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Number 1

LIE ALGEBRAS AND FLEXIBLE LIE-ADMISSIBLE ALGEBRAS

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ABOUT THE TOPIC

Lie's theory, with its diversification into algebras, groups and geometries, constitutes one of the most fundamental branches of contemporary mathematics.

In applied mathematics, Lie's theory is equally fundamental. For instance, contemporary physical theories (such as classical mechanics, statistical mechanics and quantum mechanics) constitute different realizations of Lie's theory beginning from their most fundamental dynamical part, the time evolution.

This book deals with a generalization of Lie algebras (beyond grading-supersymmetric extensions) which was proposed by A. A. Albert in 1948 under the name of Lie—admissible algebras, and subsequently developed by a number of mathematicians and theoreticians.

The contents of this book is therefore of fundamental relevance for pure as well as applied mathematics. On the former grounds, the Lie—admissible algebras permit the broadening of the mathematical structure of all branches of mathematics dealing with Lie algebras. On the latter grounds, the Lie—admissible algebras permit the generalization of physical theories for a deeper understanding of nature.

The book is authored by one of the foremost mathematical leaders in the field.

The book is indispensable for all mathematicians interested in fundamental advances, as well as for all theoreticians interested in the broadening of the structure of contemporary physical theories.

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TO

Karen, Peggy, Jane, and Michael

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PREFACE

Following the long history of Lie algebras in physics, and since the introduction of Jordan algebras by physicist P. Jordan, there have been considerable efforts to generalize the formalism of quantum mechanics by means of other nonassociative algebras, such as Jordan, noncommutative Jordan, and octonion algebras.

Since A. A. Albert introduced Lie–admissible algebras in 1948, very little has been known about the structure of these algebras until the recent emergence of Lie–admissible algebras as a fundamental methodological tool in theoretical physics and mechanics. As far as I am aware, R. M. Santilli is the first physicist who became interested in Lie-admissible algebras. In an article written in 1967, he states: "As is known, there have been attempts to introduce new algebraic structures in physics other than Lie algebras (L.A.). One of the most interesting attempts is the Jordan investigation on the r–number algebras today called (commutative) Jordan algebras (C.J.A.), which however have not been successful in their physical applications. We personally think that a possible reason for this disappointment in elementary particle physics may be the want of L.A. content in the C.J.A. In other words, L.A. should not be abandoned, but might be expanded. For instance, the validity of L.A. for free particles is well known. It may be interesting to investigate the possible validity of new algebraic structures of an interacting or decaying region but only in such a way that the standard procedures corresponding to the free states remain unchanged, that is, preserving in any case a well–defined L.A. content."

Both nonflexible and flexible Lie–admissible algebras arise in classical and quantum mechanics as a generalization of conventional mechanics. However, the general Lie–admissible algebras are too broad and diversified to provide at this moment a fruitful structure theory. On the other hand, the structure of certain classes of flexible Lie–admissible algebras is closely related to the theory of Lie algebras of characteristic zero. Especially, the structure and representation of semisimple Lie algebras of characteristic zero serve as a reasonable model in the classification of certain classes of simple flexible Lie–admissible algebras.

This monograph is based on lectures delivered for a first year graduate course at Seoul National University while I was visiting under the project of the SNU–U.S.AID Graduate Program for Basic Sciences in 1979–1980. My intention was to provide elements of Lie algebras and flexible Lie-admissible algebras. The subject matter in Lie algebras was designed to set up the groundwork for the structure of certain classes of flexible Lie–admissible algebras rather than to provide a comprehensive account of the general theory of Lie algebras. Realizing that the majority of the audience had different motivations and backgrounds, and had no experience in Lie algebras, I tried to convey the material in a self–contained manner, without unnecessary difficulty. Accordingly, the amount of material in Lie algebras grew up to a moderate book length.

The treatment of Lie algebras followed the three well–known books: N. Jacobson, Lie algebras (Interscience, New York, 1962); D. J. Winter, Abstract Lie algebras (MIT Press, 1972); J. E. Humphreys, Introduction to Lie algebras and representation theory (Springer–Verlag, New York, 1972). The basic classical theorems, including the theorems of Weyl and Levi, are drawn from Winter's book which provides quick access to the classification of split semisimple Lie algebras of characteristic zero. The classification and construction of simple Lie algebras were treated as in Jacobson's book. The discussion for root systems followed the classical approach rather than the abstract one. The exposition of representation theory was based on Humphreys' book.

I had to omit many standard topics in Lie algebras, such as cohomology, theorems of Ado and Iwasawa, classification over non–algebraically closed fields, character formulas, and multiplicity formulas, which the interested reader can pursue in the books of Jacobson and Humphreys, cited above. For the discussion of certain flexible Lie–admissible algebras of prime characteristic in Chapter 7, I stated some elementary results on classical Lie algebras without proof which can be found in G. B. Seligman's book, Modular Lie algebras (Springer–Verlag, New York, 1967).

Chapters 6 and 7 are devoted to the structure of some special classes of flexible Lie–admissible algebras. The main objective in Chapter 6 is to classify finite–dimensional flexible Lie–admissible algebras over an algebraically closed field of characteristic zero whose associated Lie algebras are reductive. The central idea for this classification is the notion of adjoint operators which was first introduced by E. P. Wigner in 1941, for the Lie algebra of the SU(2) group. The general case of adjoint operators was later studied in particle physics, notably by Okubo. The main result is that simple Lie algebras of type A_n $(n \geq 2)$ alone allow a non–zero symmetric adjoint operator and otherwise, all adjoint operators are multiples of the adjoint mapping. Thus, only simple Lie algebras of type A_n $(n \geq 2)$ result in new simple flexible Lie–admissible algebras.

Chapter 7 is concerned with the structure of flexible Lie–admissible algebras of arbitrary characteristic. It is shown that the algebras turn out to be Lie algebras when their associated Lie algebras are classical in the sense of Seligman or generalized Witt algebras. In the final section, we investigate some basic structure of the mutation of an associative algebra which originated from Santilli's generalization of classical and quantum mechanics. When the mutation parameters are invertible, flexibility in a mutation of an associative algebra is virtually equivalent to all other nonassociative identities.

The material presented here is far from complete. I have had to omit many recent advances made for the structure of flexible Lie–admissible algebras which is at present quite diversified in nature. It is hoped that a collective work on Lie–admissible algebras will be published in a readable form in the near future.

In writing this monograph, I am indebted to a number of people. I should like to express my gratitude to R. M. Santilli who first brought to my attention the relevance of Lie–admissible algebras to physics in 1977. Since that time, his continual encouragement has been most influential in pursuing the study of Lie–admissible algebras. It is his suggestion to publish this monograph. I also owe a great debt to S. Okubo for making available the current developments in physics relating to Lie–admissible algebras and for many invaluable communications. The majority of material in Chapters 6 and 7 is drawn from joint works with Okubo. I wish to thank my friend and teacher M. L. Tomber who first aroused my interest in Lie–admissible algebras. I thank G. M. Benkart and J. M. Osborn for numerous conversations which have been immensely helpful in writing the last two chapters.

To the SNU–U.S.AID Graduate Program for Basic Sciences and the Department of Mathematics of Seoul National University who arranged my visit during which the majority of this monograph was written, I express my sincere thanks for financial support and generous hospitality. I would like to acknowledge also occasional support from DOE contract DE–AC02–80ER10651, and extensions A001, and A002. Special thanks are also due to J. S. Cross for a careful reading of the manuscript. It is a pleasure to acknowledge the great help provided by the editorial staff of the Hadronic Press.

I am, of course, solely responsible for the errors or shortcomings that remain.

November 1, 1982
Cedar Falls

Hyo C. Myung

1. NONASSOCIATIVE ALGEBRAS

1.1. Basic definitions

An (nonassociative) <u>algebra</u> A over a field F is a vector space over F with a multiplication $A \times A \to A$, denoted by xy, such that

$$(\alpha x + \beta y)z = \alpha(xz) + \beta(yz),$$

$$z(\alpha x + \beta y) = \alpha(zx) + \beta(zy).$$

$\alpha, \beta \in F$, $x, y, z \in A$.

Denote the associator (x,y,z) and commutator $[x,y]$ in A by

$$(x,y,z) = (xy)z - x(yz),$$

$$[x,y] = xy - yx.$$

If A is finite-dimensional over F, let $u_1, \ldots, u_n$ be a basis for A. Let

$$u_i u_j = \sum_{k=1}^{n} \gamma_{ij}^{k} u_k \, . \tag{1.1}$$

The n^3 constants $\gamma_{ij}^{k} \in F$ are called the <u>structure constants</u> for A, which determine a unique element in the space $F^{n^3} = F \times \ldots \times F$ (n^3 times). Conversely any element $(\gamma_{ij}^{k}) \in F^{n^3}$ determines a unique nonassociative algebraic structure on the underlying vector space A via (1.1). Thus the set of algebras with underlying vector space A over F is identified with F^{n^3}.

<u>Problem 1.1.1.</u> Determine all algebras of dimension 1 or 2.

The definitions of the terms, such as left or right ideal (two-sided)ideal, homomorphism, kernel, quotient algebra, isomorphism theorems, and direct sum in an algebra can be stated exactly the same as in an associative algebra. An element $1 \in A$ is called a <u>unit element</u> for A if $1x = x1 = x$ for $x \in A$. For an algebra A over F, let $A_1 = F \oplus A$ be the vector space direct sum of F and A. Define a multiplication in A_1 by

$$(\alpha + a)(\beta + b) = \alpha\beta + (\beta a + \alpha b + ab), \tag{1.2}$$

$\alpha, \beta \in F$, $a, b \in A$. Then A_1 becomes an algebra over F with unit element $1 \in F$.

Let A, B be algebras over F and let $B \otimes_F A = B \otimes A$ be the tensor product of A and B. Defining a multiplication in $B \otimes A$ by

$$(y_1 \otimes x_1)(y_2 \otimes x_2) = (y_1 y_2) \otimes (x_1 x_2), \ x_i \in A, \ y_i \in B$$

makes $B \otimes A$ into an algebra over F. If B has 1, $1 \otimes A$ is a subalgebra of $B \otimes A$ which is isomorphic to A and is identified with A. If A and B are finite-dimensional over F,

$$\dim_F B \otimes A = (\dim B)(\dim A).$$

In particular, if $B = K$ is an extension field of F, then in $A_K \equiv K \otimes A$ we identify x with $1 \otimes x$, $x \in A$, and A_K becomes an algebra over K via

$$\alpha(\Sigma\, \beta_i x_i) = \Sigma\, \alpha\beta_i x_i \ ,$$

$\alpha, \beta_i \in K$, $x_i \in A$. If A is finite-dimensional over F and $u_1, \ldots, u_n$ is a basis for A over F, it is readily seen that it is also a basis for A_K over K and so $\dim_K A_K = n$. A_K is called the <u>scalar extension</u> of A to K. While the scalar extension A_K is often useful, it should be stressed that some algebraic properties in A may be collapsed in A_K. An algebra A is called <u>simple</u> if $AA \neq 0$ and A has no proper iedals.

<u>Exercise 1.1.1.</u> Give an example of an algebra A such that A is simple over F but A_K is not over K for some extension K of F.

1.2. Modules

Definition 1.2.1. Let S be a set. An S-module over F is a vector space V over F together with a mapping $V \times S \to V$, denoted by $(x,s) \to xs$, such that $(\alpha x + \beta y)s = \alpha(xs) + \beta(ys)$, $\alpha, \beta \in F$, $x,y \in V$, $s \in S$. //

For an S-module V over F and a subset $W \subset V$ we let WS be the subspace of V spanned by xs, $x \in W$, $s \in S$. An S-submodule of V is a subspace W of V such that $WS \subset W$. If W is an S-submodule of V, the vector space V/W becomes an S-module over F via $(x + W)s = xs + W$, $x \in V$, $s \in S$. An S-homomorphism from an S-module V into an S-module V' over F is a linear transformation $f: V \to V'$ over F such that $f(xs) = f(x)s$, $x \in v$, $s \in S$. The isomorphism theorems for S-modules are straightforward generalizations of the usual ones.

Definition 1.2.2. (1) An S-complement of an S-submodule W of V is an S-submodule W' of V such that $V = W \oplus W'$.

(2) An S-module V is S-completely reducible if $VS = V$ and every S-submodule of V has an S-complement.

(3) V is S-irredcible if $VS = V$ and V has no proper S-submodules. //

Lemma 1.2.1. Let V be S-completely reducible and let W be an S-submodule of V. Then W and V/W are S-completely reducible.

Proof. (1) Let W_0 be an S-submodule of W and let W_0' be an S-complement of W_0 in V, so that $V = W_0 \oplus W_0'$. One sees that $W = W \cap (W_0 \oplus W_0')$ $= W_0 \oplus (W \cap W_0')$. Since $VS = WS \oplus W'S = V = W \oplus W'$, $WS = W$. Thus W is S-completely reducible.

(2) Let $\overline{V} = V/W$ and let $\overline{P}$ be an S-submodule of $\overline{V}$. If P denotes the inverse image of $\overline{P}$ under the natural S-homomorphism : $V \to \overline{V}$, $V = P \oplus P'$ for an S-submodule P' of V. Thus $\overline{V} = \overline{P} \oplus \overline{P}'$ since $P \supset W$. Clearly $VS = V$ implies $\overline{V}S = \overline{V}$ and so $\overline{V}$ is S-completely reducible. //

Lemma 1.2.2. Every nonzero S-completely reducible module V has a nonzero S-irreducible S-submodule.

Proof. Pick an $x \neq 0$ in V. If every nonzero S-submodule of V contains x, the proof is done. If there is an S-submodule not containing x, by Zorn's lemma one chooses a maximal S-submodule W of V such that $x \notin W$. Thus $V = W \oplus W'$, W' an S-complement, and $W' \neq 0$. Then W' is S-irreducible, since if not, $W' = W_1 \oplus W_2$ for some proper S-submodules W_1, W_2 (Lemma 1.2.1); but then $x \in (W + W_1) \cap (W + W_2)$ by the maximality of W, so that $x \in W$, a contradiction. //

<u>Theorem 1.2.3.</u> The following are equivalent.

1) V is S-completely reducible ;

2) $V = \Sigma V_i$ for some family $\{V_i\}$ of S-irreducible S-submodules of V ;

3) $V = \Sigma \oplus V_i$ for some family $\{V_i\}$ of S-irreducible S-submodules of V.

<u>Proof.</u> 1) $\Rightarrow$ 2): Assume V is S-completely reducible and let $W = \Sigma V_i$ where $\{V_i\}$ is the collection of all S-irreducible S-submodules of V. Let $V = W \oplus W'$, W' an S-complement of W. Since W' has no nonzero S-irreducible S-submodules, by Lemmas 1.2.1 and 1.2.2, $W' = \{0\}$. Thus $V = \Sigma V_i$.

2) $\Rightarrow$ 3) : Let $\{W_i\}$ be a family of S-submodules. Call $\{W_i\}$ <u>direct</u> if ΣW_i is direct. Now, assume $V = \Sigma V_i$ for some family $\{V_i\}$ of S-irreducible S-submodules of V. By Zorn's lemma, one can choose a maximal direct family $\{V_k\}$ of S-submodules from the family $\{V_i\}$. Thus $V = \Sigma \oplus V_k$, for otherwise $V_i \not\subset \Sigma \oplus V_k$ for some i . But then $V_i \cap \Sigma \oplus V_k \neq V_i$ and so $V_i \cap \Sigma \oplus V_k = \{0\}$ since V_i is S-irreducible. Thus $V_i \oplus \Sigma \oplus V_k$ is direct and this contradicts the maximality of $\{V_k\}$.

3) $\Rightarrow$ 1) : Let $V = \Sigma \oplus V_i$ for some family of S-irreducible S-submodules of V. Let W be an S-submodule of V. By Zorn's lemma, one can pick a maximal S-submodule W' with $W \cap W' = 0$. For each i, $(W \oplus W') \cap V_i = 0$ or V_i. In the first case, $W \oplus W' \oplus V_i$ is direct and so $W \cap (W' \oplus V_i) \neq 0$, so $V_i = \{0\}$ by the maximality of W'. In the second case, $V_i \subset W \oplus W'$ and so $V = W \oplus W'$. //

1.3. Jordan and Fitting decompositions

If V is an S-module over F, then for $T \varepsilon S$, T_V denotes the linear transformation of V defined by $xT_V = xT$, $x \varepsilon V$. We say that $T \varepsilon S$ (or T_V) is split over F if the eigenvalues of T_V are in F or equivalently the minimum polynomial of T_V is factored over F into linear factors. Also, S is split over F if every element in S is split over F.

Definition 1.3.1. For $T \varepsilon S$ and $\alpha \varepsilon F$, let $V_\alpha(T) = \{x \varepsilon V | \ x(T_V - \alpha)^n = 0 \text{ for some } n > 0\}$. //

<u>Theorem 1.3.1</u>. Let V be a finite-dimensional S-module over F. If $T \varepsilon S$ is split over F then

$$V = \Sigma_{\alpha \varepsilon F} \oplus V_\alpha(T) .$$

<u>Proof</u>. Let $f(X)$ be the minimum polynomial of T_V and let $f(X) = \Pi(X - \alpha_i)^{e_i}$ with $\alpha_i \varepsilon F$ distinct. Setting $h_i(X) = f(X)/(X - \alpha_i)^{e_i}$, since the $h_i(X)$ are relatively prime, one finds polynomials $g_i(X)$ such that $\Sigma\, g_i(X)h_i(X) = 1$. Let

$$V_i = Vh_i(T_V) .$$

Since $x = \Sigma\, (xg_i(T_V))\, h_i(T_V)$ for $x \varepsilon V$, we have $V = \Sigma\, V_i$. Also, $V_i \subset V_{\alpha_i}(T)$ since $V_i(T_V - \alpha_i)^{e_i}$ $= Vh_i\,(T_V)(T_V - \alpha_i)^{e_i} = Vf(T_V) = \{0\}$.
Thus $V = \Sigma\, V_{\alpha_i}(T)$ and it remains to show that the sum is direct. Let $x \varepsilon V_{\alpha_i}(T) \cap \Sigma_{j \neq i} V_{\alpha_j}(T)$. Then the ideal $J = \{h(X) \varepsilon F[X] \mid xh(T_V) = 0\}$ contains $(X - \alpha_i)^{k_i}$ and $\Pi_{j \neq i}(X - \alpha_j)^{k_j}$ for some k_i, k_j. Since these polynomials are relatively prime, $J = F[X]$ and $1 \varepsilon J$, so $x = x1 = 0$. //

<u>Remark 1.3.1</u>. In fact we have shown that $Vh_i(T_V) = V_{\alpha_i}(T)$, $i = 1,2,\ldots$. //

Definition 1.3.2. Let $\dim V < \infty$ and let T be an element in $\mathrm{Hom}\, V$ which is split over F. Then the semisimple part of T is the element $T_s \in \mathrm{Hom}\, V$ such that $T_s \mid_{V_\alpha(T)} = \alpha I_{V_\alpha(T)}$ for $\alpha \in F$. The nilpotent part of T is $T_n = T - T_s$. The decomposition $T = T_s + T_n$ is called the Jordan decomposition of T. If $T = T_s$, T is called semisimple. If $T = T_n$, T is called nilpotent. //

Let W be a T-stable subspace of V. It is evident that $W = \Sigma\, W_\alpha(T)$ where $W_\alpha(T) = V_\alpha(T) \cap W$. Hence if T is semisimple on V then so is $T|_W$ on W.

Since $xT_n = x(T - \alpha)$, $x \in V_\alpha(T)$, T_n and T_s stablize each $V_\alpha(T)$ and so T_n and T_s commute. Note that T is nilpotent if and only if $T^n = 0$ for some $n > 0$.

Lemma 1.3.2. Let $T \in \mathrm{Hom}\, V$ be split over F. Let $T = S + N$, where S is semisimple, N is nilpotent, and $SN = NS$. Then $S = T_s$ and $N = T_n$.

Proof. Since $SN = NS$, N and S stablize each $V_\alpha(T)$. If $xS = \beta x$ for $x \in V_\alpha(T)$, $xN = x(T - \beta)$ and so S has only one eigenvalue α on $V_\alpha(T)$. Since S is semisimple, this implies $S \mid_{V_\alpha(T)} = \alpha I_{V_\alpha(T)}$, $\alpha \in F$ and $S = T_s$. Thus $N = T - S = T - T_s = T_n$. //

Theorem 1.3.3. Let $T \in \mathrm{Hom}\, V$ be split over F. Define $\mathrm{ad}\, T \in \mathrm{Hom}\,(\mathrm{Hom}\, V)$ by

$$S\, \mathrm{ad}\, T = [S,T] = ST - TS, \quad S \in \mathrm{Hom}\, V .$$

Then $\mathrm{ad}\, T$ is split over F, and $(\mathrm{ad}\, T)_s = \mathrm{ad}\, T_s$ and $(\mathrm{ad}\, T)_n = \mathrm{ad}\, T_n$.

Proof. First, note that $\mathrm{ad}\,[S,T] = [\mathrm{ad}\, S, \mathrm{ad}\, T]$. Thus $\mathrm{ad}\, T_s\, \mathrm{ad}\, T_n = \mathrm{ad}\, T_n\, \mathrm{ad}\, T_s$ since $T_s T_n = T_n T_s$. In view of Lemma 1.3.2, it suffices to show that $\mathrm{ad}\, T_s$ is split over F and semisimple, and $\mathrm{ad}\, T_n$ is nilpotent. One can choose a basis $e_1, \ldots, e_n$ for V consisting of eigenvectors for T_s, so that $e_i T_s = \alpha_i e_i$ for $1 \leqq i \leqq n$. Let $\{E_{ij} \mid 1 \leqq i, j \leqq n\}$ be the basis for $\mathrm{Hom}_F V$ such that $e_k E_{ij} = \delta_{ik} e_j$ where δ_{ik} is the Kronecker delta. Since $T_s = \Sigma\, \alpha_i E_{ii}$, we have $E_{ij}\, \mathrm{ad}\, T_s = (\alpha_j - \alpha_i) E_{ij}$. Thus $\mathrm{ad}\, T_s$ is split over F and semisimple. Let $T_n^{\,q} = 0$. Since $Y\,(\mathrm{ad}\, T_n)^m = \Sigma\, T_n^r Y T_n^{m-r}$, $(\mathrm{ad}\, T_n)^{2q} = 0$ and $\mathrm{ad}\, T_n$ is nilpotent. //

Definition 1.3.3. Let $T \in \mathrm{Hom}_F V$, $\dim V < \infty$. Let $V_0 = V_0(T) = \bigcup_{i=0}^{\infty} \ker T^i$ and $V_* = V_*(T) = \bigcap_{i=1}^{\infty} VT^i$. Then V_0 and V_* are called the Fitting 0-component and 1-component of V with respect to T. //

<u>Fitting Lemma 1.3.4</u>. For any $T \in \mathrm{Hom}_F V$, $V = V_0 \oplus V_*$, V_0 and V_* are T-stable, and T is bijective on V_*.

<u>Proof</u>. Note that $V \supset VT \supset VT^2 \supset \ldots$ and $\ker T \subset \ker T^2 \subset \ldots$ Since $\dim V < \infty$, there exist p,q such that $VT^p = VT^{p+1} = \ldots$ and $\ker T^q = \ker T^{q+1} = \ldots$ If $t = \max(p,q)$ then $V_0 = \ker T^t$ and $V_* = VT^t$. For $x \in V$, one gets $sT^t = yT^{2t}$ for some $y \in V$ and $x = (x - yT^t) + yT^t$ with $x - yT^t \in V_0$ and $yT^t \in V_*$, so $V = V_0 + V_*$. If $z \in V_0 \cap V_*$, $z = uT^t$, $u \in V$, and $0 = zT^t = uT^{2t}$. Since $\ker T^t = \ker T^{2t}$, $uT^t = z = 0$ and so $V = V_0 \oplus V_*$. Since $V_*T = VT^{t+1} = VT^t = V_*$, T is surjective on V_* and so bijective on V_*. //

Let $f(X)$ be the characteristic polynomial of T on V and let $f(X) = \Pi f_i(X)^{e_i}$ be the prime factorization of $f(X)$ with $f_1(X) = x^{e_1}$. Put

$$V_i = \{x \in V \mid xf_i(T)^k = 0 \text{ for some } k > 0\}.$$

Then, as in the proof of Theorem 1.3.1, we see that

$$V = \Sigma \oplus V_i . \tag{1.3}$$

The decomposition (1.3) is called the <u>primary decomposition</u> of V relative to T. It is routine to check that T is bijective on each V_i for $i \geqq 2$, $V_1 = V_0(T)$ and

$\Sigma_{i \geq 2} V_i = V_*(T)$. Since $V = V_0 \oplus V_*$ is direct, $f(X)$ is the product of the characteristic polynomials of T on V_0 and V_* . Thus $\dim V_0 = e_1$, the multiplicity of the eigenvalue 0 in $f(X)$. //

1.4. Derivations

Let A be an algebra over F . For $a \in A$, let L_a and R_a be the elements in $\mathrm{Hom}_F A$ defined by $xL_a = ax$, $xR_a = xa$ for $x \in A$. We call L_a and R_a the left and right multiplications in A by a . The adjoint mapping $\mathrm{ad}\, a$ by a is the element in $\mathrm{Hom}\, A$ defined by $x\, \mathrm{ad}\, a = [x,a] = xa - ax$, $x \in A$. If S is a subset of A , denote $L_S = \{L_a \mid a \in S\}$.

For an algebra A over F , denote by A^- the algebra with multiplication $[x,y] = xy - yx$ defined on the same vector space as A . If char $F \neq 2$, define A^+ as the algebra over F with multiplication $x \cdot y = \frac{1}{2}(xy + yx)$, called the Jordan product, with the same underlying space as A .

<u>Definition 1.4.1</u>. An algebra A over a field F with multiplication $[xy]$ is called a Lie algebra if A satisfies

1) $[xx] = 0, \ x \in A$,
2) the Jacobi identity $[[xy]z] + [[yz]x] + [[zx]y] = 0$ $x,y,z \in A$. //

Note that if char $F \neq 2$, $[xx] = 0$ is equivalent to the <u>anticommutative law</u> $[xy] = -[yx]$. Various types of Lie algebras will be discussed later.

<u>Definition 1.4.2</u>. An algebra A is said to be Lie-admissible if the associated algebra A^- is a Lie algebra, that is, A^- satisfies the Jacobi identity

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0 . \quad //$$

It is readily seen that any Lie and associative algebras are Lie-admissible. Various classes of (nonassociative) Lie-admissible algebras will be explored later. To express the Lie-admissible condition into a more convenient form, we introduce the notation

$$S(x,y,z) \equiv (x,y,z) + (y,z,x) + (z,x,y)$$

where (x,y,z) is the associator of x,y,z in A. Then, by a direct computation, one checks that the identity

$$[xy,z] + [yz,x] + [zx,y] = S(x,y,z) \tag{1.4}$$

holds in any algebra A. Thus A satisfies the identity

$$S(x,y,z) - S(x,z,y) = [[x,y],z] + [[y,z],x] + [[z,x],y].$$

Therefore, we have

Lemma 1.4.1. An algebra A over F is Lie-admissible if and only if A satisfies $S(x,y,z) = S(x,z,y), x,y,z \in A$. //

While the algebraic origin of Lie-admissible algebras arises as a byproduct of the study of nonassociative algebras defined by identities, the analytic orgin of Lie-admissibility stems from a nonassociative product formed in terms of partial differential equations in a space of differentiable functions which are defined on a C^{∞}-manifold. Lie-admissible algebras have been utilized to construct a nonassociative quantization of forces or couplings not derivable from a potential. Thus Lie-admissible algebras have direct physical relevances in both classical and quantum mechanics (see R.M. Santilli, "Lie-admissible approach to the hadronic structure," Vol. 1,2, Hadronic Press, Nonantum, Mass. 1979).

It has been observed that the general Lie-admissible algebras are too broad tc obtain a fruitful structure theory (see H.C. Myung, "On nonflexible Lie-admissible algebras", Hadronic J. 1,(1978), 1021-1143).

<u>Definition 1.4.3</u>. A Lie-admissible algebra A is called flexible Lie-admissible if A is a flexible algebra, that is, A satisfies the flexible law

$$(xy)x = x(yx) \; . \qquad // \tag{1.6}$$

Note that the associative and Lie algebras are flexible Lie-admissible. The flexible law is also written as $(x,y,x) = 0$ which is equivalent to

$$(x,y,z) = -(z,y,x) \tag{1.7}$$

if char $F \neq 2$. The study of flexible Lie-admissible algebras was first initiated by A.A.Albert ("Power-associative rings" Trans. Amer. Math. Soc. 64(1948), 552-597), and the structure of these algebras have been investigated by Laufer and Tomber, Myung, and Okubo (for a review, see H.C.Myung, "Lie-admissible algebras", Hadronic J. 1(1978), 169-193). Applications of flexible Lie-admissible algebras to physics have been recently pointed out in particular reference to a generalization of the Heisenberg equation by a number of authors (see Santilli's monographs cited above; C.N. Ktorides, H.C. Myung and R.M. Santilli, "Elaboration of the recently proposed test of Pauli's principle under strong interactions", Phys. Rev. D22 (1980), 892-907; H.C. Myung and R.M. Santilli, "Further studies on the recently proposed experimental test of Pauli's exclusion principle for the strong interactions", Proc. of the 2nd Workshop on Lie-admissible Formulations, held at Harvard University, August 1979, Hadronic J. 3(1979), 194-255, S. Okubo, "Non-associative quantum mechanics via flexible Lie-admissible algebras, Proc. of the 3rd Workshop on Current Problems in High Energy Particle Theory, held at Florence, Italy, 1979, Edit. R. Casabuni, G. Domokos and S.K. Domokos, John Hopkins Univ. Press, (1979), 103-120, and "A generalization of Hurwitz theorem and flexible Lie-admissible algebras", Proc. of the 2nd Workshop on Lie-admissible Formulations, Hadronic J. 3(1979), 1-52).

Definition 1.4.3. Let A be an algebra over a field F. Then an element $D \in \mathrm{Hom}_F A$ is called a derivation of A if

$$(xy)D = (xD)y + x(yD) \ , \quad x,y \in A \ . \tag{1.8}$$

Denote by $\mathrm{Der}\, A$ the set of derivations of A. //

In terms of the left and right multiplications L_x and R_x, (1.8) is expressed as

$$L_x D = L_{xD} + DL_x \quad \text{or} \quad L_{xD} = [L_x , D] , \tag{1.9}$$

$$R_y D = DR_y + R_{yD} \quad \text{or} \quad R_{yD} = [R_y , D] \tag{1.10}$$

for $x,y \in A$. Since the Jacobi identity in A^- is equivalent to the fact that all adjoint mappings $ad\ x$ are derivations of A^-, in view of (1.9) or (1.10) A is Lie-admissible if and only if

$$ad[x,y] = [ad\ x, ad\ y] \tag{1.11}$$

for all $x,y \in A$. In this case, notice that $ad\ x$ is not necessarily a derivation of A (why?) .

Suppose that A is an algebra over a field F of char $\neq 2$. For $x \in A$, define $T_x \in \text{Hom}\ A$ by $T_x = \frac{1}{2}(R_x + L_x)$. We contend that the following identities are equivalent.

$$(xy)x = x(yx) \text{ ; the flexible law ,} \tag{1.12}$$

$$(x,y,z) + (z,y,x) = 0 , \tag{1.13}$$

$$(x,y,z) + (z,y,x) + (x,z,y) + (y,z,x) = (y,x,z) + (z,x,y) , \tag{1.14}$$

$$[x,y.z] = [x,y].z + y.[x,z] . \tag{1.15}$$

As noted earlier, (1.12) $\Leftrightarrow$ (1.13) . Clearly, (1.13) $\Leftrightarrow$ (1.14), while (1.14) $\Leftrightarrow$ (1.12) with $x = z$. By a direct expansion, one sees (1.14) $\Leftrightarrow$ (1.15) . Since (1.15) is to say that each $\text{ad } x$ is a derivation of A^+ , we can state

<u>Lemma 1.4.2</u>. Let $\text{char } F \neq 2$. Then A is flexible if and only if $\text{ad } A \subseteq \text{Der } A^+$ if and only if $T_{[x,y]} = [T_x, \text{ad } y]$ $x,y \in A$. //

<u>Theorem 1.4.3</u>. Let A be an algebra over a field F of $\text{char} \neq 2$. Then A is flexible Lie-admissible if and only if $\text{ad } A \subseteq \text{Der } A$.

<u>Proof</u>. If A is flexible and Lie-admissible, then, in view of (1.11) and Lemma 1.4.2, we have

$$\text{ad}[x,y] + 2T_{[x,y]} = [\text{ad } x, \text{ad } y] + [2T_x, \text{ad } y]$$

which implies

$$R_{[x,y]} = [R_x, \text{ad } y] \tag{1.16}$$

for $x,y \in A$, so, by (1.10), $\text{ad } A \subseteq \text{Der } A$. If $\text{ad } A \subseteq \text{Der } A$, $\text{ad } A \subseteq \text{Der } A^-$ and A is Lie-admissible. Also, (1.16) implies the flexible law with $x = y$. //

<u>Theorem 1.4.4.</u> For any algebra A over F, $\operatorname{Der} A$ is a subalgebra of $(\operatorname{Hom}_F A)^-$. That is, $\operatorname{Der} A$ is closed under the Lie product $[D,E] = DE - ED$.

<u>Proof.</u> By the Jacobi identity and (1.9),

$$[L_x,[D,E]] = [[L_x,D],E] + [D,[L_x,E]]$$

$$= [L_{xD},E] + [D,L_{xE}]$$

$$= L_{xDE} - L_{xED}$$

$$= L_{x[D,E]} \quad . \qquad //$$

<u>Lemma 1.4.5.</u> For $D \in \operatorname{Der} A$ and $\alpha,\beta \in F$,

$$(xy)(D - \alpha - \beta)^n = \sum_{m=0}^{n} \binom{n}{m} x(D-\alpha)^m \, y(D-\beta)^{n-m}$$

for all positive integers n and $x,y \in A$.

<u>Proof.</u> By induction, if $n = 1$,

$$(xy)(D - \alpha - \beta) = (xD)y - \alpha(xy) + x(yD) - \beta(xy)$$

$$= x[y(D - \beta)] + [x(D - \alpha)]y \; .$$

Assume $(xy)(D - \alpha - \beta)^{n-1} = \sum \binom{n-1}{m} x(D - \alpha)^m \, y(D - \beta)^{n-m-1}$.

Then $(xy)(D - \alpha - \beta)^n$

$$= \sum \binom{n-1}{m} x(D - \alpha)^{m+1} \, y(D - \beta)^{n-(m+1)}$$

$$+ \sum \binom{n-1}{m} x(D - \alpha)^m \, y(D - \beta)^{n-m}$$

$$= \Sigma \binom{n}{m} x(D - \alpha)^m \, y(D - \beta)^{n-m}$$

since $\binom{n-1}{m-1} + \binom{n-1}{m} = \binom{n}{m}$ for $n > m$. //

<u>Corollary 1.4.6. (Leibniz's rule)</u>. For $D \in \text{Der } A$,

$$(xy)D^n = \Sigma \binom{n}{m} xD^m \, yD^{n-m} . \qquad //$$

Let $p > 0$ be a prime. Since $p \mid \binom{p}{i}$ for $1 \leqq i < p$, we have from Corollary 1.4.6

<u>Corollary 1.4.7</u>. If $\text{char } F = p > 0$ then $D^p \in \text{Der } A$ for $D \in \text{Der } A$. //

<u>Corollary 1.4.8</u>. Let A be a finite-dimensional algebra over F. Let $D \in \text{Der } A$. Then $A_\alpha(D) \, A_\beta(D) \subseteq A_{\alpha+\beta}(D)$ for $\alpha, \beta \in F$ and $A_0(D)$ is a subalgebra of A. If D is split over F, then $A_0(D)A_*(D) \subseteq A_*(D)$ and $A_*(D)A_0(D) \subseteq A_*(D)$.

<u>Proof</u>. Let $x \in A_\alpha(D)$, $y \in A_\beta(D)$ and pick an r such that $x(D - \alpha)^r = y(D - \beta)^r = 0$. Letting $n = 2r$ in Lemma 1.4.5, $(xy)(D - \alpha - \beta)^n = 0$ and $A_\alpha(D)A_\beta(D) \subseteq A_{\alpha+\beta}(D)$. So, $A_0(D)$ is a subalgebra of A. If D is split over F, by the remark following the primary decomposition we have $A_*(D) = \Sigma_{\alpha \neq 0} A_\alpha(D)$ and the second assertion follows from this. //

<u>Lemma 1.4.9</u>. Let A be finite-dimensional over F and let $D \in \mathrm{Hom}_F A$ be semisimple and split over F. Then $D \in \mathrm{Der}\, A$ if and only if $A_\alpha(D)A_\beta(D) \subseteq A_{\alpha+\beta}(D)$ for all $\alpha, \beta \in F$.

<u>Proof</u>. Let $x \in A_\alpha(D)$, $y \in A_\beta(D)$. If $xy \in A_{\alpha+\beta}(D)$ then since D is semisimple, $(xy)D = (\alpha + \beta)xy = (\alpha x)y + x(\beta y) = (xD)y + x(yD)$. Since $A = \Sigma_\alpha A_\alpha(D)$, $D \in \mathrm{Der}\, A$. The converse follows from Corollary 1.4.8. //

<u>Corollary 1.4.10</u>. Let $\dim A < \infty$ and let $D \in \mathrm{Der}\, A$ be split over F. Then D_s and D_n are in $\mathrm{Der}\, A$.

<u>Proof</u>. If D is split over F, so is D_s. Since $A = \Sigma_\alpha \oplus A_\alpha(D) = \Sigma_\alpha \oplus A_\alpha(D_s)$ and $A_\alpha(D) \subset A_\alpha(D_s)$, $A_\alpha(D) = A_\alpha(D_s)$. Thus $D_s \in \mathrm{Der}\, A$ by Lemma 1.4.9, and $D_n = D - D_s \in \mathrm{Der}\, A$. //

<u>Lemma 1.4.11</u>. Let $D \in \mathrm{Der}\, A$ be nilpotent and let $\mathrm{char}\, F = 0$. Then $\exp D = e^D = \Sigma_{m=0}^{\infty} D^m/m!$ is an automorphism of A.

<u>Proof</u>. Let n be such that $D^{n+1} = 0$. Then

$$xe^D ye^D = \sum_{m=0}^{2n} \sum_{i=0}^{m} \frac{xD^i}{i!} \frac{yD^{m-i}}{(m-i)!} .$$

By Leibniz's rule

$$(xy)e^D = \sum_{m=0}^{n} (xy) \frac{D^m}{m!} = \sum_{m=0}^{n} \sum_{i=0}^{m} \frac{1}{i!} \binom{m}{i} xD^i yD^{m-i}$$

$$= \sum_{m=0}^{2n} \sum_{i=0}^{m} \frac{xD^i}{i!} \frac{yD^{m-i}}{(m-i)!}$$

since $xD^i = yD^i = 0$ for $i > n$. //

1.5. Solvability and radical

Let A be an algebra over F. For subspaces B, C of A, denote by BC the subspace of A spanned by bc, $b \in B$, $c \in C$.

Definition 1.5.1. The subalgebras $A^{(i)}$, $i \geq 0$, are recursively defined by $A^{(0)} = A$, $A^{(i)} = A^{(i-1)}A^{(i-1)}$ for $i > 1$. For a subalgebra B of A, $B^{(i)}$ is similarily defined. Then B is called solvable if $B^{(i)} = 0$ for some $i > 0$. //

Note that if A is Lie or associative, $A^{(i)}$ is an ideal of A by the Jacobi identity.

Lemma 1.5.1. Let B be an ideal of A. Then A is solvable if and only if B and A/B are solvable.

Proof. Clearly $(A/B)^{(i)} = (A^{(i)} + B)/B$ and $B^{(i)} \subseteq A^{(i)}$. Thus if A is solvable then so are B and A/B. If B and A/B are solvable, choose i, j such that $(A/B)^{(i)} = 0$ and $B^{(j)} = 0$. Thus $A^{(i)} \subseteq B$ and $A^{(i+j)} = (A^{(i)})^{(j)} = 0$. //

As an immediate consequence of Lemma 1.5.1, we have

Corollary 1.5.2. If B is a solvable ideal of A and C is a solvable subalgebra then $B + C$ is solvable. //

If B and C are maximal solvable ideals of A, then by Corollary 1.5.2, $B + C$ is solvable, so $B = B + C = C$. Therefore, if A is finite-dimensional, A contains a unique maximal solvable ideal which is called the (solvable) radical of A and denoted by Rad A.

Definition 1.5.2. An algebra A over F is called semisimple if A has no nonzero solvable ideal. //

If A is finite-dimensional, then $A/\text{Rad } A$ is semisimple and A is semisimple if and only if Rad $A = 0$. Denote by $M(A)$ the (associative) subalgebra of $\text{Hom}_F A$ generated by L_x, R_x, $x \in A$.

Lemma 1.5.3. A is simple if and only if A is $M(A)$-irreducible. A is the direct sum of simple ideals of A if and only if A is $M(A)$-completely reducible.

Proof. The first assertion as well as one direction of the second is obvious. Let A be $M(A)$-completely reducible. Then $A = \Sigma \oplus A_i$ where the A_i are $M(A)$-irreducible $M(A)$-submodules of A. So, A_i is an ideal of A. Note that $A_iA_j = 0$ for $i \neq j$ since $A_iA_j \subset A_i \cap A_j$. Thus an ideal of A_i is an ideal of A and $A_i^2 = A_iA \neq 0$ since $A_iM(A) = A_i$. Hence each A_i has no proper ideal and $A_i^2 = A_i$, so A_i is simple. //

Corollary 1.5.4. Let $\dim A < \infty$ and let $A = \Sigma \oplus A_i$ with A_i simple ideals of A. Then every ideal B of A is a sum of some A_i's. In particular, the A_i are the only simple ideals of A.

Proof. By Lemma 1.5.3, A is $M(A)$-completely reducible. Thus B has an $M(A)$-complement C, so that C is an ideal of A and $A = B \oplus C$. Since $A = A^2$, $A = BA \oplus CA$ and $B = BA = \Sigma BA_i$. Since $B \cap A_i$ is an ideal of A_i, $B \cap A_i = 0$ or A_i. Thus B is the sum of those A_i's with $B \cap A_i = A_i$ ($B \cap A_i = A_i$ implies $BA_i = A_i$). //

If A is a direct sum of simple ideals, A is semisimple by Corollary 1.5.4. The converse is not true; however, we will see in the next section that the converse holds when A has a suitable nondegenerate bilinear form.

1.6. Algebras with invariant forms

Let $(\, , \,)$ be symmetric bilinear form on a vector space V over F. For $S \subseteq V$, define $S^{\perp} = \{x \in V \mid (x,y) = 0 \text{ for all } y \in S\}$. Call $V^{\perp}$ the <u>radical</u> of V. If $V^{\perp} = 0$, $(\, , \,)$ is <u>nondegenerate</u>. If we define $(\, , \,)'$ on $V/V^{\perp}$ by $(x + V^{\perp}, y + V^{\perp})' = (x,y)$, $x,y \in V$, $(\, , \,)'$ is nondegenerate on $V/V^{\perp}$.

<u>Definition 1.6.1</u>. Let A be an algebra over F. An invariant form $(\, , \,)$ on A is a symmetric bilinear form on A satisfying the associative law

$$(xy,z) = (x,yz), \quad x,y \in A . \tag{1.17}$$

If $(\, , \,)$ is nondegenerate, call A a symmetric algebra.

Note that if $(\, ,)$ is an invariant form on A, then $A/A^{\perp}$ is a symmetric algebra. Observe that if B is an ideal of A, so is $B^{\perp}$ by (1.17).

<u>Theorem 1.6.1(Dieudonné)</u>. Let A be a finite-dimensional symmetric algebra over a field F such that there is no nonzero ideal B with $B^2 = 0$. Then $A = \Sigma \oplus A_i$ where the A_i are simple ideals of A and symmetric with $(A_i, A_j) = 0$ for $i \neq j$.

<u>Proof</u>. Let $(\, , \,)$ be a nondegenerate invariant form on A. Let B be any ideal of A. First we show that $C = B \cap B^{\perp} = 0$. Since $C \subseteq B$ and $CA \subseteq B^{\perp}$, by (1.17) $(xy,z) = (x,yz) = 0$ for $x,y \in C$ and $z \in A$. Thus $C^2 \subseteq A^{\perp} = 0$ and $C^2 = 0$, so $C = 0$. Since $\dim A = \dim B + \dim B^{\perp}$, this proves $A = B \oplus B^{\perp}$. Hence A is $M(A)$-completely reducible and by Lemma 1.5.3 $A = A_1 \oplus \dots \oplus A_n$ where the A_i are simple ideals of A. Since $(A_i \, , A_j) = (A_i^2 \, , A_j) = (A_i \, , A_iA_j) = 0$ for $i \neq j$, each A_i is symmetric too. //

If $A = \Sigma \oplus A_i$ where the A_i are simple symmetric ideals of A then A is semisimple by Corollary 1.5.4 and is symmetric.

<u>Exercise 1.6.1</u>. Let A be a finite-dimensional symmetric algebra with $(\, , \,)$ over a field F of char $\neq 2$. Prove :

(1) If A satisfies third-power-associativity $xx^2 = x^2x$ for all $x \in A$ then A is flexible.

(2) If char $F \neq 2,3,5$ and A satisfies $xx^2 = x^2x$ and $x^2x^2 = xx^3$ for all $x \in A$ then A satisfies the Jordan identity

$$(x^2y)x = x^2(yx) \quad . \tag{1.18}$$

A flexible algebra satisfying (1.18) is called a noncommutative Jordan algebra

Exercise 1.6.2. Let char $F \neq 2$. An algebra A over F is called Jordan-admissible if the algebra A^+ is a Jordan algebra ; that is, A^+ satisfies $x^2 \cdot (y \cdot x) = (x^2 \cdot y) \cdot x$. Prove that A is a noncomutative Jordan algebra if and only if A is flexible Jordan-admissible.

Definition 1.6.2. Let L be a finite-dimensional Lie algebra over F and V be a finite-dimensional vector space over F. Then a representation f of L acting on V is a homomorphism of L into $(\mathrm{Hom}_F V)^-$. That is, $f([xy]) = [f(x), f(y)]$, $x, y \in L$. Also, define $(x,y) = (x,y)_f = \mathrm{Tr} f(x) f(y)$ and call (x,y) the trace form of L with respect to f. The mapping $\mathrm{ad} : L \to (\mathrm{Hom}\, L)^-$ defined by $x \to \mathrm{ad}\, x$ becomes a representation acting on L, called the adjoint representation of L. The trace form $K(\, , \,)$ of L with respect to ad is called the Killing form of L. //

Lemma 1.6.2. The trace form $(\, , \,) = (\, , \,)_f$ is an invariant form on L.

Proof. $([x,y], z) = \mathrm{Tr}\, f([xy])\, f(z)$

$$= \mathrm{Tr}\, [f(x), f(y)]\, f(z)$$

$$= \mathrm{Tr}\, f(x)\, f(y)\, f(z) - \mathrm{Tr}\, f(y)\, f(x)\, f(z)$$

$$= \mathrm{Tr}\, f(x)\, f(y)\, f(z) - \mathrm{Tr}\, f(x)\, f(z)\, f(y)$$

$$= \mathrm{Tr}\, f(x)\, [f(y), f(z)]$$

$$= (x, [yz]) \;. \qquad //$$

Lemma 1.6.3. Let the Killing form $K(\;,\;)$ be nondegenerate on L. Then $L = \Sigma \oplus L_i$ where the L_i are simple ideals of L and $K(\;,\;)$ is nondegenerate on each L_i. In particular, L is semisimple.

Proof. Let B be an ideal of A with $B^2 = 0$. Then $B \text{ ad } B \text{ ad } L = 0$ and $L \text{ ad } B \text{ ad } L \subset B$. Thus $K(b,x) = \mathrm{Tr}(\text{ad } b \text{ ad } x) = 0$ for all $b \in B$, $x \in L$ and so $B \subset L^{\perp} = 0$. The result then follows from Theorem 1.6.1. //

2. POLYNOMIAL MAPPINGS

2.1. The Zariski topology

Let V and V' be vector spaces over a field F and let $\dim V = m$, $\dim V' = n$. Let $\{e_i\}$ and $\{e'_j\}$ be basis for V and V', respectively. A mapping $f : V \to V'$ is called a polynomial mapping with respect to $\{e_i\}$, $\{e'_j\}$ if

$$f(\Sigma\, \alpha_i e_i) = \Sigma\, f_j(\alpha_1, \ldots, \alpha_m) e'_j ,$$

where $f_j \in F[X_1, \ldots, X_m]$, $j = 1, 2, \ldots, n$. A polynomial mapping $f : V \to F$ is called a polynomial function on V. Note that $\mathrm{Hom}_F\,(V, V')$ consists of polynomial mappings. Denote by $F[V]$ the set of polynomial functions on V. Then $F[V]$ becomes a commutative associative algebra over F with multiplication defined by $(fg)(x) = f(x)g(x)$, $x \in V$.

Henceforth we assume that F is infinite. Thus $F[V]$ is an F-algebra which is an integral domain.

If $f \varepsilon F[V]$, define $V_f = \{x \varepsilon V \mid f(x) \neq 0\}$.

Then $V_1 = V$, $V_0 = \phi$, and $V_{f_1} \cap \cdots \cap V_{f_n} = V_{f_1 f_2 \cdots f_n}$.

The <u>Zariski topology</u> in V is the topology for V having the V_f, $f \varepsilon F[V]$, as a basis of open sets.
Note that this topology is not Hausdorff. Let P be a subset of $F[X_1,\ldots,X_m]$ and identify V with F^m.
Let

$$\mathfrak{Z}(P) = \{x = (x_1,\ldots,x_m) \varepsilon V \mid f(x) = 0 \text{ for all } f \varepsilon P\}.$$

Then the Zariski closed sets in V are the sets $\mathfrak{Z}(P)$.
Let I be the ideal of $F[X_1,\ldots,X_m]$ generated by P.
Clearly $\mathfrak{Z}(P) = \mathfrak{Z}(I) = \mathfrak{Z}(\{f_1,\ldots,f_r\})$ by the Hilbert Basis Theorem, where $f_1,\ldots,f_r \varepsilon I$. Thus the closed sets are the affine varieties in V over F.

<u>Lemma 2.1.1</u>. Every nonempty open set in V is dense in V.

<u>Proof</u>. Let U_1 and U_2 be nonempty open sets in V. It is enough to show $U_1 \cap U_2 \neq \phi$. Assume $U_i \neq V$, $i = 1,2$, and let f_1, f_2 be nonconstant polynomials such that $U_i \supset V_{f_i}$.

Then $V_{f_1 f_2} = V_{f_1} \cap V_{f_2} \neq 0$ since $f_1 f_2 \neq 0$ and F is infinite. Thus $U_1 \cap U_2 \neq \phi$. //

Lemma 2.1.2. Any polynomial mapping of V into V' is continuous with respect to the Zariski topology.

Proof. Let $\dim V = m$ and $\dim V' = n$, and let $f : V \to V'$ be a polynomial mapping. Let S be a closed subset of V' , so $S = \mathcal{Z}(\{g_1,\ldots,g_r\})$. Let $f(x) = \Sigma_j f_j(\alpha_1,\ldots,\alpha_m)\, e_j'$, where $x = \Sigma \alpha_i e_i \in V$.
Put

$$h_i(X_1,\ldots,X_m) = g_i(f_1(X_1,\ldots,X_m),\ldots,f_n(X_1,\ldots,X_m)) ,$$

$$i = 1,2,\ldots,r .$$

It is routine to check that $f^{-1}(S) = \mathcal{Z}(\{h_1,\ldots,h_r\})$ and hence $f^{-1}(S)$ is closed. //

2.2. Differentials

Let $\{e_1,\ldots e_m\}$ and $\{e_1',\ldots,e_n'\}$ be basis for V and V', respectively.

<u>Definition 2.2.2</u>. Let $p : V \to V'$ be a polynomial mapping. Define the mapping $\sigma_p : F[V'] \to F[V]$ by $(\sigma_p(f))(x) = f(p(x))$, $x \in V$, $f \in F[V']$. //

Clearly, σ_p is an F-algebra homomorphism. Let π_i be the element in $F[V]$ given by $\pi_i(\Sigma\alpha_j e_j) = \alpha_i$. Call π_i the i-th <u>proiection</u>. Then $F[V]$ is generated by $\pi_1,\ldots,\pi_m$, i.e.,

$$F[V] = F[\pi_1,\ldots,\pi_m] \; . \tag{1.21}$$

For each $y \in V$, define the mapping $\tau_y : F[V] \to F$ by $\tau_y(f) = f(y)$, $f \in F[V]$. Then τ_y is an F-algebra homomorphism. Any F-algebra homomorphism $\lambda: F[V] \to F$ is described in this way. Indeed, let $\alpha_i = \lambda(\pi_i)$, $i = 1,\ldots,m$, and let $y = \Sigma\alpha_i e_i$. By (1.21) any $f \in F[V]$ is of the form $f = f(\pi_1,\ldots,\pi_m)$ and hence $\lambda(f) = f(\alpha_1,\ldots,\alpha_m) = \tau_y(f)$. Also, $\tau_x = \tau_y$ if and only if $x = y$. Thus we have

<u>Lemma 2.2.1</u>. For any F-algebra homomorphism $\lambda : F[V] \to F$, there is a unique $y \in V$ such that $\tau_y = \lambda$. //

<u>Lemma 2.2.2</u>. Let F be a perfect field. Let $p : V \to V'$ be a polynomial mapping. If $d_a p$ is surjective for some $a \in V$ then $\sigma_p : F[V'] \to F[V]$ is an isomorphism.

Proof. Let $\dim V = m$ and $\dim V' = n$. Write $\beta = (\beta_1,\ldots,\beta_m)$ and $X = (X_1,\ldots,X_m)$, so that $g(\beta) = g(\beta_1,\ldots,\beta_m)$, $g \in F[X] = F[X_1,\ldots,X_m]$. Suppose that σ_p is not injective. Then, there is an $f = f(Y_1,\ldots,Y_n) \neq 0$ such that $\sigma_p(f) = 0$, so that

$0 = f(p(y)) = f(p_1(\beta),\ldots,p_n(\beta))$ for all $y = \Sigma\, \beta_i e_i \in V$ where $p(y) = \Sigma\, p_i(\beta)e_i'$. Since F is infinite, this implies $f(p_1(X),\ldots,p_n(X)) = 0$ and so $p_1(X),\ldots,p_n(X)$ are algebraically dependent. Let f be of minimal degree giving the algebraic dependence. By the chain rule we have

$$0 = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial Y_j}\right)(p_1(X),\ldots,p_n(X))\frac{\partial p_j}{\partial X_i}\ , \quad i = 1,\ldots,m . \qquad (1.22)$$

Since $d_a p$ is surjective, $n \leq m$ and the matrix $(\partial p_j/\partial X_i)$ has rank n. Thus the system (1.22) has the only trivial solution,

$$\frac{\partial f}{\partial Y_j}(p_1(X),\ldots,p_n(X)) = 0\ , \quad j = 1,2,\ldots,n .$$

Since $\deg\, (\partial f/\partial Y_j) < \deg f$, it follows that $\partial f/\partial Y_j = 0$ for $j = 1,\ldots,n$. Thus, if $\operatorname{char} F = 0$, f is a nonzero constant, which is absurd. If $\operatorname{char} F = p > 0$, f is a polynomial in $Y_1^p,\ldots,Y_n^p$ and so $f = g^p$ since F is perfect. This contradicts the minimality of f. //

2.3. Extension of homomorphisms

Theorem 2.3.1. Let A' be an F-algebra which is an integral domain, K be an algebraically closed extension field of F and R be an F-subalgebra of A' containing 1. Let $S = R[z_1, \ldots, z_n]$, $z_i \in A'$. Let f be any nonzero element in S. Then there exists a nonzero element $g \in R$ such that if $\delta : R \to K$ is any F-algebra homomorphism with $g\delta \neq 0$, then δ can be extended to an F-algebra homomorphism $\tau : S \to K$ such that $f\tau \neq 0$.

Proof.(Chevalley). To proceed by induction, let $S = R[z]$.

Case 1 : z is transcendental over R. Since $f \in R[z]$, $f = a_0 + \ldots + a_k z^k$ with $a_k \neq 0$. Put $g = a_k$ and let $\delta : R \to K$ be an F-algebra homomorphism with $g\delta \neq 0$. Let $\bar{f}(X) = a_0\delta + \ldots + (a_k\delta) X^k \in K[X]$. Then $\bar{f}(X) \neq 0$ and there is an $\alpha \in K$ such that $\bar{f}(\alpha) \neq 0$. Since z is transcendental over R, the mapping $\tau : R[z] \to K$ defined by $(\Sigma\, b_i z^i)\tau = \Sigma\, (b_i\delta)\alpha^i$ gives an F-algebra homomorphism such that $\tau|_R = \delta$ and $f\tau = \bar{f}(\alpha) \neq 0$

<u>Case 2</u> : z is algebraic over R . Let $p(x)$ be a polynomial in $R[X]$ of minimal degree with $p(z) = 0$ and let $p(X) = a_0 + \dots + a_n X^n$ with $a_n \neq 0$. Then $p(X) = a_n h(X)$ where $h(X)$ is monic in $Q[X]$, Q the quotient field of R . Thus $h(X)$ is the minimum polynomial of z over Q . Let $k(X) = b_0 + \dots + b_m X^m$ be a minimum polynomial of f over R .
Then $b_0 \neq 0$ and set $g = ab_0 \neq 0$. We show that g is a desired element. Let $\delta : R \to K$ be an F-algebra homomorphism with $g\delta \neq 0$. Let $\alpha \in K$ be a root of $p^\delta(X) = a_0\delta + \dots + (a_n\delta)X^n \neq 0$. For each $q(z) = \Sigma\, q_i z^i \in R[z]$, define $\tau : R[z] \to K$ by

$q(z)\tau = q^\delta(\alpha) = \Sigma\, (q_i\delta)\alpha^i$. Then $\tau\,|_R = \delta$. We show that τ is well-defined. Suppose that $q(z) = 0$ for some $q(X) \in R[X]$. Then $q(x) = c(X)h(X)$, $c(X) \in Q[X]$ and so

$$a_n q(X) = c(X)a_n h(X) = c(X)p(X) . \qquad (1.23)$$

Letting $c(X) = c_0 + c_1 X + \dots + c_t X^t$ and $q(X) = q_0 + q_1 X + \dots + q_s X^s$, one computes from (1.23)

$$a_n q_s = a_n c_t \Rightarrow q_s = c_t \in R ,$$

$$a_n q_{s-1} = c_{t-1}a_n + c_t a_{n-1} \Rightarrow c_{t-1}a_n \in R ,$$

$$a_n q_{s-2} = c_{t-2}a_n + c_t a_{n-2} + c_{t-1}a_{n-1} .$$

Multiplying the last equation by a_n, $c_{t-2}a_n^2 \in R$ and continuing this we have that $c_{t-i}a_n^i \in R$, $i = 1,2,\ldots,t$. Since $a_n \in R$, this gives $c_j a_n^{t+1} \in R$ for all $j = 0,1,\ldots,t$. Thus

$$a_n^{t+2}q(X) = a_n^{t+1}c(X)p(X) = \bar{c}(X)p(X) \in R[X]$$

and $[a_n^{t+2}q(z)]\,\tau = (a_n\delta)^{t+2}(q^\delta(\alpha)) = \bar{c}^\delta(\alpha)p^\delta(\alpha) = 0$.

Since $a_n\delta \neq 0$, $q^\delta(\alpha) = q(z)\tau = 0$, so τ is well-defined. Evidently, $f\tau \neq 0$ since $b_0 \neq 0$.

Suppose that the result holds for $n - 1$ and let $B = R[z_n]$. Then $S = B[z_1,\ldots,z_{n-1}]$. Let $f \neq 0$ be in S. Then there is a $v \neq 0$ in B such that if $\rho : B \to K$ is any F-algebra homomorphism with $v\rho \neq 0$, ρ can be extended to an F-algebra homomorphism $\tau : S \to K$ with $f\tau \neq 0$. The case $n = 1$ applied to B with $v \neq 0$ gives the desired homomorphism τ. //

<u>Theorem 2.3.2.</u> Let F be algebraically closed and perfect. Let $p : V \to V'$ be a polynomial mapping such that $d_a p$ is surjective for some $a \in V$. For any nonzero $f \in F[V]$, there exists a nonzero element $g \in F[V']$ such that $p(V_f) \supseteq V'_g$, or equivalently, there exists $0 \neq g \in F[V']$ such that if $g(y) \neq 0$ then $p(x) = y$ for some $x \in V$ with $f(x) \neq 0$.

Proof. Let $R = F[V']\sigma_p$ where $\sigma_p : F[V'] \to F[V]$ is an F-algebra isomorphism by Lemma 2.2.2. Then $S \equiv F[V] = F[\pi_1,\ldots,\pi_m] = R[\pi_1,\ldots,\pi_m]$. By Theorem 2.3.1, there exists a nonzero $g' \in R$ such that any homomorphism $\delta : R \to F$ with $g'\delta \neq 0$ is extended to an F-algebra homomorphism $\tau : S \to F$ such that $f\tau \neq 0$.
Choose a $g \neq 0$ in $F[V']$ with $g\sigma_p = g'$. Let $y \in V'$ be any element such that $g(y) \neq 0$, i.e., $y \in V'_g$.
Define $\delta : R \to F$ by $(h\sigma_p)\delta = h(y)$, $h \in F[V']$.
Then δ is an F-algebra homomorphism with $g'\delta \neq 0$.
Let τ be an extension of σ to an F-algebra homomorphism : $S \to F$ such that $f\tau \neq 0$. By Lemma 2.2.1 $\tau = \tau_x$ for some $x \in V$, so that $f\tau = \tau_x(f) = f(x) \neq 0$.
For any $h \in F[V']$,

$$(h\sigma_p)\tau = (h\sigma_p)\delta = h(y)$$

$$= (h\sigma_p)\tau_x = h\sigma_p(x) = h(p(x)) .$$

Thus $p(x) = y$ and $y \in p(V_f)$. //

Exercise 2.3.1. Let K be an algebraically closed field containing F and let $F[X] = F[X_1,\ldots,X_n]$ be the polynomial ring in $X_1,\ldots,X_n$ over F . Let S be a subset of $F[X]$. The set $\mathfrak{Z}_K(S) = \{v \in K^n | f(v) = 0$ for all $f \in S\}$ is called an (Zariski) F-closed set in K^n

If I is an ideal of $F[X]$, the radical $\sqrt{I}$ of I is defined by $\sqrt{I} = \{f \in F[X] \mid f^m \in I \text{ for } m > 0\}$. Clearly, $\sqrt{I}$ is an ideal of $F[X]$ containing I. Using Theorem 2.3.1 prove the Hilbert Nullstellensatz : For any ideal I of $F[X]$, $\sqrt{I} = \mathcal{I}(\mathcal{Z}_K(I))$ where $\mathcal{I}(E) = \{f \in F[X] \mid f(v) = 0 \text{ for all } v \in E\}$ for a subset E of K^n.

3. LIE ALGEBRAS OF CHARACTERISTIC 0

3.1. Introduction

The theory of Lie algebras of characteristic 0 is an outgrowth of the Lie theory of continuous groups in which local problems concerning Lie groups are reduced to corresponding problems on Lie algebras. During the development of the structure of Lie algebras for many years, Lie algebras brought applications to many branches of mathematics, such as group theory, differential geometry, differential equations, topology, and physics. Besides being useful in many parts of mathematics, the theory of Lie algebras is the most widely and successfully studied area of nonassociative algebras, mainly because of the elegance and completeness of the structure and representation theories for semisimple Lie algebras of characteristic 0 .

In this chapter, we briefly discuss the classical theorems on Lie algebras which are essential for the general structure, the classification and representations of semisimple Lie algebras of characteristic 0. The development of the material is designed to set up the groundwork for the structure of flexible Lie-admissible algebras rather than to provide the general theory of Lie algebras for a comprehensive account.

<u>Definition 3.1.1</u>. Let B, S be subsets of a Lie algebra L over a field F. The centralizer of S in B is the set $C_B(S) = \{x \in B \mid [xS] = 0\}$. The centralizer of S is $C_L(S)$ and the center of L is $C(L) = C_L(L)$. For a subspace B of L, the normalizer of B (in L) is the set $N_L(B) = N(B) = \{x \in L \mid [xB] \subseteq B\}$. //

<u>Definition 3.1.2</u>. A Lie module V for L is an L-module over F such that $v[xy] = (vx)y - (vy)x$, $v \in V$, $x,y \in L$. If $f : L \to (\mathrm{Hom}_F\, V)^-$ is a representation of L, then V together with the module operation $vx = vf(x)$ for $v \in V$, $x \in L$ is a Lie module for L. The Lie module obtained in this way is called the Lie module afforded by f. By an irreducible representation f of L, we mean that the L-module V afforded by f is L-irreducible.

Conversely, if V is a Lie module for L over F, then the mapping $f : L \to (\mathrm{Hom}_F\, V)^-$ defined by $vf(x) = vx$ for $x \in L$ becomes a representation of L acting on V, which is called the representation afforded by V. //

Henceforth, for a Lie algebra L, all L-modules are referred to as Lie modules for L. Note that the kernel of the ad representation of L is the center $C(L)$. Also, a subalgebra B of L is an ideal of $N_L(B)$, and if B and C are ideals of L then so is $[BC] = [CB]$. These are consequences of the Jacobi identity.

Let V be an L-module and suppose that V is also a Lie algebra such that $[vv']x = [(vx)v'] + [v(v'x)]$. $v,v' \in V$, $x \in L$. Define a product in $L \oplus V$ by

$$[x + v,\ x' + v'] = [xx'] + (vx' - v'x + [vv'])$$

for $x,x' \in L$ and $v,v' \in V$. Then $L \oplus V$ becomes a Lie algebra. Indeed, clearly $L \oplus V$ is anticommutative. If $x \in L$, $R_x|_L$ and $R_x|_V$ are derivations. Thus, to see $R_x \in \mathrm{Der}\,(L \oplus V)$, one checks

$$[[v,y],x] = (vy)x = (vx)y + v[yx]$$

$$= [[v,x],y] + [v,[y,x]]$$

for $v \in V$, $x,y \in L$. For $v \in V$, $R_v|_V$ is clearly a derivation and it remains to check that

$$[[x,y],v] = -v[xy] = -(vx)y + (vy)x = [[x,v],y] + [x,[y,v]]$$

and likewise R_v acts on $v'y$ as a derivation. Thus $R_{L\oplus V} \subseteq \mathrm{Der}\,(L \oplus V)$ and $L \oplus V$ is a Lie algebra which is called the <u>split extension</u> of L by V. In particular, the split extension of $\mathrm{Der}\,L$ by L is called the <u>holomorph</u> of L. The Lie algebra $L \oplus V$ with $[VV] = 0$ is the split <u>null</u> extension of L by V.

3.2. <u>Nilpotent Lie algebras</u>

The structure of Lie algebras can be described in terms of certain nilpotent subalgebras (Cartan subalgebra). In this section we develop some fundamental properties of nilpotent Lie algebras. Here we assume that the characteristic of F is arbitrary.

<u>Definition 3.2.1</u>. For a Lie algebra L, the descending central series $L \supseteq L^2 \supset \ldots \supset L^n \supset \ldots$ is recursively defined by $L^1 = L$, $L^{i+1} = [L^i L]$, $i = 0,1,2,\ldots$

If $L^i = 0$ for some i, L is called nilpotent. //

The derived series $L^{(i)}$ and the descending central series L^i are analogous to those of a group. If B is an ideal of L, then $B^{(i)}$ and B^i are ideals of L. Note also that any homomorphic images of nilpotent Lie algebras are nilpotent. By induction, one easily sees that $[L^i L^j] \subset L^{i+j}$ for integers $i, j \geq 0$. This implies that the products of n elements of L in any association are contained in L^n. In particular, $L^{(n)} \subset L^{2^n}$ and hence any nilpotent Lie algebra is solvable ; however the convers is not true (why ?). If C is the center of L and L/C is nilpotent then L is nilpotent since $L^i \subset C$ implies $L^{i+1} = 0$. Since C is the kernel of ad, we have

<u>Lemma 3.2.1</u>. L is nilpotent if and only if ad L is nilpotent. //

A stronger version of Lemma 3.2.1 is the celebrated theorem of Engel that L is nilpotent if and only if ad L consists of nilpotent linear transformations, which we show below.

<u>Definition 3.2.2</u>. The ascending central series of L is the sequence $C^i(L)$, $i \geq 0$, defined recursively by

$$C^0(L) = 0,\ C^i(L) = \{x \in L \mid [xL] \subset C^{i-1}(L)\} \qquad i \geq 1 .$$

Obviously, $C^1(L) = C(L)$ and $C^i(L)/C^{i-1}(L)$

$$= C(L/C^{i-1}(L)) \ . \qquad //$$

<u>Lemma 3.2.2</u>. $L^{i+1} = 0$ if and only if $C^i(L) = L$ for $i \geqq 0$.

<u>Proof</u>. If $L^{i+1} = 0$, $L^i \subset C^1(L)$ and so $L^{i-1} \subset C^2(L)$, $L \subset C^i(L)$. If $C^i(L) = L$, $L^2 \subset C^{i-1}(L)$ and continuing this gives $L^{i+1} \subset C^0(L) = 0$. //

<u>Definition 3.2.3</u>. Let N be a subalgebra of L . Then $C_L^i(N)$ is the series of subspaces of L defined inductively by

$$C_L^0(N) = 0 \ , \ C_L^i(N) = \{x \in L \mid [xN] \subset C_N^{i-1}(N)\} \ .$$

Clearly, $C^i(L) = C_L^i(L)$. //

<u>Lemma 3.2.3</u>. Let T be a nilpotent linear transformation on V . If $W \subset U$ are T-stable subspaces of V then $W \subset U$ can be refined as $W = U_0 \subset U_1 \subset \ldots \subset U_{r-1} \subset U_r = U$ with $U_i T \subset U_{i-1}$, $i = 1,\ldots,r$.

<u>Proof</u>. If we let $\overline{U} = U/W$, T induces a nilpotent transformation $\overline{T}$ on $\overline{U}$ by $(a + W)\overline{T} = aT + W$, $a \in U$. Thus $\overline{U}\,\overline{T}^r = \overline{0}$ for some $r > 0$ and, letting $\overline{U}_i = \overline{U}\,\overline{T}^{r-i}$, $i = 0,1,\ldots,r$, we have $\overline{0} \subset \overline{U}_1 \subset \ldots \subset \overline{U}_r$

$= \overline{U}$ with $\overline{U}_i \overline{T} = \overline{U}_{i-1}$. Let U_i be the inverse image of $\overline{U}_i$ by the natural homomorphism. Then $W \subset U_1 \subset \dots \subset U_r =$ with $U_i T \subset U_{i-1}$, as desired. //

<u>Theorem 3.2.4</u>. Let N be a subalgebra of L such that $\mathrm{ad}_L N$ consists of nilpotent linear transformations. Then $C_L^i(N) = L$ for some i .

<u>Proof</u>. If $\dim N = 1$, it is trivial. Assume that the result holds for such subalgebras of dimension $< \dim N$. Let H be a maximal proper subalgebra of N . Then there exists an $m > 0$ such that $C_L^m(H) = L$, so $C_N^m(H) = N$ since $C_L^m(H) \cap N = C_N^m(H)$. Thus one can choose a j such that $C_N^j(H) \subset H$ but $C_N^{j+1}(H) \not\subset H$, and let $x \in C_N^{j+1}(H) - H$. Then $Fx \oplus H$ is a subalgebra of N since $[xH] \subset H$, so $N = Fx + H$ and H is an ideal of N . We show by induction on i that $C_L^i(H)$ is ad x-stable. Thus, if $C_L^{i-1}(H)$ is ad x-stable, $y \in C_L^i(H)$ and $h \in H$, then

$$[[yx]h] = [[yh]x] + [y[xh]]$$

$$\in [C_L^{i-1}(H)x] + [C_L^i(H)[xh]] \subset C_L^{i-1}(H)$$

since $[xH] \in H$, so $y \, \mathrm{ad}\, x \in C_L^i(H)$. Finally, since ad x is nilpotent and stablizes the spaces in the chain

$0 = C_L^0(H) \subset \dots \subset C_L^m(H) = L$, by Lemma 3.2.3 this chain has a refinement $0 = B^0 \subset B^1 \subset \dots \subset B^n = L$ such that $[B^i x] \subset B^{i-1}$. Moreover, $[B^i H] \subset B^{i-1}$ since $[C_L^i(H)H] \subset C_L^{i-1}(H)$. If follows that $[B^i N] \subset B^{i-1}$ for $1 \leq i \leq n$. Thus $C_L^n(N) = L$. //

If L is nilpotent, clearly ad x is nilpotent for all $x \in L$. If ad L consists of nilpotent transformations then by Theorem 3.2.4 $L = C_L^i(L) = C^i(L)$ for some i and by Lemma 3.2.2 L is nilpotent. Therefore we have the following theorem of Engel.

Theorem 3.2.5 (Engel). L is nilpotent if and only if ad L consists of nilpotent transformations. //

Remark. Let A be an algebra over F . For a subalgebra B of A and a positive integer n , define B^n as the linear span of all product of n elements in B in all possible associations. Then B is called nilpotent if $B^n = 0$ for some $n > 0$. We have observed that a nilpotent Lie algebra is also nilpotent in this sense. An algebra A is called power-associative if the subalgebra generated by $x \in A$ is associative for every $x \in A$, or equivalently, $x^m x^n = x^{m+n}$ for all integers $m,n \geq 0$, $x \in A$. Then, an element $x \in A$ is said to be nilpotent if $x^m = 0$ for some $m > 0$, and

A is called nil if every element is nilpotent. An anticommutative algebra A is trivially nil with $x^2 = 0$, $x \varepsilon A$. An algebra A is called alternative if it satisfies the alternative laws $x^2y = x(xy)$ and $yx^2 = (yx)x$ for all $x,y \varepsilon A$. Unlike anticommutative algebras (so Lie algebras), for other nonassociative algebras such as Jordan or alternative algebras A, the nilpotence of x, L_x and R_x are equivalent. Moreover, if A is finite-dimensional (char $F \neq 2$), solvability, nilpotence and nility of A are equivalent (R.D.Schafer, "An introduction to nonassociative algebras", Academic Press, N.Y., 1966). Therefore, for finite-dimensional alternative and Jordan algebras, Engel's Theorem holds. A.A.Albert conjectured in 1948 that the solvability and nilpotence are equivalent in a commutative power-associative algebra. However, D.Suttles disproved this conjecture in 1972 by constructing a 5-dimensional commutative power-associative algebra A which is solvable but not nilpotent. The algebra A has a basis $\{a,b,c,d,e\}$ with multiplication given by $ab = c$, $ac = d$, $ae = -c$, $bc = e$, $bd = c$ and all other products are 0. It is easily seen that $A^2A^2 = 0$ but A is not nilpotent. Subsequently, M. Gerstenhaber and H. C. Myung showed that dimesion 5 is least possible by demonstrating that any commutative power-associative nil algebra of dimension $\leqq 4$ is nilpotent ("On commutative power-associative nilalgebras of low dimension", Proc. Amer. Math. Soc. 48(1975), 29-32). In particular, Suttles' example shows that Engel's Theorem does not hold for commutative algebras. //

If one takes $N = L$ in Theorem 3.2.4 then the proof establishes the following result.

Corollary 3.2.6. A maximal proper subalgebra of a nilpotent Lie algebra L is an ideal of L of codimension 1. //

Definition 3.2.4. Let N be a set and let B be an N-module. Then $B_0^i(N)$ is defined recursively by

$$B_0^0(N) = 0 \,, \quad B_0^i(N) = \{x \in B \mid xN \subset B_0^{i-1}(N)\} \,. \qquad //$$

If N is a subalgebra of L and $B = L$ then $B_0^i(\mathrm{ad}\, N) = C_L^i(N)$. In the following we prove two variants of Engel's Theorem.

Theorem 3.2.7. Let V be a finite-dimensional vector space over F. Let N be a subalgebra of $(\mathrm{Hom}_F\, V)^-$ consisting of nilpotent linear transformations. Then $V_0^i(N) = V$ for some i. Thus, relative to a suitable basis for V, the subalgebra N is represented by nil triangular matrices.

Proof. Since $y(\text{ad }x)^n$ is a linear combination of terms $x^m yx^{n-m}$ for $x, y \in \text{Hom}_F V$, $x^{2n} = 0$ implies $(\text{ad }x)^n = 0$. Thus, $\text{ad}_N N$ consists of nilpotent linear transformations. Let $L = N \oplus V$ be the split null extension of N by V. Then $\text{ad}_L N$ also consists of nilpotent linear transformations. Thus by Theorem 3.2.4 $L = C_L^i(N) = L_0^i(N)$ for some i, where $L_0^i(N) = L_0^i(\text{ad } N)$. Since $V \subset L$, it follows by induction that $V \cap L_0^i(N) = V_0^i(N)$, $i = 0,1,2,\ldots$. Therefore, $V = V_0^i(N)$, and this gives a chain $V_0^1(N) \subset \ldots \subset V_0^{i-1}(N) \subset V_0^i(N) = V$ such that $V_0^k(N)N \subset V_0^{k-1}(N)$, $k = 1,2,\ldots,i$. Hence one can choose a basis for V relative to which N is represented by nil triangular matrices. //

Theorem 3.2.8. (Engel). Let $f : L \to (\text{Hom}_F V)^-$ be a representation where V is finite-dimensional. Suppose that $f(x)$ is nilpotent for $x \in L$. Then there exists a nonzero vector $v \in V$ such that $vf(x) = 0$ for all $x \in L$.

Proof. In Theorem 3.2.6, take $f(L) = N$. Since $V = V_0^i(N)$ for some i, $V_0^1(N) \neq 0$ and any nonzero vector in $V_0^1(N)$ gives the desired condition. //

Definition 3.2.5. Let B be a finite-dimensional L-module over F. For a function $a : L \to F$, define

$$B_a(L) = \{x \in B \mid x(T - a(T))^{n(T)} = 0 \text{ for } T \in L\} .$$

If L is a nilpotent Lie algebra and $B_a(L) \neq 0$, then $B_a(L)$ is called the weight space for L in B with respect to a and a is called a weight of L in B. Note that $B_a(L) = \bigcap_{T \in L} B_{a(T)}(T)$ for any function $a : L \to F$. Let $B_*(L) = \Sigma_{T \in L} B_*(T)$. Then $B_0(L)$ and $B_*(L)$ are called the Fitting components of B with respect to L. //

Theorem 3.2.9. Let B be a finite-dimensional L-module over F. Suppose that L is nilpotent and that each $x \in L$ is split over F. Then B is a direct sum $B = \Sigma_a \oplus B_a(L)$ of weight spaces for L in B and each $B_a(L)$ is an L-submodule of B.

Proof. The proof is by induction on $\dim B$. If $\dim B = 1$, there is nothing to prove. One may take L to be a subalgebra of $(\mathrm{Hom}_F B)^-$. Then $ad_L x$ is nilpotent for $x \in L$. Thus $0 = (ad_L x)_s = ad_L x_s$ by Theorem 1.3.3 and $[y, x_s] = 0$ for $x, y \in L$. If x_s for each $x \in L$ acts on B as a single scalar $a(x)$, then $B = B_a(L)$ since $x - a(x) = x_n$. Thus we may assume that there is an x_s which is not a scalar on B.

Hence $B = \Sigma \oplus B_{\alpha_i}(x_s)$ and $B_{\alpha_i}(x_s) \equiv B_i \neq 0$, $1 \leqq i \leqq m$ with $m > 1$, so $\alpha_1, \ldots, \alpha_m$ are distinct.

Since $[L, x_s] = 0$, each B_i is L-stable and by induction applied to B_i, we have $B_i = \Sigma_a \oplus B_{ia}(L)$ where $B_{ia}(L)$ is L-stable. Since $\Sigma_i B_{ia}(L) \subset B_a(L)$, $B = \Sigma_a B_a(L)$. Note that $B_i = B_{\alpha_i}(x_s) = B_{\alpha_i}(x)$. Since the α_i are distinct, it follows from this that if $B_a(L) \neq 0$, $a(x) = \alpha_i$ for some i since $a(x)$ is an eigenvalue of x. Thus $B_a(L) \subset B_i$ for some i and moreover it is easily seen that $B_a(L) = B_{ia}(L)$. //

In view of Theorem 3.2.9, if B is finite-dimensional, there are only finitely many weights for L in B.

<u>Theorem 3.2.10</u>. Let N be a nilpotent subalgebra of L and let B be a finite-dimensional L-module. Regard L as an N-module via ad. Then $B_a(N)L_b(N) \subset B_{a+b}(N)$ for all functions $a,b : N \to F$.

<u>Proof</u>. Let $\overline{B} = L \oplus B$ be the split null extension of L by B. Regard $\overline{B}$ as an N-module via ad. Since L and B are N-submodules of $\overline{B}$, $\overline{B}_a(N) = L_a(N) \oplus B_a(N)$. Noting that $[\overline{B}_a(N), \overline{B}_b(N)] \subset \overline{B}_{a+b}(N)$ by Corollary 1.4.8, we have

$$[\overline{B}_a(N), \overline{B}_b(N)] = [L_a(N)L_b(N)] + B_a(N)L_b(N) + B_b(N)L_a(N)$$

$$\subset L_{a+b}(N) \oplus B_{a+b}(N) .$$

Since B is an ideal of $\overline{B}$, this in particular implies $B_a(N)L_b(N) \subset B_{a+b}(N)$. //

<u>Exercise 3.2.1</u>. Let N be a nilpotent ideal in L and let B be a finite-dimensional L-module. Prove that $B_a(N)$ is an L-submodule for every $a : N \to F$.

3.3. <u>Cartan subalgebras</u>

The Cartan subalgebras of a Lie algebra L are certain nilpotent subalgebras that are central objects for the structure of L . We show here that they exist if F is infinite and that the decomposition of L into weight spaces for a split Cartan subalgebra provides a rough description of a multiplication table for L .

<u>Definition 3.3.1</u>. A subalgebra H of L is a Cartan subalgebra (CSA) of L if H is nilpotent and $H = L_0(\mathrm{ad}\, H)$, the Fitting 0-component of L relative to $\mathrm{ad}\, H$. //

The following characterization of a CSA is often convenient.

<u>Theorem 3.3.1</u>. A subalgebra H of L is a CSA of L if and only if H is nilpotent and $H = N_L(H)$.

<u>Proof</u>. Let $N = N_L(H)$. Suppose that H is a CSA of L , so $H = L_0(\text{ad } H)$. It follows that $N(\text{ad } x)^n = 0$ for $x \in H$ and $n > 0$, since $N \text{ ad } x \subset H$ for $x \in H$. Thus $N \subset L_0(\text{ad } H) = H$ and $N = H$. For the converse, show that if $H \subsetneq L_0(\text{ad } H)$, $H \subsetneq N$. Let $L_0 = L_0(\text{ad } H)$. Since ad H stablizes L_0 and H , and ad $x (x \in H)$ is nilpotent on L_0 and H , ad H gives rise to a Lie algebra of nilpotent linear transformations on $L_0/H \neq 0$. Thus by Engel's Theorem 3.2.8 there exists $\overline{0} \neq x + H \in L_0/H$ such that $(x + H) \text{ ad } H = 0$ and hence $x \in N$ but $x \notin H$, i.e., $H \subsetneq N$. //

<u>Exercise 3.3.1</u>. Show that a CSA H of L is a maximal nilpotent subalgebra of L .

<u>Definition 3.3.2</u>. An element $h \in L$ such that $\dim L_0(\text{ad } h)$ is minimal is called a regular element. Denote by L_{reg} the set of regular elements in L . //

Note that $\dim L_0(\text{ad } h) \geqq 1$ since $[hh] = 0$ and that $L_{reg} \neq \phi$. Henceforth we assume that F is infinite.

Theorem 3.3.2. If $x \in L_{reg}$ then $H = L_0(\mathrm{ad}\ x)$ is a CSA of L .

Proof. Let $L = H \oplus L_*(\mathrm{ad}\ x)$ be the Fitting decomposition of L relative to $\mathrm{ad}\ x$. Let $L_* = L_*(\mathrm{ad}\ x)$. By Corollary 1.4.8 $L_*\ \mathrm{ad}\ H \subset L_*$. Thus if we let $f(y) = \det(\mathrm{ad}\ y|_{L_*})$ for $y \in H$, then f is a polynomial function on H , and $H_f = \{y \in H | f(y) \neq 0\}$ is a nonempty Zariski open set in H since $\mathrm{ad}\ x$ is nonsingular on L_* by Lemma 1.3.4 and so $f(x) \neq 0$. Hence H_f is dense in H . Since $L = L_* \oplus H$ and $\mathrm{ad}\ y$ stablizes H and L_* for $y \in H$, it follows that $L_0(\mathrm{ad}\ y) \subseteq L_0(\mathrm{ad}\ x) = H$ for $y \in H_f$. (Note that $\mathrm{ad}\ y$ is nonsingular for $y \in H_f$) . Since $\dim H$ is minimal, $L_0(\mathrm{ad}\ y) = H$ for all $y \in H_f$. Thus if $\dim H = n$, $(\mathrm{ad}\ y|_H)^n = 0$ for $y \in H_f$. If we let $g(y) = (\mathrm{ad}\ y|_H)^n$ for $y \in H$, g is a polynomial mapping of H into $\mathrm{Hom}_F H$ and so is continuous by Lemma 2.1.2. Since each point is closed in $\mathrm{Hom}\ H$, $\{y \in H | g(y) = 0\}$ is a closed subset of H containing H_f and coincides with H since H_f is dense in H . Thus $(\mathrm{ad}\ y|_H)^n = 0$ for all $y \in H$ and by Engel's Theorem H is nilpotent. Since $x \in H$, this implies that $H = H_0(\mathrm{ad}\ H) \subset L_0(\mathrm{ad}\ H) \subset L_0(\mathrm{ad}\ x) = H$. Thus $L_0(\mathrm{ad}\ H) = H$ and H is a CSA of L . //

Corollary 3.3.3. Let H be a CSA of L. If $a \in H \cap L_{reg}$ then $H = L_0(\mathrm{ad}\ a)$.

Proof. Since H is nilpotent, $H \subset L_0(\mathrm{ad}\ a)$. But by Theorem 3.3.2 $L_0(\mathrm{ad}\ a)$ is a CSA of L and $H = L_0(\mathrm{ad}\ a)$ since a CSA of L is a maximal nilpotent subalgebra of L (Exercise 3.3.1). //

In view of Corollary 3.3.3, if two CSA of L have a regular element in common, they coincide.

Corollary 3.3.4. If F is infinite, L has a CSA.

Proof. Since $L_{reg} \neq \phi$, the result follows from Theorem 3.3.2. //

Remark. It can be shown that any solvable Lie algebra over an arbitrary field F has a CSA (D.J. Winter, "Abstract Lie algebras", MIT Press, Cambridge, MA, 1972). Also, C.W.Barnes has shown that if F has at least $\dim L$ elements, L has a CSA ("On Cartan subalgebras of Lie algebras", Math. Z., 101(1967), 350-355). //

Theorem 3.3.5. If F is infinite, L_{reg} is a Zariski open set in L.

Proof. Note that $L_{reg} \neq \phi$ and $x \in L_{reg}$ if and onl[y] if $\dim L_*(\mathrm{ad}\ x)$ is maximal since $L = L_0(\mathrm{ad}\ x) \oplus L_*(\mathrm{ad}\ x)$

Let $s = \max\{\dim L_*(\mathrm{ad}\ x) \mid x \in L\}$ and let $n = \dim L$. Then $L_*(\mathrm{ad}\ x) = L(\mathrm{ad}\ x)^n$ (see the proof of Fitting Lemma 1.3.4). Thus $x \in L_{reg}$ if and only if $s = \mathrm{rank}(\mathrm{ad}\ x)^n$ if and only if $\det_B M(x) \neq 0$ for some $s \times s$ minor $M(x)$ of $(\mathrm{ad}\ x)^n$ where B is a fixed basis for L. Since the mapping $x \to \det_B M(x)$ is a polynomial function on L, it is continuous and $\{x \in L \mid \det_B M(x) \neq 0\}$ is open in L. Let $M_1(x), \ldots, M_r(x)$ be all the $s \times s$ minors of $(\mathrm{ad}\ x)^n$ and let $U_i = \{x \in L \mid \det_B M_i(x) \neq 0\}$, $i = 1, 2, \ldots, r$. Then $L_{reg} = U_1 \cup \ldots \cup U_r$ and L_{reg} is open. //

Since L_{reg} is dense in L, almost all elements in L are regular. If H is a CSA of L, we always regard L as an H-module via the ad representation.

<u>Definition 3.3.3</u>. We say that a CSA H of L is split over F if $\mathrm{ad}_L\ x$ is split over F for $x \in H$. Also, L is split over F if L has a split CSA over F. If H is a CSA of L, a root of H in L is a function $a : H \to F$ such that $L_a(H) (= L_a(\mathrm{ad}\ H)) \neq 0$. //

Thus, the roots of H in L are the weights of H in L where L is regarded as an H-module via ad. We will show later that the roots of a split CSA are linear functions if the characteristic is 0.

<u>Theorem 3.3.6</u>. Let H be a split CSA of L. Then

1) $[L_a(H)\ L_b(H)] \subset L_{a+b}(H)$ for all $a,b : H \to F$;

2) $[H\ L_a(H)] \subset L_a(H)$ for all $a : H \to F$ and if $a \neq 0$ then $[H\ L_a(H)] = [L_a(H)\ H] = L_a(H)$;

3) $K(L_a(H), L_b(H)) = 0$ for $a,b : H \to F$ with $a + b \not\equiv 0$;

4) $L = \Sigma_a \oplus L_a(H)$ where a ranges over the roots of H in L ;

5) $K(H, L_a(H)) = 0$ for all $a : H \to F$ with $a \neq 0$.

<u>Proof</u>. 1) and the first part of 2) are consequences of Theorem 3.2.10 while 4) follows from Theorem 3.2.9. For the second part of 2), let $h \in H$ with $a(h) \neq 0$. If $x \text{ ad } h = 0$ for some $x \in L_a(H)$, then $x(\text{ad } h - a(h))^n = 0$ implies $a(h)^n x = 0$ or $x = 0$, so $\text{ad } h$ is injective. For 3), let $a + b \not\equiv 0$ and let $L = \Sigma_c \oplus L_c(H)$. Let $B = \bigcup_c B_c$ be a basis for L such that B_c is a basis of $L_c(H)$. For $x \in L_a(H)$ and $y \in L_b(H)$, $L_c(H) \text{ ad } x \text{ ad } y \subset L_{a+b+c}(H)$ where $L_{a+b+c}(H) = 0$ or $c \neq a + b + c$. Thus the matrix of $\text{ad } x \text{ ad } y$ relative to B is

$$\begin{bmatrix} 0 & * & & * \\ * & 0 & & \\ & & \ddots & \\ * & & & 0 \end{bmatrix} .$$

Hence $\mathrm{Tr}(\mathrm{ad}\ x\ \mathrm{ad}\ y) = K(x,y) = 0$. 5) is a special case of 3). //

3.4. Solvable Lie algebras

In this section we assume that B is a finite-dimensional L-module over a field F of characteristic 0 .

Theorem 3.4.1. Let $h \in L$ be such that $h \in (L_0(\mathrm{ad}\ h))^{(1)}$. Then $B = B_0(h)$.

Proof. We may assume that F is algebraically closed. Indeed, for the scalar extension B_K of B to the algebraic closure K of F we have $B \cap (B_K)_0(h) = B_0(h)$. Let $h = \Sigma[x_i y_i]$ where $x_i, y_i \in L_0(\mathrm{ad}\ h)$. By Theorem 3.2.10 $B_\alpha(h)L_0(\mathrm{ad}\ h) \subset B_\alpha(h)$ where $B = \Sigma_\alpha \oplus B_\alpha(h)$. Hence each $B_\alpha(h)$ is stable under x_i, y_i .

Let $f : L \to (\mathrm{Hom}_F\ B)^-$ be the representation of L afforded by B. Then

$$f(h)|_{B_\alpha(h)} = \Sigma f([x_i y_i])|_{B_\alpha(h)} = \Sigma [f(x_i), f(y_i)]|_{B_\alpha(h)}$$

and hence $\mathrm{Tr}\ f(h)|_{B_\alpha(h)} = \alpha \cdot \dim B_\alpha(h) = 0$. Since $\mathrm{char}\ F = 0$, this implies $\alpha = 0$ unless $B_\alpha(h) = 0$, so $B = B_0(h)$. //

<u>Corollary 3.4.2</u>. If $\mathrm{ad}\ h$ is nilpotent and $h \in L^{(1)}$ then $B = B_0(h)$. //

<u>Theorem 3.4.3</u>. Let L be solvable. Then $B = B_0(L^{(1)})$.

<u>Proof</u>. Suppose not and take a counterexample with $\dim L + \dim B$ minimal. Choose n maximal such that $L^{(n)} \neq 0$, and let $A = L^{(n)}$. Then A is an abelian ideal of L. If $n = 0$, $L^{(1)} = 0$ and so $B_0(L^{(1)}) = B$, contrary to the supposition. Thus $n \geqq 1$. Also $L\ \mathrm{ad}[xy] \subset L^{(n)}$ for $x,y \in L^{(n-1)}$ and hence $L(\mathrm{ad}[xy])^2 = 0$ since A is abelian. Therefore $L = L_0(\mathrm{ad}[xy])$ and by Corollary 3.4.2 $B = B_0([xy])$ for $x,y \in L^{(n-1)}$. Since A is abelian, this implies that $B = B_0(A)$. Let $W = \{v \in B | vA = 0\}$. Since $B = B_0(A) \neq 0$, $W \neq 0$ by Engel's Theorem 3.2.8.

Suppose $W = B$. Then $BA = 0$ and B is regarded as an L/A-module with $v(x + A) = vx$, $x \in L$ and $v \in B$. By the minimality of $\dim B + \dim L$, we must have $B = B_0((L/A)^{(1)})$ and this implies $B = B_0(L^{(1)})$, a contradiction. If $W \neq B$ then since $(WL)A \subseteq (WA)L + W[LA] = 0$, W is L-stable and B/W becomes an L-module with $(v + W)x = vx + W$, $v \in B$, $x \in L$. Thus $W = W_0(L^{(1)})$ and $B/W = (B/W)_0(L^{(1)})$ by the minimality of $\dim B + \dim L$. For $v \in B$, $(v + W)x^m = 0$ for some $m > 0$ and $vx^m \in W$, so $vx^{m+n} = 0$ for $x \in L^{(1)}$. Hence $B = B_0(L^{(1)})$, a contradiction. //

<u>Corollary 3.4.4</u>. Let L be solvable. Then the set $N = \{x \in L \mid B = B_0(x)\}$ is an ideal of L which contains $L^{(1)}$.

<u>Proof</u>. By Theorem 3.4.3, $L^{(1)} \subset N$. Let H be a maximal subalgebra of L such that $L^{(1)} \subset H$ and $B = B_0(H)$. Thus, in view of Theorem 3.2.7, $B_0^m(H) = B$ for some $m > 0$. We show that $N = H$, whence N is an ideal of L since $[NL] \subseteq L^{(1)} \subset N$. So, let $x \in N$ be any element and consider the series $0 = B_0^0(H) \subset \ldots \subset B_0^m(H) = B$. For $v \in B_0^i(H)$,

$(vx)H \subset (vH)x + v[Hx] \subset B_0^{i-1}(H)x + vH$ since $[Hx] \subset L^{(1)} \subset H$.

Thus by induction we see that each $B_0^i(H)$ is x-stable. By Lemma 3.2.3 this series has a refinement $0 = B^0 \subset B^1 \subset \ldots \subset B^n = B$ such that $B^i x \subset B^{i-1}$ and $B^i H \subset B^{i-1}$ (since $B_0^i(H)H \subset B_0^{i-1}(H)$) . Hence $B^i(Fx + H) \subset B^{i-1}$ for all i and $B(Fx + H)^n = 0$, so $B = B_0(Fx + H)$. By the maximality of H we have $H = Fx + H$ and $H = N$. //

Theorem 3.4.5.(Lie). Let L be a solvable Lie algebra over F and let $B \neq 0$ be a finite-dimensional irreducible L-module over F . Suppose that L has a split CSA H such that every $x \in H$ is split on B over F . Then $\dim B = 1$.

Proof. In view of Theorem 3.4.3, $B = B_0(L^{(1)})$ and so by Theorem 3.2.7 $B_0^1(L^{(1)}) \equiv W \neq 0$. Since $L^{(1)}$ is an ideal of L , W is an L-submodule and hence $W = B$ by the L-irreducibility of B . Thus $BL^{(1)} = 0$ and B is regarded as an A-module with $v(x + L^{(1)}) = vx$, $v \in B$ and $x \in L$, where $A = L/L^{(1)}$. Put $L_* = \Sigma_{a \neq 0} L_a(\mathrm{ad}\, H)$. Then by Theorem 3.3.6 $L = H \oplus L_*$ and $[HL_*] = L_* = [L_*H]$. Thus $L = H + L^{(1)}$ and $A = (H + L^{(1)})/L^{(1)}$, so every element in A is split on B over F . By Theorem 3.2.9 $B = \Sigma \oplus B_a(A)$ where each $B_a(A)$ is an A-submodule. But then since

$BL^{(1)} = 0$, every A-submodule of B is an L-submodule of B, so $B = B_a(A)$ for $B_a(A) \neq 0$. Now, let $x \in A$ be any element. Then there is a $v \neq 0$ in B such that $vx = a(x)v$ and so $B_x \equiv \{v \in B | vx = a(x)v\} \neq 0$. It follows that B_x is an A-submodule of B, since A is abelian. Thus $B_x = B = B_a(A)$ for $x \in A$ and this happens only if $\dim B = 1$, since B is A-irreducible. //

Definition 3.4.1. Let V be an S-module where S is a set. An S-composition series of V is a sequence of S-submodules of V

$$0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n = V$$

such that each V_i/V_{i-1} is S-irreducible. //

If V is a finite-dimensional S-module, V has an S-composition series. Indeed, let V_m be a proper S-submodule of V of maximal dimension. Then $0 \subset V_m \subset V$ and continuing this with V_m, one arrives at an S-composition series.

Let L be a solvable subalgebra of $(\mathrm{Hom}_F B)$ such that each $x \in L$ is split on B over F [illegible] B is a finite-dimensional vector space over F. [illegible] that each $ad_L x$ is split over F by Theorem [illegible]

Then B has an L-composition series

$$0 = B_0 \subsetneqq B_1 \subsetneqq \cdots \neq B_m = B \ .$$

Since each $B_i/B_{i-1} \neq 0$ is L-irreducible, by Theorem 3.4.5 $\dim B_i/B_{i-1} = 1$, $i = 1,2,\ldots,m$. Thus $\dim B_i = i$, $i = 1,2,\ldots,m$. Therefore, one can choose a basis for B relative to which the matrix of each $x \varepsilon L$ is in an upper triangular form :

$$\begin{bmatrix} \alpha_1 & & & * \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_m \end{bmatrix} . \qquad (1.24)$$

We refer to this situation as the <u>simultaneous triangulability</u> of L. We state this as

<u>Theorem 3.4.6. (Lie)</u>. Let B be a finite-dimension vector space over a field F of characteristic 0. Suppose that L is a "split" solvable subalgebra of $(\mathrm{Hom}_F\ B)^-$. Then L is simultaneously triangulable. //

<u>Corollary 3.4.7</u>. Let L be a "split" nilpotent Lie algebra and let B be a finite-dimensional L-module. Then each weight of L in B is linear and vanishes on $L^{(1)}$. In particular, each root of a split CSA of a Lie algebra is linear. (char $F = 0$).

Proof. By Theorem 3.2.9 $B = \Sigma_a \oplus B_a(L)$ where each $B_a(L)$ is an L-submodule of B and the only eigenvalue of $x|_{B_a(L)}$ is $a(x)$ for $x \in L$. Thus by Theorem 3.4.6 the matrix (1.24) of $x|_{B_a(L)}$ has the form

$$\begin{bmatrix} a(x) & & * \\ & \ddots & \\ 0 & & a(x) \end{bmatrix}$$

for $x \in L$. Thus there is an element $v \neq 0$ in $B_a(L)$ such that $vx = a(x)v$ for all $x \in L$. Hence

$$v(x+y) = a(x+y)v = vx + vy = [a(x) + a(y)]v ,$$

$$v(\alpha x) = a(\alpha x)v = \alpha(a(x))v \quad \text{and} \quad v[xy] = (vx)y - (vy)x$$

$$= [a(x)a(y) - a(y)a(x)]\, v = a([xy])v , \quad \text{so} \quad a([xy]) = 0 ,$$

$\alpha \in F$, $x,y \in L$. . //

Corollary 3.4.8. Let L be a split solvable subalgebra of $(\mathrm{Hom}_F\, B)^-$. Then $L^{(1)}$ is nilpotent.

Proof. By Theorem 3.4.6 $L^{(1)}$ is simultaneously nil triangulable. //

Corollary 3.4.9. If L is solvable then $L^{(1)}$ is nilpotent.

Proof. We may assume that F is algebraically closed, since solvability is preserved under scalar extension. Since $\mathrm{ad}\, L$ is a solvable subalgebra of $(\mathrm{Hom}\, L)^-$, $[\mathrm{ad}\, L, \mathrm{ad}\, L] = \mathrm{ad}\, L^{(1)}$ is nilpotent and so is $L^{(1)}$ by Lemma 3.2.1. //

Corollary 3.4.10. Let L be the same as in Corollary 3.4.8. Then L has a common eigenvector $v \varepsilon B$, that is, $vx = \lambda(x)v$, $x \varepsilon L$, $\lambda(x) \varepsilon F$.

Proof. This is clear from Theorem 3.4.6. //

3.5. Conjugacy of Cartan subalgebras

In this section we assume that F is of characteristic 0. If H is a split CSA of L then by Theorem 3.3.6 L is expressed as a direct sum of root spaces for H in L;

$$L = H \oplus L_\alpha \oplus L_\beta \oplus \ldots \oplus L_\rho \qquad (1.25)$$

where $L_\alpha = L_\alpha(H)$ and $\alpha, \beta, \ldots, \rho$ are the nonzero roots of H in L.

We refer to (1.25) as the <u>Cartan decomposition</u> of L relative to H. Note also that the roots of H in L are linear (Corollary 3.4.7) and so polynomial functions on H. Let

$$H^0 = \{h \in H \mid \alpha(h)\beta(h) \ldots \rho(h) \neq 0\} .$$

Then H^0 is a nonempty Zariski open set in H. Letting $L_* = L_\alpha + \ldots + L_\rho$, L_* is ad h-stable for $h \in H$ and if $h \in H^0$ then all the eigenvalues of $\text{ad } h|_{L_*}$ are nonzero and so $\text{ad } h|_{L_*}$ is nonsingular. Therefore we have $H = L_0(\text{ad } h)$ for $h \in H^0$. Conversely, if $H = L_0(\text{ad } h)$ for $h \in H$ then $\text{ad } h|_{L_*}$ is nonsingular since $L = H \oplus L_*$, and hence $h \in H^0$. Thus we have

<u>Lemma 3.5.1</u>. $H^0 = \{h \in H \mid \text{ad } h|_{L_*} \text{ is nonsingular}\}$

$$= \{h \in H \mid H = L_0(\text{ad } h)\} . \quad //$$

If ad x is nilpotent for $x \in L$ then exp ad x is an automorphism of L. Denote by $\text{Aut}_e(L)$ the group of automorphisms of L generated by exp ad x for $x \in L$ where ad x is nilpotent. Each element in $\text{Aut}_e(L)$ is called an <u>invariant automorphism</u> of L. For a nonzero root α of H, let $x \in L_\alpha$. Since $L_\beta(\text{ad } x)^n \in L_{\beta+n\alpha}$ and there are only finitely many roots of H, $L_\beta(\text{ad } x)^n = 0$ for some $n > 0$ and any root β of H.

Thus $\exp \operatorname{ad} L_\alpha \subset \mathrm{Aut}_e(L)$ for nonzero roots α of H.

<u>Theorem 3.5.2</u>. Let L be a finite-dimensional Lie algebra over an algebraically closed field F of characteristic 0. Let H and H_1 be any CSA of L. Then there exists an invariant automorphism η of L such that $H = H_1\eta$.

<u>Proof</u>. Let $L = H + L_\alpha + L_\beta + \dots + L_\rho$ be the Cartan decomposition of L relative to H and let $L_* = L_\alpha + L_\beta + \dots + L_\rho$. Let $\{h_1, \dots, h_\ell, e_{\ell+1}, \dots, e_n\}$ be a basis for L where $\{h_1, \dots, h_\ell\}$ is a basis of H and the e_j consist of basis of $L_\alpha, \dots, L_\rho$. For indeterminates $X_1, X_2, \dots, X_n$, let

$$(\Sigma_i X_i h_i)\exp(\operatorname{ad} X_{\ell+1}e_{\ell+1}) \dots \exp(\operatorname{ad} X_n e_n)$$

$$= \Sigma_i p_i(X_1, \dots, X_n)h_i + \Sigma_j p_j(X_i, \dots, X_n)e_j .$$

Then the mapping $p : L \to L$ defined by

$$p(\Sigma \alpha_i h_i + \Sigma \alpha_j e_j) = \Sigma p_i(\alpha_1, \dots, \alpha_n)h_i + \Sigma p_j(\alpha_1, \dots, \alpha_n)e_j$$

is a polynomial mapping. For any elements $h_0 \in H^0 = \{x \in H \mid \alpha(x) \dots \rho(x) \neq 0\}$ and $e \in L_*$ with $e = \Sigma \alpha_j e_j$ we have

$$p(h_0 + t(h + e)) = (h_0 + th)\exp(\operatorname{ad} t\alpha_{\ell+1}e_{\ell+1}) \dots \exp(\operatorname{ad} t\alpha_n e_n)$$

$$\equiv h_0 + th + t[h_0 e] \qquad (\mathrm{mod}\ t^2)$$

where $h \in H$ and t is an indeterminate. But then the Taylor formula applied to $p(h_0 + t(h + e))$ implies that $(d_{h_0}p)(h + e) = h + [h_0 e]$. Since $\mathrm{ad}\ h_0|_{L_*}$ is nonsingular by Lemma 3.5.1, this shows that $d_{h_0}p$ is nonsingular on L and so is surjective. Since $H^0 + L_*$ is a nonempty open set in L , by Theorem 2.3.2 $p(H^0 + L_*) \supset U$ for some open set $U \neq \phi$ in L , so $U \subset H^0 \mathrm{Aut}_e(L)$. The same argument applied to H_1 assures that $U_1 \subset H_1^0 \mathrm{Aut}_e(L)$ for some nonempty open set U_1 . Since $U \cap U_1 \neq \phi$, we see that $h = k\eta$ for some $h \in H^0$, $k \in H_1^0$ and $\eta \in \mathrm{Aut}_e(L)$. But by Lemma 3.5.1, $H = L_0(\mathrm{ad}\ h) = L_0(\mathrm{ad}\ (k\eta)) = L_0(\mathrm{ad}\ k)\eta = H_1\eta$. //

Theorem 3.5.2 in particular implies that all CSA of L have the same dimension. Thus by Corollary 3.3.3 and Lemma 3.5.1 we have

<u>Corollary 3.5.3</u>. $H \cap L_{reg} = H^0$. //

3.6. Cartan's criteria

We discuss important criteria for solvability and semisimplicity of L in terms of a trace form. These criteria give the basis for the structure of semisimple Lie algebras. We assume that all L-modules are finite-dimensional over a field F of characteristic 0 and f denotes the representation of L afforded by B.

Theorem 3.6.1. Let x, y, h be elements in L such that $[xy] = h$, $x \in L_{-\alpha}(\text{ad } h)$ and $y \in L_{\alpha}(\text{ad } h)$ where $\alpha \in F$. Suppose that $B \neq B_0(h)$. Then $\operatorname{Tr} f(h)^2 \neq 0$ and $\operatorname{Tr} f(h)^2 = r\alpha^2$ for a positive rational number r. In particular, $\alpha \neq 0$.

Proof. As in Theorem 3.4.1. we assume that F is algebraically closed. Let $B = \Sigma \oplus B_\beta(h)$. Then β is the only eigenvalue of $f(h)|_{B_\beta(h)}$ if $B_\beta(h) \neq 0$.

Letting $W_\beta = \sum_{i \in Z} \oplus B_{\beta+i\alpha}(h)$, by Theorem 3.2.10 W_β is stable under x and y. Thus

$$0 = \operatorname{Tr}[f(x), f(y)]\big|_{W_\beta} = \operatorname{Tr} f(h)|_{W_\beta} = \Sigma d_i(\beta + i\alpha)$$

where $d_i = \dim B_{\beta+i\alpha}$. Hence $\beta = -\dfrac{\Sigma\, i d_i}{\Sigma\, d_i}\alpha$

and $\beta = r_\beta \alpha$ for some rational r_β . Since $B \neq B_0(h)$, $\dim B_\beta(h) \neq 0$ for some $\beta \neq 0$ and, for such β , $r_\beta \neq 0$ and so $\alpha \neq 0$. Since β^2 is the only eigenvalue of $f(h)^2|_{B_\beta(h)}$, we now have $\operatorname{Tr} f(h)^2 = \Sigma_\beta\, d_\beta \beta^2 = \Sigma d_\beta (r_\beta \alpha)^2$ where $d_\beta = \dim B_\beta(h)$. //

Let K be an extension field of F and L_K be the scalar extension of L to K . Then we note that $L_K^{(i)} = (L_K)^{(i)}$ and so L is solvable if and only if L_K is. Note also that f has a unique extension f_K to L_K , which is afforded by B_K , and that $\ker f_K = (\ker f)_K$.

<u>Theorem 3.6.2. (Cartan's criterion for solvability)</u>. Let $(x,y) = \operatorname{Tr} f(x)f(y)$. Then L is solvable if and only if $\ker f$ is solvable and $(x,y) = 0$ for all $x,y \in L^{(1)}$.

<u>Proof</u>. By the foregoing remark, we can assume that F is algebraically closed. If L is solvable, $\ker f$ and $f(L)$ are solvable. Since $f(L)$ is a solvable subalgebra of $(\operatorname{Hom}_F B)^-$, by Lie's Theorem 3.4.6 $f(L)$ is simultaneously triangulable. Thus, relative to some basis for B , the elements of $f(L)^{(1)} = f(L^{(1)})$ are upper triangular nilpotent matrices, so $(x,y) = \operatorname{Tr} f(x)f(y) = 0$ for $x,y \in L^{(1)}$.

Conversely, suppose that $\ker f$ is solvable and $(x,y) = 0$ for $x,y \in L^{(1)}$. We proceed by induction on $\dim L$. If $\dim L = 1$, the assertion is trivial. It suffices to show that $f(L)$ is solvable since $f(L) \cong L/_{\ker f}$ and $\ker f$ is solvable (Lemma 1.5.1). Thus, we assume that L is a subalgebra of $(\mathrm{Hom}_F\, B)^-$ and $f(x) = x$ for $x \in L$. Let H be a CSA of L and let $L = H + \Sigma_{\alpha\neq 0} L_\alpha$ be the Cartan decomposition of L relative to H. Let $x \in L_{-\alpha}$, $y \in L_\alpha$, $h = [xy]$. Then $[xy] = h \in H$ and since $h \in L^{(1)}$, $(h,h) = \mathrm{Tr}\, h^2 = 0$. Thus by Theorem 3.6.1 $B = B_0(h)$ and h is nilpotent. So, $[L_{-\alpha}L_\alpha] \subset N \equiv \{x \in H | x$ is nilpotent$\}$ for all roots α. Now, by Corollary 3.4.4, N is an ideal of H. Letting $J = N + \Sigma_{\alpha\neq 0}L_\alpha$, J is an ideal of L since $[HJ] \subset J$ and $[L_{-\alpha}L_\alpha] \subset N$. If $J \neq L$, by induction J is solvable and $L/J = (H + J)/J \cong H/_{H \cap J} = H/_N$ is solvable, so is L. If $J = L$ then $H = N$ and x is nilpotent for $x \in H$, so $\mathrm{ad}\, x$ is nilpotent by Theorem 1.3.3. Thus $L = L_0(\mathrm{ad}\, H) = H$ since H is a CSA. //

Notice that, in Theorem 3.6.2, $(x,y) = 0$ for $x,y \in L^{(1)}$ if and only if $(x,x) = 0$ for $x \in L^{(1)}$.

<u>Theorem 3.6.3 (Cartan's criterion for semisimplicity)</u>. If L is semisimple then the trace form of any 1 - 1 representation of L is nondegenerate.

L is semisimple if and only if the Killing form is nondegenerate.

<u>Proof</u>. If f is a $1-1$ representation of L, let $(x,y) = \mathrm{Tr}\, f(x)f(y)$. Thus $\ker f = 0$, and $L^{\perp} = \{x \in L \mid (x,y) = 0 \text{ for } y \in L\}$ is an ideal of L. Since $(x,y) = 0$ for $x,y \in (L^{\perp})^{(1)}$, by Theorem 3.6.2. $L^{\perp}$ is solvable. Hence if L is semisimple, $(\ ,\ )$ is nondegenerate. Note that if L is semisimple, ad representation of L is $1-1$. The converse follows from Lemma 1.6.3. //

<u>Corollary 3.6.4</u>. Let J be an ideal of L. Then L/J is semisimple if and only if $\mathrm{Rad}\, L \subset J$.

<u>Proof</u>. If L/J is semisimple, then $(J + \mathrm{Rad}\, L)/J$ is a solvable ideal of L/J and so $\mathrm{Rad}\, L \subset J$. Suppose that $\mathrm{Rad}\, L \subset J$. Let $\overline{L} = L/\mathrm{Rad}\, L$ and $\overline{J} = J/\mathrm{Rad}\, L$. Then $L/J \cong \overline{L}/\overline{J}$ and since $\overline{L}$ is semisimple by Lemma 1.5.1, the Killing form on $\overline{L}$ is nondegenerate by Theorem 3.6.3. Lemma 1.6.3 then assures that $\overline{L}$ is a direct sum of simple ideals $\overline{L}_i$ of $\overline{L}$. On the other hand, by Corollary 1.5.4 $\overline{J}$ is a sum of some $\overline{L}_j$'s and hence $\overline{L}/\overline{J}$ is a direct sum of simple ideals. In fact, $\overline{L}/\overline{J}$ is a direct sum of simple ideals $(\overline{L}_i + \overline{J})/\overline{J}$ where $\overline{L}_i \not\subset \overline{J}$. Thus L/J is semisimple. //

Corollary 3.6.5. Any homomorphic image of a semisimple Lie algebra is semisimple.

Proof. Let $f : L \to f(L)$ be a homomorphism of a semisimple Lie algebra. Then $f(L) \cong L/_{\ker f}$ is semisimple by Corollary 3.6.4. //

Let $\{x_1, x_2, \ldots, x_n\}$ be a basis of L. Then we note that the Killing form $K(\ ,\ )$ of L is nondegenerate if and only if the matrix $(K(x_i, x_j))$ is nonsingular. For an extension field K of F, since $K(\ ,\ )$ is regarded as the Killing form on L_K and $\{x_1, \ldots, x_n\}$ is a basis for L_K, by Theorem 3.6.3 we have

Corollary 3.6.6. If L is a semisimple Lie algebra over F (of char 0), then any scalar extension L_K of L is semisimple. //

3.7. The theorems of Weyl and Levi

In this section we prove two important theorems on the existence of complements. One is the Weyl's theorem