IL NUOVO CIMENTO Vol. 5 A, N. 4 21 Ottobre 1971

Representation Theory of a New Relativistic Dynamical Group (*).

J. J. Aghassi, P. Roman and R. M. Santilli
Department of Physics, Boston University - Boston, Mass.

(ricevuto il 9 Dicembre 1970)

Summary. — Via the method of induced representations, all irreducible unitary projective representations of the recently introduced new relativistic dynamical group $\mathfrak{G}_5$ are deduced and classified. An explicit form of the transformation law is given. The properties of the corresponding infinite-dimensional basis functions are studied. It is shown that in the limiting case of $l=\infty$ (corresponding to $\tilde{\mathfrak{G}}_5 \to \mathfrak{G}_5$) the infinite spin-tower representations become reducible and decompose into irreducible representations of the Poincaré group. The reduction of the direct product of two irreducible unitary ray representations of $\tilde{\mathfrak{G}}_5$ is studied. The Clebsch-Gordan coefficients are computed. Finally, some comments on the physical interpretation of the results are given.

1. – Introduction.

In a previous publication (¹) we introduced a new symmetry group (denoted by $\mathfrak{G}_5$) for relativistic dynamics. This group acts on the Cartesian product space $E_{3,1}\times E_1$, where $E_{3,1}$ is the Minkowski space with points x^μ and E_1 is a one-dimensional manifold with points denoted by u. As was indicated in ref. (¹) and discussed in greater detail in a subsequent publication (²), the new kinematical variable u must be interpreted as the proper time. The defining trans-

(*) Work supported by the U.S. Air Force under Grant No. AFOSR-67-0385B.
(¹) J. J. Aghassi, P. Roman and R. M. Santilli: *Phys. Rev. D*, **1**, 2753 (1970).
(²) J. J. Aghassi, P. Roman and R. M. Santilli: *Journ. Math. Phys.*, **11**, 2297 (1970).

formations [3] of $\mathfrak{G}_5$ are

$$x'_\mu = \Lambda^\nu_\mu x_\nu + b_\mu u + a_\mu\,, \qquad u' = u + \sigma\,. \tag{1.1}$$

Here Λ^ν_μ is a restricted Lorentz matrix, a_μ a constant translation vector, σ a constant scalar. The transformations associated with the constant vector b_μ are analogous to the boost (velocity) transformations of the nonrelativistic Galilei group. We call these the « zest » transformations. In obvious notation, the structure of $\mathfrak{G}_5$ is as follows:

$$\mathfrak{G}_5 = \{T^a_4 \times T^\sigma_1\} \otimes \{T^b_4 \otimes SO_{0(3,1)}\}\,, \tag{1.2}$$

where $\times$ and $\otimes$ denote direct and semi-direct products, respectively. Thus, $\mathfrak{G}_5$ contains as a subgroup both the restricted Poincaré group and the nonrelativistic Galilei group. Moreover, $\mathfrak{G}_5$ is a group extension [4] of the restricted Lorentz group $SO_{0(3,1)}$. From these comments it follows that, on the one hand, our $\mathfrak{G}_5$ is a natural generalization of the Poincaré group and, on the other hand, it is also a natural generalization of the nonrelativistic Galilei group.

In ref. [1] it was pointed out that for the use in relativistic *quantum* mechanics, the central extension [4] of the covering group of $\mathfrak{G}_5$ by a one-dimensional Abelian phase group T^θ_1 must be used. This new relativistic quantum-mechanical dynamical group will be denoted by $\tilde{\mathfrak{G}}_5$ and its structure is

$$\tilde{\mathfrak{G}}_5 = \{T^a_4 \times T^\sigma_1\} \otimes \{T^b_4 \otimes (SL_{2,c} \times T^\theta_1)\}\,, \tag{1.3}$$

where $SL_{2,c}$ appears as the covering of $SO_{0(3,1)}$.

The generators of $\tilde{\mathfrak{G}}_5$ are denoted by $J_{\mu\nu}$, P_μ, Q_μ, S and they generate the subgroup $SL_{2,c}$, T^a_4, T^b_4, T^σ_1, respectively. Since we shall not need them in this paper, we do not write out here the Lie algebra [5] in full. But we recall the most important relation, *viz.* [6]

$$[P_\mu, Q_\nu] = -\,i g_{\mu\nu} l^{-1}\,. \tag{1.4}$$

Here the real constant l has the dimension of length and its appearance is connected with the phase group T^θ_1.

---

(3) In a recent private communication M. NOGA (Purdue University) gave an alternative derivation of our group, emphasizing that it is actually the dynamical group of the standard equation of motion in relativistic mechanics. See also ref. (76).

(4) See ref. (1), Appendix A.

(5) See ref. (1), eqs. (3.7) through (3.12).

(6) We use the Minkowski metric $g_{00} = -g_{kk} = 1$. Note, incidentally, that the full carrier space $E_{3,1} \times E_1$ is not a metric space.

The Casimir operators of $\tilde{\mathfrak{G}}_5$ are

$$\mathscr{D} = P_\mu P^\mu + 2l^{-1} S \,, \tag{1.5a}$$

$$\mathscr{J} = \tfrac{1}{2} T_{\mu\nu} T^{\mu\nu} \,, \tag{1.5b}$$

$$\mathscr{K} = \tfrac{1}{4} \varepsilon_{\mu\nu\varphi\sigma} T^{\mu\nu} T^{\varphi\sigma} \,, \tag{1.5c}$$

where

$$T_{\mu\nu} \equiv J_{\mu\nu} - l M_{\mu\nu} \tag{1.6}$$

with

$$M_{\mu\nu} \equiv P_\mu Q_\nu - P_\nu Q_\mu \,. \tag{1.6a}$$

Of course, in addition to $\mathscr{D}$, $\mathscr{J}$, $\mathscr{K}$, the operator $l1$ is also an invariant of our group. As is well known (see, for example, ref. [7]) this leads to a superselection rule.

In ref. [1] we showed that $X_\mu \equiv -lQ_\mu$ is a perfectly acceptable relativistic space-time position operator [8] and $\mathscr{M}^2 \equiv -2l^{-1}S$ is a nontrivial relativistic mass operator. S also plays the role of an evolution operator with respect to proper time. Some other physical consequences of $\tilde{\mathfrak{G}}_5$ were also explored [1,2], and finally we showed [2] that $\tilde{\mathfrak{G}}_5$ is the contracted limit of the covering of the connected component of the inhomogeneous de Sitter group $ISO_{3,2}$.

The main purpose of the present paper is to study in detail and with sufficient mathematical rigor the representations of $\tilde{\mathfrak{G}}_5$. We find this study crucial, because all further applications of $\tilde{\mathfrak{G}}_5$ depend critically on the thorough understanding of the representations [9]. Apart from this, the representation theory of $\tilde{\mathfrak{G}}_5$ merits study from the purely mathematical point of view. The group has a sufficiently interesting structure (cf. (1.3)) and the mathematics involved is far from being trivial. It is true that there are some similarities with the nonrelativistic Galilei group, but in the present case the little group (see Subsect. **2**.4) is noncompact; this makes the theory quite involved.

In Sect. **2** and **3** we systematically derive all irreducible unitary projective representations of $\mathfrak{G}_5$, in an explicit form. In Sect. **4** we study separately the $l = \infty$ limiting case, which corresponds [10] to replacing $\tilde{\mathfrak{G}}_5$ by $\mathfrak{G}_5$. In Sect. **5** we give a discussion of the products of representations and their reduction. This turns out to be a rather involved problem. In Sect. **6** we discuss additional features of our group, pointing out also some problematic aspects.

---

(7) V. BARGMANN: *Ann. Math.*, **59**, 1 (1954).
(8) In this respect, see also J. E. JOHNSON: *Phys. Rev.*, **181**, 1755 (1969); L. CASTELL: *Nuovo Cimento*, **49** A, 285 (1967).
(9) Among other things, we have in mind the establishing of wave equations for arbitrary spin.
(10) See ref. [1], Appendix C.

The main mathematical tool used in this paper will be the method of induced representations, developed by MACKEY (11). Actually, some parts of our calculations parallel rather closely the work of VOISIN (12), who used Mackey's method to study the ray representations of the nonrelativistic Galilei group (13).

2. – Some algebraic preliminaries.

2·1. *Factor system.* – Let us represent a generic element g of $\tilde{\mathfrak{G}}_5$ by

$$g = (\exp[i\theta];\ \sigma, a, b, \Lambda)\,, \tag{2.1}$$

where σ, a, b, Λ stand for the parameters in (1.1) and θ is the phase associated with the T_1^θ subgroup. As we already stated in ref. (1), the composition law of $\tilde{\mathfrak{G}}_5$ can be written as

$$\begin{aligned} g_2 g_1 &\equiv (\exp[i\theta_2];\ \sigma_2, a_2, b_2, \Lambda_2)(\exp[i\theta_1];\ \sigma_1, a_1, b_1, \Lambda_1) = \\ &= (\omega(g_2, g_1)\exp[i(\theta_2+\theta_1)];\ \sigma_2+\sigma_1,\ a_2+\Lambda_2 a_1+\sigma_1 b_2,\ b_2+\Lambda_2 b_1,\ \Lambda_2\Lambda_1)\,. \end{aligned} \tag{2.2}$$

Here

$$\omega(g_2, g_1) \equiv \exp[if(g_2, g_1)] \tag{2.3}$$

is a phase factor (f is real), called the factor system (14), which arises from the scalar extension of $\mathfrak{G}_5$ to $\tilde{\mathfrak{G}}_5$. Its appearance in (2.2) has deep implications for the representation theory of $\tilde{\mathfrak{G}}_5$. Let us consider a homomorphism

$$g \to \mathcal{U}_g \tag{2.4}$$

from $\tilde{\mathfrak{G}}_5$ to a family of unitary operators. The multiplication law for these

(11) A very readable account of this powerful tool can be found in G. W. MACKEY: *Induced Representations of Groups and Quantum Mechanics* (New York, 1968). A shorter, but more rigorous summary is given in G. W. MACKEY: *Group representations in Hilbert space*, which is the Appendix in I. E. SEGAL: *Mathematical Problems in Relativistic Physics* (New York, 1963). The latter contains also a bibliography of original publications.

(12) J. VOISIN: *Journ. Math. Phys.*, **6**, 1519, 1822 (1965).

(13) An alternative, somewhat more intuitive treatment of the ray representations of the nonrelativistic Galilei group was given by J.-M. LÉVY-LEBLOND: *Journ. Math. Phys.*, **4**, 776 (1963). Some parts of our calculations are analogous to those of LÉVY-LEBLOND.

(14) See Appendix A of ref. (1).

operators corresponding to the composition law (2.2) is

$$\mathscr{U}_{g_2}\mathscr{U}_{g_1} = \omega(g_2, g_1)\mathscr{U}_{g_2 g_1}\,. \tag{2.5}$$

As follows from the general theory of nontrivial central extension of groups (7), the phase ω in (2.5) is essential and cannot be eliminated by a redefinition $\mathscr{U}_g \to \tau(g)\mathscr{U}_g$, $|\tau(g)| = 1$, of the operators $\mathscr{U}_g$.

The explicit determination of ω is done by applying (2.5) onto the state vector of the one-dimensional representation. This will be shown at the end of the Appendix. The result of the calculation is that f (defined by (2.3)) is given by

$$f(g_2, g_1) = -l^{-1}(b_2 \Lambda_2 a_1 + \tfrac{1}{2} b_2^2 \sigma_1)\,, \tag{2.6}$$

where l is the constant appearing in (1.4) and (1.5*a*). Thus, f or ω depends only on the translation part of g_1 and on the «homogeneous» part of g_2.

Finally, we note that the unit element of $\tilde{\mathfrak{G}}_5$ is $(1; 0, 0, 0, 1)$ and hence the inverse element g^{-1} is given by

$$g^{-1} = (\exp[-i(\theta + \hat{f})]; -\sigma, -\Lambda^{-1}(a - b\sigma), -\Lambda^{-1}b, \Lambda^{-1})\,, \tag{2.7}$$

where

$$\hat{f} \equiv f(g, g^{-1}) = -l^{-1}(-ba + \tfrac{1}{2}b^2\sigma)\,. \tag{2.7a}$$

The unitary representations of $\tilde{\mathfrak{G}}_5$ furnished by the homomorphism (2.4) and the multiplication law (2.5) (with ω given by (2.3) and (2.6)) are called *unitary projective* (or *ray*) *representations* (7). It is these ray representations (which cannot be reduced to the true representations of $\mathfrak{G}_5$) that will be constructed in the following. The first step in this program is the decomposition of $\tilde{\mathfrak{G}}_5$ into the semi-direct product of a suitably chosen invariant Abelian subgroup N and a remainder H. The coset space $\Gamma = \tilde{\mathfrak{G}}_5/H$ will then be taken as the repsentation space.

2'2. *Invariant Abelian subgroup.* – Consider the invariant Abelian subgroup

$$N = T_1^\sigma \times T_4^a \tag{2.8}$$

of $\mathfrak{G}_5$ and introduce the notation

$$H = \{T_4^b \otimes (SL_{2,C} \times T_1^\theta)\}\,. \tag{2.9}$$

Then $\tilde{\mathfrak{G}}_5$ can be written as the semi-direct product

$$\tilde{\mathfrak{G}}_5 = N \otimes H\,. \tag{2.10}$$

The semi-direct product structure is realized by the automorphisms π_h of N,

$$(2.11)\qquad n \to \pi_h(n) \equiv hnh^{-1}\,, \qquad n \in N\,, \qquad h \in H\,.$$

Indeed, the mapping

$$(2.12)\qquad h \to \pi_h\,, \qquad h \in H\,,$$

defines a homomorphism of H into the group of all automorphisms of N. Thus, every element g of $\tilde{\mathfrak{G}}_5$ can be uniquely represented by a pair,

$$(2.13)\qquad g = (n; h)\,, \qquad n \in N\,, \qquad h \in H\,,$$

in terms of which the composition law of $\tilde{\mathfrak{G}}_5$ becomes

$$(2.14)\qquad g_2 g_1 = (n_2; h_2)(n_1; h_1) = \left(n_2 \pi_{h_2}(n_1);\, h_2 h_1\right)\,.$$

We now turn to the irreducible unitary representations of the Abelian group N. They are, of course, one dimensional, and have the form $\mathscr{U}_n = \exp[i(r\sigma + pa)]1$. Here r is a real scalar and p a real four-vector (15). For convenience (and to emphasize the dual role of the parameters σ, a and representation labels r, p), we introduce the notation

$$(2.15)\qquad (\sigma, a|r, p) = \exp[i(r\sigma + pa)]\,.$$

The pair of labels $[r, p]$ is called the *character* of the representation.

The set of all representations $(\sigma, a|r, p)$ forms a group $\hat{N}$, usually called the *character group* of N. For each $h \in H$, the automorphism π_h defines a one-to-one mapping of $\hat{N}$ into itself, because the transformed form of (2.15) induced by π_h is again a unitary irreducible representation of N, so that it belongs to $\hat{N}$.

2'3. *Orbits.* – Let $n = (1; \sigma, a, 0, 1) \in N$ and let $h = (\exp[i\theta]; 0, 0, b, \Lambda) \in H$. The automorphism (2.11) is explicitly given by

$$(2.16)\qquad n \to \pi_h(n) = \left(\omega(h, n);\, \sigma, \Lambda a + b\sigma, 0, 1\right)\,,$$

where, according to (2.3) and (2.6),

$$(2.16a)\qquad \omega(h, n) = \exp[-il^{-1}(b\Lambda a + \tfrac{1}{2}b^2\sigma)]\,.$$

---

(15) Since S and P_μ are the generators of T_1^σ and T_4^a, respectively, the numbers r and p_μ are obviously the corresponding eigenvalues.

Because of the dualism between (σ, a) and (r, p) in (2.15), eq. (2.16) implies that the character $[r; p]$ evaluated at $\pi_h(n)$ is equal to a transformed character $[r'; p']$ evaluated at n. In other words,

$$h(\sigma, a|r, p)\,h^{-1} = \omega(h, n)(\sigma, \Lambda a + b\sigma|r, p) \equiv (\sigma, a|r', p')\,. \tag{2.17a}$$

In a similar manner we get

$$h^{-1}(\sigma, a|r, p)\,h = \omega(h^{-1}, n)(\sigma, \Lambda^{-1}a - \Lambda^{-1}b\sigma|r, p) \equiv (\sigma, a|r'', p'')\,. \tag{2.17b}$$

Thus, h and h^{-1} induce automorphic transformations of the characters. We write, somewhat symbolically,

$$h^{-1}[r, p] \equiv [r', p'] = \left[r + pb - \frac{1}{2l}\,b^2, \Lambda^{-1}\left(p - \frac{1}{l}\,b\right)\right], \tag{2.18a}$$

$$h[r, p] \equiv [r'', p''] = \left[r - \Lambda pb - \frac{1}{2l}\,b^2, \Lambda p + \frac{1}{l}\,b\right]. \tag{2.18b}$$

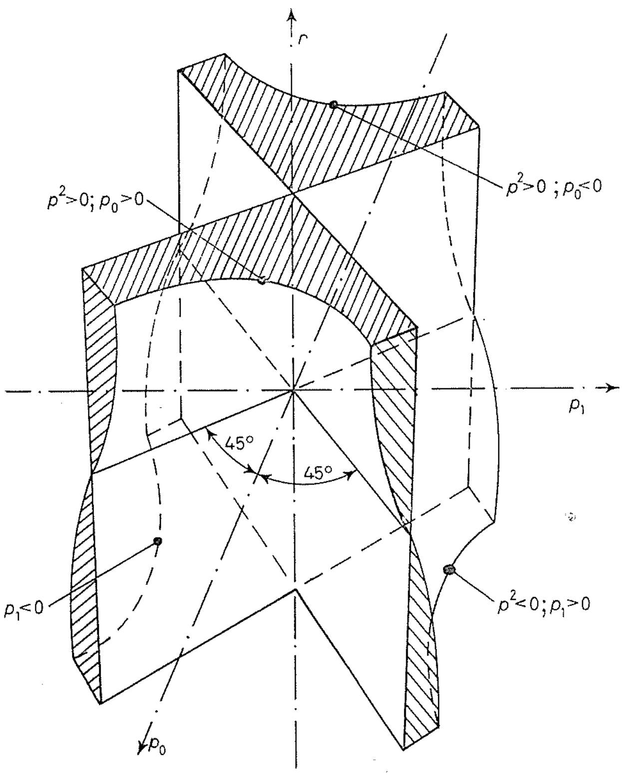

Fig. 1. – The orbit surface $p^2 + 2l^{-1}r = \mathscr{D}$ for $\mathscr{D} = 0$, $l < 0$. Two spatial directions (p_2 and p_3) are suppressed. Cuts parallel to the (p_0, p_1) plane give all Poincaré orbits.

It is easily verified that under these transformations the value of the invariant $\mathscr{D}$ (cf. eq. (1.5*a*)) is left unchanged, *i.e.*

$$p^2 + 2l^{-1}r = p'^2 + 2l^{-1}r' = p''^2 + 2l^{-1}r'' = \mathscr{D} . \tag{2.19}$$

Thus, the automorphisms π_h of N define *orbits* in N. Each orbit is characterized by some standard character $[\hat{r}, \hat{p}]_{\mathscr{D}l}$. (The subscripts $\mathscr{D}$ and l were used to emphasize that each orbit is fixed when the invariants $\mathscr{D}$ and l are given. In the following, however, we shall suppress these subscripts.) If $[r_1, p_1]$ and $[r_2, p_2]$ belong to the same orbit then there exists an element $h \in H$ such that

$$[r_2, p_2] = h[r_1, p_1] . \tag{2.20}$$

The orbit (2.19) is graphically represented in Fig. 1.

2'4. *Stability group.* – By definition [11], the *stabilizer* (*stability group*, or *little group*) H_0 of an orbit $[\hat{r}, \hat{p}]$ is a subgroup of H such that for every element $h_0 \in H_0 \subset H$, any given point of the orbit remains fixed, *i.e.*

$$h_0[\hat{r}, \hat{p}] = [\hat{r}, \hat{p}] . \tag{2.21}$$

From (2.18*a*) and (2.21) we obtain the conditions

$$\hat{r} = \hat{r} - \Lambda\hat{p}b - \frac{1}{2l}\, b^2 , \tag{2.22a}$$

$$\hat{p} = \Lambda\hat{p} + \frac{1}{l}\, b . \tag{2.22b}$$

These are not independent, because using (2.22*b*) in (2.22*a*), we simply get

$$\Lambda\hat{p}b + \frac{1}{2l}\, b^2 = \frac{l}{2}\, [(\Lambda\hat{p} + l^{-1}b)^2 - \hat{p}^2] = 0 . \tag{2.23}$$

This tells us that, selecting an arbitrary Λ, we only have to choose

$$b = l(\hat{p} - \Lambda\hat{p}) , \tag{2.24}$$

whence both conditions become satisfied. Thus, an arbitrary element $h_0 \in H_0$ has the form

$$h_0(\theta; \Lambda) \equiv (\exp[i\theta]; 0, 0, l(\hat{p} - \Lambda\hat{p}), \Lambda) . \tag{2.25}$$

We easily verify the combination law

$$h_0(\theta''; \Lambda'')\, h_0(\theta'; \Lambda') = h_0(\theta'' + \theta'; \Lambda''\Lambda') . \tag{2.26}$$

Thus, we see that the little group is isomorphic to the direct product of a phase group with an $SL_{2,C}$ group ([16]):

$$H_0 \approx T_1^\theta \times SL_{2,C}\,. \tag{2.27}$$

At this point we wish to make a remark. Instead of (2.8), we could have chosen the *maximal* Abelian subgroup of $\tilde{\mathfrak{G}}_5$, *viz.* $T_1^\theta \times T_1^\sigma \times T_4^a$. Then, instead of (2.9), we would have had $T_4^b \otimes SL_{2,C}$, but (2.10) would still have held. If we had done so, the characters would have been $\exp[i(\beta\theta + r\sigma + pa)] \equiv (\theta, \sigma, a|\beta, r, a)$, and the parameter β would have explicitly occurred in the transformations (2.18*a*), (2.18*b*). The orbits would no longer have been the invariants (2.19) of $\tilde{\mathfrak{G}}_5$; instead we would have had $p^2 + 2l^{-1}\beta^{-1}r = \text{const}$ as their equation ([17]). On the other hand, (2.25) would have been replaced by the simpler term $h_0(\Lambda) = (1; 0, 0, l(\hat{p} - \Lambda\hat{p}), \Lambda)$, and the little group would have been simply $H_0 \approx SL_{2,C}$. However, we feel that our treatment is more satisfactory, because, as pointed out above, our orbits have a more direct interpretation. The inconvenience of having a somewhat more complicated little group is only trivial, since (2.27) is a *direct* product.

Returning to our main subject, we now wish to find a relation between elements of our little group (2.27), and arbitrary elements of H. Let us consider an orbit $[\hat{r}, \hat{p}]$ and choose for a specific point $[r, p]$ of this orbit an element $h_{r,p}$ of H such that ([18])

$$h_{r,p}[\hat{r}, \hat{p}] = [r, p]\,. \tag{2.28}$$

Let now $h(\theta; b, \Lambda) \equiv (\exp[i\theta]; 0, 0, b, \Lambda)$ be an arbitrary element of H and write

$$h^{-1}(\theta; b, \Lambda)[r, p] = [r', p']\,, \tag{2.29}$$

where the r.h.s. is given by eq. (2.18*a*). Combining the last two equations, we get the identity

$$h^{-1}_{r',p'}\, h^{-1}(\theta; b, \Lambda)\, h_{r,p}[\hat{r}, \hat{p}] = [\hat{r}, \hat{p}]\,,$$

([16]) From (2.25) it is clear that the $SL_{2,C}$ which appears in H_0 is *not* the $SL_{2,C}$ subgroup of $\tilde{\mathfrak{G}}_5$. We shall come back to this point later, in Sect. **3**·5.

([17]) In his work on the ray representations of the nonrelativistic Galilei group, VOISIN actually proceeds in a manner as now sketched, and obtains the orbits $E - \boldsymbol{p}^2/2Mq_0 = \text{const}$ (cf. eq. (14) of ref. ([12]), first paper), instead of the more desirable $E - \boldsymbol{p}^2/2M = \text{const}$ paraboloids. At a later point, he then sets $q_0 = 1$ which, even though it seems to be an artificial choice, apparently does not lead to loss of generality.

([18]) In view of (2.20), we are assured of the existence of such an element of H. Actually, $h_{r,p}$ is not even unique.

which, because of (2.21), implies that the product of the three elements on the l.h.s. is an element of the little group H_0. We denote this particular element by h_0^{-1}, so that we have

$$h_0 = h_{r,p}^{-1} h(\theta; b, \Lambda) h_{r',p'} . \tag{2.30}$$

Conversely, we have

$$h(\theta; b, \Lambda) = h_{r,p} h_0 h_{r',p'}^{-1} . \tag{2.31}$$

The meaning of this equation is that given an arbitrary element $h \in H$, it can be expressed in terms of *some* element $h_0 \in H_0$ associated with the orbit $[\hat{r}, \hat{p}]$. The transformer $h_{r,p}$ is defined by (2.28). The point $[r', p']$ that labels the transformer on the right is related to $[r, p]$ by eq. (2.18*a*).

2'5. *Representation space.* – We choose for our representation space the coset space

$$\Gamma = \tilde{\mathfrak{G}}_5 / H . \tag{2.32}$$

In view of eq. (2.10), Γ is isomorphic to the Abelian group N. Consequently, there is a one-to-one correspondence between the elements of Γ and the characters $[r, p]$ of the orbits $[\hat{r}, \hat{p}]$ in $\hat{N}$. Therefore, the elements of Γ can be labeled by the character of an orbit $[\hat{r}, \hat{p}]$, *i.e.* by all numbers r, p which obey the relation

$$p^2 + 2l^{-1} r = \hat{p}^2 + 2l^{-1} \hat{r} = \mathscr{D} .$$

The basis of our representation will be a set of functions $\psi_\xi(r, p)$, which depend on the character $[r, p]$ as specified above. The additional collective label ξ stands to distinguish further the states within a given representation. (Thus, in general, the functions ψ_ξ are vector valued.) We introduce in Γ an invariant measure by defining

$$\mathrm{d}\Omega(r, p) = \mathrm{d}r\, \mathrm{d}^4 p\, \delta(p^2 + 2l^{-1} r - \mathscr{D}) , \tag{2.33}$$

and require that the functions ψ_ξ belong to the space $\mathscr{L}^2(\Gamma, \mathrm{d}\Omega)$ of square-integrable functions. Thus, our representation space is the Hilbert space $\mathscr{H}(\Gamma)$ with the inner product defined by

$$\langle \psi | \phi \rangle = \int \mathrm{d}r\, \mathrm{d}^4 p\, \delta(p^2 + 2l^{-1} r - \mathscr{D})\, \psi_\xi^\dagger(r, p) \phi_\xi(r, p) . \tag{2.34}$$

Here summation over ξ is understood.

If $\mathscr{U}_n$ is a unitary representation of N, the basis ψ_ξ will transform under the action of this representation according to

$$\mathscr{U}_n \psi_\xi(r, p) = \exp[i(r\sigma + pa)]\, \psi_\xi(r, p)\,. \tag{2.35}$$

Our problem is now to find the transformation law of the basis under the unitary irreducible projective representations $\mathscr{U}_g$ of $\mathfrak{G}_5$.

3. – The unitary irreducible projective representations of $\mathfrak{G}_5$.

In order to check the applicability of the subsequent mathematical construction, let us summarize the relevant properties of our group $\tilde{\mathfrak{G}}_5$:

i) $\tilde{\mathfrak{G}}_5$ is a separable locally compact group [19],

ii) it can be written as the semi-direct product of the invariant Abelian subgroup $N' = T_1^\sigma \times T_4^a \times T_1^\theta = N \times T_1^\theta$ and the noninvariant subgroup $H' = T_4^b \otimes \otimes SL_{2.C}$,

iii) both factors N' and H' are closed subgroups [20].

As MACKEY has shown [11], the fulfilment of these criteria ensures that the method of induced representations will furnish all irreducible representations of the group if the irreducible representations of the stabilizer are known.

3·1. *Representations of* H_0. – We start with the study of the representations of the stability group H_0. Because of its direct-product structure (exhibited by (2.27)), all irreducible representations of H_0 will have the form

$$\mathscr{U}_{h_0} = \exp[i\beta\theta] D(\hat{h}_0)\,. \tag{3.1}$$

Here, the first factor (with β arbitrary and real) is a representation of T_1^θ and the second factor stands for a representation of $SL_{2,C}$. In view of (2.27) and (2.26), the group element $\hat{h}_0$ in the argument of D denotes the element (2.25) with θ set equal to zero, *i.e.*

$$\hat{h}_0 = h_0(0; \Lambda) = (1;\, 0,\, 0,\, l(\hat{p} - \Lambda\hat{p}), \Lambda)\,. \tag{3.1a}$$

---

(19) This is obvious from the parametrization of the group elements.

(20) Since N is Abelian, its closedness is obvious. To see that H is closed, we note that the first factor in (2.9) is isomorphic to the covering of a Poincaré group which is known to be closed, and the second factor is an Abelian phase group, likewise closed.

The irreducible representations D of $SL_{2,C}$ are well known [21]. They are labeled by a pair of indices [22] k, c and we shall write D^{kc} to symbolize a specific irreducible representation. There are the following cases:

a) $k=0$, $c=1$. This is the *trivial one-dimensional* (and obviously unitary) representation.

b) k and c simultaneously integral or half-integral [23] and $|c|>|k|$. These representations are *finite dimensional and nonunitary.*

c) $k=0, \frac{1}{2}, 1, \ldots$ and $c=i\varphi$, with $-\infty<\varphi<+\infty$. These representations are infinite dimensional and unitary. They are said to belong to the *principal series.*

d) $k=0$, and c is a real number such that $0<c<1$. These are also infinite-dimensional unitary representations and are said to belong to the *supplementary series.*

We shall not be interested in the nonunitary representations of case *b*), and will discuss the simple case of the representation *a*) separately in the Appendix. For the unitary infinite-dimensional representations *c*) and *d*), each state of a given representation is characterized by two numbers [24] s and s_3. For any given representation D^{kc}, s can take on the infinite sequence of discrete values

$$s=k,\; k+1,\; k+2,\; \ldots \tag{3.2a}$$

For any specified s, the s_3 then can assume the $2s+1$ values

$$s_3=-s,\; -s+1,\; \ldots,\; s-1,\; s\,. \tag{3.2b}$$

We mention that the representations D^{kc} and D^{-k-c} are equivalent. Finally, we recall that the representation D^{k-c} is conjugate to D^{kc}.

3·2. *Induced representations of* $\tilde{\mathfrak{G}}_5$. – We are now in a position to determine the labels which are needed to specify a representation of $\tilde{\mathfrak{G}}_5$. They are as

---

(21) See, for example, I. M. GELFAND, R. A. MINLOS and Z. YA. SHAPIRO: *Representations of the Rotation and Lorentz Group* (New York, 1963), especially p. 200 and p. 188. See also M. A. NAIMARK: *Linear Representations of the Lorentz Group* (New York, 1964).

(22) The numbers k and c are related to the Casimir operators of $SL_{2,C}$; see eqs. (3.26) and (3.27) below.

(23) Case *a*) is a special case of Case *b*), but for obvious reasons has been treated separately.

(24) These numbers are related to the Casimir operators occurring in the chain $SL_{2C}\supset SU_2\supset SO_2$.

follows:

i) an arbitrary real number l,

ii) an arbitrary real number $\mathscr{D}$,

iii) two numbers k and c (as given above).

Here l and $\mathscr{D}$ are necessary to specify an orbit, and k, c are needed to specify the representation of the little group associated with the orbit ([25]). A representation of $\tilde{\mathfrak{G}}_5$ will be denoted by the symbol $(l|\mathscr{D}, k, c)$. Each state of a given representation is labeled (apart from r and p, selecting a point of the orbit) by two supplementary labels ([26]) s and s_3. For the relevant unitary irreducible representations (Cases c) and d) above) the possible values of s and s_3 are given by ([27]) (3.2a), (3.2b). In view of these comments, the complete labeling of the basis functions $\psi_\xi(r, p)$ (introduced in Subsect. **2**·5) belonging to an irreducible representation of $\tilde{\mathfrak{G}}_5$ is given by the notation

$$\psi = \psi^{l\mathscr{D}kc}_{ss_3}(r, p) .$$

Let us now consider a representation $\mathscr{U}_{h_0} = \exp[i\beta\theta]D^{kc}(\hat{h}_0)$ of H_0. The transformation law of our basis under $\mathscr{U}_{h_0}$ is

$$\mathscr{U}_{h_0}\psi^{l\mathscr{D}kc}_{ss_3}(r, p) = \exp[i\beta\theta][D^{kc}(\hat{h}_0)]_{s's'_3;\, ss_3}\psi^{l\mathscr{D}kc}_{s's'_3}(r, p) , \tag{3.3}$$

where summation over s' and s'_3 is understood.

Next we consider a homomorphism $h \to \mathscr{U}_h$ from the subgroup H to a set of unitary operators. On account of eq. (2.31) we can write

$$\mathscr{U}_h = \mathscr{U}_{h_{rp}h_0h^{-1}_{r'p'}} . \tag{3.4}$$

Since the functions $\psi(r, p)$ carry the representation $\mathscr{U}_h$, we have (omitting for a moment the super- and subscripts of ψ)

$$\mathscr{U}_h\psi(r, p) \equiv \psi(h^{-1}[r, p]) = \psi(h_{r'p'}h_0^{-1}h^{-1}_{rp}[r, p]) = \psi(h_{r'p'}h_0^{-1}[\hat{r}, \hat{p}]) \equiv \mathscr{U}_{h_0h^{-1}_{r'p'}}\psi(\hat{r}, \hat{p}) .$$

([25]) The additional real number β, that occurs in (3.1) and which, in addition to k and c, is needed to specify a representation of the little group, is immaterial because $\exp[i\beta\theta]$ will only be an arbitrary phase factor multiplier in the representation of $\tilde{\mathfrak{G}}_5$, cf. eq. (3.6) below.

([26]) Thus, the additional « collective label » ξ which was introduced in Subsect. **2**·5, corresponds to the pair s, s_3.

([27]) Thus, these labels run through a set of discrete, integer or half-integer numbers.

In the next to the last step we used (2.28). Now, we have [28]

$$\mathscr{U}_{h_0 h^{-1}_{r'p'}} = \mathscr{U}_{h_0}\mathscr{U}_{h^{-1}_{r'p'}} .$$

Hence, we can continue the previous equation as

$$\mathscr{U}_{h_0 h^{-1}_{r'p'}}\psi(r, p) = \mathscr{U}_{h_0}\mathscr{U}_{h^{-1}_{r'p'}}\psi(\hat{r}, \hat{p}) = \mathscr{U}_{h_0}\psi(h_{r'p'}[r, p]) =$$
$$= \mathscr{U}_{h_0}\psi(r', p') = \exp[i\beta\theta]D^{kc}(\hat{h}_0)\psi(r', p') .$$

In the penultimate step we used again (2.28) and in the last step we utilized (3.3). Thus, in detail we have the transformation law

$$\text{(3.5)} \qquad \mathscr{U}_h \psi^{l\mathscr{D}kc}_{ss_3}(r, p) = \exp[i\beta\theta][D^{kc}(\hat{h}_0)]_{s's'_3;\, ss_3}\psi^{l\mathscr{D}kc}_{s's'_3}(r', p') .$$

In view of (2.28) and (2.18a), the arguments r' and p' are explicitly

$$\text{(3.5}a\text{)} \qquad r' = r + pb - \frac{1}{2l}b^2 , \qquad p' = \Lambda^{-1}\left(p - \frac{1}{l}b\right) .$$

Finally, let us consider an arbitrary group element $g \in \tilde{\mathfrak{G}}_5$. Because of (2.10), we have the unique decomposition [29] $g = nh$. In the homomorphism $g \to \mathscr{U}_g$ this means that [30] $\mathscr{U}_g = \mathscr{U}_n\mathscr{U}_h$. The action of $\mathscr{U}_h$ is given by (3.5), and the action of $\mathscr{U}_n$ is shown in (2.35). Thus, putting these together, we obtain the transformation law for the irreducible unitary projective representations of $\mathfrak{G}_5$ as follows [31,32]:

$$\text{(3.6)} \qquad \mathscr{U}_g \psi^{l\mathscr{D}kc}_{ss_3}(r, p) = \exp[i(\beta\theta + r\sigma + pa)][D^{kc}(\hat{h}_0)]_{s's'_3;\, ss_3}\psi^{l\mathscr{D}kc}_{s's'_3}(r', p') .$$

We note that the unitarity of the representation is, of course, meant with respect to the inner product in the Hilbert space $\mathscr{H}(\Gamma)$ of $\mathscr{L}^2(\Gamma, \mathrm{d}\Omega)$ integrable functions, as defined by (2.34). That is, for $\psi, \phi \in \mathscr{L}^2(\Gamma, \mathrm{d}\Omega)$,

$$\text{(3.7)} \qquad \langle\psi|\phi\rangle = \int \mathrm{d}r\, \mathrm{d}^4p\, \delta\left(p^2 + \frac{2}{l}r - \mathscr{D}\right)(\psi^{l\mathscr{D}kc}_{ss_3}(r, p))^\dagger \phi^{l\mathscr{D}kc}_{ss_3}(r, p)$$

---

(28) Since the group elements $h \in H$ have no translational part ($a = \sigma = 0$), no phase factor will occur in the composition law of the representation operators $\mathscr{U}_H$.

(29) As is well known, this is a consequence of the representation (2.13) and composition law (2.14) of semi-direct–product groups.

(30) We do not have a phase factor, because the group element h has no translational part.

(31) Equation (3.6) has been already given, without proof and without detailed discussion, in Appendix C of our first paper, ref. (1).

(32) We remark that, as the reader will easily verify, we would get eq. (3.6) in an unchanged form if we had used the maximal Abelian subgroup $T^\theta_1 \times T^\sigma_1 \times T^a_4$ instead of (2.8).

(summation over s, s_3 understood). The unitarity follows trivially from that of D^{kc}. Furthermore, we emphasize that in consequence of Mackey's theorems ([11]), our construction (3.6) gives all unitary irreducible representations, up to equivalence, because, as pointed out at the beginning of Sect. **3**, all necessary criteria are satisfied.

For clarity's sake we summarize the notation in (3.6). $\mathscr{D}$ and l are invariants of $\tilde{\mathfrak{G}}_5$. The numbers k, c are the labels of the unitary irreducible representations of the $SL_{2,C}$ part of H_0. The labels s, s_3 characterize the state within each representation of $\tilde{\mathfrak{G}}_5$, together with the labels r, p. The transformed r' and p' are given by (3.5a). The element $\hat{h}_0$ is given by h_0 with zero phase, where

$$h_0 = h^{-1}_{r,p}\, h(\theta\,;\, b, \Lambda)\, h_{r',p'}\,, \tag{3.8}$$

as follows from (2.31). Here, in turn, $h_{r,p}$ is defined by (2.28), $[\hat{r}, \hat{p}]$ being an arbitrary point on the orbit selected by l and $\mathscr{D}$.

Finally, we note that the constant β in (3.6) is completely arbitrary. Since our representations are ray representations, the β may be taken to be one, without any loss of generality.

It appears from (3.8) as though the dependence of the operator $D^{kc}(\hat{h}_0)$ on the parameters of the group was rather complicated. However, we may take advantage of the arbitrariness of $[\hat{r}, \hat{p}]$ and simplify this dependence considerably. Let us choose, in particular, $\hat{r} = l/2\mathscr{D}$, $\hat{p} = 0$. (This is certainly a point of the orbit defined by l and $\mathscr{D}$.) Then eq. (2.38) reads

$$h_{r,p}[l/2\mathscr{D}, 0] = [r, p]$$

and one easily verifies that the simplest ([33]) solution is

$$h_{r,p} \;= (1\,;\, 0, 0, lp, 1)\,. \tag{3.9a}$$

Similarly,

$$h_{r',p'} = (1\,;\, 0, 0, lp', 1)\,, \tag{3.9b}$$

with p' being given, of course, by (3.5a). Using (3.9a), (3.9b) in (3.8), one easily finds

$$\hat{h}_0 = (1\,;\, 0, 0, 0, \Lambda)\,. \tag{3.9c}$$

Thus, $\hat{h}_0$ can be taken to be *a pure $SL_{2,C}$ transformation of $\tilde{\mathfrak{G}}_5$*. We can re-

---

([33]) Cf. footnote ([18]).

write (3.6) in the *final form*

$$(3.10)\qquad \mathscr{U}_g \psi^{l\mathscr{D}kc}_{ss_3}(r, p) = \exp[i(\beta\theta + r\sigma + pa)][D^{kc}(\Lambda)]_{s's'_3;\, ss_3} \psi^{l\mathscr{D}kc}_{s's'_3}(r', p')\,,$$

where r', p' are given by (3.5*a*).

3·3. *An equivalence theorem.* – As is well known (7), the concept of equivalence for projective representations is somewhat different from that which applies for true representations. Because of the appearance of ω in the composition law (2.5), we must consider $\mathscr{U}'_g$ unitarily equivalent to $\mathscr{U}_g$ if

$$(3.11)\qquad \mathscr{U}'_g = \alpha(g)\, V\mathscr{U}_g V^{-1}\,,$$

where V is a unitary operator and α is a complex function of modulus one.

Let us define now an operator V by setting

$$(3.12)\qquad V\psi^{l\mathscr{D}kc}_{ss_3}\psi(r, p) = \psi^{l\mathscr{D}kc}_{ss_3}\left(r + \frac{l}{2}\mathscr{D}, p\right) \equiv \hat{\psi}^{l\mathscr{D}kc}_{ss_3}(r, p)\,.$$

We have, using (3.7) and (3.12), and setting $\tilde{r} = r + l/2\mathscr{D}$,

$$\langle\psi|\phi\rangle = \int \mathrm{d}r\, \mathrm{d}^4p\, \delta\left(p^2 + \frac{2}{l}\, r - \mathscr{D}\right) \psi^\dagger(r, p)\phi(r, p) \equiv$$

$$\equiv \int \mathrm{d}\tilde{r}\, \mathrm{d}^4p\, \delta\left(p^2 + \frac{2}{l}\,\tilde{r} - \mathscr{D}\right) \psi^\dagger(\tilde{r}, p)\phi(\tilde{r}, p) = \langle\hat{\psi}|\hat{\phi}\rangle = \langle V\psi|V\phi\rangle\,.$$

Hence, V is a unitary operator, and provides an isomorphic mapping from the Hilbert space $\mathscr{H}(\Gamma)$ to the Hilbert space $\mathscr{H}_0(\Gamma)$. The latter is the same set of functions but is equipped with the measure

$$(3.13)\qquad \mathrm{d}\Omega = \mathrm{d}r\, \mathrm{d}^4p\; \delta(p^2 + 2l^{-1}r)$$

instead of (2.33). This implies that *the unitary representation* $(l|\mathscr{D}, k, c)$ *is unitarily equivalent to the representation* $(l|0, k, c)$. Actually, it is easily seen from (3.12) and (3.10) that if $\mathscr{U}_g$ is a representation in $\mathscr{H}(\Gamma)$, then

$$(3.14)\qquad \mathscr{U}'_g = \exp\left[i\,\frac{l}{2}\,\sigma\mathscr{D}\right] V\mathscr{U}_g V^{-1}$$

is a representation in $\mathscr{H}_0(\Gamma)$, which, in view of (3.11), bears out our statement in detail.

Therefore, without loss of generality, we can restrict ourselves to representations with $\mathscr{D}=0$. The label $\mathscr{D}$ may be omitted ([34]).

3.4. *Conjugate representations.* – The complex conjugate of eq. (3.10) is

$$\overline{\mathscr{U}}_g\,\bar{\psi}^{l\mathscr{D}kc}_{ss_3}(r,p)=\exp\left[i(-\beta\theta-r\sigma-pa)\right]\left[D^{k-c}(\Lambda)\right]_{s's'_3;\,ss_3}\bar{\psi}^{l\mathscr{D}kc}_{s's'_3}(r',p')\,, \tag{3.15}$$

where the bar means complex conjugation and where we took cognizance of $\bar{D}^{kc}=D^{k-c}$. On the other hand, let us consider the representation $(-l|\mathscr{D},k,-c)$ of $\tilde{\mathfrak{G}}_5$. Denoting the unitary operator which corresponds to a group element g in *this* representation by $\check{\mathscr{U}}_g$, we have

$$\check{\mathscr{U}}_g\,\psi^{-l\mathscr{D}k-c}_{ss_3}(r,p)=\exp\left[i(\beta\theta+r\sigma+pa)\right]\left[D^{k-c}(\Lambda)\right]_{s's'_3;\,ss_3}\psi^{-l\mathscr{D}kc}_{s's'_3}(\check{r},\check{p})\,, \tag{3.16}$$

where now

$$\check{r}=r+pb+\frac{1}{2l}b^2\,,\qquad \check{p}=\Lambda^{-1}\left(p+\frac{1}{l}b\right). \tag{3.16a}$$

If we introduce the operator A defined by

$$A\psi^{l\mathscr{D}kc}_{ss_3}(r,p)=\bar{\psi}^{l\mathscr{D}kc}_{ss_3}(-r,-p)\equiv\check{\psi}^{l\mathscr{D}kc}_{ss_3}(r,p)\,, \tag{3.17}$$

then, using (3.15) and (3.5a), we easily find that

$$(\overline{\mathscr{U}}_g\check{\psi}(r,p)=\overline{\mathscr{U}}_g\,\bar{\psi}(-r,-p)=\exp\left[i(-\beta\theta+r\sigma+bp)\right]\left[D^{k-c}(\Lambda)\right]\check{\psi}(r',p') \tag{3.18}$$

with

$$r'=r+pb+\frac{1}{2l}b^2\,,\qquad p'=\Lambda^{-1}\left(p+\frac{1}{l}b\right). \tag{3.18a}$$

Thus, comparing with (3.16) and (3.16a), we can write

$$\exp\left[2i\beta\theta\right]\overline{\mathscr{U}}_g\,\check{\psi}(r,p)=\check{\mathscr{U}}_g\,\check{\psi}(r,p)\,. \tag{3.19}$$

Using (3.17), we find that

$$\overline{\mathscr{U}}_g=\exp\left[-2i\beta\theta\right]A\check{\mathscr{U}}_gA^{-1}\,. \tag{3.20}$$

([34]) See, however, our subsequent discussion of the reduction of products of representations, Subsect. 5.2. Furthermore, the equivalence theorem obviously holds true only as long as l is finite.

([35]) For simplicity, we suppress in this calculation all labels.

Thus, according to (3.11), the representations $\overline{\mathscr{U}}_g$ and $\check{\mathscr{U}}_g$ are equivalent [36] in the sense of ray representations. We write symbolically

$$\overline{(l|\mathscr{D}, k, c)} \approx (-l|\mathscr{D}, k, -c)\,. \tag{3.21}$$

It also follows from the above discussion that the basis functions $\psi_{ss_3}^{-l\mathscr{D}k-c}$ of the $(-l|\mathscr{D}, k, -c)$ representation can be expressed in terms of the basis functions $\psi_{ss_3}^{l\mathscr{D}kc}$ of the $(l|\mathscr{D}, k, c)$ representation. We have

$$\psi_{ss_3}^{-l\mathscr{D}kc}(r, p) = A\psi_{ss_3}^{l\mathscr{D}kc}(r, p) = \bar{\psi}_{ss_3}^{l\mathscr{D}kc}(-r, -p)\,. \tag{3.22}$$

Finally, by checking the inner product, we easily verify that $\langle\psi|\phi\rangle = \langle A\phi|A\psi\rangle$, so that A is antilinear unitary.

3·5. *Some properties of the basis functions.* – For subsequent physical applications, it will be useful to summarize the effect of some operators of $\tilde{\mathfrak{G}}_5$ on the basis functions. First, it is obvious that the ψ are eigenfunctions of P_μ and of S, and we have [37]

$$P_\mu\psi_{ss_3}^{l\mathscr{D}kc}(r, p) = p_\mu\psi_{ss_3}^{l\mathscr{D}kc}(r, p)\,, \tag{3.23}$$

$$S\psi_{ss_3}^{l\mathscr{D}kc}(r, p) = r\psi_{ss_3}^{l\mathscr{D}kc}(r, p)\,. \tag{3.24}$$

Next we recall that the Casimir operators of $\tilde{\mathfrak{G}}_5$ are the operators $\mathscr{D}$, $\mathscr{J}$, $\mathscr{K}$ as given by (1.5*a*)-(1.5*c*). Hence, we have [38], from (1.5*a*),

$$(P_\mu P^\mu + 2l^{-1}S)\,\psi_{ss_3}^{l\mathscr{D}kc}(r, p) = \mathscr{D}\psi_{ss_3}^{l\mathscr{D}kc}(r, p)\,. \tag{3.25}$$

From (1.5*b*) and (1.5*c*) we obtain

$$\tfrac{1}{2}T_{\mu\nu}T^{\mu\nu}\psi_{ss_3}^{l\mathscr{D}kc}(r, p) = (k^2 + c^2 - 1)\,\psi_{ss_3}^{l\mathscr{D}kc}(r, p)\,, \tag{3.26}$$

$$\tfrac{1}{4}\varepsilon_{\mu\nu\varphi\sigma}T^{\mu\nu}T^{\varphi\sigma}\psi_{ss_3}^{l\mathscr{D}kc}(r, p) = 2ikc\,\psi_{ss_3}^{l\mathscr{D}kc}(r, p)\,, \tag{3.27}$$

respectively. Here we used the facts that $\mathscr{J}$ and $\mathscr{K}$ are Casimir operators of [39] $SL_{2,C}$ and that our basis carries a representation of $SL_{2,C}$. (The relation

---

(36) On the other hand, it must be emphasized that the conjugate representation $\overline{(l|\mathscr{D}, k, c)}$ is *not* equivalent to the original $(l|\mathscr{D}, k, c)$ representation unless $c = 0$.

(37) This follows from the fact that the representation space Γ is isomorphic to N. It is also consistent, naturally, with the realizations $P_\mu = i\partial_\mu$, $S = i\partial_u$ in configuration space, cf. ref. (1). This can be seen by taking the Fourier transforms of (3.23) and (3.24).

(38) Equation (3.25) follows also from (3.23), (3.24) and (2.19).

(39) In ref. (1) we showed that the operators $T_{\mu\nu}$ generate an $SL_{2,C}$ algebra.

of the eigenvalues of $SL_{2,C}$ Casimir operators to the labels k, c of the representation can be found, for example, in ref. ([21]).) Incidentally, eqs. (3.26) and (3.27) tell us that *the $SL_{2,C}$ part of our little group H_0 is generated by the operators $T_{\mu\nu}$* (defined by (1.6)).

Furthermore, in view of the meaning of the state labels s and s_3 (cf. footnote ([24])) and the algebraic properties of $T_{\mu\nu}$, we also have the relations:

$$T^2 \psi^{l\mathscr{D}kc}_{ss_3}(r, p) = s(s+1)\psi^{l\mathscr{D}kc}_{ss_3}(r, p)\,, \tag{3.28}$$

$$T_{12}\psi^{l\mathscr{D}kc}_{ss_3}(r, p) = s_3\psi^{l\mathscr{D}kc}_{ss_3}(r, p)\,. \tag{3.29}$$

In eq. (3.28) we used the notation ([40])

$$\boldsymbol{T} = (T_{23}, T_{31}, T_{12})\,. \tag{3.30}$$

Equations (3.23) through (3.29) exhibit clearly the mathematical meaning of all labels relative to the basis functions of our representation. However, we must make one additional comment. Equations (3.23) and (3.24) *do not hold for the conjugate representations*. Indeed, from (3.22) we find

$$P_\mu \psi^{-l\mathscr{D}k-c}_{ss_3}(r, p) = P_\mu \bar{\psi}^{l\mathscr{D}kc}_{ss_3}(-r, -p) = -p\bar{\psi}^{l\mathscr{D}kc}_{ss_3}(-r, -p) = -p\psi^{-l\mathscr{D}k-c}_{ss_3}(r, p)\,.$$

Thus, (3.23) and (3.24) must be replaced by

$$P_\mu \psi^{-l\mathscr{D}k-c}_{ss_3}(r, p) = -p\psi^{-l\mathscr{D}k-c}_{ss_3}(r, p)\,, \tag{3.31}$$

$$S\psi^{-l\mathscr{D}k-c}_{ss_3}(r, p) = -r\psi^{-l\mathscr{D}k-c}_{ss_3}(r, p)\,. \tag{3.32}$$

This is not too surprising since it is consistent with the definition of the orbit, which is now given by $(-p)^2 + 2(-l)^{-1}(-r) = \mathscr{D} = p^2 + 2l^{-1}r$, that is *the $\mathscr{D}$ has the same value as for the original $(l|\mathscr{D}kc)$ representation* ([41]). (This is how the representation $(-l|\mathscr{D}, k, -c)$ was defined.)

The essence of this Subsection can be summarized as follows. Our basis for the representations of $\tilde{\mathfrak{G}}_5$, to be called henceforth the *canonical basis*, is distinguished by having taken the set of *mutually commuting state labeling operators* to be the set

$$\mathscr{C} = \{P_\mu, S, \boldsymbol{T}^2, T_3\}\,. \tag{3.33}$$

([40]) Incidentally, if we introduce alongside of $\boldsymbol{T}$ the quantity $\boldsymbol{V} = (T_{01}, T_{02}, T_{03})$, then we may write $\mathscr{I} = \boldsymbol{T}^2 - \boldsymbol{V}^2$ and $\mathscr{K} = \frac{1}{2}\boldsymbol{T}\boldsymbol{V}$.

([41]) Equation (3.25) does not change: replacing l by $-l$ both in the operator acting on ψ and in the label of ψ leaves the equation unchanged.

Corresponding to this choice, we have the chain of reduction of $\tilde{\mathfrak{G}}_5$

$$\tilde{\mathfrak{G}}_5 \Rightarrow T_1^\sigma \otimes T_4^a \otimes SL_{2,C;T} \supset T_1^\sigma \otimes T_4^a \otimes SU_{2;T} \supset T_1^\sigma \otimes T_4^a \otimes SO_{2;T} . \tag{3.34}$$

(Here the subscript T reminds us that the corresponding groups are generated by the operators $T_{\mu\nu}$, $\boldsymbol{T}$, $T_3 \equiv T_{12}$.) The choice of $\mathscr{C}$ and the corresponding choice of the reduction chain *define* a basis ([42]), via the following equations ([43]):

$$(P^2 + 2l^{-1} S)\, \psi_{ss_3}^{l\mathscr{D}kc}(r, p) = \mathscr{D}\psi , \tag{3.35a}$$

$$\tfrac{1}{2} T_{\mu\nu} T^{\mu\nu} \psi_{ss_3}^{l\mathscr{D}kc}(r, p) = (k^2 + c^2 - 1)\, \psi_{ss_3}^{l\mathscr{D}kc}(r, p) , \tag{3.35b}$$

$$\tfrac{1}{4} \varepsilon_{\mu\nu\varphi\sigma} T^{\mu\nu} T^{\varphi\sigma} \psi_{ss_3}^{l\mathscr{D}kc}(r, p) = 2ikc\psi_{ss_3}^{l\mathscr{D}kc}(r, p) , \tag{3.35c}$$

$$\boldsymbol{T}^2 \psi_{ss_3}^{l\mathscr{D}kc}(r, p) = s(s+1)\, \psi_{ss_3}^{l\mathscr{D}kc}(r, p) , \tag{3.35d}$$

$$T_3 \psi_{ss_3}^{l\mathscr{D}kc}(r, p) = s_3 \psi_{ss_3}^{l\mathscr{D}kc}(r, p) , \tag{3.35e}$$

$$P_\mu \psi_{ss_3}^{l\mathscr{D}kc}(r, p) = p_\mu \psi_{ss_3}^{l\mathscr{D}kc}(r, p) , \tag{3.35f}$$

$$S\psi_{ss_3}^{l\mathscr{D}kc}(r, p) = r\psi_{ss_3}^{l\mathscr{D}kc}(r, p) . \tag{3.35g}$$

For the reader's convenience, we quote here the well-known corresponding case for the Poincaré group. The set of commuting operators is usually taken to be $\{P_\mu, W_3\}$, where

$$W_\mu = -\tfrac{1}{2} \varepsilon_{\mu\nu\varphi\sigma} J^{\nu\varphi} P^\sigma . \tag{3.36}$$

The corresponding chain of reduction is (for $P^2 \equiv m^2 > 0$)

$$\mathscr{P} \Rightarrow T_4^a \otimes SU_{2;J} \supset T_4^a \otimes SO_{2;J} . \tag{3.37}$$

The basis is defined by

$$P^2 \psi_{s_3}^{m^2 s}(p) = m^2 \psi_{s_3}^{m^2 s}(p) , \tag{3.38a}$$

$$W^2 \psi_{s_3}^{m^2 s}(p) = -m^2 s(s+1)\, \psi_{s_3}^{m^2 s}(p) , \tag{3.38b}$$

$$P_\mu \psi_{s_3}^{m^2 s}(p) = p_\mu \psi_{s_3}^{m^2 s}(p) , \tag{3.38c}$$

$$W_3 \psi_{s_3}^{m^2 s}(p) = m s_3 \psi_{s_3}^{m^2 s}(p) . \tag{3.38d}$$

([42]) Obviously, other possibilities for defining a basis exist. Our choice is directed by convenience.

([43]) For the conjugate representation, (3.35*f*), (3.35*g*) must be changed, cf. (3.31), (3.32).

Finally, for comparison, we also quote here the case of the nonrelativistic Galilei group. The set of commuting operators is $\{\boldsymbol{P}, H, \boldsymbol{F}^2, F_3\}$, where [44]

$$\boldsymbol{F} = \boldsymbol{J} - \underline{\boldsymbol{X}} \times \boldsymbol{P} . \tag{3.39}$$

The corresponding chain of reduction is

$$\tilde{\mathfrak{G}}_4 \Rightarrow T_1^\tau \otimes T_3^a \otimes SU_{2;F} \supset T_1^\tau \otimes T_3^a \otimes SO_{2;F} . \tag{3.40}$$

The basis is defined by (for $m > 0$)

$$\left(\frac{\boldsymbol{P}^2}{2m} - H\right) \psi_{s_3}^{m\mathscr{B}s}(E, \boldsymbol{p}) = \mathscr{B}\psi_{s_3}^{m\mathscr{B}s}(E, \boldsymbol{p}) , \tag{3.41a}$$

$$\boldsymbol{F}^2 \psi_{s_3}^{m\mathscr{B}s}(E, \boldsymbol{p}) = s(s+1)\,\psi_{s_3}^{m\mathscr{B}s}(E, \boldsymbol{p}) , \tag{3.41b}$$

$$F_3 \psi_{s_3}^{m\mathscr{B}s}(E, \boldsymbol{p}) = s_3 \psi_{s_3}^{m\mathscr{B}s}(E, \boldsymbol{p}) , \tag{3.41c}$$

$$P_k \psi_{s_3}^{m\mathscr{B}s}(E, \boldsymbol{p}) = p_k \psi_{s_3}^{m\mathscr{B}s}(E, \boldsymbol{p}) , \tag{3.41d}$$

$$H\psi_{s_3}^{m\mathscr{B}s}(E, \boldsymbol{p}) = E\psi_{s_3}^{m\mathscr{B}s}(E, \boldsymbol{p}) . \tag{3.41e}$$

4. – The special case of $l = \infty$.

In the preceding analysis we, of course, assumed that l is finite. In the limiting case $l \to \infty$, several changes occur. First, as was already pointed out in ref. [1], the crucial commutator (1.4) changes to [45]

$$[P_\mu, Q_\nu] = 0 . \tag{4.1}$$

Furthermore, the factor system $\omega(g_1, g_2)$ becomes 1 (cf. (2.3) and (2.6)), so that the central extension becomes trivial [1], and the group is

$$\lim_{l\to\infty} \tilde{\mathfrak{G}}_5 = T_1^\theta \times \mathfrak{G}_5 . \tag{4.2}$$

The Casimir operators of this group are [46]

$$C_1 = P_\mu P^\mu , \tag{4.3a}$$

$$C_2 = W_\mu W^\mu , \tag{4.3b}$$

---

(44) For details, see, for example, our summary in Appendix B of ref. (1).

(45) This does not mean that we « lose quantum mechanics ». Since $X_\mu = -lQ_\mu$, we still have $[P_\mu, X_\nu] = -ig_{\mu\nu}$.

(46) It is easy to check that C_1 and C_2 commute now with all generators. That there are no other Casimir invariants can be checked by considering the contraction (see ref. (2)) of the $ISO_{0(3,2)}$ Casimir operators in the limit $l \to \infty$.

where W_μ is the Pauli-Lubanski vector (cf. (3.36)). Actually, C_1 arises directly from eq. (1.5a) when $l \to \infty$, so that we have

$$P_\mu P^\mu = \mathscr{D} . \tag{4.4}$$

These simple observations suggest that, when $l = \infty$, the unitary irreducible ray representation $(l|\mathscr{D}, k, c)$ will become reducible and decomposes into irreducible unitary representations of the Poincaré group (47). We shall study this below in detail.

First, we realize that the construction (3.10) of Sect. **3** still holds and we have

$$\mathscr{U}_g \psi^{\infty\mathscr{D}kc}_{ss_3}(r, p) = \exp[i(\beta\theta + r\sigma + pa)][D^{kc}(\Lambda)]_{s's'_3;\, ss_3} \psi^{\infty\mathscr{D}kc}_{s's'_3}(r', p') , \tag{4.5}$$

where now, instead of (3.5a), we have

$$r' = r + pb , \qquad p' = \Lambda^{-1} p . \tag{4.5a}$$

Furthermore, the measure in (3.23) changes too, and we have

$$\mathrm{d}\Omega^\infty = \mathrm{d}r\, \mathrm{d}^4 p\, \delta(p^2 - \mathscr{D}) , \tag{4.6}$$

i.e. r and p become unrelated (48); the value of r is arbitrary.

Next, we note that the equation of the orbit is, on account of (4.4), $p^2 = \mathscr{D}$, as in the Poincaré group. Hence, there will be four different types of little groups for our $\mathfrak{G}_5$. We have

Case I)	$\mathscr{D} = m^2 > 0$,	$\hat{p} = (\sqrt{m^2}, 0, 0, 0)$,	$\mathscr{L}_{\text{I}} = SU_2$,
Case II)	$\mathscr{D} = \mu^2 < 0$,	$\hat{p} = (0, 0, \lvert\sqrt{\mu^2}\rvert, 0)$,	$\mathscr{L}_{\text{II}} = SL_{2,R}$,
Case III)	$\mathscr{D} = 0$,	$\hat{p} = (1, 0, 0, 1)$,	$\mathscr{L}_{\text{III}} = \tilde{E}_2$,
Case IV)	$\mathscr{D} = 0$,	$\hat{p} = (0, 0, 0, 0)$,	$\mathscr{L}_{\text{IV}} = SL_{2,C}$.

In each case we listed, besides the invariant $\mathscr{D}$, a typical character vector of the orbit, and showed the corresponding little group (49) $\mathscr{L}$.

---

(47) This is so because C_1 and C_2 are invariants of the Poincaré group, generated by P_μ and $J_{\mu\nu}$.

(48) In passing we note here that, in consequence of (4.6), we cannot apply the reasoning of Subsect. **3**·3, so that $\mathscr{D}$ becomes an essential label of the representation and cannot be shifted to zero.

(49) We note that $SL_{2,R}$ is the covering of $SO_{2,1}$ and $\tilde{E}_2$ denotes the covering of the two-dimensional Euclidean group.

From the foregoing it follows that the irreducible unitary representations of $\mathfrak{G}_5$ are characterized by the same labels as are those of the Poincaré group. Corresponding to the four cases above, we then have the representations

$$(4.7)\qquad \begin{cases} \text{Case I)} & (m^2;\, s), \quad \text{when } \mathscr{D}>0\,, \\ \text{Case II)} & (\mu^2;\, l,\, s), \quad \text{when } \mathscr{D}<0\,, \\ \text{Case III)} & (0;\, \varphi,\, \lambda), \quad \text{when } \mathscr{D}=0\,, \\ \text{Case IV)} & (0;\, k,\, c) \quad \text{when } \mathscr{D}=0\,,\ p=0\,. \end{cases}$$

In Case I), s is the label for the irreducible unitary representation of the SU_2 little group, and analogous statements hold for the other cases [50].

We now see that (as was already mentioned in footnote [48]), when $l=\infty$, the value of $\mathscr{D}$ is essential if one wishes to recover from the decomposition of the representation (4.5) all four classes of the possible representations that may occur.

Before going into details of the reduction process, we wish to dispose briefly of the physically uninteresting Case IV). Since here not only $\mathscr{D}=0$ but also each component of $\hat{p}$ is null, these representations are not connected with particles or tachyons and will not be studied in the following. However, it is interesting to note that in Case IV) the representation (4.5) happens to be irreducible and we have, symbolically,

$$(4.8)\qquad (\infty|0,\, k,\, c)_{\hat{p}=0} = \exp\,[ir\sigma]\times(0;\, k,\, c)\,.$$

The basis functions are $\psi^{kc}_{ss_3}(r,\, p)$, and the Hilbert space of the representation is defined with respect to the degenerate measure given by

$$(4.8a)\qquad \mathrm{d}\Omega^{\infty}_{\mathrm{IV}} = \mathrm{d}r\,\mathrm{d}^4p\,\delta(p_0)\,\delta(p_1)\,\delta(p_2)\,\delta(p_3)\,.$$

In the following we shall study, by a somewhat intuitive but quite direct method, the decomposition of the representation (4.5) into irreducible components, for the Cases I), II), III).

4·1. *Algebraic preliminaries.* – Let $\mathscr{L}$ be any one of the little groups of the classes I), II), or III), and let $\hat{p}$ be a corresponding character. Let us define

---

[50] An easily readable survey of the representations in question is given, for example, by H. Joos: *Forts. Phys.*, **10**, 65 (1962) or in the article by T. D. Newton: in *The Theory of Groups in Classical and Quantum Physics*, Vol. **1**, edited by T. Kahan (Edinburgh, 1965).

the three transformations Λ_p, $\Lambda_{p'}$, $\Lambda \in SL_{2,C}$ by setting

$$(4.9) \qquad \Lambda_p^{-1}\hat{p} = p\,, \qquad \Lambda_{p'}^{-1}\hat{p} = p'\,, \qquad \Lambda^{-1}p = p'\,.$$

Combing these definitions, we find that

$$\hat{p} = \Lambda_{p'}\Lambda^{-1}\Lambda_p^{-1}\hat{p}\,.$$

This implies that any element L of $\mathscr{L}$ can be written as

$$(4.10a) \qquad L = \Lambda_p \Lambda \Lambda_{p'}^{-1}\,.$$

Conversely, any given $\Lambda \in SL_{2,C}$ can be expressed in terms of some $L \in \mathscr{L}$ as

$$(4.10b) \qquad \Lambda = \Lambda_p^{-1} L \Lambda_{p'}\,.$$

Consider now two elements Λ_p and Λ of $SL_{2,C}$. The corresponding representatives in some unitary irreducible representation of $SL_{2,C}$ are $D(\Lambda_p)$ and $D(\Lambda)$. Using (4.10b) we then find that

$$(4.11) \qquad D(\Lambda_p)D(\Lambda) = D(L)D(\Lambda_{p'})\,.$$

At this point we note that in our previous work in Sect. **3**, we used the representations D of $SL_{2,C}$ in an SU_2 basis, *i.e.* the basis (3.35), where the states are labeled by s and s_3, corresponding to the reduction chain containing SU_2 and SO_2. For our present purposes, however, we must introduce a basis which fits the little group ([51]). This, in turn, is determined by the value of $\mathscr{D}$, as was discussed above. Thus, in Case I) ($\mathscr{D} > 0$), we again use the basis corresponding to the chain $SL_{2,C} \supset SU_2 \supset SO_2$. In Case II) ($\mathscr{D} < 0$) we use the basis corresponding to $SL_{2,C} \supset SL_{2,R} \supset SO_2$. In Case III) ($\mathscr{D} = 0$), we use the basis corresponding to $SL_{2,C} \supset \tilde{E}_2 \supset SO_2$. For the sake of uniform notation, we shall indicate the state labels by the generic symbols β, γ. (In Case I), $\beta = s =$ fixed, $\gamma = s_3$. For the other two cases, see, for example, ref. ([50]).)

Now, starting with the functions $\psi_{\beta\gamma}^{\infty\mathscr{D}kc}$ of the appropriate basis, let us define the new set of functions ([52])

$$(4.12) \qquad \varphi_{\beta\gamma}^{\alpha}(r, p) = [D^{kc}(\Lambda_p)]_{\beta'\gamma';\beta\gamma}\,\psi_{\beta'\gamma'}^{\infty\mathscr{D}kc}(r, p)\,.$$

Here α refers to the labels necessary to characterize the representations of the Poincaré group, corresponding to Cases I), II), III) (determined by $\mathscr{D}$), see (4.7).

([51]) It must be pointed out, however, that the representations using different bases are unitarily equivalent.

([52]) As always, summation over β' and γ' is understood.

In the following we show that *the subspaces of functions* $\varphi^{\alpha}_{\beta\gamma}$ *with given* α *are invariant subspaces.* This then tells us that, as claimed, for $l=\infty$ the $(\infty|\mathscr{D}, k, c)$ representations are reducible and reduce into components which are, essentially, irreducible representations of the group $T^{\theta}_{1}\times T^{\sigma}_{1}\times\mathscr{P}$, where $\mathscr{P}$ is the Poincaré group.

4.2. *Irreducible components of* $(\infty|\mathscr{D}, k, c)$. – From (4.5) and (4.12) we get

$$\begin{aligned}\mathscr{U}_g\varphi^{\alpha}_{\beta\gamma}(r, p) &= [D^{kc}(\Lambda_p)]_{\beta'\gamma';\beta\gamma}\mathscr{U}_g\psi^{\infty\mathscr{D}kc}_{\beta'\gamma'}(r, p) =\\ &= \exp[i(\beta\theta+r\sigma+pa)][D^{kc}(\Lambda_p)]_{\beta'\gamma';\beta\gamma}[D^{kc}(\Lambda)]_{\beta''\gamma'';\beta'\gamma'}\psi^{\infty\mathscr{D}kc}_{\beta''\gamma''}(r', p') =\\ &= \exp[i(\beta\theta+r\sigma+pa)][D^{kc}(L)]_{\beta'\gamma';\beta\gamma}[D^{kc}(\Lambda_{p'})]_{\beta''\gamma'';\beta'\gamma'}\psi^{\infty\mathscr{D}kc}_{\beta''\gamma''}(r', p')\,.\end{aligned}$$

In the last step, we used (4.11). Using on the r.h.s. the definition (4.12), we finally have

$$\mathscr{U}_g\varphi^{\alpha}_{\beta\gamma}(r, p) = \exp[i(\beta\theta+r\sigma+pa)[D^{kc}(L)]_{\beta'\gamma';\beta\gamma}\varphi^{\alpha}_{\beta'\gamma'}(r', p')\,. \tag{4.13}$$

Here, clearly, $D^{kc}(L)$ is an irreducible unitary representation of the little group $\mathscr{L}$.

Thus, the $\varphi^{\alpha}_{\beta\gamma}$ with fixed α span an irreducible unitary representation of $\mathfrak{G}_5$. The transformation law in this representation is the same as for the group [53] $T^{\sigma}_{1}\times\mathscr{P}$, except for the fact that φ depends not only on p, but also on r, and the latter must be also transformed, according to (4.5a).

For illustration, let us consider Case I) ($\mathscr{D}=m^2>0$) in some detail. The elements of the little group SU_2 are the Lorentz transformations Λ restricted to pure spatial rotations R. The matrix elements of the $SL_{2,c}$ representation become diagonal with respect to the SU_2 subgroup when Λ is restricted to R, and we have [54,55]

$$[D^{kc}(R)]_{s's'_3;\,ss_3} = \delta_{s's}[D^{s}(R)]_{s'_3;s_3}\,, \tag{4.14}$$

where D^s are the familiar «rotation» matrices. Equation (4.13) then reads [56]

$$\mathscr{U}_g\varphi^{m^2s}_{s_3}(r, p) = \exp[i(\beta\theta+r\sigma+pa)][D^{s}(R)]_{s'_3;\,s_3}\varphi^{m^2s}_{s'_3}(r', p')\,. \tag{4.15}$$

([53]) The constant β is arbitrary and the factor $\exp[i\beta\theta]$ can now be omitted.

([54]) This well-known result can be proved easily by using, for instance, the Ström basis, cf. S. Ström: *Ark. Fys.*, **34**, 215 (1967) and earlier papers quoted therein.

([55]) Note that the matrices of the restriction of the representation become independent of k, c.

([56]) The subscript s on φ is irrelevant and can be suppressed. The properties of the basis functions are those given by eq. (3.38).

Here $R = \Lambda_p \Lambda \Lambda_{p'}^{-1}$, where, in turn, Λ_p is determined by $\Lambda_p p = \hat{p}$. In detail, this means $(\Lambda_p)^{0\mu} p_\mu = \sqrt{m^2}$, $(\Lambda_p)^{i\mu} p_\mu = 0$.

The result of these considerations can be summarized symbolically by writing

$$(4.16) \qquad (\infty|\mathscr{D}, k, c)_{\mathrm{I}} = \exp[ir\sigma] \times \bigoplus_{s=k}^{\infty} (m^2; s),$$

where $m^2 = \mathscr{D}$. Incidentally, this result may be also derived by decomposing the basis $\psi_{ss_3}^{\infty \mathscr{D} kc}$ into irreducible components with respect to the basis of the little group:

$$(4.17) \qquad \psi_{ss_3}^{\infty \mathscr{D} kc}(r, p) = \sum_{s'=k}^{\infty} \sum_{s_3'=-s}^{s} G_{s_3 s_3'}^{ss'}(k, c)\, \psi_{s_3'}^{m^2 s'}.$$

Formulae analogous to (but more complicated than) eqs. (4.16), (4.17) hold for Cases II) and III).

5. – Product of two representations.

In this Section we consider the direct product of two irreducible unitary projective representations $(l_2|\mathscr{D}_2, k_2, c_2) \otimes (l_1|\mathscr{D}_1, k_1, c_1)$, and we shall decompose it into irreducible components.

5·1. *Definition of the direct product.* – Let $\tilde{\mathfrak{G}}_5^{l_1}$ and $\tilde{\mathfrak{G}}_5^{l_2}$ be two $\tilde{\mathfrak{G}}_5$ groups with constants $l = l_1$ and $l = l_2$, respectively ([57]). According to (2.10) we have

$$(5.1) \qquad \tilde{\mathfrak{G}}_5^{l_i} = N^{l_i} \otimes H^{l_i}, \qquad i = 1, 2,$$

with N and H defined as in (2.8), (2.9). Consider the coset space

$$(5.2) \qquad \Gamma_{12} = (\tilde{\mathfrak{G}}_5^{l_1} \otimes \tilde{\mathfrak{G}}_5^{l_2})/(H^{l_1} \otimes H^{l_2})$$

and the corresponding invariant measure defined by

$$(5.3) \qquad \mathrm{d}\Omega_{12} = \mathrm{d}\Omega_1(r_1, p_1)\, \mathrm{d}\Omega(r_2, p_2) = \\ = \mathrm{d}r_1\, \mathrm{d}^4 p_1\, \mathrm{d}r_2\, \mathrm{d}^4 p_2\, \delta\left(p_1^2 + \frac{2}{l_1} r_1 - \mathscr{D}_1\right) \delta\left(p_2^2 + \frac{2}{l_2} r_2 - \mathscr{D}_2\right).$$

Here $[r_i, p_i]$ are the characters in $\hat{N}^{l_i}$, $i = 1, 2$, respectively.

We now define the Hilbert space $\mathscr{L}^2(\Gamma_{12}, \mathrm{d}\Omega_{12})$ which is spanned by the

([57]) We assume $l_1 \neq 0$, $l_2 \neq 0$.

functions [58]

$$\psi^{l_1\mathscr{D}_1k_1c_1}_{s_1m_1}(r_1, p_1)\otimes\psi^{l_2\mathscr{D}_2k_2c_2}_{s_2m_2}(r_2, p_2)\,.$$

Next, let us restrict $\tilde{\mathfrak{G}}_5^{l_1}\otimes\tilde{\mathfrak{G}}_5^{l_2}$ to its « diagonal » subgroup, spanned by the elements

$$(g_1, g_2)_L = \{(\exp[i\theta_1];\sigma, a, b, \Lambda)_{l_1}, (\exp[i\theta_2];\sigma, a, b, \Lambda)_{l_2}\}\,. \tag{5.4}$$

It is not difficult to show that this group is isomorphic to a $\tilde{\mathfrak{G}}_5^L$ group with [59]

$$L=\frac{l_1l_2}{l_1+l_2}\,. \tag{5.5}$$

Let

$$\Gamma_{12}^L=\tilde{\mathfrak{G}}_5^L/H^L \tag{5.6}$$

be the restriction of Γ_{12} to the coset space of $\tilde{\mathfrak{G}}_5^L$. Then the representation $(l_2|\mathscr{D}_2, k_2, c_2)\otimes(l_1|\mathscr{D}_1, k_1, c_1)$ on Γ_{12}^L can be explicitly written as

$$\begin{aligned}\mathscr{U}_{(g_1.g_2)_L}\{\psi^{l_1\mathscr{D}_1k_1c_1}_{s_1m_1}(r_1, p_1)\otimes\psi^{l_2\mathscr{D}_2k_2c_2}_{s_2m_2}(r_2, p_2)\} &=\\ =\exp[i(\theta_1+\theta_2)\beta+(r_1+r_2)\sigma+(p_1+p_2)a]\cdot&\\ \cdot[D^{k_1c_1}(\Lambda)]_{s_1'm_1';\,s_1m_1}[D^{k_2c_2}(\Lambda)]_{s_2'm_2';\,s_2m_2}\psi^{l_1\mathscr{D}_1k_1c_1}_{s_1'm_1'}(r_1', p_1')\otimes\psi^{l_2\mathscr{D}_2k_2c_2}_{s_2'm_2'}(r_2', p_2')&\,,\end{aligned} \tag{5.7}$$

where

$$\begin{cases} r_i' = r_i+p_ib-\dfrac{1}{2l_i}b^2\,,\\[2mm] p_i'=\Lambda^{-1}\left(p_i-\dfrac{1}{l_i}b\right), \end{cases}\qquad i=1,2\,. \tag{5.8}$$

5·2. *Reduction of the measure.* – We wish to reduce the measure (5.3) with respect to the total four-momentum p_1+p_2. To this end, we introduce, instead of $[r_1, p_1]$ and $[r_2, p_2]$, the new characters

$$R=r_1+r_2\,,\qquad P=p_1+p_2 \tag{5.9a}$$

and

$$U=(l_1r_1-l_2r_2)/(l_1+l_2)\,,\qquad T=(l_1p_1-l_2p_2)/(l_1+l_2)\,. \tag{5.9b}$$

---

[58] For ease of notation, we shall henceforth denote the « spin component » s_3 by m, so that we have m_1 and m_2, respectively, as labels on the two ψ's.
[59] We recall that l^{-1} enters as a factor in the definition of the phase, so that it is l^{-1} rather than l which behaves additively, *i.e.* $L^{-1}=l_1^{-1}+l_2^{-1}$.

From (5.8) it follows that

$$(5.10)\qquad P^2 + 2L^{-1}R \equiv \mathscr{D}$$

is invariant under the transformation

$$(5.11)\qquad \begin{cases} R' = r_1' + r_2' = R + Pb - (2L)^{-1}b^2 \,, \\ P' \equiv p_1' + p_2' = \Lambda^{-1}(P - L^{-1}b) \,. \end{cases}$$

Thus, the new set $[R, P]$ of characters is suitable for the decomposition of the product representation.

The orbits

$$(5.12)\qquad p_i^2 + 2l_i^{-1}r_i = \mathscr{D}_i \,, \qquad i = 1, 2,$$

can be expressed as

$$(5.13)\qquad \begin{cases} 2l_1^{-1}r_1 + (L^2/l_1^2)P^2 + (2L/l_1)PT + T^2 = \mathscr{D}_1 \,, \\ 2l_2^{-1}r_2 + (L^2/l_2^2)P^2 - (2L/l_2)PT + T^2 = \mathscr{D}_2 \,, \end{cases}$$

so that we obtain for the new set $[T, U]$ of characters the restrictions

$$(5.14a)\qquad T^2 = \alpha\mathscr{D} + \beta_1\mathscr{D}_1 + \beta_2\mathscr{D}_2 \,,$$

$$(5.14b)\qquad U + \gamma_1 PT + \gamma_2 T^2 = \delta_1\mathscr{D}_1 - \delta_2\mathscr{D}_2 \,.$$

Here we have used the notations

$$(5.15)\qquad \begin{cases} \alpha = -l_1 l_2/(l_1 + l_2)^2 \,, \quad \beta_1 = L/l_2 \,, \quad \beta_2 = L/l_1 \,, \\ \gamma_1 = L \,, \quad \gamma_2 = \frac{1}{2}(l_1 - l_2) \,, \quad \delta_1 = l_1^2/2(l_1 + l_2) \,, \quad \delta_2 = l_2^2/2(l_1 + l_2) \,. \end{cases}$$

We can now transform the volume element (5.3) by means of (5.9*a*), (5.9*b*), (5.10) and (5.14*a*), (5.14*b*) into the form

$$(5.16)\qquad \mathrm{d}\Omega_{12}(R, P;\, U, T) = \mathrm{d}\mathscr{D}\,\mathrm{d}R\,\mathrm{d}^4P\,\mathrm{d}U\,\mathrm{d}^4T\,\cdot$$
$$\cdot\,\delta\left(P^2 + \frac{2}{L}R - \mathscr{D}\right)\delta(T^2 - \alpha_1\mathscr{D} - \beta_1\mathscr{D}_1 - \beta_2\mathscr{D}_2)\,\delta(U - \gamma_1 PT + \gamma_2 T^2 - \delta_1\mathscr{D}_1 - \delta_2\mathscr{D}_2)\,.$$

Here, because of (5.10), $\mathscr{D}$ varies from $-\infty$ to $+\infty$.

Equation (5.16) tells us that the direct-product representation can be decomposed into irreducible components $(L|\mathscr{D}, k, c)$ with respect to the characters $[R, P]$. More specifically, the Hilbert space $\mathscr{H}\big[(l_1|\mathscr{D}_1, k_1, c_1) \otimes (l_2|\mathscr{D}_2, k_2, c_2)\big]$ can be represented as a direct integral of Hilbert spaces $\mathscr{H}\big[(l|D, k, c)\big]$ of the irreducible components. From now on, we shall consider only the principal series representations of $SL_{(2,C)T}$. It is then convenient to introduce the notation $\varphi = -ic$ (φ real, $-\infty < \varphi < +\infty$) and then our preceding statement can be expressed by the formula

$$(5.17)\qquad \mathscr{H}[(l_1|\mathscr{D}_1, k_1, \varphi_1)\otimes(l_2|\mathscr{D}_2, k_2, \varphi_2)] = \bigoplus_k \int_{-\infty}^{+\infty} \mathrm{d}\mathscr{D} \int_{-\infty}^{+\infty} \mathrm{d}\varphi\, \mathscr{H}_\varepsilon[(L|\mathscr{D}, k, \varphi)]\,.$$

The decomposition is done as a direct-sum decomposition relative to all admissible values of the discrete label k, and a direct-integral decomposition relative to the continuous labels $\mathscr{D}$ and φ. The subscript ε refers to possible multiplicities.

At this point we recall that in each representation $(l_i|\mathscr{D}_i, k_i, \varphi_i)$ the label $\mathscr{D}_i$ ($i = 1, 2$) is inessential, because (cf. Sect. **3**·3) they are ray equivalent to $(l_i|0, k_i, \varphi_i)$. However, the label $\mathscr{D}$ in the irreducible components of the product representation is essential, since by (5.14a) it is connected to T^2. But, of course, we may still choose $\mathscr{D}_1 = \mathscr{D}_2 = 0$. Then (5.14$a$) becomes

$$(5.18)\qquad \mathscr{D} = \alpha^{-1} T^2 = -(l_1 p_1 - l_2 p_2)^2 / l_1 l_2\,,$$

and (5.14b) simplifies correspondingly. The meaning of $\mathscr{D}$ is evident from (5.10). The total invariant mass square M^2 being $(p_1 + p_2)^2 \equiv P^2$, we must write

$$(5.19)\qquad M^2 = \mathscr{D} - \frac{2}{L} R\,,$$

in contrast to the « free-particle representations », where $M^2 = -2l^{-1}r$. Using, as we decided, $\mathscr{D}_1 = \mathscr{D}_2 = 0$, eq. (5.19) becomes, explicitly,

$$(5.20)\qquad M^2 = -(l_1 p_1 - l_2 p_2)^2 / l_1 l_2 - 2(l_1 + l_2)(r_1 + r_2)/l_1 l_2\,.$$

To obtain further insight into the significance of $\mathscr{D}$, we consider the special case $l_1 = l_2 = l$. Then $L = l/2$, and (5.18) gives

$$(5.21)\qquad \mathscr{D} = -(p_1 - p_2)^2\,.$$

Thus, in this case $\mathscr{D}$ can be interpreted as the negative of the squared momentum transfer [60]. Another important special case is $l_1 = -l_2 = l$. Then $L = \infty$ and

$$(5.22)\qquad \mathscr{D} = (p_1 + p_2)^2\,,$$

---

(60) That is, the Mandelstam variable t.

whence now $\mathscr{D}$ can be interpreted as the squared c.m. energy [61]. Equations (5.19), (5.22) give, in the present case, the invariant

$$\mathscr{D} = P^2 = M^2 . \tag{5.22a}$$

The components $(L|\mathscr{D}, k, \varphi) \equiv (\infty|M^2, k, \varphi)$ in (5.17) are now no longer irreducible, and a further reduction, as discussed in Sect. **4**, must be performed.

5·3. *Clebsch-Gordan coefficients.* – Let

$$\psi_{sm}^{L\mathscr{D}k\varphi;A}(R, P) \in \mathscr{H}_s\big[(L|\mathscr{D}, k, \varphi)\big]$$

be the basis of the irreducible component in the direct product $(l_1|\mathscr{D}_1, k_1, \varphi_1) \otimes \otimes (l_2|\mathscr{D}_2, k_2, \varphi_2)$. The superscript A stands for the set of labels l_i, $\mathscr{D}_i$, k_i, φ_i, $i = 1, 2$. We define the Clebsch-Gordan coefficients of $\tilde{\mathfrak{G}}_5$ by writing

$$\psi_{sm}^{L\mathscr{D}k\varphi;A}(R, P) = \int \mathrm{d}\Omega'_{12} \left\langle \begin{matrix} RP \\ T \end{matrix} \middle| \begin{matrix} L\ \mathscr{D}\ k\ \varphi \\ sm \end{matrix} \middle| \begin{matrix} l_1\ \mathscr{D}_1\ k_1\ \varphi_1 \\ s_1 m_1 \end{matrix} \middle| \begin{matrix} l_2\ \mathscr{D}_2\ k_2\ \varphi_2 \\ s_2 m_2 \end{matrix} \middle| \begin{matrix} r'_1 p'_1 \\ r'_2 p'_2 \end{matrix} \right\rangle \cdot \\ \cdot \{\psi_{s_1m_1}^{l_1\mathscr{D}_1k_1\varphi_1}(r'_1, p'_1) \otimes \psi_{s_2m_2}^{l_2\mathscr{D}_2k_2\varphi_2}(r'_2, p'_2)\} , \tag{5.23}$$

where $\mathrm{d}\Omega_{12}$ is given by (5.16). We determine the CG coefficient by a procedure somewhat similar to that used in the corresponding problem for the Poincaré group [62].

First we recall that

$$\mathscr{U}_{g_L} \psi_{sm}^{L\mathscr{D}k\varphi}(R, P) = \exp\,[i(\theta\beta + R\sigma + Pa)][D^{k\varphi}(\Lambda)]_{s'm';sm}\, \psi_{s'm'}^{L\mathscr{D}k\varphi}(R', P') , \tag{5.24}$$

where $g_L = (g_1, g_2)_L$ (cf. (5.4)), and where R', P' are given by (5.11). Taking, in particular,

$$(g_1, g_2)_L = \{(\exp\,[i\theta_1];\, \sigma, a, 0, 1)_{l_1}, (\exp\,[i\theta_2];\, \sigma, a, 0, 1)_{l_2}\} ,$$

and applying the corresponding operator $\mathscr{U}_{g_L}$ to eq. (5.23), we see that the CG coefficient vanishes unless [63] $\theta = \theta_1 + \theta_2$, $R' = R$, $P' = P$ (and hence $\mathscr{D}' = \mathscr{D}$). Thus, the CG coefficient must contain the factor

$$\delta(\mathscr{D}' - \mathscr{D})\, \delta(R' - R)\, \delta^4(P' - P) .$$

---

(61) That is, the Mandelstam variable s.

(62) Cf. the article by P. MOUSSA and R. STORA: in *Lectures in Theoretical Physics*, Vol. **7** A (Boulder, Colo, 1965); H. JOOS: *Forts. Phys.*, **10**, 65 (1962); A. J. MACFARLANE: *Journ. Math. Phys.*, **4**, 490 (1962).

(63) Notation: $R, P, \mathscr{D}$ are defined by (5.9a), (5.10) and the corresponding $R', P', \mathscr{D}'$ arise by changing $r_i \to r'_i$, $p_i \to p'_i$.

At this point we must decide what particular coupling scheme we wish to use. We will choose one which is analogous to the l-s coupling for the Poincaré group.

Let us consider the group element

$$\tilde{g} = (1;\, 0,\, 0,\, \tilde{b},\, \tilde{\Lambda})\,, \tag{5.25}$$

where $\tilde{b}$, $\tilde{\Lambda}$ are determined by the requirement that the mapped characters $p_1' \equiv \tilde{p}_1$ and $p_2' \equiv \tilde{p}_2$ (calculated according to (5.8)) are such that [64]

$$\begin{cases} \tilde{p}_1^k = \tilde{p}_2^k = 0 & \text{for } k = 1, 2, 3\,, \\ \tilde{p}_1^0 + \tilde{p}_2^0 \neq 0\,, & l_1\tilde{p}_1^0 - l_2\tilde{p}_2^0 \neq 0\,. \end{cases} \tag{5.26}$$

Applying now $\mathscr{U}_{\tilde{g}}$ to (5.23), and using (5.24), (5.7), we obtain the transformation of the CG coefficient to our special frame (5.26). We find

$$\left\langle \begin{matrix} RP \\ T \end{matrix} \middle| \begin{matrix} L\,\mathscr{D}\,k\,\varphi \\ sm \end{matrix} \middle| \begin{matrix} l_1\,\mathscr{D}_1\,k_1\,\varphi_1 \\ s_1 m_1 \end{matrix} \middle| \begin{matrix} l_2\,\mathscr{D}_2\,k_2\,\varphi_2 \\ s_2 m_2 \end{matrix} \middle| \begin{matrix} r_1' p_1' \\ r_2' p_2' \end{matrix} \right\rangle = \tag{5.27}$$

$$= [D^{k_1\varphi_1}(\tilde{\Lambda})]_{s_1'm_1';\,s_1m_1}[D^{k_2\varphi_2}(\tilde{\Lambda})]_{s_2'm_2';\,s_2m_2}\cdot \left\langle \begin{matrix} RP \\ T \end{matrix} \middle| \begin{matrix} L\,\mathscr{D}\,k\,\varphi \\ sm \end{matrix} \middle| \begin{matrix} l_1\,\mathscr{D}_1\,k_1\,\varphi_1 \\ s_1' m_1' \end{matrix} \middle| \begin{matrix} l_2\,\mathscr{D}_2\,k_2\,\varphi_2 \\ s_2' m_2' \end{matrix} \middle| \begin{matrix} \tilde{r}_1 \tilde{p}_1 \\ \tilde{r}_2 \tilde{p}_2 \end{matrix} \right\rangle .$$

Now we rewrite our product basis in terms of quantities referred to the special frame (5.26). We have

$$\psi^{l_1\mathscr{D}_1k_1\varphi_1}_{s_1'm_1'}(r_1', p_1') \otimes \psi^{l_2\mathscr{D}_2k_2\varphi_2}_{s_2'm_2'}(r_2', p_2') = \tag{5.28}$$

$$= [D^{k_1\varphi_1}(\tilde{\Lambda}^{-1})]_{s_1''m_1'';s_1'm_1'}[D^{k_2\varphi_2}(\tilde{\Lambda}^{-1})]_{s_2''m_2'';s_2'm_2'}\{\varphi^{l_1\mathscr{D}_1k_1\varphi_1}_{s_1''m_1''}(\tilde{r}_1, \tilde{p}_1) \otimes \varphi^{l_2\mathscr{D}_2k_2\varphi_2}_{s_2''m_2''}(\tilde{r}_2, \tilde{p}_2)\}\,.$$

If we use the composition law for the $SL_{2,c}$ representations [65] and take notice of the transformation (5.9a), (5.9b) to the new characters, we can write

$$\varphi^{l_1\mathscr{D}_1k_1\varphi_1}_{s_1m_1}(\tilde{r}_1, \tilde{p}_1) \otimes \varphi^{l_2\mathscr{D}_2k_2\varphi_2}_{s_2m_2}(\tilde{r}_2, \tilde{p}_2) = \tag{5.29}$$

$$= \bigoplus_{\tilde{k}} \int\limits_{-\infty}^{+\infty} \mathrm{d}\Omega(\tilde{\varphi}, \tilde{k})\, \langle k_1\varphi_1 s_1 m_1;\, k_2\varphi_2 s_2 m_2 | \tilde{k}\tilde{\varphi}\tilde{s}\tilde{m}\rangle^*\, \psi^{L\tilde{\mathscr{D}}\tilde{k}\tilde{\varphi};\tilde{\Lambda}}_{\tilde{s}\tilde{m}}(\tilde{R}, \tilde{P};\, \tilde{U}, \tilde{T})\,.$$

---

[64] It is not difficult to see that one can find (infinitely many) values for $\tilde{b}$ and $\tilde{\Lambda}$ so that (5.26) is satisfied.

[65] R. L. Anderson, R. Racka, M. A. Rashid and P. Winternitz: two papers in *Journ. Math. Phys.*, in press (1970). We are much obliged to Dr. P. Winternitz for having made available to us the galley proofs of these papers prior to publication.

Here

$$\text{(5.29a)} \qquad \begin{cases} \tilde{R} = \tilde{r}_1 + \tilde{r}_2\,, \\ \tilde{P} = \tilde{p}_1 + \tilde{p}_2 = (\tilde{p}_1^0 + \tilde{p}_2^0,\, 0,\, 0,\, 0)\,, \\ (l_1 + l_2)\tilde{T} = (l_1\tilde{p}_1^0 - l_2\tilde{p}_2^0,\, 0,\, 0,\, 0)\,, \\ \mathscr{D} = \tilde{P}^2 + 2L^{-1}\tilde{R}\,. \end{cases}$$

Furthermore,

$$\text{(5.30a)} \qquad \mathrm{d}\Omega(\tilde{\varphi}, \tilde{k}) \equiv \mathrm{d}\tilde{\varphi}(\tilde{\varphi}^2 + 4\tilde{k}^2)\,.$$

The $\tilde{k}$ summation extends over all $\tilde{k}$ such that

$$\text{(5.30b)} \qquad \tilde{k} + k_1 + k_2 \equiv n_1 = 0, 1, 2, \ldots\,.$$

The $SL_{2,\sigma}$ CG coefficient (whose complex conjugate appears in (5.29)) can be written as [65]

$$\text{(5.30c)} \qquad \langle k_1\varphi_1 s_1 m_1;\, k_2\varphi_2 s_2 m_2 | k\varphi s m\rangle = (s_1 m_1,\, s_2 m_2 | s m)\, X(k_1\varphi_1 s_1,\, k_2\varphi_2 s_2,\, k\varphi s)\,,$$

where the first factor is an SU_2 CG coefficient and X vanishes unless

$$\text{(5.30d)} \qquad |s_1 - s_2| \leqslant s \leqslant s_1 + s_2\,.$$

The orthogonality relation reads

$$\text{(5.30e)} \qquad \sum_{s_1 m_1}\sum_{s_2 m_2} \langle k_1\varphi_1 s_1 m_1;\, k_2\varphi_2 s_2 m_2 | k\varphi s m\rangle^* \langle k_1\varphi_1 s_1 m_1;\, k_2\varphi_2 s_2 m_2 | k'\varphi' s' m'\rangle = \\ = (\varphi^2 + 4k^2)^{-1}\,\delta(\varphi - \varphi')\,\delta_{kk'}\,\delta_{ss'}\,\delta_{mm'}\,.$$

Using (5.27), (5.29), (5.30) we can now rewrite (5.23) as

$$\text{(5.31)} \qquad \psi_{sm}^{L\mathscr{D}k\varphi;\,A}(R, P) = \int \mathrm{d}\tilde{\mathscr{D}}\,\mathrm{d}\tilde{R}\,\mathrm{d}^4\tilde{P}\,\mathrm{d}^4\tilde{T}\,\delta(\tilde{T}^2 - \tilde{\mathscr{C}})\,\cdot \\ \cdot\left\langle \begin{matrix} RP \\ T \end{matrix} \middle| \begin{matrix} L\,\mathscr{D}\,k\,\varphi \\ sm \end{matrix} \middle| \begin{matrix} l_1\,\mathscr{D}_1\,k_1\,\varphi_1 \\ s_1 m_1 \end{matrix} \middle| \begin{matrix} l_2\,\mathscr{D}_2\,k_2\,\varphi_2 \\ s_2 m_2 \end{matrix} \middle| \begin{matrix} \tilde{r}_1\tilde{p}_1 \\ \tilde{r}_2\tilde{p}_2 \end{matrix} \right\rangle \cdot \\ \cdot\left\{ \bigoplus_{\tilde{k}} \int \mathrm{d}\Omega(\tilde{\varphi}, \tilde{k})\, \langle k_1\varphi_1 s_1 m_1;\, k_2\varphi_2 s_2 m_2 | \tilde{k}\,\tilde{\varphi}\,\tilde{s}\,\tilde{m}\rangle\, \hat{\psi}_{\tilde{s}\tilde{m}}^{L\tilde{\mathscr{D}}\tilde{k}\tilde{\varphi};\,\tilde{A}}(\tilde{R}, \tilde{P};\, \tilde{T}) \right\}\,,$$

where

$$\text{(5.31a)} \qquad \tilde{\mathscr{C}} = \alpha\tilde{\mathscr{D}} + \beta_1\mathscr{D}_1 + \beta_2\mathscr{D}_2$$

and

$$(5.31b)\qquad \hat{\psi}^{\underset{\sim}{L}\tilde{\mathscr{D}}\tilde{k}\tilde{\varphi};\tilde{A}}_{\underset{\sim}{s}\underset{\sim}{m}}(\tilde{R},\tilde{P};\tilde{T})\equiv$$
$$=\int \mathrm{d}\tilde{U}\,\delta(\tilde{U}+\gamma_1\tilde{P}\tilde{T}+\gamma_2\tilde{T}^2-\delta_1\mathscr{D}_1-\delta_2\mathscr{D}_2)\,\psi^{\underset{\sim}{L}\tilde{\mathscr{D}}\tilde{k}\tilde{\varphi};\tilde{A}}_{\underset{\sim}{s}\underset{\sim}{m}}(\tilde{R},\tilde{P};\tilde{U},\tilde{T})\,.$$

Next we note that, as seen from (5.31), the function $\hat{\psi}$ is invariant under homogeneous Lorentz transformations on $\tilde{T}$. We introduce the new variable ([66])

$$(5.32)\qquad v=\tilde{T}/\sqrt{\tilde{T}^2}=\tilde{T}/\sqrt{\tilde{\mathscr{C}}}\,,$$

and expand $\hat{\psi}$ on the hyperboloid $v^2=1$ in terms of the basis $f^{k\varphi}_{sm}$ of irreducible unitary representations of the principal series of $SL_{2,C}$. We write

$$(5.33)\qquad \hat{\psi}^{\underset{\sim}{L}\tilde{\mathscr{D}}\tilde{k}\tilde{\varphi};\tilde{A}}_{\underset{\sim}{s}\underset{\sim}{m}}(\tilde{R},\tilde{P};v)=\bigoplus_{k=0}^{\infty}\int_{-\infty}^{+\infty}\mathrm{d}\Omega(\hat{\varphi},\hat{k})\,f^{\widehat{k\varphi}}_{\widehat{sm}}(v)\,K^{\underset{\sim}{L}\tilde{\mathscr{D}};\tilde{k}\tilde{\varphi};\widehat{k\varphi};\tilde{A}}_{\underset{\sim}{s}\underset{\sim}{m};\widehat{sm}}(\tilde{R},\tilde{P})\,.$$

Here, the basis f satisfies the orthonormality relation

$$(5.33a)\qquad \int\frac{\mathrm{d}^3v}{2v_0}\,f^{k\varphi}_{sm}(v)\,\overline{f^{k'\varphi'}_{s'm'}}(v)=(\varrho^2+4k^2)^{-1}\delta(\varphi'-\varphi)\,\delta_{k'k}\,\delta_{s's}\,\delta_{m'm}\,.$$

The coefficients K in (5.33) transform under $SL_{2,C}$ as the product of two basis functions of the irreducible representations of $SL_{2,C}$. Hence, K can be reduced (according to the pattern (5.30)) and we get

$$(5.34)\qquad K^{\underset{\sim}{L}\tilde{\mathscr{D}};\tilde{k}\tilde{\varphi};\widehat{k\varphi};\tilde{A}}_{\underset{\sim}{s}\underset{\sim}{m};\widehat{sm}}(\tilde{R},\tilde{P})=$$
$$=\bigoplus_{k'}\int_{-\infty}^{+\infty}\mathrm{d}\Omega(\varphi',k')\langle\tilde{k}\,\tilde{\varphi}\,\tilde{s}\,\tilde{m};\hat{k}\,\hat{\varphi}\,\hat{s}\,\hat{m}|k'\varphi's'm'\rangle^*\,\psi^{\underset{\sim}{L}\tilde{\mathscr{D}}k'\varphi';\tilde{A}}_{s'm'}(\tilde{R},\tilde{P})\,,$$

where the sum ranges over all k' values such that

$$(5.34a)\qquad k'+\tilde{k}+\hat{k}\equiv n_2=0,1,2,\ldots\,.$$

Combining now (5-34), (5.31b) and (5.31), and using the orthogonality condiditions (5.33a) and (5.30e), we finally obtain the *explicit form of our CG*

([66]) Note that the δ-function in (5.31) restricts $\tilde{T}^2$ to be $\tilde{\mathscr{C}}$. Furthermore $\tilde{T}^2>0$, because of (5.29a).

coefficients:

$$\left\langle \begin{matrix} RP \\ v \end{matrix} \middle| \begin{matrix} L\mathscr{D}\,k\varphi \\ sm \end{matrix} \middle| \begin{matrix} l_1\,\mathscr{D}_1\,k_1\varphi_1 \\ s_1 m_1 \end{matrix} \middle| \begin{matrix} l_2\,\mathscr{D}_2\,k_2\varphi_2 \\ s_2 m_2 \end{matrix} \middle| \begin{matrix} r_1'p_1' \\ r_2'p_2' \end{matrix} \right\rangle = \tag{5.35}$$

$$= \delta(\mathscr{D}'-\mathscr{D})\,\delta(R'-R)\,\delta^4(P'-P)[D^{k_1\varphi_1}(\tilde{\Lambda}^{-1})]_{s_1'm_1';\,s_1m_1}[D^{k_2\varphi_2}(\tilde{\Lambda}^{-1})]_{s_2'm_2';s_2m_2}\cdot$$

$$\cdot\langle k_1\varphi_1 s_1' m_1';\, k_2\varphi_2 s_2' m_2' | \hat{k}\,\hat{\varphi}\,\hat{s}\,\hat{m}\rangle\langle \hat{k}\,\hat{\varphi}\,\hat{s}\,\hat{m};\, k_3\varphi_3 s_3 m_3 | k\varphi s m\rangle \overline{f^{k_3\varphi_3}_{s_3m_3}}(v)\,.$$

If desired, (5.30c) may be used to split off SU_2 CG coefficients.

The reader may find it instructive to compare (5.35) with the corresponding result for the Poincaré group ([62]).

5'4. *Reduction of the product representation.* We are now in a position to reduce the scalar product in $\mathscr{H}\big[(l_1|\mathscr{D}_1, k_1, \varphi_1)\otimes(l_2|\mathscr{D}_2, k_2, \varphi_2)\big]$. We define, naturally, the scalar product in $\mathscr{H}_s\big[(L|\mathscr{D}, k, \varphi)\big]$ by

$$(\phi, \psi)_L \equiv \int \mathrm{d}R\,\mathrm{d}^4P\,\delta\left(P^2 + \frac{2}{L}R - \mathscr{D}\right)[\phi^{L\mathscr{D}k\varphi}_{sm}(R, P)]^\dagger \psi^{L\mathscr{D}k\varphi}_{sm}(R, P)\,. \tag{5.36}$$

Using then (5.30), (5.33), (5.34) (as applied to a general reference system) and taking note of (5.33a), (5.30e), we obtain, after a lengthy but straightforward calculation,

$$(\phi, \psi)_{l_1l_2} \equiv \tag{5.37}$$

$$\equiv \int \mathrm{d}\Omega_{12}\{\phi^{l_1\mathscr{D}_1k_1\varphi_1}_{s_1m_1}(r_1, p_1)\otimes\phi^{l_2\mathscr{D}_2k_2\varphi_2}_{s_2m_2}(r_2, p_2)\}^\dagger\{\psi^{l_1\mathscr{D}_1k_1\varphi_1}_{s_1m_1}(r_1, p_1)\otimes\psi^{l_2\mathscr{D}_2k_2\varphi_2}_{s_2m_2}(r_2, p_2)\} =$$

$$= \bigoplus_{\hat{k}}\bigoplus_{k_3}\bigoplus_{k}\int_{-\infty}^{+\infty}\mathrm{d}\mathscr{D}\int_{-\infty}^{+\infty}\mathrm{d}\Omega(\hat{k},\hat{\varphi})\,\mathrm{d}\Omega(k_3,\varphi_3)\,\mathrm{d}\Omega(k,\varphi)(\phi,\psi)_L.$$

The ranges of summations are

$$\begin{cases} \hat{k} = n_1 - (k_1 + k_2)\,, & n_1 = 0, 1, 2, \ldots, \qquad k_3 = 0, 1, 2, \ldots, \\ k = n_2 - (\hat{k} + k_3) = n_2 - n_1 + [(k_1 + k_2) - k_3]\,, & n_2 = 0, 1, 2, \ldots. \end{cases} \tag{5.37a}$$

In view of (5.37), we can formally write *the decomposition of direct products* as

$$(l_1|\mathscr{D}_1, k_1, \varphi_1)\otimes(l_2|\mathscr{D}_2, k_2, \varphi_2) = \tag{5.38}$$

$$= \bigoplus_{\hat{k}}\bigoplus_{k_3}\bigoplus_{k}\int_{-\infty}^{+\infty}\mathrm{d}\mathscr{D}\int_{-\infty}^{+\infty}\mathrm{d}\Omega(\hat{k},\hat{\varphi})\,\mathrm{d}\Omega(k_3,\varphi_3)\,\mathrm{d}\Omega(k,\varphi)(L|\mathscr{D}, k,\varphi)\,.$$

We also see now that the multiplicity label ε (which we introduced in eq. (5.17)) corresponds to the set (n_1, n_2, k_3).

6. – Concluding remarks.

It was our intention to keep this paper on a purely mathematical level. The complexity of the problems we dealt with fully warrants such an attitude. The length of this study prevents us from also presenting here a detailed analysis of the physical consequences of our mathematical results, and now we only make some general remarks, so as to avoid any possible misinterpretation.

Equations (3.10) and (3.35) tell us that the unitary irreducible ray representations correspond to states which are infinite spin towers ([67]). The minimal value of spin is, in a given tower, given by k. This observation furnishes the physical meaning of k. In order to find the meaning of c, we note that physical particle states must be normalizable. Now, in view of (3.35), our basis states have the form $\psi^{l\mathscr{D}kc}_{ss_3}(r, p) = f^{kc}_{ss_3}\varphi^{l\mathscr{D}}(r, p)$, where f is a basis function of $SL_{(2,C)T}$. It is well known ([21]) that unless $c = 0$, f has only a delta normalization. However, for $c = 0$, one has simply ([68])

$$\int \mathrm{d}^4 v\, f^k_{ss_3}(v)\, f^{k'}_{s's'_3}(v) = \delta_{kk'}\,\delta_{ss'}\,\delta_{s_3s'_3}\,.$$

Thus, for admissible physical representations ([69]), we must take $c = 0$. Incidentally, this consideration also tells us that all our spin towers will be in the principal series ([70]) (the supplementary series has $c \neq 0$). Of course, in the direct product of two $c = 0$ representations we will have all possible φ-values (cf. (5.38)). However, scattering states need not be normalizable, so we have no contradiction.

Concerning $\mathscr{D}$, the equivalence theorem of Subsect. **3**˙3 tells us that we may take $\mathscr{D} = 0$ for the single-particle representations (when $l \neq \infty$). Thus, we are left with two significant labels (l, k). This is quite similar to both the Poincaré and the nonrelativistic Galilei frameworks.

Inspection of the basis-defining eqs. (3.35a)-(3.35g) reveals that the basis can be factorized as

$$\psi^{l\mathscr{D}kc}_{ss_3}(r, p) = f^{kc}_{ss_3}\cdot\varphi^{l\mathscr{D}}(r, p)\,.$$

([67]) As was discussed in ref. ([1]), $T_{\mu\nu}$ is the internal spin part of $J_{\mu\nu}$. Thus, $\boldsymbol{T}^2$ represents the square of spin angular momentum.

([68]) An example of a basis satisfying this relation has been given in ref. ([1]), Sect. **5**.

([69]) Of course, since p has a continuous spectrum, we still will have a $\delta(p-p')$ in the full orthonormality relation. But, as in the Poincaré case, this can be handled easily by an enclosing box or by wave packets.

([70]) On the other hand, for spin zero, we also have besides the spin tower $(l|0, k = 0, 0)$, the one-dimensional (relative to spin) scalar representation $(l|0, 0, 1)$. This is discussed in the Appendix.

This implies that in this basis there is no connection between the spin and the orbit equation. Hence, each irreducible tower is infinitely mass degenerate. However, it is possible that by symmetry breaking or otherwise this degeneracy could be removed and a nontrivial mass spectrum obtained.

The physical meaning of l is naturally connected to the fact that subspaces with different value of l^{-1} are incoherent ([71]). In particular, there will operate a superselection rule between any representation with $l^{-1} \neq 0$ and $l^{-1} = 0$. As we saw in Sect. **5**, the latter do not give rise to towers. Hence, it is very tempting to associate $l^{-1} = 0$ representations with leptons. We then automatically have a very desirable superselection rule between leptons and particles with $l^{-1} \neq 0$. Furthermore, let us associate baryons with $l^{-1} \neq 0$ and suppose that every baryon tower has the same $l^{-1} = l_0^{-1}$. From (5.5) we then see that systems with n baryons have $l^{-1} = n l_0^{-1}$. Thus, we could have a superselection between states with different baryon number. Finally, we observe that antiparticles can be associated with conjugate representations: if a particle is in the $(l|0, k, 0)$ tower, its antiparticle is in the $\overline{(l|0, k, 0)} \approx (-l|0, k, 0)$ tower ([72]). This, then, extends the superselection rule so that antibaryons are also correctly encompassed.

We conclude with one important comment. Inspection of the momentum transformation law (3.5a) or of the orbit shown in Fig. 1 reveals that there are zest transformations in T_4^b which can connect states with $p^2 > 0$, $p^2 = 0$, $p^2 < 0$ to one another. Consequently, besides normal particles with $p^2 \geqslant 0$, we also have tachyons ([73]) with $p^2 < 0$. In the following we show how on the basis of some physical considerations we may avoid the situation where an object, which appears to one observer as a normal particle, could appear to another observer as a tachyon.

We define a class of « inertial frames » which are distinguished by the requirement that every point of the frame F is stationary with respect to internal development, $x^\mu = \text{const}$, *i.e.*

$$\mathrm{d}x^\mu/\mathrm{d}u = 0 \,. \tag{6.1}$$

All such frames are related to each other by the subgroup $\mathscr{P} \times T_1^\sigma$, so that if $p^2 > 0$ in one of them, it will be the same in all. Now consider a zested

---

([71]) As pointed out in ref. ([1]), we have a superselection rule with respect to l. This is so because the center of the Lie algebra contains a multiple of the identity operator.

([72]) This interpretation, based on Subsect. **3**·5, is further substantiated by noting that in configuration space $\psi^{-l}(x, u) = \bar{\psi}^l(-x, -u)$, *i.e.* particle-antiparticle conjugation corresponds to total inversion and complex conjugation.

([73]) A simple review on tachyons, including references to the original literature, is given by O.-M. BILANIUK and E. C. SUDARSHAN: *Phys. Today*, **22**, No. 5, 43 (1969). See also, *Phys. Today*, **22**, No. 12, 47 (1969).

frame $\tilde{F}$. We have

$$\tilde{x}_\mu = x_\mu + b_\mu u\,. \tag{6.2}$$

For the velocity of this frame relative to an inertial frame we easily get, taking (6.1) into account,

$$\tilde{v}_k \equiv \mathrm{d}\tilde{x}_k/\mathrm{d}\tilde{x}_0 = b_k/b_0\,. \tag{6.3}$$

Since for any *physical* (material) observer his frame's velocity must be less than the light velocity ($|v_k|<1$), we see that for physically admissible observers $b_0^2 - b_k^2 \equiv b^2 > 0$. Furthermore, (6.2) gives

$$b_0 = d\tilde{x}_0/du\,. \tag{6.4}$$

Since, by definition as an increment of an independent variable, $\mathrm{d}u > 0$, and since for a *physical* observer the sense of time-flow ($\mathrm{d}\tilde{x}_0$) must be positive, it follows that $b_0 > 0$. Thus *all physical observers are related to arbitrary inertial frames by zests which are positive timelike*, $b^2 > 0$, $b_0 > 0$. Now, from (3.5*a*), in the frame $\tilde{F}$ we have

$$\tilde{p}^2 = \left(p - \frac{1}{l}\,b\right)^2 = p^2 - \frac{2}{l}\,pb + \frac{1}{l^2}\,b^2\,. \tag{6.5}$$

If we admit only $l<0$ representations [74], this $\tilde{p}^2 > 0$ whenever $p^2 > 0$. Hence, to physically realizable inertial observers a normal particle will never appear as a tachyon [75].

We would like to add that, in view of the importance of the question, further investigations on the above and related problematic aspects are in progress. For example, one may investigate nonlinear realizations of $\tilde{\mathfrak{G}}_5$ in *Minkowski* space, which may shed additional light on the problems involved.

Finally, we mention that after this work was completed a paper by NOGA [76] appeared, in which, from different considerations, he also arrived at the problematic aspects of the mass spectrum and of the tachyon states.

---

[74] There is no problem with the conjugate representations. Even though $\overline{(l|k)} \approx (-\,l|k)$, for these p has an opposite sign relative to the $(l|k)$ representation, cf. (3.31).

[75] In order to ensure also that $p^2 = 0$ particles (« luxons ») remain unchanged for $\tilde{F}$ (*i.e.* that $\tilde{p}^2 = 0$), we must demand that luxons belong to $l = \infty$ representations. This is quite agreeable, since there do not seem to be spin-tower objects with zero mass.

[76] M. NOGA: *Phys. Rev. D*, **2**, 304 (1970).

APPENDIX

In this Appendix we study, by using a direct elementary method in configuration space, the scalar (one-dimensional) representation $(l|0, 0, 1)$ of $\mathfrak{G}_5$.

As was shown in ref. [1,2], the scalar wave equation in configuration space is

$$\left(\Box - i\frac{2}{l}\,\partial_u\right)\varphi(x; u) = 0 . \tag{A.1}$$

The solutions are subject to the square integrability condition

$$\|\varphi\|^2 \equiv \int |\varphi(x; u)|^2\, \mathrm{d}^4x < \infty . \tag{A.2}$$

If we go from a frame (x, u) to a frame (x', u'), the transformed wave equation must read

$$\left(\Box' - i\frac{2}{l}\,\partial'_u\right)\varphi'(x'; u') = 0 . \tag{A.3}$$

Here

$$\varphi'(x'; u') = \mathscr{U}_g\varphi(x; u) , \tag{A.4}$$

where $g \in \mathfrak{G}_5$ and $\mathscr{U}_g$ is a unitary operator. Hence, $\|\varphi'\| = \|\varphi\|$, which means that $|\varphi'(x'; u')| = |\varphi(x; u)|$, so that

$$\varphi'(x'; u') = \exp[iF(x, u)]\varphi(x; u) . \tag{A.5}$$

In order to find $\Box'$ and ∂'_u, we note that the inverse of (1.1) is

$$\begin{cases} x_\mu = (\Lambda^{-1})^\nu_\mu(x'_\nu - b_\nu u' + b_\nu\sigma - a_\nu) ,\\ u \;= u' - \sigma . \end{cases} \tag{A.6}$$

Hence, from the identities

$$\frac{\partial}{\partial x'_\nu} = \frac{\mathrm{d}x_\mu}{\mathrm{d}x'_\nu}\frac{\partial}{\partial x_\mu} + \frac{\mathrm{d}u}{\mathrm{d}x'_\nu}\frac{\partial}{\partial u} \quad \text{and} \quad \frac{\partial}{\partial u'} = \frac{\mathrm{d}u}{\mathrm{d}u'}\frac{\partial}{\partial u} + \frac{\mathrm{d}x_\mu}{\mathrm{d}u'}\frac{\partial}{\partial x_\mu} ,$$

we easily get

$$\partial'^\nu = (\Lambda^{-1})^\nu_\mu\partial^\mu , \qquad \partial'_u = \partial_u - (\Lambda^{-1})^\nu_\mu b_\nu\partial^\mu . \tag{A.7}$$

Substituting (A.7) and (A.5) into (A.3) we obtain

$$i\varphi\Box F + \Box\varphi - \varphi\,\partial_\mu F\,\partial^\mu F + 2i\,\partial_\mu F\,\partial^\mu\varphi - \\ - i\frac{2}{l}\left[i\varphi\,\partial_u F + \partial_u\varphi - i\varphi(\Lambda^{-1})^\nu_\mu b_\nu\partial^\mu F - (\Lambda^{-1})^\nu_\mu b_\nu\,\partial^\mu\varphi\right] = 0 .$$

Using (A.1), we thus have the conditions on F

$$\Box F = 0\,, \tag{A.8a}$$

$$\partial_\mu F = -l^{-1}(\Lambda^{-1})^\nu_\mu b_\nu\,, \tag{A.8b}$$

$$\partial_u F = -(2l)^{-1} b_\mu b^\mu\,. \tag{A.8c}$$

Because of (A.8*b*), eq. (A.8*a*) is automatically satisfied. The solution of the remaining two equations is easily found to be ([77])

$$F = -l^{-1}(\Lambda^{-1}bx + \tfrac{1}{2}b^2 u)\,. \tag{A.9}$$

Thus, from (A.4), (A.5) and (A.9) we get the *explicit transformation law* in configuration space:

$$\varphi(x;\,u) \to \mathscr{U}_g\varphi(x;\,u) = \exp[-il^{-1}(\Lambda^{-1}bx + \tfrac{1}{2}b^2u)]\varphi(x;\,u)\,. \tag{A.10}$$

We now wish to pass to «momentum space». We define the Fourier transform

$$\psi(r,\,p) = \int \exp[i(ru + px)]\varphi(x;\,u)\,\mathrm{d}u\,\mathrm{d}^4x\,. \tag{A.11}$$

This implies

$$\mathscr{U}_g\psi(r,\,p) = \int \exp[i(ru' + px')]\varphi'(x';\,u')\,\mathrm{d}u'\,\mathrm{d}^4x'.$$

Using (1.1) and (A.10), we then obtain, with the help of (A.11),

$$\mathscr{U}_g\psi(r,\,p) = \exp[i(r\sigma + pa)]\,\psi(r',\,p')\,, \tag{A.12}$$

where

$$r' = r + pb - \frac{1}{2l}\,b^2\,, \qquad p' = \Lambda^{-1}\left(p - \frac{1}{l}\,b\right). \tag{A.12a}$$

The transformation law (A.12) is precisely the special case of eq. (3.10) pertinent to the scalar representation.

Finally, we wish to use (A.12) to compute the factor system (2.3). We can write (disregarding the irrelevant $\exp[i\beta\theta]$ factors)

$$\begin{aligned}\mathscr{U}_{g_2}[\mathscr{U}_{g_1}\psi(r,\,p)] &= \exp[i(r\sigma_2 + pa_2)]\mathscr{U}_{g_1}\psi(r',\,p') = \\ &= \exp\left[i\left[r\sigma_2 + pa_2 + \left(r + pb_2 - \frac{1}{2l}\,b_2^2\right)\sigma_1 + \Lambda_2^{-1}\left(p - \frac{1}{l}\,b_2\right)a_1\right]\right]\psi(r'',\,p'')\,,\end{aligned} \tag{A.13}$$

([77]) We disregard an immaterial additive constant.

where

$$r'' = r + pb_2 - \frac{1}{2l}\,b_2^2 + \Lambda_2^{-1}\left(p - \frac{1}{l}\,b_2\right)b_1 - \frac{1}{2l}\,b_1^2\,,$$

$$p'' = \Lambda_1^{-1}\left[\Lambda_2^{-1}\left(p - \frac{1}{l}\,b_2\right) - \frac{1}{l}\,b_1\right].$$

On the other hand,

$$\mathscr{U}_{g_2 g_1}\psi(r, p) = \exp\left[i[r(\sigma_2 + \sigma_1) + p(a_2 + \Lambda_2 a_1 + \sigma_1 b_2)]\right]\psi(r'', p'')\,. \tag{A.14}$$

where we took cognizance of the composition law of the parameters σ, a, b, Λ; as given by (2.2). Recalling now (2.5) and using (A.13), (A.14), we immediately obtain $\omega(g_2, g_1)$ in the form as given by (2.3), (2.6). *q.e.d.*

● RIASSUNTO (*)

Tramite il metodo delle rappresentazioni indotte si deducono e classificano tutte le rappresentazioni proiettive unitarie irriducibili del nuovo gruppo dinamico relativistico $\mathfrak{G}_5$, introdotto di recente. Si dà un forma esplicita della legge di trasformazione. Si studiano le proprietà della corrispondente funzione di base ad infinite dimensioni. Si mostra che nel caso limite $l = \infty$ (corrispondente a $\tilde{\mathfrak{G}}_5 \to \mathfrak{G}_5$) le infinite rappresentazioni della torre di spin diventano riducibili e si decompongono in rappresentazioni irriducibili del gruppo di Poincaré. Si studia la riduzione del prodotto diretto di due rappresentazioni radiali unitarie irreducibili di $\tilde{\mathfrak{G}}_5$. Si calcolano i coefficienti di Clebsch-Gordan. Infine si fanno alcuni commenti sull'interpretazione fisica dei risultati.

(*) *Traduzione a cura della Redazione.*

Теория представлений для новой релятивистской динамической группы.

Резюме (*). — Используя метод индуцированных представлений, выводятся и классифицируются все неприводимые унитарные проективные представления недавно введенной новой релятивистской динамической группы $\mathfrak{G}_5$. Приводится явная форма для закона преобразования. Исследуются свойства функций, соответствующих бесконечномерному базису. Показывается, что в предельном случае $l=\infty$ (соответствующем $\tilde{\mathfrak{G}}_5 \to \mathfrak{G}_5$) представления бесконечной спиновой башни становятся приводимыми и разлагаются на неприводимые представления группы Пуанкаре. Исследуется приведение прямого произведения двух неприводимых унитарных лучевых представлений $\tilde{\mathfrak{G}}_5$. Вычисляются коэффициенты Клебша-Гордана. В заключение делаются некоторые замечания относительно физической интерпретации полученных результатов.

(*) *Переведено редакцией.*