizes only one symmetry, Santilli's covering relativity 3.5 characterizes an infinite family of covering symmetries all admitting Galilei's symmetry as particular case.*

This is another uncompromisable point, for the evident reason that, again, if one insists in selecting only one Lie-isotopic symmetry, excessive restrictions on the physical systems follow, with the consequential problems related to excessive approximations recalled earlier.

A further aspect where preconceived ideas may lead to misconceptions is the customary linear structure of relativity transformations in contemporary physics. *The abandonment of linearity in favor of nonlinear relativity transformations is another uncompromisable point for a more adequate representation of Nature.* In fact, the insistence in preserving linearity for all possible relativities of Newtonian mechanics directly implies, again, the acceptance of the perpetual motion in our environment. An inspection of the various examples of Lie-isotopic groups [1], [15] reveal that they are in fact, generally nonlinear. Santilli's Relativity 3.5 therefore characterizes generally nonlinear symmetry transformations. A most intriguing aspect is that *all these nonlinear transformations can be cast into an isotopically linear form* (§2.4), which is essentially achieved by incorporating all nonlinear terms in the isotopic unit, thus leaving the structure of the theory formally linear. The physical and mathematical implications of this property are also intriguing although they are more transparent in the operator formulation of the theory.

Still another aspect deserving a comment is the routine tendency to characterize relativities via the so-called *manifest symmetries* [15], i.e., symmetries that can be essentially identified with a visual inspection. This is of course the case for the simple systems of Galilei's Relativity. When considering physically more complex systems, this attitude too must be abandoned, again, as a condition for a more adequate representation of physical reality. In fact *the Lie-isotopic symmetries are, in general, nonmanifest.* This point was illustrated in the original proposal [1] by showing that some of the relativity transformations are so complex, to be characterized by transcendental functions.

The reader should keep in mind that the convergence of power- series expansions is established (under the assumed topological restrictions) for the isotopically lifted Poincaré-Birkhoff-Witt theorem (§2.2). As a result, all possible Lie-isotopic groups (3.123) admit convergent and explicitly computable, finite, transformations. Thus, *Santilli's methods always permit the explicit computation of the covering symmetry transformations, from the sole knowledge of the old transformations and of the generalized Lie tensor $\Omega^{\mu\nu}$ representing the nonhamiltonian forces.* The point is that the reader should not expect simple, easily computable symmetry transformations for rather complex physical systems.

Still another point deserving an epistemological comment is the vexing problem of inertial reference frames. As well known, contemporary relativities are specifically restricted to inertial frames. But these frames do not exist in our Earthly environment, nor are they expected to be available in the future, owing to the lack of inertial character of our Solar system as well as our Galaxy. Owing to this occurrence, *Santilli's Relativity is specifically conceived for noninertial reference frames, as stressed since the original proposal [1].* More specifically, Relativity 3.5 is restricted, by construction, to the actual reference frame $\vec{r}$ of the observer which is essentially noninertial. The covering relativity then maps noninertial frames into noninertial frames. This is another uncompromisable point for attempting a better representation of physical reality. In fact, the insistence in preserving inertial frames would imply, as a consequence, the admission of only linear transformations. In turn, this would imply again the acceptance of perpetual-motion approximations, thus preventing a more adequate representation of physical reality.

Numerous epistemological aspects (such as the apparent characterization of a privilege reference frame, that at rest with the medium in which motion occurs) will not be considered here because not yet sufficiently investigated in the current literature, to our best knowledge.

In summary, the assumption of the equations of motion as the fundamental quantities of the theory implies all the epistemological consequences considered here, such as: the need for an infinite family of relativity transformations one per each individual system; the intrinsic nonlinearity of the relativity transformations, although expressible in a formally isotopic-linear form; the general nonmanifest character of the relativity symmetries; and the intrinsically noninertial character of the covering theory.

It is remarkable that, despite all these profound differences, *Galilei's and Santilli's Relativities coincide at the abstract, coordinate-free level.* In fact, under the assumed topological restrictions, the Galilei group (3.89) and its covering (3.123) are locally isomorphic [15].

3.4 Lie-Isotopic Generalization of Einstein's Special Relativity

3.4.1 Introductory Remarks

The construction of the Lie-isotopic generalization of *Einstein's Special Relativity* is another central objective of Santilli's studies under the following major structural conditions:

- The generalized relativity should recover the Galilei- isotopic relativity (§3.3) under the nonrelativistic limit (or group contraction);
- The generalized relativity should be a covering of the conventional one in the sense identified earlier (see the end of §1.3); and, last but not least,
- The generalized relativity should be admitted, locally, by a conceivable Lie-isotopic generalization of Einstein's Gravitation (see next section).

The generalized relativity verifying the above conditions shall be called hereon *Santilli's Special Relativity* (or *Santilli's Isospecial Relativity* when emphasis is needed on its isotopic character). Its mathematical foundations are those submitted in memoir [1] of 1978, as reviewed in §2. The physical foundations are essentially a relativistic generalization of the Galilean one, also submitted in memoir [1]. The generalized relativity was formally submitted in a paper of 1983 [18], following the completion of the studies on: the space-time formulation of the Lie-isotopic symmetries (§2.4); the isotopic generalization of the group of rotations (§3.2); and the isotopic generalization of the Galilei Relativity (§3.3).

Important foundations of the generalized relativity are also submitted in paper [14] and monograph [16] which preceded ref. [18]. The Isospecial Relativity then reached its final form for classical non-linear, non-Hamiltonian and non-local systems in memoir [24c]. Additional aspects are studied in paper [27]. Operator formulations have been presented in Refs. [25], [27] and [28].

Of utmost importance for the new relativity is Theorem 2.9 (which is indeed quoted in page 549 pf ref. [18]). In fact, the generalized relativity is ultimately a realization of this theorem, as the reader will see.

To emphasize the speculative nature of the studies, the reader should be aware that the physical departures of Santilli's from Einstein's Special Relativity are rather deep, inasmuch as each and every law of the old relativity is replaced with a covering law. As an illustration, the new relativity predicts the existence of physical conditions (within hyperdense hadronic matter) under which massive, physical, ordinary particles can (locally) attain speeds higher than that of light in vacuum (hereinafter indicated with c_o).

To emphasize the thrilling aspect of the covering relativity, the reader should be equally aware that, after careful examination, we have found no experimental, phenomenological or other evidence capable of disproving the novel predictions. On the contrary, all available phenomenological information (e.g., that on the anomalous dependence of the mean life of unstable hadrons with speed) appear to confirm the novel predictions quite clearly, including that of causal physical speeds higher than c_o. The predictions had simply escaped other research on Lorentz noninvariance because of the lack of rigorous mathematical tools capable of constructing a covering relativity.

Needless to say, the resolution of the validity or invalidity of the new relativity will occur at some future time via direct experiments on fundamental space-time symmetries. The need for conducting these crucial tests, which have been proposed since quite some time but essentially ignored until now, will be stressed at the end of this chapter (§3.5.18).

A true understanding and appraisal of the new relativity requires the mind to be free of preconceived ideas, essentially established by prolonged use, rather than real physical support. In approaching Santilli's Special Relativity, the reader is urged to abandon the central physical arena of Einstein's Special Relativity (motion of point-like particles in vacuum), in favor of a much more complex physical reality (e.g., extended wavepackets moving within hyperdense media composed of wavepackets of other particles). No experimental, theoretical or epistemological information accumulated throughout this century on Einstein's Special Relativity is therefore applicable to Santilli's much more complex physical setting. New studies, specifically tailored for the new relativity, must therefore be conducted.

During the preparation of this review, we had access to the files of the Institute for Basic Research in Cambridge, Massachusetts, which include a number of virtually completed, yet unsubmitted manuscripts by Santilli following works [14], [18]. In fact, manuscripts [24–28], available since 1985, were released for printing in conjunction with this review. It is appropriate here to stress that this section contains no new results besides those already published in the quoted literature. We were authorized to use the unpublished manuscripts only to gain insights for a more mature presentation of published material.

Owing to the novelty of the new relativity, and despite a number of independent contributions that have already appeared in the literature (reviewed later on), a number of truly intriguing and fundamental problems remain open to this writing at the classical level (let alone the corresponding operator level for particle physics), such as: the proof that Santilli's Special Relativity recovers the Galilei-isotopic relativity under the nonrelativistic limit; the construction of the representation theory of the Lorentz-isotopic group (only the fundamental representation has been achieved until now); the isotensorial products of these isorepresentations for the treatment of composite systems; etc.

This review will achieve a primary objective if it succeeds in stimulating this much needed independent research.

3.4.2 Foundations of Einstein's Special Relativity

As clearly stated in the historical contributions by Lorentz, Poincaré, Einstein, Minkowski, and others (see, e.g., ref. [94] and quoted historical literature), the body of formulations today known as *Einstein's Special Relativity* was conceived for the description of:

1. particles which can be effectively considered as being point- like,
2. while moving in vacuum (empty space) conceived as homogeneous and isotropic; and
3. under the conditions that the setting is classical (i.e., the action $A \gg \hbar$) and gravitational effects are ignorable(i.e., the space has null curvature).

The above conditions clearly include the electromagnetic interactions of charged particles in vacuum, as well as a vast number of other cases of physical relevance.

The relativity is based on the form-invariance of the following separation in Minkowski space $M(x, \eta, \mathbf{R})$

$$\begin{aligned}
\eta &= (\eta_{\mu\nu}) = \mathrm{diag}(1,1,1,-1) , && \text{(3.162.a)}\\
x^2 = x^t \eta x &= x^\mu \eta_{\mu\nu} x^\nu = x^1x^1 + x^2x^2 + x^3x^3 - x^4x^4 , && \text{(3.162.b)}\\
c_o &= \text{speed of light in vacuum} , && \text{(3.162.c)}\\
x^4 &= c_o t, \quad \mu, \nu = 1,2,3,4 , && \text{(3.162.d)}
\end{aligned}$$

under the largest possible group of *linear* transformations. This yields the celebrated *Lorentz transformations*, e.g., for motion along the third space component

$$x^{1\prime} = x^1 ,$$

$$x^{2\prime} = x^2 ,$$

$$x^{3\prime} = \gamma(x^3 - vt), \quad \gamma = (1-\beta^2)^{-1/2}, \quad \beta = v/c_o ,$$

$$x^{4\prime} = \gamma(x^4 - \beta x^3/c_o) . \tag{3.163}$$

The relativity constructed via Lorentz transformations characterizes well known physical laws, such as: the relativistic composition of speeds

$$V_{tot} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c_o^2}}, \tag{3.164}$$

with consequential impossibility for causal physical signal and/or processes to exceed the speed of light in vacuum (under conditions 1, 2, 3 above); the constancy of c_o for all observers; the time dilatation

$$\Delta t = x_2^{4\prime} - x_1^{4\prime} = \gamma \Delta t_o = \frac{x_{20}^4 - x_{10}^4}{(1-\beta^2)^{1/2}}; \tag{3.165}$$

the Lorentz contraction

$$\Delta l = x_2^3 - x_2^3 = \sqrt{1-\beta^2}\Delta l_o = \sqrt{1-\beta^2}(x_{20}^3 - x_{10}^3) , \tag{3.166}$$

the Doppler's effect and related aberration

$$\omega' = \omega\gamma(1 - \beta\cos\alpha) ,$$

$$\cos\alpha' = (\cos\alpha - \beta)(1 - \beta\cos\alpha) ; \tag{3.167}$$

and other laws.

Owing to incontrovertible experimental confirmations, *Santilli [18] assumed that Einstein's Special Relativity is exact under conditions 1, 2, and 3 above.* The same assumption is evidently embraced in this review.

3.4.3 Survey of Lorentz Noninvariance Research

Ref. [18] begins with a review of independent research on conceivable conditions under which the conventional Lorentz symmetry is not expected to be *exact.* The understanding tacitly assumed hereon is that its *approximate* character remains out of the question.

Authoritative doubts on the exact validity of the Lorentz symmetry under physical conditions different than those conceived by Lorentz, Poincaré, and Einstein have been expressed since the early part of this century. For instance, in regard to the interior of strongly interacting particles, Fermi [95] clearly expressed in 1949

> *"doubts as to whether the usual concepts of geometry hold for such small region of space."*

The legacy of Fermi and other Fathers of contemporary physics was based on the expected *nonlocal* nature of the strong interactions (§1.3) which implies a breakdown of the mathematical foundations of the Lorentz symmetry (e.g., its topology), let alone its physical properties.

The above legacy remained unanswered for decades, until systematic and quantitative studies were initiated in the '60s.

Consider an unstable hadron moving in a particle accelerator. Its center-of-mass motion must strictly obey Einstein Special Relativity because motion occurs in vacuum under long range electromagnetic interactions. The actual size of the hadron is therefore ignorable and all Einstenean conditions 1, 2 and 3 (§3.4.2) are met.

Deviations from the special relativity (and the Lorentz symmetry) are conceivable only in the *interior* of the particle. One of the most direct ways in which such possible interior deviations can manifest themselves to the outside is via deviations from the prediction of Einstein Special Relativity regarding the behavior of the mean life τ with the speed of the hadron, i.e., via deviation from the Einstenian law originating from Eq. (3.165)

$$\tau = \tau_o \gamma; \quad \gamma = \sqrt{1 - v^2/c_o^2}. \tag{3.168}$$

The initiation of quantitative *phenomenological* studies on the above *"Lorentz noninvariance"* are usually associated in the literature with the research by Blockhintsev [96], Redei [97], and others who suggested a modification of law (3.168) of the type

$$\tau = \tau_o \gamma (1 + 10^{25} \gamma^2 a_o^2) , \tag{3.169}$$

where a_o is a fundamental length.

Numerous additional studies followed along similar lines in various branches of physics. For instance, Kim [98] provided, via the use of quantum field theory, specific percentage predictions of deviations from law (3.168) at a number of different speeds.

A considerable phenomenological study was conducted by Nielsen, Picek and others (see ref. [99] and quoted papers) for the weak decay of hadrons within the context of unified gauge theories. In these studies deviations from the Lorentz symmetry occur in the Higgs sector of spontaneous symmetry breaking. The use of available experimental information then leads to the following modification of the Minkowski metric [*loc. cit*]

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu} = \eta_{\mu\nu} - \chi_{\mu\nu},$$

$$\chi = \mathrm{diag}(\frac{1}{3}\alpha, \frac{1}{3}\alpha, \frac{1}{3}\alpha, \alpha), \tag{3.170}$$

with generalized mean life

$$\tau = \tau_o \gamma (1 + \frac{4\alpha\gamma^2}{3}), \tag{3.171}$$

where the Lorentz asymmetry parameter α assumes for *pions* the value

$$\alpha = (-3.79 \pm 1.37) \times 10^{-3}, \tag{3.172}$$

and for *kaons*

$$\alpha = (0.61 \pm 0.17) \times 10^{-3}, \tag{3.173}$$

with weighted average

$$\alpha = (0.54 \pm 0.17) \times 10^{-3}. \tag{3.174}$$

A first experimental study regarding the anomalous energy-dependence of the mean-life as well as of other parameters of the $K^o - \overline{K}^o$ system was conducted by Aronson *et al.* [100]. The data which were obtained from a series of regeneration experiments at Fermilab (in the energy range $E_K = 30 - 100\mathrm{GeV}$) specifically indicate that the values of the mass difference $\Delta m = m_L - m_S$, the lifetime τ_S, the CP-violation parameters $|\eta_{+-}|$ and $\tan\phi_{+-}$ as determined in the $K^o - \overline{K}^o$-system rest frame, depend on the velocity of this rest frame with respect to the laboratory. The authors arrived at the conclusion that the experimental results, if correct, cannot be ascribed to an interaction of kaons with an electromagnetic, hypercharge, or gravitational field, or to the scattering of kaons from stray charges or cosmic neutrinos. In order to describe the anomalous behavior of these four parameters, denoted by χ, they introduced the slope parameters $b_\chi^{(N)}$ defined by

$$\chi = \chi_0(1 + b_\chi^{(N)}\alpha^N),$$
$$\alpha = E_K/m, \quad N = 1, 2, \tag{3.175}$$

and presented an elaborated analysis of the origin of these $b_\chi^{(N)}$. We note that Eq. (3.175) exhibits in fact, up to a factor γ, Blokhintsev-Redei-like behavior as it was described earlier for the lifetimes of unstable particles, Eq. (3.169).

A second experimental study was conducted by Grossman *et al.* [10] who verified the Einsteinian behavior of the $K^o - \overline{K}^o$ system for energies from 100 to 400 GeV, in disagreement with the results of the preceding experiment [100]. However, experiments [100] and [101] refer to different ranges of energies (30-100 GeV for the former and 100 to 400 GeV for the latter). As a result, none of them can disprove the other. Also, experiment [101] was conducted under considerable theoretical assumptions in the data elaboration, such as the assumption of a rest frame in which there is no CP violation, in disagreement with the results by Kim [98].

The only possible scientific conclusion at this time is therefore that the situation is unsettled, and the problem of the behavior of the meanlife of unstable hadrons with speed fundamentally open on both theoretical and experimental grounds, and that it will remain unsettled until resolved by a comprehensive series of experiments covering the entire range of energies from 30 to 400 GeV and more, as well as of more direct type without excessive theoretical assumptions in the data elaboration.

The preceding phenomenological papers [96–101] were studied by Cardone, Mignani and Santilli [102] who reached the following results:

1. The available phenomenological data on the behavior of the meanlife with speed *exclude* rather convincingly that the geometrization of the interior of hadrons can be exactly done via constants. Generalized metrics such as those by Nielsen and Picek [99] must therefore be solely considered as a first approximation.

2. The generalized metric apparently holding in the interior of hadrons is expected to have a *nonlinear dependence on the velocities* exactly as stated in ref. [100]. Moreover, such a nonlinear dependence should be

considered, in turn, as a mere *approximation of the expected, ultimate, nonlocal character of the structure of hadrons*, as originally stressed in ref. [2].

3. If, indeed, properties 1) and 2) above are confirmed by future data, they may imply a reinspection of the meanlife of unstable hadrons at rest, as currently provided by the Particle Data Group. In fact, these meanlifes *are not* measured at rest, but at different speeds. The meanlifes are then scaled down to at rest condition, but under the *assumption* that their behavior with speeds is exactly Einsteinian. Deviations from this Einstenian character then imply a revision of the meanlifes at rest currently known. In particular, available phenomenological data appear to support an *increase* of the mainlifes of current mainlifes of the Particle Data Group with about 60% probability.

Cardone, Mignani and Santilli [102] then concluded their studies by showing the *compatibility* of the seemingly discordant measures [100],[101] via the representation of the K^o particle as an isotopic minkowski space, thus providing additional motivations for the open character of the problem, and for the need of additional, comprehensive tests.

In regard to *theoretical* studies on Lorentz asymmetry, the literature is rather vast indeed and only a few representative contributions can be indicated here.

Gasperini has conducted a number of investigations such as: the ultra-relativistic particle motion within the context of gauge theories, with local broken gauge symmetry [103]; the possible breaking of the Lorentz symmetry in the very early stages of the universe [104]; the possible origin of Lorentz asymmetry from strong gravity [105] (see also papers [106]); besides specific studies via Santilli's Lie-isotopic (and Lie-admissible) techniques we shall review later on.

The conceivable Lorentz noninvariance of the primordial fluid was also studied by Rosen [107].

Ellis *et al.* [108], Zee [109] and others have studied the hypothesis of a possible decay of the proton from the viewpoint of Lorentz noninvariance within the context of grand unified theories. In particular, these authors have essentially confirmed Fermi's statement of some four decades earlier to the effect that in the small region in the interior of the proton "anything" can happen.

Aringazin and Asanov [110] have studied the gravitational and other consequences for a possible, local, Lorentz noninvariance from the viewpoint of the *Finsler geometry* [111, 112].

In regards to efforts for the construction of a possible generalization of Einstein Special Relativity besides those of ref. [18], the most notable theory is provided by *Bogoslovski's Special Relativity* [113], which is based on the following Finslerian generalization of the Minkowski metric for homogeneous but anisotropic spaces

$$x^\mu g_{\mu\nu} x^\nu = x^\mu(-\nu^\alpha \eta_{\alpha\beta} x^\beta)^2/(-x^\rho \eta_{\rho\sigma} x^\sigma)^r \eta_{\mu\nu} x^\nu, \qquad (3.176)$$

where $(\nu^0) = (\nu_{0,1}), \nu^2 = 0$ is a vector along the direction of anisotopy and r is a scalar parameter. Bogoslowsky's generalization of the Lorentz transformations are given by expressions of the type

$$\begin{aligned} x^{1\prime} &= x', \\ x^{2\prime} &= x^2, \\ x^{3\prime} &= \tilde{\gamma}(x^3 - \beta x^4), \quad \beta = v/c_0 , \\ x^{4\prime} &= \tilde{\gamma}(x^4 - \beta x^3), \end{aligned} \qquad (3.177)$$

where the new parameter

$$\beta = \nu/c_o , \quad \tilde{\gamma} = \gamma[1 - v/c_o/(1 + v/c_o)]^{r/2} , \qquad (3.178)$$

characterizes the Lorentz asymmetry.

In this way, Bogoslovski constructed a bona-fide generalization of the Lorentz group, although the methods were those of the conventional Lie's theory, and the relationship to the Lorentz group remained unknown.

Yet another generalization of the Lorentz transformations is that provided by Edwards [114] and, independently, by Strel'tsov [115], which can be written

$$\begin{aligned} x^{1\prime} &= x^2, \\ x^{2\prime} &= x^2, \\ x^{3\prime} &= \gamma\{[1 + \frac{1}{2}(\frac{1}{c_1^o} - \frac{1}{c_2^o})v]x^3 - vx^4\}, \\ x^{4\prime} &= \gamma\{[1 + \frac{1}{2}(\frac{1}{c_1^o} - \frac{1}{c_2^o})v]x^4 - \frac{v}{c_1^o c_2^o}x^3\}, \quad x^4 = t , \end{aligned} \qquad (3.179)$$

with related invariant

$$x^\mu g_{\mu\nu} x^\nu = x^1 x^1 + x^2 x^2 + x^3 x^3 - x^4[c_1^o c_2^o - (\frac{c_2^o}{c_1^o} - \frac{c_1^o}{c_2^o})\frac{x^3}{x^4}]x^4, \qquad (3.180)$$

where γ has the conventional value, and c_1^o, c_2^o represent the speeds of light in opposite space directions.

The *Edwards-Strel'tsov transformations* are clearly based on a possible anisotopy of time and recover the conventional Lorentz transformations for $c_1^o = c_2^o = c_o$.

A comprehensive presentation of the above (and other) topics can be found in a recent monograph by Logunov [116]. More recent research by Strel'tsov can be found in ref. [117], which include an extension of the anisotopy to the space components. Further work on anisotropy deserving a mention is that by Ikeda [118].

The above outline of research on Lorentz non-invariance (outside Lie-isotopic studies), even though far from complete, is sufficient for the scope of this book. In fact, as we shall see, *all the models reviewed in this section (and more) shall result to be particular cases of Santilli's Special Relativity* [18], trivially, because of the arbitrariness of the generalized metric g appearing in the isounit $\hat{I} = g^{-1}$, as shown by Aringazin [119].

The objective of ref. [18] was, however, not limited to the construction of a covering relativity that could unify all available research. An additional objective was to prove that, *under suitable topological restrictions, the Lorentz symmetry can be proved to be still exact, of course, when realized at the covering isotopic level.*

We shall now enter into our presentation of the new relativity beginning with the arena of its physical applicability. We shall then review the generalizations of the Minkowski space identified in ref. [18] and subdivide them into three classes owing to the variety of physical possibilities. A review of the new relativity will then follow.

3.4.4 Arena of Applicability of the Generalized Relativity

The earlier, well written, treatises on Einstein's Special Relativity stressed explicitly its conception and limited applicability to point-like particles (see, e.g., the *title* of Chapter VI of ref. [120]). Unfortunately, this sound scientific attitude was terminated in more recent times, perhaps because of the overwhelming successes of the relativity for electromagnetic interactions.

In a series of articles [1], [2], [3], [4] (as well as in monograph [5]), Santilli brought back to the attention of the physical community the intrinsic limitations 1 and 2 of §3.4.2 of Einstein's Special Relativity, and the existence of physical conditions beyond those of the original conceptions, under which the applicability of the relativity is questionable.

By continuing his studies on the Galilean setting, Santilli [18] submitted a generalization of Einstein Special Relativity for the description of closed-isolated systems of:

1' extended-deformable particles which cannot be effectively approximated as being point-like;

2' when moving in a physical medium which is generally inhomogeneous and anisotropic;

3' under the condition that quantum mechanical effects are ignorable ($A \gg \hbar$), and gravitational profiles are absent (null curvature);

under the further condition that the generalized relativity is a covering of the conventional one, i.e., it recovers the latter identically when physical conditions 1'), 2') and 3') above recover 1), 2) and 3) of §3.4.2.

As the reader can see and as expected, conditions 1'), 2') and 3') above are a relativistic generalization of conditions 1'), 2') and 3') of §3.3.3 for the Galilean framework. They have been specifically and primarily conceived for the representation of hadrons as closed-isolated systems of extended-deformable particles whose constituents possess extended wavepackets moving within a medium composed of other wavepackets (the "hadronic medium" [2]). Nevertheless, conditions 1'), 2') and 3') above apply also to a variety of classical cases such as: motion of light in liquids (Cherenkov light); motion of charged particles in metals (e.g., the motion of electrons in metals, possibly along Graneau's [121] formulation of the Ampère-Newman electrodynamics); interior problems of planets (e.g., Jupiter) with locally varying angular momentum and other physical quantities; redshift of light propagating within inhomogenous and anisotopic media (as existing around quasars); etc.

The reader should be aware that Santilli conceived his Lie-isotopic relativity specifically for *closed-isolated* systems. This is a consequence, on one side, of assuming the total physical energy H as the generator of the time evolution (as in Einstein's case) and, on the other side, of the Lie character of the theory, that is, of the antisymmetry of the Lie-isotopic product $[A\hat{,}B] = -[B\hat{,}A]$. Under these conditions, the only possibility for the total energy is that of being conserved according to the familiar rule

$$i\dot{H} = [H\hat{,}H] \equiv 0, \quad H = T + V. \tag{3.181}$$

For the case of systems that are *open*, for which $\dot{H} = f(t) \neq 0$, Santilli submitted in the final part of monograph [16] a further generalization of his Lie-isotopic relativity, this time of Lie-admissible character with product $(A\hat{,}B) = ARB - BSA$ which is neither symmetric nor antisymmetric, $(A\hat{,}B) \neq \pm(B\hat{,}A)$. In this case the total energy can indeed be the generator of the time evolution as well as be nonconserved

$$i\dot{H} = (H\hat{,}H) = f(t) \neq 0. \tag{3.182}$$

The covering Lie-admissible relativity reduces to the Lie-isotopic one under the condition

$$(A\hat{,}B)_{\dot{H}=0} \equiv [A\hat{,}B]. \tag{3.183}$$

This review is restricted to the Lie-isotopic case. The reader should be aware that, as it is the case for the Birkhoffian mechanics [15], *the Lie-isotopic theory can also represent open systems.* In this case the generator of the time evolution is the *Birkhoffian* $B \neq H$ with rule

$$i\dot{H} = [H\hat{,}B] = f(t) \neq 0. \tag{3.184}$$

A knowledge of these structural foundations is essential for a true understanding of the following review, and will be tacitly assumed hereon.

Notice, as stressed earlier, that the space (empty space) remains perfectly homogeneous and isotropic. The fundamental inhomogeneity and anisotropy of Santilli's Relativities originates from the physical medium in which motion occurs.

3.4.5 Isotopic Generalizations of the Minkowski Space

The next step of ref. [18] is the construction of suitable generalizations of the Minkowski space capable of: a) representing the generally inhomogeneous and anisotropic character of the theory; b) admitting the conventional Minkowski space as a particular case; and c) allowing a formally isolinear theory while the underlying transformations are intrinsically nonlinear.

From hereon, we shall call *Santilli's spaces* all generalizations of the Minkowski space obeying conditions a), b) and c) above. Due to the large variety of admitted cases, these spaces will be divided below into three classes of increasing complexity and methodological needs.

The main idea of ref. [18] is that, in the transition from empty space to a physical medium, the Minkowski metric $\eta_{\mu\nu}$ is generalized ("mutated" [16]) in a form $g_{\mu\nu}$ verifying conditions a) and b) above. The generalized metric $g_{\mu\nu}$ is assumed to be Hermitean, nonsingular and sufficiently smooth but otherwise with an arbitrary dependence in all needed local quantities, such as: space-time coordinates x and velocities $\dot{x}$ (see below); index of refraction n; density μ; temperature τ; etc.

$$g_{\mu\nu} = g_{\mu\nu}(x; \dot{x}; n; \mu; \tau; ...). \qquad (3.185)$$

which can be interpreted as a geometrization of the physical properties inherent to the medium considered.

The Hermiticity and smoothness of $g_{\mu\nu}$ implies the existence of its reduction to the canonical (diagonal) form

$$g = \text{diag}(g_{11}, g_{22}, g_{33}, g_{44}) \stackrel{\text{def}}{=} T\eta \stackrel{\text{def}}{=} \theta\eta\theta \ , \qquad (3.186)$$

which is the only one considered in ref. [18] as well as in this review.

The condition of nonsingularity implies the existence everywhere of the inverse T^{-1} which, as now familiar, is the generalized unit $\hat{I} = T^{-1}$ of the theory. The form $g = \theta\eta\theta$ shall be of use in gravitational studies (§3.5).

We remain with the central condition c) of achieving isolinearity. This is achieved in ref. [18] via the techniques of "hadronic mechanics" (§1.3). Let $\mathbf{R}$ be the field of real numbers and let $M(x, \eta, \mathbf{R})$ be the Minkowski space.

> *DEFINITION 3.6 [18]:* Santilli's spaces $\hat{M}(x, g, \hat{\mathbf{R}})$ *are given by all possible isotopes of the Minkowski space* $M(x, \eta, \mathbf{R})$ *where: the space-time coordinates* x *remain unchanged; the metric* η *is generalized into Hermitean, nonsingular and sufficiently smooth, but otherwise arbitrary forms* g *with a dependence on all needed local quantities* $g = g(x; \dot{x}; n; \mu; T; ...)$*; and* $\hat{\mathbf{R}}$ *is the isotope of* $\mathbf{R}$ *characterized by (see Eq. (1.38) for the complex case)*
>
> $$\hat{\mathbf{R}} = \{\hat{N} | \hat{N} = N\hat{I}, N \in \mathbf{R}, \hat{I} = T^{-1}\}. \qquad (3.187)$$

The lifting $\mathbf{R} \rightarrow \hat{\mathbf{R}}$ allows the achievement of isolinearity, as per condition c) above (§2.4). In fact, the linear transformations

$$M(x, \eta, \mathbf{R}) : x' = Ax, \qquad (3.188)$$

are now lifted into the isotransformations

$$\hat{M}(x, g, \hat{\mathbf{R}}) : x' = A * x \stackrel{\text{def}}{=} ATx, \qquad (3.189)$$

which are formally linear, yet intrinsically nonlinear because of the general dependence

$$A * x = AT(x; \dot{x}; , ...)x \ . \qquad (3.190)$$

Scalar values on $M(x,\eta,\mathbf{R})$ are in $\mathbf{R}$,

$$M(x,\eta,\mathbf{R}) : x^2 = x^\mu \eta_{\mu\nu} x^\nu \stackrel{\text{def}}{=} x^t \cdot x \in \mathbf{R}\ , \tag{3.191}$$

while scalar values on $\hat{M}(x,g,\hat{\mathbf{R}})$ are in $\hat{\mathbf{R}}$

$$\hat{M}(x,\eta,\hat{\mathbf{R}}) : x^{\hat{2}} = x^\mu g_{\mu\nu} x^\mu \hat{I} \stackrel{\text{def}}{=} x^t \circ x \in \hat{\mathbf{R}}\ . \tag{3.192}$$

It is this interplay between the isotopic transformation theory and isoscalars that ensures isolinearity. For details, the reader is recommended to consult Myung and Santilli [36].

It is intriguing to note that, *without a lifting of the field* $\mathbf{R} \rightarrow \hat{\mathbf{R}}$ *(jointly with that of the Minkowski space and of the Lie product), the generalized relativity of ref. [14] would have been mathematically inconsistent.*

In practical calculations, the lifting $\mathbf{R} \rightarrow \hat{\mathbf{R}}$ can be ignored, as it is the case for hadronic mechanics, because of a property similar to Eq. (1.40) where the measured numbers are the conventional ones. In fact, by keeping into account the multiplication in $\hat{\mathbf{R}}$

$$\hat{N}_1 * \hat{N}_2 \stackrel{\text{def}}{=} \hat{N}_1 T \hat{N}_2 = N_1 N_2 \hat{I} = \widehat{N_1 N_2}, \tag{3.193}$$

the scalar action $\hat{\mathbf{R}} * \hat{M}$ coincided with the conventional one $\mathbf{R} \times M$

$$\hat{N} * x \equiv Nx. \tag{3.194}$$

After clarifying the above mathematical structures, ref. [18] makes certain assumptions that are embraced hereon. In essence, we shall deal with three quantities:

A–*Fourvectors.* Their components are the same as those in M, but their scalar value is given by the contraction in $\hat{M}$, i.e., $x^2 = x^\mu g_{\mu\nu} x^\mu$, with the clear understanding that the correct form is (3.192). The terms *isofourvector* or *isocoordinates* shall be sometimes used to prevent confusion with the conventional case.

B–*Threevectors.* Their components are the same as those in the isotope $\hat{E}(\vec{r},g,\hat{\mathbf{R}})$ of the Euclidean space $E(\vec{r},\delta,\mathbf{R})$ used for the isorotation theory (§3.2) and the generalized Galilean relativity (§3.3). The contraction ("square") of three vector is then given by $\vec{r}^{\hat{2}} = r^i g_{ij} r^j$ with the understanding that a law of type (1.187) is more rigorous. Again the term *isovector* may be occasionally used to stress the departure from conventional Euclidean space.

C–*Scalars.* These are ordinary numbers $N \in \mathbf{R}$, with the understanding that a more rigorous form is that of Equation (3.187). We shall at times call an *ordinary* number an *isoscalar* to stress the tacit assumption of structure (3.187).

Santilli's spaces as per Definition 3.6 above are rather numerous indeed. We shall therefore introduce the following classification.

DEFINITION 3.7: Santilli's spaces $\hat{M}(x, g, \hat{\mathbf{R}})$ are classified into

- **Spaces of Class I,** *denoted $\hat{M}_I(x, g, \hat{\mathbf{R}})$, when the metric g preserves the topological properties of the Minkowski space, i.e., it is of the particular type*

$$g = \mathrm{diag}(b_1^2, b_2^2, b_3^2, -b_4^2), \quad T = \mathrm{diag}(b_1^2, b_2^2, b_3^2, b_4^2)\ ,$$

$$b_\mu^2 > 0, \quad \mu = 1, 2, 3, 4\ , \tag{3.195}$$

and the space has null curvature, i.e., the Christoffel symbols *of the second kind are identically null*

$$\Gamma^\rho_{\mu\nu} \stackrel{\text{def}}{=} \frac{1}{2} g^{\rho\sigma}(g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}) \equiv 0,$$

$$g_{\mu\sigma,\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\nu}, \quad \mu, \nu, \rho = 1, 2, 3, 4; \tag{3.196}$$

- **Spaces of class II,** *denoted $\hat{M}_{II}(x, g, \hat{\mathbf{R}})$ when they are still flat, i.e.,*

$$\Gamma^\rho_{\mu\nu} \equiv 0, \rho = 1, 2, 3, 4, \tag{3.197}$$

but the generalized metric g loses, in general, the topological properties of the Minkowski metric; and

- **Spaces of Class III,** *denoted $\hat{M}_{III}(x, g, \hat{\mathbf{R}})$ when they are curved, i.e.,*

$$\Gamma^\rho_{\mu\nu} \neq 0. \tag{3.198}$$

For the purpose of achieving a covering of Einstein's Special Relativity, Santilli restricted the presentation of ref. [18] to spaces of the first class, $\hat{M}_I(x, g, \hat{\mathbf{R}})$. In fact, the reader now familiar with the Lie-isotopic theory can expect that the assumption of spaces $\hat{M}_I(x, g, \hat{\mathbf{R}})$ assures the admission of the conventional theory as a particular case, while the Lie-isotopic covering of the Lorentz symmetry is expected to be isomorphic to the conventional one (see below).

Intriguingly, the spaces $\hat{M}_I(x, g, \hat{\mathbf{R}})$ are sufficient to unify *all* research on Lorentz noninvariance reviewed in §3.4.3, as we shall see.

The reader should be aware that the Lie-isotopic generalization of the Lorentz symmetry holds also for an unrestricted metric g, including the case when g is Riemannian or of more general gravitational nature.

An advance knowledge of this point is essential for the reader's understanding of the continuity of thought in the transition from the relativistic framework of this section to the gravitational context of the next section. Note that the contact interactions due to motion in resistive media are generally independent of the coordinates x, but dependent on the velocities $\dot{x}$ and other quantities. A considerable class of physical conditions under consideration in this paper therefore verifies the condition of null curvature, e.g., (3.196), via the stronger conditions

$$\frac{\partial g_{\mu\nu}}{\partial x^\rho} \equiv 0, \rho = 1, 2, 3, 4. \tag{3.199}$$

Unless otherwise specified, metrics of this latter type are assumed hereon in the section.

Notice also the enclosure properties

$$\hat{M}_I \subset \hat{M}_{II} \subset \hat{M}_{III}, \tag{3.200}$$

which illustrate the possibilities of increasing generalizations offered by the Lie-isotopic theory.

A final comment on the definition of the isounit $\hat{I}$ is in order. In this section we have assumed the *conventional Minkowski space* as our original space with metric η. The isotopic spaces are then characterized by the deformation T of metric η, as per Eq. (3.186), with particular case (3.195). The above assumption evidently anticipates the construction of the Lorentz-isotopic symmetry via the use of the generators and parameters of the original Lorentz symmetry.

Under these assumptions, the isounit must be the inverse of the deformation element T, $\hat{I} = T^{-1}$, while the assumption $\hat{I} = g^{-1} = (T\eta)^{-1}$ would lead to inconsistent results. Note that for Santilli's spaces of Class I, the above assumptions imply that *the isounit is positive definite.*

An alternative would be to assume as original space the *Euclidean space in four dimension* with metric $\delta = \mathrm{diag}(1, 1, 1, 1)$, $E(x, \delta, \mathbf{R})$. Santilli's spaces $\hat{M}(x, g, \hat{\mathbf{R}})$ can also be interpreted as isotopes of $E(x, \delta, \mathbf{R})$. However, in this case, the deformation of the original metric δ is given by g itself, $g = T\delta = g$. As a consequence, in this latter case the isounit must be given by the inverse of the full metric, $\hat{I} = g^{-1}$, and the assumption $\hat{I} = T^{-1}$, where T is given by Eq. (3.195), would be inconsistent.

This second assumption implies that the *Lorentz-isotopic* symmetry must be constructed by using the generators and the parameters of the *rotational* group in four dimension, $O(4)$, much along the construction of the $\hat{O}(2.1)$ isotope from the original $O(3)$ symmetry of §3.2.

In conclusion, *one could construct the conventional Minkowski space* $M(x, \eta, \mathbf{R})$ *as an isotope of* $E(x, \delta, \mathbf{R})$, *and then build Santilli's spaces* $\hat{M}(x, g, \hat{\mathbf{R}})$ *as isotopes of* $M(x, \eta, \mathbf{R})$. *Alternatively, one could ignore the Euclidean spaces* $E(x, \delta, \mathbf{R})$, *and build the isotopes* $\hat{M}(x, g, \hat{\mathbf{R}})$ *of* $M(x, \eta, \mathbf{R})$. The reader should be aware that, in his original derivation [18], Santilli followed the former approach, evidently to provide a deeper illustration of the possibilities of his isotopies. In this monograph we shall follow instead the second approach mainly for simplicity of presentation.

3.4.6 Physical Interpretation of the Generalized Metric

Before passing to the review of the generalized relativity, it may be recommendable to point-out the physical meaning of the generalized metrics of Santilli's spaces. Stated differently, our problem is to clarify the fate of light when dealing with physical media because, after all, these media are generally opaque to light.

The space components of metric (3.186)

$$(g_{ij}) = \mathrm{diag}(g_{11}, g_{22}, g_{33}) \tag{3.201}$$

is the metric of the isotope $\hat{E}(r, g, \hat{\mathbf{R}})$ of the Euclidean space $E(r, g, \mathbf{R})$, Eq. (3.28) hereinafter assumed as being dimensionless. It remains fundamentally unchanged in the transition to Santilli's $(3+1)$-dimensional isotopic spaces. All the physical consideration on metric (3.201) of §3.2 therefore apply for this section. For example metric (3.201) can represent a deformation of the particle considered caused by external forces, the inhomogeneity and anisotropy of the medium considered, etc. (See Appendix C for applications.)

Almost needless to say, the Lie-isotopic generalization $\hat{O}(3)$ of the group of rotations $O(3)$ reviewed in §3.2 is a central part of the new relativity, and its knowledge shall be tacitly implied hereon.

The remaining component g_{44} of metric (3.186) must evidently be also dimensionless, and we shall put for isospaces $\hat{M}_I$

$$g_{44} \stackrel{\text{def}}{=} -b_4^2(\dot{x};n;\mu;\tau;...), \quad x^4 = c_0 t, \quad c^2 = b_4^2 c_0^2. \tag{3.202}$$

Our problem is the physical interpretation of the "velocity" $c = b_4 c_0$.

The simplest possible cases are those of fluids transparent to light, such as water. In these cases c clearly represents the speed of light in that particular medium, according to the familiar law $c = c_o/n < c_o$, where n is the index of refraction. Note that, at a deeper study, n is not a constant, but possesses a rather complex functional dependence precisely of type (3.202).

At times, fluids may be opaque to light but not to other wavelengths. In this case Santilli [*loc. cit.*] suggests the replacement of light with any electromagnetic wave that can propagate within the physical medium considered.

Nevertheless, physical media are generally opaque to all electromagnetic waves. This is the case of metals (whether solid or liquid), or more complex media such as the structure of nuclei, of hadrons or of collapsing stars. Evidently, no electromagnetic wave can classically propagate within these media in a conventional sense (the propagation of virtual or physical photons is excluded here because of quantum mechanical nature).

In these more general cases, *the quantity $c = b_4 c_0$ generally represents a purely geometrical object without necessarily representing a physical, actual speed.* The above conclusion can be best reached by considering spaces $\hat{M}_{III}$. In this case we are dealing with curved spaces in which each element of the metric, including g_{44}, has a purely geometrical interpretation, as familiar in the theory of gravitation. This situation is merely extended by the notion of isotopy also to flat spaces of type $\hat{M}_I$.

The occurrence can be illustrated by considering the medium composed by one kaon, and Nielsen-Picek's generalization (3.170) of the Minkowski metric, i.e.,

$$g = (1 - \frac{1}{3}\alpha, \quad 1 - \frac{1}{3}\alpha, 1 - \frac{1}{3}\alpha, -(1+\alpha)), \quad x^4 = c_0 t, \tag{3.203}$$

in which case

$$\begin{aligned} c^2 = c_o^2(1+\alpha) > c_o^2\,, \\ \alpha = (0.54 \pm 0.17) \times 10^{-3}\,. \end{aligned} \tag{3.204}$$

The best conceivable interpretation of component (3.204) is that it is a purely geometrical quantity. In fact, we know at this time of no electromagnetic wave that can classically propagate through a kaon.

Note that *value (3.204) characterizes a speed c higher than the speed of light in vacuum* c_o. *But, as correctly stated in refs. [14], [18], this does not necessarily mean the existence of physical speeds within the kaons higher than* c_o. The latter problem can only be investigated later on when reviewing the characterization provided by the generalized relativity of the maximal speed of massive, physical particles within the kaon structure.

As a further comment, note that the explicit form of the generalized metric must be obtained from experimental, phenomenological, or other considerations, but it cannot possibly be predicted by the generalized relativity owing to the endless variety of possible media. This can be illustrated by comparing Nielsen-Picek's metric for the kaons, Eq. (3.204), with that of the pions, Eq. (3.172), in which case

$$c^2 = c_o^2(1+\alpha) < c_o^2,$$

$$\alpha = (-3.79 \pm 1.37) \times 10^{-3}. \tag{3.205}$$

Thus, in the transition from kaons to pions, the Lorentz asymmetry parameter α changes not only in *value*, but also in *sign*. With the advancement of our knowledge, *one should therefore expect in general different metrics for different hadrons*, with a complexity predictably increasing with mass (evidently because the size of hadrons does not increase with mass thus resulting in an increase of density with mass). Highly complex metrics for superdense hadronic matter as occurring, say, in the core of collapsing stars, are then conceivable as limit cases.

As a final comment, it should be stressed that generalized metrics of the Nielsen-Picek type (3.203) with *constant* elements are potentially misleading, unless taken in their proper perspective. In fact, as stressed in §3.4.3, they constitute *first approximations* of metrics that are expected to be intrinsically *nonlinear in the velocity*, where the nonlinearity is, in turn, expected to be an approximation of the ultimate, *nonlocal* nature of the interior dynamical problem [102].

3.4.7 Lie-isotopic Generalization of the Lorentz Symmetry

We shall now review a central part of refs. [18], [26], the Lie-isotopic generalization of the conventional Lorentz group, in its broadest applicable form, that for spaces of *gravitational* types $\hat{M}_{III}(x, g, \hat{\mathbf{R}})$ with diagonal separation

$$x^2 = x^\mu g_{\mu\nu} x^\nu = x^1 g_{11} x^1 + x^2 g_{22} x^2 + x^3 g_{33} x^3 + x^4 g_{44} x^4, \quad x^4 = c_0 t, \tag{3.206.a}$$

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}) \neq 0. \tag{3.206.b}$$

Regrettably, we have not been authorized to present unpublished extensions of the Lorentz isotopy available in Santilli's manuscripts, e.g., for non-diagonal metrics which are applicable to gravitational theories more readily than separation (3.206).

Nevertheless, spaces $\hat{M}_{III}$ are sufficient for our objective: show that the Lie-isotopic theory allows the construction, apparently for the first time, of the symmetry transformations in their explicit form for arbitrary *gravitational* theories, let alone for arbitrary flat deformations of the Minkowski metric [26].

For clarity of notations, let us first review the structural foundations of the Lorentz group. Consider the *linear* transformations in Minkowski space M

$$x' = \Lambda x, \quad x'^t = x^t \Lambda^t. \tag{3.207}$$

Under the condition that they leave invariant the conventional separation (3.162), one obtains the familiar rules

$$\begin{aligned} \Lambda^t \eta \Lambda &= \Lambda \eta \Lambda^t = \eta^{-1} \\ \Lambda^{\mu\alpha} \eta_{\mu\nu} \Lambda^{\nu\beta} &= \eta^{-1}_{\alpha\beta}, \\ \det(\Lambda) &= \pm 1, \end{aligned} \tag{3.208}$$

which characterize the *six-parameter Lorentz group* on M, usually denoted $O(3.1)$, with familiar components $O^\uparrow_\pm(3.1)$ and $O^\downarrow_\pm(3.1)$ of which $O^\uparrow_+(3.1)$ forms a *connected Lie transformation group*, i.e., it verifies the conditions

$$\begin{aligned} \Lambda(u)\Lambda(-u) &= I = \mathrm{diag}(1,1,1,1), \\ \Lambda(u)\Lambda(u') &= \Lambda(u+u'), \\ \Lambda(0) &= I, \end{aligned} \tag{3.209}$$

where the six parameters $u = \{\vec{\theta}, \vec{w}\}$ represents the three *Euler angles* $\vec{\theta}$ and the three parameters $\vec{w}$ of the *Lorentz boosts.*

The remaining components form a group only when combined to $O_+^\uparrow(3.1)$ owing to the presence of the *discrete transformations (inversions)*

$$
\begin{aligned}
Px &= P(\vec{r},t) = (-\vec{r},t), \\
\mathrm{T}x &= \mathrm{T}(\vec{r},t) = (\vec{r},-t), \\
PTx &= PT(\vec{r},t) = (-\vec{r},-t).
\end{aligned}
\tag{3.210}
$$

Let $\vec{J}$ and $\vec{M}$ be the generators of $O_+^\uparrow(3.1)$ in their fundamental representation, e.g., that of ref. [122], p. 40 (see also Eqs. (3.8) for the space generators)

$$
J_1 = J_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
J_2 = J_{31} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
J_3 = J_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
M_1 = M_{14} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},
$$

$$
M_2 = M_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},
$$

$$
M_3 = M_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.211}
$$

The structural foundations of the connected Lorentz group $O_+^\uparrow(3.1)$ are then given by the now familiar forms:

A) The *Enveloping Associative Algebra* $\xi(\mathbf{O}_+^\uparrow(3.1))$ characterized by the ordered, infinite dimensional basis

$$\xi(\mathbf{O}_+^\uparrow(3.1)) : I, \quad X_k, \quad X_iX_j, \quad X_iX_jX_k, \dots ,$$

$$i \leq j, \quad i \leq j \leq k ,$$

$$X = \{\vec{J}, \vec{M}\}; \quad i, j, k = 1, 2, 3; \tag{3.212}$$

B) The *Lie Group* $O_+^\uparrow(3.1)$, characterized by convergent infinite series in $\xi(\mathbf{O}_+^\uparrow(3.1))$ here formally written

$$O_+^\uparrow(3.1) : \Lambda(\vec{\theta}, \vec{w}) = (\Pi_{k=1}^3 \exp(J_k\theta_k)|_\xi)(\Pi_{k=1}^3 \exp(M_k w_k)|_\xi), \tag{3.213}$$

C) The *Lie algebra* $\mathbf{O}_+^\uparrow(3.1)$ characterized by the familiar commutation rules in the neighborhood of the identity $I \in \xi(\mathbf{O}_+^\uparrow(3.1))$

$$[J_i, J_j] = -\varepsilon_{ijk} J_k,$$

$$[M_i, M_j] = +\varepsilon_{ijk} J_k = -\eta_{44}\epsilon_{ijk} J_k ,$$

$$[J_i, M_j] = -\varepsilon_{ijk} M_k , \tag{3.214}$$

where the product is, of course, the simplest conceivable Lie product of matrices A, B

$$[A, B]_\xi = AB - BA. \tag{3.215}$$

The second-order *Casimir invariants* are then given by the familiar expressions

$$C_1 = \vec{J}^2 - \vec{M}^2 = \sum_{k=1}^{3} (J_k J_k + \frac{1}{\eta_{44}} M_k M_k) = -3I,$$

$$C_2 = \vec{J} \cdot \vec{M} = \sum_{k=1}^{3} J_k M_k = 0, \tag{3.216}$$

where one should keep in mind that the selected basis verifies the property

$$J_k^t = -J_k, \quad M_k^t = M_k. \tag{3.217}$$

We now pass to the review of the isotopic lifting of the Lorentz group $O(3.1)$ which is one of the central objectives of this review, and which was presented for the first time in ref. [18]. The isotopy will now appear trivial to the reader with some familiarity with the techniques; yet it's mathematical and physical implications are far from trivial.

The first step is to lift the linear transformation theory on M, Eq. (3.207), into its isotopic generalization on space $\hat{M}_{III}$ with separation (3.206)

$$x' = \hat{\Lambda} * x \stackrel{\text{def}}{=} \hat{\Lambda} T(x; \dot{x}; n; \mu; \tau; , ...)x \ ,$$

$$x'^t = x^t * \hat{\Lambda}^t \stackrel{\text{def}}{=} x^t T \hat{\Lambda}^t. \tag{3.218}$$

The emerging transformations are generally *nonlinear*, although *isotopically linear.*

The second step is to impose the form-invariance of separation (3.206.a) which yields the isotopic conditions

$$x'^t \circ x' = x'^t g x' = x^t * \hat{\Lambda}^t g \hat{\Lambda}^t * x = x^t g x = x^t \circ x \ , \tag{3.219}$$

which can be explicitly written

$$\hat{\Lambda}^t g \hat{\Lambda} = \hat{\Lambda} g \hat{\Lambda}^t = g^{-1},$$

$$\det(\hat{\Lambda}) = \pm \det(\hat{I}), \tag{3.220}$$

and constitute a clear isotopy of conventional conditions (3.208).

The theory reviewed in §2, particularly the isotopy of Lie's theorems (§2.3), ensures that the transformations $\hat{\Lambda}$ preserve the six parameters $u = \{\vec{\theta}, \vec{w}\}$ of the original transformations $\Lambda(u)$, and form a *six- parameter, connected, Lie-isotopic transformation group on* $\hat{M}_{III}$, i.e., they verify the isotopic group laws

$$\hat{\Lambda}(u) * \hat{\Lambda}(-u) = \hat{I},$$

$$\hat{\Lambda}(u) * \hat{\Lambda}(u') = \hat{\Lambda}(u') * \hat{\Lambda}(u) = \hat{\Lambda}(u + u'),$$

$$\hat{\Lambda}(0) = \hat{I}. \tag{3.221}$$

The explicit form of the isotopic lifting of the Lorentz group is then provided by the isotopes of structures A), B) and C) above. For later needs, we introduce the following redefinition of the basis

$$\hat{X}_k = \{\hat{J}_k, \hat{M}_k\} \ ,$$

$$\hat{J}_1 = g_{22}^{-1/2} g_{33}^{-1/2} J_i; \quad \hat{J}_2 = g_{11}^{-1/2} g_{33}^{-1/2} J_3; \quad \hat{J}_3 = g_{11}^{-1/2} g_{22}^{-1/2} J_3 ,$$

$$\hat{M}_k = g_{kk}^{-1/2} M_k, \quad k = 1,2,3 . \tag{3.222}$$

We then have the following results of ref. [18].

A') the *isotopic lifting* $\hat{\xi}(\mathbf{O}_+^\uparrow(3.1))$ *of the enveloping algebra* $\xi(\mathbf{O}_+^\uparrow(3.1))$ characterized by the infinite (ordered) isotopic basis (§2.2)

$$\hat{\xi}(\mathbf{O}_+^\uparrow(3.1)) : \hat{I}, \quad \hat{X}_i, \quad \hat{X}_i * \hat{X}_j; \quad \hat{X}_i * \hat{X}_j * \hat{X}_k, \dots ,$$

$$i \leq j, \quad i \leq j \leq k , \tag{3.223}$$

$$\hat{I} = T^{-1} .$$

B') The *isotopic lifting* $\hat{O}_+^\uparrow(3.1)$ *of the connected Lorentz group* $O_+^\uparrow(3.1))$ characterized by convergent infinite power series expansions in $\hat{\xi}(\mathbf{O}_+^\uparrow(3.1))$ here symbolically written

$$\hat{O}_+^\uparrow(3.1) : \hat{\Lambda}(\vec{\theta}, \vec{w}) = (\prod_{k=1}^{3} {}^* \exp(\hat{J}_k \theta_k)|_{\hat{\xi}}) * (\prod_{k=1}^{3} {}^* \exp(\hat{M}_k w_k)|_{\hat{\xi}}), \tag{3.224}$$

which can be reformulated in $\xi(\mathbf{O}_+^\uparrow(3.1))$ for computational facility

$$\begin{aligned} \hat{O}_+^\uparrow(3.1) : \hat{\Lambda}(\vec{\theta}, \vec{w}) &= (\Pi_{k=1}^3 \exp(\hat{J}_k g \theta_k)|_\xi)(\Pi_{k=1}^3 \exp(\hat{M}_k g w_k)|_\xi)\hat{I} \\ &\stackrel{\text{def}}{=} \exp A(\theta, w)\hat{I}; \end{aligned} \tag{3.225}$$

C') The *isotopic lifting* $\hat{\mathbf{O}}_+^\uparrow(3.1)$ *of the Lie algebra* $\mathbf{O}_+^\uparrow(3.1)$, which is characterized by the isocommutation rules

$$[\hat{J}_i \hat{,} \hat{J}_j] = -\varepsilon_{ijk} \hat{J}_k,$$

$$[\hat{M}_i \hat{,} \hat{M}_j] = -g_{44} \varepsilon_{ijk} \hat{J}_k,$$

$$[\hat{J}_i \hat{,} \hat{M}_j] = -g_{jj}^{-1/2} \varepsilon_{ijk} \hat{M}_k, \tag{3.226}$$

where the Lie product is now less trivial than (3.215)

$$\begin{aligned} [A \hat{,} B] &\stackrel{\text{def}}{=} [A, B]_{\hat{\xi}} = A * B - B * A, \\ &= ATB - BTA. \end{aligned} \tag{3.227}$$

The *isocenter* of the algebra is now given by the *isotopic Casimir operator* of the first order, $\hat{I}$, and those of the second-order

$$\hat{C}_1 = \hat{J}^{\hat{2}} + \frac{1}{g_{44}}\hat{M}^{\hat{2}} = \sum_{k=1}^{3}(\hat{J}_k g \hat{J}_k + \frac{1}{g_{44}}\hat{M}_k T \hat{M}_k) = -3\hat{I},$$

$$\hat{C}_2 = \hat{J} * \hat{M} = \sum_{k=1}^{3}(\hat{J}_k T \hat{M}_k) = 0. \tag{3.228}$$

As expected for mathematical consistency, the values of the Casimir invariants are isoscalar, i.e., elements of $\hat{\mathbf{R}}$ and not ordinary scalars (this clarifies the need for the lifting $\mathbf{R} \rightarrow \hat{\mathbf{R}}$ pointed out in §3.4.5).

The isotope $\hat{O}(3.1)$ of the entire Lorentz group $O(3.1)$ is achieved by including [18] the *isodiscrete transformations* (or *isoinversions*)

$$\hat{P} * x = \hat{P} * (\vec{r}, t) = P(\vec{r}, t) = (-\vec{r}, t),$$

$$\hat{\mathrm{T}} * x = \hat{\mathrm{T}} * (\vec{r}, t) = \mathrm{T}(\vec{r}, t) = (\vec{r}, -t),$$

$$\begin{aligned} \widehat{P\mathrm{T}} * x &= \hat{P} * \hat{\mathrm{T}} * x = \hat{\mathrm{T}} * \hat{P} * x \\ &= P\mathrm{T}x = P\mathrm{T}(\vec{r}, t) = (-\vec{r}, -t), \end{aligned} \tag{3.229}$$

with explicit realization

$$\hat{P} = P\hat{I}, \quad \hat{\mathrm{T}} = \mathrm{T}\hat{I}, \quad \widehat{P\mathrm{T}} = (P\mathrm{T})\hat{I}\ . \tag{3.230}$$

The above results then leads to the following property which is the most important application of Theorem 2.9.

Theorem 3.5 [18] *The Lie-isotopic generalization* $\hat{O}(3.1)$ *on spaces* $\hat{M}_{III}(x, g, \hat{\mathbf{R}})$ *with metric (3.206) of the Lorentz group* $O(3.1)$ *on* $M(x, \eta, \mathbf{R})$, *hereinafter called Santilli's (or Lorentz-isotopic) group leaves form-invariant, by construction, the separation in* $\hat{M}_{III}(x, g, \hat{\mathbf{R}})$, *i.e.,*

$$\hat{O}(3.1) : x^t \circ x = x^t g(x) x \equiv x'^t \circ x' = x'^t g(x(x'))x'\ ,$$

$$x' = \hat{\Lambda}(\vec{\theta}, \vec{w}) * x = \hat{\Lambda}(\vec{\theta}, \vec{w}) T x\ . \tag{3.231}$$

An inspection of the results then leads to the following.

Corollary 3.5.1 [18]: *The process of Lie-isotopy is insensitive as to whether Santilli's spaces* $\hat{M}(x, g, \hat{\mathbf{R}})$ *are flat or curved.*

An inspection of isotopic expressions (3.224) or (3.225) yields the following additional property.

Corollary 3.5.2 [18]: *The isotopic transformations (3.224) can be explicitly computed from the sole knowledge of the conventional Lorentz generators* $\vec{J}$ *and* $\vec{M}$ *in their fundamental (4x4) representation and the generalized metric* g.

In fact the assumed topological restrictions on g assure the existence of an isotopic Poincaré-Birkhoff-Witt theorem (§2.2). The proof of the convergence of exponentials (3.224) to a finite form is then reduced to the proof of the convergence of the conventional exponentials (3.213).

Corollary 3.5.3 [18],[26]: *The Lie isotopic theory allows the explicit construction of the form-invariant transformations not only for flat generalizations* $\hat{M}_I(x, g, \hat{\mathbf{R}})$ *and* $\hat{M}_{II}(x, g, \hat{\mathbf{R}})$ *of the Minkowski space* $M(x, \eta, \mathbf{R})$, *but also for all permitted gravitational models on* $\hat{M}_{III}(x, g, \hat{\mathbf{R}})$, *whether of conventional or generalized type (see next section).*

An inspection of isotopic commutation rules (3.226) and the use of the theory of §2.4 (see also the classification of the isotopes $\hat{O}(3)$ of §3.2) leads to the following additional property.

Lemma 3.4 [26]: *The isotopic groups* $\hat{O}(3.1)$ *on* $\hat{M}_{III}(x, g, \hat{\mathbf{R}})$ *are generally* nonisomorphic *to* $O(3.1)$. *Depending on the assumed metric and its topology,* $\hat{O}(3.1)$ *can be isomorphic to any six-parameter group of Cartan's classification, i.e.,* $O(3.1)$, *or* $O(2.2)$, *or* $O(4)$ *or other groups.*

The reader should note the appearance of the *structure functions* of §2.3 in isocommutation rules (3.226). Remarkably, Santilli identified the need to replace the structure constants with structure functions on pure mathematical grounds, while studying the isotopic generalization of Lie's Second Theorem [1]. This was several years before the essential appearance of these functions in actual models.

Recall that the central idea of the Lie-isotopic generalization of a given Lie symmetry is to leave unchanged the parameters and the generators of the theory, and generalize instead the Lie product. In the preceding analysis, Santilli left the parameters of the Lorentz group unchanged under the lifting, but changed the generators via redefinitions (3.222). This was done to reach form (3.226) of the isocommutation rules which is more suitable for the proof of the isomorphism of $\hat{O}(3.1)$ with $O(3.1)$ of the next section.

The reformulation in terms of the original basis (3.211) is straightforward. Consider that basis in the form $M_{\mu\nu}$, and recall that their commutation rules are given by

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -\eta_{\alpha\gamma}M_{\beta\delta} + \eta_{\alpha\delta}M_{\beta\gamma} + \eta_{\beta\gamma}M_{\alpha\delta} - \eta_{\beta\delta}M_{\alpha\gamma},$$
$$\eta = \mathrm{diag}(1,1,1,-1). \tag{3.232}$$

It is then simple to show that, under isotopic lifting, we have the isocommutation rules

$$[J_i \hat{,} J_j] = -\varepsilon_{ijk}g_{kk}J_k \ ,$$
$$[M_i \hat{,} M_j] = -g_{44}\varepsilon_{ijk}J_k \ ,$$
$$[J_i \hat{,} M_j] = -g_{jj}\varepsilon_{ijk}M_k \ , \tag{3.233}$$

which can be written in the unified notation [26]

$$[M_{\alpha\beta} \hat{,} M_{\gamma\delta}] = -g_{\alpha\gamma}M_{\beta\delta} + g_{\alpha\delta}M_{\beta\gamma} + g_{\beta\gamma}M_{\alpha\delta} - g_{\beta\delta}M_{\alpha\gamma},$$
$$g = \mathrm{diag}(g_{11}, g_{22}, g_{33}, g_{44}). \tag{3.234}$$

Note that, despite the similarities of rules (3.232) and (3.234), the algebras are not generally isomorphic because of the possible different topologies of the metrics η and g. Also, the reader should keep in mind that isocommutation rules (3.234) occur for a generally *curved* space, although of the isotopic form $\hat{M}_{III}$.

The extension of the results to the *Poincaré algebra* (also called the *inhomogeneous Lorentz algebra* $\mathbf{P}(3.1) = \mathbf{O}(3.1) \oplus \mathbf{T}(3.1)$, where $\mathbf{T}(3.1)$ is the Lie algebra of the group of translations in conventional Minkowski space, has been investigated by Santilli in ref. [26] for the case when g does not depend on space-time coordinates x. Consider the isocomposition of a Lorentz-isotopic transformation $\hat{\Lambda}$ and of an isotranslation $\hat{T}$ on spaces $\hat{M}_{III}$

$$\{\hat{\Lambda}, \hat{T}\} * x = \hat{\Lambda} * x + a,$$
$$a = (a^\mu) = (\vec{a}, a^4) = \text{const.}, \tag{3.235}$$

by keeping in mind that the product of two such transformations $\{\hat{\Lambda}_1, \hat{T}_1\}$ and $\{\hat{\Lambda}_2, \hat{T}_2\}$ follows the isotopic rule

$$\{\hat{\Lambda}_1, \hat{T}_1\} * \{\hat{\Lambda}_2, \hat{T}_2\} = \{\hat{\Lambda}_1 * \hat{\Lambda}_2, \hat{T}_1 + \hat{\Lambda}_2 * \hat{T}_2\}. \tag{3.236}$$

Let P_μ be the generators of translations in *conventional* Minkowski space (recall that the $M_{\mu\nu}$ generators are also the conventional ones). Then, the *Lie-isotopic generalization of the Poincaré algebra* is given by the isocommutation rules [26]

$$\hat{\mathbf{P}}(3.1): \begin{array}{rl} [M_{\alpha\beta}\hat{,}M_{\gamma\delta}] & = -g_{\alpha\gamma}M_{\beta\delta} + g_{\alpha\delta}M_{\beta\gamma} + g_{\beta\gamma}M_{\alpha\delta} - g_{\beta\delta}M_{\alpha\gamma}, \\ [M_{\alpha\beta}\hat{,}P_\gamma] & = g_{\beta\gamma}P_\alpha - g_{\alpha\gamma}P_\beta, \\ [P_\alpha\hat{,}P_\beta] & = 0, \end{array} \tag{3.237}$$

(where the reader should keep in mind the diagonality of the P's).

Again, despite the similarity of isoalgebra (3.237) with the conventional one, the algebras $\hat{\mathbf{P}}(3.1)$ and $\mathbf{P}(3.1)$ are generally nonisomorphic.

We are now in a position to elaborate in more details the comments at the end of §3.4.5 regarding the appropriate selection of the isounit. In the original proposal [18], Santilli selected for isounit the inverse of the full metric, $\hat{I} = g^{-1}$, because he constructed his isospaces and isotopic $\hat{O}(3.1)$ in their most general possible form, i.e., as isotopies of the conventional Euclidean space in four dimension and of the conventional orthogonal group $O(4)$, respectively. In fact, the generators of $\hat{O}(3.1)$ in paper [18] *are not* those of the noncompact $O(3.1)$ group, but those of the *compact* $O(4)$ group. The conventional Minkowski space and the conventional Lorentz groups are therefore particular isotopies in Santilli's general construction of ref. [18], of $E(x, \delta, \mathbf{R})$ and $O(4)$.

In this monograph we have elected, for simplicity, to construct the isotopic spaces and groups as mutations of the conventional Minkowski space and Lortentz group $O(3.1)$, respectively. Since the basic metric is now assumed to be the Minkowski one η, the isounit must be given by the inverse of the mutation T of η, i.e., $\hat{I} = T^{-1}$, $g = T\eta$ (§3.4.5). For consistency, however, we had to assume as basic generators, those of the *noncompact* $O(3.1)$, Eq. (3.211), along paper [26].

The two constructions are manifestly equivalent. As stated by Santilli himself [26], the latter construction is preferable for possible applications in particle physics because of the positive-definite character of the isounit and, thus, the more transparent possibility of preserving the abstract axioms of relativistic quantum mechanics on $\hat{M}_I$.

The extension of the above result to the lifting $\hat{P}(3.1)$ of the Poincaré group $P(3.1)$ is straightforward [26] and essentially provided by the semidirect product of the isotopic group $\hat{O}^\uparrow_+(3.1)$, Eq. (3.224), times the isotopic group of translations

$$\hat{T}(3.1) : \hat{T}(a) = \exp(P^{\mu}\eta_{\mu\nu}a^{\nu})|_{\hat{\xi}} = \hat{I}\exp(P^{\mu}g_{\mu\nu}a^{\nu})|_{\xi}, \tag{3.238}$$

with similar procedures for the inclusion of the isoinversions. For these and other aspects we refer the reader for brevity to the locally quoted paper.

In order to identify the isotopic Casimir operators, we have to review the means of lowering and raising the indices of the various quantities [26] which follow conventional geometrical (e.g., affine) approaches. Let $|g|$ be the determinant of g, and introduce the contravariant metric tensor $g^{\mu\nu}$ defined by

$$g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}_{\nu}, \tag{3.239}$$

with solution

$$|g^{\mu\nu}| = |g_{\mu\nu}|^{-1}, \tag{3.240}$$

i.e., because of the diagonal character of the considered metrics,

$$g^{\mu\nu} = g^{-1}_{\mu\nu}. \tag{3.241}$$

Then, the covariant (contravariant) vectors $x_{\mu}(P^{\mu})$ are characterized by relations of the type

$$x_{\mu} = g_{\mu\nu}x^{\nu}, \quad x^{\mu} = g^{\mu\nu}x_{\nu}, \quad P_{\mu} = g_{\mu\nu}P^{\nu}, \quad P^{\mu} = g^{\mu\nu}P_{\nu}, \tag{3.242}$$

and verify the identities

$$\begin{aligned} x^{\mu}g_{\mu\nu}x^{\nu} &= x_{\mu}g^{\mu\nu}x_{\nu} = x^{\mu}x_{\mu} = x_{\mu}x^{\mu} \\ &= x^{i}g_{ij}x^{j} + x^{4}g_{44}x^{4}, \end{aligned} \tag{3.243}$$

as the reader can verify.

Similarly, for tensors we have the raising of the indices

$$M^{\mu\nu} = g^{\mu\alpha}g^{\mu\beta}M_{\alpha\beta}, \tag{3.244}$$

with similar forms for other cases.

After these preliminaries, one can introduce the *isotopic generalization of the Pauli-Lubanski four-vector* (or *Pauli-Lubanski isovector*) on isotopic spaces $\hat{M}_{III}$ [26]

$$W_{\mu} = \frac{1}{2}\varepsilon_{\mu\alpha\beta\gamma}M^{\alpha\beta} * P^{\gamma}, \tag{3.245}$$

which verifies the properties

$$[M_{\alpha\beta}\hat{,}W_\gamma] = g_{\beta\gamma}W_\alpha - g_{\alpha\gamma}W_\beta,$$

$$[P_\alpha\hat{,}W_\beta] = 0, \qquad (3.246)$$

where use has been made of the isotopic rule [36]

$$[A\hat{,}B * C] = [A\hat{,}B] * C + B * [A\hat{,}C]. \qquad (3.247)$$

It is then easy to see that the *center of the Poincaré-isotopic algebra* $\hat{\mathbf{P}}(3.1)$ is given by the isounit $\hat{I} = g^{-1}$, the quantities in $\hat{\mathbf{R}}$

$$P^{\hat{2}} = (P^\mu g_{\mu\nu} p^\nu)\hat{I} = (P^i g_{ij} P^j + P^4 g_{44} P^4)\hat{I},$$

$$W^{\hat{2}} = (W^\mu g_{\mu\nu} W^\nu)\hat{I} = (W^i g_{ij} W^j + W^4 g_{44} W^4)\hat{I}, \qquad (3.248)$$

as well as any of their iso-combinations.

The deviations of the numerical values of the above isocasimirs from the conventional ones is the mathematical foundation of Santilli's concept of "mutation" of elementary particle when immersed within dense hadronic matter [2]. This important new concept is illustrated in Appendix C via the isotopic lifting of field equations [27], i.e., field equations that are covariant under $\hat{P}(3.1)$.

3.4.8 Lie-isotopic Generalization of Einstein's Special Relativity

Following ref. [18], we now restrict the analysis to Santilli's spaces of the first class, Eq. (3.195), with fourth component (3.202), i.e.,

$$\hat{M}_I(x, g, \hat{\mathbf{R}}) : x^2 = x^\mu g_{\mu\nu} x^\nu = x^1 b_1^2 x^1 + x^2 b_2^2 x^2 + x^3 b_3^2 x^3 - x^4 b_4^2 x^4,$$

$$x^4 = c_0 t, \quad \partial b_\mu / \partial x = 0, \quad b_\mu > 0, \quad c = c_0 b_4\ , \qquad (3.249)$$

where the diagonal elements have a positive-definite character in the considered region of isospacetime. The subsequent property follows from Theorem 2.9 and the preservation by spaces $\hat{M}_I$ of the topological character of the conventional Minkowski space.

Theorem 3.6 [18]: *The Lorentz group $O(3.1)$ on Minkowski space $M(x, \eta, \mathbf{R})$ and all possible isotopes $\hat{\mathbf{O}}(3.1)$ on spaces $\hat{M}_I(x, g, \hat{\mathbf{R}})$ are (locally) isomorphic, and they coincide at the abstract, realization-free level.*

To put it differently, *there exists only one abstract Lorentz group*, say $\mathcal{O}(3.1)$, that realized in terms of a Lie algebra with abstract product $ab-ba$, where "ab" is an unspecified Lie-admissible product in a coordinate free form. Then there exist infinite varieties or realizations $\hat{O}(3.1)$ in terms of the product $ATB - BTA$ where $g = T\eta$ possesses the same topological structure of the Minkowski metric. Finally, there exists the "simplest possible realization" $O(3.1)$, that of the contemporary literature with trivial Lie product $AB - BA$ and the Minkowski invariant. All these different realizations are geometrically equivalent, and algebraically isomorphic.

Owing to the convergence of exponentials (3.225) (Lemma 3.3), the isotopic transformations can be easily computed for each given generalized metric g (which is the only unknown of the expansions).

In the case of motion along the third axis and for arbitrary elements b_k^2, c^2, exponential (3.225) on $\hat{M}_I$ yields the following generalization of the Lorentz transformations (3.163) introduced in ref. [14]

$$x' = \hat{\Lambda} * x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(wc) & -\frac{c}{b_3}\sinh(wc) \\ 0 & 0 & -\frac{b_3}{c}\sinh(wc) & \cosh(wc) \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}, \qquad (3.250)$$

which can be written more explicitly

$$\begin{aligned} x^{1\prime} &= x^1 , \\ x^{2\prime} &= x^2 , \\ x^{3\prime} &= \hat{\gamma}(x^3 - \beta x^4), \quad \beta x^4 = vt , \\ x^{4\prime} &= \hat{\gamma}(x^4 - \hat{\beta} x^3) , \end{aligned} \qquad (3.251)$$

where

$$\begin{aligned} \beta^2 &= v^2/c_0^2, \quad \hat{\beta}^2 = v b_3^2 v / c_0 b_4^2 c_0 && (3.252.a) \\ \gamma &= (1-\beta^2)^{-1/2}, \quad \hat{\gamma} = (1-\hat{\beta}^2)^{-1/2} && (3.252.b) \\ \cosh(wc) &= \hat{\gamma}, \quad \sinh(wc) = \hat{\beta}^2 \hat{\gamma} && (3.252.c) \\ b_k &= b_\mu(\dot{x};\eta;\mu;\tau;\cdots), \quad c = c(\dot{x};n;\mu;\tau;\cdots) , && (3.252.d) \end{aligned}$$

The nonlinearity of Santilli's transformations is then evident. The verification that they do leave form-invariant separation (3.249) is a simple but instructive exercise for the interested reader. The computation of different forms for different explicit expressions of the metric is also trivial. Finally, the inclusion of the conventional Lorentz transformations, Eq. (3.163), as a particular case is also evident. The "direct universality" of Santilli's transformations will be discussed in §3.4.16.

Needless to say, the transformations acting in the three-space $\{x^1, x^2, x^3\}$ are the *isorotations* of §3.2. Eqs. (3.251) provide an example of *isoboosts.* The *isoinvertions* have been reviewed in §3.4.7.

> *DEFINITION 3.8:* Santilli's transformations *(also called* Lorentz-isotopic transformations*) shall be called of the* first, second *or* third class, *depending on whether they leave invariant the separation of spaces of the first, second and third class (Definition 3.7), respectively. Their most general form is characterized by arbitrary superpositions of isorotations, isoboosts and isoinversions.*

We now come to a central point of this review.

> *DEFINITION 3.9:* Santilli's Relativity of the First Class, *or* Santilli's Special Relativity, *is the generalization of Einstein's Special Relations characterized by the Lorentz-isotopic transformations of the first class on $\hat{M}_I(x, g, \hat{\mathbf{R}})$.*

The following property can be easily proven.

Theorem 3.7 [18]: *Santilli's Special Relativity is a covering of Einstein's Special Relativity in the sense that*

a *the generalized relativity is constructed with mathematical methods (the Lie-isotopic theory) structurally more general than those of the conventional relativity (Lie's theory in its simplest possible realization);*

b *the generalized relativity describes physical conditions (extended-deformable particles moving within inhomogeneous and anisotropic media) which are structurally more general than those of the conventional relativity (point-like particles moving in vacuum); and*

c *the generalized relativity*

- *[c-1] contains the conventional relativity as a particular case;*
- *[c-2] can approximate the conventional relativity as close as desired, evidently for $g \approx \eta$; and*
- *[c-3] recovers the conventional relativity identically for $g \equiv \eta$,* $\hat{I} = I = \mathrm{diag}(1, 1, 1, 1)$.

The explicit construction of the generalized relativity will be reviewed in the next subsections.

> DEFINITION 3.10: *Santilli's Relativities of the Second and Third Class are those characterized by Lorentz-isotopic transformations of the second and third class on* $\hat{M}_{II}(x, g, \hat{\mathbf{R}})$ *and* $\hat{M}_{III}(x, g, \hat{\mathbf{R}})$*, respectively.*

Note that these more general relativities *do not* constitute, in general, "coverings" of Einstein's Special Relativity (in the sense of Theorem 3.7) because the generalized metrics g *do not* admit, in general, the Minkowski metric as a particular case.

Also, note that the isotope $\hat{O}(3.1)$ of these broader relativities *is not* necessarily isomorphic to $O(3.1)$, as indicated earlier.

The generalization of Class II is important to achieve the desired unity of physical and mathematical thought. In fact, one can find in the literature several studies on conceivable generalizations of the special relativity, still on flat spaces, but with a topology different than that of the Minkowski metric. All these studies are then unified by Santilli's Relativity of the Second Class. As an example, Recami and Mignani [123] have introduced the *superluminal transformations*

$$x'^{\hat{2}} = x'^{t} g x' = -x^2 = -x^t \eta x, \tag{3.253}$$

which are evidently transformations in $\hat{M}_{II}(x, g, \hat{\mathbf{R}})$. Note that Recami-Mignani's transformations provide the generalization to $(3 + 1)$-dimension of the notion of *isotopic dual* introduced in §3.2 (Definition 3.1) for the case of the isotopic lifting of rotations. The reformulation of transformations (2.353) in terms of the Lie-isotopic theory is therefore recommended.

We should also indicate that the notion of isogroup duality (Definition 3.1), leads to the *duality of isorelativities.* In fact, a *subclass* of Santilli's Relativities of Class II is characterized by families of metrics which can be divided into the two classes $g = \pm T\eta$, $T > 0$, with corresponding isolorentz groups $\hat{O}(\pm, 3, 1)$.

It is evident that the relativity with $g = +T\eta$ is the one of this section, and that with $g = -T\eta$, is its isotopic dual, the isolorentz groups $\hat{O}(+, 3, 1)$ and its dual $\hat{O}(-, 3, 1)$ being isomorphic to each other and to the conventional group $O(3.1)$.

Santilli's Special relativity is specifically restricted to the case $g = +T\eta$, $T > 0$, (Definition 3.9) because it is conceived for all possible mutations T of the Minkowski metric η which are practically realizable, those preserving the topological character of the original metric. Santilli's Relativity with $g = -T\eta, T > 0$, is a sort of "mirror image" of the Special Relativity in the mathematical space with metric $\eta' = -\eta$.

Recami-Mignani transformations [123] can then be reinterpreted as mapping the relativity with $g = +T\eta$, into that with $g = -T\eta$, and viceversa. In this sense, they would be *isosuperluminal*, that is, beyond the maximal causal speed of Santilli's Special Relativity (see next subsection).

The generalized relativity of Class III is of gravitational character and, as such, will be discussed in the next section.

In closing, we note that no lifting of the Lorentz group with isotopies different than those of ref. [18] has been investigated until now, to our best knowledge. We are referring to isotopies of the associative enveloping algebra with product of the type (1.10) i.e., $A * B = WAWBW$, $W^2 = W$, or combinations of isotopies (1.4), (1.5) and (1.10) with product of the type $A * B = \alpha WAWTWBW$, $\alpha \in \mathbf{R}$, $W^2 = W$, $T = T^\dagger$. The reader should however be warned about the general loss of the unit under the latter isotopies, with evidently deep implications for Lie's theory which are absent in isotopy $A * B = ATB$, $\hat{I} = T^{-1}, g = T\eta$.

3.4.9 Maximal Speed of Massive Particles within Physical Media

In Einstein's Special Relativity, the maximal speed of a massive particle (or of a causal, physical signal) is that of light in vacuum. It is characterized by the infinitesimal separation in Minkowski space M

$$ds^2 = dx^i \delta_{ij} dx^j - dx^4 dx^4, \quad dx^4 = c_0 dt \ , \tag{3.254}$$

when of null value,

$$d\vec{r} \cdot d\vec{r} - dt c_o^2 dt = 0 \ , \tag{3.255}$$

resulting in the value

$$\vec{V}^2_{Max} = (\frac{d\vec{r}}{dt})^2 = c_o^2 \ , \tag{3.256}$$

which also provides the *fundamental invariant* of the theory.

In Santilli's Special Relativity, the infinitesimal separation is defined on isotopic spaces $\hat{M}_I$, and is given by

$$ds^2 = dx^k b_k^2 dx^k - dx^4 b_4^2 dx^4. \tag{3.257}$$

The case of null separation

$$dr^k b_k^2 dr^k - dt c^2 dt = 0, \quad c = b_4 c_0 \ , \tag{3.258}$$

then yields the expression

$$\frac{dr^k}{dt} b_k^2 \frac{dr^k}{dt} = c^2(\dot{x}; n; \mu; \tau; ...) \,, \tag{3.259}$$

which leads to the following fundamental

POSTULATE 3.1 [14],[18]. *The maximal possible speed*

$$\vec{V}_{Max} = |\frac{d\vec{r}}{dt}|_{Max} \stackrel{\text{def}}{=} C(\dot{x}; n; \mu; \tau; ...), \tag{3.260}$$

predicted by Santilli's Special Relativity for massive physical particles (or causal signals) propagating within physical media (§3.3.6) can be higher equal or smaller than the speed of light in vacuum c_o

$$V_{Max} = C \gtreqless c_o, \tag{3.261}$$

depending on the particular physical conditions at hand.

To illustrate this postulate, it is sufficient to consider the case of an isotropic Euclidean space for which Equation (3.259) becomes

$$\frac{dr^i}{dt} \delta_{ij} \frac{dr^j}{dt} = \frac{c^2}{b^2} = c_0^2 \frac{b_4^2}{b^2}, \quad b_1 = b_2 = b_3 = b > 0 \,. \tag{3.262}$$

The maximal speed C is then given in this case by

$$V_{Max} = |\frac{d\vec{r}}{dt}|_{Max} = C = c_0 \frac{b_4}{b} \,. \tag{3.263}$$

The existence in Nature of causal physical signals propagating faster than light in vacuum, which was postulated in ref. [14], has a number of independent, although preliminary, confirmations. Consider for instance, Nielsen's mutation of the Minkowski metric Eqs. (3.170), i.e.,

$$g = \text{diag}(1 - \frac{1}{3}\alpha, 1 - \frac{1}{3}\alpha, 1 - \frac{1}{3}\alpha, -(1 + \alpha)). \tag{3.264}$$

Then, Santilli's maximal speed C for the case of *kaons*, Eq. (3.173), is given by

$$\alpha = (0.61 \pm 0.17) \times 10^{-3} > 0,$$

$$C = c_o \frac{1 + \alpha}{1 - \frac{1}{3}\alpha} > c_o, \quad c = c_0(1 + \alpha) > c_0 \,, \tag{3.265}$$

and does indeed result to be *higher* than c_o. For the case of the *pions*, Eq. (3.172), we have instead

$$\alpha = (-3.79 \pm 1.37) \times 10^{-3} < 0,$$

$$C = c_o \frac{1+\alpha}{1-\frac{1}{3}\alpha} < c_o, \quad c = c_0(1+\alpha) < c_0 \,, \tag{3.266}$$

i.e., the maximal possible causal speed is *smaller* than c_o (recall from §3.4.6 that light itself cannot propagate within such hyperdense media). Eqs. (3.265) and (3.266) provide clear illustrations of Postulate 3.1 for the cases of maximal speeds higher and lower than the speed of light.

At a deeper analysis, *all studies on Lorentz noninvariance reviewed in §3.4.3 generally admit maximal possible speeds higher than that of light in vacuum.* This is the case of the studies by Blockintsev [96], Redei [97], Kim [98], the various works by Nielsen and collaborators [99], Aronson *et al.* [101], and others.

In general, *all modifications/mutations of the Minkowski metric must necessarily result in an alteration of the maximal speed of causal signals, trivially, because the space remains flat. The emerging new maximal speed can then be, depending on the conditions considered, higher, equal or smaller than the speed of light, exactly along Santilli's Postulate 3.1.* The above property was studied in details by de Sabbata and Gasperini [124] who, stimulated by Santilli's paper [14], computed the maximal possible speed within hadronic matter via the use of gauge theories, resulting again in a maximal speed which is higher than c_o. These latter calculations are reviewed in Appendix B.

Additional, independent evidence, again purely preliminary, in support of Postulate 3.1 is given in astrophysics by certain galactic conditions under which ordinary matter appears to propagate faster than c_o.

More specifically, Santilli postulated the following cases [14]:

a) *Nuclear structure*, in which case the maximal speed is expected to be *generally lower* than c_o;

b) *Hadronic structure*, in which case the maximal speed is expected to be *generally higher* than c_o; and

c) *Superdense star structure* (e.g., the core of a collapsing star) in which case the maximal possible speed is expected to be much higher than c_o and, under suitable limit conditions, even infinite.

The physical basis for the above expectations is provided by the interactions at the foundation of the studies on Lie-isotopy: the contact, nonhamiltonian interactions experienced by particles when moving within physical media. In fact, as stressed earlier, these interactions are of *instantaneous* character by conception and, as such, substantially outside Einstein's Special Relativity. Furthermore, the interactions are of *nonpotential* nature also by assumption. Therefore, conventional relativistic considerations regarding the energy needed for the acceleration of the particles simply do not apply. A new physical horizon, beyond that of Einstein, then emerges quite clearly.

When a particle is under the joint action of conventional forces (e.g., electromagnetic, or weak, or strong, or gravitational), *plus* the additional contact forces due to motion within a medium, the emerging maximal speeds is then expected to be precisely along Postulate 3.1.

The postulated increase of the maximal speed in passing from nuclear to star conditions is suggested by the progressive increase of the contact nonhamiltonian interactions. In fact, the condition of mutual wave overlapping of the constituents of nuclei (protons and neutrons) are minimal and estimated of the order of 10^{-3} nucleon's volumes from values of nuclear volumes as compared to the volumes of the charge distributions of the nucleon [35].

In the transition to the structure of hadrons, such as the protons and neutrons themselves, the conditions of mutual overlapping of the wavepackets of the constituents increase substantially to about 100% of the wavepackets which are approximately the same for all particles and equal to the range of the strong interactions (1F). Finally, in the transition to the core of stars undergoing gravitational collapse, we have not only 100% overlapping of the wavepackets of the constituents, but also their compression. A progressive increase of the maximal possible speed is then consequential. This is, in essence, the central physical idea of the postulate submitted in ref. [14].

It should be stressed, to avoid misrepresentations, that *particles propagating at speeds higher than c_o are not tachyons when dealing with Santilli's Special Relativity, but ordinary physical particles.* In fact, the conventional notion of tachyons demands propagation in vacuum, being strictly referred to the conventional special relativity. In Santilli's case we have motion within physical media, thus resulting in a different notion of tachyons, here called *isotachyons*, as conjectural particles traveling faster than the speed C of Postulate 3.1.

Another point that should be stressed to minimize misconceptions is that *the notion of maximal causal speed in Einstein's special relativity is an absolute constant, the invariant c_o that applies everywhere in space-time.*

In Santilli's Special Relativity, instead, the notion of the maximal causal speed is a strictly local invariant that can be generally defined only in the neighborhood of a point or at best in small regions of space (e.g., the interior of a kaon). In fact, as indicated in Eq. (3.260), the maximal speed C is a generally *nonlinear* function of the local quantities as an approximation of the expected, ultimate, *nonlocal* structure of matter (§3.4.6).

3.4.10 Isotopic Generalization of the Light Cone

Another important concept introduced in ref. [18] is the generalization of the conventional notion of "light cone" caused by isotopic liftings of the space.

> *DEFINITION 3.11 [18],[26]: The* isolight cone, *or* hypersurface of maximal speed of massive particles, *Eq. (3.261), is the deformation of the light cone caused by the lifting of the Minkowski space $M(x, \eta, \mathbf{R})$ into Santilli's isotope of the first class, $\hat{M}_I(x, g, \hat{\mathbf{R}})$, and divides the isospace itself into the following three regions:*
>
> 1. isotime-like region, *when the separation is negative- definite;*
> 2. isonull region *when the separation is null; and*
> 3. isospace-like region *when the separation is positive definite. (See Fig. 4 for more details.)*

Specifically, suppose that the observer is at the origin of the isotopic space $\hat{M}_I$. Let x_1 and x_2 be two *isoevents* in $\hat{M}_I$. Then their separation $x = x_1 - x_2$ can be

$$\textit{Isotime-like} \text{ when } x^2 < 0,$$
$$\textit{Isonull} \text{ when } x^2 = 0,$$
$$\textit{Isospace-like} \text{ when } x^2 > 0, \tag{3.267}$$

where

$$x^2 = x^\mu g_{\mu\nu} x^\nu = x^i b^2 \delta_{ij} x^j - x^4 b_4^2 x^4,$$
$$x = x_1 - x_2, \quad x^4 = c_0 t\,. \tag{3.268}$$

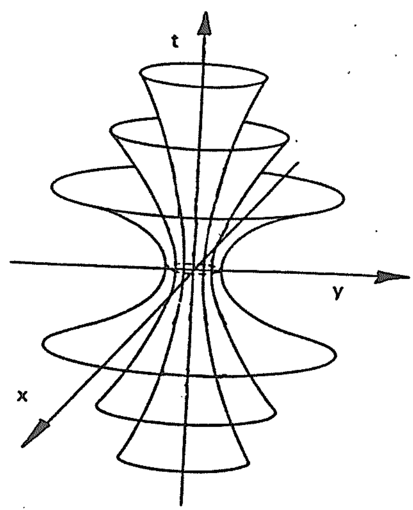


FIGURE 10. A reproduction of Fig. 1 of ref. [26] depicting the deformation (mutation) of the light cone caused by contact, instantaneous, null range interactions experienced by (extended) particles when moving within a physical medium. The deformed cones characterize the maximal speed of causal signals (e.g., the maximal speed of a massive ordinary particle) which results to be equal, greater or smaller than $c_o = c_{vacuum}$, depending on the physical conditions at hand. All available phenomenological information is encouragingly favoring the hypothesis of physical speed within hadronic matter higher than c_o. In fact, all research on Lorentz "noninvariance" reviewed in §3.4.3 favor a deformation of the Minkowski metric in the interior of hadrons. In turn, such a deformation necessarily implies an alteration of the maximal speed of massive particle [18]. The maximal speed can then be bigger or smaller than c_o depends on the circumstances. And in fact, as illustrated in the text, the Nielsen-Picek deformation of the Minkowski metric [99] implies that the maximal causal speed is smaller than c_o inside the pions, but it becomes higher than c_o inside the kaons exactly along the

two corresponding, deformed cones of the figure. This appears to indicate an increase of the maximal speed with the density, thus supporting the general calculation of the maximal speed provided in Appendix B, of course in a preliminary way. The reader should be aware that [26] Einstein's Special Relativity remains strictly valid in the arena of its original conception (motion of point-particles in vacuum). Nevertheless, when considering fundamentally different physical conditions, deviations are not only expected, but actually necessary to achieve compatibility with available phenomenological information (§3.4.3). Finally, such deviations are referred, specifically, to the Special Relativity and not to the Lorentz symmetry which remains exact in the interior of hadrons although at the covering Lie-isotopic level.

Throughout this section we shall assume for simplicity (but without loss of generality) that the metric of the isotopic Euclidean space is of the type

$$g_{ij} = b^2 \delta_{ij}, \quad b > 0 . \tag{3.269}$$

Seven possible classes of physical media were identified in ref.s [18,26] via the following analysis. First, Santilli writes the isoinvariant in $\hat{M}_I$ in the special form under assumption (3.269)

$$x^\mu g_{\mu\nu} x^\nu = \frac{1}{n_1^2}\mathbf{r}\cdot\mathbf{r} - \frac{1}{n_2^2} t c_0^2 t , \tag{3.270}$$

for which we have the expressions

$$V_{\mathrm{Max}} = \frac{c}{b} = \frac{b_4}{b} c_0 = \frac{n_1}{n_2} c_0 , \tag{3.271.a}$$

$$c = b_4 c_0 = \frac{c_0}{n_2} , \tag{3.271.b}$$

where n_1 and n_2 have a *local* dependence on all possible local quantities

$$n_k = n_k(\dot{x}; n; \mu; \tau; \cdots), \quad k = 1, 2 . \tag{3.272}$$

Three primary cases can then be identified, depending on whether, locally,

$$n_1 = n_2 , \tag{3.273.a}$$

$$n_1 < n_2 , \tag{3.273.b}$$

$$n_1 > n_2 . \tag{3.273.c}$$

The following seven cases of physical relevance follow from the behavior of the disequalities (3.273) with respect to one, according to:

SANTILLI'S CLASSIFICATION OF THE GEOMETRIZATION OF PHYSICAL MEDIA [18,26]:

CLASS 1: $n_1 = n_2 = 1$, $V_{\text{Max}} = c = c_0$.
This is evidently the case of Einstein's Special Relativity which, as now familiar, is contained in its entirety as a particular case of the covering Santilli's Special Relativity (see also the remaining parts of this section).

CLASS 2: $1 < n_1 = n_2$, $V_{\text{Max}} = c_0$, $c < c_0$.
This is the case of the propagation of light and particles within transparent fluids, such as water, a typical example being given by the *Cherenkov light.* In this case, as experimentally established a long time ago, light propagates at a speed *smaller* than c_0 according to the familiar rule

$$c = \frac{c_0}{n} < 0 \, , \tag{3.274}$$

where $n = n_1 = n_2$ is the familiar index of refraction. But ordinary particles, such as electrons, can propagate in water at speeds *higher* than c, which are exactly the condition of Class 2 above.

Note that this is the simplest conceivable isotopic lifting of Einstein's Special Relativity, given by the *scalar isotopy* of Minkowski invariant

$$x^{\mu} g_{\mu\nu} x^{\nu} = \frac{1}{n^2} x^{\mu} \eta_{\mu\nu} x^{\nu} \, . \tag{3.275}$$

We can therefore say that the scalar isotopy of the Minkowski invariant represents the transition from motion in vacuum (Einstein's conditions) to motion within a transparent, homogeneous and isotropic fluid (Santilli's conditions of Class 2).

CLASS 3: $1 > n_1 = n_2$, $V_{\text{Max}} = c_0$, $c > c_0$.
No known example exists in this class to our knowledge. Santilli conjectured as a probable candidate *superconductivity*, in which electrons can indeed attain the speed of light in vacuum, and the quantity c does not represent a physical speed, but merely a geometrization of the medium in which electrons move.

This is the last case that can be represented via scalar isotopy (3.275). As such, its dynamical implications (i.e., deviations from Einstein's laws) are minimal, as we shall illustrate later on in this section.

CLASS 4: $1 < n_1 < n_2$, $V_{\mathrm{Max}} < c_0$, $c < c_0$.
In this case, the medium is generally opaque to all electromagnetic waves. The quantity c therefore represents a geometrization of the medium and not necessarily a physical speed. The maximal causal speed is then the maximal possible speed of massive physical particles (remember that no neutrino is admitted at this classical level) which is given by $V_M < c_0$ because of "drag" and other dynamical effects caused by motion within the medium considered.

The best available illustration of this case is given by Nielsen-Picek metric for the case of the pions, Eq. (3.266), with the understanding that the medium in the interior of a pion can be geometrized via a constant only as a crude approximation [102].

Another possibility conjectured by Santilli [26] for Class 4 is the geometrization of ordinary conductors in non-superconductivity conditions, such as ordinary metals. In fact, the medium is manifestly opaque to all electromagnetic waves, while the impossibility for electrons to attain the speed c_0 exactly is plausible on numerous counts. Note that the lifting of the Special Relativity for $n_1 \neq n_2$ is no longer trivial, i.e., it cannot be expressed via the simple scalar isotopy (3.275), thus requiring structural changes of the theory, which will be more transparent later on in the study of the remaining postulates of the covering relativity.

It should be finally stressed that, from Class 4 on, the opacity of the medium is intended in general, with the understanding that particular cases of media transparent to electromagnetic waves are expected to exist.

The best example identified in the literature is the *medium surrounding quasars and astrophysical bodies.* These media are necessarily inhomogeneous (e.g., because their density tends to zero with the distance) and anisotropic (e.g., because the medium is spinning along a preferred direction in space).

Yet, these media allow the propagation of light from the interior of the astrophysical bodies to the empty space beyond the medium itself and, as such, they are transparent to electromagnetic waves. As a result, these media *cannot* be represented with scalar isotopies of type (3.275), and require the full use of Santilli's Special Relativity.

The only geometrization of astrophysical, transparent media currently under study is that by Mignani [125], which belongs exactly to Santilli's Class 4 as we shall see better later on when studying Postulate 3.4 on the redshift. Nevertheless, other astrophysical media transparent to electromagnetic waves belonging to some of the remaining geometrical classes are possible, and they will be implied later on.

CLASS 5: $n_1 < n_2 < 1$, $V_{\text{Max}} < c_0$, $c > c_0$.
This is the geometrization of the *interior dynamics of nuclei* originally suggested in ref. [14], and subsequently elaborated in ref.s [18] and [26]. According to this view, nuclear matter causes a geometrical value c higher than the corresponding value c_0 in the absence of matter. The maximal causal speed, that of the nucleon constituents, is then expected to be *smaller* than c_0, as suggested by the excellent approximations of nuclear problems provided by nonrelativistic theories.

Note that, according to this view, the constituents of nuclei (protons and neutrons) *cannot* attain the speed c_0 even under infinite energies. We should again recall that the geometrization is purely classical. As such, nuclei are completely opaque to electromagnetic waves (because virtual or real photons emitted in the nuclear structure are excluded), and the maximal possible speed is that of the nuclear constituents (because of neutrinos and other operator-particles effects are excluded).

We should finally recall that the existence of media of Class 5 which are transparent to electromagnetic waves (e.g., X rays) are not excluded.

CLASS 6: $n_1 > n_2 > 1$, $V_{\text{Max}} > c_0$, $c < c_0$.
No example is known to this writing for this class. Santilli [26] conjectured that, while *hyperdense* astrophysical, transparent

media belong to Class 4, *rarefied*, transparent, inhomogeneous and anisotropic media could belong to Class 6.

In this case, $c < c_0$ would be the actual, physical speed of the electromagnetic waves in the medium while the maximal possible speed of physical particles is conceivably higher than c_0 because of very high turbulences in the medium which provide exactly the contact interactions capable of "breaking the barrier" of the speed of light in vacuum.

CLASS 7: $1 > n_1 > n_2$, $V_{\mathrm{MAX}} > c_0$, $c > c_0$.
This is the case submitted in ref. [14] for the *structure of hadrons at large*, as well as for all remaining cases of hadronic media of density higher than that of hadrons, such as the *core of collapsing stars, supernova explosions*, etc.

In all these cases: a) the media are opaque to light; as a result, the quantity c has a geometrical meaning, and *does not* represent a physical speed; b) the medium is manifestly inhomogeneous (e.g., because of local variations of the density of the overlapping wavepackets) and anisotropic (e.g., because of the spin), thus requiring the use of the full Santilli's Special Relativity; and c) the maximal causal speed is the V_{Max} of the hadronic constituents which is generally higher than the speed of light in vacuum (thus allowing rather intriguing possibilities, e.g., of achieving a *true confinement* of quarks [44], i.e., a confinement with an explicitly computed probability of tunnel effects of free quarks which results to be identically null).

This is, by far, the most intriguing possibility predicted by Santilli's Special Relativity (the "breaking of the barrier" of the speed of light in vacuum by causal, physical signals), which appears to be confirmed by the phenomenological studies on the deviations from Einstein's Special Relativity of the behavior of the meanlife of unstable hadrons with energies, ref.s [96–102]. As an example, Nielsen-Picek metric for the kaon, Eq. (3.265), is a direct illustration of Santilli's geometrization of Class 7, and the same situation is expected for all remaining hadrons, owing to their higher density. For explicit calculations pertaining to Class 7, see Appendix B.

It should be mentioned for completeness that later clasification bring the total number of physical media to nine, depending on the comparative values of 1. n_1 and n_2. This essentially add two types, the first with $n_1 < n_2$ and $n_2 = 1$, which is contained inm Class 4, and the second with $n_1 > n_2$ and $n_2 = 1$, which is contained in Class 6.

The predictions of Class 2 (Cherenkov light) appear to be confirmed by available experimental information; the predictions of Class 7 (motion inside hadrons) appear to have indirect phenomenological confirmations from the elaboration of available data on the behavior of the mean life of unstable hadrons at different energies (§3.4.9 and Appendix B); the predictions of intermediary cases are plausible, but not sufficiently investigated as of this writing.

The understanding is that the future, final resolution of the validity or invalidity of above predictions will require direct fundamental experiments (§3.5.18).

One point is important for this review. Despite the lack of final resolution, there is no experimental or other evidence available at this time that can disprove the prediction of the generalized relativity, to our best knowledge. In fact, as stressed earlier, no information on Einstein's Special Relativity can possibly be applied to the much more complex physical conditions of Santilli's covering.

3.4.11 Isotopic Composition of Speeds

The use of successive Lorentz-isotopic transformations (3.251) yields, after some algebra, the following isotopic covering of Einstein law of composition of velocities, Eq. (3.164)

$$v_{tot} = \frac{v_1 + v_2}{1 + \frac{v_1 b^2 v_2}{c^2}} \,. \tag{3.276}$$

POSTULATE 3.2 [18]: The invariant speed *is not, in general, that of light, but the maximal speed of propagation of massive particles* $V_{\mathrm{Max}} \gtrless c_o$.

In fact, if one assumes in Eq. (3.276) $v_1 = v_2 = c$, one obtains the *noninvariant* condition

$$V_{tot} = \frac{2c}{1 + b^2} \neq c. \tag{3.277}$$

On the contrary, if one assumes $v_1 = v_2 = V_{Max} = c/b$, one obtains the *invariant* relation

$$V_{tot} = \frac{2c/b}{2} = \frac{c}{b} = V_{Max}. \qquad (3.278)$$

Note that, for the case of Einstein's Special Relativity, we trivially have the identity $V_{Max} = c = c_o$, thus recovering the familiar invariance of c_o. The nontriviality of Santilli's Special Relativity is the capability to show that, in actuality, the invariant quantity is V_{Max} and *not* the speed of light.

As indicated in ref. [18], Postulate 3.2 is verified by the Cherenkov light (in which V_{Max} is precisely c_o), and appears to be plausible for other cases. After all, *when light propagates at speeds smaller than c_o, those speeds cannot be the invariant of the theory.* Postulate 3.2 is equivalently reached when no electromagnetic wave can propagate at all within the medium considered. Any consistent relativity must, under these conditions, provides the invariance only of the maximal speed of propagation of causal signals.

3.4.12 Isotopic Generalization of Time Dilation and Lorentz Contraction

The generalization of Einstenian laws (3.165) and (3.166) provided by Santilli's Special Relativity can be directly read from the Lorentz-isotopic transformations (3.251).

POSTULATE 3.3 [18]: The dependence of time intervals with speed follows the law of isotopic time dilation

$$\Delta t = \hat{\gamma}\Delta t_o = \frac{\Delta t_o}{(1 - \frac{vb^2v}{c^2})^{1/2}}, \qquad (3.279)$$

while space intervals follow the law of isotopic space contraction

$$\Delta l = (1 - \hat{\beta}^2)^{1/2}\Delta l_o = (1 - \frac{vb^2}{c^2})^{1/2}\Delta l_o. \qquad (3.280)$$

As indicated in §3.4.9, isotopic law (3.279) appears to be confirmed by all available phenomenological elaborations of the dependance of the mean life of unstable hadrons at different speeds, although still in a preliminary way because of the lack of direct experiments. In the final part of §3.5 we shall show that isotopic law (3.279) has a truly crucial character for fundamental experiments, such as the resolution of the validity or invalidity of the locally Lorentz character of current theories.

3.4.13 Isotopic Generalization of the Doppler's Effect

The generalization of the Doppler's effect for motion within physical media (assumed in this section to be transparent to light) is straightforward, and was worked out in detail in ref. [26].

The fundamental physical assumptions are that the medium considered is *inhomogeneous* (e.g., because of the variation of the density with the local coordinated), and *anisotropic* (e.g., because of the spinning of the medium, thus creating a preferred direction in space).

A typical example is the hyperdense transparent medium existing around quasars and other astrophysical objects.

The dynamics in the medium considered *cannot* be reduced to a scalar isotopy of the Minkowski metric of type (3.275). The inomogenuity and anisotropy of the medium then render *necessary* the inapplicability of Einstein's Special Relativity for the exact calculation of the Doppler's shift. The only scientifically debatable issue is the identification of the appropriate *generalization* of Einstenian Doppler's shift law.

The solution proposed by Santilli [26] (see also ref.s [24]) is based on his geometrization of the class of media considered, Class 4 or 6, and is reviewed below.

The "plain wave" form of the electromagnetic waves on Santilli's isotopic space $\hat{M}_I$ can be written

$$\hat{\psi}(x) = A\exp(ik^{\mu}\eta_{\mu\nu}x^{\nu})|_{\hat{\xi}} = A\exp(ik^{\mu}g_{\mu\nu}x^{\nu}|_{\xi}\hat{I}, \tag{3.281}$$

where one can recognize the familiar expansions in the isoenvelope $\hat{\xi}$ as well as the expansion in the original envelope ξ. The isounit $\hat{I}$ shall be ignored hereon for simplicity (see §3.4.5 for comments in this respect).

The k-isovector in Eq. (3.281) is an isonull vector with components

$$k = (\vec{k}, \frac{w}{c}), \quad k^4 = \frac{w}{c} = \frac{2\pi}{\lambda},$$

$$k^{\mu}g_{\mu\nu}k^{\nu} = \vec{k}b^2\vec{k} - w^2 = 0, \tag{3.282}$$

where w is the wave frequency and $\vec{k}$ is the wave vector. As a first step, we have assumed the three-space with metric $g_{ij} = b^2\delta_{ij}$.

Suppose that light propagates along the third axis and i is detected by two observers S and S', e.g., one at rest with the source of the wave and one in motion with respect to it at relative speed $\vec{v}_j$ along x^3. Suppose also that $\vec{k}$ makes an angle α with the x^3-axis in frame S, $k^3 = |\vec{k}| = \frac{w}{c}\cos\alpha$. Let ω', $\vec{k}'$, and α' be the corresponding quantities in frame S'.

Santilli's Special Relativity requires the form invariance of the "isowave" (3.281), i.e.,

$$k^{\mu\prime} g_{\mu\nu} x^{\nu\prime} \equiv k^{\mu} g_{\mu\nu} x^{\nu}, \tag{3.283}$$

under which

$$\begin{aligned} k^{1\prime} &= k^1, \quad k^{2\prime} = k^2, \\ k'^3 &= \hat{\gamma}(k^3 - \beta k^4) = |k'|\cos'\alpha, \\ k'^4 &= \hat{\gamma}(k^4 - \hat{\beta}k^3) = \frac{\omega'}{c}. \end{aligned} \tag{3.284}$$

Elementary algebra then leads to the following

POSTULATE 3.4 [26]: The Doppler's frequency shift for electromagnetic waves propagating within physical media (e.g., transparent fluids) follows the isotopic law

$$\omega' = \omega\hat{\gamma}(1 - \hat{\beta}\cos\alpha), \quad \hat{\gamma} = (1 - \hat{\beta}^2)^{-1/2}, \quad \hat{\beta}^2 = \frac{vb^2v}{c^2}, \tag{3.285}$$

with isotopic aberration rule

$$\cos\alpha' = (\cos\alpha - \hat{\beta})/(1 - \hat{\beta}\cos\alpha). \tag{3.286}$$

Postulate 3.4 was recommended by Santilli for the study of possible revisions of current views on quasar redshift of two types. A first correction, expected to be of dominan quantitative character, for the propagation of light within the inhomogeneous and anisotropic media surrounding the quasars. A second correction, of quantitatively smaller implications, is expected by the fact that space can be considered empty, and thus *exactly* homogeneous and isotropic, only locally in our planetary system. When large interplanetary distances are considered, space is far from empty, being filled up with energy (light), dust, elementary particles, etc. It is not inconceivable that these actual physical characteristics of space, when properly treated in a quantitative way, result in an (expectedly small) additional correction to Einstein's law for the Doppler's redshift.

The above suggestions were worked out by Mignani [125] who recalls the unplausibility of the current interpretation of the quasar redshift, because it requires the motion of quasars in vacuum, thus *under strict Einstenian conditions*, at speed much higher than the speed of light in vacuum, thus in *violation of Einstenian laws under Einstenian conditions* (which are strictly prohibited in Santilli's Special Relativity). In fact, the use of the Einstenian redshift law now calls for quasars speeds in vacuum of the order of $10c_0$ and higher, which are manifestly unplausible, as stressed by H. Arp [126].

Mignani pointed out that the use of Santilli's geometrization of inhomogeneous and anisotropic transparent media surrounding quasars, with consequential Postulate 3.4 on the Doppler's shift, permits in principle the complete interpretation of the quasars redshift as due to the propagation of light within such media, while the quasars can remain *at rest* with respect to the associated galaxies, exactly as suggested by Arp's astrophysical measures [126].

Stated in different terms, Mignani [125] showed that the "Doppler's redshift" may not be "Doppler's" at all, in the sense that there may be no motion at all, and at best "partially Doppler's", in the sense that it may be due to motion only in part.

In particular, Mignani, *loc.cit.*, computed Santilli's parameters $k = \frac{b}{b_4} = \frac{n_2}{n_1}$ for several different galaxies, evidently as an average of parameters (3.272) over the entire medium surrounding the quasar. By assuming that the entire shift is due to propagation in the medium considered, Mignani obtained the following values: $k = 31.91$ for quasars $OB1$; $k = 20.25$ for quasar $BSQ1$; $k = 87.98$ for quasar 68; etc.

A number of comments are in order. The first point that should be remarked is that Mignani's calculations [125] avoid the violation of Einstein's relativity under Einstein's conditions which is prohibited in all Santilli's contributions and evidently adopted in this monograph.

Second, Mignani's calculations provide a clear illustration of Santilli's geometrization of the hyperdense, transparent, medium around quasars as being of Class 4, evidently because it emerges that $n_1 < n_2$ and, more specifically, $1 < n_1 < n_2$ for *all* quasars considered, as the reader can verify.

The second, quantitatively smaller corrections to the Einstenian redshift law expected from the lack of empty character of space was not computed in paper [125], and it is recommended for study by interested researchers.

Even though all calculations conducted by Mignani lead to a characterization of Class 4, the possible existence of very low density, extremely turbulent atmospheres around astrophysical objects leading to a characterization of Class 6, should not be excluded. It is hoped that additional astrophysical calculations along Mignani's line are conducted so as to clarify, in due time, the existence or lack of existence of physical media of Class 6.

As indicated earlier, the numerical values obtained by Mignani for the ratio $k = n_2/n_1$ should be interpreted as average of k over the entire atmosphere surrounding a quasar. But, as indicated by Eq. (3.272), the quantity k is a rather complex function of the local variables, $k = k(x; n; \mu; \tau; \cdots)$. Thus, the study of the average itself leading to Mignani's numerical results is important, and it is also recommended to the interested researcher.

Finally, we should note that the generalization of the Einstenian Doppler's shift law occurs only when there is a space-time anisotropy for which $b/b_4 \neq 1$. For instance, in the case of propagation of light in water (Cherenkov light), we have isotopy (3.275) under which isotopic laws (3.285) and (3.286) coincide with Einstenian laws (3.167). Santilli [26] reached in this way the important conclusion that *the Doppler's shift for light in water follows the conventional Einstenian law* [26].

3.4.14 Isotopic Generalization of Relativistic Kinematics

We now pass to the review of the isotopic generalization of the conventional kinematics for one particle according to ref. [26], which, as the reader may readily predict, is the basis for the isotopic lifting of field equations outlined in Appendix C.

The generalization is mathematically quite simple. Nevertheless, its physical implications are far reaching. It is recommendable to mention at this point the fact that, in some of his last papers, P. A. M. Dirac [54] proposed a generalization of his celebrated equation which results to be precisely of isotopic type [27], that is, of a quite simple generalized mathematical structure. Nevertheless, the spin of the represented particle is altered from the value 1/2 of the conventional equation into the value 0 of the isotopic form, as we shall review in Appendix C.

Introduce the infinitesimal invariant in isotopic space $\hat{M}_I$

$$ds^2 = -dx^\mu g_{\mu\nu} dx^\nu = dt c^2 dt - dx^i g_{ij} dx^j,$$

$$g_{ij} = b^2 \delta_{ij}, \quad c = b_4 c_0 , \tag{3.287}$$

from which one can write

$$-\frac{dx^\mu}{ds} g_{\mu\nu} \frac{dx^\mu}{ds} = 1. \tag{3.288}$$

We now define as *iso-four-velocity* the vector on $\hat{M}_I$

$$u^\mu = \frac{dx^\mu}{ds}. \tag{3.289}$$

To compute the components of u^μ, we can write from Eq. (3.288)

$$(\frac{dt}{ds})^2(c^2 - \frac{dx^i}{dt}g_{ij}\frac{dx^j}{dt}) = 1, \qquad (3.290)$$

from which we have the fourth component

$$u^4 = \frac{dt}{ds} = \hat{\gamma}c,$$
$$\hat{\gamma} = (1-\beta^2)^{-1/2},$$
$$\hat{\beta}^2 = \frac{v^i g_{ij} v^j}{c^2}. \qquad (3.291)$$

The space components are then given by

$$u^k = \frac{dx^k}{ds} = \frac{dt}{ds}\frac{dx^k}{dt} = \hat{\gamma}cv^k. \qquad (3.292)$$

We now define as *iso-four-momentum* on $\hat{M}_I$ the four-vector

$$p^\mu = m_o u^\mu, \quad p = (m_o\hat{\gamma}c\vec{v}, m_o\hat{\gamma}c). \qquad (3.293)$$

By recalling the lowering and raising of the indices in $\hat{M}_I$ of §3.4.7, we then have the fundamental property

$$\begin{aligned} p^\mu g_{\mu\nu} p^\nu = p_\mu g^{\mu\nu} p_\nu = p^\mu p_\mu = p_\mu p^\mu &= m_o^2\hat{\gamma}^2c^2v^ig_{ij}v^j - m_o^2\hat{\gamma}^2c^4 \\ &= m_o^2c^4\hat{\gamma}^2(\frac{v^ig_{ij}v^j}{c^2} - 1) \\ &= -m_o^2c^2 = -m_0^2b_4^2c_0^2\,, \qquad (3.294) \end{aligned}$$

that is, *the value* $m_oc_o^2$ *of the conventional relativity is replaced by* m_oc^2 *of the generalized relativity.*

By keeping the conventional assumption

$$p^4 = \frac{E}{c}, \qquad (3.295)$$

where E is the energy of the particle, we have from Eq. (3.294)

$$p^4c^2p^4 - \vec{p} * \vec{p} = E^2 - p^ig_{ij}p^i = m_o^2c^4, \qquad (3.296)$$

with the following consequences.

POSTULATE 3.5 [26]: The mass m_o of a particle moving within a physical medium varies with speed according to the isotopic law

$$m = m_o \hat{\gamma},$$
$$\hat{\gamma} = (1 - \hat{\beta}^2)^{1/2},$$
$$\hat{\beta} = \frac{v^i g_{ij} v^j}{c^2}, \quad (3.297)$$

and its equivalent value of the energy for at rest conditions is given by the isotopic mass-energy relation

$$E = m_o c^2 \gtrless m_o c_o^2, \quad (3.298)$$

where c is the speed of the light (or electromagnetic wave) within the medium considered, when admissible, or a geometrical quantity characterized by the medium itself as in Classes 1–7, §3.4.10.

Note that for the case of water represented by invariant (3.275), Eq.s (3.297) coincide with the conventional ones, trivially, because in this case $b/c = c_o$. Also, in this case $\hat{p^2} = \frac{1}{n^2} p^2$ and the conventional Einstenian expression $E = m_o c_o^2$ can be recovered. In order to have a nontrivial departure from the conventional relativity one must have an isotopic generalization of the Minkowski metric *other than its scalar isotopy* $g = \frac{1}{n^2}\eta$.

For the case of Nielsen-Picek metric for kaons, Eq. (3.170), one has

$$m = m_o (1 - \frac{v^2}{c_o^2} \frac{1 - \frac{1}{3}\alpha}{1 + \alpha})^{-1/2},$$

$$E = m_o c_o^2 (1 + \alpha) > m_o c_o^2,$$
$$\alpha = (0.61 \pm 0.17) \times 10^{-3}, \quad (3.299)$$

that is, the mass m of the particle *is not* infinite at speed c_o (because the particle in that metric can exceed c_o, see §3.4.9) and the energy equivalent at rest is *higher* than that predicted by Einstein's Special Relativity.

Conceivably, the above deviations from the conventional predictions are suitable for experimental resolutions in favor or against the predictions of Postulate 3.5.

It should be stressed that isotopic laws (3.299) are referred, specifically, to a mesonic constituent and not to the particle as a whole.

Note that the fundamental isoinvariant (3.294) is the central starting point of the Lie-isotopic generalization of classical (and operator) field theory. See in this respect Appendix C.

An important particularization of Santilli's Special Relativity with minimal deviations from the conventional setting has been worked out by Animalu [127]. We regret to be unable to review it here for brevity.

3.4.15 Relativistic, Closed, Nonhamiltonian-Birkhoffian Systems

We now review the studies presented in ref. [26] on the relativistic extension of the Galilei-isotopic notion of closed, nonhamiltonian systems.

Consider the *iso-four-force* on isotopic space $\hat{M}_I$, which is given by the Minkowski force

$$K = (K^{\mu}) = (\vec{K}, \frac{1}{c^2 u^4} K^i g_{ij} R^i), \tag{3.300}$$

referred to the isotopic contraction on $\hat{M}_I$. This means that K^{μ} is no longer orthogonal to the four velocity u^{μ} (§3.4.13) on the conventional Minkowski space, i.e., $K^{\mu}\eta_{\mu\nu}u^{\nu} \neq 0$. We have instead the isotopically lifted property

$$K^{\mu} g_{\mu\nu} u^{\nu} = K_{\mu} g^{\mu\nu} u_{\nu} = K^{\nu} u_{\mu} = K_{\mu} u^{\mu} = 0. \tag{3.301}$$

The dynamic equations for one particle can therefore be written

$$m_o \frac{du^{\mu}}{ds} = K^{\mu}. \tag{3.302}$$

The space component is given by

$$m_o \frac{d\vec{u}}{ds} = m_o \frac{dt}{ds}\frac{d\vec{u}}{dt} = m_o \hat{\gamma} c \frac{d\vec{u}}{dt} = \vec{K}. \tag{3.303}$$

thus yielding the following relationship

$$\vec{F}_{Newton} = \frac{1}{\hat{\gamma} c} \vec{K}_{Santilli}. \tag{3.304}$$

As now familiar, forces (3.304) are classified into self- adjoint (SA) and non-self-adjoint (NSA) depending on whether they are derivable or not from a potential [4].

We reach in this way the following "relativistic" generalization of the Galilei-isotopic systems (3.93) of N particles with Hamiltonian and nonhamiltonian forces [58]

$$m_o^a \frac{du_a^{\mu}}{ds} = K_{aSA}^{\mu} + K_{aNSA}^{\mu},$$

$$a = 1,2,\ldots,N,$$

$$\frac{d}{ds}P^{\mu}_{tot} = \frac{d}{ds}(\sum_{a=1}^{N} p^{\mu}_{a}) = 0,$$

$$\frac{d}{ds}M^{\mu\nu}_{tot} = \frac{d}{ds}(\sum_{a=1}^{N} M^{\mu\nu}_{a}) = 0,$$

$$\dot{x}^{\mu}_{a} g_{\mu\nu} \dot{x}^{\nu}_{a} = -1, \tag{3.305}$$

where the ten conservation laws for the P^{μ}_{tot} and $M^{\mu\nu}_{tot}$ are the relativistic version of the ten Galilean conservation laws of Eq. (3.93) and the last equations are the conventional constraints for relativistic theories, evidently expressed in the generalized metric.

Equations (3.305) constitute systems of $4N$ ordinary differential equations with $N + 10$ subsidiary constraints, which can be interpreted as algebraic constraints in the $4N$ components of K^{μ}_{NSA}. Thus, for $N \geq 3$, system (3.305) admit an infinite variety of solutions, the case $N = 2$ being a special one, exactly as it happened in the Galilean case (§3.3).

Systems (3.305) are of central relevance for Santilli's Special Relativity, not only classically, but also operationally. In fact, the systems are proposed as a classical limit of structure models of hadrons, in which each extended-deformable constituent moves within a medium with metric $g_{\mu\nu}$ geometrizing the wavepackets of the other constituents. As presented, systems (3.305) are based on the requirement that *the center of mass of the system, when seen from an outside observer, obeys Einstein's Special Relativity, while having a manifestly generalized internal structure obeying Santilli's Special Relativity.*

A large variety of generalizations, implementations and modifications of systems (3.305) are conceivable. Stronger requirements may be expressed by the more restrictive systems [26]

$$m^{a}_{o}\frac{du^{\mu}_{a}}{ds} = K^{\mu}_{aSA} + K^{\mu}_{aNSA},$$

$$a = 1,2,\ldots,N,$$

$$\frac{d}{ds}P^{\mu}_{tot} = 0, \frac{d}{ds}M^{\mu\nu}_{tot} = 0, \dot{x}^{\mu}_{a} g_{\mu\nu} \dot{x}^{\nu}_{a} = -1,$$

$$\frac{d}{ds}(P^{\mu}\eta_{\mu\nu}P^{\nu}) = 0,$$

$$\frac{d}{ds}(W^{\mu}\eta_{\mu\nu}W^{\nu}) = 0, \quad W_{\alpha} = \frac{1}{2}\varepsilon_{\alpha\mu\nu\lambda}M^{\mu\nu}_{tot}P^{\lambda}_{tot},$$

$$\dot{X}^{\mu}_{tot}\eta_{\mu\nu}\dot{X}^{\nu}_{tot} = -1, \quad X^{\mu}_{tot} = \sum_{a=1}^{N} x^{\mu}_{a}, \tag{3.306}$$

where one can see, not only the ten conventional, relativistic conservation laws, but also the condition that total quantities can be well defined in the conventional Minkowski space. For these stricter systems, *the generalized internal structure is not detectable from the outside, trivially, because external observers can only detect total quantities, and such quantities are constrained to a conventional Minkowski space.*

Note that, despite their restrictive character, systems (3.306) remain consistent for sufficiently large N. In fact, the total number of constraints is $N+13$. By assuming that the self- adjoint forces are conventionally assigned, one remains with a total of $4N+4$ free functions, the $4N$ components of K^{μ}_{NSA} and the four diagonal functions of the metric g. Solutions then exists, again, for $N \geq 3$, and they are expected for $N = 2$.

Following ref. [26], we shall now present a special class of closed nonhamiltonian systems, those verifying by construction the Poincaré-isotopic symmetry. In this case the ten conservation laws for P^{μ}_{tot} and M^{μ}_{tot} are guaranteed by the symmetry itself (see Theorem 3.3).

Recall that the kinetic energy of each particle is given by

$$E^{a}_{kin} = m^{a}c^{2} = m^{a}_{o}\hat{\gamma}c^{2}, \tag{3.307}$$

and, for "nonrelativistic" conditions, does indeed recover the corresponding Galilei-isotopic counterpart (except for a scale term)

$$E^{a}_{kin} \approx Em^{a}_{o}c^{2}(1+\frac{1}{2}\hat{\beta}^{2}) = E^{a}_{o} + \frac{1}{2}m^{a}_{o}v^{i}g_{ij}v^{j}. \tag{3.308}$$

Suppose that the particles are under the action of an electromagnetic field generated by the other particles, and represented with the four potential $A^{\mu}_{a}(x)$. Then, the correct form of the potential energy in $\hat{M}_I$ must be written

$$U(x,\dot{x}) = \sum_{a}\frac{e_a}{c}A^{\mu}_{a}(x)g_{\mu\nu}\dot{x}^{\nu}_{a}. \tag{3.309}$$

To add the contact nonhamiltonian forces, one needs only to generalize conventional, relativistic, variational principles into those of Birkhoffian type, where the Birkhoffian is the conventional total Hamiltonian properly formulated on $\hat{M}_I$.

The simplest class of such systems is a relativistic generalization of the Birkhoffian systems used to identify realizations of the isotopic group of rotations $\hat{O}(3)$, Eqs. (3.57). By assuming for conventional Hamiltonian on $\hat{M}_I$ expressions of the type (see, e.g., ref. [128], p. 127)

$$H = \sum_a \frac{1}{2\lambda_a} p_a^\mu g_{\mu\nu} p_a^\nu - \sum_a m_a c^2 + U(x, \dot{x}) + \frac{1}{2}\sum_a \lambda_a, \tag{3.310}$$

where the λs are multipliers, the desired particularization of systems (3.305) can be represented by the generalized action [26]

$$\hat{A} = \int_{s_1}^{s_2} [\sum_a p_a^\mu g_{\mu\nu}(p)\dot{x}_a^\nu + \sum_a \Gamma_a \hat{\lambda}_a - H] ds. \tag{3.311}$$

The symplectic tensor of the theory is then given by

$$\Omega_{ij} = \begin{pmatrix} 0 & -g_{\mu\nu} \\ g_{\mu\nu} & 0 \end{pmatrix}, \quad i, j = 1, 2, ..., 8, \tag{3.312}$$

with corresponding, Lie-isotopic counterpart

$$\Omega^{ij} = \begin{pmatrix} 0 & g_{\mu\nu}^{-1} \\ -g_{\mu\nu}^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & g^{\mu\nu} \\ -g^{\mu\nu} & 0 \end{pmatrix}, \tag{3.313}$$

and generalized Poisson brackets

$$[A \hat{,} B] = \frac{\partial A}{\partial x^\mu} g_{\mu\nu}^{-1} \frac{\partial B}{\partial p^\nu} - \frac{\partial B}{\partial x^\mu} g_{\mu\mu}^{-1} \frac{\partial A}{\partial p^\nu}. \tag{3.314}$$

The departure of the generalized from the conventional brackets is a direct representative of the non-self-adjoint forces, as the reader is urged to verify via a study of monograph [15]. Systems (3.311) then assume the Birkhoffian form

$$\dot{x}_a^\mu = g^{\mu\nu} \frac{\partial H}{\partial p_a^\nu}, \quad \dot{p}_a^\mu = -g^{\mu\nu} \frac{\partial H}{\partial x_a^\nu},$$
$$\dot{\lambda}_a = \frac{\partial H}{\partial \Gamma_a}, \quad \dot{\Gamma}_a = -\frac{\partial H}{\partial \lambda_a}, \tag{3.315}$$

where, as now familiar, the last two sets of equations represent the subsidiary constraints $\dot{x}_a * \dot{x}_a = -1$.

Note the manifest Poincaré-isotopic invariance of Pfaffian action (3.311), with consequential conservation laws. Note also that the conserved quantities are the *conventional* ones because Santilli's Lie-isotopic theory leaves unchanged the parameters and generators of the original symmetry.

Almost needless to say, several refinements of systems (3.305), (3.306) or (3.315) are possible, most notably, that via Dirac's theory of relativistic systems with constraints. For brevity, we must refer the interested reader to paper [26] for a discussion of these and other aspects. For the purpose of this paper it has been sufficient to review that:

1. the relativistic generalization of Galilei-isotopic, closed, nonhamiltonian systems can be consistently formulated in isospaces $\hat{M}_I$;

2. such systems are not only consistent, but generally admit infinite varieties of different solutions; and

3. the systems admit a representation in terms of (relativistic) Birkhoff's equations which allows the identification of the generalized metric of the theory from given nonhamiltonian forces via the use of the techniques of monograph [15].

The above results are sufficient for the limited scope of this review.

In closing, we would like to mention another important consequence of Santilli's Special Relativity, that of being able to bypass the so-called *No-Interaction Theorem* of Einstein's Special Relativity (see, e.g., ref. [128]). The theorem essentially states that, under certain quite plausible, Lorentz-covariant conditions, *systems of particles that are in nontrivial mutual interactions are incompatible with Einstein's Special Relativity.* For the isotopic setting we have instead the following property.

Theorem 3.8 [26]: *(The No No-Interactions Theorem). Systems of particles on isospace $\hat{M}_I$ which verify Santilli's Special Relativity (i.e., are covariant under the Lorentz-isotopic group) cannot be reduced to a free form unless the generalized relativity is reduced to the conventional Einstenian relativity (and the isospace $\hat{M}_I$ is reduced to the conventional Minkowski space $\hat{M}$, except for a possible scalar isotopy).*

The proof of the theorem [26] is based on Gasperini's [46] Lie-isotopic generalization of Einstein's Gravitation and, in particular, the property that, under a nontrivial isotopy of the Minkowski metric, the motion *cannot* be reduced to a geodesic one (i.e., it is irreducibly nongeodesic), thus establishing the existence of irreducible nontrivial interactions.

The main conceptual foundations of this important result is the following. Recall that the Birkhoffian formulations are based on a nontrivial generalization of the conventional canonical action into a Pfaffian form. Now, at the conventional (relativistic) level, there exist canonical transformations capable of reducing the system to its free, and therefore noninteraction, form.

At the level of Santilli's Special Relativity, the situation is different. In fact, the use of the transformation theory can at best reduce the Birkhoffian to the "free" form, but the system remains interacting owing to the remaining nontrivial Pfaffian terms.

To put it differently, the conventional transformation theory can at best eliminate the potential-Hamiltonian forces, but not the contact, nonhamiltonian forces owing to their representation by the generalized unit (and related Lie-isotopic brackets) which evidently remain unaffected by the transformation theory. This yields Theorem 3.8 above.

In conclusion, *particles obeying Santilli's Special Relativity and which therefore admit a generalized isounit $\hat{I}$, cannot be free.* This important property has truly fundamental implications, especially in particle physics, as we hope to indicate in a subsequent review on "hadronic mechanics" (§1.3).

3.4.16 The Direct Universality of the Lorentz-Isotopic Symmetry

As the reader familiar with the Lie-isotopic techniques can now predict, *Santilli's Lie-isotopic generalization of the Lorentz symmetry is directly universal, i.e., capable of including all possible cases of noninvariance or generalizations considered until now (universality) without any need of the transformation theory (direct universality).*

This property is a direct consequence of the arbitrariness of the metric g in the Lorentz-isotopic symmetry. It is the "relativistic" counterpart of the direct universality of Birkhoffian mechanics and its Galilean-isotopic relativity [15].

As an illustration, the generalized relativity on isotopic spaces $\hat{M}_I$ includes, as particular cases, *all* models of Lorentz noninvariance reviewed in §3.4.3.

This latter property has been studied in detail by Aringazin [119] for the case of (1 + 1)-dimensional spaces with components x^3 and x^4, and the assumption that the quantity b_3 of the Lorentz-isotopic transformations (3.251) does not depend on the local coordinates, but only on the velocities v, thus allowing power series expansion of the type

$$b_3(v) = 1 + \lambda_o + \lambda_1\gamma + \lambda_2\gamma^2 + \ldots, \tag{3.316}$$

where the λ's are much smaller than one, and the quantity γ is the conventional relativistic one. By putting $c = 1$ for convenience, Aringazin expresses behavior (3.279) of the mean life of unstable hadrons in the form

$$\tau = \tau_o\gamma\{1 + \lambda_o\gamma^2 + \lambda_1(1 + \lambda_o)\gamma^3 + [\frac{\lambda_1^2}{2} + \lambda_2(1 + \lambda_o)]\gamma^4 + \ldots\}, \quad (3.317)$$

which evidently includes behavior (3.169) by Blockhintsev [96] and Pecei [97], as well as behavior (3.171) by Nielsen and collaborators [99].

The case of behavior (3.175) by Aronson *et al.* [101] for the mean life is also a subcase of Aringazin's expansion (3.317). The corresponding behavior for other parameters of the $K^0 - \overline{K}^0$-system, such as the mass difference, were obtained by Aringazin via a Lie-isotopic lifting of the field equations compatible with the assumed structure of the metric underlying Eq. (3.317). For brevity, we refer the interested reader to ref. [119].

Finslerians spaces are also a subcase of the isotopic spaces and are obtained, trivially, by factorizing the anisotropic term from all terms of the metric $g = \text{Diag.}(\vec{b}^2, c^2)$, as the reader can verify.

In case a model of Lorentz noninvariance breaks the topology of the Minkowski metric but it is still flat, the isotopic spaces $\hat{M}_{II}$, are needed. The lifting of the Lorentz symmetry is in fact unaffected by this generalization, as indicated earlier. This broader class includes models such as Recami-Mignani's superluminal invariants (3.253), ref. [123], and others.

The illustration of the direct universality of the Lorentz-isotopy with other available models is left to the interested reader.

3.4.17 Reconstruction of the Exact Lorentz-Isotopic Symmetry when Conventionally Broken

One of the most important properties of Santilli's Special Relativity is that of being able to reconstruct as exact, at the isotopic level, space-time symmetries that are conventionally broken (see fundamental Theorem 2.9).

The direct consequence is that all statements of "Lorentz noninvariance" or "breaking of the Lorentz symmetry" available in the contemporary literature are generally incorrect on strict technical grounds.

As an example, consider the phenomenological studies by Nielsen and Picek [99]. They result in generalized metric (3.170), i.e.,

$$g = \text{diag}(1 - \frac{1}{3}\alpha, 1 - \frac{1}{3}\alpha, 1 - \frac{1}{3}\alpha, -(1 + \alpha)) = T\eta\,, \quad (3.318)$$

which is clearly "Lorentz noninvariant" but only when the symmetry is realized in its simplest possible way, that via the trivial Lie product $AB - BA$.

If the Lorentz symmetry is realized instead in a more general way, via Santilli's isotopic products $A * B - B * A = ATB - BTA$, with the metric g given by form (3.318), then the Lorentz symmetry remains exact (Theorem 3.6).

A similar situation occurs for all cases of "Lorentz noninvariance" reviewed in §3.4.3 Consider, for instance, the generalization of Einstein's Special Relativity proposed by Bogoslovski [113] for Finslerian invariants of type (3.176). Since the topological character of the conventional Minkowski metric is preserved (for positive-definite anisotropic terms), Theorem 3.5 applies and the abstract Lorentz symmetry in Bogoslovski's Special Relativity remains exact.

The implications of the above results are far reaching. *A central result of the Lie-isotopic studies is that, by no means, the Lorentz symmetry is "broken" and therefore "abandoned." Instead, it is preserved in full, although realized in its most general possible form.*

As a consequence, all *the "deviations" from conventional laws expressed by Postulates 3.1-3.5 are deviations from the Einstenian realization of the Lorentz symmetry, and not from Lorentz symmetry which remains exact.*

3.4.18 Epistemological Comments

A few epistemological comments are important to illustrate in more depth the physical departures of Santilli's Special Relativity from the Einstenian one.

As now familiar, the generalized relativity has been constructed with the objective of admitting the Galilei-isotopic relativity as particular case for "nonrelativistic" speeds. As a result, all the epistemological comments of §3.3.9 on the Galilei-isotopic relativity apply, of course, in their "relativistic" generalization. Traditionally, the (conventional) Lorentz symmetry has been assumed as the fundamental symmetry of Nature. The metric (or the equations of motion) have then been restricted to comply with such a symmetry. Santilli advocates the reverse attitude: *one should assume as fundamental physical information the metric (or equations of motion) as provided by experimental, phenomenological or other evidence, and then seek the generalized relativity capable of leaving that metric (or equations of motion) invariant.* The insistence on the assumption of the conventional Lorentz symmetry and Minkowski metric as the fundamental quantities would directly imply disagreements with available phenomenological information, besides forcing the excessive approximations of physical reality indicated in §3.3 (Perpetual-motion approximations in interior dynamics, etc.).

As for the Galilean case, relativistic studies have been essentially restricted until now to only one symmetry, the Lorentz symmetry. *Santilli's Special Relativity characterizes, instead, an infinite number of different symmetry transformations, each of which is a covering of the conventional Lorentz symmetry.* This is evidently due to the infinite variety of possible metrics g. If only one symmetry is imposed, whether conventional or generalized, a substantial limitation on the representational capability of physical reality would follow.

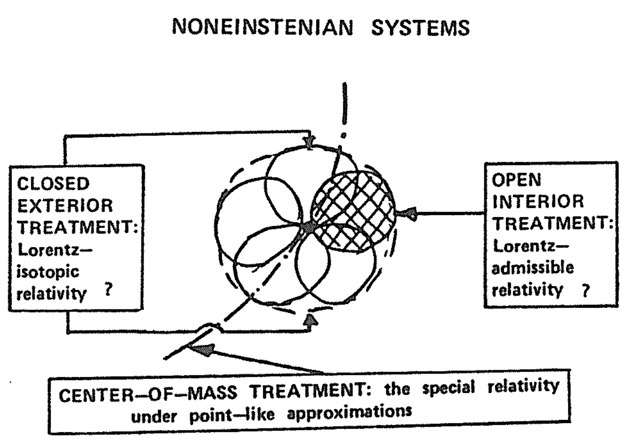


FIGURE 11. A reproduction of Fig. 2 of ref. [26] depicting the various descriptions that may eventually result to be needed for the dynamical behavior of a hadron. First, we have the description of the center-of-mass behavior of the particle under external, action-at-a-distance interactions, say, when moving in a particle accelerator. This first description strictly obeys Einstein's Special Relativity. Second, we have the description of the structure of the particle when inspected from an outside observer. In this case, Santilli's "Lorentz-isotopic relativity" reviewed in this section is recommended because the generalized unit of the theory allows the representation of nonlocal, integrodifferential internal forces due to mutual wave overlapping of the hadronic constituents, all in a way compatible with total, conventional,

conservation laws. Finally, we have a third description, that of one individual constituent when considering all the others as external. The relativity suggested for this latter viewpoint is of "Lorentz-admissible" type along monograph [21], that is, with an algebraic structure capable of directly representing the nonconservation of the physical quantities of the constituent. Note that in the atomic structure one single relativity, Einstein's Special Relativity, is fully sufficient for the description of both, an atom as a whole and one of its peripheral electrons. This is due to the fact that the electrons have stable orbits under potential forces. The conventional Lie-Hamiltonian structure is then sufficient. In the transition to the structure of hadrons, the situation is expected to be different because individual constituents are in generally nonconservative conditions under the action of nonhamiltonian forces. A description of non-Lie type is then recommendable. We hope to review the above aspects in a subsequent review of "hadronic mechanics."

The abandonment of linearity in favor of intrinsically nonlinear, but formally isolinear transformations, is another condition for a more adequate representation of Nature. Again, the insistence on the linearity of the transformations would imply another substantial limitation on the representational capability, with consequential excessive approximation of Nature.

Also, as in the Galilean-isotopic case, *Santilli's Lorentz-isotopic symmetries are generally nonmanifest; yet they can be explicitly computed from the sole knowledge of the new metric and the old Lorentz symmetry.*

Finally, *Santilli's Special Relativity has been conceived to map noninertial frames into noninertial frames, because inertial frames are a conceptual abstraction that cannot be realized in experiments.* From a different viewpoint, the insistence on the preservation of inertial frames would imply only linear symmetry transformations, with the consequential excessive approximation of Nature indicated earlier.

Despite all the above differences with the conventional case, it is remarkable that *Santilli's and Einstein's Special Relativities coincide at the abstract, realization-free level.* By keeping in mind the large variety of different particular cases admitted by the Lie-isotopic theory, this latter property provides genuine hopes for a true, ultimate unity of mathematical and physical thought.

We regret being unable to review a considerable number of contributions, all conceived in a way entirely independent from the Lie-isotopic theory, but which eventually result to be a particularization of the Lie-isotopic lifting of the Special Relativity. We limit ourselves to mention Preparata works [129] which are based on the idea of a possible anisotropy in the interior of hadrons with intriguing implications. Quite clearly, Preparata's research, in its classical formulation, is a particular case of Santilli's Special Relativity which, as now familiar, deals with the most general possible class of anisotropy *and* inhomogeneity in the interior of hadrons. Intriguingly, as now predictable, Preparata's anisotropy *does not* imply the necessary violation of the Lorentz symmetry, which is expected to be recovered as an exact symmetry at the level of Lie-isotopic formulations (Theorem 3.6). Needless to say, the establishing of the property requires a form of statistical averaging [130] of Preparata's space or other approaches capable of reducing the quantum field theoretical setting of ref. [129] to a primitive, classical, anisotropic framework.

Similarly, we are unable to review a rather considerable number of additional research, such as the studies by P. Bandyopadhyay and S. Roy [130], or S. Roy [131] and others.

We would be grateful to any colleague who sends to our attention (at the Institute of Basic Research, P.O Box 1577, Palm Harbor, FL 34682, U.S.A.) articles or references of papers directly or indirectly related to Santilli's Special Relativity for their possible review in a future work.

3.5 Lie-Isotopic Generalization of Einstein's Gravitation

3.5.1 Introductory Remarks

As stressed in the preceding section, when violations occur because of motion within inhomogeneous and anisotropic media, Einstein's Special Relativity must still be considered *approximately* valid.

The situation for *Einstein's General Theory of Relativity* (or *Einstein's Gravitation* for short) is different because of the existence of so many and so deep problematic aspects to create serious doubts even on its approximate validity.

The literature on these problematic aspects accumulated throughout this century is so vast that it cannot possibly be reviewed here. We shall simply limit ourselves to a review of those problematic aspects that have a direct relevance for our objective: a review of the Lie-isotopic generalization of Einstein's Gravitation.

With reference to monograph [5], a necessary condition [64] for understanding the problematic aspects, as well as for avoiding potentially major misrepresentations, is a return to the old separation of (any) theory of gravitation into

A) *The interior problem.* This is essentially the theory of gravitation applicable to the interior of the minimal surface (or sphere, for simplicity) containing all matter (thus including the atmosphere, when it exists).

B) *The exterior problem.* This is essentially the theory of gravitation applicable to the exterior of the above identified surface (or sphere).

The best way to illustrate the distinctions between the above two problems is by observing their physical differences as they occur in Nature.

Consider the motion of a *test particle* in a given gravitational field, say, that of Jupiter. When considering the *exterior* problem, motion occurs in vacuum (empty space), in which case the actual size of the particle is ignorable. We can then effectively deal with a point-like test particle moving in vacuum under a gravitational field, with consequential local conservation laws, e.g., that of the angular momentum.

When the same test particle enters the *interior* problem, the situation is different because we now have motion within a physical medium such as Jupiter's atmosphere. Under these conditions, the actual size of the particle is no longer ignorable, but must be properly represented to avoid excessive approximations of physical reality. We therefore have motion of an extended particle within a physical medium, with the consequential contact, nonhamiltonian forces we have encountered at each level of our analysis (Newtonian, relativistic and, therefore, gravitational), with all their particular physical implications that are simply absent in the exterior problem (inapplicability of the notion of potential, null range, deviations from local conservation laws, etc.).

At a deeper analysis, the distinction between the exterior and the interior problem is even deeper than that. In fact, the interior problem includes not only the long range interactions (electric, magnetic and gravitational), and the contact nonhamiltonian interaction indicated above, but also the short range interactions that are typical of the structure of matter (such as the weak, nuclear and strong interactions). By comparison, the exterior problem includes only the long range interactions without any contact or quantum mechanical effect.

The distinction between the interior and the exterior problem was well known soon after the inception of Einstein's Gravitation but it has been ignored in more recent times, thus reaching the condition of (most of) the contemporary literature in which no mention is made of such distinction.

This is regrettable owing to the incontrovertible experimental evidence establishing the physical differences of the motion of a test particle in the exterior and in the interior problem.

The distinction under consideration is crucial for the physical applications of the Lie-isotopic theory, consistently, at all levels of study, from the Newtonian to the relativistic and to the gravitational level. In fact, the distinction was brought back by Santilli in 1978 [2] with the notion of closed nonhamiltonian systems (§3.3), and then extended to the relativistic context (§3.4). As now familiar, these systems obey conventional relativities for the exterior dynamics, but require a structurally more general description for the interior problem.

It is then natural to expect that a similar distinction plays a fundamental role in the Lie-isotopic formulation of gravity.

A similar distinction also exists in the structure model of hadrons [*loc. cit.*] according to the "hadronic generalization of quantum mechanics" (§1.3) in which, again, conventional quantum mechanical laws and relativities apply in the exterior problem, while structurally more general laws and relativities apply in the interior dynamics. In turn, this dichotomy opens up a truly new frontier of possible advances, we hope to present in a subsequent review, such as: achievement of a true confinement of quarks (with an identically null probability of tunnel effects for free quarks); identification of the quark constituents with physical, ordinary particles; etc.

As it occurred in the preceeding parts of this review, in this section we shall ignore operator profiles and restrict the analysis only to classical aspects. Thus, all short range quantum mechanical interactions of the interior problem will be only marginally indicated without treatment.

We shall now begin with a review of some of the problematic aspects of Einstein's Gravitation; identify what we call an "ideal" theory of gravitation; and then pass to a review of Gasperini [81,82,83,84] and Santilli [5,16,18,26] work on the Lie-isotopic General Relativity.

3.5.2 Problematic Aspects of Einstein's Gravitation for the Interior Problem

Einstein's Gravitation is based on a geometry, the Riemannian geometry, which is *local* and differential. Santilli [5],[16] points out that such a geometry is fundamentally incompatible with the interior problem of celestial bodies, because of the incontrovertible *nonlocal* nature of the forces for the interior dynamics, as well as the ultimate nonlocal nature caused by the mutual penetration and overlapping of the wavepackets of particles in the core of the celestial body considered (Fig. 1). The use of a suitable nonlocal integrodifferential generalization of the Riemannian geometry is therefore advocated as the fundamental mathematical tool for a more adequate treatment of the interior problem.

The above occurrance leaves open the problem whether the Riemannian geometry can be at least *approximately* valid for the interior problem. The answer to this question appears also to be negative.

A known way to approximate contact nonlocal interactions experienced by an extended object moving, say, within a gas, is via *power-series expansions in the velocities,* as well known in Newtonian mechanics.

In this way, *locality* is regained in first approximation; yet the power series in the velocities allows a quantitative treatment of the conditions considered. Santilli [*loc. cit.*] contends that the Riemannian geometry does not allow the representation of a sufficiently high value of the power of the velocities, thus preventing a nontrivial, quantitative treatment of the interior dynamical conditions (this is also known in the specialized literature as the *Cartan legacy*, that is, the inability of the Riemannian geometry to recover all possible Newton's equations of motion under PPN approximations, see ref.[6]).

Another central property of Einstein Gravitation is its intrinsically homogeneous and isotropic character. Santilli [*loc.cit.*] contends that such character is in violation of incontrovertible physical evidence for the interior problem (only). In fact, interior motions are not in empty space, but occur within the physical medium constituted by the celectial body itself. In turn, such medium is, in general, inhomogeneous and anisotropic. As an example, the density of Jupiter manifestly increases with the decrease of the distance from the center.

Owing to this occurence, Santilli [26] advocates the construction of a gravitational theory for the interior problem capable of representing motion within generally inhomogeneous and anisotropic material media. The understanding is that space itself remains homogeneous and isotropic, exactly as in the Newtonian and relativistic cases.

Another central feature of Einstein's Gravitation is its *local Lorentz character.* Santilli [*loc.cit.*] contends that this character too is violated in classical mechanics by incontrovertible physical evidence. In fact, the local Lorentz character implies, in particular, the local rotational symmetries, as well known. Santilli therefore suggests the observation of dynamical systems in the interior problem of our Earth, such as satellites during re-entry with their continuously decaying angular momentum; the vortices in Jupiter's atmosphere with their continuously varying angular momentum; etc. All these systems constitute incontrovertible physical evidence of the breaking of the (conventional) rotational symmetry in our classical environment. The violation of the local Lorentz symmetry is then consequentive.

In the words of the quoted author, *the insistence in the acceptance of Einstein's Gravitation for the interior problem directly implies the acceptance of the perpetual motion in our environment.* In turn, the acceptance of excessive approximations of Nature, inevitably raises ethical issues (which are not considered in this review). Needless to say, the selection of the appropriate theory capable of avoiding the above perpetual-motion approximations is open to scientific debate, and the Lie-isotopic solution reviewed below is presented only as one possibility. But the insufficiency of Einstein's Gravitation for the interior problem should not be questioned to avoid issues of scientific ethics.

It should be stressed that the objective is strictly *classical*, in the sense that it consists in identifying a classical theory of gravitation for the interior problem which allows local nonconservation laws, of course, in a way compatible with total conservation laws as well as with the exterior treatment of gravitation (see below). Once this physical reality of our direct observation is represented, and only then, the study of ossible operator/particles interpretations has a sound scientific value. But, again, the ignorance of the classical representation of our direct physical reality of the local nonconservation laws of the interior dynamics and its reduction to theoretical assumptions at an operator/particle level would also raise issues of scientific ethics.

The customary attitude when facing systems with varying angular momentum is that such breaking of the rotational symmetry is "illusory" in the sense that, when the interior system considered is reduced to its elementary particle constituents, the rotational symmetry is regained in full.

The suggestive "journey without return" in the Solar system [21] shows that such an attitude itself is "illusory". In fact, a mathematical proof of the contention would require that an object such as a satellite during re-entry, with its continuously decaying angular momentum (and noncanonical, nonhamiltonian time evolution) is reducible to a finite number of elementary particles all possessing a locally conserved angular momentum (and thus obeying a unitary, Hamiltonian time evolution). Such a proof evidently does not exist. Santilli contends that the proof is impossible for numerous technical reasons, such as the impossibility for a finite number of unitary, Hamiltonian time evolutions to reconstruct a classical, noncanonical, nonhamiltonian time evolution.

Again, in the words of the quoted author, the insistence in the capability to resolve classical violations of the rotational symmetry at the particle level without rigorous mathematical proofs, constitutes such an approximation of Nature to shift the issue from a technical to an ethical context.

The next problematic aspect (which is also linked to the above "illusory" reduction) is the now vexing impossiblity of achieving a consistent quantum mechanical formulation of Einstein's gravitation, despite serious and protracted efforts. Santilli [*loc.cit.*] claims that this additional problematic aspect is due to the intrinsic property of Einstein's Gravitation of admitting a null Hamiltonian. As a result, a "true" quantization of the theory (i.e. a unique quantization without ambiguities) is expected to be quite difficult if not impossible to achieve owing to the intrinsically Hamiltonian character of quantum mechanics.

In view of this occurance, and because of the evident need that any future theory of gravitation must eventually admit a consistent operator formulation on Hilbert spaces, Santilli [*loc.cit.*] suggests that a more adequate theory of gravitation for the interior problem (which is not expected to be Hamiltonian because of the power series expansions in the velocities indicated earlier) should admit a consistent Birkhoffian representation via a generalize Pfaffian action principle (§1.3). Once such a non-null structure has been identified, "hadronization" without ambiguities becomes at least conceivable (see also §1.3).

Finally, in regards to the problem of quantization/hadronization, Santilli [*loc.cit.*] does not see the need, or even the consistency, of a quantum mechanical formulation for interplanetary distances, but only locally, in the interior problem. To put it differently, the conceivable operator formulation of gravity is recommended for the interior, but not for the exterior problem. This is trivially due to the evidence that quantum mechanical effects are manifestly present in the interior problem and manifestly absent in the exterior one.

For a technical understanding of the above comments, we urge the reader to acquire a knowledge of the techniques of variational selfadjointness [4] and understand occurrences such as the violation of the integrability conditions for the existence of a Hamiltonian in the frame of the observer for systems experiencing forces with an arbitrary dependence on the velocities, while such a representation always exists, (directly universality) for the Birkhoffian covering of Hamiltonian mechanics [16].

3.5.3 Problematic Aspects of Einstein's Gravitation for the Exterior Problem Caused by the Lack of Source

The problematic aspects for the interior problem reviewed above are *absent* or otherwise inapplicable to the exterior problem of gravity.

In fact, the Riemannian geometry is evidently applicable to the exterior problem because of the absence of all the contact, nonlocal, integrodifferential effects of the interior dynamics. Also, the medium of the exterior problem, empty space, can be well assumed to be homogeneous and isotropic. Finally, since motion occurs in empty space with evident local conservation laws, the local Lorentz character of the theory is also applicable. Santilli [16,26] therefore advocates a gravitational theory for the exterior problem that is Riemannian and locally Lorentz in character.

Despite that, Einstein's Gravitation for the exterior problem remains still affected by such fundamental problematic aspects to raise serious doubts on its approximate validity.

A first problematic aspect relevant for this review was identified also by Santilli in 1974 [132]. It consists of an apparently irreconcilable incompatibility of Einstein's field equations for the exterior problem with the electromagnetic structure of matter. Intriguingly, the resolution of these problematic aspects offers intriguing possibilities for achieving a "grand unification" of all known interaction as we shall indicate below. To put it differently, the current difficulties in achieving a complete unification of all interactions appears to be due to the lack of acknowledgement of the irreconcilable incompatability of Einstein's Gravitation and Maxwell electromagnetism identified in ref. [132].

Consider a celestial body with null total electromagnetic phenomenology, i.e., null total charge, null total electric and magnetic dipole moments, etc. Under these assumptions, Einstein's field equations for the *interior problem* are given by the familiar form

$$G_{\mu\nu} \stackrel{\text{def}}{=} R_{\mu\nu} - \frac{1}{2} g_{uv} R = \frac{8\pi G}{c_0^4} M_{\mu\nu}^{\text{Mat}} , \tag{3.319}$$

where $G_{\mu\nu}$ is Einstein's tensor, and $M_{\mu\nu}^{\text{Mat}}$ is the energy-momentum tensor of matter. For the *exterior problem* the equations acquire the familiar form

$$G_{\mu\nu} = 0 , \tag{3.320}$$

which represent the essence of Einstein's Gravitation, namely, *the gravitational field of a celestial body with null total electromagnetic phenomenology is characterized by pure geometry without source.*

Santilli [132] essentially shows that *the above purely gravitational feature of Einstein's exterior gravitation is incompatible with the charge structure of the body considered in an apparently irreconcilable way* (see Fig. 12). As well known, matter is composed of atoms which, even though neutral, are composed of charged particles in highly dynamical conditions, the peripheral electrons and the elementary charged constituents of protons and neutrons. It is the *dynamical* condition of the charged consituents of matter that results into a total non-null electromagnetic field outside the body considered, contrary to Eq.s (3.320). (See Figure 6). Owing to the importance of this point for the analysis of this paper, let us review its essential aspects.

As a first step, Santilli computes the total electromagnetic field outside a π^0 under the assumption that it is a bound state of a generic "parton" and an "antiparton" (say a quark-antiquark system, or equivalently in Santilli's approach, an "eleton"-"antieleton" system) of charges (+q,-q). The analysis is purely classical and relativistic. Also, it is based on the conventional Maxwell's theory of electromagnetism in flat space-time via the use of the (advanced and retarded) Lienard-Wieckert potential at a point x of the Minkowski space M

$$ {}_qA^\mu_m(x) = -q\frac{v_m}{d_{nm}}, m = \text{Adv.}, \text{Ret.}, \tag{3.321}$$

where the v's are the velocities and the d's are the distances between the charges.

Under the approximation of a point-like structure of the π^0 constituents and of their absence of magnetic moments (spin zero), the potential of the system at an exterior point in Minkowski space M is given by

$$\begin{aligned} {}_qA^\mu_{\pi^0}(x) &= -q\sum_{nm}\epsilon_n\epsilon_m C_{nm}\frac{v^\mu_{nm}}{d_{nm}} \\ &= -q\left\{\left[C_{+\text{Ret}}\frac{v^\mu_{+\text{Ret}}}{d_{+\text{Ret}}} - C_{+\text{Adv}}\frac{v^\mu_{+\text{Adv}}}{d_{+\text{Adv}}}\right]\right. \\ &\quad \left. - \left[C_{-\text{Ret}}\frac{v^\mu_{-\text{Ret}}}{d_{-\text{Ret}}} - C_{-\text{Adv}}\frac{v^\mu_{-\text{Adv}}}{d_{-\text{Adv}}}\right]\right\} = \sum_{nm} C_{nm}A^\mu_{nm}(x), \end{aligned} \tag{3.322}$$

where: $\epsilon_\mu = -1$ for positive charges; $\epsilon = +1$ for negative charges; the C's are (at this point) arbitrary constants verifying the properties

$$\begin{aligned} C_{+\text{Ret}} + C_{+\text{Adv}} &= 1\,, \\ C_{-\text{Ret}} + C_{-\text{Adv}} &= 1\,, \end{aligned} \tag{3.323}$$

and

$$A^{\mu}_{nm}(x) = -q\epsilon_n\epsilon_m \frac{v^{\mu}_{nm}}{d_{nm}} \ . \tag{3.324}$$

The exterior energy-momentum tensor is then given by

$$\begin{aligned} {}_{1q}T^{\alpha\beta}_{\pi^0} \ = \ & \frac{q^2 c_0^4}{4\pi} \sum_{nn'} \Bigg\{ \frac{1}{{d_n}^6}[c_0^2 {D_n}^{\alpha} {D_n}^{\beta} + (D\cdot v)_n \{D^{\alpha}, v^{\beta}\}_n \\ & - \frac{(1-\delta_{nn'})}{{d_n}^3 d^3_{n'}}[(D_n \cdot v_{n'})\{D^{\alpha}_{n'}, {v_n}^{\beta}\} - \frac{1}{2}(v_n \cdot v_{n'})\{{D_n}^{\alpha}, D^{\beta}_{n'}\} \\ & - \frac{1}{2}(D_n \cdot D_{n'})\{{v_n}^{\alpha}, v^{\beta}_{n'}\}] - \frac{1}{2} g^{\alpha\beta} \frac{(D_n \cdot v_n)^2}{{d_n}^6} \\ & - g^{\alpha\beta} \frac{(1-\delta_{nn'})}{{d_n}^3 d^3_{n'}} [(D_n \cdot D_{n'})(v_n \cdot v_{n'}) - (D_n \cdot v_{n'})(D_{n'} \cdot v_n)] \Bigg\} \ , \end{aligned} \tag{3.325}$$

where

$$\begin{aligned} \{A^{\alpha}, B^{\beta}\} &= A^{\alpha}B^{\beta} + B^{\beta}A^{\alpha} \ , \\ D^{\alpha}_{nm} &= X^{\alpha} - Y^{\alpha}_{nm} \ . \end{aligned} \tag{3.326}$$

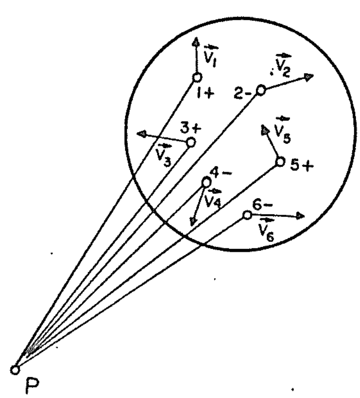


FIGURE 12. A reproduction of Fig. 1, p. 111, ref. [132] presenting a schematic view of a celestial body with null total charge as a "gas" of charged particles in highly dynamical conditions. Even though the total charge is

zero, the total electromagnetic field is nowhere null in both the interior and the exterior problem. Explicit calculations show that such a field has a value so high to account, in principle, for the gravitational mass of the body (Assumption 3.1). These results establish an incompatibility between Einstein's Gravitation and Maxwell's Electromagnetism in the sense that the latter theory predicts the existence of a large, first order source due to the charged structure of matter which is simply lacking in the former theory. Ref. [132] evidently embraces Maxwell's electromagnetism and suggests a revision of Einstein's gravitation with the inclusion of a nowhere null source tensor. Besides the resolution of the above inconsistency, the revision also offer's the possibility of resolving the vexing problem of "unification" of the gravitational and electromagnetic fields via their "identification" in the sense of Assumption 3.1.

When the magnetic moments μ of the constituents (in a singlet ground state) are included, we have an additional potential outside the π^0 system given by

$$\begin{aligned} {}_{\mu}A^{\alpha}_{m}(x) &= [\, {c_0}^3 \frac{\mu^{\alpha\beta} D_\beta}{d^3} + c_0 (D \cdot a) \frac{\mu^{\alpha\beta} D_\beta}{d^3} - \\ &- c_0 \frac{\dot{\mu}^{\alpha\beta} D_\beta}{d^2} \,]_{r=r_m} \,, \end{aligned} \tag{3.327}$$

with additional total electromagnetic field

$$ {}_{\mu}F^{\alpha\beta} = {}_{\mu}F^{\alpha\beta}_{1/D^3} + {}_{\mu}F^{\alpha\beta}_{1/D^2} + {}_{\mu}F^{\alpha\beta}_{1/D} \tag{3.328}$$

where

$$ {}_{\mu}F^{\alpha\beta}_{1/D^3} = \frac{2c_0^3}{d^3} \mu^{\alpha\beta} - \frac{3c_0^5}{d^5} (\mu^{\alpha\rho} D^\beta - \mu^{\beta\rho} D^\alpha) D_\rho \,, \tag{3.329}$$

$$\begin{aligned} {}_{\mu}F^{\alpha\beta}_{1/D^2} &= -\frac{2c_0}{d^2} \mu^{\alpha\beta} + \frac{2c_0}{d^3} (D \cdot a) \mu^{\alpha\beta} \\ &- \left(\frac{6c_0^3}{d^4} + \frac{6c_0^3}{d^5} (D \cdot a) \right) (\mu^{\alpha\rho} D^\beta - \mu^{\beta\rho} D^\alpha) D_\rho \\ &+ \frac{3c_0^3}{d^4} (\mu^{\alpha\beta} D^\beta - \mu^{\alpha\beta} D^\alpha) D_\rho + \frac{c_0}{d^3} (\mu^{\alpha\rho} a^\beta - \mu^{\beta\rho} a^\alpha) D_\rho \\ &+ \frac{c_0}{d^3} (\mu^{\alpha\rho} D^\beta - \mu^{\alpha\rho} D^\alpha) v_\rho + \frac{2c_0}{d^3} (\mu^{\alpha\rho} v^\beta - \mu^{\beta\rho} v^\alpha) D_\rho, \\ {}_{\mu}F^{\alpha\beta}_{1/D} &= \left[\frac{c_0 (a \cdot D)}{d^4} - \frac{3c_0 (D \cdot a)^2}{d^5} \right] (\mu^{\alpha\rho} D^\beta - \mu^{\beta\rho} D^\alpha) D_\rho \end{aligned}$$

$$+\frac{3c_0(D\cdot a)}{d^4}(\mu^{\alpha\rho}D^\beta-\mu^{\beta\rho}D^\alpha)D_\rho-\frac{c_0}{d^3}(\ddot{\mu}^{\alpha\rho}D^\beta-\ddot{\mu}^{\beta\rho}D^\alpha)D_\rho.$$

After a judicious handling of the advanced and retarded component, the corresponding energy-momentum tensor is then given by

$$\begin{aligned}\Theta_{\pi^0}^{\alpha\beta} &= \frac{1}{4\pi}\left(F_{\pi^0}^{\alpha\mu}F_{\pi^0\mu}^{\beta}+\frac{1}{4}\eta^{\alpha\beta}F_{\pi^0}^{\mu\nu}F_{\pi^0_{\mu\nu}}\right), \\ F_{\pi^0}^{\alpha\beta} &= {}_qF^{\alpha\beta}+{}_\mu F^{\alpha\beta}.\end{aligned} \tag{3.330}$$

The volume integral of the $0-0$ component of the above tensor then characterizes the electromagnetic contribution to the gravitational mass of the π^0,

$$m_{\pi^0}^{Elm}=\frac{1}{c_0^2}\int\Theta_{\pi^0}{}_0^0dv\,. \tag{3.331}$$

Explicit calculations [132] show that the above value of $m_{\pi^0}^{Elm}$ is very close to the rest mass of the π^0. Under certain velocity-dependent corrections (caused by the deep wave overlapping of the π^0 constituent), the total rest mass of the π^0 can be reached both, via Schrödinger's type equations [2,25,28] as well as via purely electromagnetic contributions [132].

Clearly, such a large, first-order value of the electromagnetic field in the exterior of the π^0 is incompatible with Einstein's field equations (3.320). Note that the π^0 was selected because, (as it is the case for the celestial body considered) it has null total charge as well as null total electric and magnetic moments.

The extrapolation of the analysis to a massive body is conducted in ref. [132] in sequential steps. First, the problem of the neutron n is considered under the assumption (rather generally accepted nowaday) that quarks are not elementary but have a structure resulting from a suitable bound state of a yet unknown number of elementary charges.

This results into an energy-momentum tensor for the neutron of the type

$$\begin{aligned}\Theta_{\mu\nu}^{Elm} &= \frac{1}{4\pi}\left(F_{\mu\alpha}F_{\nu}^{\alpha}+\frac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right), \\ F_{\mu\nu}^{Elm} &= \sum_{q,\mu=1}^{N}\left({}_qF_{\mu\nu}^{Elm}+{}_\mu F_{\mu\nu}^{Elm}\right),\end{aligned} \tag{3.332}$$

where the sum goes over all elementary constituents. For a sufficiently high number of such constituents under sufficiently high dynamical conditions, volume intergral (3.331) for the case at hand acquires, again, such a high value to be able to account, in principle, for the entire gravitational mass of the particle.

The extension to the proton is trivial inasmuch as it implies a simple increase of the sum in Eq. s (3.331). The extension to atoms and molecules then follows the same pattern along Eq.s (3.332), and with similar results.

Notice that $\Theta_{\mu\nu}^{Elm}$ cannot be reduced to zero unless one alters the structure of matter, e.g., by forcing all charges to be at rest and sufficiently close to each other.

As a result of this analysis, Santilli [loc.cit.] concludes that *any massive celestial body with null total electromagnetic phenomenology has a sizable, first-order, energy-momentum tensor* $\Theta_{\mu\nu}^{Elm}$ *due to the electromagnetic structure of matter which is nowhere reducible to zero.*

Under the classical approximation here considered (i.e., short range, weak and strong interactions are ignored), the following hypothesis was formulated.

> ASSUMPTION 3.1 [132] (*Strong Assumption*):
> *The gravitational mass of any massive body is entirely due to the electromagnetic field of its charged constituents.*

The "*weak assumption*" (which is the minimal possible under the calculations of ref. [132]) is that the gravitational field of a massive body is substantially, but not entirely due to the electromagnetic field of the charged constituents (because of the additional short range fields of the weak and strong interactions).

On the contrary, no contribution to the gravitational field is admitted in Einstein's Gravitation, evidently because, under such a contribution, the r.h.s. of Eq.s (3.320) cannot be null.

In different terms, *the gravitational field is nowhere sourceless*, because the only possibility to render integral (3.332) null is to work-out an *ad hoc*, profound modification of Maxwell electrodynamics which would undoubtedly result to be contrary to experimental evidence.

Under the classical approximation indicated earlier, ref. [132] therefore submitted the following reformulation of Einstein's field equations for both the *exterior* and the *interior* problem

$$G_{\mu\nu} = \frac{8\pi G}{c_0^4}\Theta_{\mu\nu}^{Elm} , \tag{3.333}$$

where $\Theta_{\mu\nu}^{Elm}$ is precisely the energy-momentum tensor produced by all charged constituents of matter.

As a consequence of the above results, Santilli put the foundation for a possible genuine resolution of the vexing problem of the "*unification*" of the gravitational and the electromagnetic fields, and replace it with the "*identification*" of the gravitational field with the electromagnetic field of the matter constituents. The understanding is that contributions from short term (weak and strong) interactions must be expected from a more appropriate formulation of a gravitational theory (see Figure 13).

Stated differently, the ultimate objective of ref. [132] was to conduct a study on the *origin* of the gravitational field along the hypothesis underlying Eq.s (3.332),

$$M_{\mu\nu}^{Mat} \equiv \Theta_{\mu\nu}^{Elm}. \tag{3.334}$$

After all, the use of mass terms is nothing but an expression of our ignorance of the dynamical structure originating the mass.

A number of approximate expressions for the tensor $\Theta_{\mu\nu}^{Elm}$ are computed in ref. [132]. The value of the tensor $M_{\mu\nu}^{Mat}$ itself can evidently be assumed as an approximation of $\Theta_{\mu\nu}^{Elm}$, provided that its dependence in space is assumed to be equal to that of $\Theta_{\mu\nu}^{Elm}$, and the tensor can be rendered nowhere null.

A number of conceivable experiments to test the expected gravitational character of the energy of electric or magnetic nature (which have not been conducted to this day, to our best knowledge) were also formulated in ref. [132] (see below).

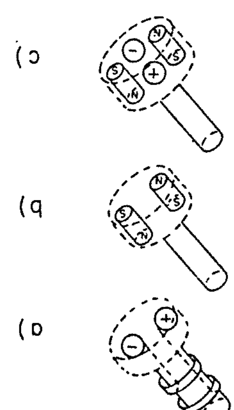


FIGURE 13. A reproduction of Figure 7 of ref. [132] depicting the "electromagnetic heads" of the proposed experiments. All gravitational theories predict that any electromagnetic field generates a gravitational field via mass (3.331). This prediction has not been experimentally verified until

now. For this reason, Santilli suggested back in 1974 [132] the conduction of this fundamental test in a number of ways. The first proposal was to test the prediction itself in its most direct possible way, via the use of the largest available sources of electromagnetic fields, e.g., those of the large magnets available in a number of laboratories. The test can be conducted via available neutron interferomenter techniques and/or gravity meters of high sensitivity by measuring first the background with the magnetic field off, and then the gravitational field following the activation of the magnetic field. To our best knowledge this first fundamental experiment is indeed feasible nowayday (and of rather contained cost) because, on one side, neutron interferometric techniques have reached a very high degree of sensitivity, while, on the other side, we have available very large sources of magnetic field. Regretably, this proposal has remained ignored by the experimental community, to our best knowledge, despite its manifestly fundamental nature, e.g., for the possible resolution of the vexing open problem of "unification" of the electromagnetic and gravitational fields, e.g., via Santilli's hypothesis of their "identification" (Assumption 3.1). The second test suggested in ref. [132] is a deeper refinement of the first test and considerably more difficult in practical realization. An inspection of energy-momentum tensor (3.330) under potential (3.332) indicate that a significant part of gravitational mass (3.331) is due to the dynamical conditions of the charges. The second test under consideration was intended precisely to test the contributions to the gravitational field originating from the dynamical conditions of the charges. For that purpose, Santilli suggested the measurement of the gravitational field (also via neutron interferometric techniques) produced by the "electromagnetic heads" of the figure which, as one can see, essentially consist of opposite charges and magnetic moments in extremely high rotational conditions so as to reproduce the conditions of the structure of matter as close as possible. Apparently, this second test was not feasible back in 1974 (Santilli, private communication) owing to a number of limitations such as: the impossibility to reach sufficiently high angular momentum, and electromagnetic fields. Nevertheless, this second class of experiments can well be within practical feasibility nowadays owing to the advancements in technology that have occurred in the meantime (e.g., in superconductors). Whether along the lines of proposals [132] or any other approach, the above tests are strongly recommended here for consideration by experimenters in the field.

We would like to present now a few informal comments on the problem of "grand unification". To begin, the problem is enlarged by Santilli's studies because, in addition to the conventional, electromagnetic, weak, strong and gravitational interactions, any unification must also include the additional class of contact, nonhamiltonian interactions due to wave-overlappings (§1.3).

The only way known for the incorporation of the latter interactions is via Lie-isotopic techniques. Thus, *the structure of the "grand unification" emerging from these studies is given by an isotopic lifting of current, unified, gauge theories, plus Assumption 3.1 on the identification of the gravitational and electromagnetic fields.* In fact, the current unification of weak and electromagnetic interactions is expected to be isomorphically lifted under isotopy. Santilli's Strong Assumption 3.1 (or the weaker assumption of ref. [132]), may then permit the identification (rather than unification) of electromagnetic and gravitational fields, while the isotopy of the underlying gravitational-gauge structure opens up the possibility of an unambiguous hadronization into an operator form (via the existence of a non-null Birkhoffian). This, in turn, opens up the possiblity of incorporating strong interactions, e.g., because the isotopy, per se, is a direct representative of the contact nonhamiltonian component of the strong interactions.

Needless to say, we are referring here to mere unverified possibilities currently under investigations (see, e.g., D. Rapoport-Campodonico's [133] studies on isotopic unification via stochastic techniques, and Santilli's forthcoming works; the isotopy of gauge theories is reviewed in Appendix A).

As a final comment, the reader should be aware that the extension of the analysis to a celestial body with a non-null electromagnetic phenomenology is simple. Einstein's field equations assume in this case the familiar form

$$G_{\mu\nu} = \frac{8\pi G}{c_0^4}\left(M_{\mu\nu}^{Mat} + t_{\mu\nu}^{Elm}\right) , \tag{3.335}$$

where $t_{\mu\nu}^{Elm}$ is the electromagnetic tensor solely due to the *total* electromagnetic quantities (and, as such, itsvalue is much smaller than that of $\Theta_{\mu\nu}^{Elm}$). Eq.s (3.335) are trivially contained as a particular case of Eq.s (3.333), i.e. $\Theta_{\mu\nu}^{Elm}$ is inclusive, by construction, also of total electromagnetic effects.

3.5.4 Problematic Aspects of Einstein's Gravitation for the Exterior Problem Caused by the Lack of Stress-Energy Tensor.

By far the leading expert on the problematic aspects of Einstein Gravitation forthe exterior problem is H. Yilmaz. We list here only some of his papers, ref.s [134] through [144]. Yilmaz's analysis is related to that by Santilli

(§3.5.3), although it is based on different physical motivations. In fact, Yilmaz advocates a generalization of Einstein's field equations (3.320) for the *exterior* problem of the type

$$G_{\mu\nu} = \frac{8\pi G}{c_0^4} t_{\mu\nu}^{\text{Grav}} , \tag{3.336}$$

where $t_{\mu\nu}^{\text{Grav}}$ is the stress-energy tensor of the gravitational field, i.e., a tensor physically and mathematically different than $\Theta_{\mu\nu}^{Elm}$ of Eq.s (3.333).

Yilmaz's motivations for Eq.s (3.336) are numerous and can be only summarily reviewed here. First, he shows that, when the stress-energy tensor is absent, the Newtonian limit of Einstein's Gravitation is unable to recover the Galilean description of the planetary system, because it recovers instead the so-called Hooke's mechanics (in which the Sun has infinite inertia and there is no principle of action and reaction). When the stress-energy tensor is however present, this problem is apparently resolved.

Furthermore, Yilmaz [*loc.cit*] shows that, in the absence of the stress-energy tensor, Einstein's gravitation is appartently unable to recover the energy-momentum conservation law of the Special Relativity.

More seriously, Yilmaz additionally shows that, under the absence of the stress-energy tensor, Einstein's Gravitation is indeed capable of representing the celebrated 43" of advancement of the perihelion of Mercury, but serious problematic aspects exist for a consistent representation of the Newtonian 532" because of the strict Hamiltonian character of Newton's laws.

By implementing Einstein's equations with the stress-energy tensor, Yilmaz has constructed a generalization of the theory, hereinafter referred to as *Yilmaz's Theory of Gravitation for the exterior problem* with rather remarkable possibilities, such as:

a) compatibility with the Newtonian description of the planetary motion;

b) compatibility with the relativistic description; and

c) compatibility with the quantum-mechanical formulation.

In particular, Yilmaz's theory appears to be consistent with currently available experimental evidence.

The latest development in the space-time theory of gravitation is the rediscovery of Freud's identity and its application to the problem of exterior field equations. Freud's identity was originally found by P. Freud [145] in 1939. It was mentioned by W. Pauli [94] in 1958 in the "Notes" section of the Dover edition of his famous 1921 work and by J. Weber [146] in his 1961 work on gravity waves but no systematic application to the problem of field equations was made.

Recenly H. Yilmaz [143,144] pointed out that the existence of two independent identities (that of Bianchi and of Freud) creates severe restrictions on the possible form of the field equations. This is the problem of overdetermination in the presence of multiple conditions and the consequences are quite dramatic: *In order for the field equations to be compatible with two identities one must add in the exterior problem the stress-energy tensor $t_{\mu\nu}^{\rm Grav}$ on the right hand side with unit cofficient $\lambda t_{\mu}^{\rm Grav}, \lambda = 1$.*

Otherwise the field equations either have no solutions or only solutions which are trivial (for example, only a 1-body solution if $\lambda \neq 1$) or solutions which are non-unique (for example, a linearly accelerating frame depending on a parameter $\epsilon = \pm\sqrt{(1-\lambda)}$ which is double valued).

Compatibility requires $\lambda = 1$, so that in this case one has non-trivial N-body solutions and, at the same time, all solutions are unique since $\epsilon = \pm\sqrt{(1-\lambda)} = 0$. Furthermore, Yilmaz demonstrates that only when $\lambda = 1$ (that is, only when the field equations are compatible with the two identities) that the theory is experimentally viable.

An example of this is that, unless $\lambda = 1$, there are no N-body solutions, hence the N-body equations of motion cannot be constructed other than (possibly) by putting the second, third, etc. bodies by hand. But then the theory becomes a test body theory and cannot predict the 532" per century N-body part of the perihelion advance of Mercury, since test particles cannot interact with each other.

These points were already made by H. Yilmaz [144] before the rediscovery of Freud's identity but now with their exact derivation using that identity, the results become quite strong.

Yilmaz's theory itself is not immune from criticisms. For instance, Eq.s (3.336) are unable to account for the first-order tensor $\Theta_{\mu\nu}^{Elm}$ of Section 3.5.3 owing to the different structures of the two tensors (one traceless and the other not). As a consequence, the electromagnetic field originating from the charged structure of matter and propagating in the exterior problem is not explicitely represented in Yilmaz's theory.

This indicates that, even though Yilmaz's criticisms of Einstein's Gravitation appear to be valid, and his objectives a), b) and c) above are equally valid, his theory might need further generalizations to achieve compatibility with other aspects, such as the origin of the gravitational field itself, and the unification of all interactions.

It is possible to show that Yilmaz's arguments persist after the addition of a traceless tensor to his stress-energy tensor. On these grounds, the *field equations advocated by Santilli [26] for the exterior gravitational problem* are given by

$$G_{\mu\nu} = \frac{8\pi G}{c_0^4}\left(\Theta_{\mu\nu}^{Elm} + t_{\mu\nu}^{\text{Grav}}\right) , \tag{3.337}$$

and their physical origin can be motivated as follows.

A primary emphasis of this review is the *identification of the origin of any tensor that is needed in the r.h.s of the field equations for the exterior problem.* This emphasis has been the guide for the presentation of the electromagnetic tensor $\Theta_{\mu\nu}^{Elm}$. As a result, we cannot escape the problem of the possible origin of Yilmaz's stress-energy tensor $t_{\mu\nu}^{Grav}$. To put it differently, if a clear origin of such a tensor can be identified, its place in the r.h.s. of the field equations becomes incontrovertible irrespective of any of the advantages reviewed earlier in this section.

Santilli [26] contends that Yilmaz's tensor $t_{\mu\nu}^{Grav}$ sees its origin in the short range (weak and strong) interactions at the foundations of the structure of matter. More explicitly, he recalls that all fields of the elementary constituents of matter are expected to contribute to the total gravitational mass. Of these fields, the electromagnetic fields reviewed in §3.5.3 is accountable for the $\Theta_{\mu\nu}^{Elm}$ tensor which, as such, is traceless. The remaining weak and strong fields are responsible for an additional tensor. It is easy to see that such tensor cannot be traceless and, thus, it can well be Yilmaz's stress-energy tensor $t_{\mu\nu}^{Grav}$.

The position assumed in this monograph is that the correct field equations for the exterior problem are expected to be Eq.s (3.337), with the understanding that the contribution to the gravitational field by $t_{\mu\nu}^{Grav}$ is expected to be smaller than that of $\Theta_{\mu\nu}^{Elm}$.

In summary, Santilli's [132] and Yilmaz's [134–144] studies indicate the existence, in the r.h.s. of the gravitational field equations for the exterior problem, of a nowhere null tensor whose traceless part represents the contributions from the electromagnetic structure of matter, and the remaining part originates from the short-range interactions at the nuclear and hadronic levels. Since the former tensor $\Theta_{\mu\nu}^{\text{Elm}}$ is of the order of magnitude of the conventional mass tensor $M_{\mu\nu}^{\text{Matter}}$ and, as such, it is expected to provide the conventional predictions of the theory. The second tensor $t_{\mu\nu}^{\text{Grav}}$ is then expected to provide the additional ones suggested by Yilmaz.

The point which should be stressed here is that we can merely indicate the *plausibility* of Eq.s (3.337) and of the physical origin of their terms, although the *validity* of the theory is unknown at this writing owing to the lack of specific quantitative studies.

We would like to close this section with the indication that, except for a few comments, we shall not be unable to consider orsion in our gravitational analysis. Nevertheless, the reader should be aware that most of the aspects considered below can be equivalently treated with torsion, in such an effective way that torsion is often considered as a measure of the *departure* from conventional Einsteinian settings. For in depth treatment of torsion within an isotopic gravitational context, we refer the interested reader to the studies by Rapoport-Campodonico [52].

3.5.5 Some Desirable Features for a Generalized Theory of Gravitation

By combining the various problematic aspects of Einstein's Gravitation, Santilli [16,26] advocates the construction of a suitably generalized theory of gravitation having the following primary features.

INTERIOR PROBLEM

1. The generalized theory should represent motion within a generally inhomogeneous and anisotropic material medium. The understanding is that space itself remains homogeneous and isotropic.

2. The generalized theory should be based on a nonlocal, integrodifferential generalization of the Riemannian geometry in order to account for the nonlocal forces experienced by an extended test particle moving within the medium composed by all the other particles. If a local-differential approximation is assumed (via power series expansions in the velocities), the generalized theory should be able to produce under the PPN approximation *all* possible Newtonian equations of motion, with an arbitrary functional dependence on the velocities (the essentially nonselfadjoint forces of ref. [4]).
3. The generalized theory should be able to represent local deviations from the conventional rotational and Lorentz symmetry, in order to avoid perpetual motion approximations, as evident in the classical physical reality of the interior problem.

4. Despite all the above departures from the conventional Einstein's Gravitation, the generalized theory should be locally Lorentz-isotopic (§3.4) and, in particular, the local Lorentz-isotopic symmetry should be isomorphic to the abstract Lorentz symmetry on isotopic spaces $\hat{M}_{\text{III}}$ (§3.4). This latter requirement evidently demands the realization of the preceding characteristics via a Lie-isotopic generalization of Einstein's Gravitation.
5. The generalized theory should admit a non-null Birkhoffian representation via a nontrivial, Pfaffian generalization of the canonical action principle. Furthermore, such a representation should permit an unambiguous "hadronization" of the theory into an operator form on Hilbert spaces (§1.3).

EXTERIOR PROBLEM

6. Along the lines of the Galilean (§3.3) and relativistic (§3.4) closed nonhamiltonian systems, the generalized theory is expected to be a theory with subsidiary constraints to ensure the validity of conventional total conservation laws, as well as to ensure any needed additional feature.
7. The generalized theory is expected to be purely Riemannian in the exterior geometrical character and, therefore, should possess the local, conventional, Lorentz character in the exterior problem;
8. The generalized theory should incorporate the electromagnetic tensor originating from the charged structure of matter, as well as the stress-energy tensor of the gravitational field.
9. Last, but not least, the generalized theory must be compatible with all available experimental data on gravitation, for both the exterior and the interior problems.

It should be stressed that a gravitational theory satisfying all the above requirements does not exist at this writing, to our best knowledge. Nevertheless, major advances have been made along these lines, as we shall report in the rest of this section.

The foundations of the studies are provided by Santilli's identification of the apparent electromagnetic origin of the gravitational field (§3.5.3), the Lie-isotopic generalization of the conventional Lie's theory (Section 2); the Lie-isotopic generalization of Galilei Relativity (§3.3) and of Einstein's Special Relativity (§3.4); as well as the formulation of the Lorentz-isotopic symmetry for generally curved isotopic spaces $\hat{M}_{\text{III}}$ (§3.4.7).

Following these lines Gasperini [81–84] constructed, for the first time, a step-by-step Lie-isotopic generalization of Einstein's Gravitation which possesses precisely a local Lorentz-isotopic character. He then presented numerous developments, particularizations and examples.

Santilli [26] reinspected Gasperini's theory, by making a number of additional contributions, such as: the restriction of the isotopy to the interior problem only in order to recover the conventional homogenuity and isotropy of space as well as the conventional Riemannian geometry for the exterior problem; by restricting the Lie-isotopic theory in the interior problem to be locally isomorphic to the abstract Lorentz symmetry, so that this fundamental symmetry is not lost, but only realized in its most general possible way; and by presenting additional contributions reviewed below.

Santilli then resumed his studies of interior gravitation where he made his most significant mathematical and physical contributions encompassing all preceding results. In fact, in memoir [24d] Santilli introduced the isotopic generalization of the affine geometry and of the Riemannian geometry for the most general known formulation of non-linear, non-Lagrangian and non-local/integral interior gravitation, and constructed a generalization of Gasperini formulation in his Riemannian-isotopic geometry.

As stressed earlier, the most visible evidence on the ultimate non-local nature of interior problems is gravitation itself, say, for a star undergoing gravitational collapse, in which we have not only total mutual penetration of the wavepackets of the constituents, but also their compression in large numbers within a small region of space. Under these conditions, the emergence of interior non-linear, non-local and non-Lagrangian interactions is simply beyond any scientific doubt.

Now, the conventional Riemannian geometry is strictly local and differential. As such, it cannot provide a representation of non-local interior gravitational models. Because of this, Santilli constructed a step-by-step generalization of the Riemannian geometry based on *Riemannian-isotopic spaces* (which essentially are Minkowski-isotopic spaces of the third class). This included a generalization of the basic unit in which all non-local interactions are embedded, and the consequential generalization of the totality of the structure of the Riemannian geometry, such as metric, Christoffel's symbols, curvature, torsion, identities, etc.

Of utmost importance is Santilli's isotopic generalization of the conventional notions of parallel transport and geodesic motion. In fact, Santilli essentially proved that, in the transition from motion in vacuum to motion within an inhomogeneous and anisotropic medium, the axiomatic structure of parallelism and geodesic is preserved in its entirety, thus allowing an ultimate geometric unification between conventional and isotopic relativities which, in our view, is the ultimate mathematical and physical result achieved by Santilli in his sequence of studies.

Regrettably, we cannot review here the *Riemannian-isotopic geometry* to avoid a prohibitive length. We are therefore forced to content ourselves for brevity with an intermediate presentation, that on a conventional Riemannian space. The understanding is that our presentation is merely preparatory to *Santilli's Isogravitation* on a full Riemannian-isotopic space [24d].

The generalized theory of gravitation which emerges from the above studies shall be referred hereon as the *Gasperini-Santilli General Relativity* (or the *Gasperini-Santilli Gravitation*).

In the remaining parts of this section we shall review the rudiments of this novel theory, point out which of the above requirements 1-9 is verified, and identify some of the open problems.

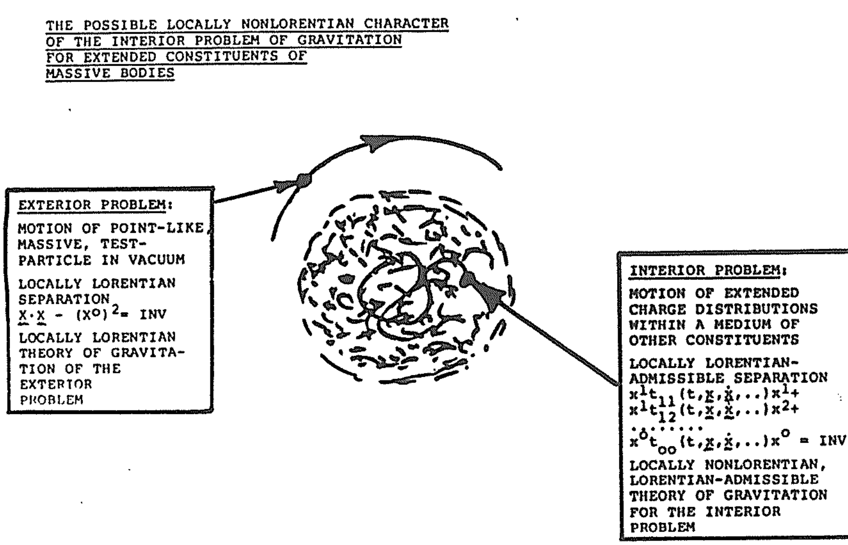


FIGURE 14. A reproduction of Figure 5.7 of ref. [16] illustrating the essential aspects of the "ideal" theory of gravitation reviewed in Section 3.5.5: the physical differences between the exterior and the interior dynamics, with motion of test particles in vacuum (empty space) in the former case,

and motion within a physical medium in the latter case. For these reasons, ref.s [16] and [26] advocate the use of the conventional local, Lorentzian theory of gravitation for the exterior problem. The reader should be aware that the gravitational theory for the interior problem advocated in ref. [16] is a covering of the Lie-isotopic theory reviewed in this section, owing to its Lie-admissible character. As it happens for other levels of study (see, e.g., Figure 3), the Lie-isotopic theory puts the emphasis on total, conventional, conservation laws under generalized internal structures. The still more general Lie-admissible approach essentially represents one individual test particle when the rest of the system is external, thus resulting in an open, nonconservative, system requiring the Lorentz-admissible generalization [26] of the Lorentz-isotopic symmetry of this review.

Remarkably, the Gasperini-Santilli Gravitation we shall review hereon is only a particular case of a more general theory of gravitation of the covering *Lie-admissible* (rather than Lie-isotopic) character for the study of *open* gravitational problems, which has been independently investigated by Gasperini [147],Santilli [16], P.F. Gonzalez-Diaz [148], A. Jannussis and collaborators [149] and others. This more general approach will not be reviewed (although we hope to review it in a future work).

In Section 1.2 we quoted Bruck's statement to the effect that the notion of algebraic isotopy is "so natural to creep in unnoticed". In this section it is appropriate to quote Gasperini's words (ref. [81], p. 652): "This (Lie-isotopic) generalization (of Einstein's gravitation) is so natural to appear nearly trivial. However, its physical implications are rather deep", as we shall see.

3.5.6 Lie-isotopic Lifting of Einstein's Gravitations without Source

In three pioneering papers of 1984 Gasperini [81],[82],[83] presented a Lie-isotopic generalization of Einstein's Gravitations for the case without energy-momentum tensor of matter (see later on for the case with source tensor).

The starting point is the formulation of conventional gravitational theories as gauge theories with local Lorentz invariance [150]. Gasperini first shows that conventional gauge theories admit a consistent (and intriguing) Lie-isotopic generalization, as reviewed in Appendix A. The Lie-isotopic lifting of gravitation was presented subsequent to such gauge isotopy.

Consider Einstein's Gravitation for the simplest possible case, that without matter and field equations (3.320). Reformulate such theory in the gauge language [150–152].

Let P_a and M_{ab} be the conventional generators of the local Poincaré symmetry, where small Latin indices denote anholonomic Lorentz indices. Denote the usual frame and connection one-form with

$$V^a = V^a_\mu dx^\mu, w^{ab} = w^{ab}_\mu dx^\mu, \tag{3.338}$$

respectively where small Greek indices denote Lorentz indices in our space-time (as in the preceding sections of this work).

The standard "potential" of Einstein's Gravitation can then be written

$$h = h^A X_A = V^a P_a + w^{ab} M_{ab}, \tag{3.339}$$

where capital indices $A, B, \ldots$ run over the set $(a, ab, \ldots)$.

Along the lines of Santilli's Lie-isotopic theory [1], Gasperini [81,82] leaves the parameter and the generators of the theory unchanged, but submits the various composition laws to a lifting characterized by generally different isotopic elements ${T_A}^B$ for different generators. Conventional potential (3.339) then becomes under lifting

$$\begin{aligned} \hat{h} &= h^A {T_A}^B X_B = V^a {T_a}^b P_b + V^a {T_a}^{bc} M_{bc} + \\ &+ w^{ab} {T_{ab}}^c P_c + w^{ab} {T_{ab}}^{cd} M_{cd} \ . \end{aligned} \tag{3.340}$$

Suppose that the isotopic elements ${T_A}^B$ are constant matrices, which commute with the Poincaré generators and among themselves. The isotopic curvature can be expressed in terms of the generalized components of the potential $\hat{h}^A = h^B {T_B}^A = \{\hat{V}^a, \hat{w}^{ab}\}$ according to the expressions

$$\begin{aligned} \hat{V}^a &= h^A {T_B}^a = V^b {T_b}^a + w^{bc} {T_{bc}}^a \ , \\ \hat{w}^{ab} &= h^B {T_B}^{ab} = V^c {T_c}^{ab} + w^{cd} {T_{cd}}^{ab} \ . \end{aligned} \tag{3.341}$$

Using the standard commutation rules of the Poincaré algebra, one obtains then the same structure equations as in general relativity [151]

$$\hat{R}^a = d\hat{V}^a + {\hat{w}^a}_b \wedge \hat{V}^b \ , \tag{3.342}$$

and

$$\hat{R}^{ab} = d\hat{w}^{ab} + {\hat{w}^a}_c \wedge \hat{w}^{cb} \ , \tag{3.343}$$

defining the isotopic torsion $\hat{R}^a$, and curvature $\hat{R}^{ab}$, in terms of the isotopic potentials $\hat{h}^A = \left\{\hat{V}^a, \hat{w}^{ab}\right\}$.

Imposing the constraint $\hat{R}^a = 0$ as in general relativity, the group manifold procedure prescribes then for this theory the standard Einstein action expressed in this case with the generalized variables $\hat{R}^A$ and $\hat{h}^A$, i.e.

$$\hat{S} = \frac{1}{4k} \int \hat{R}^{ab}(\hat{w}) \wedge \hat{V}^c \wedge \hat{V}^d \epsilon_{abcd} , \qquad (3.344)$$

where ϵ_{abcd} is the totally antisymmetric symbol, and $k = 16\pi G/c_0^4$ is the usual Newton coupling constant.

In this way, Gasperini [loc.cit] achieves a result analogous to those reached in Sections 3.3 and 3.4, namely, that *the conventional Einstein's Gravitation and its image under isotopic lifting coincide at the level of abstract, realization-free formulations.*

Despite these similarities, and exactly as it happens for the Galilei-isotopic and the Lorentz-isotopic cases, the physical differences between the conventional and the isotopically lifted theory of gravitation are rather deep.

In order to identify these differences, the isotopically lifted theory must be explicitly worked out and expressed in terms of the *conventional* potential $h^A = \left\{V^a, w^{ab}\right\}$ for, again, these mathematical symbols represent physical quantities that remain unaffected by the lifting.

It then follows that the geometrical structure underlying the isotopically lifted theory is more general than that of the conventional theory, as we shall see below.

3.5.7 Isotopic Origin of Torsion

Gasperini [82] first illustrated the physical differences between the conventional and the isotopically lifted theory by showing that the former is a torsion free theory, while the latter is, intrinsically, a gravitation theory with torsion. In turn, the appearance of torsion is of fundamental nature, inasmuch as it allows the possibility of attempting the resolution of at least some of the problematic aspects of Einstein's Gravitation recalled earlier (Sect. 3.5.4).

Consider the simple isotopic lifting defined by

$$T_{ab}{}^{c} = 0 = T_c{}^{ab} , \qquad (3.345)$$

$$T_{ab} = \eta_{bc} T_a{}^c \neq \eta_{ab} \quad , \quad T_{ab}{}^{cd} = \delta_a{}^c \delta_b{}^d ,$$

where $T_{ab} = T_{ba}$ is a symmetric constant matrix, and $\eta_{ab} = \text{diag}(1,1,1,\text{-}1)$ is the Minkowski metric.

In this case the isotopic potentials (3.341) become

$$\hat{V}^a = V^b T_b{}^a \quad , \quad \hat{w}^{ab} = w^{ab} , \tag{3.346}$$

and the isotopic structure equations are

$$\begin{aligned} \hat{R} &= R^b T_b{}^a = dT^a + w^a{}_b \wedge T^b , \\ \hat{R}^{ab}(\hat{w}) &= R^{ab}(w) = dw^{ab} + w^a{}_c \wedge w^{cb} . \end{aligned} \tag{3.347}$$

Therefore, the connection and the curvature are not lifted (for the simple case considered); however, according to Eq. (3.347), the connection is defined in terms of a generalized vierbein field $\hat{V}^a_\mu$

$$\hat{V}^a_\mu = V^b_\mu T_b{}^a = T^a_\mu . \tag{3.348}$$

The action (3.344) for this isotopic theory of gravity becomes

$$\hat{S} = \frac{1}{4k} \int R^{ab}(w) \wedge T^c \wedge T^d \epsilon_{abcd} . \tag{3.349}$$

By introducing explicitly holonomic indices, we have $R^{ab} = R_{\mu\nu}{}^{ab} dx^\mu \wedge dx^\nu$, and $dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta = d^4x \epsilon^{\mu\nu\alpha\beta}$. By using the properties of the totally antisymmetric symbols Gasperini rewrites this action in the (perhaps more familiar) tensor language

$$\begin{aligned} \hat{S} &= \frac{1}{4k} \int d^4x \sqrt{-g} \{ \frac{1}{2} R\phi^2 - \frac{1}{2} R T_\alpha{}^\beta T_\beta{}^\alpha - 2 R_\nu{}^\alpha T_\alpha{}^\nu \phi + \\ &+ \ 2 \ R_\nu{}^\alpha T_\alpha{}^\beta T_\beta{}^\nu + R_{\mu\nu}{}^{\alpha\beta} T_\alpha{}^\mu T_\beta{}^\nu \} , \end{aligned} \tag{3.350}$$

where

$$R_{\mu\nu}{}^{\alpha\beta} = 2 V^\alpha_a V^\beta_b (\partial_{[\mu} w_{\nu]}{}^{ab} + w_{[\mu}{}^{ac} w_{\nu]c}{}^b) , \tag{3.351}$$

is the usual curvature tensor for the connection w, $R_{\mu\nu} = R_{\mu\alpha\nu}{}^\alpha$ is the Ricci tensor, $R = R_\mu{}^\mu$ the curvature scalar, and $g = \det g_{\mu\nu} = (\det V^a_\mu)^2$, where

$$g_{\mu\nu} = V^a_\mu V^b_\nu \eta_{ab} , \tag{3.352}$$

is the world metric tensor. Finally, Lorentz indices are holonomized by means of V^a_μ, for example $T_\mu{}^\nu = V^a_\mu V^\nu_b T_a{}^b$, and

$$\phi = g^{\mu\nu} T_{\mu\nu} = \eta^{ab} T_{ab} \tag{3.353}$$

is the trace of the isotopic element.

Comparing the action (3.350) of the isotopic theory with the usual Einstein action

$$S = \frac{1}{4k}\int R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} = \frac{1}{k}\int d^4x\sqrt{-g}R \ , \tag{3.354}$$

the coupling constant of the usual gravitational Lagrangian is renormalized, $R/k \rightarrow R/k'$ [82], where

$$k' = \frac{2k}{\phi^2 - T_\alpha{}^\beta T_\beta{}^\alpha} \ , \tag{3.355}$$

as expected in general when performing the isotopic lifting of a gauge theory (See Appendix A). Also, the isotopic element is coupled, in a strongly non-minimal way, to the curvature tensor, thus introducing additional terms to the Lagrangian besides the scalar curvature. Notice that these new terms are all proportional to the scalar curvature, if the isotopic element is the same for all the generators of the group or, in other words, if T_{ab} is proportional to η_{ab}.

By varying the isotopic action (3.349) with respect to the frames V^a, Gasperini obtains the modified Einstein's field equations in vacuum

$$R^{ab}(w) \wedge V^i T_i{}^c T_k{}^d \epsilon_{abcd} = 0 \ , \tag{3.356}$$

or, in the usual notation,

$$\begin{aligned} 4\phi T_\alpha{}^\nu G_\nu{}^\beta &= -2\ R\ T_\alpha{}^\nu T_\nu{}^\beta - 4\ R_\mu{}^\nu T_\nu{}^\mu T_\alpha{}^\beta + \\ &+4\ R_\mu{}^\nu T_\nu{}^\beta T_\alpha{}^\mu + 4\ T_\alpha{}^\mu T_\mu{}^\nu R_\nu{}^\beta\ , +4\ R_{zmu\lambda}{}^{\nu\beta} T_\nu{}^\mu T_\alpha{}^\lambda \ , \end{aligned} \tag{3.357}$$

where

$$G_\nu{}^\beta = R_\nu{}^\beta - \frac{1}{2}R\delta_\nu{}^\beta \tag{3.358}$$

is the usual Einstein's tensor (remember that in the standard theory the vacuum field equations are simply $G_\alpha{}^\beta = 0$).

By varying (3.349) with respect to the connection w^{ab} one obtains the expected constraint on the isotopic torsion, i.e.

$$\hat{R}^a = \nabla T^a = dT^a + \omega^a{}_b \wedge T^b = 0 \ , \tag{3.359}$$

where ∇ denotes the Lorentz covariant exterior derivative. From this equation Gasperini is led to the remarkable result that, *even considering the* *lifting of a torsionless theory (such as general relativity), the connection ω^{ab} of the isotopic theory contains a generally nonvanishing torsion part.*

Consider in fact the following decomposition of the isotopic element:

$$T_a{}^b = \varphi\delta_a{}^b + \tau_a{}^b , \qquad (3.360)$$

where 4φ is the trace and $\tau_a{}^b = T_a{}^b - \varphi\delta_a{}^b$ the tracefree part of $T_a{}^b$. The isotopic frame then becomes

$$\hat{V}^a = T^a = V^b_\mu T_b{}^a dx^\mu = \varphi V^a + \tau^a , \qquad (3.361)$$

and the isotopic structure eq. (3.359) can be rewritten

$$dV^a + \omega^a{}_b \wedge V^b + \varphi^{-1}\{d\tau^a + \omega^a{}_b \wedge \tau^b\} = 0 , \qquad (3.362)$$

from which

$$\hat{R}^a = \nabla\hat{V}^a = -\varphi^{-1} \nabla \tau^a \neq 0 , \qquad (3.363)$$

where R^a is the usual torsion two-form relative to the standard frame. Gasperini [82] therefore reaches the following important conclusion.

> *The Lie-isotopic lifting of a Riemannian geometry induces even in the absence of matter, a Riemann-Cartan [153–157] geometrical structure, with the isotopic element acting as a source of torsion.*

The connection ω can be explicitly calculated in terms of V and τ solving Eq. (3.359) which can be written explicitly as

$$C_{bc}{}^a + \frac{1}{2}\omega_b{}^a{}_c - \frac{1}{2}\omega_c{}^a{}_b + Q_{bc}{}^a = 0 , \qquad (3.364)$$

where

$$C_{bc}{}^a = V_b{}^\mu V_c^\nu \delta_{[\mu} V^a_{\nu]} \qquad (3.365)$$

are the usual Ricci rotation coefficients, and $Q_{bc}{}^a$ are the components of the torsion tensor

$$\begin{aligned} Q_{bc}{}^a &= V_b{}^\mu V_c^\nu \varphi^{-1}\{\partial_{[\mu} V_{\nu]}{}^i \tau_i{}^a + \omega_{[\mu 1}{}^a{}_i V_{\nu]}{}^k \tau_k{}^i\} \\ &= \varphi^{-1}\{C_{bc}{}^i \tau_i{}^a + \frac{1}{2}\omega_b{}^a{}_i \tau_c{}^i - \frac{1}{2}\omega_c{}^a{}_i \tau_b{}^i\} , \end{aligned} \qquad (3.366)$$

(remember that $\tau_a{}^b$ is a constant matrix).

By cyclic permutation of indices in Eq.(3.363), using the metricity property [151] $\omega^{cab} = \omega^{c[ab]}$, one obtains then

$$\omega_{acb} = \gamma_{acb} + K_{acb} \ , \tag{3.367}$$

where γ_{acb} is the usual Riemannian part of the connection

$$\gamma_{acb} = C_{bca} - C_{cab} - C_{abc} \ , \tag{3.368}$$

and K_{acb} is the *contorsion tensor* [153,154]

$$K_{acb} = Q_{bca} - Q_{cab} - Q_{abc} \ . \tag{3.369}$$

This isotopic theory can be interpreted [82] then as an *Einstein-Cartan theory for gravity coupled nonminimally to a symmetric second-rank tensor, which is a source of torsion according to Eq.s(3.366).*

3.5.8 Modified Field Equations with Torsion

Another direct way for showing the differences between the conventional and the isotopically lifted gravitation identified by Gasperini [82,83] is to work-out explicitly the field equations, and show that they do not coincide with the pure geometrical equations (3.320) but exhibit a first-order non-null tensor on the right hand side. This result is implicit in Eq.s (3.357). We shall derive it again for clarity following ref. [83].

As now known, the second structure equation (3.347) defining the curvature two-form is not modified by the lifting, i.e. $\hat{R}^{ab}(\hat{\omega}) = R^{ab}(\omega) = R_{\mu\nu}{}^{ab} dx^\mu \wedge dx^\nu$. The action (3.349) for the isotopic theory becomes then, using (3.361)

$$\hat{S} = \frac{1}{4} \int R^{ab} \wedge (V^c \wedge V^d \varphi^2 + 2\varphi V^c \wedge \tau^d + \tau^c \wedge \tau^d)\epsilon_{abcd}. \tag{3.370}$$

By introducing explicitly holonomic indices, we have $dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta = d^4x\epsilon^{\mu\nu\alpha\beta}$, and using the properties of the totally antisymmetric symbols we can rewrite this action in the more familiar tensor language

$$\hat{S} = \int d^4x\sqrt{-g}[R\varphi^2 - 2\varphi R_{\mu\nu}\tau^{\nu\mu} + 2R_{\mu\nu}\tau^\nu{}_\alpha\tau^{\alpha\mu} - \frac{1}{2}R\tau_{\alpha\beta}\tau^{\beta\alpha} + R_{\mu\nu\alpha\beta}\tau^{\alpha\mu}\tau^{\beta\nu}], \tag{3.371}$$

where: square brackets denote antisymmetrization;

$$R_{\mu\nu}{}^{\alpha\beta}(\omega) = V_a^\alpha V_b^\beta 2(\partial_{[\mu}\omega_{\nu]}{}^{ab} + \omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b) \tag{3.372}$$

is the usual curvature tensor, constructed from the Riemann-Cartan connection ω; $R_{\mu\nu} = R_{\mu\alpha\nu}{}^{\alpha}$ is the Ricci tensor; $R = R_{\mu\nu}{}^{\mu\nu}$ is the scalar curvature (flat indices are holonomized by means of $V^{a}{}_{\mu}$, for example $\tau_{\mu\nu} = V^a_\mu V^b_\nu \tau_{ab}$); finally, $g = \det g_{\mu\nu} = (\det V^a_\mu)^2$, and

$$g_{\mu\nu} = V^a_\mu V^b_\nu \eta_{ab}, \eta = \text{diag}\,(1,1,1,-1) \tag{3.373}$$

is the world metric tensor.

Santilli's Lie-isotopic theory becomes then, in this case, an Einstein-Cartan theory for gravity coupled, in a strongly nonminimal way, to a symmetric second-rank tensor.

The variation of the isotopic action in the form (3.370) with respect to ω^{ab}, gives the expected constraint on the isotopic torsion,

$$\hat{R}^a = \nabla \hat{V}^a = 0. \tag{3.374}$$

By varying Eq.(3.370) with respect to the frames V^a, we obtain the modified Einstein field equations

$$(\varphi^2 R^{ab} \wedge V^c + \varphi R^{ab} \wedge \tau^c)\epsilon_{abcd} + (\varphi R^{ab} \wedge V^c \tau_d{}^k + R^{ab} \wedge \tau^c \tau_d{}^k)\epsilon_{abck} = 0\ , \tag{3.375}$$

or, in the usual notations

$$G_\alpha{}^\beta = \varphi^{-1}(F_\alpha{}^\beta - \tau_\alpha{}^\nu G_\nu{}^\beta) + \varphi^{-2}\tau_\alpha{}^\nu F_\nu{}^\beta, \tag{3.376}$$

where $G_\alpha{}^\beta$ is Einstein's tensor and

$$F_\alpha{}^\beta = R_\alpha{}^\nu \tau_\nu{}^\beta + \tau_\alpha{}^\nu R_\nu{}^\beta - \frac{1}{2} R \tau_\alpha{}^\beta - R_\nu{}^\mu \tau_\mu{}^\nu \delta_\alpha{}^\beta + R_{\mu\alpha}{}^{\nu\beta} \tau_\nu{}^\mu. \tag{3.377}$$

In the same way, the variation of the action (3.370) with respect to φ and τ gives the equations for the isotopic element. The kinetic terms for these fields are obtained inserting, into the definition of curvature (3.372), the explicit expression for ω^{ab}.

Notice that Eq. (3.363) can be solved by an iterative procedure, under the hypothesis that the isotopic element T_{ab} induces small deviations from the original geometrical structure, i.e. $T_{ab} - \eta_{ab} = \xi_{ab}$, with $|\xi_{ab}| \ll 1$ (a sort of weak-field approximation). To the first order in $\xi, \hat{R}^a$ is then equivalent to

$$dV^a + \omega^a{}_b \wedge V^b + d\xi^a = 0, \tag{3.378}$$

and this equation can be easily solved for ω to obtain the first-order isotopic contribution to the connection.

The connection between Gasperini's modification (3.376) of Einstein's field equations (3.320) and Yilmaz's modification (3.336) is remarkable. In fact, Yilmaz's stress-energy tensor $t_{\mu\nu}^{\rm Grav}$ is contained in the right hand side of Eq.s (3.376). A study of this important, yet unexplored aspect, is recommended here to interested researchers.

3.5.9 Isotopic Generalization of the Equations of Motion

To clarify better the generalized theory, Gasperini [82,83] provides the explicit calculation of the generalized equations of motion.

As is well known, the equations of motion in a gravitational theory should be obtained as a consequence of the energy-momentum conservation, which follows from the contracted Bianchi identities and from the field equations with matter sources for the interior problem.

In general relativity, the contracted Bianchi identity is given by

$$G^{\mu\nu};_{\nu} = 0 \,, \tag{3.379}$$

where a semicolon denotes the usual covariant derivative in terms of the holonomic connection $\Gamma_{\mu\nu}{}^{d}$. The field equations are given by

$$G^{\mu\nu} = \frac{k}{2}\Theta^{\mu\nu} \,, \tag{3.380}$$

where $\Theta^{\mu\nu}$ is the (symmetric) matter energy-momentum tensor. The conservation equations which follow from the above equations are given by

$$\Theta^{\mu\nu}_{;\nu} = 0 \,, \tag{3.381}$$

and can be written explicitly (remembering that in this case the connection reduces to the Christoffel coefficients) as

$$\partial_\nu(\sqrt{-g}\Theta^{\mu\nu}) + \left\{ {\mu \atop \nu\alpha} \right\} \sqrt{-g}\Theta^{\alpha\nu} = 0 \,. \tag{3.382}$$

By integrating this conservation law over the world tube of the test particle, following Papapetrou's method [125], defining

$$m\ u^\mu u^\nu = \frac{dt}{ds}\int d^3x\sqrt{-g}\Theta^{\mu\nu} \,, \tag{3.383}$$

and developing in power series the gravitational field, one gets in first approximation (pole-particle) the *geodesic equations of motion*

$$\frac{dp^\mu}{ds} + \left\{ {\mu \atop \nu\alpha} \right\} p^\alpha \mu^\nu = 0 \,, \tag{3.384}$$

where $p^\mu = m\, u^\mu$ is the momentum and u^μ the four-velocity of the test body (§3.4.13).

In the Einstein-Cartan theory [153–157] we have two Bianchi identities, one for the curvature

$$\nabla R^{ab} = 0\,, \tag{3.385}$$

and one for the torsion

$$\nabla R^a = R^a{}_b \wedge V^b\,. \tag{3.386}$$

By introducing holonomic indices, and contracting the first identity, we obtain, instead of Eq. (3.387), the following one

$$G^{\mu\nu}_{;\nu} = -2\, Q_\nu G^{\mu\nu} - 2\, Q_\nu{}^{\mu\alpha} G_\alpha{}^\nu + S_{\alpha\beta\nu} R^{\mu\nu\alpha\beta}\,, \tag{3.387}$$

and contracting the Bianchi identity for the torsion one obtains

$$G_{[\alpha\beta]} = S^\nu_{\alpha\beta;\nu} + 2\, Q_\nu S_{\alpha\beta}{}^\nu\,, \tag{3.388}$$

where $Q_\nu = Q_{\nu\alpha}{}^\alpha$, $S_{\alpha\beta\nu}$ is the so-called modified torsion tensor

$$S_{\alpha\beta}{}^\nu = Q_{\alpha\beta}{}^\nu + \partial^\nu_\alpha Q_\beta - \partial^\nu_\beta Q_\alpha, \tag{3.389}$$

and the covariant derivative now must be expressed in terms of the Riemann-Cartan connection.

Using the field equations of the theory

$$\begin{aligned} G_{\mu\nu} &= \frac{k}{2}\Theta_{\mu\nu}\,, \\ S_{\mu\nu\alpha} &= k\sigma_{\mu\nu\alpha}\,, \end{aligned} \tag{3.390}$$

where $\Theta_{\mu\nu}$ is the (generally nonsymmetric) canonical energy-momentum tensor, and $\sigma_{\mu\nu\alpha}$ the canonical spin density tensor, one obtains, from Eq. (3.387), the following generalized conservation law

$$\Theta^{\mu\nu}_{;\nu} + 2\, Q_\nu \Theta^{\mu\nu} + 2\, Q_\nu{}^{\mu\alpha}\Theta_\alpha{}^\nu - \sigma_{\alpha\beta\nu} R^{\mu\nu\alpha\beta} = 0\,. \tag{3.391}$$

Writing explicitly the covariant derivative, and separating the symmetric and antisymmetric part of $\Theta^{\mu\nu}$, this equation reduces to [158]

$$\begin{aligned} &\delta_\nu(\sqrt{-g}\Theta^{\mu\nu}) + \sqrt{-g}\left\{ {\mu \atop \nu\alpha} \right\}\Theta^{(\alpha\nu)} + \sqrt{-g} K^\mu{}_{\nu\alpha}\Theta^{[\alpha\nu]} = \\ &= \sqrt{-g}\sigma_{\alpha\beta\nu} R^{\mu\nu\alpha\beta}\,, \end{aligned} \tag{3.392}$$

and integrating this conservation law, as before, one can obtain the generalized equation of motion for a test particle in the Einstein-Cartan theory.

Notice that the antisymmetric part of $\Theta^{\mu\nu}$ can be expressed as a funciton of the spin density. For a spinless test particle one has then $\sigma_{\mu\nu\alpha} = 0$ and $\Theta_{[\mu\nu]} = G_{[\mu\nu]} = 0$, so that the conservation law reduces to the Riemannian one (3.390) and we obtain again a geodesical motion, as noticed first by Hehl [159].

The isotopic theory of gravity has the same geometrical structure as an Einstein-Cartan theory, as shown in Section 3.5.7, in which torsion is produced by the isotopic element $T_a{}^b$. Using the decomposition (3.361), the isotopic structure (3.347) can be written

$$\begin{aligned} R^{ab} &= d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \, , \\ R^a &= dV^a + \omega^a{}_b \wedge V^b \, , \end{aligned} \tag{3.393}$$

where

$$R^a = -\phi^{-1}\{C_{bc}{}^i \tau_{ia} + \omega_{[b}{}^{ai}\tau_{c]i}\} V^b \wedge V^c \, . \tag{3.394}$$

By taking the Lorentz exterior covariant derivative, we get the same Bianchi identities as in the Einstein-Cartan case

$$\begin{aligned} \nabla R^{ab} &= 0 \, , \\ \nabla R^a &= R^a{}_b \wedge V^b \, , \end{aligned} \tag{3.395}$$

and then, contracting indices, we are led to Eq. (3.387), (3.388) as before.

In order to obtain the equations of motion, however, it is necessary to introduce field equations relating Einstein tensor and the torsion tensor of the isotopic theory to the matter sources, so that the corresponding conservation equations for energy-momentum and angular momentum can be written.

To this aim, the isotopic theory for pure gravity considered until now must be completed by introducing a term in the isotopic action coupling the matter sources to gravity. As the full theory must be based on a Lie-isotopic algebra, also the matter fields, in general, will be coupled to the operator defining the isotopic lifting.

It should be stressed, therefore, that consistent equations of motion can be formulated only in the framework of a complete Lie-isotopic theory, including matter sources besides the gravitational field.

Such a theory will be reviewed later on. However, even in the simple case in which the matter Lagrangian does not contain explicitly the isotopic element (and then the source of gravity is simply the usual canonical stress-energy tensor $\Theta^{\mu\nu}$) the equations of motion of the isotopic theory are different from the ones of the Einstein-Cartan theory considered previously. Consider in fact the isotopic field equation (3.376) and, to simplify notations, let us denote with $\Lambda_\alpha{}^\beta$ the isotopic correction to the Einstein tensor, i.e.

$$\begin{aligned}\Lambda_\alpha{}^\beta &= \varphi^{-1}(R_\alpha{}^\nu \tau_\nu{}^\beta - R_\mu{}^\nu \tau_\nu{}^\mu \delta_\alpha{}^\beta + R_{\mu\alpha}{}^{\nu\beta}\tau_\nu{}^\mu) \\ &+ \varphi^{-2}\tau_\alpha{}^\nu F_\nu{}^\beta \, . \end{aligned} \tag{3.396}$$

Suppose that, introducing matter, the isotopic field equations (3.376), i.e.

$$G_\alpha{}^\beta = \Lambda_\alpha{}^\beta \, , \tag{3.397}$$

are modified as follows

$$G_\alpha{}^\beta = \Lambda_\alpha{}^\beta + \frac{k}{2}\Theta_\alpha{}^\beta \, , \tag{3.398}$$

and that torsion is related to spin according to the usual Eq. (3.390). We then have

$$G_{\alpha;\beta}^{\beta} = \frac{k}{2}\Theta^{\alpha\beta}{}_{;\beta} + \Lambda_\alpha{}^\beta{}_{;\beta} \, , \tag{3.399}$$

and from the contracted Bianchi identity (3.395) we obtain the following conservation equations

$$\begin{aligned}&\Theta^{\mu\nu}{}_{;\nu} + 2Q_\nu \Theta^{\mu\nu} + 2Q_\mu{}^\beta \Theta_\alpha{}^\nu - \sigma_{\alpha\beta\nu}R^{\mu\nu\alpha\beta} = \\ &-\frac{2}{k}(\Lambda^{\mu\nu}{}_{;\nu} + 2Q_\nu \Lambda^{\mu\nu} + 2Q_\nu{}^{\mu\alpha}\Lambda_\alpha{}^\nu) \, , \end{aligned} \tag{3.400}$$

which differ from Eq.(3.391) because of the $\Lambda^{\mu\nu}$ terms representing the contributions due to the coupling of gravity to the isotopic tensor.

Again the connection of the above results with Yilmaz's [134] theory of gravitation is remarkable. In fact, tensor (3.396) is evidently inclusive of Yilmaz's stress-energy tensor.

We can therefore say that *the isotopic generalization of Einstein's gravitation naturally produces Yilmaz's stress-energy tensor.*

The connection of the above results with Santilli's [132] identification of the gravitational field with the electromagnetic field of matter constituents is also intriguing. In fact, the tensor $\Theta_{\mu\nu}$ of Eq.s (3.398) can be interpreted as Santilli's electromagnetic tensor $\Theta_{\mu\nu}^{\mathrm{Elm}}$ of Eq.s (3.333).

In summary, the isotopic generalization of gravity does indeed offer genuine hopes of achieving all conditions 1 through 9 of Sections 3.5.5 for an "ideal" theory of gravitation, as we shall see better later on.

3.5.10 Deviation from Geodesic Motion of the Isotopically Lifted Gravitation

Gasperini [83] then passes to the identification of another important aspect of the generalized theory of gravitation, the irreducible lack of (conventional) geodesic character. In turn, this property is at the foundation of the proof of the "No no-interaction theorem" considered in Section 3.4.14, as well as of several other implications of the generalized theory.

The equations of motion of the isotopic theory are not geodesics even in the case of spinless test particles. In this case, in fact, we have from (3.398)

$$\Theta_{[\alpha\beta]} = -\frac{2}{k}\Lambda_{[\alpha\beta]}\,, \tag{3.401}$$

and Eq. (3.400) becomes

$$\begin{aligned}\partial_\nu(\sqrt{-g}\Theta^{(\mu\nu)}) + \sqrt{-g}\left\{ {\mu \atop \nu\alpha} \right\}\Theta^{(\alpha\nu)} &= \frac{2}{k}\partial_\nu(\sqrt{-g}\Lambda^{[\mu\nu]})\\ +\frac{2}{k}\sqrt{-g}(k^\mu{}_{\nu\alpha}\,\Lambda^{[\alpha\nu]} - \Lambda^{\mu\nu}{}_{;\nu} - 2Q_\nu\,\Lambda^{\mu\nu} - 2Q_\nu{}^{\mu\alpha}\,\Lambda_\alpha{}^\nu)\,. \end{aligned}\tag{3.402}$$

Suppose that the deviations of $T_a{}^b$ from the identity are very small. We can put $\varphi \simeq 1$ and $\tau \ll 1$ and, neglecting terms which are quadratic in τ, conservation eq. (3.410) reduces to

$$\partial_\nu(\sqrt{-g}\Theta^{(\mu\nu)}) + \sqrt{-g}\left\{ {\mu \atop \nu\alpha} \right\}\Theta^{(\alpha\nu)} + \sqrt{-g}\Phi^\mu = 0\,, \tag{3.403}$$

where, to the first order in τ,

$$\begin{aligned}\sqrt{-g}\Phi^\mu &= \frac{2}{k}\sqrt{-g}\Lambda^{(\mu\nu)}{}_{;\nu} =\\ &\frac{2}{k}\partial_\nu(\sqrt{-g}\Lambda^{(\mu\nu)}) + \frac{2}{k}\sqrt{-g}\left\{ {\mu \atop \nu\alpha} \right\}\Lambda^{(\alpha\nu)}\,,\end{aligned}\tag{3.404}$$

and

$$\Lambda^{\mu\nu} = R^{\mu\alpha}\tau_\alpha{}^\nu - R_\alpha{}^\beta\tau_\alpha{}^\nu - R_\alpha{}^\beta\tau_\beta{}^\alpha g^{\mu\nu} + R^{\alpha\mu\beta\nu}\tau_{\beta\alpha}\,. \tag{3.405}$$

The integration of this equation, according to the standard procedure, shows that *the path of a test body in this isotopic theory deviates from a geodesic, and it is described by the equation*

$$\frac{dp^\mu}{ds} + \left\{ {\mu \atop \nu\alpha} \right\}p^\alpha p^\nu - F^\mu = 0\,, \tag{3.406}$$

where, to first order in τ,

$$F^{\mu} = \frac{dt}{ds}\int d^3x\sqrt{-g}\Phi^{\mu} =$$
$$= \frac{2}{k}\frac{d}{ds}\int d^3x\sqrt{-g}\,\Lambda^{\mu 4} + \frac{2}{4}\left\{ {\mu \atop \nu\alpha} \right\}\frac{dt}{dS}\int d^3x\sqrt{-g}\Lambda^{(\alpha\nu)}\ , \quad (3.407)$$

is the isotopic 4-force acting on a spinless test particle.

Notice that in global flat space ($g_{\mu\nu} = \eta_{\mu\nu}$ everywhere) this force is vanishing. However, it cannot be locally eliminated, because of the curvature dependent terms contained in $\Lambda^{\mu 4}$ appearing in the first integral; F^{μ} is then similar, in this respect, to the spin-curvature forces [158–159] which break the validity of the equivalence principle in its strong form.

As pointed out by Santilli [26], Eq.s (3.406) are the gravitational extensions of the Lorentz-isotopic dynamics, Eq.s (3.305). The gravitational formulation of closed nonselfadjoint systems of N particles each moving within a medium composed by the remaining particles is then characterized by [26]

$$\left\{ \begin{array}{l} \frac{dp^{\mu}_k}{ds} + \left\{ {\mu \atop \nu\alpha} \right\} p^{\alpha}_k u^{\nu}_k = F^{\mu}_{\text{kNSA}}, \quad k = 1, 2, ..., N \\ \Theta^{\mu\nu}{}_{,\nu} = 0 \end{array} \right. \ , \quad (3.408)$$

where one recognizes the *conventional* total conservation laws (see, e.g., ref. [160]) as subsidiary constraints.

Also, Eq.s (3.406) cleary establish the No No-interaction Theorem of Section 3.4.14, trivially, because the nongeodesic forces cannot be eliminated in a Lie-isotopic theory of gravitation. Thus, a nontrivial isotopic lifting always implies the existence of nontrivial interactions.

This result is trivial if one keeps in mind the arena under consideration here: test particles moving within a physical medium in the interior problem of gravitation. In fact, a test particle cannot be reduced to free conditions when moving, say, in Jupiter's atmosphere.

As a final remark, the reader should be aware that the nongeodesic forces F_{NSA} have been derived here for the case of the simplest possible realization of the isotopic elements $T_a{}^b$. When such elements acquire a nontrivial functional dependence, say in the velocities (see Sect. 3.5.10), the nongeodesic forces also acquire a nontrivial functional dependence. It is at this more general level that the nonselfadjointness of the nongeodesic forces emerges more clearly.

3.5.11 Restriction of the Isotopically Lifted Gravitation to the Interior Problem

We now pass to the review of other important aspects of the generalized theory: the extension by Gasperini [82,83] to isotopic elements with a nontrivial functional dependence, and its use by Santilli [26] for the restriction of the lifting to the interior problem of gravitation.

As shown in Section 3.4.11, the algebraic structure of the Poincaré group is preserved not only in the case of a constant matrix $T_a{}^b$, but also in the case of variable isotopic elements. Consider, as an example, the following particular form [82]

$$T_a{}^b = \text{diag}\ (f_1(x_1), f_2(x_2), f_3(x_3), f_4(x_4)) \ , \tag{3.409}$$

where $f_i(x_i)$ are four scalar functions, each depending only on the corresponding coordinate. Assume the general expression for the isotopic structure (3.340), where X_A are the Poincaré generators, $X_A = \{P_a, M_{ab}\}$. As the isotopic elements (3.409) commute with rotations, but not with translations, the isotopic curvature reduces in this case to

$$\begin{aligned} \hat{R} &= \hat{R}^A X_A = \{d\hat{h}^A + \frac{1}{2} f_{BC}{}^A \hat{h}^B \wedge \hat{h}^C\} X_A + \\ &+ V^a \wedge V^b T_a{}^c [P_c, T_b{}^d] P_d \ , \end{aligned} \tag{3.410}$$

where $\hat{h}^A + \{\hat{V}^a, w^{ab}\}$ are the components of the isotopic potential. We obtain then, from (3.418) the following isotopic structure equations

$$\begin{aligned} \hat{R}^a &= d\hat{V}^a + w^a{}_b \wedge \hat{V}^b + V^b \wedge V^c T_b{}^d [P_d, T_c{}^a] \ , \\ \hat{R}^{ab} &= dw^{ab} + w^a{}_c \wedge w^{cb} \ . \end{aligned} \tag{3.411}$$

But since $[P_{d'} T_c{}^a] \propto \partial_d T_c{}^a$, it is easy to see that for the particular form (3.409) of the isotopic elements, one has

$$T_{[b|}{}^d \partial_d T_{|c]}{}^a = 0 \ . \tag{3.412}$$

In this case the generalized structure (3.518) reduces simply to Eq. (3.411). The algebraic and geometrical structure of the Lorentz group is preserved, and an isotopic gravitational theory can be formulated following exactly the same formalism of Section 3.5.7, with the only difference that there is an additional contribution to torsion due to the derivative of $T_a{}^b$.

Another very simple, but interesting, lifting in terms of a not constant isotopic element is obtained starting again with the form (3.345) of the isotopic operator, and putting

$$T_a{}^b = \varphi \delta_a{}^b , \tag{3.413}$$

where φ is a scalar field, $\varphi = \varphi(x)$.

Following the same procedure as before, we obtain again Eq.(3.411). This time, however, the last term of the isotopic torsion is not vanishing, and we have a theory with a new algebraic structure, different from general reltivity. In particular, $\hat{V}^a = \varphi V^a$ and the isotopic torsion (3.410) becomes explicitly

$$\hat{R}^a = d\varphi \wedge V^a + \varphi R^a + \varphi^d \varphi \wedge V^a , \tag{3.414}$$

where $R^a + \nabla V^a$ is the usual torsion two form relative to the standard frame V^a. In this case the choice of a standard gravitational action (3.344) is no longer justified, as the underlying geometrical structure is changed, and the problem of finding an appropriate action to formulate a consistent Lie-isotopic theory in this case is presently open.

Santilli [26] reinspected the above findings by Gasperini and pointed out that *the isotopic elements $T_a{}^b$ represent the deviations from the conventional, local Minkowski space caused by motion of a test particle within the physical medium of the interior problem* as per Eq. (3.186), where $(T_a^b) = \theta$ (see below). As a consequence, the functional dependence of the elements $T_a{}^b$ is expected to be, in general, not only the local coordinates x of the test particle, but also the velocities $\dot{x}$, density μ of the interior medium, temperature T, and any other needed physical quantity, according to geometrization (§3.4.10)

$$T_a{}^b = T_a{}^b(x, \dot{x}, \mu, T, ...) . \tag{3.415}$$

In particular, the dependence on the local coordinates could be indirect, e.g., via a dependence of the density and temperature on the distance r from the center of the system, i.e.,

$$T_a{}^b = T_a{}^b(\dot{x}, \mu(x), T(x), ...) , \tag{3.416}$$

but without a direct dependence on x.

In different terms, the most important functional dependence of the isotopic elements is in the *velocities* because, *when a particle is at rest with respect to the interior medium, the contact nonhamiltonian forces are null.* The second dominant functional dependence is on the density because, again, *when such density is null, the contact nonhamiltonian forces are also null.* Santilli therefore suggested the following form of the isotopic elements

$$\begin{aligned}\left(T_a{}^b\right) &= \text{diag}\,(f_1, f_2, f_3, f_4)\\ &\quad f_a = f_a(\dot{x}, \mu, T), \quad a, b = 1, 2, 3, 4\ ,\end{aligned} \tag{3.417}$$

where the local dependence of the density and temperature on the distance r from the center is ignored for first, local approximations.

The important aspect is that elements (3.417) commute, locally, with all generators of the Poincaré algebra, by therefore putting the foundations for regaining the exact (but isotopic),local, Poincaré symmetry, as we shall see better in the next section.

Once the isotopic elements are interpreted as representing the deviations caused by the interior physical media from the dynamics of the exterior problem, it then follows, as a consequence, that they must reduce to the identity in the exterior problem itself. This leads to *the subsidiary constraint (or conditions) imposed by Santilli [26] on all Lie-isotopic generalizations of Einstein's Gravitation*

$$T_a{}^b\|_{r>R} = \delta_a{}^b\ , \tag{3.418}$$

where R is the radius of the sphere of the interior problem, and $r = |\ \vec{x}\ |$ is the distance of the considered local point from the center.

As now familiar from the work by Gasperini [82], *when $T_a{}^b = \delta_a{}^b$ the conventional gravitational theory is recovered in its entirety.* In this way Santilli ensures the existence of a generalized geometry for the interior problem of gravitation, while ensuring the preservation of conventional geometries for the exterior problem, exactly along the preceding occurences at the Newtonian (§3.3) and relativistic (§3.4) levels.

It should be stressed, however, that, even though the *geometry* for the exterior problem is the conventional one, the *field equations* are expected to be different than those by Einstein, Eq.s (3.320), because of their sourceless character which is incompatible with the charged structure of matter (§3.5.3).

In summary, we shall hereon assume the following *realization of the isotopic elements*

$$\begin{aligned}(T_a{}^b) &= \text{Diag}\,(f_1, f_2, f_3, f_4), f_a = f_a(\dot{x}, \mu, T)\ ,\\ &\quad T_a{}^b\Big|_{r>R} = \delta_a{}^b\ ,\end{aligned} \tag{3.419}$$

where the second conditions can be verified either with a discontinuous function (say, a step function) or with a smooth functional behaviour, depending on the physical conditions at hand.

More specifically, suppose that the celestial body considered has no atmosphere. Then, the transition from the interior to the exterior problem is discontinuous and a step function is appropriate. Suppose instead, that the body has an atmosphere with a density continuously going to zero with the increase of r. A correspondingly smooth realization of conditions (3.419b) is then needed.

The equations of motion for isotopic elements (3.419) are expected to be the same as those of Section 3.5.9. Nevertheless, specific studies to this effect are absent at this moment, to our knowledge.

Santilli [26] finally suggested the *use of integrodifferential realizations of the isotopic elements, as a way to represent more closely the nonlocal nature of the contact interactions experienced by the test particle.*

This yields an intriguing geometrical structure. Recall that all available geometries are essentially local in character because the topology most known until now is local in nature. A bona-fide generalization of a geometry into a nonlocal/integrodifferential form therefore requires a generalization of the background topology into a suitable nonlocal form, which has not yet been accomplished by mathematicians in a final form applicable to physics, to our best knowledge.

Santilli's Lie-isotopic lifting appears to be able to bypass these topological problems and yield a genuine, mathematically consistent nonlocal/integrodifferential geometry. The idea is so natural to "creep in unnoticed". The mechanism is essentially based in incorporating all nonlocal/integrodifferential terms in the isotopic unit (or elements) of the theory. But Lie's theory leaves such unit unaffected. Thus, conventional, local topologies can be used, while the emerging geometrical context is intrinsically nonlocal.

As indicated in Section 3.5.5, if local realizations of the isotopic elements are desired, one can obtain them via power series expansions in the velocities. As a result, *the velocity-dependence of the isotopic element is, in general, arbitrary*, and depends on the considered conditions at hand, including the value of the speed itself. In fact, as now familiar in engineering (but equally so in physics), contact forces of test particles in Earth's atmosphere (say, rockets or satellites) may reach powers in the three-velocity as high as the 10-th.

3.5.12 The Locally Lorentz-Isotopic Character of the Generalized Theory

We now review a central aspect of the Lie-isotopic generalization of Einstein's Gravitation, its local Lorentz-isotopic character identified by Gasperini [82], and its restriction to a form isomorphic to the Lorentz group by Santilli [26]. An important property of the isotopic theory of gravitation is therefore that the local Lorentz symmetry, rather than being "violated" in the interior problem, is instead realized in its most general possible form.

For clarity, let us recall the definition of Santilli's spaces of the first, second and third class $\hat{M}_I, \hat{M}_{II}, \hat{M}_{III}$ (§3.4.5). In essence, $\hat{M}_I$ is a space with null curvature equipped with isotopic metric (3.195) which is topologically equivalent to the Minkowski metric; $\hat{M}_{II}$ is an isospace also with null curvature, but the topological equivalence of the metric with the Minkowski metric is generally lost; finally, $\hat{M}_{III}$ is a generally curved isospace.

Recall also from Section 3.4.7 that the Lie-isotopic lifting of the Lorentz symmetry is formulated for space $\hat{M}_{III}$, although it evidently admits formulations in the simpler spaces $\hat{M}_{II}$ and $\hat{M}_I$. In all these liftings, the isomorphism of the Lie-isotopic Lorentz group with the conventional group is ensured when the isounit is positive-definite.

The applications of these results to any theory of gravitation are at least twofold [26]. First, as anticipated in Section 3.4.7, Santilli's lifting of the Lorentz symmetry provides means for the explicit construction of the generalized transformations leaving invariant the metric g of the curved space. As the reader will recall, this is achieved via the sole knowledge of the new metric g and use of expansions (3.224).

This first step is applicable to *any* theory of gravitation, (whether Lie-isotopic or not, and Riemannian or not) and we shall symbolically write

$$\begin{pmatrix} \text{Flat theory} \\ \text{Lorentz Symmetry } 0(3.1) \\ \text{Minkowski space } M(x,n,\mathbf{R}) \end{pmatrix} \rightarrow \begin{pmatrix} \text{Curved gravitational theory} \\ \text{Lorentz} - \text{isotopic symmetry } \hat{0}(3.1) \\ \text{Santilli space } \hat{M}_{III}(x,g,\hat{\mathbf{R}}) \end{pmatrix}. \tag{3.420}$$

Secondly, the lifting is applicable to the *tangent space* of *any* gravitational theory (Lie-isotopic, Riemannian-Cartan, affine, etc.), in which the local symmetry of the tangent space is no longer the conventional Lorentz symmetry. Again, the methods provide the means for the explicit construction of the generalized symmetry transformations in their explicit form, via the sole knowledge of the generalized metric $\hat{\eta}$ and we shall symbolically write

$$\begin{pmatrix} \text{Flat theory} \\ \text{Lorentz symmetry } 0(3.1) \\ \text{Minkowski space } M(x,\eta,\mathbf{R}) \end{pmatrix} \to \begin{pmatrix} \text{Flat tangent space} \\ \text{Lorentz} - \text{isotopic symmetry } \hat{0}(3.1) \\ \text{Santilli space } \hat{M}_{III}(x,\hat{\eta},\hat{\mathbf{R}}) \, , \end{pmatrix} , \tag{3.421}$$

where the generalized tangent space is assumed to be generally curved.

But all gravitational theories are two-metric theories, one metric for the curved space and one metric for the tangent space. Whenever the tangent space is a generalization of the Minkowski space, Santilli's isotopy of the Lorentz symmetry applies again, thus allowing the construciton of the explicit symmetry transformations of the tangent space. Furthermore, the techniqes show that, in this latter case, the deviations of the generalized tangent metric from the conventional Minkowski one may only be apparent, in the sense that the Lorentz symmetry can still be exact at the Lie-isotopic level (see also the Finslerian treatment of tangent spaces [110,112]).

We now pass to the inspection of the Lie-isotopic generalization of Einstein's gravitation. First, the Lorentz-isotopic symmetry is applicable to the curved space via lifting (3.420).

Second, Gasperini [82] identified the local Lorentz-isotopic character of the theory (or, more accurately, constructed the generalized theory in such a way to be locally Lorentz-isotopic). In fact, via the use of lifting (3.350) and vierbein (3.351), Gasperini obtains a generalized theory with the tangent space characterized by the metric

$$\hat{\eta}^{ab} = \eta^{cd} T_c{}^a T_d{}^b \, . \tag{3.422}$$

The Lie-isotopic character of the theory is then evident, with the identification of the metric $g = T\eta$ of Eq. (3.186) with $T(\eta^{cd} T_c{}^a T_d^b)$.

Santilli [26] reinspected Gasperini's results and introduced the *restriction of the isotopic metric* $\hat{\eta}$ *to be topologically equivalent to the Minkowski metric*, and we shall symbollically write

$$\hat{\eta} \approx \text{Diag}\ (b_1^2, b_2^2, b_3^2, -b_4^2), \quad b_\mu^2 > 0, \quad \mu = 1,2,3,4. \tag{3.423}$$

The above restriction essentially ensures that the isotopic Lorentz symmetry and the conventional one are isomorphic.

Restriction (3.423) is formulated for curved tangent spaces $\hat{M}_{III}$. Nevertheless, flat tangent isospaces $\hat{M}_I$ are generally sufficient for practical applications of the isotopic theory of gravity (see the examples later on in this section). In this case, restriction (3.423) implies the equivalence of metric (3.422) with the diagonal form (3.195), i.e.

$$\hat{\eta}^{ab} \equiv \text{Diag}\ (b_1^2, b_2^2, b_3^2, -b_4^2), \quad b_\mu^2 > 0\ , \tag{3.424}$$

which is characterized by lifting (3.350) via the explicit form of the isotopic elements

$$(T_a{}^b) = \text{Diag}\ (b_1, b_2, b_3, b_4), \quad b_a > 0\ . \tag{3.425}$$

The local isomorphism of the Lorentz-isotopic and the conventional symmetry is then ensured by Theorem 3.5. All the examples to be reviewed later on are particular cases of isotopic elements (3.425).

In summary, the Lie-isotopic gravitation is a two-metric theory as it happens for all gravitational theories. The primary difference with conventional theories is that the metric of the tangent space (for the interior problem only) is generalized (as it happens already at the level of Finslerian approaches [110,112]). However, this does not imply a breaking of the local Lorentz symmetry, but its preservation as an exact symmetry, although realized in its most general possible form.

Also, the two metrics are not independent, but rigidly related. In fact, according to Eq.s (3.350) and (3.351), the tangent space metric (3.422) is defined via the isotopic elements of the algebraic (Lorentz) isotopy which is coupled nonminimally to the gravitational metric according to the rules [60]

$$\hat{\eta}^{ab} = g^{\mu\nu} \hat{V}_\mu^a \hat{V}_\nu^b = g^{\mu\nu} V_\mu^c T_c^a V_\nu^d T_\alpha^b\ . \tag{3.426}$$

In different terms, the Lie-isotopic lifting of Einstein's gravitation produces a form of quasi-Riemannian gauge theory with a tangent space group other than the Lorentz group (in conventional realization), and that group results to be the Lie-isotopic Lorentz group.

In this generalized geometrical framework, gravitation can be interpreted as a deviation of the world manifold from Santilli's tangent isospace $\hat{M}_{III}$. This allows Gasperini [81,82] to reach the following, important, additional result

ISOTOPIC PRINCIPLE OF EQUIVALENCE: Gravitational effects may locally disappear when the metric of the space-time manifold approaches the metric of the tangent, Santilli's isospace $\hat{M}_{III}$, i.e. for

$$g^{\mu\nu} \to \hat{\eta}^{\mu\nu}\ . \tag{3.427}$$

In fact, it can be seen from Eq.s (3.347), that the isotopic connection $\hat{\omega}$ can be locally eliminated by putting $V^a_\mu = \delta^a_\mu$. A free falling observer in flat media defined by $V^a_\mu = \delta^a_\mu$ will no longer represent an inertial frame for the Lie-isotopic theory. In this system, force fields are the physical manifestation of the Lorentz-isotopic symmetry (see the deviation from geodesic motion of §3.5.10). Also, the deviation of the isotopic metric $\hat{\eta}$ from the conventional Minkowski metric η is a measure of the "breaking" of the (conventional) Lorentz symmetry.

To understand the generalized theory of gravitation we are here formulating (see below for a more accurate definition), the reader should think of a test particle that begins its motion in the *exterior* problem. In this case motion occurs in empty space, and the metric of the tangent space is the conventional Minkowski metric. The geometry for the exterior problem is then the conventional one, but the field equations are not expected to be Einstein's Eq.s (3.320) because of the lack of electromagnetic source (Section 3.5.3).

Suppose now that the same test particle moves into the *interior* problem (say, Jupiter's upper atmosphere). Then the particle experiences velocity-dependent, contact forces which imply a necessary deviation from the conventional Minkowski metric of the tangent space. The Lie-isotopic generalization of Einstein's Gravitation is then activated. The new physical features (the generally inhomogeneous and anisotropic character of the medium, the velocity-dependence of the forces, etc.) are represented precisely by generalized action functional (3.370).

As far as the local symmetry of the tangent space is concerned, the contact interactions of the interior problem do generate a deformation of the Minkowski metric, but the deformation is not such to alter the topological character of the original Minkowski metric, in the sense that the topological structure Diag (1,1,1, -1) *cannot* be deformed into an inequivalent topology, say, of the type (1,1,-1,-1). As a result, the Lorentz symmetry remains exact, although at the covering isotopic level.

In closing, we would like to indicate that S. Weinberg [155] has proposed a quasi-Riemannian theory of gravity with a tangent space symmetry other than the Lorentz symmetry. It would be interesting to identify the possible connections between Weinberg's and Gasperini's works.

Similarly, C. Wetterich [156] has proposed a vierbein of the type

$$g^{\mu\nu} e^i_\mu e^j_\nu \neq \eta^{ij} , \tag{3.428}$$

starting from a different physical motivation, within the context of multidimensional, chiral, fermionic theories.

Also, Rosen [157] has formulated a bimetric theory in which one of the two metrics describes gravitation, and the other describes a generally curved background associated to a fundamental reference frame, a preferred rest frame of the universe. The connections with the Lie-isotopic theory of gravitation are remarkable and deserving a study. In fact, the former metric can describe gravitation in both theories, and the second metric could be associated, in the Lie-isotopic theory, with a priveleged reference frame at rest with the medium in which motion of the interior problem occurs, as suggested in ref. [1]. Note that liftings (3.420) and (3.421) apply to both metrics of *Rosen's Gravitation* and that, under restriction (3.424), the exact nature of the (abstract) Lorentz symmetry persists.

Studies directly related to the Lie-isotopic lifting of Einstein's Gravitation have been conducted by Nishioka [162,163]. In the first paper, one can find a Lie-isotopic formulation of Maxwell electromagnetism and a Lie-isotopic formulation of gravitational, electromagnetic and scalar fields. The second paper deals with the connection of the Lie-isotopic lifting of the Riemannian manifolds with the Lyra and Weyl Manifolds.

3.5.13 Gasperini-Santilli Gravitation

We shall now summarize all the preceding results of this section and present the essential aspects of *Gasperini-Santilli General Relativity* (or *Gravitation*, for short).

As now familiar, the conventional Einstein's General Relativity can be formulated as a gauge theory for the Poincaré group. The fundamental variables of the theory are then the frames V^a and the connection ω^{ab}. Using the algebra of the Poincaré generators, one obtains the usual structure equations defining the torsion R^a and the curvature R^{ab}. The simplest action, including matter sources minimally coupled to gravity, can be written as

$$S = \int (\frac{1}{4} R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} - \frac{1}{3} \Theta_a \wedge V^a), \tag{3.429}$$

where Θ_a is the canonical energy-momentum three-form, whose explicit expression, in terms of the canonical energy-momentum tensor Θ^{ab}, is

$$\Theta^a = \Theta^{a\mu} \epsilon_{\mu\nu\alpha\beta} dx^\nu \wedge dx^\alpha \wedge dx^\beta \ , \tag{3.430}$$

and $\epsilon_{\mu\nu\alpha\beta}$ is the totally antisymmetric symbol.

As reviewed earlier, a Lie-isotopic theory of gravity can be formulated by introducing a generalized frame $T^a = V^b T_b{}^a$ where $T_a{}^b$ is an isotopic operator which defines the lifting of the vierbein field. The structure equations are then

$$\begin{aligned} \hat{R}^a &= dT^a + \omega^a{}_b \wedge T^b \,, \\ \hat{R}^{ab} &= d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \,. \end{aligned} \tag{3.431}$$

In this way, one is led to the following Lie-isotopic action without source

$$\hat{S} = \frac{1}{4} \int \hat{R}^{ab} \wedge T^c \wedge T^d \epsilon_{abcd} \,. \tag{3.432}$$

The geometric and algebraic structure of general relativity is preserved; however, the gravitational gauge fields are T^a and ω^{ab}, instead of V^a and ω^{ab}.

It seems therefore natural, in the framework of such Lie-isotopic gravitational theory, to introduce matter according to the formal prescription (3.390) supposing that the canonical stress tensor Θ_{ab} is minimally coupled not to V^a, but to the generalized isotopic frames T^a.

The *Gasperini [81–84]-Santilli [16,26,132] Gravitation* can then be defined by the following equations

$$\begin{aligned} \hat{S} &= \int (\frac{1}{4} \hat{R}^{ab} \wedge T^c \wedge T^d \epsilon_{abcd} - \frac{1}{3} \Theta_a^{\text{Elm}} \wedge T^a) \,, && (3.433.a) \\ T_a{}^b &= T_a{}^b(\dot{x}, \mu, T, ...) \,, && (3.433.b) \\ T_a{}^b\Big|_{r>R} &= 0 \,, && (3.433.c) \\ \hat{\eta}^{ab} &= \eta^{cd} T_c{}^a T_d{}^b \approx \text{Diag}\,(b_1^2, b_2^2, b_3^2, -b_4^2), b_a > 0 \,, && (3.433.d) \\ \Theta_{ab,c}^{\text{Elm}} &= 0 \,, && (3.433.e) \end{aligned}$$

where: Eq. (3.433a) is Gasperini's [82] isotopic action with Santilli's [132] hypothesis on the electromagnetic origin of the gravitational field, of course, in this first classical approximation; the second and third equations represent the assumed functional dependence of the isotopic elements and their restriction to the interior problem only [26]; Eq.s (3.433d) represents the restriction of admissible metrics for the tangent isospace to be topologically equivalent to the Minkowski metric, so as to preserve the exact character of the Lorentz symmetry; and Eq.s (3.433e) represent the *conventional* conservation laws as *subsidiary constraints* to isotopic action (3.433a). Note that the condition of topological equivalence, Eq.s (3.433d) implies the validity of all topological properties of the Lie-isotopic theory, such as the sufficient smoothness of functional dependence (3.433b) or the invertibility of the elements $T_a{}^b$.

The variation of action (3.433a) with respect to T^a gives the isotopic generalization of Einstein's field equations

$$\frac{1}{2}\hat{R}^{ab} \wedge T^c \epsilon_{abcd} = \frac{1}{3}\Theta_d^{\text{Elm}} . \tag{3.434}$$

Note that the same equation can be obtained also by using the definition $T^a = V^b T_b{}^a$ and performing the variation with respect to V^a, because $T_a{}^b$ is invertible.

The variation with respect to the connection ω^{ab} (supposing that we are considering unpolarized macroscopic matter, i.e. that ω^{ab} is not explicitly contained in the matter part of the action) gives the isotopic generalization of the usual torsion equations

$$\frac{1}{2}\hat{R}^a \wedge T^b \epsilon_{abcd} = 0. \tag{3.435}$$

In order to obtain a solution of the field equations for the interior problem it is convenient to rewrite the isotopic equations (3.433) in the usual tensor language, introducing explicitly holonomic indices. Using the decomposition $T = V + \tau$ for the isotopic frames, Eq.s (3.434) yield the *Gasperini-Santilli field equations for the interior problem*

$$G_\alpha{}^\beta = \Theta_\alpha^{\beta\text{elm}} + G_\alpha{}^\nu \tau_\nu{}^\beta - R_\mu{}^\nu \tau_\nu{}^\mu \delta_\alpha{}^\beta + R_\nu{}^\beta \tau_\alpha{}^\nu + R_{\mu\alpha}{}^{\nu\beta} \tau_\nu{}^\mu, \tag{3.436}$$

where:

$$R_{\mu\nu\alpha}{}^\beta = \delta_\nu \Gamma_{\nu\alpha}{}^\beta - \delta_\nu \Gamma_{\mu\alpha}{}^\beta + \Gamma_{\mu\rho}{}^\beta \Gamma_{\nu\alpha}{}^\rho - \Gamma_{\nu\rho}{}^\beta \Gamma_{\mu\alpha}{}^\rho \tag{3.437}$$

is the Riemann curvature tensor;

$$\Gamma_{\mu\nu}{}^\alpha = \frac{1}{2} g^{\alpha\beta}(\partial_\nu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}) \tag{3.438}$$

are the Christoffel symbols; $R_\mu{}^\nu = R_{\mu\alpha}{}^{\nu\alpha}, R = R_\mu{}^\mu$;

$$G_\alpha{}^\beta = R_\alpha{}^\beta - \frac{1}{2}\delta_\alpha{}^\beta R \tag{3.439}$$

is Einstein's tensor; and finally, $\tau_\mu{}^\nu$ is the traceless part of the tensor $T_a{}^b$.

The *Gasperini-Santilli field equations for the exterior problem* are given by

$$G_{\alpha\beta} = \Theta^{\text{Elm}}_{\alpha\beta} \,. \tag{3.440}$$

The above theory can be trivially extended with the addition of Yilmaz's stress-energy tensor $t^{\text{Grav}}_{\mu\nu}$ as in Eq. (3.337), in which case we shall call it t he *Gasperini-Santilli-Yilmaz Theory of Gravitation.*

To avoid possible misrepresentations of the above equations, the reader should recall that, in the conventional theory for the exterior problem, the matter tensor $M_{\alpha\beta}$ is null, and all sources resulting from nonull total values of the charge and of the magnetic moments are represented via an additional tensor $t^{\text{Elm}}_{\alpha\beta}$. In Santilli identification of the gravitational and electromagnetic field, $M_{\alpha\beta} \approx \Theta^{\text{Elm}}_{\alpha\beta}$, such an additional tensor is redundant because the contributions $t^{\text{Elm}}_{\alpha\beta}$ are automatically produced by the contributions of each individual charged consituent of the body considered.

It is easy to see that the Gasperini-Santilli-Yilmaz Gravitation does indeed verify most of the conditions set forth in Section 3.5.5, with the understanding that considerable additional research remains to be done.

To begin, the background (empty) space remains homogeneous and isotropic, as represented by the local Minkowski metric η of Eq.s (3.433d). Nevertheless, the geometry of the interior problem is generally inhomogeneous and anisotropic, as represented by the metric $\hat{\eta}$ of the tangent isospace $\hat{M}_{III}$.

The theory is, in the interior problem, essentially noninvariant under local, conventional, Lorentz transformations. This is a necessary condition to represent local variations from conventional conservation laws (in a way compatible with the total conservation laws) and avoid perpetual-motion approximations. Nevertheless, the theory is invariant under the local, Lorentz-isotopic symmetry (§3.4). Furthermore, under condition (3.433d), this symmetry results to be isomorphic to the conventional one.

In the transition to the exterior problem, condition (3.433c) ensures the recovering of the conventional Riemannian geometry with a conventional, local, Lorentz symmetry.

However, unlike Einstein's Gravitation, the Gasperini-Santilli-Yilmaz theory exhibits in the exterior problem a nowhere null source tensor of the gravitational field, thus allowing the compatibility of the gravitational theory with the charged and stress-energy structure of matter. Gravitation is then nowhere reducible to pure geometry, but it is generated, in a classical approximation, by the contributions of all interactions of the constituents of matter.

Finally, the *conventional* conservation laws (see, e.g., ref. [160], Section IV-20) are imposed as subsidiary constraints in order to achieve a gravitational counterpart of the notion of closed nonhamiltonian systems, which we have already encountered at the Newtonian (§3.3) and relativistic (§3.4) levels. As the reader will recall, *conventional* total conservation laws are imposed as subsidiary constraints in all these systems.

The reader should also recall that, at the Newtonian and relativistic levels, the systems considered admit algebraic solutions, that is, the number of constraints represented by total conservation laws results to be less in number than the total number of internal nonselfadjoint forces (for $N \geq 3$). The systems therefore admit particular cases in which total conservation laws are automatically satisfied without being bona-fide subsidiary constraints (see, Eq.s (3.96) and following comments). As expected, exactly the same situation occurs at the gravitational level, e.g., because the number of subsidiary constraints for total conservation laws is less than the number of isotopic elements.

We therefore expect the existence of explicit models of the Gasperini-Santilli-Yilmaz Gravitation in which the conventional total conservation laws are automatically verified without being genuine subsidiary constraints to the isotopic action (3.433a). Nevertheless, in general, Eq.s (3.433e) are indeed bona-fide subsidiary constraints to action (3.433a), exactly as it happens at the Newtonian and relativistic levels.

In this way, the Gasperini-Santilli-Yilmaz Gravitation verifies most of the conditions 1-9 of an "ideal" gravitational theory introduced in Section 3.5.5, with the understanding that so much remains to be investigated.

An explicit example verifying all conditions (3.433) is presented in the subsequent sections via a small constant deformation of the Minkowski metric in the interior problem.

Without any claim of completeness, we point out below the following open problems.

The explicit construction of a general example verifying all conditions (3.431) is recommended. In particular: the example should exhibit a nontrivial functional dependence of the isotopic elements at least in the velocities and/or in the density of the interior medium; the Lorentz-isotopic symmetry transformations for the related tangent space should be explicitly computed via the techniques of Section 3.4; the verification of conventional total conservation laws should be explicitly proven; and the covering nature of the gravitational model over the corresponding relativistic and Newtonian models should be studied as an important element for completing the classical study of closed nonselfadjoint systems.

An important point in the explicit construction of a model of gravitation (3.433) is that Santilli's electromagnetic tensor for the case of the interior problem *is not* the conventional one of Maxwell theory on curved spaces [160], but requires a construction on an isotopically lifted space along Nishioka lines [161].

The possible existence of a (noncanonical) *Birkhoffian representation* of the interior problem should also be investigated because of the possibility of allowing an unambiguous "hadronization" into an operator form on Hilbert spaces, along the lines of Section 1.3, Eq (1.58) and following.

In turn, the existence of a consistent "hadronization" could allow the identification of the possible short range contributions to the "origin" of the gravitational field, i.e. those of weak, nuclear and strong character.

Another problem that remains open is the resolution of the issue of "unification" of the gravitational and electromagnetic fields, as attempted in most of the literature, versus the "identification" of the gravitational and electromagnetic fields advocated by Santilli (§3.5.3).

A further problem that remains open is the study whether the Gasperini-Santilli Gravitation is capable of representing all experimental data in gravitation.

Despite the existence of these open (and rather intriguing) problems, we are unaware of any experimental, phenomenological or other information that may disprove the Gasperini-Santilli Gravitation for the interior problem. This is evidently due to the fact that *all* classical information accumulated during this century on gravitation is strictly related to the exterior problem and certainly not applicable, say, in the interior of a star. *We therefore know of no criticism that can be moved against the Gasperini-Santilli interior gravitation.* As a matter of fact, all available information favors the generalized relativity over the conventional one. We are referring, classically, to the incontrovertible experimental evidence of local interior departures from the conventional rotational and Lorentz symmetries, versus the perpetual-motion approximation implied by the conventional theory. At the microscopic level, all available phenomenological information also favors a departure from the Minkowski metric in the interior of hadrons, as reviewed in Section 3.4.3. Needless to say, all this information is merely preliminary. The final resolution of the issue is evidently of experimental nature, and will occur only after the conduction of the fundamental tests of space-time symmetries recommended in Section 3.5.18 (see Fig.6).

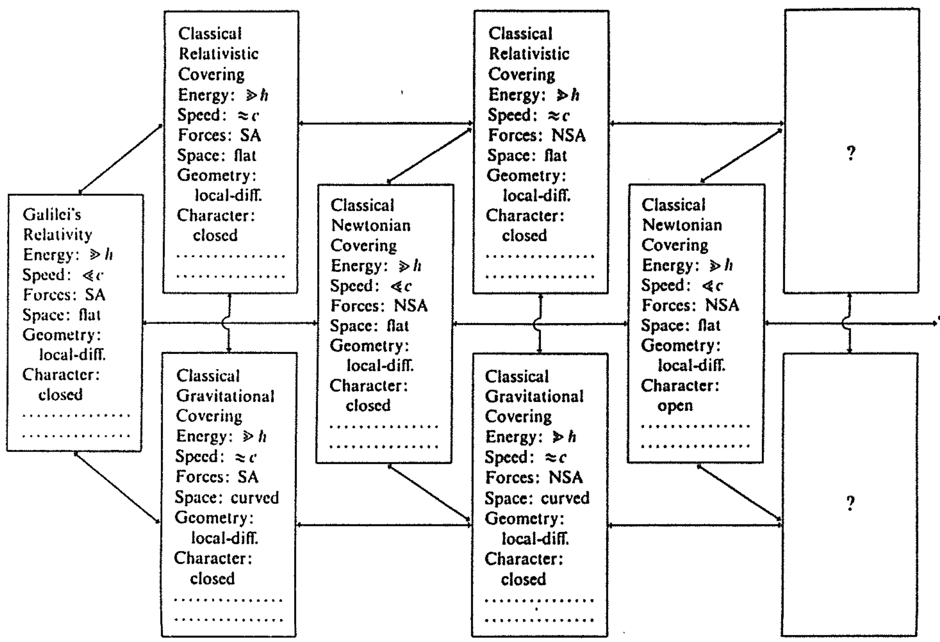


FIGURE 15. A reproduction of Figure 6.1 page 250 of monograph [15] (see also the more general Figure 1, page xvii of monograph [16]) expressing Santilli's view of the lack of existence of terminal physical theories. In fact, starting from conventional relativities, the figure includes the generalized relativities reviewed in this work, and indicates the yet more general relativites that are already conceivable at this time, although not technically realizable because of insufficient mathematical formulations (e.g., insufficient topologies).

As far as the exterior problem is concerned, the generalized theory provides no numerical alteration of the results achieved by Einstein's Gravitation, trivially, because $\Theta^{\rm Elm}_{\mu\nu} \approx M_{\mu\nu} + t^{\rm Elm}_{\mu\nu}$. Notice also that, when seen from the outside, the Gasperini-Santilli gravitation coincides with the conventional one because of the subsidiary constraints (3.433e). As a matter of fact, the restriction of the isotopy to the interior problem, and the conventional total conservation laws as subsidiary constraints are imposed [26] precisely for the purpose of avoiding any quantitative differentiation in the exterior problem between the isotopically lifted and the conventional theory. As a result, *we know of no criticism that can be moved against the Gasperini-Santilli exterior problem* unless exactly the same criticism applies also to the conventional theory (see Yilmaz's criticisms of Section 3.5.4).

In closing, we would like to mention the fact that *the Gasperini-Santilli Gravitation is a genuine covering of Einstein's Gravitation* in the sense of ref. [1], that is:

a) The generalized theory is constructed via mathematical methods (Lie-isotopy) more general than those of the conventional theory;
b) the generalized theory describes physical conditions (contact nonhamiltonian interactions) more general than those of the conventional theory; and
c) the generalized theory admits the conventional one as a particular case when all isotopic elements are everywhere equal to the identity.

The authors would be grateful to any colleagues bringing to their attention (at the address of the Institute for Basic Research, P.O. Box 1577, Palm Harbor, FL. 34682, USA, Fax: 813-934-9275) any information, whether in favor or against, related to the theory of gravitation here considered.

3.5.14 An Example of Isotopic Interior Problem

We now consider, as an example, a neutron star which, for simplicity, is assumed to be spherical, homogeneous and isotropic. In his original proposal [82], Gasperini worked out an example of the isotopically lifted gravity which is directly applicable to the interior problem of such a neutron star. For simplicity, we shall ignore herinafter the source terms and restrict our attention to the pure contribution from the isotopy. Its generalization to include the source terms was worked out by Gasperini in the subsequent paper [84].

A very simple parametrizaton of Lorentz "non-invariance" formulations has been suggested by Nielsen and Picek [99] in terms of the following generalized metric tensor (§3.4.3)

$$\hat{\eta}^{ab} = \eta^{ab} - \chi^{ab}, \quad \hat{\eta} \in \hat{M}_I \,. \tag{3.441}$$

Under the assumed rotational invariance, χ_{ab} is a symmetric traceless tensor, defined in terms of only one constant parameter α,

$$\begin{aligned}\chi_{ab} &= \frac{\alpha}{3}\,\mathrm{diag}\,(1,1,1,3) = \\ &= \frac{\alpha}{3}(\eta_{ab} + 2\delta_{ab})\,. \end{aligned} \tag{3.442}$$

If the metric (3.441) is interpreted as the metric of Santilli's isospace $\hat{M}_I$ (for a different interpretation see however ref. [105]), one can formulate a Lie-isotopic theory of gravity based on this metric, by introducing the isotopic element

$$T_a{}^b = \mathrm{diag}\,\left(\sqrt{1-\frac{\alpha}{3}}, \sqrt{1-\frac{\alpha}{3}}, \sqrt{1-\frac{\alpha}{3}}, \sqrt{1+\alpha}\right)\,, \tag{3.443}$$

for which

$$\begin{aligned}\eta^{cd}T_c{}^a T_d{}^b &= \mathrm{diag}\,\left(1-\frac{\alpha}{3}, 1-\frac{\alpha}{3}, 1-\frac{\alpha}{3}, -(1+\alpha)\right) \\ &= \eta^{ab} - \chi^{ab}\,. \end{aligned} \tag{3.444}$$

The underlying assumption is that the neutron star has the same density and interior problem, say, of kaons. The gravitational field is however modified, as we can see explicitly by considering the vacuum field equations of the isotopic theory.

Suppose that the deviations from the conventional Lorentz symmetry are very small, i.e. $\alpha \ll 1$ (in ref. [99] the value $\alpha \leq 10^{-3}$ has been obtained from experimental data relative to the charged pion and kaon decays). Gasperini then evaluates the isotopic corrections to the usual Einstein theory to the first order in α.

In this approximation we have

$$\begin{aligned}T_{ab} &\simeq \mathrm{diag}\,\left(1-\frac{\alpha}{6}, 1-\frac{\alpha}{6}, 1-\frac{\alpha}{6}, -(1+\frac{\alpha}{2})\right) \\ &= \eta_{ab} - \frac{1}{2}\chi_{ab}\,. \end{aligned} \tag{3.445}$$

The vacuum field equations (3.376), neglecting α^2 terms, and putting $\varphi \simeq 1$ and $\tau_{ab} \simeq \frac{1}{2}\chi_{ab}$, become

$$G_a{}^b = \frac{1}{2}R_a{}^c\chi_c{}^b - \frac{1}{2}R_c{}^d\chi_d{}^c\delta_a{}^b + \frac{1}{2}R_{ca}{}^{db}\chi_d{}^c\,, \tag{3.446}$$

from which, using the explicit expression (3.442) for χ_{ab}, we have again to first order in α,

$$G_{ab} = \frac{\alpha}{3}(R_a{}^c\delta_{cb} - R^{cd}\delta_{cd}\eta_{ab} + R_{cabd}\delta^{cd}) . \tag{3.447}$$

Notice the explicit breaking of the conventional Lorentz symmetry, corresponding to the contraction of the curvature with the Kronecker tensor δ_{ab}, instead of that with the Minkowski metric η_{ab}.

Notice also the reconstruction of the exact Lorentz symmetry at the Lie-isotopic level because metric (3.441) verifies conditions (3.433d).

Other isotopic corrections to Einstein's equations are due to the non-Riemannian part of the connection, contained implicitly in the curvature terms. The first order contribution to the torsion can be obtained from Eq.s (3.366), putting $\varphi \simeq 1$ and $\tau_{ab} \simeq \chi_{ab}/2$:

$$Q_{bc}{}^a = \frac{1}{2}[\gamma_{bc}{}^i\chi_{ia} + \frac{1}{2}\gamma_{ba}{}^i\chi_{ci} - \frac{1}{2}\gamma_{ca}{}^i\chi_{bi}] , \tag{3.448}$$

where γ_{abc} is the Riemannian part of the connection, defined in Eq. (3.368). Using Eq. (3.367), one obtains the isotopic connection to the first order in α

$$\omega_{acb} = \gamma_{acb} + K_{acb} , \tag{3.449}$$

where

$$\begin{aligned} K_{acb} \quad = \quad & \frac{\alpha}{3}[\gamma_{bc}{}^i\delta_{ai} - \gamma_{cb}{}^i\delta_{ai} + \gamma_{ba}{}^i\delta_{ci} \\ & -\gamma_{ca}{}^i\delta_{bi}] . \end{aligned} \tag{3.450}$$

Again the presence of the Kronecker symbol denotes the deviation from the conventional Lorentz symmetry. Notice that in this particular case the non-Riemannian part of the connection is nonvanishing only if at least one of the indices of K_{abc} is equal to four, otherwise $K_{abc} = 0$ because of the metricity of the Riemannian connection, $\gamma_{bca} = -\gamma_{bac}$.

Finally, using the definition of curvature (3.351) applied to the connection (3.349) one finds, to the first order in α,

$$\begin{aligned} R_{\mu\nu}{}^{ab} \quad = \quad & R_{\mu\nu}{}^{ab} + \partial_{[\mu}K_{\nu]}{}^{ab} + \partial_{[\mu}{}^{ac}K_{\nu]c}{}^b \\ & +K_{[\mu}{}^{ac}\gamma_{\nu]c}{}^b , \end{aligned} \tag{3.451}$$

where $R_{\mu\nu}{}^{ab}$ denotes the usual curvature tensor for the Riemannian part of the connection. By using contraction to obtain the Ricci tensor and the scalar curvature, one gets

$$\begin{aligned} G_a{}^b &= G_a{}^b + \partial_{[a} K_{c]}{}^{bc} - \frac{1}{2}\partial_{[c} K_{d]}{}^{cd} \delta_a{}^b - \\ &- \gamma_{[i}{}^{ic} K_{j]c}{}^j \delta_a{}^b + \gamma_{[a}{}^{bi} K_{c]i}{}^c + K_{[a}{}^{bi} \gamma_{c]i}{}^c \, . \end{aligned} \tag{3.452}$$

Combining Eq.s (3.447) and (3.452) we have the explicit expression for the first order isotopic corrections to Einstein's field equations $G_{ab} = 0$

$$\begin{aligned} G_{ab} &= -\partial_{[a} K_{c]b}{}^c + \frac{1}{2}\partial_{[c} K_{d]}{}^{cd} \eta_{ab} + \\ &+ \gamma_{[i}{}^{ic} K_{j]c}{}^j \eta_{ab} - \gamma_{[a|b}{}^i K_{c]i}{}^c - K_{[a|b}{}^i \gamma_{c]i}{}^c + \\ &+ \frac{\alpha}{3}(R_a{}^c \delta_{cb} - R^{cd}\delta_{cd}\eta_{ab} + R_{cadb}\delta^{cd}) \, , \end{aligned} \tag{3.453}$$

where the contorsion K is given by Eq.s (3.450).

Note that the considered neutron star has a discontinuous transition from the interior to the exterior problem, as far as matter density is concerned. A step function is in this case appropriate for the realization of condition (3.433c).

The above example is useful to illustrate the local symmetry of the theory. In fact, the *conventional*, local Lorentz symmetry is manifestly broken. But metric (3.441) preserves the topological character of the Minkowski metric, thus verifying Theorem 3.5 and conditions (3.433c). The Lorentz symmetry therefore remains exact.

The next issue illustrated by the above model is the *dynamics* of the tangent isospace $\hat{M}_I$, that is, which is the local invariant, and what is the maximal possible speed of causal signals. These questions are answered by Santilli's Special Relativity (§3.4). The reader should recall that the value of the fourth component of metric (3.441) is greater than the speed of light in vacuum, $\hat{\eta}^{44} = c_0(1+\alpha) > c_0, c_0 = 1, \alpha > 0$. Nevertheless, $\hat{\eta}^{44}$ *is not* the invariant of the theory, which is given instead by the maximal speed of causal signals (§3.4.9), and this speed, for the model of neutron star considered, results to be greater than c_0 (see Appendix B for more details).

The reader should be aware that the above results, *are not* peculiar for the Nielsen-Picek metric, because any modification of the Minkowski metric necessarily implies a change in the maximal speed of causal signals [18],[26].

In summary, the example illustrates that *the Gasperini-Santilli gravitation exhibits deviations in the tangent space of the interior problem from Einstein's Special Relativity, but the Lorentz symmetry remains exact.* In turn, this occurrence is important for the formulation of experiments, as suggested in the closing Section 3.5.18.

3.5.15 An Example of Isotopic Equations of Motion

Gasperini [84] continued the example reviewed in the preceding section by working out explicitly the isotopic equations of motion for the case of Nielsen-Picek metric (3.441) on Santilli's space $\hat{M}_I$. We reproduce the example below because of its value for interior problems, such as its direct applicability to the neutron star of the preceding section.

As stressed earlier the equations of motion for a test particle in a given external Lie-isotopic gravitational field are to be obtained by integrating the conservation laws of the energy and angular momentum. This follows from the field equations and the Bianchi identities of the Lie-isotopic theory.

By taking the covariant exterior derivative of the isotopic equations (3.434), we obtain the isotopic Bianchi identities

$$\begin{aligned} \nabla \hat{R}^a &= \hat{R}^a{}_b \wedge T^b , \\ \nabla \hat{R}^{ab} &= 0. \end{aligned} \tag{3.454}$$

Using these relations, the exterior covariant derivative of the field equations (3.434) and (3.435) gives the conservation laws of the isotopic theory, for the energy-momentum

$$\nabla \Theta_d = \frac{3}{2} \hat{R}^{ab} \wedge R^c \epsilon_{abcd} , \tag{3.455}$$

and for the angular momentum

$$\frac{1}{2} \hat{R}^a{}_k \wedge T^k \wedge T^b \epsilon_{abcd} = 0 , \tag{3.456}$$

where the symbol "Elm" has been dropped for simplicity. To first order in α, the torsion is vanishing, $\hat{R}^a = 0$, the connection is Riemannian, and condition (3.455) is reduced simply to $\nabla \Theta^d = 0$, which, introducing holonomic indices, can be rewritten as

$$\Delta_\nu \Theta^{\mu\nu} \equiv \Theta^{\mu\nu}_{;\nu} = 0 , \tag{3.457}$$

where a semicolon denotes the covariant derivative performed with the Christoffel symbols.

The second isotopic equation of conservation (3.456) provides informations on the antisymmetric part of the canonical stress tensor. In fact, Eqs. (3.456) and (3.434) imply

$$\Theta^a \wedge T^b - \Theta^b \wedge T^a = 0, \tag{3.458}$$

and can be rewritten in the tensor language as

$$\Theta^{\mu\nu} - \Theta^{\nu\mu} = \Theta^{\nu\alpha}\tau_\alpha{}^\mu - \Theta^{\mu\alpha}\tau_\alpha{}^\nu. \tag{3.459}$$

Expanding $\Theta^{\mu\nu}$ in power series in the parameter α, and using the explicit form of $\tau_\mu{}^\nu$, one obtains, to first order in α

$$\begin{aligned} \Theta^{\mu\nu} &= \Theta^{(\mu\nu)} + \Theta^{[\mu\nu]}, \\ \Theta^{[\mu\nu]} &= \frac{\alpha}{3}[\Theta^{(4\nu)}\delta_4{}^\mu - \Theta^{(4\mu)}\delta_4{}^\nu] , \end{aligned} \tag{3.460}$$

where round and square brackets denote respectively symmetrization and antisymmetrization.

Therefore, the isotopic conservation equations are different from the corresponding general relativistic one because, in the isotopic theory, the canonical energy-momentum is no longer a symmetric tensor, even in the case of spinless matter.

Using the properties of the Christoffel connection, Eq.s (3.457) become

$$\begin{aligned} &\partial_\nu\left(\sqrt{-g}\Theta^{(\mu\nu)}\right) + \Gamma_{\nu\alpha}{}^\mu\sqrt{-g}\Theta^{(\alpha\nu)} \\ &+\partial_\nu\left(\sqrt{-g}\Theta^{[\mu\nu]}\right) = 0. \end{aligned} \tag{3.461}$$

Integrating this conservation law over an isospacelike section (§3.4.11) of the world tube of the test particle, performing a multipole expansion of the gravitational field inside the particle according to Papapetrou's method [158], multiplying by dt/ds, and using Eq.s (3.460), one gets, in first approximation

$$\begin{aligned} &\frac{d}{ds}\int d^3x'\sqrt{-g}[\Theta^{(\mu 4)}(1-\frac{4}{3}\alpha) + \frac{4}{3}\alpha\Theta^{(44)}\delta_4{}^\mu] \\ &+\Gamma_{\nu\alpha}{}^\mu\frac{dt}{ds}\int d^3x'\sqrt{-g}\Theta^{(\alpha\nu)} = 0. \end{aligned} \tag{3.462}$$

Finally, defining

$$m_0 u^\mu u^\nu = \frac{dt}{ds}\int d^3x'\sqrt{-g}\Theta^{(\mu\nu)}, \tag{3.463}$$

where m_0 is the mass, and $u^\mu = dx^\mu/ds$ is the four-velocity of the test body, we obtain the following equations of motion, for a Lie-isotopic theory of gravity in which the deviation from the Lorentz symmetry is parametrized by the metric of Nielsen and Picek

$$(1-\frac{4}{3}\alpha)\frac{d^2x^\mu}{ds^2} + \frac{4}{3}\alpha\delta_4{}^\mu\frac{d^2x^4}{ds^2} + \Gamma_{\nu\alpha}{}^\mu\frac{dx^\nu}{ds}\frac{dx^\alpha}{ds} = 0. \tag{3.464}$$

In the limit $\alpha \to 0$, we recover the usual geodesic equations, as expected.

The above example of equations of motion is important to illustrate another aspect of the generalized theory, the local deviation from geodesic motion in the interior problem (only), which is the crucial condition for the representation of local internal deviations from conservative conditions, and interior trajectories of perpetual-motion type.

The example also illustrates the "No No-Interaction Theorem" of Sect. 3.4.15. In fact, for $\alpha \neq 0$, trajectory (3.464) *cannot* be reduced to a conventional geodesic motion. As a result, the test particle under consideration is experiencing a nowhere reducible, nontrivial interaction.

As a final comment, the reader should be aware that Eq.s (3.464) have primarily *local* meaning, that is, they provide a first approximation of the motion of the test particle in the neighborhood of a given point of isospace $\hat{M}_I$. In fact, as stressed in §3.4, the isotopic metric $\hat{\eta}$ is expected to be a nonlinear and nonlocal function of the local quantities.

3.5.16 An Example of Isotopically Lifted Orbital Motion for the Interior Problem

Gasperini [84] continued his example with the explicit calculation of the modification to orbital trajectories caused by isotopy, again, for the case of Nielsen-Picek parametrization of local Lorentz "noninvariance". We review this additional development below because of its practical value for explicit calculations regarding the interior problem.

The reader should be aware that Gasperini presented the isotopy of orbital trajectories for the *exterior* problem [84]. The objective was that of ascertaining possible upper limits to Nielsen-Picek Lorentz-asymmetry parameter α that could be established by the precession of the perihelion of Mercury.

Following Santilli's analysis [26], such objective is no longer realizable, even though the orbital equations reviewed below remain valuable for the *interior* problem.

To put it differently, the Gasperini-Santilli Gravitation as per Eq.s (3.433) recovers the conventional Riemannian geometry in the exterior problem and, thus, admits *conventional* orbital motion in the exterior problem.

Using spherical polar coordinates, and inserting the Christoffel symbols corresponding to the metric (3.441) the isotopic equations of motion (3.464) for $x^2 = \theta$ are satisfied by assuming that the orbit is confined to the equatorial plane (as in general relativity), so we can put everywhere $\theta = \pi/2 =$

const. The remaining equations for $x^1 = r, x^3 = \varphi$, and $x^4 = t$ become then

$$(1 - \frac{4}{3}\alpha)\ddot{r} + \frac{1}{2}\frac{d\lambda}{dr}\dot{r}^2 - re^{-\lambda}\dot{\varphi}^2 + \frac{1}{2}\frac{dv}{dr}e^{\nu-\lambda}\dot{t}^2 = 0,$$
$$(1 - \frac{4}{3}\alpha)\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0, \quad \ddot{t} + \dot{v}\dot{t} = 0\,, \tag{3.465}$$

where a dot now denotes derivative with respect to s.

The last two equations can be easily integrated, and we obtain

$$\begin{aligned} r^{2(1+\frac{4}{3}\alpha)}\dot{\varphi} &= h^{1+\frac{8}{3}\alpha}, \\ e^{\nu}\dot{t} &= k, \end{aligned} \tag{3.466}$$

where h and k are two integration constants. Notice that Kepler's second law is modified, as the areal velocity is no longer constant but becomes a function of the radial coordinate, $r^2\dot{\varphi} \propto r^{-8\alpha/3}$ (if $\alpha \neq 0$); this is a consequence of the generalized angular momentum conservation law of the isotopic theory.

Using Eqs. (3.466), the radial equation (3.456) can be rewritten

$$\begin{aligned} &\frac{d}{d\varphi}\Big\{(1-2mu)^{-1}h^2u'^2(hu)^{\frac{16}{3}\alpha} \\ &+ \; h^2u^2(hu)^{\frac{16}{3}\alpha} - k^2(1-2mu)^{(1+\frac{2}{3}\alpha)}\Big\} \\ &= \; \alpha\left\{(1-2mu)^{-1}\frac{4}{3}\frac{d}{d\varphi}(h^2u'^2) \right. \\ &\left. \quad +\frac{8}{3}\frac{d}{d\varphi}(h^2u^2)\right\}, \end{aligned} \tag{3.467}$$

where $u = r^{-1}$ and a prime denotes derivatives with respect to φ. Notice that a circular orbit of constant radius, $u' = 0$, is still a possible solution of the isotopic equation (3.467). Supposing $u' \neq 0$, the equation of the orbit becomes, to first order in α,

$$\begin{aligned} &u''(hu)^{\frac{16}{3}\alpha} - \frac{4}{3}\alpha u'' + \mu(hu)^{\frac{16}{3}\alpha}(1-2mu) \\ &+ \; \frac{8}{3}\alpha u'^2u^{-1} + mu'^2(hu)^{\frac{16}{3}\alpha}(1-2mu)^{-1} \\ &- \; \frac{k^2m}{h^2}(1+\frac{2}{3}\alpha)(1-2mu)^{-(1+\frac{2}{3}\alpha)} = 0\,. \end{aligned} \tag{3.468}$$

An approximate solution of this equation can be obtained by using an iterative procedure (as in general relativity), putting

$$u \simeq \overset{(0)}{u} + \overset{(1)}{u}, \tag{3.469}$$

where $\overset{(0)}{u}$ is the unperturbed Newtonian solution, obtained putting $\alpha = 0$ and neglecting the relativistic contributions. We suppose then that the isotopic corrective terms, of order $\alpha\ \overset{(0)}{u}$, are not larger than the terms representing general relativistic corrections ($\sim m\ \overset{(0)}{u}{}^2$) due to the curvature of the world manifold, and that both contributions are included in $\overset{(1)}{u}$.

Putting the $\alpha = 0$, from Eq.s (3.467) we have

$$\begin{aligned} &\left(1 - 2m\ \overset{(0)}{u}\right)^{-1} h^2\ \overset{(0)}{u'}{}^2 + h^2\ \overset{(0)}{u}{}^2 \\ &- k^2\left(1 - 2m\ \overset{(0)}{u}\right)^{-1} = -1, \end{aligned} \tag{3.470}$$

and Eq. (3.468) for $\overset{(0)}{u}$ is reduced to

$$\overset{(0)}{u''} + \overset{(0)}{u} - \frac{m}{h^2} = 0. \tag{3.471}$$

The well-known Newtonian solution

$$\overset{(0)}{u} = \frac{m}{h^2}[1 + e\cos(\varphi - \varphi_0)] \tag{3.472}$$

represents an ellipse with eccentricity e, semimajor axis a, and semilatus rectum L related by

$$L = a(1 - e^2) = \frac{h^2}{m}\,. \tag{3.473}$$

To first order in α, neglecting terms of order higher than

$$\alpha\ \overset{(0)}{u} \sim \frac{\alpha}{L} \text{ and } m\ \overset{(0)}{u}{}^2 \sim \frac{m}{L^2}, \tag{3.474}$$

we have, from (3.468) and (3.469), the following equation for $\overset{(1)}{u}$

$$\begin{aligned} &\overset{(1)}{u''} + \overset{(1)}{u} - 3m\ \overset{(0)}{u}{}^2 + \frac{4}{3}\alpha\ \overset{(0)}{u} - 2\alpha\frac{m}{h^2} \\ + \ &\frac{8}{3}\alpha\ \overset{(0)}{u'}{}^2\overset{(0)}{u}{}^{-1} + \frac{16}{3}\alpha\frac{m}{h^2} ln\left(h\ \overset{(0)}{u}\right) = 0. \end{aligned} \tag{3.475}$$

Using now the explicit expression (3.472) for $\overset{(0)}{u}$, we can expand the last two terms for orbits with small eccentricity, neglecting terms of order e^2 and higher. We can also neglect the terms representing constant corrections, as they do not produce observable effects, keeping only those corrections whose contibution increases continuously at each revolution. Equation (3.475) is reduced then to

$$\overset{(1)}{u}'' + \overset{(1)}{u} = \left(\frac{6m^3}{h^4} - \tfrac{20}{3}\,\alpha \frac{m}{h^2} \right) e \cos(\varphi - \varphi_0) \,, \tag{3.476}$$

and the solution is

$$\overset{(1)}{u} = \left(3\frac{m^2}{h^2} - \tfrac{10}{3}\alpha \right) \frac{m}{h^2} e \sin(\varphi - \varphi_0) \,. \tag{3.477}$$

To first order in α, including also general-relativistic effects, Gasperini obtains the approximate equation of the orbit

$$u \simeq \overset{(0)}{u} + \overset{(1)}{u} = \frac{m}{h^2} \left[1 + \cos(\varphi - \varphi_0 - \Delta\varphi_0)\right] , \tag{3.478}$$

where

$$\Delta\varphi_0 = \left(\frac{3m^2}{h^2} - \tfrac{10}{3}\alpha \right) \varphi \,, \tag{3.479}$$

is the precession of the orbit per unit revolution angle φ. After a full revolution ($\varphi = 2\pi$), the perihelion shift is then

$$\Delta\varphi_0 = \frac{6\pi m}{L} \left(1 - \tfrac{10}{9}\alpha \frac{L}{M} \right) . \tag{3.480}$$

The above equations, again, are valid for the interior problem only of Gasperini-Santilli Gravitation with Nielsen-Picek tangent metric. They essentially provide a quantitative, although approximate model of deviations from conventional Einsteinian equations that are expected for realistic conditions in the interior motion. Equivalently, Eq.s (3.478) provide a first approximation of the physical differences existing between the motion of a test particle in the exterior case (motion in vacuum) and in the interior case (motion within a physical medium).

The above approximation with a small and constant isotopy of the Minkowski metric, $\eta \to \hat{\eta}$, could however have astrophysical applications, e.g., for a body orbiting inside the atmosphere of another body.

3.5.17 An Isotopic Generalization of the Schwarzschild Metric

Gasperini [84] finally computed explicitly a Lie-isotopic generalization of the Schwarzschild metric for the case of Nielsen-Picek parametrization (3.441) of local Lorentz "noninvariance". It should be stressed from the outset that *the isotopic metric provided below is not an exact solutions of Eq.s (3.433)*, but it provides a solution only on a first approximation.

Nevertheless, the emerging model naturally applies to the interior of the neutron star of the preceding sections. As one can see, the findings are significant to illustrate the profound physical implications of the Lie-isotopic generalization of Einstein's Gravitation, e.g., for the problem of black holes. The reader should be aware that similar results are obtained via any theory capable of representing the local nonconservation of angular momentum and other quantities in the interior problem. To put it differently, the Schwarzschild metric is a by-product of the perpetual motion approximation in the interior problem implied by Einstein's Gravitation. If physical reality is admitted and quantitatively represented, say, for the vortices in Jupiter's atmosphere with a varying angular momentum, a departure from Scwarzschild metric is unavoidable. A suitable revision of the conventional notions of "singularities" and "black" (or "brown") holes is then expected.

The usual procedure to obtain a static and spherically symmetric solution of the gravitational equations is to introduce spherical polar coordinates $\{r, \theta, \varphi\}$, so that the proper-time interval can be written in the "standard form"

$$ds^2 = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \tag{3.481}$$

where ν and λ are functions only of r (notice that a static and isotopic metric like (3.481) can be a solution of the isotopic equations (3.447) because, using the metric of Nielsen and Picek as the metric of the local Santilli space, we are considering a deviation from the conventional Lorentz invariance which is still rotationally invariant).

The explicit computation of the Christoffel symbols and of the curvature tensor for the metric (3.481) shows that $R_\alpha{}^\beta = 0$ if $\alpha \neq \beta$, that $R_2{}^2 = R_3{}^3$, and that the only nonvanishing components of $R_{4\alpha}{}^{4\beta}$ are

$$\begin{aligned} R_{41}{}^{41} &= R_4{}^4 + \frac{1}{r}\frac{dv}{dr} e^{-\lambda}, \\ R_{42}{}^{42} &= R_{43}{}^{43} = -\frac{1}{2r}\frac{dv}{dr} e^{-\lambda}. \end{aligned} \tag{3.482}$$

In this case the isotopic equations (3.447) are reduced to only three independent equations:

$$\frac{1}{2}\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}+\frac{1}{4}\left(\frac{dv}{dr}\right)^2-\frac{1}{4}\frac{dv}{dr}\frac{dv}{dr}\frac{d\lambda}{dr}=0,$$

$$\frac{1}{2}\frac{d^2v}{dr^2}+\frac{1}{4}\left(\frac{dv}{dr}\right)^2-\frac{1}{r}\frac{d\lambda}{dr}-\frac{1}{4}\frac{dv}{dr}\frac{d\lambda}{dr}=-\frac{2}{3}\frac{\alpha}{r}\frac{dv}{dr},$$

$$l^{-\lambda}\left[1+\tfrac{1}{2}r\left(\frac{dv}{dr}-\frac{d\lambda}{dr}\right)\right]-1=\frac{\alpha}{3}r\frac{dv}{dr}e^{-\lambda}. \tag{3.483}$$

By subtracting (3.483b) from (3.483a) and integrating, we get

$$\nu(1-\tfrac{2}{3}\alpha)+\lambda=\beta, \tag{3.484}$$

where β is an integration constant; Eq. (3.483c) gives then

$$r\ e^{\nu(1-\frac{2}{3}\alpha)-\beta}=r\ +\ \text{const.} \tag{3.485}$$

We choose, as usual, this second integration constant equal to $-2m = -2GM$ (where G is the Newton constant and M the mass of the central source) in order to obtain the Newtonian gravitational potential in the weak field limit; combining (3.484) and (3.485), we obtain then, to first order in α,

$$\begin{aligned} e^{-\lambda} &= 1-2\frac{m}{r}, \\ e^{\nu} &= \left(1-2\frac{m}{r}\right)^{1+\frac{2}{3}\alpha} e^{\beta(1+\frac{2}{3}\alpha)}. \end{aligned} \tag{3.486}$$

The value of β could be determined by imposing the boundary condition that, in the limit in which the gravitational field is vanishing, the metric must reduce to the isotopic form (3.441). In this case one can easily obtain, to first order, $\beta = ln(1+\alpha)$ (modulo a suitable renormalization of the constant value of the light velocity in vacuum). Notice, however, that the requirement of spherical symmetry is not sufficient to determine univocally the choice of the time coordinate in the proper-time interval (3.481) and, in view of the general covariance of the theory, we are free to define a new coordinate $t' = f(t)$, where f is an arbitrary function of t only.

Using this freedom we can choose then the time coordinate so as to eliminate the exponential factor on the right-hand side of Eq. (3.486b) or, in other words, to put the integration constant $\beta = 0$. Therefore

$$e^{\nu}=\left(1-2\frac{m}{r}\right)^{1+\frac{2}{3}\alpha}. \tag{3.487}$$

In this way Gasperini obtains the static, spherically symmetric approximate solution of the isotopic equations (3.436), to first order in α,

$$\begin{aligned} ds^2 &= -\left(1 - 2\frac{m}{r}\right)^{1+\frac{2}{3}\alpha} dt^2 + \frac{dr^2}{1 - 2m/r} \\ &+ r^2(d\theta^2 + \sin^2\theta d\varphi^2). \end{aligned} \tag{3.488}$$

The usual Schwarzschild solution can be recovered in the limit $\alpha \to 0$, corresponding to an exterior motion.

Isotopic lifting (3.488) of the Schwarzschild metric is significant for a number of aspects. In fact, it opens up a problem, unexplored until now, regarding the implications for black holes caused by the isotopic lifting of gravitation.

More specifically, we are referring to the identification of the departures from the Schwarzschild metric and gravitational singularities in general caused by a bona-fide representation of the physical conditions of the interior problem with metrics less approximated than (3.441). All available studies are essentially based on the same theory for both the exterior and interior problem without any treatment of their physical differences.

Note also that the Lorentz-isotopic theory can provide the explicit form of the symmetry transformations, not only for the isotopically lifted metric (3.488), but also for the conventional Schwarzschild metric. Their explicit construction is another interesting open problem we recommend for consideration by interested researchers.

3.5.18 Some Overdue Fundamental Experiments

A primary purpose of this review is to recommend the conduction of truly fundamental experiments, that is, of experiments on fundamental physical laws, rather than tests on secondary aspects and, therefore, of secondary relevance. We shall review below a few basic tests which have been suggested for quite some time, but have remained, regrettably, ignored by the experimental community and are now substantially overdue.

As well known, a truly considerable number of experiments have been suggested to test Einstein's Special Relativity. Regrettably, we cannot possibly review them here. In fact, for brevity, we shall consider only the tests that are of fundamental character, as well as directly related to Lie-isotopic techniques.

The first basic tests of this work were reviewed in Section 3.5.3 and they can be expressed as follows:

FUNDAMENTAL TESTS I: Measure the prediction of gravitational theories that any electromagnetic field is a source of a gravitational field (Figure 13)

The experiments have been lingering in the literature on gravitation since the early stages of the theory. In 1974, Santilli [132] brought them back to the attention of the experimental community by recommending first the measure of the gravitational field which is expected to be produced by large magnets as available at several laboratories. This first test is well within current ranges of sensitivity of neutron interferometric techniques. Secondly, Santilli suggested the conduction of deeper tests to measure the contribution to the gravitational field caused by the *dynamical conditions* of charges and/or magnetic moments (see Figure 13 for a summary and ref. [132] for details). These additional tests, apparently, were not feasible in 1974 because of limitations on the sensitivity of gravity meters, on one side, versus limitations for reaching sufficiently high electric and/or magnetic fields and sufficiently high rotational conditions. Nevertheless, the experiments may well be within practical realization nowaday, owing to advances in superconductivity and other fields.

The fundamental nature of Tests I is manifest. For instance, the tests could well allow the resolution of the vexing problem of "unification" of the gravitational and electromagnetic fields originating in the charge structure of matter along Santilli's hypothesis of their "identification" (3.334), i.e.

$$M_{\mu\nu}^{\mathrm{Grav}} \approx \Theta_{\mu\nu}^{\mathrm{Elm}} , \tag{3.489}$$

In turn, such an identification would open the door to realistic possibilities of achieving a unification, not only of the electromagnetic and weak interactions (as permitted by the conventional Lie theory), but also of the strong and gravitational interactions, as conceivable under isotopic lifting of gauge theories (See Appendix A).

There is also little doubt that Fundamental Tests I are now grossly overdue.

The second group of experiments reviewed in this work consist of the following.

FUNDAMENTAL TESTS II: Measure the deformation/rotational-assymmetry/magnetic-moment-mutation which is expected for neutrons (or any other hadron) under sufficiently intense (e.g. nuclear) external fields (Figure 6).

The tests have been conducted by Rauch and his collaborators (see ref. [88] and quoted papers) via neutron interferometric techniques up to 1978, but regrettably halted since that year.

The latest available experiments tested the spinorial symmetry of neutrons via two complete spin flips while the (low energy) neutron beam is under the action of an external nuclear field (i.e. the spin flips occur while neutrons are under external nuclear forces). The best available measure are 715.87 ± 3.8 which, as such, do not include (within the limits of the experimental error) the 720 deg needed to establish the exact nature of the spinorial symmetry.

A preliminary, but full and direct representation of the above deviations from the exact $SU(2)$ symmetry has been reached by Santilli [27] via the *iso-Dirac's equation*, i.e., the isotopic generalization of the conventional Dirac's equation which is invariant under the Poincaré-isotopic group of Section 3.4.

Fundamental Tests II shall be considered in detail in a separate review on the "hadronic generalization of quantum mechanics" (Sect. 1.3), owing to their essential operator nature on Hilbert spaces. Here, we limit ourselves only to indicate the manifest plausibility of the violation of the conventional rotational symmetry in particle physics. In fact, perfectly rigid, spherical, charge distributions (3.1), i.e.

$$r^t \delta r = xx + yy + zz = 1 , \tag{3.490}$$

do not exist in Nature, but admit instead deformations, e.g., of the ellipsoidic type (3.3), i.e.

$$\begin{aligned} r^t g r &= x b_1^2 x + y b_2^2 y + z b_3^3 z = 1 , \\ & \quad b_k > 0, \quad k = 1, 2, 3 , \end{aligned} \tag{3.491}$$

with manifest breaking of the rotational symmetry. The deformation is measurable because it implies a (necessary) alteration of the magnetic moment which, in turn, is measurable via the test of the spinor 2π-symmetry of neutrons under external nuclear interactions.

It should be stressed that the tests have been fully within current experimental feasibility since quite some time, as well known.

The fundamental nature of Tests II is incontrovertible. After all, the rotational symmetry is at the foundation of quantum mechanics and all of particle physics. It is a truism to say that deviations from the rotational symmetry, when experimentally established, could stimulate a new scientific renaissance. In particular, the mutation of metric (3.490) into form (3.491) is a clear case of isotopy and, as such, it provides one of the most important applications of Santilli Lie-isotopic generalization of the group of rotations (§3.2). This implies, in particular, that the rotational symmetry remains *exact* at the isotopic level. Only its *conventional* realization is violated by deformations (3.491) (see Appendix C).

Fundamental Tests II are also grossly overdue. In fact, the only available tests are those by Rauch and collaborators [88]. In particular they show a violation of about 1% (outside statistical errors). Lacking the final experimental resolution of the issue one way or the other, the entire branch of particle physics dealing with the rotational symmetry is now in a state of "suspended animation".

The situation becomes unreassuring and acquires nonscientific (e.g., ethical) overtones if one notes that all experiments currently preferred in particle physics, besides costing substantially more than the relatively inexpensive Tests II, are of manifest, less, comparative relevance. These ethical aspects have been pointed out by Santilli [163], and are not reviewed here. We restrict ourselves only to indicate that, when fundamental tests remain ignored for protracted periods of time, scientists in good faith should expect the emergence of ethical issues.

But the fundamental tests primarily suggested in this review are the following.

> *FUNDAMENTAL TESTS III: Measure the local validity or invalidity of Einstein's Special and General Relativities within hadronic matter via the measure of the behaviour of the mean life of unstable hadrons at different energies (Figure 16).*

The test in this case are numerous and consist, more specifically, of the *measure of the behaviour of the mean life of (at least) pions and kaons at a sufficient number of different speeds to allow the verification or the disproof of the Einsteinian law (3.165),*

$$\tau = \tau_0 \gamma = \tau_0 (1 - v^2/c_0^2)^{-1/2} \,. \tag{3.492}$$

As reviewed in Section 3.4, possible deviations from law (3.492) should follow Santilli's isotopic law (3.279), i.e.,

$$\tau = \tau_0 \hat{\gamma} = \tau_0 \left(1 - \frac{v b^2 v}{c^2}\right)^{-1/2} , \tag{3.493}$$

which, as shown by Aringazin [119], unifies all known or otherwise conceivable models of "Lorentz noninvariance" reviewed in Section 3.4.3.

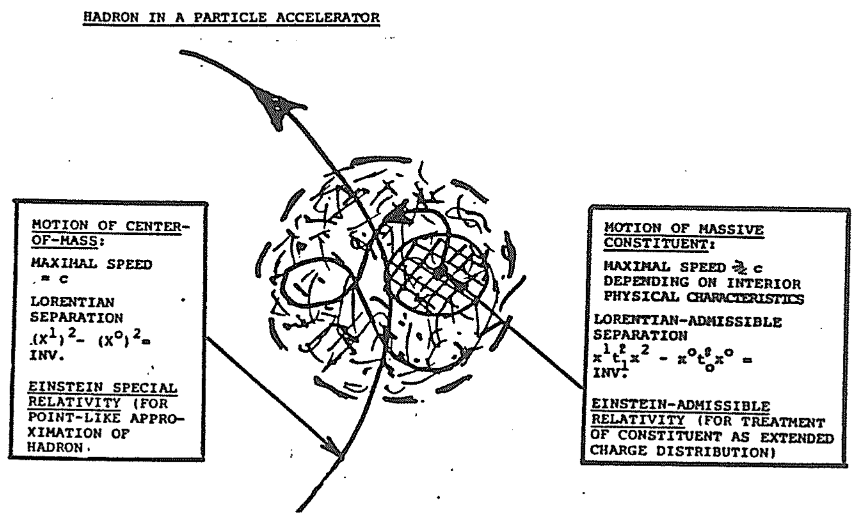


FIGURE 16. A reproduction of Figure 5.6, page 592 of monograph [16] summarizing the conceptual basis of the fundamental experimental tests recommended in this work: measure the behaviour of the mean life of unstable hadrons while moving at different energies in a particle accelerator. The center-of-mass motion strictly obeys Einstein's Special Relativity. Nevertheless, the interior dynamics is fundamentally noneinsteinian. This is due to the fact that the former dynamical evolution occurs in vacuum, while the latter deals with motion of extended wavepackets (the hadronic constituents) moving within a physical medium composed of other constituents, thus resulting in *nonlocal* forces under which Einstein's Special Relativity is well known to fail. The compatibility of the above two different dynamical conditions has been proved at all levels of description, that is, at the Galilean (§3.3), Relativistic (§3.4), Gravitational (§3.5) and operator levels (§1.3), and it is now an established fact. The only known way according to which deviations from Einsteinian laws in the interior dynamics manifest themselves in the exterior one is via departures from the Einsteinian law of time dilation. As a consequence, the proposed tests are the most fundamental

ones conceivable at this time, inasmuch as they probe the local Einsteinian or noneinstenian behaviour of the ultimate structure of matter. It is very regrettable that the tests, proposed in the literature for decades, have been ignored by experimentalists until now.

For instance, Nielsen and Picek [99] have reached modification (3.170) of the Minkowski metric in the interior of pions via the use of currently available phenomenological information

$$\begin{aligned} g &= (1-\frac{1}{3}\alpha, 1-\frac{1}{3}\alpha, 1-\frac{1}{3}\alpha, -(1+\alpha)) , \\ & \alpha = (-3.79 \pm 1.37) \times 10^{-3} , \end{aligned} \tag{3.494}$$

which, when plotted in law (3.493) yields the non-Einstenian behavior

$$\tau = \tau \left[1 - \frac{v(1-\frac{1}{3}\alpha)^2 v}{c_0^2(1+\alpha)^2}\right] . \tag{3.495}$$

Similarly, for the case of kaons, Nielsen and Picek have reached modification (3.170) of the Minkowski metric with

$$\alpha = (+0.54 \pm 0.17) \times 10^{-3} , \tag{3.496}$$

with corresponding non-Einstenian behavior (3.495).

Numerous, additional, essentially similar, quantitative predictions also exist in the literature as reviewed in Section 3.4.3.

Let us also recall alternative law (3.171) proposed by Nielsen and Picek, i.e.,

$$\tau = \tau_0 \gamma \left(1 + \frac{4\alpha\gamma^2}{3}\right) , \tag{3.497}$$

which is however reducible to Santilli's unified form (3.493) as shown by Aringazin [119].

Note not only the *different value* of the "Lorentz-asymmetry" parameter α but also its *different sign* in the transition from pions to kaons. This confirms, quite eloquently the need to conduct Tests III for at least pions and kaons.

It should be stressed again that isotopic metrics (3.494) with constant values provide only a first approximation of the geometry in the interior of hadrons. In fact, the isotopic metrics are expected to be *nonlinear* [101] in the velocity, in turn, as an approximation of the ultimate *nonlocal* [102] structure of hadrons (Fig. 16). Quantitatively too, metrics (3.494) are expected not to be representative. In fact, while these metrics deal with deviations of the order of 10^{-3}, Kim [98] predicted deviations of the order of 2.6% at 150 GeV and 14.3% at 400 GeV. For the best available predictions of violations, we refer the reader to ref.s [101,102].

In conclusion, by no means metrics (3.494) should be taken as an indication of the lack of feasibility of the Fundamental Tests III, because of their very low deviation from the Minkowski metric. In fact, metrics (3.494) merely provide *first approximations for low speeds*, which, as such, have only an indicational value. On the contrary, the feasibility of Fundamental Tests III should be considered as fully and completely within current range, because of the predictions of large numerical percentage of departures from law (3.493) that are readily attainable today in particle accelerators [101,102].

The reader should keep in mind the truly fundamental implications of modifications (3.493) of Einsteinian law (3.492), as represented by Santilli's Special Relativity (Sect. 3.4). For instance, the maximal speed of a physical, massive, particle (or causal signal) is *smaller* than c_0 for the interior of pions, but *bigger* than c_0 for the interior of the heavier kaons and, expectedly, of all remaining (still heavier) hadrons. In turn, the possibility for causal signals of surpassing the speed of light in vacuum, if experimentally established, would have truly deep implications throughout all of particle physics, by offering intriguing and still unexplored possibilities (e.g., the achievement of a true confinement of quarks with null probability of tunnel effects [44]). Finally, the experimental verification of deviations (3.493) would establish the need for a Lie-isotopic generalization of Einstein's Gravitation for the interior problem at the *operator/particle level.* In turn, this would have at least two-fold implications. First, the occurrence would be a rather natural, particle-image of the established classical violations of Einstein's Special Relativity in the interior dynamics, such as satellites during re-entry with a continuously decaying angular momentum. Secondly, the occurrence considered would finally remove the current, rather widespread belief that the classical violations of the Special Relativity are resolvable via the reduction of the classical object to its elementary particle constituents (see Sect. 3.5.3 for the lack of technical feasibility of such a belief).

The tests for unstable leptons, such as the muons, are recommended but positively not in lieu of the above tests for hadrons. In fact, the problem whether leptons are elementary or composite is still basically unsolved. If they are indeed elementary, then they are expected to obey law (3.492) exactly, thus leaving the issue under consideration here (local Einstenian character of *strong interactions*) fundamentally open. At the same time, if the tests are conducted for unstable leptons, and they show violation, this would be indirect experimental evidence of their composite structure.

It should be stressed here that Fundamental Tests III are quite simple and fully feasible nowadays. In fact, they require relatively low energies, and as such, they are realizable in all available particle accelerators throughout the world. Also, the tests are of very moderate costs, particularly when compared to the costs of the current search for heavy mesons and other contemporary particle experiments. Finally, Tests III require no theoretical elaboration of the results, trivially, because they have simply to measure a time at a given speed. As such, they are intrinsically model-independent (a feature rather rare in contemporary particle experiments).

The motivation for the conduction of the suggested tests are simply compelling, because of their number and diversification. They are at the very foundation of the Lie-isotopic theory, and, as such, have been reviewed throughout this work. We simply recall here:

1. The incontrovertible experimental evidence requiring a deep overlapping of the wavepackets of the constituents of unstable hadrons (Figure 1) with consequential nonlocal nature of the strong interactions. In turn, such a nature implies a necessary violation of Einstein's Special Relativity, as well known since quite some time;

2. All phenomenological calculations of the mean life conducted until now show clear violation and none of them recovers Einsteinian law (3.492). We are referring here to the predictions by Blockhintsev [96], Redei [97], Kim [98], Nielsen and Picek [99], Huerta-Quintanilla and Lucio [100], Aronson, Bock, Cheng and Fishbach [101], Cardone, Mignani and Santilli [102] and several others;

3. The incontrovertible experimental evidence of the violation of Einstein's Special Relativity in *classical* macroscopic dynamics of interior problems recalled earlier;

as well as other motivations.

The contributions made by the Lie-isotopic theory at the various levels considered (Newtonian, relativistic and gravitational) are numerous, such as:

a) The proof of the compatibility of deviations (3.493) for the interior problem, with the exact character of Einstein's Special Relativity for the center-of-mass motion of the unstable hadrons (see Figure 16) achieved, as now familiar, with the notion of closed nonhamiltonian systems;

b) The construction of genuine covering relativities at all levels of study which do not leave the "broken" context mathematically and physically undefined, but replace it with covering, explicitly computable symmetries unifying all available generalizations;

c) The clarification that, contrary to popular belief, the Lorentz symmetry remains exact under generalized law (3.493). As a result, all predictions of violations of ref.s [96] through [101] must be referred, specifically, to Einstein's Special Relativity and *not* to the Lorentz symmetry;

and numerous additional contributions reviewed in this work.

The fundamental experiments under consideration have already been recommended for decades, but regrettably, they have not been conducted until now.

For instance, the paper by Kim [98] originated as a preprint at SLAC back in 1977. The paper by Huerta-Quintanilla and Lucio [100] originated as a preprint at FERMILAB. Santilli [163] conducted a rather considerable effort in the period 1978-1981 at various laboratories in the USA and abroad (see ref. [163], Vol. I, Sect. X, Vol. II, Section XII and Vol. III, Sect. XXXIII) to recommend the conduction of Fundamental Tests III but this effort too resulted in no actual conduction of the tests.

In particular, Kim [98] concludes his analysis with the statement that Tests III are such to "*deserve a serious, unprocrastinable study*". Santilli states on p. 1977 of ref. [6] that

> "*Until the validity or invalidity of Einstein's Special Relativity for strong interactions has been experimentally resolved, all theoretical studies on hadrons and all experiments in strong interactions will remain of conjectural character*".

Note the conjectural character of the *experiments* on strong interactions in the absence of Tests III. In fact, Einstein's Special Relativity is a central component of the data elaborations of experiments on strong interactions. If deviations of type (3.493) do occur, a corresponding alteration of the data elaboration is evident, and equally evident is the alteration of the experimental results.

This occurrence is clearly illustrated by the fact that, in case deviations of type (3.493) do occur, this will imply an increase of the meanlife of unstable hadrons at rest as currently provided by the Particle Data Group [102]. In fact, these meanlifes are evidently not computed at rest, but at given different speeds, and then extrapolated at rest by using precisely Einstenian law (3.492). Any deviation from the latter law will then predictably imply a revision of all "experimental numbers" based in the (generally tacit) assumption of its exact validity. The same nonreassuring situation occurs for virtually all experimental in strong interactions, which illustrate Santilli's statement quoted above.

In short, *the entire, theoretical and experimental knowledge on strong interactions is kept in a state of "suspended animation" by the lack of Tests III, and this situation will persist until the tests are finally conducted.* The economical, let alone scientific implications for any additional deferral of the tests are then evident.

This situation is regrettable, not only for the experimental community, but also for the entire physics community, world-wide. The lack of conduction of the tests has essentially left the foundations of contemporary physics in a state of "limbo", with no resolution one way or the other, and with manifest implications beyond those of scientific ethics.

As well known, *physics is a discipline with an absolute standard of value: the experiments.* Lacking a direct experimental verification, physical theories remain conjectural no matter how old, and no matter how important they are.

Experiments themselves have their own standard of value: the more fundamental the test, the more relevant is its conduction as compared to lesser fundamental tests. It is in the tradition of physics to measure and then measure again physical quantities. And in fact, the mean life of unstable hadrons has been measured a truly considerable number of times, and additional tests are scheduled for a refinement of available data (see, e.g., ref. [164] and quoted papers). But then, by comparison, the conduction of new fundamental tests of these mean lifes at different speeds is manifestly more important than the refinement of already established data. By no means are we against new measurements of the mean life of unstable hadrons at rest, because *all* feasible experiments must be supported. We are merely stressing a known absolute standard of value among various experiments.

The reason for priority on fundamental tests are evident and well known. Refinements of available data can at best imply refinements of available theories. But new, fundamental tests of the type recommended here have, by comparison, potentially far greater implications, no matter whether the results are in favor or against Einstein's Special Relativity.

In the final analysis, ref.s [96] through [102], by no means, recommend the test of the violation of Einstein's Special Relativity. On the contrary, they simply require its verification in new physical areas, such as in the interior of hadrons, in the tradition of physics: via experiments, rather than conjectural-theoretical work.

When the above scientific scene is put all together, including:

- the manifestly fundamental character of the experiments;
- the clear plausibility of the violations;
- the rigorous mathematical structure of the proposed covering theories;
- the clear feasibility of the experiments with currently available equipments and technology;
- their moderate costs when compared to other, lesser relevant tests;
- the truly historical implications of the results, whether in favor or against old doctrines;

and many additional motivations, the conduction of Fundamental Tests III becomes simply compelling.

A primary hope of this review is that experimentalists will understand this scenario, and finally conduct the much overdue tests.

APPENDIX A:
LIE-ISOTOPIC LIFTING OF GAUGE THEORIES

Gauge theories, within the context of the conventional formulation of Lie's theory, have been instrumental for an important physical achievement: the unification of electromagnetic and weak interactions (see, e.g., ref. [165] and quoted papers).

The Lie-isotopic covering of the above theories appears to offer realistic possibilities of advances, in due time, toward a much broader unification which is inclusive of the strong as well as the gravitational interactions. This possibility is a central theme of a subsequent possible review on Santilli's "hadronic generalization of quantum mechanics".

At this point we merely limit ourselves to mention that these advances toward a "true grand unification" are made conceivable by the following elements reviewed in this work: the novel representational capabilities of Lie-isotopic theories offered by the isounit $\hat{I} = T^{-1}$, Eq. (1.35); the additional degrees of freedom offered by the isotopic element G of the underlying Hilbert space, Eq. (1.49); and, last but not least, the hypothesis of "identification" of the gravitational field with the electromagnetic field of matter constituents, Eq. (3.334).

In this appendix we shall review the pioneering works of 1983 by M. Gasperini [85],[86] who formulated, for the first time, the Lie-isotopic generalization of gauge theories. We shall also review important advances achieved subsequently on the subject by M. Nishioka [161,162,166,167] and G. Karayannis and A. Jannussis [168–171]. The analysis shall remain essentially classical as in the rest of this review. All major operator aspects are deferred to the possible subsequent review of "hadronic mechanics." Additional important research by Nishioka, Karayannis, Jannussis *et al.* on isotopic gauge theories will be reviewed in Appendix C following the introduction of isotopic field equations.

By following Gasperini's original presentation [85] as close as possible, we shall first review, for notational convenience, the notion of compact gauge group, present its Lie-isotopic covering, identify some of the physical implications and then conclude with a review of additional advances.

Suppose we have a field theory invariant under some compact Lie group G of global transformations, which can be represented via the transformations

$$\psi' = U\psi \,, \tag{A.1}$$

where

$$U = e^{-i\theta^k X_k} \tag{A.2}$$

θ^k is a set of constant real parameters, X_k is a matrix representation of the generators of the group satisfying the rules

$$[X_i, X_j] = i\ c_{ij}{}^k X_k\ , \tag{A.3}$$

and $c_{ij}{}^k$ are the structure constants of the Lie algebra of G. The infinitesimal form of the transformation (A.1) is

$$\delta\psi = -i\varepsilon^k X_k \psi\ , \tag{A.4}$$

where ε^k are the infinitesimal parameters corresponding to θ^k.

Notice that the representation matrices of the transformations are unitary

$$U^+ U = I, \qquad [U^+, U] = 0\ , \tag{A.5}$$

and the basic invariant of the theory is $\psi^+\psi = \psi'^+\psi'$.

If the global symmetry is enlarged to a local symmetry, i.e. if we consider transformations with space-time dependent parameters, $\theta^k = \theta^k(x)$, then the theory is no more invariant, in general. The invariance is restored if the partial derivative of the matter field, $\delta_\mu\psi$, is replaced with the covariant derivative

$$D_\mu\psi = (\partial_\mu - igA^k_\mu X_k)I\psi\ , \tag{A.6}$$

where g is the group coupling constant, and the gauge potential $A_\mu = A^k_\mu X_k$ is a vector field with values in the Lie algebra of G. Its transformation properties are fixed by imposing that $D_\mu\psi$ transforms like ψ, that is

$$D'_\mu U\psi = UD_\mu\psi\ . \tag{A.7}$$

We obtain then

$$A'^i_\mu X_i = UA^i_\mu X_i U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}\ . \tag{A.8}$$

Performing an infinitesimal transformation, i.e. putting

$$U \simeq I - i\varepsilon^k X_k \quad , \quad U^{-1} \simeq I + i\varepsilon^k X_k\ , \tag{A.9}$$

and using the commutation relations (A.3), we can obtain, from Eq. (A.8), the infinitesimal gauge transformations for the potential vector

$$\delta A^i_\mu = -\frac{1}{g}\partial_\mu\varepsilon^i + c_{jk}{}^i\varepsilon^j A^k_\mu\ . \tag{A.10}$$

Finally, we must complete the field theory by adding a dynamical term for the gauge potential. To this aim, one defines the Yang-Mills field strengths $F_{\mu\nu}$ as follows

$$F_{\mu\nu}\psi = F^i_{\mu\nu}X_i\psi = -\frac{1}{ig}[D_\mu, D_\nu]\psi \ , \tag{A.11}$$

that is, using Eq.s (A.6) and (A.3),

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\ c_{jk}{}^i A^j_\mu A^k_\nu \ . \tag{A.12}$$

Its transformation law can be obtained from Eq.s (A.11) and (A.7)

$$F'_{\mu\nu} = U F_{\mu\nu} U^{-1} \ , \tag{A.13}$$

and then we can construct the following gauge-invariant kinetic term

$$Tr(F_{\mu\nu}F^{\mu\nu}) = Tr(F'_{\mu\nu}F'^{\mu\nu}) \propto F^i_{\mu\nu}F^{i\mu\nu} \ . \tag{A.14}$$

These few basic notions on classical gauge theory are sufficient for the purpose of this review. Further details can be found by the interested reader in ref. [165].

At this point, Gasperini [85] introduces Santilli's Lie-isotopic generalization of the conventional formulation of Lie's theory (Sect. 2). For notational convenience we review the isotopy for the case at hand.

Given an invertible and hermitian operator T, the enveloping Lie algebra of a theory with associative product AB and unit I is generalized introducing the isotopic (associative) product $A * B = A\ T\ B$ and a new unity $\hat{I} = T^{-1}$, such that $A \star \hat{I} = \hat{I} \star A = A$.

As a consequence, the usual definition of hermitian conjugate, A^+, and inverse, A^{-1}, of an operator A must be replaced by the isotopic generalizations, the T-hermitian conjugate (141), i.e.,

$$A^{\hat{+}} = T^+ A^+ \hat{I} \ , \tag{A.15}$$

and the T-inverse

$$A^{-\hat{1}} = \hat{I} A^{-1} \hat{I} \ . \tag{A.16}$$

Furthermore, the T-isotope, exp A, of an exponential operator exp A, is given by Eq. (2.138), i.e.

$$\hat{e}^A = \hat{I}\ e^{TA} = e^{AT}\hat{I} \ . \tag{A.17}$$

The Lie-isotopic lifting $\hat{G}$ of the compact Lie group G is represented then by transformations (2.134), i.e.,

$$\psi' = \hat{U} \star \psi \,, \tag{A.18}$$

where

$$\hat{U} = \hat{I}\; e^{-i\theta^k \star X_k} = e^{-iX_k \star \theta^k}\,\hat{I}\,. \tag{A.19}$$

The explicit computation of its isotopic hermitian conjugate

$$\hat{U}^{\hat{+}} = T^{+} e^{i\theta^k \star X_k} T^{-1} = e^{i\theta^k \star X_k}\,\hat{I}\,, \tag{A.20}$$

and of its inverse

$$\hat{U}^{-\hat{1}} = T^{-1}T\; e^{i\theta^k \star X_k} T^{-1} = e^{i\theta^k \star X_k}\,\hat{I}\,, \tag{A.21}$$

shows that $\hat{U}$ is a T-unitary operator, since

$$\hat{U}^{\hat{+}} = \hat{U}^{-\hat{1}}\,. \tag{A.22}$$

It is important to mention that $\hat{G}$ is locally isomorphic to G if T is a positive- or negative-definite isotopic element (this result is due to Santilli's Theorem 2.9).

Another important point is that the isotopic condition of hermiticity coincides with the usual one, when the Hilbert space is generalized introducing the isotopic inner product $(a, b)^{\star} = (a, Tb)\hat{I}$ of Eq. (1.50).

Notice also that the infinitesimal form of the Lie-isotopic transformation (A.19) is given by

$$\hat{U} \simeq \hat{I} - i\varepsilon^k X_k \,, \tag{A.23}$$

and

$$\delta\psi = -iX_k \star \varepsilon^k \psi \,, \tag{A.24}$$

respectively.

At this point, Gasperini [85] introduces his Lie-isotopic generalization of a gauge theory, by following as close as possible the structure of a conventional theory.

Suppose we have a field theory invariant under the global isotopic transformations

$$\psi' = \hat{U} \star \psi \,, \tag{A.25}$$

where $\hat{U}$ is the representation of a continuous, Lie-isotopic group $\hat{G}$, and is given by Eq. (A.19). As $\hat{U}$ is a T-unitary operator,

$$\hat{U}^{\hat{+}} \star \hat{U} = \hat{I} = \hat{U} \star \hat{U}^{\hat{+}} , \tag{A.26}$$

the basic invariant of the theory is then structure (2.153), i.e.,

$$\psi^{+} \star \psi = \psi'^{+} \star \psi' = \psi^{+} \star \hat{U}^{\hat{+}} \star \hat{U} \star \psi . \tag{A.27}$$

In order to preserve invariance also under local isotopic transformations $\hat{U} = \hat{U}(x)$ (i.e. $\theta = \theta(x)$ and/or $T = T(x)$), we introduce, in analogy with ordinary gauge theory, the isotopic covariant derivative

$$\hat{D}_\mu = (\partial_\mu - igA^k_\mu \star X_k)\hat{I} , \tag{A.28}$$

and we impose the following transformation rules

$$\hat{D}'_\mu = \hat{U} \star \hat{D}_\mu \star \hat{U}^{-\hat{1}} , \tag{A.29}$$

that is

$$\hat{D}'_\mu \star \hat{U} \star \psi = \hat{U} \star \hat{D}_\mu \star \psi . \tag{A.30}$$

By using the factorization $\hat{D} = \overset{\star}{D}_\mu \hat{I}$ and $\hat{U} = \overset{\star}{U} \hat{I}$, where

$$\overset{\star}{D}_\mu = \partial_\mu - igA^k_\mu \star X_k , \tag{A.31}$$

$$\overset{\star}{U} = e^{-ie^k \star X_k} , \tag{A.32}$$

we obtain, from Eq. (A.30),

$$A'^i_\mu \star X_i = \overset{\star}{U} A^i_\mu \star X_i \overset{\star}{U}^{-1} - \frac{i}{g}(\partial_\mu \overset{\star}{U}) \overset{\star}{U}^{-1} , \tag{A.33}$$

which is *the isotopic lifting of the gauge transformations* (A.8). In order to obtain the corresponding infinitesimal transformations, we develop $\overset{\star}{U}$ as follows

$$\begin{aligned} \overset{\star}{U} &\simeq I - i\varepsilon^k \star X_k , \\ \overset{\star}{U}^{-1} &\simeq I + i\varepsilon^k \star X_k , \end{aligned} \tag{A.34}$$

and then we get, to the first order in ε,

$$\delta A^i_\mu \star X_i = \frac{1}{g}\partial_\mu(\varepsilon^i \star X_i) + i[A^j_\mu \star X_j, \varepsilon^k \star X_k] , \tag{A.35}$$

which can be written

$$A^i_\mu T X_i = -\frac{1}{g}(\partial_\mu \varepsilon^i T) X_i + i A^j_\mu \varepsilon^k T [X_j \hat{,} X_k] , \tag{A.36}$$

where we have introduced the isotopic commutators [1], i.e.

$$[X_i \hat{,} X_j] = X_i T X_j - X_j T X_i . \tag{A.37}$$

Finally, Gasperini [*loc.cit*] defines the *isotopic Yang-Mills field strengths* $\hat{F}_{\mu\nu}$ for the gauge potential as follows:

$$\begin{aligned} \hat{F}_{\mu\nu} \star \psi &= \frac{-1}{ig}[\hat{D}_\mu \hat{,} \hat{D}_\nu]\psi = \\ &= -\frac{1}{ig}(\hat{D}_\mu \star \hat{D}_\nu - \hat{D}_\nu \star \hat{D}_\mu) \star \psi , \end{aligned} \tag{A.38}$$

which transforms covariantly under an isotopic gauge transformation. In fact, from Eq.s (A.29), (A.38) he gets

$$\hat{F}'_{\mu\nu} = \hat{U} \star \hat{F}_{\mu\nu} \star \hat{U}^{\hat{-1}} . \tag{A.39}$$

Its explicit expression can be easily obtained substituting Eq. (A.28) into the definition (A.38). The result is

$$\begin{aligned} F^i_{\mu\nu} \star X_i &= (\partial_\mu A^i_\nu - \partial_\nu A^i_\mu) \star X_i + A^i_\alpha(\partial^\alpha_\nu \partial_\mu T - \\ &-\delta^\alpha_\mu \partial_\nu T) X_i - ig A^j_\mu A^k_\nu T [X_j \hat{,} X_k] . \end{aligned} \tag{A.40}$$

Equations (A.28),(A.36) and (A.40) describe the main aspects of Gasperini's *Lie-isotopic gauge theory.* It must be stressed that the isotopic generalization is simple but not trivial, as one can see. For example in Eq. (A.40) the gauge field is radically modified by the coupling to the isotopic element T.

At this point, Gasperini [*loc.cit*] passes to a preliminary physical interpretation of the results.

For this purpose, some additional information on the element T is needed. Assume the simplifying hypothesis that T is in the center of the algebra of the original Lie group G, i.e.

$$[X_i, T] = 0 , \tag{A.41}$$

since, as stressed by Santilli [1],[26], this condition is verified in several cases of physical interest.

In this case, using the commutation relations (A.3), the basic equations of the Lie-isotopic gauge theory can be rewritten as follows:

$$\hat{D}_\mu \star \psi = (\partial_\mu - igTA^k_\mu X_k)\psi \,, \tag{A.42}$$

$$\delta A^i_\mu = -\frac{T^{-1}}{g}\partial_\mu(\varepsilon^i T) + c_{jk}{}^i(\varepsilon^j T)A^k_\mu \,, \tag{A.43}$$

$$F^i_{\mu\nu} = \nabla_\mu A^i_\nu - \nabla_\nu A^i_\mu + gTc_{jk}{}^i A^j_\mu A^k_\nu \,, \tag{A.44}$$

where

$$\nabla_\mu A^i{}_\nu = \partial_\mu A^i{}_\nu - A^i{}_\alpha \Gamma_{\mu\nu}{}^\alpha \,, \tag{A.45}$$

$$\Gamma_{\mu\nu}{}^\alpha = \frac{1}{2}(\partial^\alpha{}_\mu \partial_\nu T - \delta^\alpha{}_\nu \delta_\mu T)T^{-1} \,, \tag{A.46}$$

Comparing the above isotopic equations with the corresponding equations of the conventional theory, Gasperini *interprets the isotopic theory as a gauge theory for local transformations with infinitesimal parameter* $\varepsilon'^i = \varepsilon^i T$, *and with an effective coupling constant given by*

$$g' = g\,T \,. \tag{A.47}$$

Since the isotopic element does depend, in general, on the spacetime coordinates x, the linear momentum p, the energy E, and so on, then we have a *gauge theory with a variable coupling* g'

$$g' = g'(x, p, E, ...) \,. \tag{A.48}$$

This offers rather interesting possibilities which could be connected with the phenomenon of the so called "running coupling constants" (i.e. couplings evolving as a function of the energy scale), which takes place in the framework of the grand-unified theories [165].

Furthermore, *the isotopic field strengths of Eq. (A.44) can be interpreted as the gauge field for a potential* $A^i{}_\mu$ *coupled to the geometry of an effective Riemann-Cartan space (§3.5) equipped with the antisymmetric connection* $\Gamma_{\mu\nu}{}^\alpha = -\Gamma_{\mu\nu}{}^\alpha$ *of Eq. (A.46).*

It should be stressed that the coupling (A.45) of the gauge field to the geometry is the usual "minimal coupling" obtained by replacing partial derivatives with the geometrical covariant ones. In a Riemann-Cartan space such a coupling is usually forbidden, as is well known [153] because it destroys the gauge invariance of the theory, and only more indirect interactions, such as a "semi-minimal" coupling, are allowed between torsion and the gauge potential (§3.5).

It is then remarkable that the minimal coupling of Eq. (A.45) is compatible with the invariance under isotopic gauge transformations. But even more remarkable is the fact that *the coupling of the gauge field to the geometry of a curved manifold is not only allowed, but also necessary when the isotopic lifting of a gauge theory is performed.* This may suggest, as stressed by Santilli [16], that the Lie-isotopic theory represents a promising clue towards a satisfactory quantum mechanical formulation of gravitation.

Another point worth noticing is that, by putting

$$T = I\ f^{-1}(x)\ , \tag{A.49}$$

where $f(x)$ is a scalar function, Eq.s (A.42-A.46) reduce to the same equations proposed by Hojman *et al.* [172] for the case of an abelian Lie group, and generalized by Mukku and Sayed [173] to the non-Abelian case. The latter theories, therefore, are only particular cases of the Lie-isotopic lifting of a gauge theory.

In papers [172,173], however, the modification of the gauge structure, and the explicit form of the torsion tensor (i.e. of the antisymmetric part of the connection), are introduced "ad hoc", with the only justification of allowing a gauge invariant coupling between torsion and the gauge potential. In Gasperini's theory, on the contrary, the modification of the theory, and the necessity of introducing a connection with a nonzero antisymmetric part, given in Eq. (A.46), are well justified as the consequences of an underlying isotopic algebraic structure.

Notice that by putting $T = 1$, the Lie-isotopic gauge theory reduces to the usual gauge theory. As a result, Gasperini's isotopic gauge theory is a bona-fide covering of the conventional theory, in the same way as Santilli's Relativities are a covering of the conventional ones (Sect. 3).

Gasperini then concludes paper [85] with the following words.

> *"Perhaps the most intriguing dream of contemporary physics is to describe all interactions with a unified theory. Electro-weak and strong forces have been put together into grand-unified theories [165], but it seems likely that the gravitational interaction can be included only by gauging a graded Lie group and using a graded algebra (extended supergravity theories [173]). The results, however, are not fully satisfactory up to now, as the theory*

is unable to contain the totality of existing particles, even in its maximal extension ($N = 8$). As stressed by Santilli [1], the graded Lie theory is only a particular case of the Lie-admissible theory. Therefore it is tempting to speculate that a realistic unified theory, comprehensive of all particles and forces of nature, will be reached only on the ground of Lie-admissible generalization of supersymmetry and extended supergravity."

Note the referral, specifically to the broader Lie-admissible generalization of the Lie-isotopic gauge theory, evidently for a structurally higher level of treatment.

* * * *

We now review some of the research by M. Nishioka, beginning with paper [166]. As shown in the preceding review, in his formulation of the isotopic gauge theory, Gasperini essentially selects an element T which is in the center of the algebra of the original Lie group G,

$$[X_i, T] = 0. \tag{A.50}$$

Under this hypothesis, Gasperini obtains the basic equations of the Lie-isotopic gauge theory which are reviewed here for convenience

$$\begin{aligned} \hat{D}_\mu \Psi &= (\partial_\mu - igTA^k_\mu X_k)\Psi \ , \\ \delta A_\mu &= -\frac{1}{gT}\partial_\mu(\varepsilon^i T) + c_{jk}{}^i(\varepsilon^j T)A^k_\mu \ , \\ F^i_{\mu\nu} &= \nabla_\mu A^i_\nu - \nabla_\nu A^i_\mu + gTc_{jk}{}^i A^j_\mu A^k_\nu \ , \end{aligned} \tag{A.51}$$

where

$$\begin{aligned} \nabla_\mu A^i_\nu &= \partial_\mu A^i_\nu - \Gamma_{\mu\nu}{}^\alpha A^i_\alpha \ , \\ \Gamma_{\mu\nu}{}^\alpha &= \frac{1}{2}(\delta^\alpha_\mu \partial_\nu T - \delta^\alpha_\mu \partial_\mu T)T^{-1}, \end{aligned} \tag{A.52}$$

$\hat{D}_\mu$ are the isotopic covariant derivatives, A^i_μ are the gauge potentials, and $F^i_{\mu\nu}$ are the field strengths. In the space-time with symmetric connections $\Gamma'_{\mu\nu}{}^\alpha$ consider isotopic covariant derivatives (A.51a) and the covariant derivatives (A.52a) with antisymmetric connection $\Gamma_{\mu\nu}{}^\alpha$. Then, the covariant derivatives are generalized to

$$\begin{aligned} \hat{D}_\mu * \lambda^\nu &= (\partial_\mu - \mathrm{igTA}^k_\mu X_k)\lambda^\nu + \Gamma'_{\mu\sigma}{}^\nu \lambda^\sigma, \\ V_\mu A^i_\nu &= \partial_\mu A^i_\nu - L_{\mu\nu}{}^\alpha A^i_\alpha, \end{aligned} \tag{A.53}$$

where λ^μ are the components of an arbitrary vector, and $L_{\mu\nu}{}^\alpha$ are given by

$$L_{\mu\nu}{}^\alpha = \Gamma_{\mu\nu}{}^\alpha + \Gamma'_{\mu\nu}{}^\alpha. \qquad \text{(A.54)}$$

The generalized isotopic covariant derivatives (A.53a) are equivalent to the covariant derivatives appearing in a gauge model of gravitation except for the variable coupling constant gT.

Following the well known parallelism of vectors defined by Levi-Civita in a Riemannian manifold, Eisenhart [174] gave a definition of parallelism of vectors in a general connected manifold given by

$$\lambda^\rho \left(\frac{d\lambda^\sigma}{dt} + L_{\mu\nu}{}^\sigma \lambda^\mu \frac{dx^\nu}{dt}\right) - \lambda^\sigma \left(\frac{d\lambda^\rho}{dt} + L_{\mu\nu}{}^\rho \lambda^\mu \frac{dx^\nu}{dt}\right) = 0\,, \qquad \text{(A.55)}$$

where λ^σ are the components of a vector on a curve which is the locus of points for which the coordinates x^ν are functions of a parameter t. The curves whose tangents are parallel with respect to the curves are called the *paths of the manifold.* The equations of these curves are

$$\frac{dx^\nu}{dt}\left(\frac{d^2x^\mu}{dt^2} + \Gamma'_{\sigma\rho}{}^\mu \frac{dx^\sigma}{dt}\frac{dx^\rho}{dt}\right) - \frac{dx^\mu}{dt}\left(\frac{d^2x^\nu}{dt^2} + \Gamma'_{\sigma\rho}{}^\nu \frac{dx^\sigma}{dt}\frac{dx^\rho}{dt}\right) = 0. \quad \text{(A.56)}$$

From (A.56) it is clear that all connected manifolds for which $\Gamma'_{\mu\nu}{}^\sigma$ are the same but $\Gamma_{\mu\nu}{}^\sigma$ are arbitrary, have the same paths.

The changes of connection which preserve parallelism were also investigated by Eisenhart. Nishioka [166] makes use of these results. Let $L_{\mu\nu}{}^\sigma$ and $\bar{L}_{\mu\nu}{}^\sigma$ be the coefficients of two different connections, under the condition that parallel directions along every curve in the space-time are the same for the two connections,

$$\bar{L}_{\mu\nu}{}^\sigma = L_{\mu\nu}{}^\sigma + 2\delta^\sigma_\mu \phi_\nu\,, \qquad \text{(A.57)}$$

where ϕ_ν is an arbitrary covariant vector.

If we denote the symmetric and antisymmetric parts of $\bar{L}_{\mu\nu}{}^\sigma$ by $\bar{\Gamma}'_{\mu\nu}{}^\sigma$ and $\Omega_{\mu\nu}{}^\sigma$ respectively, from (A.54) and (A.57) they are given by

$$\begin{aligned}\bar{L}_{\mu\nu}{}^\sigma &= L_{\mu\nu}{}^\sigma + 2\delta^\sigma_\mu \phi_\nu + \delta^\sigma_\nu \phi_\mu, \\ \Omega_{\mu\nu}{}^\sigma &= \Gamma_{\mu\nu}{}^\sigma + \delta^\sigma_\mu \phi_\nu - \delta^\sigma_\nu \phi_\mu.\end{aligned} \qquad \text{(A.58)}$$

From (A.52b) and (A.58) Nishioka concludes that $\Omega_{\mu\nu}{}^\sigma$ vanishes if the following relations hold

$$\partial_\mu T = -2\phi_\mu T. \qquad \text{(A.59)}$$

In this case, from (A.58a) and (A.58b) $\bar{\Gamma}'_{\mu\nu}{}^{\sigma}$ becomes

$$\bar{\Gamma}'_{\mu\nu}{}^{\sigma} = \Gamma'_{\mu\nu}{}^{\sigma} - \frac{1}{2}(\delta^{\sigma}_{\mu}\partial_{\nu}T + \delta^{\sigma}_{\nu}\partial_{\mu}T)T^{-1}, \tag{A.60}$$

and $\bar{L}'_{\mu\nu}{}^{\sigma}$ is symmetric.

If T is a function, ϕ_{μ} becomes the gradient

$$\phi_{\mu} = -\partial_{\mu} ln\sqrt{T}, \tag{A.61}$$

where we assume T is positive definite. In this case we have

$$\bar{L}^{\sigma}_{\mu\nu\rho} = L^{\sigma}_{\mu\nu\rho}, \tag{A.62}$$

because, in general, from (A.57)

$$\bar{L}^{\sigma}_{\mu\nu\rho} = L^{\sigma}_{\mu\nu\rho} + 2\delta^{\sigma}_{\mu}\left(\frac{\partial\phi_{\rho}}{\partial x^{\nu}} - \frac{\partial\phi_{\nu}}{\partial x^{\rho}}\right), \tag{A.63}$$

where $L^{\sigma}_{\mu\nu\rho}$ are the components of the curvature tensor for $L_{\mu\nu}{}^{\sigma}$.

From the above analysis, Nishioka [166] concludes that, as far as the preservation of parallelism is concerned, the symmetric connection in space-time plus the antisymmetric connection induced by the Lie-isotopic lifting become equivalent to the symmetric connection $\bar{L}'_{\mu\nu}{}^{\sigma}$ provided that (A.59) or (A.61) hold.

We now pass to the review of Nishioka's paper [162]. Santilli (Sect. 3.4) has shown that, in the framework of a Lie-isotopic theory, the conventional Lorentz symmetry should be replaced with a generalized Lorentz-isotopic symmetry whose transformations preserve a corresponding Minkowski-isotopic metric describing a generally inhomogeneous and anisotropic physical medium. Along these lines Gasperini (Sect. 3.5) has formulated a corresponding Lie-isotopic theory of gravity, i.e. a generalized gravitational theory based on an underlying Lie-isotopic algebra and has moreover suggested the formulation of a Lie-admissible theory of gravity.

In note [162] Nishioka takes a slightly different position from the above, stressing the isotopic generalization of the associative product and neglecting the isotopic lifting of other concepts or of other entities. Along this line the author gives some connections between the Lie-isotopic lifting of the space-time (Riemannian manifold) and the Lyra or Weyl manifold.

Following the parallelism of vectors defined by Levi-Civita in a Riemannian manifold, Eisenhart [174] gave a definition (A.55) of parallelism of vectors in a general connected manifold. Changes of connection which preserve parallelism were investigated also by Eisenhart. Let $L_{\mu\nu}{}^{\sigma}$ and $\bar{L}_{\mu\nu}{}^{\sigma}$ be the coefficients of two different connections. We impose the condition that parallel directions along every curve in the space-time are the same for the two connections. The condition is given by

$$\bar{L}_{\mu\nu}{}^{\sigma} = L_{\mu\nu}{}^{\sigma} + \frac{1}{2}\delta^{\sigma}_{\mu}A_{\nu}, \tag{A.64}$$

where A_{ν} is an arbitrary covariant vector. Notice that, if $L_{\mu\nu}{}^{\sigma}$ are symmetric with respect to μ and ν, then $\bar{L}_{\mu\nu}{}^{\sigma}$ are asymmetric. Notice also the introduction of an arbitrary vector A_{ν} which plays an important role later.

As usual, Nishioka assumes that in the space-times the length of the displacement vector $\xi^{\mu} = dx^{\mu}$ between two points $P(x^{\mu})$ and $P'(x^{\mu} + dx^{\mu})$ is defined by the invariant quantity

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}, \tag{A.65}$$

where $g_{\mu\nu}$ is the metric (symmetric) tensor of second rank. Nishioka's isotopic lifting of the space-time begins by introducing the isotopic associative product

$$\overline{ds^2} = g_{\mu\nu} * dx^{\mu} * dx^{\nu}, \tag{A.66}$$

where the symbol $*$ defines Santilli's product $A * B \equiv A\phi B$, and ϕ is a positive definite scalar function.

In the usual space-times the parallel transfer of a vector ξ^{μ} is given by

$$\delta\xi^{\mu} = -\Gamma_{\nu\sigma}{}^{\mu}\xi^{\nu}dx^{\sigma}, \tag{A.67}$$

where $\Gamma_{\nu\sigma}{}^{\mu}$ are the Christoffel symbols of the second kind. For the isotopic lifting of (A.67) Nishioka assumes

$$\overline{\delta\xi^{\mu}} = -L_{\nu\sigma}{}^{\mu} * \xi^{\nu} * dx^{\sigma}, \tag{A.68}$$

where $L_{\nu\sigma}{}^{\mu}$ are the coefficients of the connection in a general connected manifold.

For the parallel transfer of length, Nishioka assumes as in the Riemannian manifold that it is integrable, that is,

$$\overline{\delta(g_{\mu\nu} * \xi^{\mu} * \xi^{\nu})} = 0. \tag{A.69}$$

The above condition can be considered as the isotopic lifting of

$$\delta(g_{\mu\nu}\xi^{\mu}\xi^{\nu}) = 0. \tag{A.70}$$

From (A.68) and (A.69) Nishioka obtains the representation of $\bar{L}_{\nu\sigma}{}^{\mu}$

$$\bar{L}_{\mu\nu}{}^{\sigma} = \phi^{-2}\Gamma_{\mu\nu}{}^{\sigma} + \phi^{-1}\frac{1}{2}(\delta^{\sigma}_{\mu}A_{\nu} + \delta^{\sigma}_{\nu}A_{\mu} - g_{\mu\nu}A^{\sigma}), \qquad (A.71)$$

where $\bar{L}_{\mu\nu}{}^{\sigma}$ have been found to be symmetric with respect μ and ν.

By setting

$$\phi^{-1}A_{\mu} \equiv B_{\mu}, \qquad \phi^2 = \psi, \qquad (A.72)$$

$\bar{L}_{\mu\nu}{}^{\sigma}$ are found to have the same form as the coefficients of a connection in a manifold suggested by Lyra in 1951 [175] as a modification of the Weyl manifold, which had a defect of nonintegrability of length transfer. In Lyra's geometry ψ is called a gauge function and B_{μ} is the electromagnetic field. Although one can obtain the coefficients of connection in Weyl's geometry provided that $\phi = 1$, this is uninteresting, because it is a very special case of Lie-isotopic liftings. If one uses a new unity $\bar{I} = \phi^{-1}$, which is an important concept in Lie-isotopic theory, the coefficients of the connection in Lyra manifold $L^{(r)\sigma}_{\mu\nu}$ can be written as

$$L^{(r)\sigma}_{\mu\nu} = \bar{I}L_{\mu\nu}{}^{\sigma}, \qquad (A.73)$$

where ϕ is a gauge function and A_{μ} is an electromagnetic field. In this way, Nishioka [162] identifies a remarkable connection between the isounit $\hat{I}$, the gauge potential A_{μ} and the electromagnetic potential B_{μ}.

We now pass to the review of Nishioka's papers [167], which essentially consists of the introduction of the gauge field via the Lie-isotopic lifting of the Hilbert space (Sect 1.3) where the commutator between the isotopic element and the generators of the Lie algebra does not vanish.

Let T be an operator that is nonsingular and Hermitian. Following [38] we shall introduce the isotopic lifting $\hat{\mathcal{H}}$ of the Hilbert space $\mathcal{H}$ of quantum mechanics. Let vectors be $\hat{\varphi}, \hat{\psi}, \ldots$. The inner product will be defined via Eq. (1.49), i.e.

$$(\hat{\varphi}\hat{|}\hat{\psi}) = (\hat{\varphi}|T|\hat{\psi}) = (\hat{\varphi}|T\hat{\psi})\varepsilon C \ , \qquad (A.74)$$

and normalization

$$(\hat{\varphi}\hat{|}\hat{\varphi}) = 1, \qquad (A.75)$$

where all symbols without the upper hat denote the corresponding quantity $\mathcal{H}$.

Following ref. [1], Nishioka defines the Lie-isotopic lifting of the enveloping associative algebra of Hermitian operators A, B on C whose composition is given by the simple associative product AB, into the isotopic form

$\hat{\varepsilon}$ characterized by the product $A * B = ATB$ and the new unity $\hat{I} = T^{-1}$, $\hat{I} * A = A * \hat{I} = A$.

Following ref. [38], Nishioka defines the action of the algebra $\hat{\varepsilon}$ on the space $\hat{\mathcal{H}}$, which is characterized by the modular isotopic form $A * \hat{\psi} = AT\hat{\psi}$, as well as the *linear, Hermitian, adjoint* as follows

$$(A * \hat{\psi} \hat{|} \hat{\varphi}) = (\hat{\psi} \hat{|} A^{\dagger} * \hat{\varphi}). \tag{A.76}$$

Nishioka therefore assumes liftings (1.52) with $T = G$, in which case

$$A^{\hat{\dagger}} = A^{\dagger}. \tag{A.77}$$

Next, the *isotopic, linear, unitary* operator is defined by

$$(\hat{U} * \psi \hat{|} \hat{U} * \hat{\varphi}) = (\hat{\psi} \hat{|} \hat{\varphi}) \,, \tag{A.78}$$

which characterizes Eq.s (1.43), i.e.

$$\hat{U}^{\hat{\dagger}} * \hat{U} = \hat{U} * \hat{U}^{\dagger} = \hat{I}. \tag{A.79}$$

Santilli's Lie-isotopic lifting $\hat{G}$ of the compact group G is represented by the transformation (Sect 2.5)

$$\hat{\psi}' = \hat{U} * \hat{\psi}, \tag{A.80}$$

where $\hat{U}$ is an *isotopic, linear, unitary* operator given by

$$\hat{U} = \hat{I} \exp[-i\theta^k * X_k] = \exp[-i\theta^k * X_k]\hat{I}, \tag{A.81}$$

θ^k is a function of x, X_k is a matrix representation of the generators of the group G satisfying

$$[X_i, X_j] = i c_{ij}{}^k X_k, \tag{A.82}$$

and $c_{ij}{}^k$ are the structure constants of the Lie algebra of $\hat{G}$. If one sets

$$\varrho = \hat{\psi}^{\dagger} * \hat{\psi} \,, \tag{A.83}$$

it follows from isounitarity

$$\varrho' = \varrho, \tag{A.84}$$

that is to say

$$\hat{\psi}^{\dagger\prime} * \hat{\psi}' = \hat{\psi}^{\dagger} * \hat{\psi}. \tag{A.85}$$

Next Nishioka [*loc.cit*] introduces a Lie-isotopic lifting of the exterior derivative as follows:

$$\hat{d} = d\hat{I}, \tag{A.86}$$

where d is the ordinary exterior derivative.

The operation of $\hat{d}$ on $\hat{\psi}$ is assumed as follows:

$$\hat{d} * \hat{\psi} = d\hat{\psi}. \tag{A.87}$$

One can then operate $\hat{d}$ on ϱ by making use of (A.87)

$$\hat{d} * \varrho = (\hat{d} * \hat{\psi}^{\dagger}) * \hat{\psi} + \hat{\psi}^{\dagger}(\hat{d} * T)\hat{\psi} + \hat{\psi} * (\hat{d} * \hat{\psi}). \tag{A.88}$$

If one assumes

$$\hat{d} * T = V^{\dagger} * T + T * V, \tag{A.89}$$

where V is given by

$$V = F\hat{I}, \tag{A.90}$$

where F is a 1-form, then it follows that

$$\hat{d} * \varrho = (\hat{D} * \hat{\psi})^{\dagger} * \hat{\psi} + \hat{\psi}^{\dagger} * (\hat{D} * \hat{\psi}), \tag{A.91}$$

where $\hat{D}$ is given by

$$\hat{D} * \hat{\psi} = \hat{d} * \hat{\psi} + V * \hat{\psi}. \tag{A.92}$$

At this point Nishioka postulates that under transformations (A.80) and (A.91) should be invariant. Then

$$(\hat{d} * \varrho)' = \hat{d} * \varrho. \tag{A.93}$$

From Eq. (A.93) one has the transformation law for $\hat{D} * \hat{\psi}$

$$\hat{D}' * \hat{U} * \hat{\psi} = \hat{U} * \hat{D} * \hat{\psi}\,, \tag{A.94}$$

from which

$$\hat{D} = \hat{d} - iA^k * X_k\hat{I}, \tag{A.95}$$

where, from Eq. (A.90), F is given by

$$F = -iA^k * X_k, \tag{A.96}$$

and A^k is a 1-form.

From (A.94) one also obtains the transformation rules for A^k_μ which are defined as $A^k = A^k_\mu dx^\mu$

$$A'^i_\mu * X_i = UA^i_\mu * X_i U^{-1} - (\partial_\mu U)U^{-1} , \tag{A.97}$$

where $\hat{U} = U\hat{I}$.

A^k_μ can be identified as a gauge potential. One can then define the isotopic gauge field strengths $\hat{F}_{\mu\nu}$ for the gauge potentials as follows:

$$\hat{F}_{\mu\nu} * \hat{\psi} = i(\hat{D}_\mu * \hat{D}_\nu - \hat{D}_\nu * \hat{D}_\mu) * \hat{\psi}, \tag{A.98}$$

from Eq. (A.94) and (A.98) Nishioka derives the transformation rule for $\hat{F}_{\mu\nu}$

$$\hat{F}'_{\mu\nu} = \hat{U} * \hat{F}_{\mu\nu} * \hat{U}^{-1}, \tag{A.99}$$

where $\hat{D}_\mu$ are defined from $\hat{D} = \hat{D}_\mu dx^\mu$.

The simplifying hypothesis that T is in the center of the algebra of the original Lie group G,

$$[X_i, T] = 0 , \tag{A.100}$$

does not hold, in general, in the above analysis, because if it holds, formula (A.89) vanishes provided that the commutator between the gauge potentials and T vanishes. In the discussion reviewed above the vanishing of the commutator between the gauge potentials and T was tacitly assumed.

* * * *

We now review the research conducted by G. Karayannis and A. Jannussis in ref. [168–171] (additional research by the same authors will be reviewed in Appendix C after the introduction of the isofield theory).

Paper [168] is important for this review inasmuch as it provides a direct connection between the Gasperini-Santilli Gravitation for the interior problem (reviewed in Section 3.5) and Gasperini's isogauge theory (reviewed earlier in this Appendix). The connection is established by studying one of the simplest conceivable interior test particles: a charged particle moving in a physical medium with a velocity-dependent drag force of the type $-\gamma \vec{v}$ caused by the medium itself. A semiclassical treatment (which remains essentially valid at the pure classical level), allows the authors to reach the following results: a) via an essential use of Santilli's isotopic theory, the motion of the charged test particle under drag is isogauge invariant; b) the electromagnetic field of the test particle, when properly written in the isotopic theory, is isogauge invariant in excellent agreement with Gasperini's isogauge theory; and c) there is the natural emergence of a torsion produced precisely by the drag force due to the medium, which is in excellent agreement with the Gasperini-Santilli Gravitation for the interior problem.

The problem of finding the proper Hamiltonian that describes the motion of a particle in an electromagnetic field with quantum friction has been faced by many authors [176,177]. All efforts consist of constructing the classical Hamiltonian and its canonical quantization. This method usually leads to ambiguities related to the Heisenberg uncertainty relation, and other problems. A novel approach was pioneered in memoir [2] and based on the Lie-isotopic formalism, where the Schrödinger equation is generalized in for (1.45), i.e.

$$i\hbar\frac{\partial\Psi}{\partial t} = HT\Psi. \tag{A.101}$$

The explicit form of the operator T and the Hamiltonian H which describe the motion of a particle in an electromagnetic field with quantum friction was computed in ref. [178], resulting into

$$T = e^{\gamma t}\,, \tag{A.102}$$

and

$$\tilde{H} \equiv HT = -\frac{h^2}{2m}e^{-\gamma t \nabla^2} + \frac{ieh}{mc_0}(\bar{A}\bar{\nabla}. + \bar{V}\bar{A}.) + (V + e\phi + \frac{e^2}{2mc_0^2}A^2)e^{\gamma^t}. \tag{A.103}$$

In ref. [168], Karayannis and Jannussis prove that this result is also established by the isotopic gauge invariance principle. Consider the transformations

$$\begin{aligned} \bar{A}' &= \bar{A} + \beta(q,t)\bar{\nabla}\wedge\,, \\ \phi' &= \phi - \frac{1}{c_0}\delta(q,t)\frac{\partial\wedge}{\partial t}\,, \\ \Psi' &= e^{i\epsilon(q,t)}\Psi\,, \end{aligned} \tag{A.104}$$

for Eq. [A.101-103], and compute the functions β, δ and ϵ under the conditions of being a gauge transformation. After simple calculations the results are

$$\delta = \beta = e^{-\gamma t} \qquad \text{and} \qquad \epsilon = \frac{e}{hc}\wedge\,. \tag{A.105}$$

Then the new gauge transformation takes the form

$$\begin{aligned} \bar{A}' &= \bar{A} + e^{-\gamma t}\bar{\nabla}\Lambda , \\ \phi' &= \phi - \frac{1}{c_0} e^{-\gamma t}\frac{\partial \Lambda}{\partial t} , \\ \Psi' &= e^{\frac{ie}{\hbar c}\Lambda} \cdot \Psi . \end{aligned} \tag{A.106}$$

The conservation of probability density is given by the equation

$$\frac{\partial \rho}{\partial t} + \bar{\nabla}\bar{J} = 0 , \tag{A.107}$$

where

$$\rho = \Phi\Phi^{+} , \tag{A.108}$$

and

$$\bar{J} = \frac{ih}{2m}(\Phi \nabla \Psi^{+} - \Psi^{+}\bar{\nabla}\Psi)e^{-\gamma t} - \frac{e}{mc_0}\bar{A}\Psi\Psi^{+}. \tag{A.109}$$

From this equation we see that the expression for the current density, which is invariant under the gauge transformation (A.108b) is similar to that of a frictionless motion except for the factor $e^{-\gamma t}$ multiplying the first term of the second member.

Gauge transformation (A.106) in compact for is given by

$$A'_{\mu} = A_{\mu} + e^{-\gamma t} , \tag{A.110}$$

and it does not hold for the Razavy's case [176]. The preceding expressions for $\gamma \to 0$ are reduced to the usual ones.

Next, Karayannis and Jannussis reproduce, under certain conditions, the gauge transformation (A.106) using the Lie-isotopic formalism.

Let the Lie-isotopic gauge transformation be

$$\Psi' = \bar{U}_T * \Psi , \tag{A.111}$$

where $\bar{U}_T$ is T-unitary and the symbol $(*)$ denotes Santilli's isotope product. Equation (A.101) with the new product is written

$$ih\frac{\overrightarrow{\partial}\,\Psi}{\partial t} = \overrightarrow{H} * \Psi , \tag{A.112}$$

and the gauge-transformed equation becomes

$$\hbar\frac{\overrightarrow{\partial}\,\Psi'}{\partial t} = H^{\dagger} * \Psi' . \tag{A.113}$$

From relations (A.111-115) we get

$$H' = \hat{U}_T * H * \hat{U}_T^{-1} + \hbar \left(\frac{\partial \hat{U}_T}{\partial t} * \hat{U}_T^{-1} \hat{U}_T * \frac{\partial T^{-1}}{\partial t} * \hat{U}_T^{-1} \right) . \tag{A.114}$$

It is known from quantum mechanics that operators corresponding to observables must be Hermitian. Also, there should be a gauge invariant such that its expectation value is independent of the gauge transformation.

By expressing these requirements in the Lie-isotopic generalization of quantum mechanics *we demand the expectation value of the operator corresponding to an observable to be invariant under the isotopic gauge transformation i.e.,*

$$(\Psi, \Theta(A_\mu) * \Psi) = (\Psi', \Theta(A'_\mu) * \Psi') . \tag{A.115}$$

Since the operator $\hat{U}_T$ of the Eq. (A.111) is T-unitary we have

$$\Theta(A'_\mu) = \hat{U}_T * \Theta(A_\mu) * \hat{U}^{-1} \equiv \Theta'(A_\mu) . \tag{A.116}$$

This means that the T-gauge transformation of the operator which corresponds to an observable, generates a gauge transformation in the fields A_μ.

From equation (A.116) for

$$\hat{U}_T = e_T^{\frac{ie}{\hbar c_0} \chi} , \tag{A.117}$$

where

$$e_T^A = T^{-1} e^{TA} = e^{AT} T^{-1} , \tag{A.118}$$

we have

$$p' = \hat{U}_T * P * \hat{U}_T^{-1} = p + \left[\frac{ie}{\hbar c} X, p \right] + \dots . \tag{A.119}$$

By taking $T = e^{\gamma t}$ we get

$$p' = p - \frac{e}{c} \frac{\partial X}{\partial q} . \tag{A.120}$$

Similarly

$$A'_\mu = \hat{U}_T * A_\mu * \hat{U}^{-1} = A_\mu , \tag{A.121}$$

and for the kinetic momentum one gets

$$\pi'_x = \hat{U}_T * (p - \frac{e}{c_0} A) * \hat{U}_T^{-1} = p - \frac{e}{c_0} (A + \frac{\partial X}{\partial q}) . \tag{A.122}$$

Now since the kinetic momentum is an observable it should be gauge invariant, i.e.

$$\pi'_x = p - \frac{e}{c}A' , \tag{A.123}$$

and we have the transformation

$$\bar{A}' = \bar{A} + \bar{\nabla} X . \tag{A.124}$$

Equation (A.112) must be form invariant, where

$$\begin{aligned} H = \quad & \frac{1}{2m}(p - \frac{e}{c_0}A)^2 + V + e\Phi , \\ & \hat{U}_T = T^{-1} e^{\frac{ie}{\hbar c_0}TX} , \\ & \hat{U}_T^{-1} = e^{-\frac{ie}{\hbar c_0}XT} T^{-1} , \\ & T = e^{\gamma t} , \end{aligned} \tag{A.125}$$

and from the invariance of the operator (A.114) we get

$$H' = \frac{1}{2m}(p - \frac{e}{c_0}A')^2 + V + e[\Phi - \frac{1}{c_0}(\gamma X + \frac{\partial X}{\partial t})] . \tag{A.126}$$

Thus

$$\Phi' = \Phi - \frac{1}{c_0}(\gamma X + \frac{\partial X}{\partial t}) . \tag{A.127}$$

We see that the isotopic extension of the Schrödinger equation yields gauges transformations (A.124) and (A.122) which coincide with Eq (A.106), if $e^{\gamma t} X = \Lambda$.

Finally if we demand the operator H of Eq. (A.112) to remain T-gauge invariant, then from Eq. (A.114) we have the relation

$$\frac{\partial T^{-1}}{\partial t} = e^{-\frac{ie}{\hbar c_0}\Lambda T} \left(\frac{\partial T^{-1}}{\partial t} + \frac{ie}{\hbar c_0} T^{-1} \frac{\partial T}{\partial t} \Lambda + \frac{ie}{\hbar c_0} \frac{\partial \Lambda}{\partial t} \right) e^{\frac{ie}{\hbar c_0}T\Lambda} , \tag{A.128}$$

which connects the isotopic operator T and the gauge function Λ.

Next, Karayannis and Jannussis [168] pass to the study of the Lie-isotopic formulation of the electromagnetic field of a charged particle in dissipative conditions due to motion in a physical medium.

In conventional cases, one considers the invariance of the fields

$$\begin{aligned} \bar{B} &= \bar{\nabla} \times \bar{A} , \\ \bar{E} &= -\bar{\nabla}\Phi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t} , \end{aligned} \tag{A.129}$$

under the gauge transformation

$$\bar{A} \rightarrow \bar{A}' = \bar{A} + \bar{\nabla}\Lambda ,$$
$$\Phi \rightarrow \Phi' = \Phi - \frac{1}{c_0}\frac{\partial \Lambda}{\partial t} , \tag{A.130}$$

as a consequence of the requirement that the fields and not the potential enter the several physical processes.

If one demands the same to hold in the Lie-isotopic gauge theory, one must properly modify relations (A.129) in such a way that the invariance with respect to the new gauge transformation (A.106) is conserved. Indeed from Gasperini's work [85] reviewed earlier it follows that

$$\begin{aligned} \bar{B}T &= \bar{\nabla}X(\bar{A}T) , \\ \bar{E}T &= -\bar{\nabla}(\Phi T) - \frac{1}{c_0}\frac{\partial}{\partial t}(\bar{A}T) . \end{aligned} \tag{A.131}$$

For $T = e^{\gamma t}$ we have for the fields

$$\bar{B} = \bar{\nabla} \times \bar{A} ,$$
$$\bar{E} = -\bar{\nabla}\Phi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t} - \frac{\gamma}{c}\bar{A} , \tag{A.132}$$

which, as one can easily see, remain invariance with respect to the new gauge transformation (A.106).

Note the essential character of Santilli's Lie-isotopic theory to achieve the above results. In fact, other conventional approaches such as [177], do not allow the achievement of a gauge invariant formulation.

We now continue to a review of Karayannis and Jannussis' studies [168] on the connection between (quantum) friction and torsion for the interior problem of gravitation (sec. 3.5).

S. Hojman *et al.* [172] have developed a formalism making torsion compatible with the principles of gauge invariance and of minimal coupling. This theory leads to the following modified form of the gauge transformation of the field A_μ

$$A'_\mu = A_\mu + e^{\Phi}\partial_\mu\Lambda , \tag{A.133}$$

which depends on a scalar field (the "tlaplon" field) Φ which serves as a potential for torsion

$$T^\sigma_{\mu\nu} = \delta^\sigma_\nu\partial_\mu\Phi - \delta^\sigma_\mu\partial_\nu\Phi. \tag{A.134}$$

In the case of the electromagnetic field of a charge particle with friction we have $\Phi = -\gamma t$ and we can say that the problem of the quantum friction in the electromagnetic field is equivalent to an interrelation of the electromagnetic field A_μ, a complex field Ψ, and a scalar field $\Phi = -\gamma t$, which generates a constant torsion,

$$T^{\nu}_{4\nu} = i\frac{\gamma}{c_0} , \tag{A.135}$$

in agreement with the gauge invariance and the minimal coupling principles.

A consequence of the existence of this torsion is the appearance of Ampere's - like equivalent currents [121]

$$\begin{aligned} \bar{J}^{e}_{\gamma} &= \frac{\gamma}{4\pi}\bar{E} = \bar{\nabla} \times \bar{M}^{e} , \\ \bar{J}^{m}_{\gamma} &= -\frac{\gamma}{4\pi}\bar{B} = \bar{\nabla} \times \bar{M}^{m} , \end{aligned} \tag{A.136}$$

The components of the electromagnetic field tensor $F_{\mu\nu'}$ in case of quantum friction, are given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{for} \quad \mu,\nu \neq 4 , \tag{A.137}$$

and

$$F_{4\nu} = \partial_4 A_\nu - \partial_\nu A_4 - A_\nu T^{\nu}_{4\nu} = \partial_4 A_\nu - \partial_\nu A_4 - A_\nu \frac{i\gamma}{c} , \tag{A.138}$$

which leads again to relations (A.132). In general, Hojman's electromagnetic tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - A_\sigma(\delta^\sigma_\nu \partial_\mu \Phi - \delta^\sigma_\mu \partial_\nu \Phi) , \tag{A.139}$$

and corresponds to the special case of Gasperini's theory if we put

$$T = e^{-\Phi} . \tag{A.140}$$

Also, from Gasperini's theory [85] it results that

$$F_{\mu\nu}T = \partial_\mu(A_\nu T) - \partial_\nu(A_\mu T). \tag{A.141}$$

If we put

$$B_\mu \equiv A_\mu T , \tag{A.142}$$

and

$$H_{\mu\nu} \equiv F_{\mu\nu} T , \tag{A.143}$$

the new gauge invariant fields are derived from B_μ

$$H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \ , \tag{A.144}$$

and obey the conventional gauge transformation

$$B'_\mu = B_\mu + \partial_\mu \wedge . \tag{A.145}$$

Karayannis and Jannussis [168] conclude by noting *that Gasperini's theory implies a "lifting" of the field A_μ to B_μ which obeys the known gauge transformation.* This conclusion is in agreement with the relations (A.131).

The "lifting" of the field A_μ to $A_\mu T$ gives an electric and a magnetic current and thus acts as a part of the source of electromagnetism. Indeed from the relation

$$\bar{\nabla} \times (\bar{B}T) = \frac{1}{c_0}\frac{\partial}{\partial t}(\bar{E}T) \ , \tag{A.146}$$

we have a density of electric current related to T

$$\bar{J}^c_T = \frac{c_0}{4\pi}(\frac{1}{c_0}\bar{E}\frac{\partial T}{\partial t} + \bar{B} \times \bar{\nabla}T)T^{-1} \ , \tag{A.147}$$

and from the equation

$$\bar{\nabla} \times (\bar{E}T) = \frac{1}{c_0}\frac{\partial}{\partial t}(\bar{B}T) \tag{A.148}$$

one gets the density of magnetic current

$$\bar{J}^m_T = \frac{c_0}{4\pi}(-\frac{1}{c_0}\bar{B}\frac{\partial T}{\partial t} + \bar{E} \times \bar{\nabla}T)T^{-1} \ . \tag{A.149}$$

Relations (A.147) and (A.149) coincide exactly with those of Hojman's theory [172] for $T = e^{-\Phi}$, and constitute a particular case of the more general theories of ref. [27]. Similarly, from the other two "lifted" Maxwell equations, one finds the relations for the electric charge density and magnetic charge density respectively,

$$P^e_T = \frac{1}{4\pi}(\bar{E}.\bar{\nabla}T)T^{-1}, \tag{A.150}$$

$$P^m_T = -\frac{1}{4\pi}(\bar{B}.\bar{\nabla}T)T^{-1}. \tag{A.151}$$

For the case of quantum friction in electromagnetic field from (A.147) and (A.149) it follows that

$$\bar{J}^e_\gamma = \frac{\gamma}{4\pi}\bar{E} \,, \tag{A.152}$$

$$\bar{J}^m_\gamma = -\frac{\gamma}{4\pi}\bar{B} \,, \tag{A.153}$$

and in a region free of charge we have

$$\begin{aligned} \bar{J}^e_\gamma &\equiv \bar{\nabla} \times \bar{M}^e \,, \\ \bar{J}^m_\gamma &\equiv \bar{\nabla} \times \bar{M}^m \,. \end{aligned} \tag{A.154}$$

Thus the currents $\bar{J}^e_\gamma$ and $\bar{J}^m_\gamma$ behave like Ampere's currents [121].

Karayannis and Jannussis conclude the analysis of ref. [168] (see also ref.s [169–171]) by noting that the invariance of Eq. (A.101) under a gauge transformation leads to a new gauge transformation for the potentials A_μ and establishes, in the Lie-isotopic theory, the requirement that the expectation value of the operator corresponding to an observable must be invariant under the isotopic gauge transformation, Eq. (A.115). The new gauge transformation demands a "lifting" of the fields from A_μ to $A_\mu T$ which in the case of quantum friction takes the forms (A.132). In this way, they reach a new definition of the fields $\bar{E}$ and $\bar{B}$ from the potentials Φ and $\bar{A}$.

In Maxwell's equations the "lifting" of the fields gives an electric and a magnetic current where the corresponding relations (A.147) and (A.149) coincide exactly with those of Hojman's theory [172]. These studies lead to the conjecture that the quantum friction in the electromagnetic field generates a constant torsion between the electromagnetic field and a complex field in agreement with the gauge invariance and the minimal coupling principles. These results are also in remarkable agreement with the Gasperini-Santilli Gravitation for the interior problem §3.5).

APPENDIX B:
CALCULATION OF THE MAXIMAL SPEED OF CAUSAL SIGNALS WITHIN DENSE HADRONIC MATTER.

It is generally believed that massive physical particles (causal signal) cannot acquire speeds bigger than the speed of light in vacuum c_0. By using the Lie-isotopic theory, R.M. Santilli [14] has disproved this belief by establishing, apparently for the first time on rigorous theoretical grounds, the conceivable existence of dynamical conditions under which ordinary massive particles may indeed surpass the speed of light c_0. In turn, this result is of a manifestly fundamental nature for the Lie-isotopic studies, particularly those of operator nature on Hilbert spaces, because it opens-up possibilities that are otherwise precluded, such as the achievement of a true confinement of quarks (with identically null probability of tunnel effect [44]), as we hope to illustrate in our possible subsequent review of "hadronic mechanics".

In his courageous paper of 1982, Santilli [14] stressed that the maximal speed of a causal signal is certainly c_0 for the conditions originally conceived by Einstein (point-like particles moving in empty space under long range action-at-a-distance interactions), but not necessarily for substantially different physical conditions. In fact, he considered extended particles moving within physical media under action-at-a-distance potential forces as well as contact resistive forces caused by the medium. He pointed out that the latter forces are profoundly different than the former one, inasmuch as:

1. the formers admit potential energy, while the notion of potential has no meaning for the latters; on more technical grounds, the formers are Hamiltonian, while the latters are not because they violate the integrability conditions for the existence of a Hamiltonian in the frame of the observer [4];

2. The formers have infinite range, while the latters have zero range, being contact forces by conception; and

3. the formers are action-at-a-distance, while the latters are instantaneous (evidently from their null range).

Owing to these profound dynamical differences, Santilli [*loc.cit.*] conjectured that *the maximal speed of massive particles $V_{\max}$ while moving within a physical medium is not necessarily c_0, but can be bigger, equal or smaller than c_0 depending on the local physical conditions at hand*

$$V_{\mathrm{Max}} \gtreqless c_0 \,. \tag{B.1}$$

On the basis of his theory that the strong interactions have a component precisely of the above contact type due to the necessary condition of mutual penetration and overlapping of the wavepackets of the particles [2] (see Figure 1), Santilli [*loc.cit.*] submitted the following

> *HYPOTHESIS I: The maximal speed of nuclear constituents (protons and neutrons) is smaller than* c_0*; and*
> *HYPOTHESIS II: The maximal speed of the hadronic constituents or, in general, of a hadron within dense hadronic matter (e.g., the core of a collapsing star) can be bigger than* c_0.

The former hypothesis was formulated on the basis of the observation that nonrelativistic calculations have a truly remarkable degree of accuracy in nuclear physics. The latter hypothesis was formulated on the basis that *null range, instantaneous, forces are structurally outside the framework of Einstein's Special Relativity* and, as such, the maximal speed must be recomputed independently from conventional prescriptions. Besides, since the forces considered have no potential energy, there is no a priory technical, experimental, or conceptual information precluding the achievement of speeds beyond c_0. Needless to say, these speeds higher than c_0 should be generally conceived as being local, that is, as conceivable at one given point in space-time inside superdense hadronic matter.

In the subsequent papers [18,26], Santilli constructed his Lie-isotopic covering of Einstein's Special Relativity (§3.4) which confirmed in full Hypotheses I,II. In particular, the application of the new relativity to the Nielsen-Picek metric [99] for the interior of kaons, Eq.s (3.170), i.e.,

$$\begin{aligned}(\eta_{\mu\nu}) &= \text{Diag}(1,1,1,-1) \longrightarrow (g_{\mu\nu}) \\ &= \text{Diag}(1-\frac{1}{3}\alpha, 1-\frac{1}{3}\alpha, 1-\frac{1}{3}\alpha, -(1+\alpha)) \\ &\quad \alpha = (0.61 \pm 0.17) \times 10^{-3}\,, \end{aligned} \tag{B.2}$$

provided a direct confirmation of Hypothesis II (Section 3.4). In fact, by using Eq.s (3.263), one obtains for the above metric

$$V_{\text{Max}} = c_0 \frac{1+\alpha}{1-\frac{1}{3}\alpha} > c_0\,, \tag{B.3}$$

while the value $V_{\text{Max}} < c_0$ occurs for pions.

Papers [18,26] established the fact that, *any modification of the Minkowski metric in the interior of hadrons as suggested by the currently available phenomenology (Section 3.4.3) implies a corresponding necessary modification* *of the maximal speed of causal signals precisely along Eq. (B.1).* In this way, the Lie-isotopic generalization of the conventional Lie's theory permitted the rigorous prediction of the possibility of breaking the "barrier" of the speed of light c_0 by physical massive particles.

The above findings were confirmed by V. De Sabbata and M. Gasperini who published a paper [124] in 1982 following paper [14] providing an explicit calculation of the maximal causal speed within hadronic matter, via the use of *conventional* gauge theories. In the following we review the calculation by De Sabbata and Gasperini because particularly relevant for the line of study of this work.

In a preceding paper [181], De Sabbata and Gasperini had shown that the breaking of the $SU(2) \times U(1)$ gauge symmetry can be related to the possibility, inside hadrons, that causal signals propagate with a speed c different than c_0, much along the classical case of the Cherenkov light. This result was obtained by embedding the Yang-Mills Lagrangian in a space-time with a constant scalar curvature and allowing the maximal causal speed to be a local variable. The Higgs field was therefore introduced in a natural way into the gauge Lagrangian, and the Higgs potential can acquire a gravitational interpretation.

However, in order to reproduce the negative mass squared term of the Higgs potential, De Sabbata and Gasperini [150] were forced to introduce a space-time with a *negative* scalar curvature.

Santilli's hypothesis [14] of maximal causal speeds higher than c_0 allowed the elimination of the negative curvature, thus rendering the model more realistic. In fact, De Sabbata and Gasperini showed in the subsequent paper [124] that, by using a metric background with a nonzero cosmological constant, one can obtain the spontaneous breaking of the internal symmetry without introducing a negative curvature. This also establishes a quite intriguing link between the maximal propagation of a causal signal and the mechanism of symmetry breaking in the presence of interactions on a curved background. (The reader should note that paper [124] was written prior to Gasperini's isotopic generalization of gauge theories [85]. As a consequence, a conventional gauge theory was used in the calculations. This creates the intriguing problem, still open to our knowledge, of reinspecting the calculations via the theory of Appendix A.

Consider a space with a conformally flat metric tensor, $g_{\mu\nu} = \omega^2(t)\eta_{\mu\nu}$, and with a nonvanishing cosmological constant Λ. The gravitational Lagrangian is then given by

$$\mathcal{L}_0 = \frac{\sqrt{-g}c_0^4}{16\pi G}(R_0 - 2\Lambda), \tag{B.4}$$

where $\sqrt{-g} = \omega^4$ and $R_0 = 6\ddot{\omega}/\omega^3 c_0^2(\dot{\omega} = d\omega/dt)$. Suppose that the maximal causal speed is a variable quantity, $c_0 \to c(x,t)$ (this is to be seen only like a starting formal prescription, as we end up with a constant vacuum light velocity), and associate to c a scalar multiplet φ such that [181]

$$c^2(x,t) = \frac{G}{3v^2}|\varphi(x,t)|^2 , \tag{B.5}$$

where v is a constant velocity, introduced for dimensional reasons, which will be interpreted later on.

Putting $R = 6\ddot{\omega}\omega^3 v^2$, Lagrangian (B.4) becomes

$$\mathcal{L} = \frac{\sqrt{-g}}{8\pi}\left(\frac{R}{6}|\varphi|^2 - \frac{G\Lambda}{9v^4}|\varphi|^4\right) . \tag{B.6}$$

Complete it by adding a kinetic term for the scalar field. Then, the total Lagrangian, which can be interpreted as the Higgs Lagrangian producing spontaneous symmetry breaking, is given by

$$\mathcal{L}_T = \frac{\sqrt{-g}}{8\pi}[(D_\mu\varphi) + D^\mu\varphi - V(\varphi)], \tag{B.7}$$

where

$$V(\varphi) = \frac{R}{6}|\varphi|^2 + \frac{G\Lambda}{9v^4}|\varphi|^4 . \tag{B.8}$$

To preserve invariance under the local gauge transformations of φ, the authors used the gauge covariant derivative $D_\mu = \partial_\mu - i\alpha A_\mu^k\theta_k$, where A_μ^k are the gauge potentials, α and θ_k are, respectively, the coupling constant and the generators of the gauge group. Notice that for a positive curvature, $R > 0$, potential (B.8) has the right signs to provide a positive real mass for the scalar field after the application of the Higgs mechanism.

From Lagrangian (B.7) one obtains the following field equations for $g_{\mu\nu}$ and φ:

$$\frac{1}{12}|\varphi|^2 G_{\mu\nu} = -\frac{1}{2}\left[T_{\mu\nu}(\varphi) + g_{\mu\nu}\frac{G\Lambda}{18v^4}|\varphi|^4\right], \tag{B.9}$$

$$(D^\mu\varphi)_{|\mu} = i\alpha A_\mu^k\theta_k D^\mu\varphi + \frac{\partial V}{\partial\varphi^+}, \tag{B.10}$$

where a bar denotes the metric-covariant derivative, $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}(\varphi)$ is the so-called "improved" energy-momentum tensor of the scalar field (see for example ref. [182])

$$T_{\mu\nu}(\varphi) = (D_\mu\varphi)^+ D_\nu\varphi - \tfrac{1}{2}g_{\mu\nu}(D_\alpha\varphi)^+ D^\alpha\varphi + \tfrac{1}{6}(|\varphi|^2_{|\mu|\nu} - g_{\mu\nu}|\varphi|^{2|\alpha}{}_{|\alpha}). \quad \text{(B.11)}$$

The vacuum state is obtained for $A^k_\mu = 0$ and $\langle\varphi\rangle = \langle\varphi^+\rangle = \varphi_0$, where φ_0 is a constant value minimizing $V(\varphi)$. The field equations (B.9) and (B.10) in vacuum are reduced to

$$\langle G_{\mu\nu}\rangle = -\langle g_{\mu\nu}\rangle \frac{G\Lambda}{3v^4}\varphi_0^2, \quad \text{(B.12)}$$

$$\frac{\delta\langle V\rangle}{\delta\varphi_0} = 0\,, \quad \text{(B.13)}$$

and they both give

$$\varphi_0^2 = \frac{3v^4}{G}\frac{\langle R\rangle}{3\Lambda}, \quad \text{(B.14)}$$

where $\langle R\rangle$ is the vacuum scalar curvature. Obviously it must be $\langle R\rangle \neq 0$ in order that spontaneous symmetry breaking may occur.

Assuming in vacuum a De Sitter metric background, i.e. putting $\langle\omega\rangle = \tau/t$, where τ is the "Hubble constant", we have $\langle R\rangle = 4\Lambda = 12/a^2$, where $a = v\tau$ is the constant space-time radius of curvature of the vacuum; it follows then, from (B.5) and (B.13), that $|\varphi_0| = v^2(3/G)^{\frac{1}{2}}$, and $\langle c\rangle = v$.

Therefore, the parameter v may be interpreted as the constant value of the speed of light in vacuum, and since it depends on φ_0, its experimental value is not arbitrary, but is fixed by the spontaneous breaking of some internal symmetry. It is amusing to notice that in the absence of symmetry breaking, we have $\varphi_0 = 0$ and then, according to our model, $\langle c_0\rangle = 0$, i.e. light cannot propagate in vacuum.

In conclusion, De Sabbata and Gasperini [124] evaluated the maximal speed of causal signals inside hadronic matter, applying their model for the Higgs Lagrangian to the $SU_2 \times U_1$ gauge group of the standard Weinberg-Salam theory. In this case φ_0 must satisfy the low-energy experimental condition [181]

$$\frac{\hbar^2 v^2}{4\varphi_0^2} = \frac{G_F}{\sqrt{2}}\,, \quad \text{(B.15)}$$

where G_F is the Fermi coupling constant, and then we obtain

$$v^2 = \frac{\hbar^2}{6\sqrt{2}}\frac{G}{G_F}. \quad \text{(B.16)}$$

Since in the De Sabbata-Gasperini model the curved space-time must represent the "hadronic medium" [2], one can identify the De Sitter metric background with the hadronic "microuniverse" governed by strong interactions (see ref. [183] for an extension review of the possibility of this identification). By replacing the Newton constant G with the strong gravity coupling constant $k_f = (0.85 \cdot 10^{38})G$ (as in ref. [183]) of De Sabbata and Gasperini then reach the following value for the maximal speed of causal signals

$$v = \left(\frac{\hbar^2}{6\sqrt{2}} \frac{k_f}{G_F} \right)^{\frac{1}{2}} \simeq 75 c_0 \, , \qquad \text{(B.17)}$$

which is determined by the spontaneous breaking of the weak-interaction symmetry induced by the presence of Santilli's "hadronic medium".

APPENDIX C:
THEORY OF MUTATION OF ELEMENTARY PARTICLES AND SOME OF ITS APPLICATIONS.

In this appendix we shall provide a semiclassical review of a central physical notion of hadronic mechanics, the hypothesis that ordinary, massive, particles experience an alteration (called *mutation*) of their intrinsic characteristics in the transition from the conditions under which they have been measured until now (motion in vacuum under external electromagnetic interactions) to motion within the hyperdense hadronic medium in the interior of nuclei, hadrons, and stars.

The hypothesis was formulated by Santilli in the original proposal of hadronic mechanics [2] via the generalizations of Dirac's and other field equations of variationally nonselfadjoint type (ref. [2], §4.20, pp. 798–906) conceived as realization of the Lie-isotopic symmetries presented in the preceding memoir [1]. A comprehensive study of the notion was subsequently presented by Santilli, first, in papers [24], where the notion of isotopic lifting of the $SU(2)$-spin symmetry is studied in detail, then in paper [27] where a study of the Poincaré-isotopic symmetry is conducted, and finally in paper [28] which presents the isotopic generalization of conventional field equations as realizations of the Poincaré-isotopic symmetries. In the same paper [29], the theory of mutation of elementary particles is applied to the interpretation of Rauch's [88] experimental data on the apparent deformation/mutation of the magnetic moment of neutrons under external nuclear field, achieved via neutron interferometric techniques. The theory of mutation was subsequently used by Santilli for the achievement of a consistent representation of Rutherford's historical hypothesis according to which the neutron is a "compressed hydrogen atom", i.e., a bound state of a proton and an ordinary electron totally compressed inside the hyperdense medium in the interior of the proton, say, when in the core of a collapsing star [25,29]. Additional uses of the theory of mutation of particles are in progress, e.g., the possibility of identifying quarks with mutated forms of ordinary massive particles that are freely emitted by unstable hadrons in their spontaneous decays [44].

A central part of the appendix is a little known generalization of Dirac's equation by P.A.M. Dirac himself in two of his last papers [54], which implies the mutation of spin from $\frac{1}{2}$ to zero (for at rest conditions, exactly as needed in Rutherford's hypothesis). As proved by Santilli [28], "Dirac's generalization of Dirac's equation" possesses an essential isotopic structure and, as such, it is a particular case of the isotopic generalizations of Dirac's equations.

It is evident that we cannot possibly review here all this comprehensive research. In this appendix we shall therefore limit ourselves to a review of: 1) Santilli's theory of mutation of elementary particles achieved via an isotopic lifting of conventional field equations [29]; 2) an outline of the notions of *hadronic angular momentum and spin* achieved via the isorepresentations of isotopic $\widehat{SU}(2)$ coverings [24] of the $\hat{O}(3)$ symmetries [23]; 3) "Dirac's generalization of Dirac's equation" [54]; 4) a review of certain developments of the theory by Nishioka [184], and Janussis, Karayannis *et al.* [168–171]; and, finally, 5) a review of some of the applications, such as: direct interpretation of Rauch's interferometric measures on the apparent alteration of the magnetic moment of neutrons under external nuclear fields; technical characterization of current "hadronization models"; application of the "hadronic models of structure" of hadrons with massive physical constituents freely produced in the spontaneous decays (including Rutherford's historical hypothesis on the neutron); and an indication of the expected results in the ongoing applications to the "quark models of classification" of hadrons into families.

* * * *

The central physical notion of this appendix is the concept of *mutation of elementary particles* proposed by Santilli in his second memoir of 1978 [2]. This is an alteration of the intrinsic characteristics of a particle (rest energy, spin, magnetic and electric moments, etc.) that is conceivable under the transition from motion in vacuum (strict Einsteinian conditions), to motion within a hyperdense hadronic medium (Santilli's conditions).

Santilli proposed this concept following his isotopic (and Lie-admissible) generalization of the Galilei relativity [1] (and prior to his generalization of Einstein's relativity [14]), precisely as one way to illustrate the physical implications expected from the Galilei-isotopic (and the Lorentz-isotopic) symmetries. As well known, field equations are characterized by representations of the fundamental Galilei or Lorentz symmetry. If the latter symmetries are subjected to an isotopic generalization, he expected the characterization of different field equations which, in turn, render inevitable the alteration of the characteristics of conventional particles according to the schematic view in Fig. 17.

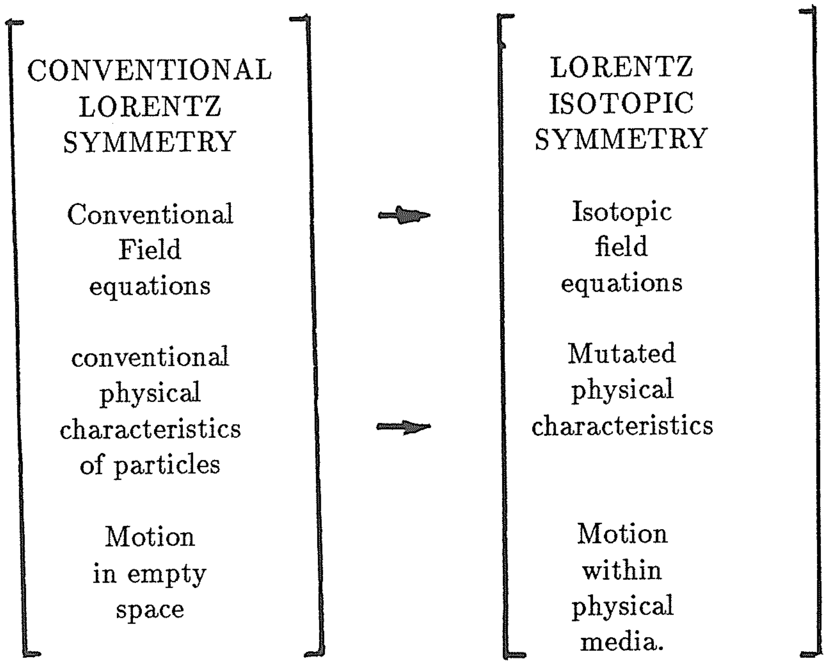


FIGURE 17: A schematic view of the origin of mutation of the intrinsic characteristics of particles, as originally conceived in ref.s [1,2].

As a result, the notion of mutations of particles is already implied by the Galilei-isotopic (or Galilei-admissible) symmetry. As a matter of fact, the notion can see its mathematical foundations in the isotopic generalization of Lie's First, Second and Third Theorems introduced in memoir [1] jointly with the consequential notion of Lie-isotopic symmetry. The Lorentz-isotopic symmetry essentially provides a technical refinement (hadronic mechanics provides yet another contribution to mutation that will be indicated later on).

More specifically, Santilli submitted in memoir [2] the notion of *eleton* as a mutated form of the conventional electron which is conceivable when the wavepacket of the particle is in a state of total immersion within hadronic matter. He then formulated the hypothesis that *eletons are the physical constituents of hadrons (or of quarks).*

The central part of memoir [2], the proposal to construct hadronic mechanics via an isotopic lifting of its enveloping algebra, was formulated precisely to achieve a quantitative representation of the notion of mutation of elementary particles at large, and of the notion of eleton in particular (mutations are necessary to achieve a consistent model of quark and/or of hadronic structure with physical already known constituents [25,29]).

Also, the construction of hadronic mechanics was suggested to achieve *a consistent model of structure of hadrons (or of quarks) as (hadronic) bound states of eletons.* In memoir [2] Santilli presented a consistent model of structure of the light mesons (which is capable of representing *all* known total characteristics of the particle, including their size). Unaware of Dirac's work [54] at that time (1978), he then suggested as a subsequent objective for hadronic mechanics the achievement of a consistent formulation of Rutherford's hypothesis of the neutron as a *compressed hydrogen atom* (in Rutherford's words). Thanks to the resolution of the problem of spin achieved by Dirac himself [54] and in ref. [24], the consistency of Rutherford's hypothesis has been recently indicated in ref.s [25,29], as we shall see in a possible separate review on hadronic mechanics.

In conclusion, *the particle characterized by the Dirac's new equation [54] turns out to be exactly one form of Santilli's eleton* with a finite discrete mutation of the spin, while intermediary forms of mutation of the eleton are provided by the isofield equation of ref. [28].

This appendix is an essential complement of the isotopic generalization of the Lorentz group of Section 3.4, because it provides a quantitative illustration of the physical implications occurring in the transition from conventional representations of the Lorentz group to isorepresentations of the Lorentz-isotopic group.

To see these novel physical results, the reader is urged to alter the conventional mental attitude (*preservation* as much as possible of established doctrines), and leave instead free course to scientific curiosity by seeking, specifically, the maximal possible *alteration* of conventional doctrines.

* * * *

We shall now review *the foundation of the isofield equations* as presented in ref. [29], i.e. *as characterized by isorepresentations of the inhomogeneous Lorentz-isotopic (Poincaré-isotopic) group* $\hat{P}(3.1)$ (§3.4.7). Let us begin by assuming the following formulation of the underlying Minkowski and Minkowski-isotopic metrics

$$\begin{aligned} \eta &= (\eta_{\mu\nu}) = \mathrm{Diag}(1,1,1,-1), \\ g &= T\eta = \mathrm{Diag.}(g_{11}, g_{22}, g_{33}, -g_{44}), \\ T &\stackrel{\text{def}}{=} \mathrm{Diag}(b_1^2, b_2^2, b_3^2 b_4^2), \quad b_\mu > 0, \\ x &= (\vec{x}, x^4) = (\vec{x}, c_0 t)\,, \end{aligned} \qquad \text{(C.1)}$$

where the b's are independent of x but can have dependences of the type $b_\mu = b_\mu(\dot{x}, \mu, \tau, ...)$. The central (classical) invariant of the theory is then isoinvariant (3.246) on the Minkowski-isotopic space $\hat{M}_I = \hat{M}_I(x, g, \hat{\mathbf{R}})$

$$\begin{aligned} p^{\hat{2}} &= p_\mu g^{\mu\nu} p_\nu = p^\mu g_{\mu\nu} p^\nu = p^\mu p_\mu = p_\mu p^\mu \\ &= p^k b_k^2 p^k - p_4^2 c_0^2 b_4^2 p_4^2 = -m_0^2 c_0^2 b_4^2 = -m_0^2 c^2 \, , \end{aligned} \qquad \text{(C.2)}$$

In this way, the (dimensionless) quantities b_μ represent the mutation of the conventional Minkowski metric suggested by the specific case at hand, such as the Nielsen-Picek mutation (3.170) for the medium constituted by the interior of pions and kaons.

The reader should be aware that the quantity "c" $= c_0 b_4$ of the theory is not necessarily the speed of light within the physical medium considered, but can be a geometrical quantity characterizing the contact, zero-range, instantaneous interactions (§3.4.6). Also, in general, $c \neq c_0 =$ speed of light in vacuum. The reader should finally recall that the unit of space $\hat{M}_I$ is the familiar isounit of §3.4,

$$\hat{1} = T^{-1} = \mathrm{Diag}(b_1^{-2}, b_2^{-2}, b_3^{-2}, b_4^{-2}) > 0 \, . \qquad \text{(C.3)}$$

In order to reach the (semiclassical) isofield equations, Santilli introduces hadronic mechanics (see the elements reviewed in Section 1.3) according to structure (1.52) assumed (for simplicity but without loss of generality) for the particular case in which $G = T$ is a Hermitean and positive-definite operator, i.e.

$$\hat{\xi} \; : \; A * B \stackrel{\text{def}}{=} ATB \, , \qquad \text{(4.a)}$$

$$\hat{\mathbf{C}} \; : \; \{\hat{c} | \hat{c} = c\hat{1}, c \in \mathbf{C}, \hat{1} = T^{-1}\} \, , \qquad \text{(4.b)}$$

$$\hat{\mathcal{H}} \; : \; \langle \phi \hat{|} \psi \rangle \stackrel{\text{def}}{=} \langle \phi | * | \psi \rangle \hat{1} = \langle \phi | T | \psi \rangle \hat{1} \in \hat{\mathbf{C}}. \qquad \text{(4.c)}$$

The modular action of an operator A of an element ψ of $\hat{\mathcal{H}}$ is then given (for necessary reasons of consistency) by the isotopic form (1.40), i.e.

$$A * \psi \stackrel{\text{def}}{=} AT\psi \, . \qquad \text{(C.5)}$$

The "hadronization" (i.e., the mapping of Birkhoffian into hadronic mechanics, see §1.3) is done according to the isorelativistic extension of rule (1.63), i.e.

$$p_\mu^{0p} * \psi = -i\partial_\mu \psi, \hbar = 1, \partial_\mu = \frac{\partial}{\partial x^\mu} . \qquad \text{(C.6)}$$

In order to understand the isofield equations, it is important to review the basic rules for properly writing equations in hadronic mechanics. In fact, misrepresentations are possible because familiar expressions such as $x_i p_j$ or $p_i p_j$, which are fully defined within the context of quantum mechanics, become intrinsically inconsistent when referred to hadronic mechanics (they violate the postulate of isolinearity because the trivial associative product of operators has no mathematical or physical meaning for isoenvelope $\hat{\xi}$, see §2.2).

The essential notions of the theory are the following:

1. *Units.* Whenever confronting generalizations of Lie's theory it is recommendable to identify first the underlying unit. In general, we have two different isounits, for the simple reason that we have two generalized structures, the iso-Minkowsky space $\hat{M}_I$, with related isounit $\hat{I}$ and the isoenvelope $\hat{\xi}$ with isounit $\hat{1}$. These two units are generally different, and we shall write

$$\hat{I}_{\text{Isosymm.}} \neq \hat{1}_{\text{Oper. Alg.}} \,. \tag{C.7}$$

   As we shall see, this is exactly the case of "Dirac's generalization of Dirac's equation" [54]. Nevertheless, in the first part of this review we shall assume the simpler case in which $\hat{I} = \hat{1} = T^{-1}$.

2. *Scalars.* Ordinary scalars $n \in \mathbf{R}$ (or $c \in \mathbf{C}$) have no mathematical sense in hadronic mechanics and must be replaced with the *isoscalars* of Eq. (C.4b), e.g. $\hat{n} = n\hat{1}$. However, as shown in Section 3.4, this way of writing scalars is purely formal and has no practical implications, because the product of the isoscalars is given by $\hat{n}_1 * \hat{n}_2 = \widehat{n_1 n_2} = n_1 n_2 \hat{1}$. As a result

$$\hat{n} * \psi \equiv n\psi \,. \tag{C.8}$$

   Hereon we shall ignore the above mathematical formality and use ordinary scalars for simplicity. Note that the elements $g_{\mu\nu}$ of the generalized metric g are, strictly speaking, isoscalars and should be written $\hat{g}_{\mu\nu} = \hat{1} g_{\mu\nu}$. However, property (C.8) allows the reduction of isotopic contractions to ordinary ones, e.g.

$$x_\mu = \hat{g}_{\mu\nu} * x^\nu \equiv g_{\mu\nu} x^\nu \,. \tag{C.9}$$

3. *Operators.* As stressed earlier, the conventional associative product is inconsistent within the context of generalized envelope $\hat{\xi}$ and must be

replaced with the more general isoproduct. For example, if x_i and p_j are operators, their "product" in $\hat{\xi}$ must be written

$$x_i^{\hat{2}} = x_i * x_i, \quad p_j^{\hat{2}} = p_j * p_j, \quad \text{etc. (no sum).} \tag{C.10}$$

The reader should also keep in mind that, from the assumption $T = G$ in Eq. (1.52), and from rule (1.51), operators that are conventionally Hermitean remain isotopically Hermitean. See §1.3 and ref.s [36-38] for details.

4. *Vectors.* Until now we have been dealing with classical vectors on isotopic generalizations of metric spaces, such as three-vectors $\vec{r}$ on iso-Euclidean spaces $\hat{E}(\vec{r}, g, \hat{\mathbf{R}})$, or four-vectors x and p on iso-Minkowski space $\hat{M}_I$. Their products are therefore characterized by the generalized contractions

$$\begin{aligned} \vec{r}^{\hat{2}} &= r^i g_{ij} r^j \quad , \vec{p}^{\hat{2}} = p_i g^{ij} p_j, \\ x^{\hat{2}} &= x^\mu g_{\mu\nu} x^\nu \quad , p^{\hat{2}} = p_\mu g^{\mu\nu} p_\nu \, . \end{aligned} \tag{C.11}$$

But the quantities "x_i" and "p_j" are now operators, that is, they acquire the additional meaning of being elements of $\hat{\xi}$ acting on iso-Hilbert space $\hat{\mathcal{H}}$. As such, contractions of the type (C.11) are no longer acceptable after hadronization, and must be replaced with the expressions

$$\begin{aligned} \vec{r}^{\hat{2}}_{op} &= \hat{g}_{ij} * r^i * r^j = g_{ij} r^i T r^j ; \vec{p}^{\hat{2}}_{op} = \hat{g}^{ij} * p_i * p_j = g^{ij} p_i T p \\ x^{\hat{2}}_{op} &= \hat{g}_{\mu\nu} * x^\mu * x^\nu = g_{\mu\nu} x^\mu T x^\nu ; p^{\hat{2}}_{op} = \hat{g}^{\mu\nu} * p_\mu * p_\nu = g^{\mu\nu} p_\mu T p_\nu \, . \end{aligned} \tag{C.12}$$

For simplicity, Santilli [28] also rewrites hadronization rule (C.6) in the form

$$p_\mu^{op} * \psi = -i \partial_\mu \psi \stackrel{\text{def}}{=} -i \hat{\partial}_\mu * \psi \stackrel{\text{def}}{=} -i \hat{1} \partial_\mu * \psi \, , \tag{C.13}$$

which allows the substitution rule

$$p_\mu^{op} \rightarrow -i \hat{\partial}_\mu = -i \hat{1} \partial_\mu \, . \tag{C.14}$$

Thus, *the fundamental operator-invariant of the isofield theory* can be written

$$\begin{aligned}\Box &\stackrel{\text{def}}{=} \hat{g}^{\mu\nu} * p_\mu * p_\nu = g^{\mu\nu} p_\mu * p_\nu = p^\mu * p_\mu \\ &= g^{\mu\nu} p_\mu T p_\nu = -g^{\mu\nu}\hat{\partial}_\mu * \hat{\partial}_\nu = -\hat{\partial}^\mu * \hat{\partial}_\mu \\ &= -g^{\mu\nu}\partial_\mu\partial_\nu \hat{1} = -\partial^\mu\partial_\mu \hat{1} \,. \end{aligned} \tag{C.15}$$

Similarly, if $\hat{\gamma}^\mu$ are matrices, their correct contraction with the operators p_μ must be written

$$\hat{\gamma}^\mu * p_\mu = \hat{g}^{\mu\nu} * \hat{\gamma}_\mu * p_\nu = g^{\mu\nu}\hat{\gamma}_\mu T p_\nu \,, \tag{C.16}$$

as correctly identified for the first time by Karayannis and Jannussis [168].

5. *Hilbert space.* The reader should finally keep in mind that the proper way of writing the norm in the underlying Hilbert space is form (C.4c). i.e. the space is a iso-Hilbert space. Thus, the conventional, linear-action of an operator, say H, on an element ψ of $\hat{\mathcal{H}}$ has no mathematical or physical meaning and must be replaced with the isolinear form (C.5). If H is (iso)Hermitean, then the (iso)eigenvalues are real [36] and we shall write

$$H * \psi = h\psi, \quad H = H^\dagger, \quad h \in \mathbf{R} \,. \tag{C.17}$$

Finally, the correct product of the element ψ and its dual $\psi^\dagger$ is given by $\psi^\dagger * \psi = \psi^\dagger T \psi$.

We are now in a position to point out the contribution provided by hadronic mechanics to Santilli's notion of mutation of elementary particles. This latter notion is intrinsic in the very basic eigenvalue equation of the theory, Eq. (C.17). Suppose that the Hermitean operator H has eigenvalue h_0 in quantum mechanics, $H\phi = h_0\phi$. Then the *same* operator H has a *different* eigenvalue h in hadronic mechanics, $H * \psi = h\psi$. The transition $h_0 \to h$ is precisely Santilli's notion of mutation because it mutates specifically, the physical characteristic h_0 [28].

When the isotopy of isospace $\hat{M}_I$ is different than that of isoenvelope $\hat{\xi}$, Eq. (C.7), we have two different, mutually compatible contributions to the notion of mutation, one originating from the fundamental Lorentz-isotopic structure, and one generated by the isotopic lifting of the enveloping operator algebra.

Once rules 1-5 above and the physical objectives of the theory are properly understood, the formulation of isofield equations is quite easy. In fact, we readily have the following *iso-Klein-Gordon field equation* [28].

$$\left(\hat{g}^{\mu\nu} * p_\mu * p_\nu + m_0^2 \hat{c}^2\right) * \psi = \left(p^\mu * p_\mu + m_0^2 \hat{c}^2\right) * \psi$$
$$= -\left(\Box - m_0^2 \hat{c}^2\right) * \psi = -\left(\partial^\mu \partial_\mu \hat{1} - m_0^2 \hat{c}^2\right) * \psi = 0. \qquad \text{(C.18)}$$

The corresponding extension to the case of a *charged particle under an external electromagnetic field while in immesion within the hadronic medium* is given by

$$[\hat{g}^{\mu\nu} * \left(p_\mu + \frac{e}{c} A_\mu\right) * \left(p_\nu + \frac{e}{c} A_\nu\right) + m_0^2 \hat{c}^2] * \psi$$
$$= -\left[\left(\hat{\partial}^\mu + i\frac{e}{c} A^\mu\right) * \left(\hat{\partial}_\mu + \frac{ie}{c} A_\mu\right) - m_0^2 \hat{c}^2\right] * \psi = 0. \qquad \text{(C.19)}$$

The *isofourcurrent* is then given by [*loc.cit*]

$$\hat{J}_\mu = \frac{1}{2im_0}\left[\psi^\dagger * \hat{\partial}_\mu * \psi - \left(\hat{\partial}_\mu * \psi^\dagger\right) * \psi\right]$$
$$+ \frac{e}{m_0 c^2} A_\mu * \left(\psi^\dagger * \psi\right), \qquad \text{(C.20)}$$

and it verifies the conventional conservation law

$$\partial^\mu \hat{J}_\mu = \hat{\partial}^\mu * \hat{J}_\mu = 0. \qquad \text{(C.21)}$$

The *isocharge density* is then given by

$$\hat{\rho} = \frac{1}{ic}\hat{J}_4 = \frac{i}{2m_0 c^2}\left(\psi^\dagger * \frac{\partial \psi}{\partial t} - \frac{\partial \psi^\dagger}{\partial t} * \psi\right)$$
$$+ \frac{e}{mc^2} A_0 * \psi^\dagger * \psi, \qquad \text{(C.22)}$$

and it is indeed conserved,

$$\frac{dQ}{dt} = \frac{d}{dt}\int \hat{\rho}\, dv = 0. \qquad \text{(C.23)}$$

The mutation of the characteristics of the particle is now evident. To begin, we have a *mutation of the rest energy*, from the value $m_o c_0^2$ for Einsteinian conditions to the value $m_0 c^2$ for Santilli conditions where the quantity c has been defined in Section 3.4.6. Suppose that ρ is the charge density for Einsteinian conditions, i.e., for $T = 1$. Under the presence of contact, zero-range, instantaneous interactions represented by the operator $T \neq 1$ the charge density assumes value (C.22). The transition $\rho \to \hat{\rho}$ is evidently a form of *charge mutation.* A similar situation occurs for the fourcurrent. The mutation of other characteristics will be considered in more details below.

The iso-Klein-Gordon equation (C.19) is invariant under the full Poincaré-isotopic group $\hat{P}$ (3.1) [28]. In particular, the wavefunction transforms as an isoscalar. In fact, by keeping in mind rules (3.218-320), it is easy to see that

$$g^{\mu\nu}\frac{\hat{\partial}}{\partial x^{\mu}} * \frac{\hat{\partial}}{\partial x^{\nu}} - m_0^2c^2 = g^{\mu\nu}\frac{\hat{\partial}}{\partial x'^{\mu}} * \frac{\hat{\partial}}{\partial x'^{\nu}} - m_0^2c^2 \,. \qquad \text{(C.24)}$$

A *plane isowave solution* is given by Eq. (3.281), i.e.,

$$\psi(x) = Ne^{ik^{\mu}g_{\mu\nu}x^{\nu}} \,, \qquad \text{(C.25)}$$

and evidently transforms as follows

$$\psi(x) = Ne^{ik^{\mu}g_{\mu\nu}x^{\nu}} = Ne^{ik'^{\mu}g_{\mu\nu}x'^{\nu}} = \psi'(x') \qquad \text{(C.26)}$$

thus proving the isoscalar nature of the equation under $\hat{P}$ (3.1).

The reader should be aware that *the iso-Klein-Gordon equation does not represent a free particle* (see the No-No-Interaction Theorem of Section 3.4.15), and, thus, solution (C.25) is not a conventional, free, plane wave solution. This is evidently due to the fact that the deformation of the metric $\eta \to g$ is per sé a representative of interactions, not of the conventional Hamiltonian-Lagrangian type, but precisely of Santilli's non-Hamiltonian type.

To put it differently, *a relativistic (massive, spin zero) particle that is truly free must obey Einstein's Special Relativity exactly and, as such, must be characterized by the conventional Klein-Gordon equation. Any deviation from this established setting caused by motion of the same particle within a physical medium or other reasons obeys the covering Santilli's Special Relativity and, as such, it can be characterized by the covering iso-Klein-Gordon equation.*

As a final comment, note the way Santilli [*loc.cit*] writes wavefunction (C.25) with the exponent given by $g_{\mu\nu}k^{\mu}x^{\nu}$ and *not* $g^{\mu\nu}k_{\mu} * x_{\nu} = g_{\mu\nu}k^{\mu}Tx^{\nu}$. This is evidently due to the fact that the quantities "k" and "x" in the *isophase* are isoscalars and not isooperators.

The construction of the remaining essential parts of the theory (e.g., the iso-Green functions) will be deferred to the possible subsequent review on hadronic mechanics. It is appropriate here to bring to the attention of the interested reader the important work by Nishioka [185] on the so-called *Dirac-Myung-Santilli delta function* (which is essentially an isotopic generalization of the structure of the conventional delta function) and which plays an essential role for the further development of the isofield theory.

We now pass to the review of the *isotopic generalization of Dirac's equation* as presented in ref. [28]. The origin of the equation is an isotopic decomposition of the fundamental second-order isoinvariant operator, Eq. (C.18). For this purpose, suppose that $\hat{\gamma}_\mu$ are 4×4 matrices. Then, the second-order isoinvariant operator can be decomposed into the isoproduct of the two first-order 4×4 operators according to the form

$$\begin{aligned} & p_\mu * p^\mu + m_0^2 \hat{c^2} \\ & = (\hat{\gamma}_\mu * p^\mu - i m_0 \hat{c}) * (\hat{\gamma}_\nu * p^\nu + i m_0 \hat{c}) \\ & = \frac{1}{2} \{\hat{\gamma}_\mu \hat{,} \hat{\gamma}_\nu\} * p^\mu * p^\nu + m_0^2 \hat{c^2} , \end{aligned} \tag{C.27}$$

which holds iff the $\hat{\gamma}_\mu$ matrices verify the laws

$$\{\hat{\gamma}_\mu \hat{,} \hat{\gamma}_\nu\} = \hat{\gamma}_\mu * \hat{\gamma}_\nu + \hat{\gamma}_\nu * \hat{\gamma}_\mu = 2 g_{\mu\nu} \hat{1} . \tag{C.28}$$

Note that the preceding law is exactly the isotopic lifting of the conditions on the conventional γ-matrices of Dirac's equation

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \eta_{\mu\nu} I . \tag{C.29}$$

The desired *isotopic lifting of Dirac's equation* is then given by

$$\begin{aligned} & (\hat{\gamma}_\mu * p^\mu + i m_0 \hat{c}) * \psi = 0 \\ & = -(i \hat{\gamma}_\mu * \hat{\partial}^\mu - i m_0 \hat{c}) * \psi = 0 . \end{aligned} \tag{C.30}$$

Introduce now the adjoint wavefunction

$$\hat{\bar{\psi}} = \psi^\dagger * \hat{\gamma}_4 , \tag{C.31}$$

then, each and every step of the theory of conventional Dirac's equations (see, e.g., ref. [186]) can be subjected to an isotopic lifting. In fact, the (iso)adjoint of Eq. (C.30) is given by

$$(i \hat{\partial}^\mu * \hat{\bar{\psi}}) * \hat{\gamma}_\mu - i m_0 c \hat{\bar{\psi}} = 0 . \tag{C.32}$$

The combination of Eq. s (C.30) and (C.32) then yields

$$(i\hat{\partial}^{\mu} * \hat{\bar{\psi}}) * \hat{\gamma}_{\mu} * \psi + \hat{\bar{\psi}} * \hat{\gamma}_{\mu} * (\hat{\partial}^{\mu} * \psi) = 0 , \tag{C.33}$$

thus allowing the introduction of the *isocurrent*

$$\hat{J}_{\mu} = ic\hat{\bar{\psi}} * \hat{\gamma}_{\mu} * \psi , \tag{C.34}$$

which is evidently conserved

$$\hat{\partial}^{\mu} * \hat{J}_{\mu} = \partial^{\mu} \hat{J}_{\mu} = 0 . \tag{C.35}$$

The *isocharge density* is then given by

$$\hat{\rho} = \frac{e}{ic}\hat{J}_4 = \hat{\bar{\psi}} * \hat{\gamma}_4 * \psi = b_4^2 \psi^{\dagger} * \psi , \tag{C.36}$$

with corresponding *isocharge*

$$Q = \int \hat{\rho} d^3x = eb_4^2 \int \psi^{\dagger} * \psi d^3x , \tag{C.37}$$

that is, it is proportionate to the isoinner product of the underlying space $\hat{\mathcal{H}}$. Note that density (C.34) is positive-definite under assumptions (C.4), i.e., an isotopic lifting of a positive-definite inner product via a positive-definite operator T. Along the same lines, it is possible to prove that $\hat{J}_{\mu}$ is (iso)Hermitean, and that its components are real.

A realization of the $\hat{\gamma}$-matrices verifying Eq.s (C.25) has been identified by Santilli [29] and it is given by

$$\hat{\gamma}_{\mu} = \tilde{\gamma}_{\mu}\hat{1} = b_{\mu}\gamma_{\mu}\hat{1} \text{ (no sum)} , \tag{C.38}$$

where the γ's are any given representation of the conventional γ-matrices [186]. Note the non-triviality of the generalization, inasmuch as the quantities b_{μ} (representing the *deviation* from the Minkowski metric caused by motion within the hadronic medium) enter directly into the structure of the $\hat{\gamma}$-matrices.

Santilli [28] then passes to the identification of the mutation of angular momentum and spin caused by isotopic lifting (C.30). First, the total angular momentum can be defined as the sum of the orbital and intrinsic angular momentum in $\hat{\xi}$

$$\hat{M}_i^T = \hat{M}_i + \hat{S}_i . \tag{C.39}$$

The orbital part is given by

$$\hat{M}_i = \int \psi^\dagger * (\varepsilon_{ijk} x_j * \frac{1}{i}\hat{\partial}_k) * \psi d^3x \ , \tag{C.40}$$

while the intrinsic part (spin) is given by

$$\hat{S}_i = \frac{1}{2}\int \psi^\dagger * (\varepsilon_{ijk}\hat{\gamma}_j * \hat{\gamma}_k) * \psi d^3x. \tag{C.41}$$

The above expressions are nothing but isotopic liftings of the corresponding equations for the conventional Dirac's setting (see, e.g., ref. [186], page 142). The corresponding densities are

$$\begin{aligned} \hat{m}_i &= \varepsilon_{ijk} x_j * \frac{1}{i}\hat{\partial}_k \ , \\ \hat{s}_i &= \frac{1}{2}\varepsilon_{ijk}\hat{\gamma}_j * \hat{\gamma}_k \ , \end{aligned} \tag{C.42}$$

with explicit form of the spin matrices

$$\begin{aligned} \hat{s}_1 &= \frac{1}{2}\hat{\gamma}_2 * \hat{\gamma}_3 = \frac{1}{2}\bar{\gamma}_2\bar{\gamma}_3\hat{1} \ , \\ \hat{s}_2 &= \frac{1}{2}\hat{\gamma}_3 * \hat{\gamma}_1 = \frac{1}{2}\bar{\gamma}_3\bar{\gamma}_1\hat{1} \ , \\ \hat{s}_3 &= \frac{1}{2}\hat{\gamma}_1 * \hat{\gamma}_2 = \frac{1}{2}\bar{\gamma}_1\bar{\gamma}_2\hat{1} \ . \end{aligned} \tag{C.43}$$

By using Eq. (C.28) the isocommutation rules for the spin matrices are readily computed, resulting in the isotopic rules [28]

$$[\hat{s}_i \hat{,} \hat{s}_j] = \hat{s}_i * \hat{s}_j - \hat{s}_j * \hat{s}_i = -\varepsilon_{ijk} g_{kk} \hat{s}_k \ , \tag{C.44}$$

which formally coincide with those of the isotopic rotational algebras $\hat{\mathbf{O}}$ (3), Eq. (3.30), but characterize instead those of the isotopic $\widehat{\mathbf{SU}}$ (2) algebra. In this way, Santilli [*loc.cit.*] reached the desired *mutation of the spin of Dirac's equation* which can be expressed via the eigenvalue equation

$$\begin{aligned} \hat{s}^{\hat{2}} * \psi &= (\hat{s}_1 * \hat{s}_1 + \hat{s}_2 * \hat{s}_2 + \hat{s}_3 * \hat{s}_3) * \psi \\ &= \frac{1}{4}(g_{11}g_{22} + g_{22}g_{33} + g_{33}g_{11})\psi \ . \end{aligned} \tag{C.45}$$

Note that $\hat{s}^{\hat{2}}$ *is not* an iso-Casimir invariant (because it is not proportional to $\hat{1}$, which is iven by expression of type (3.39) after redefinitions

(3.37). Nevertheless, $\hat{s}^{\hat{2}}$ is indeed invariant for metric of Nielsen-Picek type, Eq. (3.170), i.e., when $g_{11} = g_{22} = g_{33}(\neq g_{44})$. For infinitesimal mutations these latter conditions can always be assumed. In this case, Santilli recovered the expression [*loc.cit.*]

$$\hat{s} = \frac{1}{2} + \epsilon \quad , \quad \epsilon \approx 0 \quad , \quad g_{kk} \simeq 1 \, , \tag{C.46}$$

which is precisely the mutation of spin he submitted at his invited talk at the 1980 Clausthal's Conference on *Differential Geometric Methods in Mathematical Physics* (see Fig. 1; also Eq. (4.26), p. 1249 of ref. [11]).

The orbital angular momentum remains formally unaffected by the lifting. In fact, from hadronization rule (C.6), the *fundamental isocommutation rules* preserve the quantum mechanical values

$$\begin{aligned} &[p_i \hat{,} x_i] * \psi = (p_i * x_j - x_j * p_i) * \psi = -i\delta_{ij}\psi \, , \\ &[x_i \hat{,} x_j] * \psi = [p_i \hat{,} p_j] * \psi = 0. \end{aligned} \tag{C.47}$$

As a result, the components of the angular momentum verify the isocommutation rules,

$$[\hat{m}_i \hat{,} \hat{m}_j] * \psi = \epsilon_{ijk} \hat{m}_k * \psi \, , \tag{C.48}$$

that is, the structure constants are not modified by the isotopy under consideration [28]. In computing rules (C.47), the reader should be aware of the validity of the following properties [36]

$$[A * B \hat{,} C] = A * [B \hat{,} C] + [A \hat{,} C] * B. \tag{C.49}$$

Santilli [*loc.cit.*] then passes to the study of the transformation properties of the iso-Dirac's equation (C.30). Essentially he proves that *the equation is invariant under the full Poincaré-isotopic group* $\hat{P}(3.1)$ (§3.4.7, Eq. (3.237)) that is, the wavefunction transforms according to

$$\hat{P}(3.1) : \left\{ \begin{array}{l} \psi'(x') = \hat{S} * \psi(x) = \hat{S}(\hat{\Lambda}) * \psi[\hat{\Lambda}^{-\hat{1}} * (x - a)] \, , \\ \overline{\psi}'(x') = \overline{\psi}(x) * \hat{S}^{-1}(\hat{\Lambda}) = \overline{\psi}[\hat{\Lambda}^{-\hat{1}} * (x - a)] * \hat{S}^{-\hat{1}}(\hat{\Lambda}) \, , \\ \hat{\Lambda}^t * \hat{\Lambda} = \hat{\Lambda} * \hat{\Lambda}^t = \hat{I} = g^{-1}, \det\hat{\Lambda} = \pm \det\hat{I}. \end{array} \right. \tag{C.50}$$

The equations transform according to

$$\begin{aligned} (-i\hat{\gamma}^\mu * \hat{\partial}'_\mu + i m_0 \hat{c}) * \psi'(x') &= 0 \, , \\ \bar{\psi}'(x') * (-i\hat{\partial}'_\mu * \hat{\gamma}^\mu - i m_0 \hat{c}) &= 0 \, , \end{aligned} \tag{C.51}$$

under the particular rules

$$\hat{\Lambda}^{-\hat{1}} * \hat{\gamma}_\mu * \hat{\Lambda} * \partial'^\mu = \hat{\gamma}_\mu * \hat{\partial}^\mu ,$$
$$\hat{\Lambda}^{-\hat{1}} = \hat{\gamma}_4 * \hat{\Lambda}^\dagger * \hat{\gamma}_4. \qquad (C.52)$$

Let us review first the *transformations under isotopic rotations.* By following the isotopic lifting of conventional lines (see, again, ref. [186, pp. 162-163), it is easy to see that Eq. (C.30) is invariant under the following realization of the isotopic $\hat{SU}(2)$ group

$$\hat{SU}(2) : \hat{R}(\theta) = e^{\frac{1}{2}\hat{\gamma}_2 * \hat{\gamma}_3 * \theta_1}|_\xi e^{\frac{1}{2}\hat{\gamma}_3 * \hat{\gamma}_1 * \theta_2}|_\xi e^{\frac{1}{2}\hat{\gamma}_1 * \hat{\gamma}_2 * \theta_3}|_\xi \hat{1} , \qquad (C.53)$$

which turns out to be precisely an isospinorial covering of Eq. (3.24-25).

The nontriviality of isotopy (C.30) can now be shown in all its depth. In fact, *the invariance of the iso-Dirac equation under isotopic rotations implies a breaking of the exact spinorial character of the conventional equation [28]. This is readily proved by nothing that the components of structure (C.53) can be written in the form*

$$\hat{R}(\theta_3) = e^{\gamma_1 \gamma_2 \left(\frac{b_1 b_2}{2} \theta_3\right)} . \qquad (C.54)$$

Eq. (C.30) therefore breaks the exact spinorial character of the conventional Dirac's equation in view of the factor $b_1 b_2$. Note also that realization (C.54) holds irrespective of whether the iso-Casimiz invariant is an expression of type (C.46) or of type (3.39). This illustrates the irreducible nature of Santilli's spin mutation.

The experimental implications are also far reaching. Recall the fundamental experiment by Rauch and collaborators [88] on the spinor symmetry of neutrons when in the vicinity of nuclei. As now well known to experts of Lie-isotopic theory (see Fig. 6 of §3.2), neutrons are expected to experience a deformation of their charge distribution caused by external nuclear fields. This, in turn, would necessarily imply a mutation of the magnetic moment of the particle from the conventional value μ to a mutated value $\mu_m \neq \mu, \mu_{\hat{m}} \approx \mu$. The current experimental numbers for two complete spin flips are [88]

$$\alpha = 715.87 \pm 3.8 \text{Deg}; \alpha_{\text{Max}} = 719.67 \text{ Deg} < 720 \text{ Deg}]\alpha_{\text{Min}} = 712.07 , \qquad (C.55)$$

that is, available data DO NOT prove the exact character of the spinor symmetry for the case considered, but show a deviation of about 1%. The isotopy of Dirac's equation is capable of representing experimental data exactly. In fact, from Eq. (C.53) and (C.54) Santilli [28] reaches the values of the deformed metric

$$b_1 = b_2 \cong 1.005\ , \quad b_3 = 0.995\ , \tag{C.56}$$

which essentially shows a small deformation of a spherical charge distribution into an oblate spheroidal ellipsoid.

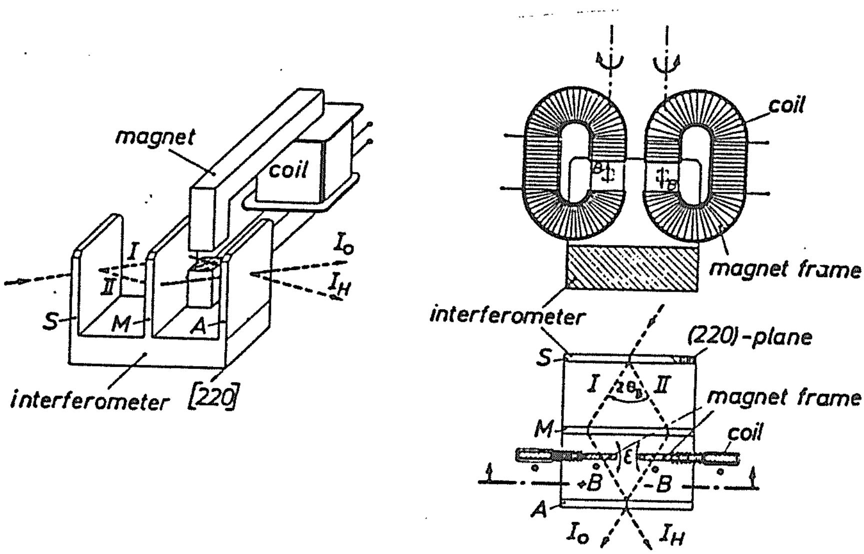


FIGURE 18: A view of the perfect crystal neutron interferometric apparatus used by Rauch and his collaborators [88] to test the spinorial symmetry of neutrons under external nuclear fields. The figure shows the electromagnetic gap in air, although that used for the latest tests was filled up with Mu-metal sheets to reduce stray fields. This turned the experiments into a test of the $SU(2)$-spin symmetry under joint external, electromagnetic and strong/nuclear interactions. Recall that all experimental information achieved during this century on elementary particles was achieved via *external electromagnetic interactions.* The historical aspect of Rauch's experiments is that they are among the first to achieve *measure under external nuclear interactions,* where the external character is evidently established by the fact that the Mu-metal sheets are fixed and external with respect to the neutron beam. The mechanism of mutation is so simple to appear

trivial, and it is due to the expected deformation of the charged distribution of neutrons (Fig. 6, p. 96) under sufficiently intense external field, which, in turn, demands a *necessary* alteration (mutation) of the magnetic moment, as established at the classical and atomic levels. Current measures (C.55) are evidently preliminary and, in need of numerous additional tests stressed in §3.5.18, such as the repetition of the test for [28]: several spin flips, various different metals filling up the electromagnetic gap, and others. Note that the *amount* of mutation of the neutron magnetic moment is unknown at this writing, and open to scientific debates, but the *existence* of the mutation under sufficiently intense external fields should be out of the question to prevent the raising of issues of scientific ethics, for the simple reason that there are no perfectly rigid objects in Nature.

At a deeper analysis, one can start with the representation of the shape of the charge distribution of a proton via hadronic mechanics provided by Nishioka and Santilli [42]. In this case, one has a shape of the type $g_{11} = g_{22} = 1$, $g_{33} = 0.60$, i.e., one has an oblate spheroidal ellipsoid already in the absence of mutation (because of the anomalous value of the magnetic moment). The mutation merely increases the oblate nature of the ellipsoid because of values (C.55), i.e., $g_{11} = g_{22} = 0.597$. Intriguingly, *the iso-Dirac's equation* does indeed reconstruct the exact (iso)spinorial symmetry [28]. In fact, for measures (C.55) and values (C.56), the total angle of rotation is exactly 720 deg., i.e.

$$\frac{1}{2} b_1 b_2 \theta |_{\theta=715.87 \text{ Deg}} = 720 \text{ Deg} . \qquad \text{(C.57)}$$

This is fully in line with all other cases of conventional symmetry breaking we have encountered throughout our analysis. In fact, as it was the case for the rotational, Galilean and Lorentz symmetries, when the symmetry is broken at the conventional level, it is exact at the Lie-isotopic level.

Paper [28] then passes to the study of the *invariance of Eq. (C.30) under iso-Lorentz transformations.* In this way Santilli reaches the following realization of $\widehat{SL}(2.C)$ *covering of the orthochronous Lorentz-isotopic group* (§3.4.7)

$$\begin{aligned} \hat{S}(w_1) &= e^{\frac{1}{2}\gamma_1 * \hat{\gamma}_4 * w_1}|_\xi \hat{1} = e^{\frac{1}{2}\bar{\gamma}_1 \bar{\gamma}_4 w_1}|_\xi \hat{1} , \\ \widehat{SL}(2.C) : \hat{S}(w_2) &= e^{\frac{1}{2}\hat{\gamma}_2 * \hat{\gamma}_4 * w_2}|_\xi \hat{1} = e^{\frac{1}{2}\bar{\gamma}_2 \bar{\gamma}_4 w_2}|_\xi \hat{1} , \\ \hat{S}(w_3) &= e^{\frac{1}{2}\hat{\gamma}_3 * \hat{\gamma}_4 * w_3}|_\xi \hat{1} = e^{\frac{1}{2}\bar{\gamma}_3 \bar{\gamma}_4 w_3}|_\xi \hat{1} , \end{aligned} \qquad \text{(C.58)}$$

where each expression evidently holds for speeds along the directions x_1, x_2, and x_3, respectively. The proof of the invariance of Eq. (C.28) under transformations (C.58) is an instructive exercise for the interested reader.

Paper [28] then passes to the study of the invariance of Eq. (C.30) under discrete transformations, the *isoinversions* (see Eq.s (3.229)). For the case of *space iso-inversions* one has

$$x' = (\vec{x}', ct') = \hat{P} * (\vec{x}, ct) = (-\vec{x}, ct) = P(\vec{x}, ct)\ , \tag{C.59}$$

where P is the ordinary space-inversion operator. In this case, the $\hat{S}$ quantities verify the conditions

$$\begin{aligned} \hat{S}^{-\hat{1}} * \hat{\gamma}_k * \hat{S} &= -\hat{\gamma}_k, k = 1, 2, 3\ , \\ \hat{S}^{-\hat{1}} * \hat{\gamma}_4 * \hat{S} &= \hat{\gamma}_4\ , \end{aligned} \tag{C.60}$$

with a solution given by the expected isotopic lifting of the conventional forms $\hat{S} = \eta_p \hat{\gamma}_4, \eta_p = \pm 1, \pm i$ where the last value originates from the condition that two space isoinversions provide the identity transformation.

For the case of the invariance of Eq. (C.30) under *time iso-inversions*, one has [28]

$$x' = (\vec{x}', ct') = \hat{T} * x = T(\vec{x}, ct) = (\vec{x}, -ct)\ . \tag{C.61}$$

The $\hat{S}$ quantities must then verify the conditions

$$\begin{aligned} \hat{S}^{-\hat{1}} * \hat{\gamma}_k * \hat{S} &= \hat{\gamma}_k, \quad k = 1, 2, 3\ , \\ \hat{S}^{-\hat{1}} * \hat{\gamma}_4 * \hat{S} &= -\hat{\gamma}_4\ , \end{aligned} \tag{C.62}$$

with solutions

$$\begin{aligned} \hat{S} &= \eta_T \hat{\gamma}_5 * \hat{\gamma}_4\ , \\ \hat{\gamma}_5 &= \hat{\gamma}_4 * \hat{\gamma}_1 * \hat{\gamma}_2 * \hat{\gamma}_3\ , \\ \eta_T &= \pm 1, \pm i\ . \end{aligned} \tag{C.63}$$

In this case too we have a simple generalization of conventional settings. In fact, *the time iso-reversal is equivalent to the operation of complex iso-conjugation* [36] which is formally identical to the conventional complex conjugation for assumptions (C.4). We can therefore write

$$\psi' = (\eta_T \hat{\gamma}_5 * \hat{\gamma}_4 * \psi)^*\ . \tag{C.64}$$

As a further comment, the reader should keep in mind that, as it was the case for Eq. (C.18), *the iso-Dirac's equation (C.30) DOES NOT represent* *a free particle.* After all, deformation (C.45) of the conventional spinoral character is due precisely to interactions which, being represented by the generalized unit of the theory, is of non-Hamiltonian (or of non-Lagrangian) type.

Next, paper [28] identifies *the mutation of the magnetic and electric dipole moments characterized by the iso-Dirac's equation* in a way parallel to the spin mutation (C.45). For this purpose, introduce the extension of Eq. (C.30) to represent a *charged isoparticle under an external electromagnetic field*

$$[\hat{\gamma}^{\mu} * (-i\hat{\partial}_{\mu} + \frac{e}{c}A_{\mu}) - im_0\hat{c}] * \psi$$
$$\stackrel{\text{def}}{=} (\hat{\gamma}^{\mu} * \pi_{\mu} - im_0\hat{c}) * \psi = 0 \ , \qquad \text{(C.65)}$$

which is *manifestly invariant under Gasperini's isogauge theory* (Appendix A).

The isocurrent remains the same as in Eq. (C.34). In particular, the isocharge is given by Eq. (C.37).

Eq (C.66) is invariant under the following *charge iso-conjugation*

$$\psi' = \eta_c^* \hat{S}_c^{-\hat{1}} * \psi, \bar{\psi}' = -\eta_c \psi^T * \hat{S}_c \ , \qquad \text{(C.66)}$$

with a solution (for the iso-Pauli's representation of the $\hat{\gamma}$-matrices)

$$\hat{S}_c = \hat{\gamma}_2 * \hat{\gamma}_4 \ , \qquad \text{(C.67)}$$

(and similar solutions for other representations).

In order to understand Eq. (C.66) and its underlying mutation of conventional quantities, one must differentiate between physical quantities that are isotopically lifted in an essential way, and those that are not. Along these lines, Santilli points out first that the *electromagnetic field is not mutated in Eq. (C.66).* This is an important property, for such a field is external and, as such, is expected to be conventional. Explicitly, the four-potentials A_{μ} are the conventional ones, and the associated *iso-electromagnetic field* coincides with the conventional one owing to the properties

$$\hat{F}_{\mu\nu} = \hat{\partial}_{\mu} * A_{\nu} - \hat{\partial}_{\nu} * A_{\mu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} = F_{\mu\nu} \ . \qquad \text{(C.68)}$$

In addition, the isotopic commutators of the π-operators coincide with the conventional commutators owing to the properties

$$[\pi_{\mu} \hat{,} \pi_{\nu}] = \frac{e}{c}\hat{F}_{\mu\nu} \equiv \frac{e}{c}F_{\mu\nu} = [\pi_{\mu}, \pi_{\nu}] \ . \qquad \text{(C.69)}$$

The quantities that are mutated in an intrinsic (non-reducible) way are the $\hat{\gamma}$-matrices, owing to their new structure (C.28). In fact, they can be written explicitly by using Eq. (C.1) and (C.38)

$$\hat{\gamma}_k = \begin{pmatrix} 0 & \tilde{\sigma}_k \\ -\tilde{\sigma}_k & 0 \end{pmatrix} \hat{1} = b_k \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \hat{1}, \hat{\gamma}_4 = \begin{pmatrix} \tilde{I} & 0 \\ & -\tilde{I} \end{pmatrix} = b_4 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \hat{1} ,$$

$$\hat{\gamma}_k = \hat{\gamma}_4 * \hat{\sigma}_k , \hat{\alpha}_k = \begin{pmatrix} 0 & \tilde{\sigma}_k \\ \tilde{\sigma}_k & 0 \end{pmatrix} \hat{1} = b_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \hat{1}, \; k = 1,2,3 , \qquad (C.70)$$

where the σ_k are the conventional Pauli's matrices. For the isometric

$$g = \mathrm{diag}(b_1^2, b_2^2, b_3^2, -b_4^{-2}), \quad b_1 = b_2 = b_3 \stackrel{\text{def}}{=} b > 0 , \qquad (C.71)$$

we can write

$$\hat{\vec{\gamma}} = b \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \hat{1} \quad , \quad \hat{\vec{\alpha}} = b \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \hat{1} . \qquad (C.72)$$

For the Nielsen-Picek generalized metric (3.170), one therefore has the appearance of the term $(1 + \frac{1}{3}\alpha)^{\frac{1}{2}}$ (representing the mutation of the space part of the Minkowski metric) directly in the structure of the $\hat{\vec{\gamma}}$-matrices.

The conventional spin tensor is then lifted into the *isospin tensor* [28]

$$\hat{\vec{\sigma}}_{\mu\nu} = \frac{1}{2}(\hat{\gamma}_\mu * \hat{\gamma}_\nu - \hat{\gamma}_\nu * \hat{\gamma}_\mu) , \qquad (C.73)$$

also in an essential way, as the reader can easily compute eplicitly.

Once the above basic concepts have been understood, the ientification of the mutation of the magnetic and electric dipole moments is quite simple. Consider the isosquare

$$\begin{aligned} &(\hat{\gamma}^\mu * \pi_\mu + i m_0 \hat{c}) * (\hat{\gamma}^\nu * \pi_\nu - i m_0 \hat{c}) \\ &= (\pi^\mu * \pi_\mu + m_0^2 \hat{c}^2) + \frac{e}{2c} \hat{\sigma}^{\mu\nu} * \hat{F}_{\mu\nu} . \end{aligned} \qquad (C.74)$$

This shows that the second order equation corresponding to Eq. (C.30) is Eq. (C.19) plus the term

$$\begin{aligned} &\frac{e}{2c} \hat{\sigma}^{\mu\nu} * \hat{F}_{\mu\nu} = \frac{e}{2c} \hat{\gamma}^\mu * \hat{\gamma}^\nu * \hat{F}_{\mu\nu} \\ &= \frac{e}{2 m_0 c} (\hat{\sigma}^k * H_k = i \hat{\alpha}^k * E_k) , \end{aligned} \qquad (C.75)$$

which is precisely the derived isotopic lifting of the conventional term.

In this way, Santilli [28] reaches the important result of identifying the *mutation of the magnetic moment and of the electric dipole moment* characterized by Eq. (C.66) which, for the general case, are given respectively by

$$\vec{\mu}_m = \frac{e}{2m_0c}\,\vec{\tilde{\sigma}}\,, \qquad \vec{m}_m = i\frac{e}{2m_0c}\,\vec{\tilde{\alpha}}\,, \tag{C.76}$$

and for the particular case of matrices (C.72), can be written

$$\vec{\mu}_m = \frac{e}{2m_0c}b\,\vec{\sigma} = \frac{e}{2m_0c_0}\frac{b}{b_4}\vec{\sigma}\,, \qquad \vec{m}_m = i\frac{e}{2m_0c}b\,\vec{\alpha} = i\frac{e}{2m_0c_0}\frac{b}{b_4}\vec{\alpha}\,, \tag{C.77}$$

where the mutation is manifestly represented by the b value.

This concludes our review of paper [28]. Additional developments on isofield equations require, a detailed study of the isorepresentations of the Poincaré-isotopic group $\hat{P}(3.1)$ (which is essentially lacking at this time), as well as the construction of the iso-Green functions and related solutions, which is a topic more appropriate to the possible subsequent review of hadronic mechanics.

On historical grounds, it should be remarked that, by no means, the hypothesis of the mutation of the magnetic moment is new. In effect it dates back to the early stages of nuclear physics [187] and emerged immediately following the availability of experimental data on total nuclear moments in the 40's. These data, as well known, show a rather sizable departure from the expected total values (which are far from being truly explained to this day). The hypothesis was subsequently abandoned, as soon as it was clear that it implies significant deviations from orthodox lines of inquiry.

Santilli's isofield equation (C.30) with its mutated values (C.77) and related experimental backing [88] offer an intriguing possibility of reinspecting the problem of the total magnetic moments of nuclei on the basis of the hypothesis that the charge distribution of nucleons and related magnetic moments are altered when these particles become members of a nuclear structure.

It is hoped that such an investigation is indeed conducted by interested physicists in the field.

* * * *

A brief review of the original submission of the hypothesis of mutation is recommendable, not only to point out its connection with the preceding results, but also because of the usefulness of the original derivation for contemporary studies, e.g., those of phenomenological hadronization.

The proposal of mutation was submitted via Eq.s (4.20.5), p. 88, ref. [2], i.e.

$$\left\{\left[\begin{pmatrix} -\gamma^\mu\partial_\mu\bar{e} & +m\bar{e} \\ \gamma^\mu\partial_\mu e & +me \end{pmatrix}_{\rm SA} - \begin{pmatrix} f_{\bar{e}} \\ f_e \end{pmatrix}^{\rm Elm}_{\rm SA} - \begin{pmatrix} f_{\bar{e}} \\ f_e \end{pmatrix}\right]\right\}_{\rm NSA} = 0\,, \qquad \text{(C.78)}$$

and is centered in any *variationally nonselfadjoint* (NSA) [4] generalization of the conventional, selfadjoint (SA) Dirac's equation, where: the γ's are the conventional gamma matrices; $f_e^{\rm Elm}$ is the conventional electromagnetic interaction, and the e's ($\bar{e}$'s) are the fields of the eleton (antieleton).

Since variational nonselfadjoint couplings are generally dependent in the velocities [4], Santilli expressed the strong couplings in the form

$$F_e = \Gamma^\mu(x,\dot{x},e,\bar{e},\cdots)\partial_\mu e\,,\ F_{\bar{e}} = \bar{\Gamma}^\mu(x,\dot{x},e,\bar{e},-)\partial_\mu\bar{e}\,, \qquad \text{(C.79)}$$

which allowed to rewrite Eq. (C.78) in the form

$$\left[\begin{pmatrix} -(\gamma^\mu+\bar{\Gamma}^\mu)\partial_\mu\bar{e} & +m\bar{e} \\ (\gamma^\mu+\Gamma^\mu)\partial_\mu e & me \end{pmatrix}_{\rm NSA} - \begin{pmatrix} f_{\bar{e}} \\ f_e \end{pmatrix}_{\rm SA}\right]_{\rm NSA} = 0\,, \qquad \text{(C.80)}$$

the assumption $\Gamma^\mu = f\gamma^\mu$, where f is a function on local coordinates and velocities, allowed to illustrate, apparently for the first time, the following mutation of spin, magnetic and electric moments (Eq.s (4.20.12) and (4.20.16) *loc. cit.*)

$$\Gamma^{\mu\nu} = f(x,\dot{x},\cdots)\sigma^{\mu\nu}\,, \qquad \text{(81.a)}$$

$$\vec{\mu'} = [1+f(x,\dot{x},\cdots)]\frac{e\hbar}{2mc_0}\vec{\sigma}\,, \qquad \text{(81.b)}$$

$$\vec{m'} = i[1+f(x,\dot{x},\cdots)]\frac{e\hbar}{2mc_0}\vec{\alpha}\,. \qquad \text{(81.c)}$$

In ref. [28] Santilli proved that original proposal (C.78) and isotopic form (C.30) are equivalent. In fact, the nonselfadjoint character of Eq. (C.78) implies that the underlying variational principle is *necessarily noncanonical-Birkhoffian* of the type for constant element, $g_{\mu\nu}$

$$\hat{A} = \int_{t_1}^{t_2} \frac{1}{2} \left[\hat{\overline{\psi}} \hat{\gamma}^\mu g_{\mu\nu} \partial^\nu \psi - (\partial^\mu \overline{\psi}) g_{\mu\nu} \gamma^\nu \psi + 2mc_0 \hat{\overline{\psi}} * \psi \right] dt \,, \tag{C.82}$$

where $g = T\eta$, $\hat{\overline{\psi}} * \psi = \hat{\overline{\psi}} T \psi$, and T is precisely the isotopic element of Eq. (C.30), with more complex expressions for non-constant elements $g_{\mu\nu}$. Equivalently, one can readily obtain Eq.s (C.80) from (C.30) by simply assuming $\hat{\gamma}^\mu = \gamma^\mu + \Gamma^\mu(x, \dot{x}, \cdots)$.

The reader can now see the relevance of the original formulation (C.78). In fact, *any generalization of conventional field equations via additive couplings of variational nonselfadjoint type implies a mutation of the intrinsic characteristics of the particle considered* [28]. The physical origin of the mutation is the loss of the canonical character of the underlying variational principle in favor of a more general Birkhoffian one, which is *necessary* under nonselfadjoint couplings within a fixed system of local coordinates. The emerging equations can then be written in the more conventional form (C.78), or in the more geometric, but equivalent form (C.30).

Oddly, a considerable number of models existing in the literature do indeed characterize mutation of the intrinsic characteristics of the represented particles without any acknowledgement of it, evidently because of the lack of the techniques for its quantitative identification. The first category is, in general, that of particles under *nonunitary* time-evolutions, as rather frequent in nuclear physics. These models require the more general Lie-admissible formulations and their mutation is not treated here (see, however, the original presentation [2] which is precisely Lie-admissible in character).

A class of systems more directly in line with the above Lie-isotopic techniques is that of the so-called *phenomenological hadronization models*, such as the generalization of Dirac's equation of ref. [188], i.e.,

$$\left[(i\gamma^\mu \partial_\mu \psi + m\psi)_{\mathrm{SA}} + i \frac{\vec{\gamma} \cdot \vec{x}}{x_4} \psi \right]_{\mathrm{NSA}} = 0 \,. \tag{C.83}$$

In fact, Aringazin [189] has shown that the above equation has precisely the isotopic structure (C.30) with the isotopic element T given by the Gaussian

$$T = N \exp(-\vec{x}^2 / x_4) \,. \tag{C.84}$$

Note also that the statements of "Lorentz noninvariance" caused by the term $\vec{\gamma} \cdot \vec{x}/x_4$ are not technically correct whenever the isotopic element T is positive-definite (Theorem 3.6). Finally, it appears recommendable to study explicitly the mutation of the intrinsic characteristics of the original particle under phenomenological hadronization (C.82), which does not appear done so far.

In summary, Santilli's theory of mutation of elementary particles suggest a reinspection of all available modifications of conventional field equations via additive couplings that violates the integrability conditions for the existence of a canonical action principle in the frame of the experimenter [4], or which possess a manifest isotopic structure, as per model (C.82). The understanding, stressed in ref. [28], is that any, selfadjoint, coupling that is added to conventional equations characterizes no mutation at all.

This reinspection is evidently recommendable to prevent scientific distortions, such as the belief that the original characteristics of particles persist under the generalizations considered, with consequential physical inconsistencies in the results.

The *direct universality of Santilli's Lie-isotopic methods* should be recalled here, to prevent the other illusory hope that nonselfadjoint generalizations of conventional field equations could escape mutations. In fact, *Birkhoffian mechanics has been proved to be directly universal* (Theorem 4.5.1, p. 54, Ref. [4]). Thus, *all* nonselfadjoint generalization of conventional field equations admit a Birkhoffian representation of type (C.82) (under sufficient topological conditions, e.g., analyticity).

The *direct universality of iso-Dirac's equation (C.30) for all conceivable generalizations of the conventional equation under additive nonselfadjoint couplings then follows under an arbitrary functional dependence of the isotopic elements* $b_\mu = b_\mu(\dot{x}, \ldots)$.

* * * *

We now pass to another central topic, Santilli's [24,25,29] characterization of the mutated angular momentum and spin (called *hadronic angular momentum and spin* to stress their applicability only for particles *inside* hadronic matter), achieved via the construction of the isorepresentations of the isotopic $\widehat{SU}(2)$ covering of the $\hat{O}(3)$ symmetries in ref. [23].

Let us begin by defining $\widehat{SU}(2)$ as the infinite family of isotopes of $SU(2)$, along the general lines of §2.4. As well known, $SU(2)$ can be interpreted as a group of unimodular isometries of the invariant in the two-dimensional complex Euclidean space $E(z, \delta\mathbf{R})$ [24,29]

$$\begin{aligned} SU(2) : z^{+}\delta z &= z_i^* \delta_{ij} z = z_1^* z_1 + z_2^* z_2^* \,, \\ \delta &= \mathrm{Diag}(1,1) > 0 \,. \end{aligned} \tag{C.85}$$

We shall therefore define the (infinite) family of $\widehat{SU}(2)$ coverings of $SU(2)$ as the Lie-isotopic groups of isounimodular isometries of the invariant in the isotopic space $\hat{E}(z, g, \hat{\mathbf{R}})$

$$\begin{aligned} \widehat{SU}(2) &: \; z^t g z = z_i^* g_{ij} z_j = z_1^* g_{11} z_1 + z_2^* g_{22} z_2 \,, \\ g^t &= g \,, \quad g = \mathrm{Diag}(g_{11}, g_{22}) > 0 \,, \\ & \quad\; g = g(t, z, z^t, \ldots) \,. \end{aligned} \tag{C.86}$$

From the analysis of Chapters 2 and 3, we therefore know that all isotopes $\widehat{SU}(2)$ are locally isomorphic to the conventional $SU(2)$ group, owing to the preservation by the isometric g of the positive-definiteness of the original metric δ (if this condition is relaxed, we would get also, as part of the family of isotopes $\widehat{SU}(2)$, groups locally isomorphic to $SU(1.1)$, which are not relevant for the physical objectives of this appendix even though mathematically significant).

Santilli realized $\widehat{SU}(2)$ with respect to the generalized unit (isounit)

$$\begin{aligned} \hat{I} &= g^{-1} = \begin{pmatrix} g_{11}^{-1} & 0 \\ 0 & g_{22}^{-1} \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} g_{22} & 0 \\ 0 & g_{11} \end{pmatrix} \,, \\ \Delta &= \mathrm{Det} g = g_{11} g_{22} > 0 \end{aligned} \tag{C.87}$$

with universal enveloping isoassociatve algebra of generators, say, $\hat{J}_k, k = 1, 2, 3,$

$$\hat{\mathcal{A}}(SU(2)) : \hat{J}_i * \hat{J}_j \stackrel{\text{def}}{=} \hat{J}_i g \hat{J}_j \,, \tag{C.88}$$

(where the preservation of the dimension 3 of the group under isotopy has been implemented from Theorem 2.9; similar implementations will be tacitly done hereon).

Algebra $\hat{\mathcal{A}}(SU(2))$ shall act on the isohilbert space with isoinner product

$$\hat{\mathcal{H}}_{\hat{J}} : \langle U \hat{|} V \rangle = \langle U | * | V \rangle = \langle U | g | V \rangle \,, \tag{C.89}$$

where the isotopic element has been assumed equal to that of the envelope to ensure the Hermiticity of the generators under lifting (Sect. 1.3). The underlying field shall be the isocomplex field.

Santilli realized $\widehat{SU}(2)$ as the Lie-isotopic group of isounitary transformations

$$\hat{U} * \hat{U}^{\dagger} = \hat{U}^{\dagger} * \hat{U} = \hat{I} ,$$
$$\hat{U} = e_{\hat{A}}^{i\hat{J}_k\theta_k} = \hat{I}e^{i\theta_k g\hat{J}_k} = e^{i\hat{J}_k g\theta_k}\hat{I} , \tag{C.90}$$

(no sum in the latter exponents), where the θ's are the conventional Euler's angles [21], under the condition of isounimodularity

$$\widehat{\mathrm{Det}}\hat{U} = \hat{I} , \tag{C.91}$$

which can hold iff

$$\mathrm{Det}\hat{U} = 1 , \tag{C.92}$$

i.e., iff

$$\mathrm{Tr}J_k g = 0 , \quad k = 1,2,3 . \tag{C.93}$$

From the local isomorphism between $\widehat{SU}(2)$ and $SU(2)$, we can write the *isocommutation relations* in the form [24,29]

$$\begin{aligned} \left[\hat{J}_i \hat{,} \hat{J}_j\right] &= \hat{J}_i * \hat{J}_j - \hat{J}_j * \hat{J}_i = \hat{J}_i g \hat{J}_j - \hat{J}_j g \hat{J}_i \\ &= i\epsilon_{ijk}\hat{J}_k , \end{aligned} \tag{C.94}$$

which are defined up to redefinitions to be reviewed later on in this appendix.

Santilli [24,29] then constructed the *isorepresentations* of $\widehat{SU}(2)$ for the general case, and then specialized them to the hadronic angular momentum and spin (the reader should be aware that, to our knowledge, this is the first attempt at constructing isorepresentations of Lie-isotopic groups in general, which are known until now only for the regular-fundamental cases).

Isocommutation rules (C.93) imply the following consequences as for the conventional case

$$\begin{aligned} \left[\hat{\vec{J}}^2 \hat{,} \hat{J}_k\right] &= \left[\hat{\vec{J}}^2 \hat{,} \hat{J}_\pm\right] = 0 , \\ \left[\hat{J}_3 \hat{,} \hat{J}_\pm\right] &= \pm\hat{J}_\pm , \\ \left[\hat{J}_+ \hat{,} \hat{J}_-\right] &= 2\hat{J}_3 , \\ \hat{\vec{J}}^2 &= \sum_{k=1}^{3} \hat{J}_k * \hat{J}_k , \\ \hat{J}_\pm &= \hat{J}_1 \pm i\hat{J}_2 . \end{aligned} \tag{C.95}$$

Let $|b^d_k\rangle$, $\quad k = 1, 2, \ldots, d$, be the d-dimensional *isobasis* of $\widehat{SU}(2)$ on $\hat{\mathcal{H}}_j$ with isoorthogonality properties

$$\langle b^d_i| * |b^d_j\rangle = \langle b^d_i|g|b^d_i\rangle = \delta_{ij} \ , \quad i, j = 1, 2, \ldots, \eta \ . \tag{C.96}$$

Note that, if $|\pm\rangle$ are the conventional basis for the 2-dimensional representation of $SU(2)$, then the above isobasis are given by

$$|b^2_1\rangle = g_{11}^{-1/2}|+\rangle \ , \ |b^2_2\rangle = g_{22}^{-1/2}|-\rangle \ . \tag{C.97}$$

If $|+\rangle$, $|0\rangle$, $|-\rangle$ are the conventional basis of the three-dimensional representation of $SU(2)$, then, from Eq. (C.89) the isobasis is given by

$$|b^3_1\rangle = g_{11}^{-1/2}|+\rangle \ , |b^3_2\rangle = g_{22}^{-1/2}|0\rangle \ , |b^3_3\rangle = g_{33}^{-1/2}|-\rangle \ , \tag{C.98}$$

and similarly for the other cases.

As in the conventional case, from Eq.s (C.94), we can diagonalize $\hat{\vec{J}}^2$ and $\hat{J}_3$. We can therefore assume the existence of the following *isoeigenvalues*

$$\hat{J}_3 * |b^d_k\rangle \stackrel{\text{def}}{=} \hat{J}_3 g|b^d_k\rangle = b^d_k|b^d_k\rangle \ , \tag{C.99}$$

where the b's must be necessarily real under the conditions assumed. Then, from Eq.s (C.94) we have, as in the conventional case,

$$\begin{aligned}
\hat{J}_\pm * |b^d_k\rangle &= |b^d_{k\pm1}\rangle \ , \\
\hat{J}_3 * |b^d_{k\pm1}\rangle &= b^d_{k\pm1}|b^d_{k\pm1}\rangle \ , \\
= \hat{J}_3 * \hat{J}_\pm * |b^d_k\rangle &= (\hat{J}_\pm * \hat{J}_3 \pm \hat{J}_\pm) * |b^d_k\rangle \ , \\
= (b^d_k \hat{J}_\pm \pm \hat{J}_\pm) * |b^d_k\rangle &= (b^d_k \pm 1)|b^d_k\rangle \ .
\end{aligned} \tag{C.100}$$

Thus,

$$b^d_{k\pm1} = b^d_k \pm 1 \ . \tag{C.101}$$

Since $\hat{J}_+ * |b^d_d\rangle = 0$, we have

$$\begin{aligned}
\hat{\vec{J}}^2 * |b^d_d\rangle &= (\hat{J}_- * \hat{J}_+ + \hat{J}_3 * \hat{J}_3 + \hat{J}_3) * |b^d_d\rangle \\
&= b^d_d(b^d_d + 1)|b^d_d\rangle \ ,
\end{aligned} \tag{C.102}$$

and similarly

$$\begin{aligned}
\hat{\vec{J}}^2 * |b^d_1\rangle &= (\hat{J}_+ * \hat{J}_- + \hat{J}_3 * \hat{J}_3 - \hat{J}_3) * |b^d_1\rangle \\
&= b^d_1(b^d_1 - 1) \ ,
\end{aligned} \tag{C.103}$$

from which

$$
\begin{aligned}
K &= b_d^d(b_d^d+1) \equiv b_1^d(b_1^d-1)\ , \\
b_1^d &= -b_d^d\ .
\end{aligned}
\tag{C.104}
$$

In this way Santilli shows that, as in the conventional case, the dimension of the isorepresentation is characterized by the familiar rule

$$
n = 2j+1\ , \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\ . \tag{C.105}
$$

The explicit form of the matrix elements was computed by lifting the conventional case. In fact, we can write for $\hat{J}_-$

$$
\begin{aligned}
&\hat{J}_- * |b_k^d\rangle = \alpha_k |b_k^d\ , \\
\alpha_k^2 &= \langle b_k^d| * \hat{J}_+ * \hat{J}_- * |b_k^d\rangle \\
&= \langle b_k^d| * (\hat{\vec{J}}^{\hat{2}} - \hat{J}_3^{\hat{2}} + \hat{J}_3) * |b_k^d\rangle \\
&= k - (b_k^d)^2 + b_k^d \stackrel{\text{def}}{=} 2b_k \quad \text{for } \eta = 2, \text{etc.}\ ,
\end{aligned}
\tag{C.106}
$$

and for $\hat{J}_+$

$$
\begin{aligned}
\hat{J}_+ * |b_k^d\rangle &= \beta_k |b_k^d\rangle \\
\beta_k^2 = \langle b_k^d| * \hat{J}_- * \hat{J}_+ * |b_k^d\rangle &= \langle b_k^d| * (\hat{\vec{J}}^{\hat{2}} - \hat{J}_3^{\hat{2}} - \hat{J}_3) * |b_k^d\rangle \\
= k - (b_k^d)^2 - b_k &\stackrel{\text{def}}{=} -2b_k \quad \text{for } \eta = 2\text{etc.}\ .
\end{aligned}
\tag{C.107}
$$

The matrix elements are then given by [24,29]

$$
\begin{aligned}
(\hat{J}_3)_{ij} &= \langle b_i^d| * \hat{J}_3 * |b_j^d\rangle\ , \\
(\hat{J}_2)_{ij} &= \frac{1}{2}\langle b_i^d| * (\hat{J}_+ + \hat{J}_-) * |b_j^d\rangle\ , \\
(\hat{J}_1)_{ij} &= \frac{i}{2}\langle b_i^d| * (\hat{J}_- - \hat{J}_+) * |b_d^k\rangle\ ,
\end{aligned}
\tag{C.108}
$$

under the subsidiary conditions (C.91) or (C.92), i.e.,

$$
\det(\hat{J}_k g) = 1\ , \quad \mathrm{Tr}(\hat{J}_k g) = 0\ , \quad k = 1,2,3\ , \tag{C.109}
$$

which essentially imply the identification of the b_k quantities with the elements g_{kk}.

For the case of the fundamental 2-dimensional representation, the above results permitted Santilli to construct the following isorepresentation [24,29]

$$
\begin{aligned}
\hat{J}_1 &= \frac{1}{2\Delta^{1/2}}\begin{pmatrix} 0 & g_{11}^{1/2} \\ g_{22}^{1/2} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & g_{22}^{-1/2} \\ g_{11}^{-1/2} & 0 \end{pmatrix} , \\
\hat{J}_2 &= \frac{1}{2\Delta^{1/2}}\begin{pmatrix} 0 & -ig_{11}^{1/2} \\ ig_{22}^{1/2} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -ig_{22}^{-1/2} \\ ig_{11}^{-1/2} & 0 \end{pmatrix} , \\
\hat{J}_3 &= \frac{1}{2\Delta^{1/2}}\begin{pmatrix} 22 & 0 \\ 0 & -g_{11} \end{pmatrix} = \frac{\Delta^{1/2}}{2}\begin{pmatrix} g_{11}^{-1} & 0 \\ 0 & -g_{22}^{-1} \end{pmatrix} . \qquad \text{(C.110)}
\end{aligned}
$$

The isoeigenvalues of $\hat{\vec{J}}^2$ and $\hat{J}_3$ are then given by [*loc. cit.*]

$$
\begin{aligned}
\hat{\vec{J}}^2 * |b_k^2\rangle &= \frac{\Delta^{1/2}}{2}\left(\frac{\Delta^{1/2}}{2}+1\right)|b_k^2\rangle , \\
\hat{J}_3 * |b_k^2\rangle &= \pm\frac{\Delta^{1/2}}{2}|b_k^2\rangle , \quad k = 1,2 , \qquad \text{(C.111)}
\end{aligned}
$$

and they coincide with the isoexpectation values under the conditions assumed, i.e.,

$$
\begin{aligned}
\widehat{\langle \vec{J}^{\hat{2}} \rangle} &= \frac{\Delta^{1/2}}{2}\left(\frac{\Delta^{1/2}}{2}+1\right) , \\
\widehat{\langle \hat{J}_3 \rangle} &= \pm\frac{\Delta^{1/2}}{2} , \quad \hbar = 1 . \qquad \text{(C.112)}
\end{aligned}
$$

We can therefore conclude by saying that *Santilli's liftings $\hat{SU}(2)$ of the $SU(2)$-spin symmetry imply the following mutation of the conventional eigenvalues*

$$
j \rightarrow \Delta^{1/2} j , \qquad \text{(C.113)}
$$
$$
j = 0, 1/2, 1, 3/2, \ldots ; \qquad \Delta = \text{Det}.g = g_{11}g_{22}\cdots g_{\eta\eta} \ ; \ \eta - 2j + 1 .
$$

As a result, *the $\widehat{SU}(2)$ symmetry is suitable for the characterization of the hadronic angular momentum and spin* [*loc. cit*].

Notice the lack of reducibility of representation (C.110) to the conventional Pauli's matrices

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \text{(C.114)}
$$

that is, the lack of existence of equivalence transformations of the type

$$\hat{J}_k = \frac{1}{2} A \sigma_k A^{-1} \tag{C.115}$$

under which the $SU(2)$ algebra of Pauli's matrices remains conventional, i.e., with the trivial Lie product $AB - BA$, and this illustrates the mathematical nontriviality of the representations of $\widehat{SU}(2)$ introduced in this section. The nontriviality of the physical implications will be illustrated shortly in this paper via the application of the theory to Rutherford's historical hypothesis.

The degrees of freedom of the hadronic $\widehat{SU}(2)$-spin should also be reviewed for completeness, because they are considerably broader than those of the conventional theory.

The $\widehat{SU}(2)$ symmetry was constructed by Santilli under the specific condition of preserving the structure constants of $SU(2)$, via isocommutation rules (C.93). This was done for the evident purpose of stressing the isomorphism of $\widehat{SU}(2)$ and $SU(2)$.

However, the general formulation of the Lie-isotopic theory, as originally proposed in memoir [1] (see Chapter 2) requires the generalization of the conventional structure constants into the *structure functions*.

We can easily generalize the preceding realization of $\widehat{SU}(2)$ into the following form [24,29]

$$[\hat{J}'_1 \hat{,} \hat{J}'_2] = i\hat{J}'_3 , \quad [\hat{J}'_2 \hat{,} \hat{J}'_3] = i\Delta \hat{J}_1 , \quad [\hat{J}'_3 \hat{,} \hat{J}'_1] = i\Delta \hat{J}'_2 , \tag{C.116}$$

where the generally nonlinear and arbitrary dependence of the determinant Δ on all the local variables illustrates the structure functions of the Lie-isotopic theory.

A repetition of the representation theory previously given then yields, after tedious but simple calculations, the following matrix forms

$$\begin{aligned} \hat{J}'_1 &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \frac{1}{2}\sigma_1 , \quad \hat{J}'_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \equiv \frac{1}{2}\sigma_2 , \\ \hat{J}'_3 &= \frac{1}{2}\begin{pmatrix} g_{22} & 0 \\ 0 & g_{11} \end{pmatrix} = \frac{\Delta}{2}\hat{I}\sigma_3 . \end{aligned} \tag{C.117}$$

The isoeigenvalue equations are then given by

$$\begin{aligned} \hat{J}'_3 * |b^2_k\rangle &= \pm\frac{\Delta}{2}|b^2_k\rangle \, , , = 1, 2 , \\ \vec{\hat{J}}^2 * |b^2_k\rangle &= \frac{\Delta}{2}\left(\frac{\Delta}{2} + 1\right)|b^2_k\rangle , \end{aligned} \tag{C.118}$$

and, again, we have the preservation of the eigenvalues as the expectation values, under the assumptions considered.

One can notice the additional mutation in the transition from Eq.s (C.111) to (C.118)

$$\frac{\Delta^{1/2}}{2} j \to \frac{\Delta}{2} j \;, \tag{C.119}$$

which is nothing but an isotopic lifting of an isotopic symmetry [24,29].

To understand the occurrence from the geometric-algebraic viewpoint, one can recall the results of §3.4 in the construction of the $\widehat{SO}(3.1)$ covering of the $SO(3.1)$ symmetry. As one can see, $\widehat{SO}(3.1)$ can be constructed in one single step as an isotopy of $SO(4)$ or, equivalently, in two steps, the first by constructing $SO(3.1)$ as an isotopy of $SO(4)$ and then $\widehat{SO}(3.1)$ as an isotopy of $SO(3.1)$.

Along similar lines, one should keep in mind the multiplicity infinite nature of all possible $\widehat{SU}(2)$ coverings of $SU(2)$, in the sense that, besides the infinite number of mutations (C.85) of the original metric (C.84), with consequential infinite numbers of $\widehat{SU}(2)$, one has in addition an infinite number of isotopies $\widehat{SU}'(2)$ of $\widehat{SU}(2)$ provided by additional mutations of the mutated metric (C.89). For illustrations regarding the different nature of the nonlinear isorotations for different isometrics, the reader is recommended to consult ref. [23].

A simple way of constructing the fundamental isorepresentations is via an isotopic lifting of the conventional technique based on the use of creation and annihilation operators. For brevity, we refer the reader to the original derivation [24] (where one can find numerous additional developments of the isotopic $\widehat{SU}(2)$ spin symmetries, such as the *iso-Clebsch-Gordon coefficients*, the *iso-Wigner coefficients*, and other topics).

Here, we limit ourselves to mentioning that the use of the isocreation and isoannihilation operators permits the reconstruction of the preceding 2×2 isorepresentations, plus the following one [24]

$$\begin{aligned} \hat{J}_1 &= \frac{1}{2}\begin{pmatrix} 0 & g_{11} \\ g_{22} & \end{pmatrix} , \quad \hat{J}_2 = \begin{pmatrix} 0 & -ig_{11} \\ ig_{22} & 0 \end{pmatrix} , \\ \hat{J}_3 &= \frac{1}{2}\begin{pmatrix} g_{11}g_{22}^2 & 0 \\ 0 & -g_{22}g_{11}^2 \end{pmatrix} , \end{aligned} \tag{C.120}$$

which verifies isocommutation rules (C.93) and isoeigenvalues (C.118), as one can verify.

Needless to say, fundamental isorepresentations (C.110), (C.117) and (C.120) are all isoequivalent.

* * * *

We now pass to the review of the last pioneering articles written by P.A.M. Dirac [54a,54b]. These articles were written in 1971 and 1972, respectively, but have remained largely ignored since that time, apparently because of their manifest lack of alignment with established doctrines of contemporary physics. In the following we shall first review the essential aspects of the articles in a way as close as possible to their original presentation (including the use of the original notation). We shall then point out the intrinsic isotopic character of the new equation which, as such, results to be incompatible with conventional quantum mechanics (because it breaks the linearity condition), while being a clear realization of the covering hadronic mechanics.

Consider two harmonic oscillators in one dimension with dynamical variables

$$\begin{aligned}&(q) = (q_a) = (q_1, p_1; q_2, p_2), a = 1,2,3,4 ,\\ &[q_a, q_b] = q_a q_b - q_b q_a = i\beta_{ab} ,\end{aligned} \tag{C.121}$$

where

$$\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} , \quad \beta^T = -\beta , \quad \beta^2 = -1 , \quad \beta^{-1} = \beta^T . \tag{C.122}$$

The generalized form of the conventional Dirac's equations proposed by Dirac himself at the very beginning of paper [54a] (Eq. 1.3) is given by

$$\left(\frac{\partial}{\partial x_0} + \alpha_r \frac{\partial}{\partial x_r} + \beta\right) q\psi = 0 , \tag{C.123}$$

where ψ is a scalar (one-dimensional) wavefunction with the dependence $\psi = \psi(x, q)$, the x's are the space-time coordinates of a (conventional) Minkowski space, q is a column matrix with the four elements (C.121), the α_r are 4×4 matrices that anticommute with each other and with β, and the product is the conventional associative product. One of the various possible realizations of the α_r presented by Dirac is given by

$$\alpha_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{C.124}$$

Assume $\alpha_0 = 1$ and $\partial_\mu = \partial/\partial x^\mu$. Then Eq. (C.123) was rewritten by Dirac in the form

$$(\alpha_\mu \partial^\mu + \beta) q\psi = 0. \tag{C.125}$$

The second-order equation characterized by the above form was worked out via the formulae

$$\begin{aligned} &P_a = (\alpha_\mu \partial^\mu + \beta)_{ab} q_b; \quad P_a \psi = 0 \ , \\ &[P_a, P_c] = i(\alpha_\mu \partial^\mu + \beta)_{ab} \beta_{bd} (\alpha_\nu \partial^\nu - \beta)_{cd} \ , \\ &= i\{(\alpha_\mu \partial^\mu + \beta)\beta(\alpha_\nu \partial^\nu - \beta)\}_{ac} \ , \end{aligned} \tag{C.126}$$

and resulted in the equation (Eq. 2.8, *loc. cit.*)

$$(\partial_\mu \partial^\mu + 1)\beta\psi = 0. \tag{C.127}$$

This allowed Dirac to prove the mathematical consistency of Eq. (C.125).

Next, Dirac identified the main law of the α matrices which resulted to be of the form

$$\alpha_\mu \beta \alpha_\nu + \alpha_\nu \beta \alpha_\mu = 2\beta \eta_{\mu\nu} \ , \tag{C.128}$$

where $\eta_{\mu\nu}$ is the conventional Minkowski metric. The *conventional* Lorentz covariance of the equation was proved via the infinitesimal transformations

$$x^*_\mu = x_\mu + a_\mu{}^\nu x_\nu \ , \tag{C.129}$$

which resulted in the transformed equation

$$\{(\alpha_\mu + \alpha_\mu{}^\nu \alpha_\nu)\partial^{\nu *} + \beta\} q\psi = 0. \tag{C.130}$$

By putting

$$N = \frac{1}{4} a^{\rho\sigma} \alpha_\rho \beta \alpha_\sigma \ , \tag{C.131}$$

Eq. (C.130) can be rewritten

$$\{\alpha_\mu (1 - \beta N)\partial^\mu + \beta(1 + N)\} q\psi = 0 \ , \tag{C.132}$$

thus resulting in the final form

$$(\alpha_\mu \partial^\mu + \beta) q^* \beta = 0 \,,$$
$$q^* = (1 - \beta N) q. \tag{C.133}$$

Dirac (*loc.cit.*) then passes to prove that his new equation (C.123) or (C.125) has only *positive energy* (normalizable) solutions. Assume $i\partial_\mu = p_\mu$. The equation can then be written

$$\{(p_0 - p_3)q_1 + (i + p_1)q_3 - p_2 q_4\}\psi = 0,$$
$$\{(p_0 - p_3)q_2 - p_2 q_3 + (i - p_1)q_4\}\psi = 0,$$
$$\{(p_0 + p_3)q_3 + (p_1 - i)q_1 - p_2 q_2\}\psi = 0,$$
$$\{(p_0 + p_3)q_4 - p_2 q_1 - (p_1 + i)q_2\}\psi = 0. \tag{C.134}$$

By applying a Lorentz transformation to the rest frame with $\vec{p} = 0$, the above equations show that p_0 can be either $+1$ or -1. Of these two possibilities only the first is normalizable because the underlying wave function has the form

$$\psi = k \exp\{-\frac{1}{2}[q_1^2 + q_2^2 + ip_1(q_1^2 - q_2^2) - 2ip_2 q_1 q_2]/(p_0 + p_3) \exp(-ip^\mu x_\mu)\} \,. \tag{C.135}$$

This established the first significant difference between the new equation and the conventional one (for which, as the reader knows well, both positive and negative energies are admitted).

But by far the biggest differences emerged for the spin. Consider the familiar total angular momentum

$$M_{\rho\sigma} = x_\rho \partial_\sigma - x_\sigma \partial_\rho - i s_{\rho\sigma}. \tag{C.136}$$

To identify the explicit form of $s_{\rho\sigma}$, Dirac considered the equation

$$[(\alpha_\mu \partial^\mu + \beta)q, a^{\rho\sigma}(x_\rho \partial_\sigma - x_\sigma \partial_\rho 2a^{\mu\sigma} \alpha_\mu \partial_\sigma q] =$$
$$[(\alpha_\mu \partial^\mu + \beta)q, w] = 2i(\alpha_\mu \partial^\mu + \beta)\beta N q, \tag{C.137}$$

which can be written

$$[(\alpha_\mu \partial^\mu + \beta)q, a^\rho(x_\rho \partial_\sigma - x_\sigma \partial_\rho) + iW] =$$
$$= -2N\beta(\alpha_\mu \partial^\mu + \beta)q \,,$$
$$a^{\rho\sigma} s_{\rho\sigma} = -W = -q^T N q = -\frac{1}{4} a^{\rho\sigma} q^T \alpha_\rho \beta \alpha_\sigma q \,. \tag{C.138}$$

As a consequence

$$s_{\rho\sigma} = -\frac{1}{8} q^T (\alpha_\rho \beta \alpha_\sigma - \alpha_\sigma \beta \alpha_\rho) q \,, \tag{C.139}$$

which, via Eq.s (C.128), becomes

$$s_{\rho\sigma} = -\frac{1}{4} q^T \alpha_\rho \beta \alpha_\sigma q + \frac{1}{4} g_{\rho\sigma} q^T \beta q = -\frac{1}{4} q^T \alpha_\rho \beta \alpha_\sigma q + \frac{1}{2} i \eta_{\rho\sigma}. \tag{C.140}$$

For $\rho, \sigma = 1, 2, 3,$, $M_{\rho\sigma}$ can be interpreted as the angular momentum, while $s_{\rho\sigma}$ is the spin. By using expression (C.124) for the $\alpha's$, the spin components can be computed explicitly,

$$s_{23} = \frac{1}{2}(q_1 q_2 + q_3 q_4) \,, \quad s_{31} = \frac{1}{4}({q_1}^2 - {q_2}^2 + {q_3}^2 - {q_4}^2) \,,$$

$$s_{12} = \frac{1}{2}(q_2 q_3 - q_1 q_4) \,, \tag{C.141}$$

and also

$$s_{01} = \frac{1}{4}(q_1^2 q_2^2 - q_3^2 + q_4^2) \,, \quad s_{02} = \frac{1}{2}(q_3 q_4 - q_1 q_2) \,,$$

$$s_{03} = \frac{1}{2}(q_1 q_3 + q_4 q_2) \,. \tag{C.142}$$

As a consequence,

$$\vec{s}^{\,2} = s_{23}^2 + s_{31}^2 + s_{12}^2 = \frac{1}{16}(q_1^2 + q_2^2 + q_3^2 + q_4^2)^2 - \frac{1}{4} = s(s+1) \,, \tag{C.143}$$

and, finally,

$$s = \frac{1}{4}(q_1^2 + q_2^2 + q_3^2 + q_4^2) - \frac{1}{2} = \frac{1}{2}(n + n') \,,$$
$$n, n' = 0, 1, 2, 3, \ldots \tag{C.144}$$

In this way, Dirac (*loc. cit*) reached the remarkable conclusion that *modification (C.123) of his celebrated equation can have only even values of spin beginning with the zero value.*

The six quantities $s_{\rho\sigma}$ provide a representation of the Lorentz group. By introducing the additional four quantities

$$s_{\mu 5} = -s_{5\mu} = \frac{1}{4} q^T \alpha_\mu q \,, \tag{C.145}$$

the ten quantities $s_{ab} = s_{ba}, a.b. = 1, 2, 3, 4, 5$, provide a representation of the $(3+2)$-dimensional De Sitter Group.

Dirac then passed to the identification of the four-current

$$J_\mu = \int \psi^\dagger q^T \alpha_\mu q \psi d^2 q \ , \tag{C.146}$$

which verifies the usual conservation law

$$\partial^\mu J_\mu = 0 \ , \tag{C.147}$$

and transforms correctly under the (conventional) Lorentz transformations

$$J'_\mu = \int \psi^\dagger q^T (\alpha_\mu + a_\mu{}^\sigma \alpha_\sigma) q \psi d^2 q = J_\mu + a_\mu{}^\sigma J_\sigma . \tag{C.148}$$

The charge density

$$J_0 = \int \psi^\dagger q^T q \psi d^2 q \tag{C.149}$$

is positive definite while the underlying wavefunction is normalized to unity according to the rule

$$\int \psi^\dagger q^T q \psi d^3 x = 1. \tag{C.150}$$

Generalization (C.148) of the charge density of the conventional Dirac's equation shows another departure from orthodox values. In fact, value (C.148) is manifestly different than the corresponding value for the conventional equation.

Dirac concluded paper [54a] with the calculation of the Fock representation of his new equation (which is not reviewed here for brevity) as well as with the warning that *any extension of Eq. (C.123) to include interactions is expected to result in inconsistencies. In fact, he showed that, by replacing p_μ with the familiar form $p_\mu + eA_\mu$, the equation is inconsistent except for the case $A_\mu = \partial S / X^\mu$ which means the absence of field.* (See below for comments.)

The subsequent paper [54b] was primarily devoted to the physical interpretation of the new equation. It turns out that the theory describes a collection of random circles covered by the point x on a sphere. This results in a random motion on said sphere. In particular, its radius is not constant but pulsates in time within given boundaries.

Paper [54b] then concludes with the proof that *the particle (in its ground state) has a zero spin for all possible values of the linear momentum.*

* * * * *

The reinterpretation of Dirac's pioneering paper [54a,54b] within the context of the Lie-isotopic theory (Santilli [28]) is quite instructive. In short, *Dirac's new equation (C.123) is characterized by an enveloping associative algebra with an essential isotopic structure. The underlying Hilbert space is the conventional Hilbert space without any isotopic generalization (see below for the consistency of such additional lifting). As a consequence, Dirac's new equation constitutes an intriguing realization of hadronic mechanics according to structure (1.46). The Minkowski space is kept unchanged in Eq. (C.123). As a consequence, we have no isotopic lifting of the Lorentz symmetry (although, again, a reformulation of the theory that shows a lifting also of the Lorentz symmetry is possible). Finally, Dirac's new equation provides an intriguing realization of Santilli's notion of mutation of the original conventional equation and related particle [2].*

The above results can be easily seen [28]. First, the isotopic element of the isoenvelope is *not* q (trivially, because this quantity is a column matrix), but β. Thus, *all* associative products must be formulated in the isotopic form, say

$$\hat{\xi} : A * B \stackrel{\text{def}}{=} A\beta B \ , \quad \hat{1} = \beta^{-1} \ . \tag{C.151}$$

It is easy to see that Dirac's new equation (C.123) does indeed verify this fundamental requirement at all levels.

Introduce the four-component column wavefunction $\phi = q\psi$ (as in the conventional equation). Then, by recalling that $\beta^{-1} = \beta^T = -\beta$, Eq. (C.123) can be readily written in the isotopic form

$$(\alpha_\mu \partial^\mu + \beta) q\psi = (\hat{\alpha}_\mu * \hat{\partial}^\mu + 1) * \phi = 0 \ ,$$
$$\hat{\alpha}_\mu = \alpha_\mu \hat{1}, \quad \hat{\partial}^\mu = \hat{\partial}^\mu \hat{1} \ , \tag{C.152}$$

which, in particular, verifies hadronization rule (C.6) (but not the conventional quantization rule).

To identify the properties of the $\hat{\alpha}$ matrices, we consider the expression for the characterization of the second-order equation (C.127)

$$\begin{aligned}&(\hat{\alpha}_\mu * \hat{\partial}^\mu + 1) * (\hat{\alpha}_\nu * \partial^\nu - 1)\\ &= \frac{1}{2}\{\hat{\alpha}_\mu \hat{,} \hat{\alpha}_\nu\} * \hat{\partial}^\mu * \hat{\partial}^\nu - \beta\\ &= -(\partial_\mu \partial^\mu + 1)\beta \ ,\end{aligned} \tag{C.153}$$

which holds iff

$$\{\alpha_\mu \hat{,} \alpha_\nu\} = \alpha_\mu * \alpha_\nu + \alpha_\nu * \alpha_\mu = -2\eta_{\mu\nu}\hat{1} \ . \tag{C.154}$$

Note that the above equations coincide with Eq. (C.128) because of the property $\beta = -\hat{1}$.

Next, since all operators belong to isoenvelope $\hat{\xi}$, so must be the case also of the angular momentum and spin. Again, it is easy to see that quantity (C.135) can be readily rewritten in the isotopic form

$$M_{\rho\sigma} = x_\rho \partial_\sigma - x_\sigma \partial_\rho - i s_{\rho\sigma} \equiv x_\rho * \hat{\partial}_\sigma - x_\sigma * \hat{\partial}_\rho - i\hat{S}_{\rho\sigma} , \qquad \text{(C.155)}$$

while the spin is in full isotopic form already as written by Dirac. In fact, Eq. (C.139) can be written

$$s_{\rho\sigma} \equiv \hat{S}_{\rho\sigma} = -\frac{1}{4}\overline{\alpha}_\rho * \overline{\alpha}_\sigma + \frac{i}{2}\eta_{\rho\sigma} , \quad \overline{\alpha}_\rho = \alpha_\rho q. \qquad \text{(C.156)}$$

The underlying Hilbert space can be equipped with the *conventional* inner product

$$\mathcal{H} : \langle\phi|\phi\rangle = \int \phi^\dagger \phi d^3x = 1 , \qquad \text{(C.157)}$$

in a way fully compatible with isoenvelope $\hat{\xi}$ (see Sect.1.3 and ref.s [36-38]). In this case, the current is given by Dirac's expression (C.146) without need of any reformulation. The same happens for other calculations based on Hilbert space formulations.

The reader should recall that, for structure (1.46) the definition of conventional and iso-Hermiticity are different. These definitions can be made to coincide with an isotopic lifting of the Hilbert space with the same isotopic element β, as in structure (1.52) with $T = G$. This reformulation of Dirac's new theory can be also readily achieved by introducing the new wavefunction

$$\hat{\phi} = \beta^{-\frac{1}{2}}\phi , \qquad \text{(C.158)}$$

under which the conventional inner product (C.157) can be reformulated into the isotopic form

$$\hat{\mathcal{H}} : \langle\phi|\phi\rangle = \int \phi^\dagger \phi d_x^3 = 1 \rightarrow \hat{\mathcal{H}} : \langle\hat{\phi}\hat{|}\hat{\phi}\rangle = \int \hat{\phi}^\dagger * \hat{\phi} d^3x = \hat{1} . \qquad \text{(C.159)}$$

The four-current (C.146) must be in this case rewritten in the different form

$$\hat{J}_\mu = \int \hat{\phi}^\dagger * \alpha_\mu * \phi d^2q . \qquad \text{(C.160)}$$

It is easy to see that the above four-current verifies too all essential requirements of J_μ and it is therefore a fully acceptable expression. In this way, Dirac's new theory can be extended to admit the same notion of Hermiticity as the conventional one (with consequential preservation of the reality of the eigenvalues).

Finally, the covariance under Lorentz-isotopic transformations can be readily reached via the trivial isotopy

$$x'_\mu = x_\mu + \hat{a}_\mu{}^\nu * x_\nu \ , \quad \hat{a}_\mu{}^\nu = a_\mu{}^\nu \hat{1} \ , \tag{C.161}$$

as the reader is encouraged to verify. It should be stressed that, in the theory under consideration here, the trivial lifting $a_\mu{}^\nu \rightarrow a_\mu{}^\nu \hat{1}$ is necessary because Dirac formulated the theory in a *conventional* Minkowski space without any deformation of the space-time metric. On the contrary, such a trivial isotopy of the Lorentz group would be inconsistent for the content of Section 3.4 owing to the necessary presence there of a nontrivial modification of the Minkowski metric (see the comments following Eq. (2.163)).

Notice also that, as formulated by Dirac, isofield equations (C.123) is a field theory with two units, the generalizaed one $\hat{1} = -\beta$ for the equations themselves, and the trivial unit 1 for the conventional Lorentz group on a conventional Minkowski space.

It should be indicated here, as stressed in ref. [28], that the full study of Dirac's generalized equation (C.123) will eventually require a lifting of all various aspects, beginning with he formulation of the theory in the isotopic space $\hat{M}_{II}(x, \beta, \hat{\mathbf{R}})$ (§3.4)

$$x^{\hat{r}} = x^\mu \beta_{\mu\nu} x^\nu = x^1 x^3 - x^2 x^4 - x^3 x^1 - x^4 x^2 \ . \tag{C.162}$$

The Lorentz symmetry $O(3.1)$ of the conventional Dirac's equation is then replaced by the Lorentz-isotopic symmetry $\hat{O}(3.1)$ characterized by isounit $\hat{Z}1 = -\beta$, which results to be isomorphic to the isotopic dual $O^d(3.1)$ (Definition 3.1) of $O(3.1)$ [28].

We are now in a position to comment on the extension of Eq. (C.123) to include interactions. Dirac [54] concluded that no interaction could be added to his generalized equation because he was treating them in the conventional associative envelope of the original equation, thus resulting in a number of inconsistencies (violoation of linearity, etc.).

However, when Eq. (C.123) is written in its correct mathematical form, the isotopic form (C.152), then interactions can be readily included as in Eq. (C.65), provided that they are treated in the isoassociative form [28].

As a matter of fact, the resolution of the difficulties for the inclusions of interacting terms is the best way to see the essential character of the isotopic structure of Dirac's generalized equation (C.123).

Finally, *the isotopic lifting from the conventional to the new Dirac's equation*

$$(\alpha_\mu \partial^\mu + 1)\phi = 0 \to (\hat{\alpha}_\mu * \hat{\partial}^\mu + 1) * \phi = 0 \,, \tag{C.163}$$

provides a beautiful illustration of Santilli's hypothesis of the mutation of spin (see, ref.s [2,11]). More specifically, lifting (C.162) illustrates the possibility that the ordinary electron can be subjected to a mutation into a particle with spin zero when passing from motion in empty space to full immersion within dense hadronic matter.

Lifting (C.162) and underlying mutation of the electron, play a fundamental role in the studies by Santilli [29] on the apparent consistency of the original Rutherford's hypothesis on the structure of the neutron (as a bound state of one proton and one electron), as outlined below.

The reader should be aware of the implications of these findings. The construction of hadronic mechanics was suggested for the specific purpose of achieving a quantitative treatment of the mutation of particles, so that the constituents of hadrons (or of quarks) can be consistently given by massive, already known particles. In turn this illustrates the profound physical implications of Santilli's Lie-isotopic theory under review in this work. The alternative of preserving conventional doctrines is well known: new hypothetical particles must be invented again to be the constituents of quarks. The possibility that particles experience an alteration of their physical characteristics when in conditions of total immersion within hadronic matter is manifestly more plausible and positively preferable to the invention of a second generation of unknown hypothetical particles. In the final analysis, Rauch's experiment on the spinor symmetry of neutrons under external fields (see Fig. 6 and ref. [88]) appears to confirm the mutation of the magnetic moment of the neutron in its current form, thus providing the possible experimental foundations to the notion of mutation.

These issues are the central objectives of a review of hadronic mechanics we hope to conduct at some future time.

* * * *

We now pass to the review of paper [168] by G. Karayannis and A. Jannussis on one of the first formulations of the isotopic lifting of field equations that appeared in the literature, that achieved via the lifting of conventional Lagrangian densities. The quoted article is important for this review because it establishes a direct link between Gasperini's isotopic gauge theory

(Appendix A) and the isotopic field equations. For additional papers by the same authors see ref.s [169-171].

For notational convenience, let us review first some essential aspects of Gasperini's theory from Appendix A. Let G be a compact gauge group. Its isotope $\hat{G}$ is characterized by the transformations

$$\Psi' = \hat{U} * \Psi \ , \tag{C.164}$$

where

$$\hat{U} = \hat{I} e^{i\Theta^{\kappa} * J_{\kappa}} = e^{-J_{\kappa} * \Theta^{\kappa}} \hat{I} \ . \tag{C.165}$$

Since $\hat{U}$ is a T-unitary operator, i.e.

$$U^{\hat{\dagger}} * \hat{U} = \hat{I} = \hat{U} * \hat{U}^{\hat{\dagger}}, \tag{C.166}$$

where

$$U^{\hat{\dagger}} \equiv T^{\dagger} U^{\dagger} \hat{I}, \tag{C.167}$$

the basic invariant is given by

$$\Psi^{\dagger} * \Psi = \Psi'^{\dagger} * \Psi'. \tag{C.168}$$

The isotopic Yang-Mills field strengths $\hat{F}_{\mu\nu}$ are defined as follows:

$$\begin{aligned} \hat{F}_{\mu\nu} * \Psi &= \frac{i}{g}[\hat{D}_{\mu} \hat{,} \hat{D}_{\nu}] * \Psi \\ &= \frac{i}{g}(\hat{D}_{\mu} * \hat{D}_{\nu} - \hat{D}_{\nu} * \hat{D}_{\mu}) * \Psi, \end{aligned} \tag{C.169}$$

which transforms covariantly under an isotopic gauge transformation, and $\hat{D}_{\mu}$ is the isotopic covariant derivative. In fact,

$$\hat{F}'_{\mu\nu} = \hat{U} * \hat{F}_{\mu\nu} * \hat{U}^{-\hat{1}}, \tag{C.170}$$

where

$$U^{-\hat{1}} \equiv \hat{T} U^{-1} \hat{I}. \tag{C.171}$$

Finally, to complete the field theory we can construct dynamical terms invariant under isotopic gauge transformations.

Karayannis and Jannussis construct in paper [168] a field theory which is invariant under Lie-isotopic local gauge transformations. The starting point is the free Dirac Lagrangian density

$$L = -\overline{\Psi}\partial\Psi - m\overline{\Psi}\Psi, \tag{C.172}$$

which is obviously invariant under global gauge transformations. As is well known, Eq. (C.172) remains invariant under local gauge transformations

$$\Psi' = e^{ig\Lambda(x)}\Psi \ , \tag{C.173}$$

if the conventional derivative, substituted from the covariant derivative, is transformed as

$$(D_\mu \Psi)' = e^{ig\Lambda(x)}(D_\mu \Psi) \ . \tag{C.174}$$

If we define

$$D_\mu \Psi = [\partial_\mu - igA_\mu(x)]\Psi \ , \tag{C.175}$$

we obtain the following transformation for the gauge fields

$$A'_\mu = A_\mu + \partial_\mu \Lambda \ . \tag{C.176}$$

Introducing the covariant derivative, the Lagrangian density (C.172) takes the form

$$\begin{aligned} L &= -\overline{\Psi} D \Psi - m\overline{\Psi}\Psi \\ &= -\overline{\Psi}\partial\Psi - m\overline{\Psi}\Psi + igA_\mu \overline{\Psi}\gamma^\mu \Psi, \end{aligned} \tag{C.177}$$

from which it is clear that the local gauge invariance leads to interacting field theories of a particular structure.

If one works out in a similar manner the Lie-isotopic gauge transformations,

$$\begin{aligned} \Psi' &= e^{igT\Lambda(x)}\Psi, \\ (\hat{D}_\mu * \Psi)' &= e^{igT\Lambda(x)}(\hat{D}_\mu * \Psi) \ , \end{aligned} \tag{C.178}$$

where

$$\hat{D}_\mu * \Psi = (\partial_\mu - igTA_\mu)\Psi \tag{C.179}$$

is the Lie-isotopic covariant derivative, then one obtains the generalized gauge transformation for the gauge fields

$$A'_\mu = A_\mu + \partial_\mu \Lambda + \Lambda \partial_\mu (\ln T; . \tag{C.180}$$

Thus, the Lie-isotopic lifting of Eq. (C.172), which is invariant under local Lie-isotopic gauge transformation, must have the form

$$\begin{aligned} \hat{L} &= -\frac{1}{2}\left[\hat{\overline{\Psi}} * \hat{\gamma}^\mu * \hat{D}_\mu * \Psi - \hat{D}_\mu * \hat{\overline{\Psi}} * \hat{\gamma}^\mu * \Psi\right] \\ &\quad - m\hat{\overline{\Psi}} * \Psi \end{aligned} \tag{C.181}$$

where

$$\hat{\overline{\Psi}} = \hat{\Psi}^{\dagger}\hat{\gamma}_0 \,, \tag{C.182}$$

and

$$\hat{\gamma}^{\mu} = \gamma^{\mu}\hat{I} \tag{C.183}$$

is the Lie-isotopic lifting of Dirac matrices, which are assumed to obey the rule

$$\hat{\gamma}^{\mu} * \hat{\gamma}^{\nu} + \hat{\gamma}^{\nu} * \hat{\gamma}^{\mu} = 2g^{\mu\nu}\hat{I}. \tag{C.184}$$

Here it is assumed that we are in a space-time with the metric tensor

$$g_{\mu\nu} = T^{-1}\eta_{\mu\nu}, \tag{C.185}$$

where $\eta_{\mu\nu}$ is the Minkowski metric.

Writing Eq. (C.181) in terms of Dirac matrices γ^{μ}, we have

$$\begin{aligned} \hat{L} \;\; = \;\; & -\frac{1}{2}\left(\overline{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - \partial_{\mu}\overline{\Psi}\gamma^{\mu}\Psi\right) \\ & -m\overline{\Psi}\Psi + igTA_{\mu}\overline{\Psi}\gamma^{\mu}\Psi \,. \end{aligned} \tag{C.186}$$

Thus, we see that by the Lie-isotopic gauge invariance we construct a gauge theory with an effective coupling constant $g' = gT$ which is a function of the space-time point where the gauge fields A_{μ} interact. This physical interpretation is analogous to that of Gasperini (Appendix A).

As is well known, repeated application of covariant derivatives will always yield covariant quantities. This fact can be used to construct a new covariant object. Thus, if we define the Lie-isotopic Yang-Mills field strengths $\hat{F}_{\mu\nu}$ for the gauge potential as in Eq. (C.169), then we have the curvature tensor

$$\hat{F}_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + (\partial_{\mu} ln\ T)A_{\nu} - (\partial_{\nu}\ ln\ T)A_{\mu} \tag{C.187}$$

for the Lie-isotopic gauge field $\hat{U}(1)$.

In the same manner we prove the Lie-isotopic lifting of the *Jacobi identity*

$$\begin{aligned} & \left[\hat{D}_{\mu}\hat{,}[\hat{D}_{\nu}\hat{,}\hat{D}_{\rho}]\right] + \left[\hat{D}_{\nu}\hat{,}[\hat{D}_{\rho}\hat{,}\hat{D}_{\mu}]\right] \\ & + \left[\hat{D}_{\rho}\hat{,}[\hat{D}_{\mu}, \hat{D}_{\nu}]\right] = 0. \end{aligned} \tag{C.188}$$

Combination of (C.187) with (C.188) leads to the following equation:

$$\hat{D}_{\mu} * \hat{F}_{\nu\rho} + \hat{D}_{\nu} * \hat{F}_{\rho\mu} + \hat{D}_{\rho} * \hat{F}_{\mu\nu} = 0. \tag{C.189}$$

Since the isotopic field strength is invariant under Lie-isotopic gauge transformations, one may replace covariant by ordinary derivatives,

$$\partial_\mu * \hat{F}_{\nu\rho} + \partial_\nu * \hat{F}_{\rho\mu} + \partial_\rho * \hat{F}_{\mu\nu} = 0, \tag{C.190}$$

and thus obtain the Lie-isotopic lifting of the *Bianchi identity*. Equation (C.190) in four dimensions is written as

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu * \hat{F}_{\rho\sigma} = 0. \tag{C.191}$$

If we define the dual of the $\hat{F}_{\rho\sigma}$ tensor as

$$\hat{G}^{\mu\nu} = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\hat{F}_{\rho\sigma}, \tag{C.192}$$

we obtain the following field equations

$$\partial_\nu \hat{G}^{\nu\mu} = \hat{G}^{\mu\nu}\partial_\nu(ln\ T). \tag{C.193}$$

This equation gives the Lie-isotopic lifting of the second pair of the Maxwell's equations, and it entails a magnetic current

$$\hat{J}^m_\mu = \hat{G}^{\mu\nu}\partial_\nu(ln\ T), \tag{C.194}$$

which, for $T = e^{-\varphi}$, coincides with the magnetic current given by Hojman et al [172]. Thus, Hojman's theory is a special case of the general Lie-isotopic gauge theory.

The field-strength tensor (C.187) can now be used to write a Lie-isotopic gauge-invariant Lagrangian density for the gauge field, as follows

$$L_G = -\frac{1}{4}\hat{T}r(\hat{F}^{\mu\nu} * \hat{F}_{\mu\nu}) \tag{C.195}$$

Here the symbol $\hat{T}r$ denotes the Lie-isotopic trace [36]. The Lagrangian density (C.195) can now be added to the previous one (C.181), so that we have obtained an interacting system of vector fields and fermion fields which is invariant under the Lie-isotopic local transformations

$$\begin{aligned} L_{tot} &= -\frac{1}{2}\left[\hat{\bar{\Psi}} * \hat{\gamma}^\mu * \hat{D}_\mu * \Psi - [\hat{D}_\mu * \hat{\bar{\Psi}} * \hat{\gamma}^\mu * \Psi\right] \\ &\quad -m\hat{\bar{\Psi}} * \Psi - \frac{1}{4}\hat{T}r(\hat{F}^{\mu\nu} * \hat{F}_{\mu\nu}). \end{aligned} \tag{C.196}$$

This interaction takes place in an effective Riemann-Cartan space, equipped with an antisymmetric connection

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\left[\delta^{\alpha}_{\mu}\partial_{\nu}(ln\ T) - \delta^{\alpha}_{\nu}\partial_{\mu}(ln\ T)\right]. \tag{C.197}$$

From the total Lagrangian density (C.196) we produce the field equation under some assumptions

$$\partial_{\lambda}\hat{F}^{\lambda\chi} = -ig\overline{\Psi}^{*}\gamma_{\kappa} * \Psi - \hat{F}^{\chi\lambda}\partial_{\lambda}(ln\ T). \tag{C.198}$$

This equation constitutes the Lie-isotopic generalization of the first pair of Maxwell's equations. The first term in the right side gives the lifting of the Dirac current, while the second term gives the Lie-isotopic electric current associated with the isotopic element T.

$$\hat{J}^{e}_{\mu} = \hat{F}^{\mu\lambda}\partial_{\lambda}(ln\ T)\ . \tag{C.199}$$

We see that the Lie-isotopic lifting of conventional gauge theories yields new currents (magnetic and electric) which act as a part of the source of electromagnetism.

The other two field equations produced from the Lagrangian density (C.196) are

$$\begin{aligned} \hat{\overline{\Psi}}\left(\gamma^{\mu}\overleftarrow{\partial}_{\mu} - m\right) &= -ig\hat{\overline{\Psi}} * \gamma^{\mu}A_{\mu}, \\ (\gamma^{\mu}\partial_{\mu} + m)\Psi &= +ig\gamma^{\mu}A_{\mu} * \Psi \end{aligned} \tag{C.200}$$

and constitute the Lie-isotopic lifting of the Dirac equation. So, once more we see that the lifting of the gauge fields is equivalent with the lifting of A_{μ} to TA_{μ} or g to Tg.

Since the tensors $\hat{F}_{\mu\nu}$ and $\hat{G}^{\mu\nu}$ are antisymmetric, the total electric and magnetic currents are conserved:

$$\begin{aligned} \partial_{\mu}\hat{J}^{e}_{\mu} &= 0, \\ \partial_{\mu}\hat{J}^{m}_{\mu} &= 0. \end{aligned} \tag{C.201}$$

The explicit forms of the electric charge and current densities are

$$\begin{aligned} \rho^{e} &= g\Psi^{\dagger} * \Psi + \hat{\overline{E}} \cdot \overline{\nabla}\ ln\ T, \\ \overline{J}^{e} &= g\Psi^{\dagger} * \hat{\overline{\alpha}} * \Psi - \hat{\overline{B}} \times \overline{\nabla}\ln\ T - \frac{1}{c_0}\hat{\overline{E}}\frac{\partial\ ln\ T}{\partial t}\ . \end{aligned} \tag{C.202}$$

These relations are those given in ref. [168] except for the first terms. In a similar way we obtain, for the magnetic charge and current densities, the relations

$$\rho^m = \hat{\overline{B}} \cdot \overline{\nabla} ln\ T,$$

$$\overline{J}^m = -\hat{\overline{E}} \times \overline{\nabla}\ ln\ T + \frac{\hat{\overline{B}}}{c_0} \frac{\partial\ ln\ T}{\partial t}\ . \tag{C.203}$$

In closing, Karayannis and Jannussis [168] quote the studies by K. Cahill and S. Ozenli [190] for which g is a metric field, and which provide gauge theories for arbitrary noncompact groups.

Again, as it is the case for Gasperini's theory, the studies by Karayannis and Jannussis [*loc. cit.*] characterize the $\hat{U}(1)$-gauge theory as a gauge theory on a curved space-time where the total magnetic charge is null, as proved in ref.s [168] and [172]. The magnetic current is therefore not expected to be observable from an outside observer. However, this does not rule out the possibility that magnetic currents could be observed in the interior problem, e.g., where the torsion (see the Lie-isotopic studies on torsion by Rapoport-Campodonico [52]) is not null, or when future experimental advances will achieve the capabilities of measures under external strong interactions.

* * * * *

We now pass to the review of some of the studies by Nishioka [184,185]. In particular, we shall review only representative articles of mainly semi-classical nature with a direct connection to the preceding content of this review. The remaining articles will be outlined in a possible subsequent review on hadronic mechanics owing to their strictly operator character.

Let us begin by reviewing paper [184] which is directly related to paper [168] by Karayannis and Jannussis previously reviewed in this appendix. As one will recall, the latter paper establishes that the isotopic lifting of $U(1)$-gauge fields is equivalent to liftings

$$A_\mu \to TA_\mu\ , \quad g \to gT \tag{C.204}$$

where $A_\mu(x)$ are the gauge fields, g is a coupling constant, and T is the isotopic element. In paper [184], Nishioka confirms this important result from a different approach. Again, for notational clarity we shall review the basic elements of the theory considered.

Consider an invertible and Hermitian operator R which may be a function of space-time. The enveloping algebra of a theory with associative product AB and unit I is generalized by introducing Santilli's product $A*B = ARB$ and a new unity $\tilde{I} = R^{-1}$ such that $A*\tilde{I} = \tilde{I}*A = A$. The elements $A, B, \ldots$ are essentially constituted by polynomials in the space-time coordinate and momentum operators.

We define the isotopic generalizations of the Hermitian conjugate $A^\dagger$ and inverse A^{-1} of an operator A via the quantities $A^{\tilde{\dagger}} = R^\dagger A^\dagger \tilde{I}$, $A^{-\tilde{1}} = \tilde{I}A^{-1}\tilde{I}$, respectively.

The Lie-isotopic lifting $\tilde{G}$ of the compact group G is represented by the following transformation of a wave function Ψ:

$$\Psi' = \tilde{U} * \Psi \tag{C.205}$$

where

$$\tilde{U} = \tilde{I}\exp(-i\Theta^k * X_k) = \exp(-X_k * \Theta^k)\tilde{I}. \tag{C.206}$$

Here Θ^k is a function of the space-time coordinates, X_k is a matrix representation of the generator of group G satisfying

$$[X_i, X_j] = ic_{ij}{}^k X_k, \tag{C.207}$$

and $c_{ij}{}^k$ are the structure constants of the Lie algebra G. It is now known that $\tilde{U}$ is an R-unitary operator, that is,

$$\tilde{U}^{\tilde{\dagger}} * \tilde{U} = \tilde{I}\ , \tag{C.208}$$

with invariant form

$$\Psi^\dagger * \Psi = \Psi'^\dagger * \Psi'\ . \tag{C.209}$$

In analogy with ordinary gauge theory, Nishioka introduces Gasperini's isotopic covariant derivatives $\tilde{D}_\mu$ by imposing the following transformation rules

$$\tilde{D}'_\mu * \tilde{U} * \Psi = \tilde{U} * \tilde{D}_\mu * \Psi, \tag{C.210}$$

where

$$\tilde{D}_\mu = (\partial_\mu - igA^k_\mu * X_k)\tilde{I} \tag{C.211}$$

and

$$\tilde{D}'_\mu = \tilde{U} * \tilde{D}_\mu * \tilde{U}^{-\tilde{1}}, \tag{C.212}$$

in which A^k_μ are gauge fields, and g is as a coupling constant.

Define the isotopic gauge field strengths $\tilde{F}_{\mu\nu}$ for the gauge fields (potentials) as follows

$$\tilde{F}_{\mu\nu} * \Psi = -\frac{1}{ig}(\tilde{D}_\mu * \tilde{D}_\nu - \tilde{D}_\nu * \tilde{D}_\mu) * \Psi\ , \tag{C.213}$$

with transformation rules

$$\tilde{F}'_{\mu\nu} = \tilde{U} * \tilde{F}_{\mu\nu} * \tilde{U}^{-\tilde{1}}. \tag{C.214}$$

Notice that the minimal coupling term in (C.181) is

$$A^k_\mu * X_k = A^k_\mu (R X_k). \tag{C.215}$$

Nishioka [*loc. cit.*] then assumes that R has the following form

$$R = TS, \tag{C.216}$$

where T is a nonsingular function of the space-time coordinates and S is an invertible and Hermitian operator independent of the space-time coordinates. Represent $A^k_\mu * X_k$ as follows

$$A^k_\mu * X_k = A^k_\mu T Y_k, \tag{C.217}$$

where $Y_k = S X_k$.

Using the discussion above, define $\dot{F}_{\mu\nu}$ and $\dot{F}^i_{\mu\nu}$ as follows

$$\begin{aligned} \tilde{F}_{\mu\nu} &= \dot{F}_{\mu\nu} * \tilde{I}, \\ \dot{F}_{\mu\nu} &= \dot{F}^i_{\mu\nu} * X_i \,. \end{aligned} \tag{C.218}$$

Then $\dot{F}^i_{\mu\nu}$ are given by

$$\dot{F}^i_{\mu\nu} = T^{-1} \dot{H}^i_{\mu\nu} \,, \tag{C.219}$$

where

$$\begin{aligned} \dot{H}^i_{\mu\nu} &= \partial_\mu B^i_\nu - \partial_\nu B^i_\nu + g s_{jk}{}^i B^j_\mu B^k_\nu, \\ B^i_\mu &= A^i_\mu T, \quad [Y_i, Y_j] = i s_{ij}{}^k Y_k \,. \end{aligned} \tag{C.220}$$

Finally, introduce a "metric tensor" defined as

$$h_{\mu\nu} = T^{-1} \eta_{\mu\nu} \,, \tag{C.221}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. Then

$$\dot{F}^{i\mu\nu} = h^{\mu\rho} h^{\nu\sigma} \dot{F}^i_{\rho\sigma} = T \eta^{\mu\rho} \eta^{\nu\sigma} \dot{H}^i_{\rho\sigma} = T \dot{H}^{i\mu\nu} \,. \tag{C.222}$$

Next rewrite $\dot{F}_{\mu\nu}$ and $\dot{F}^{\mu\nu}$ in terms of Y_i as follows:

$$\begin{aligned} \dot{F}_{\mu\nu} &= \dot{F}^i_{\mu\nu} T Y_i = \dot{H}^i_{\mu\nu} Y_i \equiv \dot{H}_{\mu\nu} \,, \\ \dot{F}^{\mu\nu} &= \dot{F}^{i\mu\nu} * X_i = \dot{F}^{i\mu\nu} T Y_i \\ &= T^2 \dot{H}^{i\mu\nu} Y_i = T^2 \dot{H}^{\mu\nu} \,. \end{aligned} \tag{C.223}$$

The Lagrangian density L for the gauge fields is then given by [156a]

$$\begin{aligned} L &= -\frac{1}{4}\sqrt{-h}\dot{T}r(\dot{F}^{\mu\nu} * \dot{F}_{\mu\nu}) \\ &= -\frac{1}{4}\dot{T}r(\dot{H}^{\mu\nu} * \dot{H}_{\mu\nu}), \end{aligned} \tag{C.224}$$

where $h = \det(h_{\mu\nu})$, the relations (C.222) and (C.223) have been used, and $\dot{T}r$ denotes the Lie-isotopic trace [36].

Before we interpret Lagrangian density (C.224), notice that $\dot{H}_{\mu\nu}$ become equivalent to the field strengths of ordinary gauge fields on changing $A_\mu \to B_\mu (= B^i_\mu Y_i)$ and $X_i \to Y_i$ and that the right-hand side is represented in the flat space (the Minkowski space).

Nishioka [*loc.cit.*] therefore concludes that the Lie-isotopic lifting of general gauge fields can be done via the following steps:

1. construct the Lagrangian density for ordinary gauge fields;
2. change the ordinary gauge fields $A_\mu \to B_\mu$ and the Lie-algebra generators $X_i \to Y_i$;
3. change the product of two field strengths to the isotopic product;
4. change the trace to the isotopic trace.

We now review Nishioka's paper [185] on the *isotopic lifting of continuity equations.* This paper is important for this review, inasmuch as it establishes that several familiar dissipative models of quantum mechanics are, in actuality, particular cases of the covering hadronic mechanics, and its underlying Lie-isotopic equations. In fact, the concept of isotopy and related generalized unit was proposed by Santilli [1] precisely to represent dissipative conditions.

Consider the following Lie-isotopic lifting of the Schrödinger equation [36]

$$i\hbar\partial_t\Psi = HG\Psi = H * \Psi, \tag{C.225}$$

and its Hermitian conjugate

$$-i\hbar\Psi^\dagger \overleftarrow{\partial}_t = \Psi^\dagger \overleftarrow{G}\overleftarrow{H} = \Psi^\dagger * \overleftarrow{H} \tag{C.226}$$

where the symbol $*$ means Santilli's product $A * B = AGB$; H is the total Hamiltonian of the system, here assumed to be iso-Hermitian; and G is the

Lie-isotopic element also assumed to be Hermitian and invertible, as well as a function of space-time. Following Ref. [36], define the G-Hermitian conjugate of Ψ as the quantity

$$\Psi^{\hat{\dagger}} = G^{\dagger}\Psi^{\dagger}\hat{I} = G\Psi^{\dagger}\hat{I}, \tag{C.227}$$

where $\hat{I}$ is the new unity, $I = G^{-1}$.

Define the Lie-isotopic lifting of the probability density as follows

$$\hat{\rho} = \hat{\Psi}^{\dagger} * \Psi. \tag{C.228}$$

Next, the Lie-isotopic lifting of a differential operator – for example, the time-differential operator ∂_t – will be defined as

$$\hat{\partial}_t = \partial_t \hat{I} \qquad \hat{\partial}_t * \Psi = \partial_t \Psi \ . \tag{C.229}$$

The derivative of $\hat{\rho}$ with respect to the time t is given by

$$\hat{\partial}_t * \hat{\rho} = \partial_t(G\Psi^{\dagger}\Psi) \ . \tag{C.230}$$

Also, assume that the Hamiltonian H has the following form:

$$H = -\frac{\hbar^2}{2m} \nabla^2 + V(x, y, z), \tag{C.231}$$

where V is Hermitian and m is the mass of the particle considered. By making use of the preceding formalism, Nishioka [*loc.cit*] obtains the equation

$$\partial_t \rho + \nabla \cdot \mathbf{J} = (\partial_t \ln G)\rho, \tag{C.232}$$

where

$$\begin{aligned} \rho &= G\Psi^{\dagger}\Psi, \\ \mathbf{J} &= \frac{\hbar}{2mi}\{(G\Psi)^{\dagger} \nabla (G\Psi) \\ &\qquad -[\nabla(G\Psi)^{\dagger}](G\Psi)\} \ . \end{aligned} \tag{C.233}$$

If G is independent of time t, Eq. (C.231) is reduced to the conventional form

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0 \tag{C.234}$$

which can be rewritten

$$\partial_t \rho + \nabla \cdot (G^2 \mathbf{J}^0) = 0 \ , \tag{C.235}$$

where

$$\mathbf{J}^0 = \frac{\hbar}{2mi}[\Psi^\dagger \bigtriangledown \Psi - (\bigtriangledown \Psi^\dagger)\Psi] \, , \tag{C.236}$$

and the notation $\mathbf{A} \cdot \mathbf{B}$ denotes the scalar product. Equation (C.234) can be written

$$\partial_t(\rho^0 \hat{I}) + \bigtriangledown \cdot \mathbf{J}^0 = -(\bigtriangledown \ln G^2) \cdot \mathbf{J}^0 \, , \tag{C.237}$$

where

$$\rho^0 = \Psi^\dagger \Psi \, . \tag{C.238}$$

As applications of (C.231) and (C.236), Nishioka considers two examples: one is a complex-potential model by Feshbach, Porter, and Weisskopf [191], the other is a model in which a particle with charge is interacting with an electromagnetic field.

The complex-potential model is based on the idea that an incident particle inside a target nucleus effectively moves in a complex potential well; the real attractive potential simply refracts the incident nucleon, while the presence of the imaginary term implies absorption of the nucleon. A simplified Hamiltonian for this case is given by [191]

$$H = -\frac{\hbar^2}{2m} \bigtriangledown^2 + V - iW \, , \tag{C.239}$$

where V, W are assumed to be real, and W is constant.

The Schrödinger equation for Hamiltonian (C.239) in conventional quantum mechanics is given by

$$i\hbar \partial_t \phi = H\phi, \tag{C.240}$$

and its Hermitian conjugate is

$$-i\hbar \phi^\dagger \overleftarrow{\partial}_t = \phi^\dagger \overleftarrow{H}^\dagger \tag{C.241}$$

with continuity equation

$$\partial_t \rho_A^0 + \bigtriangledown \cdot \mathbf{J}_A^0 = -\frac{2W}{\hbar} \rho_A^0, \tag{C.242}$$

where

$$\rho_A^0 = \phi^\dagger \phi, \quad \mathbf{J}_A^0 = \frac{\hbar}{2mi}[\phi^\dagger \bigtriangledown \phi - (\bigtriangledown \phi^\dagger)\phi]. \tag{C.243}$$

Nishioka [*loc.cit.*] then assumes that the Lie-isotopic element G is a function of time only, and that the correspondences

$$\rho \rightleftarrows \rho_A^0 \quad , \quad \mathbf{J} \rightleftarrows \mathbf{J}_A^0 \tag{C.244}$$

hold. He then obtains from (C.231) and (C.242)

$$\partial_t \ lnG \rightleftarrows -\frac{2W}{\hbar}, \tag{C.245}$$

so that

$$G \rightleftarrows \exp\left(-\frac{2W}{\hbar}t\right), \tag{C.246}$$

where one has to keep in mind that Hamiltonian (C.239) is not Hermitian.

Next, Nishioka [*loc.cit.*] considers the Hamiltonian

$$H = -\frac{\hbar^2}{2m} \nabla^2 + i\mathbf{B} \cdot \nabla + V, \tag{C.247}$$

where $\mathbf{B}$ is a vector and is assumed to be Hermitian, and V contains all interaction terms (except the second term of the right side) and is assumed to be Hermitian.

The Hamiltonian in this case is Hermitian. The Schrödinger equation is given by

$$i\hbar\partial_t\psi = H\psi, \tag{C.248}$$

and its Hermitian conjugate is

$$- i\hbar\psi^\dagger \overleftarrow{\partial}_t = \psi^\dagger \overleftarrow{H} \tag{C.249}$$

with continuity equation

$$\partial_t\rho^0_B + \nabla \cdot \mathbf{J}^0_B = \frac{2m}{\hbar^2}\mathbf{B} \cdot \mathbf{J}^0_B \ , \tag{C.250}$$

where

$$\rho^0_B = \psi^\dagger\psi, \quad \mathbf{J}^0_B = \frac{\hbar}{2mi}\left[\psi^\dagger \nabla \psi - (\nabla\psi^\dagger)\psi\right] \ . \tag{C.251}$$

Assume that G is independent of time t, and that the following correspondences hold

$$G^{-i}\rho^0 \rightleftarrows \rho^0_B, \quad \mathbf{J}^0 \rightleftarrows \mathbf{J}^0_B \ . \tag{C.252}$$

Then, from (C.237) and (C.251), Nishioka obtains the correspondence

$$- \nabla lnG^2 \rightleftarrows \frac{2m}{\hbar^2}B. \tag{C.253}$$

The correspondence between the Lie-isotopic element G and the vector potential $\mathbf{B}$ of the electromagnetic field can therefore be readily established when $\mathbf{B}$ can be represented as a gradient term.

In this way, Nishioka [*loc.cit.*] identifies the possibility of representing the electromagnetic interactions themselves via an isotopic lifting of free charged particles. This property had been established at a classical level by Santilli in monograph [15], see Example 4.1, p. 98 and following, and its operator counterpart has been established in paper [185].

As a final remark, note that the reformulation of a conventional model with conserved (Hermitean) Hamiltonian generally leads to the Lie-isotopic setting, while the reformulation of a nonconserved (non-Hermitean) Hamiltonian generally leads to the broader Lie-admissible approach.

This occurrence is, in the final analysis, a confirmation of the limitations of the Lie-isotopic theory, and of the need to enlarge it into the yet more general Lie-admissible approach, exactly along Santilli's original proposal [1].

* * * *

We would like to review now the application of the (algebraic,classical and operator) Lie-isotopic formulations to the physical arena for which they were conceived by Santilli: the structure of hadrons.

These applications are being studied according to two subsequent, compatible, stages: first, the identification of the hadronic constituents with (mutated forms of) ordinary, massive particles freely produced in the spontaneous decays; and, second, the identification of quarks with mutated forms of ordinary physical particles.

The initiation of the studies on the first stage was done in the original proposal to build hadronic mechanics [2] via the construction of a structure model of the π^0 as a hadronic bound state of one electron $e-$ and one positron e^+ mutated into the eletons $\hat{e}^\pm$ as a "compressed positronium" (*loc. cit.* §5)

$$\text{Pos.} = (e^+, e^-)_{\text{Quantum Mech.}} \longrightarrow \pi^0 = (\hat{e}^+, \hat{e}^-)_{\text{Hadr. Mech.}} \cdot \qquad \text{(C.254)}$$

It was first pointed out that the use of the covering hadronic mechanics is *necessary* for the model, not only to represent the conditions of total mutual immersion of the wavepackets of the electrons, but also to achieve a consistent bound state with real binding energy.

In fact, quantum mechanics cannot produce a consistent bound state of two electrons with rest mass 0.5 MeV of the π^0, owing to a little known property according to which the initial equation of the bound state does not generally admit real solution when $E_{\text{Tot}} >> 2E_{\text{Rest}}$ [192]. In this way, Santilli proved the *necessity* of generalizing quantum mechanics already at the level of the lightest known hadron.

Hadronic mechanics proved to be readily capable of resolving the above problem. In fact, in more recent formulations, we can say that hadronization (1.59) of *Jannussis generalized action* (3.140) for the two-body, closed, nonhamiltonian system yields the iso-Schrödinger's equations

$$\begin{aligned}
&\vec{p} * \psi = \vec{p}T\psi = -i\rho\vec{\nabla}\psi\,, \quad \rho = \frac{m}{m-g}\,, \quad \hbar = 1 \\
&i\frac{\partial}{\partial t}\psi = H * \psi = \left[\frac{1}{2\overline{m}}\hat{\Delta} + \vec{V}(\vec{r}) - i\left(\frac{\partial \hat{I}}{\partial t}\right)\log\psi\right] * \psi = E\psi \\
&\overline{m} = m/\rho\,, \quad \hat{\Delta} = \hat{\Delta}\hat{I}\,, \quad V = V\hat{I}\,,
\end{aligned} \tag{C.255}$$

that is, the Birkhoffian form of the Hamilton-Jacobi equation in the linear momentum results in a generalization of the operator form of the kinetic energy which implies a sort of "renormalization" of the mass

$$\frac{1}{2m}\vec{p}\cdot\vec{p} = -\frac{1}{2m}\Delta \rightarrow \frac{1}{2m\rho}\vec{p} * \vec{p} = -\frac{1}{2\overline{m}}\Delta\,. \tag{C.256}$$

This mechanism then permits hadronic mechanics to achieve the desired total energy because the "mass renormalization" (C.255) is such that $2E_{\text{Rest}} > E_{\text{Tot}}$, thus permitting real, negative, binding energies.

Upon achieving the resolution of this problem, Santilli proposed the structure equations for the model $\pi^0 = (\hat{e}^+, \hat{e}^-)$, which we can today write in the form (Eq.s 5.1.14, p. 836, ref. [2])

$$\begin{aligned}
i\frac{\partial}{\partial t}|\psi_{\pi^0}\rangle &= \left[-\frac{1}{2\overline{m}}\hat{\Delta} - \frac{e^{\hat{r}}}{r} + i\left(\frac{\partial \hat{I}}{\partial t}\right)\log\psi\right] * |\psi_{\pi^0}\rangle \\
&= \left[-\frac{1}{2\overline{m}}\Delta - \frac{e^{r}}{r} - V_0\frac{e^{-br}}{1-e^{-br}}\right]|\psi\rangle = E|\psi\rangle\,, \\
E^{\text{Tot}}_{\pi^0} &= 2E^{\text{Rest}}_{\hat{e}} + 2E^{\text{Kin}}_{\hat{e}} - E = 135\,MeV\,, \quad \hbar = 1\,, \\
\tau^{-1}_{\pi^0} &= 4\pi\lambda|\psi(0)|^{r}\alpha E^{\text{Tot}}_{\hat{e}} = 10^{16}sec^{-1}\,, \\
b^{-1}_{\pi^0} &= 10^{-13}\text{cm}\,, \quad \overline{m} = m/\rho\,, \quad \rho = \frac{m}{m-g}\,, \\
T &= T(0)\exp(-it|E_u|\langle u|\psi\rangle)\,, \\
T(0) &= 1 - |u\rangle\langle\psi| = T(0)^{r}\,.
\end{aligned} \tag{C.257}$$

The model resulted to be capable of representing *all* intrinsic characteristics of the π^0, such as: rest energy, meanlife, spin, charge, magnetic and electric moments, space and charge parity, and primary decay $\pi^0 \rightarrow \gamma\gamma$, the decay $\pi^0 \rightarrow e^+ + e^-(< 2 \times 10^6)$ being a tunnel effect of the electrons through the Hamiltonian and nonhamiltonian barriers of hadronic mechanics.

The above model was then extended in ref. [2] to the remaining (light) mesons, with the following results

$$
\begin{aligned}
\pi^0 &= (\hat{e}^+, \hat{e}^-) = 135 \ MeV \ , \\
\pi^\pm &= (\hat{e}^+, \hat{e}^\pm, \hat{e}^-) = (\hat{\pi}^0, \hat{e}^\pm) = 139 \ MeV \ , \\
\eta &= (2\hat{e}^+, 2\hat{e}^-) = (\hat{\pi}^0, \hat{\pi}^0) = 548 \ MeV \ , \\
K^\pm &= (2\hat{e}^+, \hat{e}^\pm, 2\hat{e}^-) = (\hat{\pi}^+, \hat{e}^\pm, \hat{\pi}^-) = 493 \ MeV \ , \\
K^0 &= (3\hat{e}^+, 3\hat{e}^-) = (\hat{K}^+, \hat{K}^-) = 497 \ MeV \ ,
\end{aligned} \tag{C.258}
$$

where a hat indicates mutation.

In this way, Santilli illustrated the purpose for which hadronic mechanics (with its underlying Lie-isotopic structure and classical Birkhoffian backing) had been conceived. In achieving the interpretation of the mesonic constituents with physical particles, Santilli was however forced to *increase the number of constituents with mass*, according to a law previously established at the nuclear and atomic levels. In fact, a primary objective of compression (C.254) is to *suppress the atomic spectrum of energy down to only one admissible level: the* π^0. This suppression was achieved in full via the use of Hulten potential which, quite remarkably, resulted in system (C.257) to admit only one admissible (real) energy level, 135 MeV.

The suppression of the atomic spectrum of energy was evidently *necessary* to avoid a host of inconsistencies, e.g., to avoid the presence of an infinite spectrum of energy near the ground state of typical atomic conception, that has no representation in the physical reality of hadrons.

In conclusion, another central result by Santilli in the original proposal [2] is that *the identification of the mesonic constituents with physical particles freely produced in the spontaneous decays demands the increase of the number of constituents with mass.* In fact, starting from two constituents for the π^0 (135 MeV), he obtained three constituents for the $\pi^\pm$ (139 MeV), four constituents for the η (548 MeV), five constituents for the $K^\pm$ (493 MeV) and six constituents for the K^0 (497 MeV).

Santilli concluded the analysis of memoir [2] with the suggestion of developing mechanics up to such level of maturity to be able to represent consistently Rutherford's historical hypothesis on the structure of the neutron as a "compressed hydrogen atom" (in Rutherford's words [193]), according to the isotopic lifting

$$\text{Hydr. At.} = (p^+, e^-)_{\text{Quantum Mech.}} \longrightarrow n = (p^+, \hat{e}^-)_{\text{Hadr. Mech}} \,. \qquad (C.259)$$

He was fully aware that Eq. (C.257) could readily represent the total energy of the neutron as a hadronic bound state of one ordinary, unmutated proton p^+, and a mutated electron $\hat{e}^-$ (which is also impossible for quantum mechanics because $E_n > E_p + E_e$). Nevertheless, the primary problem of consistency in Rutherford's compression (C.259) was the achievement of the total spin $\frac{1}{2}$ from generalized bound states of two particles each of spin $\frac{1}{2}$.

At the time of memoir [2], 1978, Santilli was unaware that the problem of spin for compression (C.259) had already been solved by P.A.M. Dirac in his generalization [54] of Dirac's equation which, as reviewed earlier, implies the mutation of spin from $\frac{1}{2}$ to zero precisely for at rest conditions. This is exactly the total angular momentum needed for the eleton $\hat{e}^-$ to achieve a consistent representation of compression (C.259).

Dirac's papers [54] were brought to the attention of Santilli by A. Kalney at a meeting of 1983 in Cambridge, MA. Lack of funds (as well as very strenuous oppositions by the local Cantabridgean physicists against the studies here reviewed [163]), forced the delay of the research in Rutherford's compression, which was resumed only in subsequent years according to the following main lines.

First, Santilli [24] constructed the infinite family of isotopes $\widehat{SU}(2)$ of the $SU(2)$-spin symmetry as briefly reviewed earlier; identified their isorepresentations; and worked out a number of generalizations of the conventional theory of spin.

Second, he then applied [25] the covering $\widehat{SU}(2)$ theory to Rutherford's compression (C.259). In essence, with reference to Eq.s (C.113), the total angular momentum of the eleton in Rutherford's compression is given by the sum of the hadronic angular momentum and spin

$$j_{\hat{e}}^{\text{Tot}} = \hat{l}_{\hat{e}} + \hat{s}_{\hat{e}} = \Delta_{\hat{l}}^{1/2} - \frac{1}{2}\Delta_s^{1/2} = 0 \,, \qquad (C.260)$$

where the Δ's are the determinants of the orbital and intrinsic isometrics.

The total angular momentum of Rutherford's historical compression then resulted to be interpretable as follows

$$j_n^{\text{Tot}} = j_p + j_{\hat{e}}^{\text{Tot}} = \frac{1}{2} + \Delta_{\hat{l}}^{1/2} - \frac{1}{2}\Delta_s^{1/2} = \frac{1}{2} \equiv j_p \qquad (C.261)$$

and holds under a constraint in the *orbital* angular momentum so simple to appear trivial (see Fig. 19).

After achieving a resolution of the problem of the spin, the quantitative representation of *all* characteristics of the neutron in compression (C.259) was straight-forward. In fact, rest energy, meanlife and charge radius of the neutron were readily represented via the simple reformulation of Eq.s 5.1.14 of ref. [2] into Eq.s (2.19) of ref. [25], i.e.

$$\begin{aligned} i\frac{\partial}{\partial t}|\psi_n\rangle &= \left[-\frac{1}{2\overline{m}}\hat{\Delta} - \frac{e^{\hat{r}}}{r} + i\left(\frac{\partial \hat{I}}{\partial t}\right)\log\psi\right] * |\psi_n\rangle \\ &= \left[-\frac{1}{2\overline{m}}\Delta - \frac{e^{r}}{r} - V_0\frac{e^{-br}}{1-e^{-br}}\right]|\psi_n\rangle = E|\psi_n\rangle\ , \\ E_n^{\rm Tot} &= E_p^{\rm Rest} + E_{\hat{e}}^{\rm Rest} + E_e^{\rm Kin} - E = 938.6\ MeV\ , \\ \tau_n^{-1} &= 4\pi\lambda|\psi(0)|^2\alpha E_{\hat{e}}^{\rm Tot} = 1.09\times 10^{-3}\ {\rm sec}^{-1}\ , \\ b_n^{-1} &= 0.8\times 10^{-13}{\rm cm}\ , \quad \overline{m} = \overline{m}_{\hat{e}} = m_e/\rho\ , \quad \hbar = 1\ , \\ T &= T(0) = T(0){\rm exp}(-it|E_u|\langle u|\psi\rangle) \\ T(0) &= 1 - |u\rangle\langle\psi| = T(0)^2\ , \end{aligned} \tag{C.262}$$

which also resulted to be, not only consistent, but also capable of suppressing the spectrum of the hydrogen atom down to only one level: the neutron.

The representation of the anomalous magnetic moment of the neutron was achieved via fundamental geometrization (C.1) [18] of the proton medium and use of Eq.s (C.77)

$$\begin{aligned} \mu_n^{\rm Tot} &= \mu_p + \mu_{\hat{e}}^{\rm Orb} + \mu_{\hat{e}}^{\rm Intr} = -1.9\frac{e}{2m_p c_o} \\ &= 2.7\frac{e}{2m_p c_0} - \left|\mu_{\hat{e}}^{\rm Orb}\right| + \left|\frac{b_3}{b_4}\frac{e}{2m_e c_0}\right| \end{aligned} \tag{C.263}$$

following explicit solution

$$\mu_{\hat{e}}^{\rm Tot} = |\mu_{\hat{e}}^{\rm Orb}| - |\mu_{\hat{e}}^{\rm Intr}| = 2.5\times 10^{-3}\mu_e\ . \tag{C.264}$$

The neutron decays $n \to p^+ + e^- + \overline{\nu}_e$ then resulted to be interpretable via the decay

$$\hat{e}^- \to e^- + \overline{\nu}_e\ , \tag{C.265}$$

i.e., *Rutherford's compression of the electron down the center of the proton may well result to be the mechanism of creation of neutrinos in Nature.* It should be understood that *neutrinos are not physical constituents of the neutroon according to model (C.262).* They are merely created when the eleton exits the hyperdense medium in the interior of the proton. In different terms, the neutrinos appear to originate at the time of relation of the constraint on the orbital motion of Rutherford's electron inside the proton.

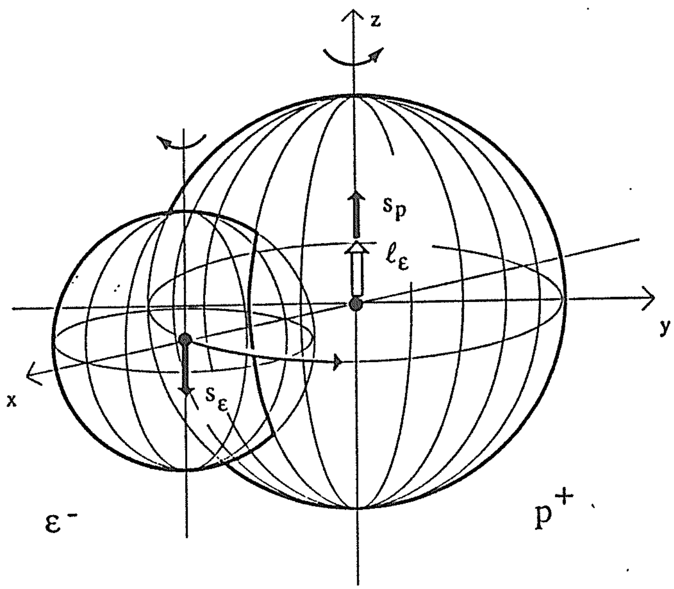


FIGURE 19. A schematic view of the structure model of the neutron $n = (p^+, \hat{e}^-)$ proposed in ref.s [25,29] along Rutherford's historical hypothesis. The proton is represented, not as an empty sphere with points in it, but as the densest object measured in laboratory until now. As such, the proton is depicted as a sphere of radius equal to that of the charge distribution (~ 1F) filled up with the wavepackets of the constituents in conditions of total mutual overlapping (because their size is also of the order of 1F). Rutherford conceived his hypothesis on the neutron as a "compressed hydrogen atom". The figure therefore depicts the initiation of Rutherford's compression of the electron inside the proton where the electron is represented by a sphere schematizing its wavepacket, and sizes are not necessarily in scale. Once this physical setting is clearly identified, it is then easy to see that the electron can only penetrate inside the proton with the relative spinning "in phase" [2] (to avoid high dissipative effects expected from the spinning of wavepackets one against the other), and with its angular mo-

mentum parallel to the spin of the proton. These initial conditions appear to be rather plausible and well founded experimentally. The final phase of the compression is conjectural as of this writing. Santilli argues that, when the electron is totally compressed inside the proton, the angular momentum is expected to coincide with the spin of the proton owing to their physical identity and, thus, to prevent inconsistencies in the mathematical treatment of the structure i.e., $\lim \hat{l}_{\hat{e}}|_{r=0} = \lim \hat{\Delta}_{\hat{l}}|_{r=0} = j_p$. This automatically allows to represent the spin of the neutron in model $n = (p^+, \hat{e}^-)$ as coinciding with that of the proton, Eq. (C.261). The representation of all other intrinsic characteristics of the neutron is then readily allowed by the techniques of hadronic mechanics. The electron, when totally immersed in the hyperdense medium in the interior of the proton, is assumed to be mutated [2] into an eleton, and acquire intrinsic characteristics different than those of the ordinary particle.

A comprehensive presentation of the above studies is presented by Santilli in paper [29]. A relativistic formulation of the model is under preparation at this writing, and it is evidently based on "Dirac's generalized equation" (C.152). As a matter of fact, we can readily state that the studies [24,25,29] were conceived by Santilli as being a nonrelativistic version of the relativistic treatment offered by Dirac.

This scientific scene can be conceptually summarized as follows. Hadronic mechanics appears indeed to be successful for the objective for which it was suggested: identification of the hadronic constituents with physical, ordinary, massive particles freely produced in the spontaneous decays. In fact, this possibility is now realistically established for all light mesons and for Rutherford's neutron. The extension of the results to the remaining hadrons via iterations of type (C.258) is then only a question of routine studies with no residual, fundamental difficulty.

The central mathematical idea is the generalization of the trivial unit of conventional space-time symmetries into the isounit $\hat{I}$ of Chapter 2 [1].

The central physical idea is the geometrization [18] of the hyperdense medium expected in the interior of hadrons via the mutation of the Minkowski metric $\eta \rightarrow g = T\eta$, $T > 0$, with consequential construction of the applicable space-time symmetry and related relativity around the generalized unit $\hat{I} = T^{-1}$.

All the results outlined in this review, including Birkhoffian mechanics, hadronic mechanics, the reduction of hadrons to two fundamental stable particles (the protons and the electrons), etc., can be all *uniquely* derived from the above generalization of the unit and its geometrical interpretation.

* * * *

We now conclude with a few comments on the problem of compatibility of the structure models of hadrons provided by hadronic mechanics and the conventional quark theories. It should be indicated that these studies are only at the beginning and, therefore, only tentative main lines can be reviewed at this time.

First, we should recall the emphasis put by Santilli since the original memoir [2] on the fact that the fundamental problem is the identification of the hadronic constituents with massive particles freely produced in the spontaneous decays. This is due to the fact that, in Santilli's view, the primary achievements of quarks theories have been in the *classification* of hadrons into family. The problem under consideration is therefore that of achieving compatibility between the *hadronic models of structure* and the *quark models of classification* [2,5].

At any rate, if one insists in preconjecturing quarks as the physical constituents of hadrons, hadronic structure models (C.257) and (C.262) are manifestly prohibited, particularly for quarks in their current conception (see below).

To understand the problem (as well as the rather intriguing possibilities for advancement the reader must abandon the current conception of hadrons in quark theories, as ideal empty spheres with points in them, and acknowledge the physical reality according to which hadrons are the densest physical media measures in laboratory by mankind.

Whether for hadronic models of structures or for the quark models of classification, the physical media in the interior of hadrons must be quantitatively represented. Again, the technical means for this treatment are today open to scientific debate. But the lack of acknowledgement of such media, and the acceptance of the underlying concept of "point-like wavepackets" of quarks constitute such a gross approximation of the physical reality to necessarily raise issues of scientific ethics [163].

The solution submitted by Santilli is the geometrical treatment of the hadronic media via the mutation of the Euclidean metric $\delta \rightarrow g = T\delta$, $T > 0$ (§2.2) with the clear understanding that there *cannot be* only one mutation for all hadrons. This is as established already at the level of first approximation for low energies by the constant mutation worked out by Nielsen and Picek, [99], Eq. (3.170), where one can see different isometrics in the transition from pions to kaons. More compelling differences emerge when the full nonlinear [101] and nonlocal [102] nature of the interior metric is taken into account.

Once the generalization of the metric for the interior of hadrons, and its generally different value for different hadrons (of the same multiplet) is understood, then and only then the study of the compatibility of the hadronic models of structure with the quark models of classification can be properly undertaken.

Along the latter lines, Mignani and Santilli [44] have first constructed the infinite family of isotopic $\widehat{SU}(3)$ generalizations/coverings of the conventional $SU(3)$ symmetries, as the isosymmetries of the complex Santilli's space in three dimension $\hat{E}(z, g, \hat{C})$ with invariant composition

$$z_i^\dagger g_{ij} z_j = \text{inv} \ , \quad g = g^\dagger = \text{Diag}(g_{11}, g_{22}, g_{33}) > 0 \ . \tag{C.266}$$

The isotopes $\widehat{SU}(3)$ were then constructed as verifying the isocommutation laws

$$\begin{aligned} \widehat{SU}(3): \qquad \left[\hat{\lambda}_i \hat{,} \hat{\lambda}_j\right] &= \hat{\lambda}_i * \hat{\lambda}_j - \hat{\lambda}_j * \hat{\lambda}_i \\ &= \hat{\lambda}_i g \hat{\lambda}_j - \hat{\lambda}_j g \hat{\lambda}_i = 2i f_{ijk} \hat{\lambda}_k \ , \end{aligned} \tag{C.267}$$

where the f's are the conventional structure constants of $SU(3)$, g is precisely the isometric of invariant (C.266), and the $\hat{\lambda}$'s are the expected generalization of the familiar λ-matrices [194].

Via a simple generalization of the isorepresentation theory of $\widehat{SU}(2)$ of papers [24,25,29], Mignani and Santilli [44] have then constructed the following fundamental isorepresentation of $\widehat{SU}(3)$

$$\begin{aligned}
\hat{\lambda}_1 &= \begin{pmatrix} 0 & g_{11} & 0 \\ g_{22} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \hat{\lambda}_2 = \begin{pmatrix} 0 & i g_{11} & 0 \\ i g_{22} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \hat{\lambda}_3 = \begin{pmatrix} g_{11} g_{22}^2 & 0 & 0 \\ 0 & -g_{22} g_{11}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\hat{\lambda}_4 &= \begin{pmatrix} 0 & 0 & \frac{g_{11} g_{22}}{g_{33}} \\ 0 & 0 & 0 \\ g_{22} & 0 & 0 \end{pmatrix} \hat{\lambda}_5 = \begin{pmatrix} 0 & 0 & -i \frac{g_{11} g_{22}}{g_{33}} \\ 0 & 0 & 0 \\ i g_{22} & & \end{pmatrix} \hat{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{g_{11} g_{22}}{g_{33}} \\ 0 & g_{11} & 0 \end{pmatrix} \\
\hat{\lambda}_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \frac{g_{11} g_{22}}{g_{33}} \\ 0 & i g_{11} & 0 \end{pmatrix}, \quad \hat{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} g_{11} g_{22}^2 & 0 & 0 \\ 0 & g_{22} g_{11}^2 & 0 \\ 0 & 0 & -2 \frac{g_{11}^2 g_{22}^2}{g_{33}} \end{pmatrix}
\end{aligned} \tag{C.268}$$

which is evidently at the foundation of the ongoing studies of compatibility of the hadronic structure models and the quark models of classification.

The following scenario can already be predicted. First, the old idea of only one $SU(3)$ symmetry has been disproved by Santilli's Lie-isotopic formulations. In fact, we can readily state today that there exist *one abstract symmetry, say, $\mathcal{SU}(3)$, with infinitely many isomorphic but physically different realizations: first the infinite isotopes $\widehat{SU}(3)$ on isospaces $\hat{E}(z, g, \hat{C})$,* *and, finally, the conventional symmetry $SU(3)$ when the simplest conceivable Lie product $AB - BA$ is assumed.*

The discovery of the infinite family of isotopes $\widehat{SU}(3)$ evidently calls for a re-inspection of quark theories from their beginning. At this moment, we can state that the first implication of the above results [44] is that the conventional concept of quarks u, d, s, requires a generalization into that of *isoquarks* $\hat{u}, \hat{d}, \hat{s}$ as characterized by the fundamental isorepresentations of $\widehat{SU}(3)$ and Santilli's theory of mutation of particles [28]. In turn, since the matrix elements g_{11}, g_{22}, g_{33} r remain totally unrestricted by the isosymmetry, they admit particular values under which the *charges of the isoquarks can be integer* [44]. In turn, this sets the foundations for the possible compatibility between hadronic models of structure of type (C.257) or (C.262) and the isoquark models of classification, whose study is under way.

In conclusion, the Lie-isotopic formulations, when applied to the lifting of the conventional quark theories, offers genuine possibilities of identifying the isoquarks $\hat{u}, \hat{d}, \hat{s}$, with suitably mutated forms of the proton $\hat{p}$ the electron $\hat{e}^-$ and the positron $\hat{e}^+$. The strict understanding is that, as stressed earlier, these studies are just at the beginning and so much remains to be done prior to the achievement of their final resolution.

* * * *

In closing, we can say that all the possibilities of identification of the hadronic constituents with physical particles outlined in this appendix are centrally dependent of the Fundamental Test reviewed in §3.5.18 on the problem whether Einstein Relativities are exactly or only approximately valid in the interior off hadrons.

In fact, if Einstein's Relativities are indeed *exactly* valid in the arena considered (i.e., the nonlocal internal effects due to mutual wave overlappings are quantitatively ignorable), then Santilli's theory of mutation of elementary particles is inapplicable, the hadronic constituents cannot be identified with ordinary massive particles freely produced in the spontaneous decays, and the hadronic constituents must then be abstract particles not detectable in laboratory, such as the quarks in their conventional conception.

On the contrary, if the non-local internal effects of hadrons will result to be quantitatively significant, and the consequential violation of Einstein's Relativities for the interior of hadrons will be experimentally established, then the mutation of elementary particles is expected to be consequential. The consistency of structure models of type (C.257) and (C.262) is then expected to follow.

In conclusion, the possible identification of the hadronic constituents with ordinary, physical, massive particles freely produced in the spontaneous decays necessarily calls for a generalization of Einstein's Special and General Relativities inside hadrons.

This review will have achieved one of its central objectives if it succeeds in indicating that the possible experimental verification of the predictions of all available phenomenological calculations [96-102] on the violation of Einstein's theories inside hadrons, rather than being a scientific drawback, opens instead the door to possible fundamental advances of clear historical proportions such as the reduction of hadrons to only two stable particles (and their antiparticles): the proton and the electron.

It is therefore hoped that this appendix elaborates the need to finally conduct the Fundamental Tests of §3.5.18 which, after having been ignored for decades, are now truly "unprocrastinable" [98].

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Needless to say, the authors are solely responsible for the contents of this work.

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