International Journal of Modern Physics A, Vol. 14, No. 14 (1999) 2205–2238

A CLASSICAL ISODUAL THEORY OF ANTIMATTER AND ITS PREDICTION OF ANTIGRAVITY

RUGGERO MARIA SANTILLI*

Institute for Basic Research,
PO Box 1577, Palm Harbor, FL 34682, USA

Received 21 April 1998
Revised 12 June 1998

An inspection of the contemporary physics literature reveals that, while matter is treated at all levels of study, from Newtonian mechanics to quantum field theory, antimatter is solely treated at the level of second quantization. For the purpose of initiating the restoration of full equivalence in the treatment of matter and antimatter in due time, and as the classical foundations of an axiomatically consistent inclusion of gravitation in unified gauge theories recently appeared elsewhere, in this paper we present a classical representation of antimatter which begins at the primitive Newtonian level with corresponding formulations at all subsequent levels. By recalling that charge conjugation of particles into antiparticles is antiautomorphic, the proposed theory of antimatter is based on a new map, called *isoduality*, which is also antiautomorphic (and more generally, anti-isomorphic), yet it is applicable beginning at the classical level and then persists at the quantum level where it becomes equivalent to charge conjugation. We therefore present, apparently for the first time, the *classical isodual theory of antimatter*, we identify the physical foundations of the theory as being the novel *isodual Galilean, special and general relativities*, and we show the compatibility of the theory with all available classical experimental data on antimatter. We identify the classical foundations of the prediction of *antigravity* for antimatter in the field of matter (or vice-versa) without any claim on its validity, and defer its resolution to specifically identified experiments. We identify the novel, classical, *isodual electromagnetic waves* which are predicted to be emitted by antimatter, the so-called *space-time machine* based on a novel non-Newtonian *geometric propulsion*, and other implications of the theory. We also introduce, apparently for the first time, the *isodual space and time inversions* and show that they are nontrivially different than the conventional ones, thus offering a possibility for the future resolution whether far away galaxies and quasars are made up of matter or of antimatter. The paper ends with the indication that the studies are at their first infancy, and indicates some of the open problems. To avoid a prohibitive length, the paper is restricted to the classical treatment, while studies on operator profiles are treated elsewhere.

1. Introduction

After being conjectured by A. Schuster in 1898, antimatter was predicted by P. A. M. Dirac[1] in the late 1920's in the *negative-energy solutions* of his celebrated

*E-mail: ibr@gte.net, http://home1.gte.net/ibr

equation. Dirac himself soon discovered that particles with negative-energy do not behave in a physical way and, for this reason, he submitted his celebrated "hole theory," which subsequently restricted the study of antimatter to the sole level of second quantization (for historical aspects on antimatter see, e.g. Ref. 2).

The above occurrence created an imbalance in the physics of this century because matter is described at all levels of study, from Newtonian mechanics to quantum field theory, while antimatter is solely treated at the level of second quantization.

To initiate the study for the future removal of this imbalance in due time, in this paper we present a theory of antimatter which has been conceived to begin at the purely classical Newtonian level, and then to admit corresponding images at all subsequent levels of study.

Our guiding principle is to identify a map which possesses the main mathematical structure of charge conjugation, yet it is applicable at all levels, and not solely at the operator level.

The main characteristic of charge conjugation is that of being *antiautomorphic* (where the term "automorphic" is referred to the map of a given space onto itself). After studying a number of possibilities, we have selected a map which is *anti-isomorphic* (where the term "isomorphic" is referred to a map from one space onto another of equivalent topological characteristics to be identified later on) applicable at all levels of study, and given by the following *isodual map* here generically expressed to an arbitrary quantity Q (i.e. a function, or a matrix or an operator)

$$Q(x, \phi, \ldots) \to Q^d(x^d, \phi^d, \ldots) = -Q^\dagger(-x^\dagger, -\phi^\dagger, \ldots)\,, \tag{1.1}$$

which, for consistency, must be applied to the totality of the mathematical structure of the conventional theory of matter, including numbers, fields, spaces, geometries, algebras, etc. This results in a new mathematics, called *isodual mathematics*, which is at the foundation of the classical isodual theory of antimatter of this paper.

Since the isodual mathematics is virtually unknown, we shall review and expand it in Sec. 2. In Sec. 3 we shall then present, apparently for the first time, the classical *isodual Galilean, special and general relativities* and show that their representation of antimatter is indeed compatible with the *totality* of available classical experimental data, those of electromagnetic nature.

In the Appendix we outline for completeness the classical *isodual Lagrangian and Hamiltonian mechanics* and the rudiments of the novel *isodual quantization* into the *isodual quantum mechanics* studied in details elsewhere jointly with the proof of the equivalence between isoduality and charge conjugation.

A basic objective of the paper is to provide classical foundations for the axiomatically consistent inclusion of gravitation in unified gauge theories of electroweak interactions recently presented elsewhere. In fact, the latter evidently require, as a pre-requisite, the achievement of a classical geometric unification of electromagnetism and gravitation for both matter and antimatter.

The rather limited existing literature in isoduality is the following. The isodual map (1.1) was first proposed by Santilli in Ref. 3 of 1985 and then remained ignored for several years. More recently, the *isodual numbers* characterized by map (1.1) have been studied in Ref. 4. The first hypothesis on the isodual theory of antimatter appeared for the operator version in Ref. 5 of 1993 which also contains an initial study of the equivalence between isoduality and charge conjugation. The fundamental notion of this study, the *isodual Poincaré symmetry* (also called the *Poincaré–Santilli isodual symmetry*), from which the entirety of the (relativistic) analysis can be uniquely derived, was submitted in Ref. 6(a) of 1993 also at the operator level. Memoir[6(b)] presents a recent systematic study of the underlying geometry.

The *isodual differential calculus*, which is fundamental for the correct formulation of dynamical equations all the way to those in curved spaces, was identified only recently in Ref. 8. A review of the *operator* profile up to 1995 is available in monograph.[9]

The prediction of the isodual theory that antimatter in the field of matter experiences antigravity was first submitted in Ref. 7(a) of 1994 which also proposed an experiment for the measure of the gravity of elementary antiparticles in the gravitational field of earth. The experiment essentially consists of comparative measurements under the gravity of collimated, *low energy* beams of positrons and electrons in *horizontal* flight on a tube with sufficiently high vacuum as well as protection from stray and patch fields and of sufficient length to permit a definite result, e.g. the view by the naked eye of the displacements due to gravity of the positron and electron beams on a scintillator at the end of the flight.

This paper is the classical counterpart of: Ref. 10(a) in which we study the operator profile with particular reference to the equivalence between isoduality and charge conjugation and the prediction of antigravity at the operator level; Ref. 10(b) in which we present the apparently first axiomatically consistent inclusion of gravity in unified gauge theories of electroweak interactions; and memoir,[11] which studies antimatter in *interior* conditions (such as the interior of an antimatter star).

An important independent contribution in the field has been made by the experimentalist A. P. Mills Jr.,[12] who has confirmed the apparent feasibility with current technology of the test of the gravity of antiparticles proposed in Ref. 7(a) via the use of electrons and positrons with energy of the order of milli-eV in horizontal flight in a vacuum tube of approximately 100 m length with a diameter and design suitable to reduce stray fields and patch effects at its center down to acceptable levels.

Additional contributions have been made by: J. V. Kadeisvili on the *isodual functional analysis* and *isodual Lie theory*;[13] Lohmus, Paal, Sorgsepp;[14(a)] Sourlas, Tsagas;[14(b)] and others.

Theoretical and experimental studies on the isodual theory of antimatter were conducted at the *International Workshop on Antimatter Gravity and Anti-Hydrogen Atom Spectroscopy*, held in Sepino, Italy, in May 1996 (see Ref. 15).

The motivations for the classical isodual theory of antimatter are rather numerous. First, there is the need indicated earlier to achieve a full equivalence in the treatment of matter and antimatter beginning at the classical level. In fact, far away galaxies and quasars may well be made up of antimatter. The absence of a classical theory of antimatter therefore implies the evident impossibility of quantitative studies of this important astrophysical issue.

Second, the current gravitational treatment of antimatter is afflicted by a number of problematic aspects. Current theories are based on only *one* map from classical to operator settings, the naive or symplectic quantization. Therefore, conventional classical representations of antimatter via positive energies *do not* yield antiparticles under quantization, but conventional particles with the mere reversal of the sign of the charge.

Third, there is a fundamental incompatibility between current theories of gravitation and unified gauge theories of electroweak interactions which is due precisely to antimatter. In fact, current gravitational theories characterize antimatter via a *positive-definite* energy–momentum tensor, while electroweak theories characterize antiparticles via *negative energy* states.

Additional motivations have been identified in Refs. 9–11. The need for a systematic study aiming at a resolution of these issues is then beyond scientific doubts.

The classical theory of antimatter proposed in this paper permits an apparent resolution of the above problematic aspects. In particular, the theory results are compatible with all known classical experimental data on antiparticles, those under electromagnetic interactions, since no conclusive experiment under gravitational interactions is available at this writing for antimatter.

Moreover, the theory proposed in this paper confirms at the classical level the prediction of Refs. 7(a) and 10(a) that *antimatter in the field of matter (or vice-versa) experiences antigravity* (defined as the reversal of the sign of the curvature tensor) in a way which by passes conventional objections.[22] In reality, as we shall see, the classical isodual theory of antimatter provides the strongest available theoretical evidence for antigravity.

The theory proposed in this paper confirms at the classical level the prediction of Ref. 10(a) according to which *antimatter emits new electromagnetic waves*, here called "isodual waves," which coincide with the conventional waves emitted by matter under all interactions, except gravitation, and can be distinguished from ordinary electromagnetic waves via discrete symmetries (Eq. (A.5)). As a consequence, if confirmed by future studies, the classical theory of antimatter proposed for the first time in this paper may one day permit quantitative theoretical and experimental studies as to whether far away galaxy or quasars are made up of matter or of antimatter.

We also point out the prediction of the so-called *space–time machine*, which is a mathematical model of a new form of non-Newtonian *geometric propulsion* in space and time as one way of illustrating the far reaching implications of the possible experimental detection of antigravity.

We finally indicate that the isodual theory of antimatter is deeply connected to a variety of pre-existing research. First, isodual particles emerge as possessing a *negative time* precisely along the historical conception by Stueckelberg for antiparticle. Moreover, the equivalence of treatment between particles and antiparticles at all levels of study can be first seen in the Stueckelberg–Feynman path integral theory.

The isodual theory emerged from the identification of negative units in the the antiparticle component of the conventional Dirac equation and the reconstruction of the theory with respect to that unit. Isoduality therefore provides a mere *reinterpretation* of Dirac's original notion of antiparticle, while leaving all numerical predictions under electroweak interactions essentially unchanged.

We then show that the isodual theory of antiparticles is deeply linked to Majorana's spinors,[26(a)] particularly in their recent formulation by Ahluwalia.[26(d)] The link is so deep that the norm of Ahluwalia's spinors for antiparticles *coincides* with that of isodual particles. Therefore, isoduality provides a mere *reinterpretation* of these results, which nevertheless implies the extension of their applicability, from the current sole level of second quantization, to first quantization, as well as to the classical level (when applicable).

The isodual discrete symmetries also turn out to be deeply linked to pre-existing studies. As an example, the parity of antiparticles originally introduced by Bargmann, Wightman and Wigner,[27(b)] when expressed in the recent formulation by Ahluwalia, Johnson and Goldman,[27(c)] turns out to be equivalent to isodual space inversions.

Despite these similarities on *physical* grounds (which are evidently expected since all theories study the same physical problem), the reader should be aware that the isodual theory of antimatter presented in this paper is *mathematically* inequivalent to pre-existing studies, as established by the fact that the latter are formulated on conventional spaces and fields, while the former is formulated on new spaces and fields.

In particular, the main novelty of this paper rests on the fundamental notion of all quantitative inquiries, the basic unit, which is assumed to be *positive* in pre-existing studies and to be *negative* in the isodual theory, as we shall see.

The paper ends with the indication of rather intriguing open problems.

2. Rudiments of Isodual Mathematics

2.1. *Isodual units, numbers, and fields*

Let $F = F(a, +, \times)$ be a conventional field of real numbers $R(n, +, \times)$, complex numbers $C(c, +, \times)$ or quaternionic numbers $Q(q, +, \times)$ with the familiar additive unit 0, multiplicative unit I, elements $a = n,\ c,\ q$, sum $a_1 + a_2$, $a + 0 = 0 + a = a$, and multiplication $a_1 \times a_2 = a_1 a_2$, $a \times I = I \times a = a$, $\forall a,\ a_1,\ a_2 \in F$.

The *isodual fields*, first introduced in Ref. 3 and then studied in details in Ref. 4, are the image $F^d = F^d(a^d, +^d, \times^d)$ of $F(a, +, \times)$ characterized by the isodual map of the unit

$$I \to I^d = -I^\dagger = -I\,, \tag{2.1}$$

which implies: *isodual numbers*

$$a^d = a^\dagger \times I^d = -a^\dagger \times I = -a^\dagger\,, \tag{2.2}$$

where † is the identity for real numbers $n^\dagger = n$, complex conjugation $c^\dagger = \bar{c}$ for complex numbers c and Hermitian conjugation $q^\dagger$ for quaternions $q^\dagger$; *isodual sum*

$$a_1^d +^d a_2^d = -(a_1^\dagger + a_2^\dagger)\,; \tag{2.3}$$

and *isodual multiplication*

$$a_1^d \times^d a_2^d = a_1^d \times I^d \times a_2^d = -a_1^\dagger \times a_2^\dagger\,; \tag{2.4}$$

under which I^d is the correct left and right unit of F^d,

$$I^d \times^d a^d = a^d \times^d I^d \equiv a^d\,, \quad \forall a^d \in F^d\,, \tag{2.5}$$

in which case (only) $\hat{I}^d$ is called *isodual unit.*

We have in this way the *isodual real field* $R^d(n^d, +^d, \times^d)$ with *isodual real numbers*

$$n^d = -n^\dagger \times I \equiv -n\,, \quad n \in R\,, \quad n^d \in R^d\,; \tag{2.6}$$

the *isodual complex field* $C^d(c^d, +^d, \times^d)$ with *isodual complex numbers*

$$\begin{aligned} c^d = -\bar{c} = -(n_1 - i \times n_2) = -n_1 + i \times n_2\,, \\ n_1,\ n_2 \in R\,, \quad c \in C\,, \quad c^d \in C^d\,; \end{aligned} \tag{2.7}$$

and the *isodual quaternionic field* which is not used in this paper for brevity.

Under the above assumptions, $F^d(a^d, +^d, \times^d)$ verifies all the axioms of a field (*loc. cit.*), although F^d and F are anti-isomorphic, as desired. This establishes that the field of numbers can be equally defined either with respect to the traditional unit $+1$ or with respect to its negative image -1. The key point is the preservation of the axiomatic character of the latter via the isoduality of the multiplication. In other words, the set of isodual numbers a^d with unit -1 and *conventional* product *does not* constitute a field because $I^d \times a^d \neq a^d$.

It is also evident that *all operations of numbers implying multiplications must be subjected for consistency to isoduality.* This implies the *isodual square root*

$$a^{d\frac{1}{2}d} = -\sqrt{-a^d}\,, \quad a^{d\frac{1}{2}d} \times^d a^{d\frac{1}{2}d} = a^d\,, \quad 1^{d\frac{1}{2}d} = i\,; \tag{2.8}$$

the *isodual quotient*

$$a^d /^d b^d = -(a^d/b^d) = -(a^\dagger/b^\dagger) = c^d\,, \qquad b^d \times^d c^d = a^d\,; \tag{2.9}$$

and so on.

A property of isodual fields of fundamental relevance for our characterization of antimatter is that *they have negative-definite norm*, called *isodual norm*[3,4]

$$|a^d|^d = |a^\dagger| \times I^d = -(aa^\dagger)^{1/2} < 0\,, \tag{2.10}$$

where $|\cdots|$ denotes the conventional norm. For isodual real numbers n^d we therefore have the isodual isonorm

$$|n^d|^d = -|n| < 0\,, \tag{2.11}$$

and for isodual complex numbers we have

$$|c^d|^d = -|\bar{c}| = -(c\bar{c})^{1/2} = -(n_1^2 + n_2^2)^{1/2}\,. \tag{2.12}$$

Lemma 2.1. *All quantities which are positive-definite when referred to fields (such as mass, energy, angular momentum, density, temperature, time, etc.) became negative-definite when referred to isodual fields.*

As recalled in Sec. 1, antiparticles have been discovered in the *negative-energy solutions* of Dirac's equation[1] and they were originally thought to evolve *backward in time* (Stueckelberg, and others, see Ref. 2). The possibility of representing antimatter and antiparticles via isodual methods is therefore visible already from these introductory notions.

The main novelty is that the conventional treatment of negative-definite energy and time was (and still is) referred to the conventional contemporary unit +1, which leads to a number of contradictions in the physical behavior of antiparticles whose solution forced the transition to second quantization.

By comparison, *the negative-definite physical quantities of isodual methods are referred to a negative-definite unit* $I^d < 0$. As we shall see, this implies a mathematical and physical equivalence between *positive-definite quantities referred to positive-definite units, characterizing matter, and negative-definite quantities referred to negative-definite units, characterizing antimatter*. These foundations then permit a novel characterization of antimatter beginning at the *Newtonian* level, and then persisting at all subsequent levels.

Definition 2.1. *A quantity is called isoselfdual when it is invariant under isoduality.*

The above notion is particularly important for this paper because it introduces a new invariance, *the invariance under isoduality*. During our study we shall encounter several isoselfdual quantities. At this introductory stage we indicate that *the imaginary number i is isoselfdual*,

$$i^d = -i^\dagger = -\bar{i} = -(-i) = i\,. \tag{2.13}$$

This property permits to understand better the isoduality of complex numbers which can be written explicitly[4]

$$c^d = (n_1 + i \times n_2)^d = n_1^d + i^d \times^d n_2^d = -n_1 + i \times n_2 = -\bar{c}\,. \tag{2.14}$$

We assume the reader is aware of the emergence here of *basically new numbers, those with a negative unit*, which have no connection with ordinary negative numbers and which are the true foundations of the proposed isodual theory of antimatter.

2.2. *Isodual functional analysis*

All conventional and special functions and transforms, as well as functional analysis at large must be subjected to isoduality for consistent applications of isodual theories, resulting in a simple, yet unique and significant *isodual functional analysis*, whose study was initiated by Kadeisvili.[13]

We here mention the *isodual trigonometric functions*

$$\sin^d \theta^d = -\sin(-\theta)\,, \qquad \cos^d \theta^d = -\cos(-\theta)\,, \tag{2.15}$$

with related basic property

$$\cos^{d\,2d} \theta^d + \sin^{d\,2d} \theta^d = 1^d = -1\,, \tag{2.16}$$

the *isodual hyperbolic functions*

$$\sinh^d w^d = -\sinh(-w)\,, \cosh^d w^d = -\cosh(-w)\,, \tag{2.17}$$

with related basic property

$$\cosh^{d\,2d} w^d - \sinh^{d\,2d} w^d = 1^d = -1\,, \tag{2.18}$$

the *isodual logarithm*

$$\log^d n^d = -\log(-n)\,, \tag{2.19}$$

etc. Interested readers can then easily construct the isodual image of special functions, transforms, distributions, etc.

2.3. *Isodual differential calculus*

The conventional differential calculus is indeed dependent on the assumed unit. This property is not so transparent in the conventional formulation because the basic unit is the trivial number +1, thus having null differential. However, the dependence of the unit emerges rather forceful under its generalization.

The *isodual differential calculus*, first introduced in Ref. 8, is characterized by the *isodual differentials*

$$d^d x^k = I^d \times dx^k = -dx^k\,, d^d x_k = -dx_k\,, \tag{2.20}$$

with corresponding *isodual derivatives*

$$\partial^d/\partial^d x^k = -\partial/\partial x^k\,, \qquad \partial^d/\partial^d x_k = -\partial/\partial x\,, \tag{2.21}$$

and other isodual properties.

Note that *conventional differentials are isoselfdual*, i.e.,

$$(dx^k)^d = d^d x^{kd} \equiv dx^k \,, \tag{2.22}$$

but *derivatives are not in general isoselfdual*,

$$(\partial f(x)/\partial x^k)^d = \partial^d f^d /^d \partial^d x^{kd} = -\partial f/\partial x^k \,. \tag{2.23}$$

Other properties can be easily derived and shall be hereon assumed.

2.4. *Isodual Lie theory*

Let **L** be an n-dimensional Lie algebra in its regular representation with universal enveloping associative algebra $\xi(\mathbf{L})$,$[\xi(\mathbf{L})]^- \approx \mathbf{L}$, n-dimensional unit $I = \mathrm{diag}(1, 1, \ldots, 1)$, ordered set of Hermitian generators $X = X^\dagger = \{X_k\}$, conventional associative product $X_i \times X_j$, and familiar Lie's Theorems over a field $F(a, +, \times)$.

The *isodual Lie theory* was first submitted in Ref. 3 and then studied in Ref. 9 as well as by other authors.[13,14] *The isodual universal associative algebra* $[\xi(\mathbf{L})]^d$ is characterized by the *isodual unit* I^d, *isodual generators* $X^d = -X$, and isodual associative product

$$X_i^d \times^d X_j^d = -X_i \times X_j \,, \tag{2.24}$$

with corresponding infinite-dimensional basis (isodual version of the conventional Poincaré–Birkhoff–Witt theorem[3]) characterizing the *isodual exponentiation* of a generic quantity A

$$e^{d^A} = I^d + A^d/^d 1!^d + A^d \times^d A^d/^d 2!^d + \cdots = -e^{A^t} \,, \tag{2.25}$$

where e is the conventional exponentiation.

The attached *isodual Lie algebra* $\mathbf{L}^d \approx (\xi^d)^-$ over the isodual field $F^d(a^d, +^d, \times^d)$ is characterized by the *isodual commutators* (*loc. cit.*)

$$[X_i^d, X_j^d]^d = -[X_i, X_j] = C_{ij}^{kd} \times^d X_k^d \,, \tag{2.26}$$

with a classical realization given in App. A.

Let G be the conventional, connected, n-dimensional Lie transformation group on a metric (or pseudometric) space $S(x, g, F)$ admitting **L** as the Lie algebra in the neighborhood of the identity, with generators X_k and parameters $w = \{w_k\}$. The *isodual Lie group* G^d [3] admitting the isodual Lie algebra $\mathbf{L}^d$ in the neighborhood of the isodual identity I^d is the n-dimensional group with generators $X^d = \{-X_k\}$ and parameters $w^d = \{-w_k\}$ over the isodual field F^d with generic element

$$U^d(w^d) = e^{d^{i^d \times^d w^d \times^d X^d}} = -e^{i\times(-w)\times X} = -U(-w) \,. \tag{2.27}$$

The *isodual symmetries* are then defined accordingly via the use of the isodual groups G^d and they are anti-isomorphic to the corresponding conventional symmetries, as desired. For additional details, one may consult Ref. 9.

In this paper we shall therefore use *Conventional Lie symmetries*, for the characterization of *matter*; and *Isodual Lie symmetries*, for the characterization of *antimatter*.

2.5. *Isodual Euclidean geometry*

Conventional (vector and) metric spaces are defined over conventional fields. It is evident that the isoduality of fields requires, for consistency, a corresponding isoduality of (vector and) metric spaces. The need for the isodualities of all quantities acting on a metric space (e.g. conventional and special functions and transforms, differential calculus, etc.) becomes then evident.

Let $S = S(x, g, R)$ be a conventional N-dimensional metric space with local coordinates $x = \{x^k\}$, $k = 1, 2, \ldots, N$, nowhere degenerate, sufficiently smooth, real-valued and symmetric metric $g(x, \ldots)$ and related invariant

$$x^2 = x^i g_{ij} x^j \,, \tag{2.28}$$

over the reals R.

The *isodual spaces*, first introduced Ref. 3, are the spaces $S^d(x^d, g^d, R^d)$ with *isodual coordinates* $x^d = x \times I^d = -x$, *isodual metric*

$$g^d(x^d, \ldots) = -g^\dagger(-x, \ldots) = -g(-x, \ldots) \,, \tag{2.29}$$

and *isodual interval*

$$\begin{aligned}(x-y)^{d2\,d} &= [(x-y)^{id} \times^d g^d_{ij} \times^d (x-y)^{jd}] \times I^d \\ &= [(x-y)^i \times g^d_{ij} \times (x-y)^{jd}] \times I^d \,,\end{aligned} \tag{2.30}$$

defined over the isodual field $R^d = R^d(n^d, +^d, \times^d)$ with the same isodual isounit I^d.

The basic space of our analysis is the three-dimensional *isodual Euclidean space*,

$$\begin{aligned} E^d(r^d, \delta^d, R^d) \colon r^d &= \{r^{kd}\} = \{-r^k\} = \{-x, -y, -z\} \,, \\ \delta^d &= -\delta = \mathrm{diag}(-1, -1, -1) \,, \\ I^d &= -I = \mathrm{diag}(-1, -1, -1) \,. \end{aligned} \tag{2.31}$$

The *isodual Euclidean geometry* is then the geometry of the isodual space E^d over R^d and it is given by a step-by-step isoduality of all the various aspects of the conventional geometry.

We only mention for brevity the notion of *isodual line* on E^d over R^d given by the isodual image of the conventional notion of line on E over R. As such, its coordinates are isodual numbers $x^d = x \times 1^d$ with unit $1^d = -1$. By recalling that the norm on R^d is negative-definite, the *isodual distance* among two points on an isodual line is also negative definite and it is given by $D^d = D \times 1^d = -D$, where D is the conventional distance. Similar isodualities apply to all remaining notions, including the notions of parallel and intersecting isodual lines, the Euclidean axioms, etc. The following property is of evident proof:

Lemma 2.2. *The isodual Euclidean geometry on E^d over R^d is anti-isomorphic to the conventional geometry on E over R.*

The *isodual sphere* is the perfect sphere on E^d over R^d and, as such, it has *negative radius*,

$$R^{d\,2d} = [x^{d\,2d} + y^{d\,2d} + z^{d\,2d}] \times I^d\,. \tag{2.32}$$

A similar characterization holds for other isodual shapes which characterize the shape of antimatter in our isodual theory.

The group of isometries of E^d over R^d is the *isodual Euclidean group* studied in Ref. 9.

2.6. *Isodual Minkowskian geometry*

The *isodual Minkowski space*, first introduced in Ref. 3, is given by

$$\begin{aligned} M^d(x^d, \eta^d, R^d)\colon x^d &= \{x^{\mu d}\} = \{x^\mu \times I^d\} = \{-r, -c_o t\} \times I\,, \\ \eta^d &= -\eta = \mathrm{diag}(-1,-1,-1,+1)\,, \\ I^d &= \mathrm{diag}(-1,-1,-1,-1)\,. \end{aligned} \tag{2.33}$$

The *isodual Minkowskian geometry*[6] is the geometry of isodual spaces M^d over R^d. It is also characterized by a simple isoduality of the conventional Minkowskian geometry and its explicit presentation is omitted for brevity.

We here merely mention the *isodual light cone*

$$\begin{aligned} x^{d\,2d} &= (x^{\mu d} \times^d \eta^d_{\mu\nu} \times^d x^{\nu d}) \times I^d \\ &= (-xx - yy - zz + tc_o^2 t) \times (-I) = 0\,. \end{aligned} \tag{2.34}$$

As one can see, the above cone formally coincides with the conventional light cone, although the two cones belong to different spaces. The isodual light cone is used in these studies as *the cone of light emitted by antimatter in empty space* (*exterior problem*).

The group of isometries of M^d over R^d is the *isodual Poincarè symmetry* $P^d(3.1) = L^d(3.1) \times T^d(3.1)$[6] and constitutes the fundamental symmetry of this paper.

It may be instructive for the reader interested in learning the new isodual theory to write down the *isodual Maxwell equations* which characterize a fundamental prediction of the theory, the *isodual electromagnetic waves* discussed later on.

2.7. *Isodual Riemannian geometry*

Consider a Riemannian space $\Re(x, g, R)$ in $(3+1)$ dimensions with basic unit $I = \mathrm{diag}(1,1,1,1)$ and related Riemannian geometry in local formulation (see, e.g. Ref. 25). The *isodual Riemannian spaces* are given by

$$\begin{aligned} \Re^d(x^d, g^d, R^d)\colon\; x^d &= \{-\hat{x}^\mu\}\,, \\ g^d &= -g(x)\,, \quad g \in \Re(x, g, R)\,, \\ I^d &= \mathrm{diag}(-1,-1,-1,-1) \end{aligned} \tag{2.35}$$

with interval $x^{2d} = [x^{dt} \times^d g^d(x^d) \times^d x^d] \times I^d = [x^t \times g^d(x^d) \times x] \times I^d$ on R^d, where t stands for transposed.

The *isodual Riemannian geometry* is the geometry of spaces $\Re^d$ over R^d, and it is also given by step-by-step isodualities of the conventional geometry, including, most importantly, the isoduality of the differential and exterior calculus.

As an example, an *isodual vector field* $X^d(x^d)$ on $\Re^d$ is given by $X^d(x^d) = -X(-x)$. The *isodual exterior differential* of $X^d(x^d)$ is given by

$$D^d X^{kd}(x^d) = d^d X^{kd}(x^d) + \Gamma^{dk}_{i\,j} \times^d X^{id} \times^d d^d x^d = DX^k(-x)\,, \qquad (2.36)$$

where the Γ^d's are the components of the *isodual connection*. The *isodual covariant derivative* is then given by

$$X^{id}(x^d)_{|^{dk}} = \partial^d X^{id}(x^d) /^d \partial^d x^{kd} + \Gamma^{dk}_{i\,j} \times^d X^{jd}(x^d) = -X^i(-x)_{|^k}\,. \qquad (2.37)$$

The interested reader can then easily derive the isoduality of the remaining notions of the conventional geometry.

It is an instructive exercise for the interested reader to work out in detail the proof of the following:

Lemma 2.3. *The isoduality of the Riemannian space $\Re(x, g, R)$ to its antiautomorphic image $\Re^d(x^d, g^d, R^d)$ is characterized by the following isodual quantities:*

$$\begin{array}{ll}
\textit{Basic unit} & I \to I^d = -I\,, \\
\textit{Metric} & g \to g^d = -g\,, \\
\textit{Connection coefficients} & \Gamma_{klh} \to \Gamma^d_{klh} = -\Gamma_{klh}\,, \\
\textit{Curvature tensor} & R_{lijk} \to R^d_{lijk} = -R_{lijk}\,, \\
\textit{Ricci tensor} & R_{\mu\nu} \to R^d_{\mu\nu} = -R_{\mu\nu}\,, \\
\textit{Ricci scalar} & R \to R^d = R\,, \\
\textit{Einstein tensor} & G_{\mu\nu} \to G^d_{\mu\nu} = -G_{\mu\nu}\,, \\
\textit{Electromagnetic potentials} & A_\mu \to A^d_\mu = -A_\mu\,, \\
\textit{Electromagnetic field} & F_{\mu\nu} \to F^d_{\mu\nu} = -F_{\mu\nu}\,, \\
\textit{Elm energy-momentum tensor} & T_{\mu\nu} \to T^d_{\mu\nu} = -T_{\mu\nu}\,.
\end{array} \qquad (2.38)$$

The reader should be aware that recent studies[6(a)] have identified the universal symmetry of conventional gravitation with Riemannian metric $g(x)$, the so-called *Poincaré-Santilli isosymmetry* $\hat{P}(3.1) = \hat{L}(3.1) \times \hat{T}(3.1)$.[6] The latter symmetry is the image of the conventional symmetry constructed with respect to the generalized unit

$$\hat{I}(x) = [T(x)]^{-1}\,, \qquad (2.39)$$

where $T(x)$ is a 4×4 matrix originating from the factorization of the Riemannian metric into the Minkowskian one,

$$g(x) = T(x) \times \eta . \tag{2.40}$$

In particular, since $T(x)$ is always positive-definite, we have the local isomorphism $\hat{P}(3.1) \approx P(3.1)$.

The same Ref. 6(a) has constructed the operator version of the *isodual Poincaré–Santilli isodual isosymmetry* $\hat{P}^d(3.1) \approx P^d(3.1)$, whose classical realization is the universal symmetry of the isodual Riemannian spaces $\Re^d$ over R^d.

In summary, the geometries significant in this paper are: the *conventional Euclidean, Minkowskian and Riemannian geometries,* which are used for the characterization of matter; and the *isodual Euclidean, Minkowskian and Riemannian geometries,* which are used for the characterization of antimatter.

The reader can now see the achievement of axiomatic compatibility between gravitation and electroweak interactions[10(a)] which is permitted by the isodual theory of antimatter. In fact, the latter is treated via negative-definite energy-momentum tensors, thus being compatible with the negative-energy antimatter solutions of electroweak interactions.

3. Classical Isodual Theory of Antimatter

3.1. *Fundamental assumption*

As it is well known, the contemporary treatment of matter is characterized by *conventional mathematics*, here referred to conventional numbers, fields, spaces, etc. with *positive unit and norm*, thus having conventional positive characteristics of mass, energy, time, etc.

In this paper we study the following:

Hypothesis 3.1. *Antimatter is characterized by the isodual mathematics, that with isodual numbers, fields, spaces, etc. thus having negative-definite units and norms. All characteristics of matter therefore change sign for antimatter represented via isoduality.*

The above hypothesis evidently provides the correct conjugation of the charge at the desired classical level. However, by no means, the sole change of the sign of the charge is sufficient to ensure a consistent classical representation of antimatter. To achieve consistency, the theory must resolve the main problematic aspect of current classical treatments of antimatter, the fact that their operator image is not the correct charge conjugation of that of matter, as evident from the existence of a single quantization procedure (Sec. 1).

It appears that the above problematic aspect is indeed resolved by the isodual theory. The main reason is that, jointly with the conjugation of the charge, isoduality also conjugates *all* other physical characteristics of matter. This implies *two* channels of quantization, the conventional one for matter and a new *isodual* *quantization* for antimatter (see App. A) such that its operator image is indeed the charge conjugate of that of matter.

In this section we shall study the physical consistency of the theory in its classical formulation. The novel isodual quantization, the equivalence of isoduality and charge conjugation and related operator issues are studied in Refs. 5 and 10.

To begin our analysis, we note that Hypothesis 3.1 removes the traditional obstacles against negative energies and masses. In fact, *particles with negative masses and energies referred to negative units are fully equivalent to particles with positive masses and energies referred to positive units.* Moreover, as we shall see shortly, particles with negative energy referred to negative units behave in a fully physical way. This has permitted the study in Ref. 10 of the possible elimination of necessary use of second quantization for the quantum characterization of antiparticles, as the reader should expect because our main objective is the achievement of equivalent treatments for particles and antiparticles at *all levels*, thus including first quantization.

Hypothesis 3.1 also resolves the additional, well known, problematic aspects of motion backward in time. In fact, *time moving backward referred to a negative unit is fully equivalent on grounds of causality to time moving forward referred to a positive unit.* This confirms the plausibility of the first conception of antiparticles by Stueckelberg and others as moving backward in time (see the historical analysis of Ref. 2), and creates new possibilities for the ongoing research on the so-called "time machine" to be studied in separate works.

In this section we construct the classical isodual theory of antimatter at the Galilean, relativistic and gravitational levels, prove its axiomatic consistency and verify its compatibility with available classical experimental evidence (that on electromagnetic interactions only). We also identify the prediction of the isodual theory that antimatter in the field of matter experiences gravitational repulsion (antigravity), and point out the ongoing efforts for its future experimental resolutions.[12,15] We finally confirm the emission by antimatter of the *isodual electromagnetic waves*, first identified at the operator level in Ref. 10(a), which coincide with the conventional waves emitted by matter under all known interactions, except gravitation. For completeness, the classical isodual Lagrangian and Hamiltonian mechanics are provided in the Appendix as the foundation of the isoquantization of the recent papers.[10]

3.2. *Representation of antimatter via the classical isodual Galilean relativity*

We now introduce the *isodual Galilean relativity* as the most effective way for the classical nonrelativistic characterization of antimatter according to Hypothesis 3.1.

The study can be initiated with the isodual representation of antimatter at the most primitive dynamical level, that of Newton's equation. Once a complete symmetry between the treatment of matter and antimatter is reached at the Newtonian level, it is expected to persist at all subsequent levels.

The conventional *Newton's equations* for a system of N pointlike *particles* with (nonnull) masses m_a, $a = 1, 2, \ldots, N$, in exterior conditions in vacuum are given by the familiar expression

$$m_a \times dv_{ka}/dt = F_{ka}(t, r, v)\,, \quad r = \{x, y, z\}\,,$$
$$a = 1, 2, \ldots, N\,, \quad v = dr/dt\,, \tag{3.1}$$

defined on the seven-dimensional Euclidean space $E_{\rm Tot}(t, r, v) = E(t, R_t) \times E(r, \delta, R_r) \times E(v, \delta, R_v)$ with corresponding seven-dimensional total unit $I_{\rm Tot} = I_t \times I_r \times I_v$, where one usually assumes $R_r = R_v$, $I_t = 1$, $I_r = I_v = {\rm diag}(1, 1, 1)$.

The *isodual Newton equations* here submitted for the representation of n pointlike antiparticles in vacuum are defined on the isodual space

$$E^d(t^d, r^d, v^d) = E^d(t^d, R^d_t) \times E^d(r^d, \delta^d, R^d) \times E^d(v^d, \delta^d, R^d)\,, \tag{3.2}$$

with total isodual unit $I^d_{\rm Tot} = I^d_t \times I^d_r \times I^d_v$, $I^d_t = -1$, $I^d_r = I^d_v = -\,{\rm diag}(1, 1, 1)$, and can be written for (nonnull) *isodual masses* $m^d_a = -m_a$)

$$m^d_a \times^d d^d v^d_{ka} /^d d^d t^d = F^d_{ka}(t^d, r^d, v^d)\,, \quad k = x,\ y,\ z,\ a = 1, 2, \ldots, N\,. \tag{3.3}$$

It is easy to see that, when projected in the original space $E(t, r, v)$, isoduality changes the sign of all physical characteristics, as expected. It is also easy to see that the above isodual equations are anti-isomorphic to the conventional forms, as desired.

We now introduce the *isodual Galilean symmetry* $G^d(3.1)$ as the step-by-step isodual image of the conventional symmetry $G(3.1)$ (see, e.g. Ref. 16). By using conventional symbols for the Galilean symmetry of a system of N particles with nonnull masses m_a, $a = 1, 2, \ldots, N$, $G^d(3.1)$ is characterized by *isodual parameters and generators*

$$\begin{aligned}
w^d &= (\theta^d_k, r^{kd}_o, v^{kd}_o, t^d_o) = -w\,,\\
J^d_k &= \sum{}_{aijk} r^d_{ja} \times^d p^k_{ja} = -J_k\,,\\
P^d_k &= \sum{}_a p^d_{ka} = -P_k\,,\\
G^d_k &= \sum{}_a (m^d_a \times^d r^d_{ak} - t^d \times p^d_{ak})\,,\\
H^d &= \frac{1}{2}^d \times^d \sum{}_a p^d_{ak} \times^d p^{kd}_a + V^d(r^d) = -H\,,
\end{aligned} \tag{3.4}$$

equipped with the *isodual commutator* (A.11), i.e.

$$\begin{aligned}
[A^d, B^d]^d = \sum{}_{a,k} [&(\partial^d A^d /^d \partial^d r^{kd}_a) \times^d (\partial^d B^d /^d \partial^d p^d_{ak})\\
&- (\partial^d B^d /^d \partial^d r^{kd}_a) \times^d (\partial^d A^d /^d \partial^d p^d_{ak})] = -[A, B]\,.
\end{aligned} \tag{3.5}$$

In accordance with rule (2.26), the structure constants and Casimir invariants of the isodual Lie algebra $G^d(3.1)$ are negative-definite. From rule (2.27), if $g(w)$ is an element of the (connected component) of the Galilei group $G(3.1)$, its isodual is characterize by

$$g^d(w^d) = e^{d^{-i^d \times^d w^d \times^d X^d}} = -e^{i\times(-w)\times X} = -g(-w) \in G^d(3.1)\,. \tag{3.6}$$

The *isodual Galilean transformations* are then given by

$$\begin{aligned} t^d &\to t'^d = t^d + t^d_o = -t'\,, \\ r^d &\to r'^d = r^d + r^d_o = -r'\,, \end{aligned} \tag{3.7}$$

$$\begin{aligned} r^d &\to r'^d = r^d + v^d_o \times^d t^d_o = -r'\,, \\ r^d &\to r'^d = R^d(\theta^d) \times^d r^d = -R(-\theta)\,, \end{aligned} \tag{3.8}$$

where $R^d(\theta^d)$ is an element of the *isodual rotational symmetry* first studied in the original proposal.[3]

The desired classical nonrelativistic characterization of antimatter is therefore given by imposing the $G^d(3.1)$ invariance of isodual equations (3.3). This implies, in particular, that the equations admit a representation via the isodual Lagrangian and Hamiltonian mechanics outlined in App. A.

We now verify that the above isodual representation of antimatter is indeed consistent with available classical experimental knowledge for antimatter, that under electromagnetic interactions. Once this property is established at the primitive Newtonian level, its verification at all subsequent levels of study is expected from mere compatibility arguments.

Consider a conventional, classical, massive *particle* and its *antiparticle* in exterior conditions in vacuum. Suppose that the particle and antiparticle have charge $-e$ and $+e$, respectively (say, an *electron* and a *positron*), and that they enter into the gap of a magnet with constant magnetic field **B**.

As it is well known, visual experimental observation establishes that particles and antiparticles under the same magnetic field have spiral trajectories of *opposite orientation*. But this behavior occurs for the *representation of both the particle and its antiparticle in the same Euclidean space*. The situation under isoduality is different, as described by the following:

Lemma 3.1. *The trajectory of a charged particle in Euclidean space under a magnetic field and the trajectory of the corresponding antiparticle in isodual Euclidean space coincide.*

Proof. Suppose that the particle has negative charge $-e$ in Euclidean space $E(r, \delta, R)$, that is, the value $-e$ is defined with respect to the positive unit $+1$ of the underlying field of real numbers $R = R(n, +, \times)$. Suppose that the particle is under the influence of the magnetic field **B**. The characterization of the corresponding antiparticle via isoduality implies the reversal of the sign of all physical quantities, thus yielding the charge $(-e)^d = +e$ in the isodual Euclidean space $E^d(r^d, \delta^d, R^d)$, as well as the reversal of the magnetic field $B^d = -B$, although now defined with respect to the negative unit $(+1)^d = -1$. It is then evident that the trajectory of a particle with charge $-e$ in the field B defined with respect to the unit $+1$ in Euclidean space and that for the antiparticle of charge $+e$ in the field $-B$ defined with respect to the unit -1 in isodual Euclidean space coincide. q.e.d.

An aspect of Theorem 3.1 which is particularly important for this paper is given by the following:

Corollary 3.1. (A) *Antiparticles reverse their trajectories when projected from their isodual space into the conventional space.*

Lemma 3.1 assures that isodualities permit the representation of the correct trajectories of antiparticles as physically observed, despite their negative energy, thus providing the foundations for a consistent representation of antiparticles at the level of *first* quantization studied in papers.[10] Moreover, Lemma 3.1 tells us that the trajectories of antiparticles may *appear* to exist in our space while in reality they may belong to an independent space, the isodual Euclidean space, coexisting with our own space.

Needless to say, the property of Corollary 3.1(A) is only a novel *mathematical* formulation of a well known physical behavior already treated in various ways, e.g. via Stueckelberg–Feynman path integrals, quantum field theory, etc.

To verify the validity of the isodual theory at the level of Newtonian laws of electromagnetic phenomenology, let us consider the *repulsive* Coulomb force among two *particles of negative* charges $-q_1$ and $-q_2$ in $E(r, \delta, R)$,

$$F = K \times (-q_1) \times (-q_2)/r \times r > 0\,, \tag{3.9}$$

where the operations of multiplication $\times$ and division $/$ are the conventional ones of the underlying field $R(n, +, \times)$. Under isoduality to $E^d(r^d, \delta^d, R^d)$ we have

$$F^d = K^d \times^d (-q_1)^d \times^d (-q_2)^d /^d r^d \times^d r^d = -F < 0\,, \tag{3.10}$$

where $\times^d = -\times$ and $/^d = -/$ are the isodual operations of the underlying field $R^d(n^d, +, \times^d)$.

But the isodual force $F^d = -F$ occurs in the isodual Euclidean space and it is therefore defined with respect to the unit -1. As a result, isoduality correctly represents the *repulsive* character of the Coulomb force for two *antiparticles* with *positive* charges.

The Coulomb force between a *particle* and an *antiparticle* can only be computed by *projecting the antiparticle in the conventional space of the particle or vice-versa.* In the former case we have

$$F = K \times (-q_1) \times (-q_2)^d/r \times r < 0\,, \tag{3.11}$$

thus yielding an *attractive* force, as experimentally established. In the projection of the particle in the isodual space of the antiparticle we have

$$F^d = K^d \times^d (-q_1) \times^d (-q_2)^d /^d r^d \times^d r^d > 0\,. \tag{3.12}$$

But this force is now referred to the unit -1, thus resulting to be again *attractive.*

In conclusion, the isodual Galilean relativity correctly represents the electromagnetic interactions of antimatter at the classical Newtonian level.

3.3. *Representation of antimatter via the isodual special relativity*

We now introduce the *isodual special relativity* as the best way to represent classical relativistic antimatter according to Hypothesis 3.1.

In essence, the conventional special relativity (see, e.g. Pauli's historical account[17]) is constructed on the fundamental four-dimensional unit of the Minkowski space $I = \text{diag}\{1,1,1\},1)$, which represents the dimensionless units of space $\{+1,+1,+1\}$, and the dimensionless unit of time $+1$, and is the unit of the Poincarè symmetry $P(3.1)$. The isodual special relativity is characterized by the map

$$I = \text{diag}(\{1,1,1\},1) > 0 \to I^d = -\,\text{diag}(\{1,1,1\},1) < 0\,. \tag{3.13}$$

namely, it is based on *negative units of space and time.* The isodual special relativity is then expressed by the isodual image of *all* mathematical and physical aspects of the conventional relativity in such a way to admit the negative-definite quantity I^d as the correct left and right unit.

This implies the reconstruction of the entire mathematics of the special relativity with respect to the single, common, four-dimensional unit I^d, including: the *isodual field* $R^d = R^d(n^d, +^d, \times^d)$ of *isodual numbers* $n^d = n \times I^d = -n \times I$ with fundamental unit $I^d = -\,\text{diag}(1,1,1=,1)$; the *isodual Minkowski space* $M^d(x^d, \eta^d, R^d)$ with isodual coordinates $x^d = x \times I^d$, isodual metric $\eta^d = -\eta$ and basic invariant over R^d

$$(x-y)^{d2d} = [(x^\mu - y^\mu) \times \eta^d_{\mu\nu} \times (x^\nu - y^\nu) \times I^d \in R^d\,; \tag{3.14}$$

the fundamental *isodual Poincarè* symmetry[6]

$$P^d(3.1) = L^d(3.1) \times^d T^d(3.1)\,, \tag{3.15}$$

where $L^d(3.1)$ is the *isodual Lorentz symmetry,* $\times^d$ is the *isodual direct product* and $T^d(3.1)$ represents the *isodual translations,* whose classical formulation is given by a simple relativistic extension of the isodual Galilean symmetry of the preceding section.

The algebra of the connected component $P_+^{\uparrow d}(3.1)$ of $P^d(3.1)$ can be constructed in terms of the isodual parameters $w^d = \{-w_k\} = \{-\theta, -v, -a\}$ and isodual generators $X^d = -X = \{-X_k\} = \{-M_{\mu\nu}, -P_\mu\}$, where the factorization by the four-dimensional unit I is understood. The isodual commutator rules are given by

$$[M^d_{\mu\nu}, M^d_{\alpha\beta}]^d = i^d \times^d (\eta^d_{\nu\alpha} \times^d M^d_{\mu\beta} - \eta^d_{\mu\alpha} \times^d M^d_{\nu\beta} - \eta^d_{\nu\beta} \times^d M^d_{\mu\alpha} + \eta^d_{\mu\beta} \times^d \hat{M}^d_{\alpha\nu}), \tag{3.16}$$

$$[M^d_{\mu\nu}, p^d_\alpha]^d = i^d \times^d (\eta^d_{\mu\alpha} \times^d p^d_\nu - \eta^d_{\nu\alpha} \times^d p^d_\mu), [p^d_\alpha, p^d_\beta]^d = 0. \tag{3.17}$$

The isodual group $P^{\uparrow d}_+(3.1)$ has a structure similar to that of Eqs. (3.6). These results then yield the following:

Lemma 3.2. *The classical isodual Poincarè transforms are given by*

$$\begin{aligned}
x^{1d\prime} &= x^{1d} = -x^1, \\
x^{2d\prime} &= x^{2d} = -x^2, \\
x^{3d\prime} &= \gamma^d \times^d (x^{3d} - \beta^d \times^d x^{4d}) = -x^{3\prime}, \\
x^{4d\prime} &= \gamma^d \times^d (x^{4d} - \beta^d \times^d x^{3d}) = -x^{4\prime}, \\
x^{d\mu\prime} &= x^{d\mu} + a^{d\mu\, d} = -x^{\mu\prime}, \\
x^{d\mu\prime} &= \pi^d \times^d x^d = -\pi \times x = (-r, x^4), \\
\tau^d \times^d x^d &= -\tau \times x = -(r, -x^4),
\end{aligned} \tag{3.18}$$

where

$$\beta^d = v^d /^d c^d_o = -\beta, \quad \beta^{d2d} = -\beta^2, \quad \gamma^d = -(1-\beta^2)^{-1/2}. \tag{3.19}$$

and the use of the isodual operations (quotient, square roots, etc.), is implied.

The *isodual spinorial covering* of the Poincarè symmetry $\mathcal{P}^d(3.1) = \mathrm{SL}^d(2.C^d) \times^d T^d(3.1)$ can then be constructed via the same methods.

The basic postulates of the isodual special relativity are also a simple isodual image of the conventional postulates. For instance, the *maximal isodual causal speed* is the speed of light in M^d, i.e.

$$V_{\max} = c^d_o = -c_o, \tag{3.20}$$

with the understanding that it is referred to a *negative-definite unit*, thus being fully equivalent to the conventional maximal speed c_o referred to a positive unit. A similar situation occurs for all other postulates.

A fundamental property of the isodual theory is the following:

Theorem 3.1. *The line elements of metric or pseudo-metric spaces are isoselfdual (Definition 2.1), i.e. they coincide with their isodual images. In particular, isoduality leaves invariant the fundamental space–time interval of the special relativity,*

$$\begin{aligned}
x^{d\,2d} &= (x^{\mu d} \times^d \eta^d_{\mu\nu} \times^d x^{\nu d}) \\
&= (-x^1x^1 - x^2x^2 - x^3x^3 - x^4x^4) \times (-I) \\
&\equiv (x^1x^1 + x^2x^2 + x^3x^3 - x^4x^4) \times I = x^2.
\end{aligned} \tag{3.21}$$

The above novel property evidently assures that conventional relativistic laws for matter are also valid for antimatter represented via isoduality, since they share the same fundamental space–time interval.

The above property illustrates that the isodual map is so natural to creep in unnoticed. The reason why, after about a century of studies, the isoduals of the Galilean, special and general relativities escaped detection is that their identification required the prior knowledge of *new numbers*, those with a negative unit.

Note that the use of the *two* Minkowskian metrics η and $\eta^d = -\eta$ has been popular since Minkowski's times. The point is that both metrics are referred to the *same* unit I, while in the isodual theory one metric is referred to the unit I on the field $R(n, +, \times)$ of conventional numbers, and the other metric is referred to the new unit $I^d = -I$ on the new field $R^d(n^d, +^d, \times^d)$ of isodual numbers $n^d = n \times I^d$.

The novelty of the isodual relativities is illustrated by the following:

Lemma 3.3. *Isodual maps and space–time inversions are inequivalent.*

In fact, space–time inversions are characterized by the change of sign $x \to -x$ *by always preserving the original metric referred to positive units*, while isoduality implies the map $x \to x^d = -x$ *but now referred to an isodual metric* $\eta^d = -\eta$ *with negative units* $I^d = -I$. Thus, *space–time inversions occur in the same space while isoduality implies the map to a different space.* Moreover, as shown by Lemma 3.2 isodualities interchange the space and time inversions.

We now introduce, apparently for the first time, the *isodual electromagnetic waves* and related *isodual Maxwell's equations*

$$\begin{aligned} F^d_{\mu\nu} = \partial^d A^d_\mu /^d \partial^d x^{\nu d} - \partial^d A^d_\nu /^d \partial^d x^{d\mu} = -F_{\mu\nu}\,, \\ \partial^d_\lambda F^d_{\mu\nu} + \partial^d_\mu F^d_{\nu\lambda} + \partial^d_\nu F^d_{\lambda\mu} = 0\,, \\ \partial^d_\mu F^{d\mu\nu} = -J^{d\nu}\,, \end{aligned} \tag{3.22}$$

which characterize the phenomenology of electromagnetic waves emitted by antimatter according to the isodual theory.

As one can verify, the isodual electromagnetic waves are essentially equivalent to the conventional waves in the sense that their behavior for antimatter is essentially the same as the corresponding behavior of the conventional electromagnetic waves for the case of matter.

Their primary differences is the behavior under *gravitation.* In fact, as we shall see, isodual electromagnetic waves are *attracted* by the gravitational field off *antimatter.* However, *isodual* waves in the gravitational field of *matter* (or vice-versa) experience a *repulsion.*

As identified earlier, *the isodual transforms and the space–time inversions are mathematically and physical different maps.* In this paper we have studied the isodual maps. The space–time inversions of the isodual electromagnetic waves will be studied in future works. Their importance is evidently due to the possible identification of physical differences between conventional and isodual electromagnetic waves which may assist in their experimental detection.

The interested reader is encouraged to verify that the physical consistency in the representation of electromagnetic interactions by the isodual Galilean relativity carries over in its entirety at the level of the isodual special relativity, thus confirming the plausibility of the isodual theory of antimatter also at the classical relativistic level.

3.4. *Representation of antimatter via the isodual general relativity*

We finally introduce the *isodual general relativity* as the most effective gravitational characterization of antimatter according to Hypothesis 3.1. The new image is also characterized by the isodual map of *all* aspects of the conventional relativity (see, e.g. Ref. 18), now defined on the isodual Riemannian spaces $\Re^d(x^d, g^d, R^d)$ of Subsec. 2.7.

The primary motivation warranting the study of the above new image of general relativity is the following. A problematic aspect in the use of the Riemannian geometry for the representation of *antimatter* is the *positive-definite energy–momentum tensor.*

In fact, such a representation has an operator image which is not the charge conjugate of that of matter, does not admit the negative-energy solutions as needed for operator treatments of antiparticles, and may be one of the reasons for the lack of achievement until now of a consistent grand unification inclusive of gravitation. After all, gauge theories are bona-fide field theories which, as such, admit both positive- and negative-energy solutions, while the contemporary formulation of gravity admits only positive-energy states, with an evident structural incompatibility.

Isoduality offers a new possibility for a resolution of these shortcomings. In fact, the isodual Riemannian geometry is defined on the isodual field of real numbers $R^d(n^d, +^d, \times^d)$ for which *the norm is negative-definite*, Eq. (2.11). As a result, *all quantities which are positive in Riemannian geometry become negative under isoduality, thus including the energy–momentum tensor.*

Explicitly, the energy–momentum tensor of the isodual electromagnetic waves, Eqs. (3.22), is given by

$$\begin{aligned} T^d_{\mu\nu} &= (4m)^{-1d} \times^d (F^d_{\mu\alpha} \times^d F^d_{\alpha_\nu} + (1/4)^{-1d} \times^d g^d_{\mu\nu} \times^d F^d_{\alpha\beta} \times^d F^{d\alpha\beta}) \\ &= -T_{\mu\nu} \,. \end{aligned} \tag{3.23}$$

As such, antimatter represented in isodual Riemannian geometry has *negative-definite energy–momentum tensor* and other physical quantities, as desired. The above occurrence is the classical foundation of the grand unified theory proposed in Ref. 10(b).

For completeness, we mention here the *isodual Einstein* equations for the *exterior gravitational problem of antimatter in vacuum*

$$G^d_{\mu\nu} = R^d_{\mu\nu} - \frac{1}{2}^d \times^d g^d_{\mu\nu} \times^d R^d = k^d \times^d T^d_{\mu\nu} \,, \tag{3.24}$$

We also mention the field equations characterized by the *Freud identity*[19] of the Riemannian geometry (reviewed by Pauli[17] and then generally forgotten)

$$R^\alpha_\beta - \frac{1}{2} \times \delta^\alpha_\beta \times R - \frac{1}{2} \times \delta^\alpha_\beta \times \Theta = U^\alpha_\beta + \partial V^{\alpha\rho}_\beta / \partial x^\rho = k \times (t^\alpha_\beta + \tau^\alpha_\beta)\,, \qquad (3.25)$$

where

$$\Theta = g^{\alpha\beta} g^{\gamma\delta} (\Gamma_{\rho\alpha\beta}\Gamma^\rho_{\gamma\beta} - \Gamma_{\rho\alpha\beta}\Gamma^\rho_{\gamma\delta})\,, \qquad (3.26)$$

$$U^\alpha_\beta = -\frac{1}{2}\frac{\partial\Theta}{\partial g^{\alpha\beta}_{|\alpha}} \hat{g}^{\alpha\beta} \uparrow_\beta\,, \qquad (3.27)$$

$$V^{\alpha\rho}_\beta = \frac{1}{2}[g^{\gamma\delta}(\delta^\alpha_\beta \Gamma^\rho_{\alpha\equiv} - \delta^\rho_\beta \Gamma^\rho_{\gamma\delta}) + (\delta^\rho_\beta g^{\alpha\gamma} - \delta^\alpha_\beta g^{\rho\gamma})\Gamma^\delta_{\gamma\delta} + g^{\rho\gamma}\Gamma^\alpha_{\beta\gamma} - g^{\alpha\gamma}\Gamma^\rho_{\delta\gamma}]\,, \qquad (3.28)$$

which indicate the apparent need for a no-where null source in the exterior problem in vacuum, contrary to Einstein's original assumption.[17] As we shall see shortly, the forgotten Freud identity appears to have a truly fundamental role for quantitative studies of antigravity.

The isodual version of Eqs. (3.25)

$$R^{\alpha d}_\beta - \frac{1}{2}^d \times^d \delta^{\alpha d}_\beta \times^d R^d - \frac{1}{2}^d \times^d \delta^{\alpha d}_\beta \times^d \Theta^d = k^d \times^d (t^{\alpha d}_\beta + \tau^{\alpha d}_\beta) \qquad (3.29)$$

are then suggested for the study of the exterior problem of antimatter in vacuum (see Ref. 11 for interior profiles).

It is instructive for the interested reader to verify that the physical consistency of the isodual theory at the preceding Galilean and relativistic levels carries over at the gravitational level, including the *attractive* character of antimatter–antimatter systems and their correct behavior under electromagnetic interactions.

Note in the latter respect that *curvature in isodual Riemannian spaces is negative-definite* (Subsec. 2.7). Nevertheless, such negative value for antimatter–antimatter systems is referred to a negative unit, thus resulting in attraction.

The universal symmetry of the isodual general relativity, the *isodual Poincaré–Santilli isosymmetry* $\hat{P}^d(3.1) \approx P^d(3.1)$, has been introduced at the operator level in Ref. 6(a). The construction of its classical counterpart is straightforward, although it cannot be reviewed here because it requires the broader *isotopic mathematics*, (that based on generalized unit), and its isodual image.

3.5. *The prediction of antigravity, isodual electromagnetic waves, and the "space–time machine"*

We close this paper with the indication that studies on antimatter have so far reaching implications, to invest in a direct or indirect way our entire mathematical and physical knowledge. At any rate, studies on antimatter are broader than those of matter evidently because the latter are included in the former, but not the other way around.

To begin, we recall that the *isodual theory of antimatter predicts the existence of antigravity* (defined as the reversal of the sign of the curvature tensor in our space–time) evidently for antimatter in the field of matter, or vice-versa.

The prediction originates at the primitive Newtonian level, persists at all subsequent levels of study,[10] and it is here identified as a consequence of the theory, without any claim on its possible validity due to the lack of experimental knowledge at this writing on the gravitational behavior of antiparticles.

In essence, antigravity is predicted by the interplay between conventional geometries and their isoduals and, in particular, by Corollary 3.1(A) according to which the trajectories we observe for antiparticles are the *projection* in our space–time of the actual trajectories in isodual space. The use of the same principle for the case of the gravitational field then yields antigravity.

Consider the Newtonian gravitational force of two conventional (thus positive) masses m_1 and m_2

$$F = -G \times m_1 \times m_2 / r \times r < 0 , \tag{3.30}$$

where the minus sign has been added for similarity with law (3.19).

Within the context of contemporary theories, the masses m_1 and m_2 remain positive irrespective of whether referred to a particle or an antiparticle. This yields the well-known *Newtonian gravitational attraction* among any pair of masses, whether for particle–particle, antiparticle–antiparticle or particle–antiparticle.

Under isoduality the situation is different. First, the particle–particle gravitational force evidently yields law (3.30). The case of antiparticle–antiparticle under isoduality yields the different law

$$F^d = -G^d \times^d m_1^d \times^d m_2^d /^d r^d \times^d r^d > 0 . \tag{3.31}$$

But this force is defined with respect to the negative unit -1. The isoduality therefore correctly represents the *attractive* character of the *gravitational* force among two *antiparticles.*

The case of particle–antiparticle under isoduality requires the *projection* of the antiparticle in the space of the particle, as it is the case for the electromagnetic interactions of Corollary 2.1(A)

$$F = -G \times m_1 \times m_2^d / r \times r > 0 , \tag{3.32}$$

which is now *repulsive*, thus illustrating the prediction of antigravity. Similarly, if we project the particle in the space of the antiparticle we have

$$F^d = -G^d \times^d m_1 \times^d m_2^d /^d r^d \times^d r^d < 0 , \tag{3.33}$$

which is also repulsive because referred to the unit -1.

We can summarize the above results by saying that *the classical representation of antiparticles via isoduality renders gravitational interactions equivalent to the* *electromagnetic ones, in the sense that the Newtonian gravitational law becomes equivalent to the Coulomb law. Note the impossibility of achieving these results without isoduality.*

The interested reader can verify the persistence of the prediction of antigravity at the relativistic and gravitational levels.

In our view, an intriguing argument favoring the existence of antigravity is given by studies on the "origin" of the gravitational field, rather than on its "description." We are here referring to theories assuming that the mass of all particles constituting a body has a primary electromagnetic origin, with second-order contributions from weak and strong interactions, as studied, e.g. in Ref. 23 for the case of the π^o meson. These theories permit the *identification* (rather than the "unification") *of the gravitational field with the fields originating mass.*[23] A primary difference is that the electromagnetic field is represented by rank-one tensorial (vectorial) equations, while the gravitational aspect of the same field is represented by rank-two tensorial equations.

The theory can be realized by identifying the t-tensor in Eqs. (3.25) with the electromagnetic field originating mass, and the τ-tensor with the weak and strong contributions. Since the latter are short range, in the exterior problem in vacuum we would only have the *identification of the gravitational and electromagnetic fields.*

In turn, the latter identification evidently implies the equivalence of the respective phenomenologies, thus including the capability of attraction and repulsion for both the electromagnetic and the gravitational fields.

Note that the above identification implies the existence of a first-order nowhere null electromagnetic source in vacuum also for bodies with null total charge (see Ref. 23 for details) which is fully in line with the field equations as characterized by the forgotten Freud identity, Eqs. (3.26)–(3.28).

A forceful nature of the above argument is due to the fact that the lack of antigravity would imply the lack of identity of gravitational and electromagnetic interactions. In turn, this would require a serious revision of the contemporary theory of elementary particles in such a way to *avoid* a first-order electromagnetic origin of their mass.

As a concrete illustration, the identification of the electromagnetic and gravitational fields in the exterior problem in vacuum was worked out in details in Ref. 23 for the case of the π^o meson. Even though its total electromagnetic data are null, this particle is made up of two opposite charges in very high dynamical conditions with respect to each other. The total electromagnetic field $t_{\mu\nu}$ in the exterior of the particle was computed via relativistic techniques (including retarded and advanced potentials) and its t_{oo}-component resulted to be so large to account for most of the rest energy of the π^o, resulting in the exterior field equations in vacuum $G_{\mu\nu} = kt_{\mu\nu}$. The additional relatively small contributions from the weak and strong interactions of the interior problem yielded the tensor $\tau_{\mu\nu}$ with interior field equations $G_{\mu\nu} = k(t_{\mu\nu} + \tau_{\mu\nu})$.

As an incidental contribution, the model yielded an explicit representation of the gravitational (exterior) mass characterized by the volume integral of t_{oo} and inertial (interior) mass characterized by the volume integral of $t_{oo} + \tau_{oo}$, as well as an explanation of their differences (due to the short range nature of the weak and strong forces).

The model also establishes the validity of the forgotten Freud identity[19] *in vacuum.* We are here referring to the emergence for the π^o of a first-order source $t_{\mu\nu}$ in its exterior *which is nowhere null even though the particle has null total change, and null multipole moments,* the extension to the $\pi^{+/-}$ protons, neutrons and all macroscopic masses being consequential. By comparison, field equations for masses with null total electromagnetic data are today written in the form $G_{\mu\nu} = 0$.

The interested reader is encouraged to inspect paper[23] and verify that the lack of existence of antigravity would imply the lack of identification of electromagnetic and gravitational fields and of their respective phenomenologies. In turn, the latter is only possible for masses with an ignorable electromagnetic origin. Still in turn, the latter requirement would imply the necessary abandonment of the entire current theory on hadrons, including the abandonment of quark theories and QCD, and their replacement with a new hypothetical theory in which the mass of all elementary particles does not possess an appreciable electromagnetic origin.

We should indicate for completeness that the identification of the gravitational and electromagnetic fields appears to be disproved by the assumption that quarks are physical constituents of hadrons owing to the known large value of their "masses." However, the latter are solely defined in the mathematical unitary space, while the only masses which can be permitted for the characterization of gravity are the eigenvalues of the second-order Casimir invariant of the Poincaré symmetry. Since the latter identification is impossible for quarks, as well known, the large values of quark "masses" is inapplicable to the above considerations on the possible identification of electromagnetism and gravitation at the level of the *structure* of matter, because the latter must solely occur *in our physical space–time* without any consideration of unitary spaces used for the hadronic classification.

Various arguments *against* the existence of antigravity exist in the literature (see, e.g. the review Ref. 22), such as those by Morrison, Schiff and Good, and others. It should be indicated that these arguments *do not* apply under isodualities owing to their essential dependence on positive units, as one can verify.

The argument against antigravity based on the positronium[22] also do not apply under isoduality, because bound systems of elementary particle–antiparticle are isoselfdual and, as such, the sign of their total energy is that of the field (observer) in which they are immersed, thus being attracted both fields of matter and antimatter, as shown in Ref. 10(a).

We can therefore state that the gravitational behavior of antiparticles is fundamentally unsettled at this writing. The true scientific resolution is evidently that via *experiments,* rather than via personal theoretical view in favor or against antigravity.

The possible alternatives are the following: (1) *Antiparticles are attracted by matter in the same way as particles,* as predicted by Einstein's field equations in vacuum, with the representation of antimatter via *positive-definite* energy–momentum tensors and the *lack* of the Freud identity; (2) *Antiparticles are repelled by matter in a way opposite to that of particles,* as predicted by the projection of exterior field equations (3.29) in the space of Eqs. (3.25), with the representation of antimatter via *negative-definite* energy–momentum tensors and the *use* of the Freud identity; (3) *Antiparticles in the field of matter are attracted, although in lesser amount as that for particles,* as predicted by certain intermediary theories (see Ref. 22 for details).

One of the first experiments on the gravity of antiparticles was done by Fairbank and Witteborn[20] via low energy positrons in *vertical* motion, although the measurements were inconclusive because of interferences from stray fields and other reasons.

Additional data on the gravity of antiparticles are those from the LEAR machine on antiprotons at CERN,[21] although these data too are inconclusive because of the excessive energy of the antiprotons as compared to the low value of gravitational effects, the sensitivity of the measures and other factors.

Santilli[7(a)] proposed the measure of the gravity of antiparticles via the use of a suitably collimated beam of very low energy positrons in *horizontal* flight in a vacuum-superconducting tube of sufficient length and diameter to yield a resolutory answer, that is, a displacement under gravity at the end of the flight up or down which is *visible by the naked eye.*

According to Mills,[12] the above experiment appears to be feasible with current technology via the use of μeV positrons and electrons in a horizontal vacuum-superconducting tube of about 100 m in length and 1 m in diameter for which stray fields and patch effects should be smaller than the gravitational deflection (if not, the problem is solved by a proportional increase of the diameter of the tube).

A number of additional experimental proposals to measure the gravity of antiparticles are available in the proceedings,[15] although their measures are more sophisticated, they require interferometric techniques, and the results are not "visible by the naked eyes" as those of test.[7(a),12]

This paper would be incomplete without the indication that the possible experimental detection of antigravity would have implications beyond our most vivid imagination.

A first illustration is given by the classical counterpart of the prediction of the "isodual photons" of Ref. 10(a), namely, the prediction (here submitted for the first time at the classical level) that *antimatter emits the novel "isodual electromagnetic waves"* with isodual fields Eqs. (3.22), which coincide with the conventional waves under all known interactions except gravitation. In fact, the isodual waves are predicted to experience a *gravitational repulsion* when in the field of matter evidently because of their negative-definite energy–momentum tensor, Eqs. (3.23). As a result, the possible existence of the novel isodual electromagnetic waves requires the existence of antigravity.

In turn, the possible experimental verification of the isodual electromagnetic waves would permit for the first time theoretical and experimental studies as to whether far away galaxies and quasars are made up of matter or of antimatter.

Yet another far reaching implication of antigravity would be the existence of a *space–time machine*,[7(b)] namely, a machine which would permit motion in space and time. Locomotion in this case is not given by the Newtonian principle of action and reaction as existing in all currently available forms of locomotions, but it is given instead by the *local alteration of the geometry of space*, called *geometric propulsion*, as permitted by sufficiently large amounts of sufficiently localized energy (see Ref. 9 for details). In turn, the modifications of space cannot occur without a corresponding alteration of time, thus implying motions in both space and time.

Geometric locomotion forward in time would be permitted by the use of sufficiently large amounts of positive energy, while that backward in time would be permitted by the use of negative energy. As far as we currently know, locomotion both forward and backward in time appears solely permitted by isoselfdual states (see Ref. 7(b) for details) and not for arbitrary matter or antimatter. Within such a setting, the so-called "barrier of the speed of light" has no mathematical or physical meaning because evidently applicable only to flat or curved space–time, while the geometric propulsion implies the local creation of a fundamentally different space–time whose maximal causal speed depends on the value of the used energy.

In short, the possible discovery of antigravity may one future day permit the conception of novel means for interstellar travel in which the speed of light is no longer a barrier.

Rather than being farfetched, it appears that one form of geometric locomotion is already realized in biological structures,[26] such as in the upward motion of the sap in very tall trees which cannot be explained via conventional means (e.g. capillary effects) due to the height. This biological locomotion occurs indeed without any Newtonian action and reaction and therefore, it is precisely a realization of the geometric propulsion herein considered.

All in all, it appears that the measure of the gravity of antiparticles has such mathematical, theoretical and experimental implications to dwarf by comparison any other possible physics experiments.

In closing we should indicate that studies on antimatter are at their first infancy, as indicated by the existence of only one meeting in the field, Ref. 15. As such, there is so much to be done.

Acknowledgments

The author would like to express his appreciation to all participants of the *International Workshop on Antimatter Gravity and Antihydrogen Atom Spectroscopy* held in Sepino, Molise, Italy, in May 1996, for invaluable critical comments. Particular thanks are also due to A. K. Aringazin, P. Bandyopadhyay, S. Kalla, J. V. Kadeisvili, N. Kamiya, A. U. Klimyk, M. Holzscheiter, J. P. Mills Jr., R. Miron,

R. Oehmke, G. Sardanashvily, H. M. Srivastava, T. Gill, Gr. Tsagas, N. Tsagas, C. Udriste and others for penetrating comments. Special thanks are finally due to D. V. Ahluwalia for an invaluable critical reading of the manuscript and for suggesting the addition of isodual space and time inversions.

Appendix A

A.1. *Isodual Lagrangian mechanics*

After having achieved in the main text the isodual theory of antimatter at the primitive Newtonian level, Eqs. (3.3) it may be of some value to outline in this appendix its analytic representation because it constitutes the foundations of the novel quantization for antimatter studied in the joint papers.[10]

A conventional (first-order) Lagrangian $L(t,x,v) = \frac{1}{2}mv^k v_k + V(t,x,v)$ on the configuration space $E(t,x,v) = E(t,R_t) \times E(r,\delta,R_r) \times E(v,\delta,R_v)$ of Newton's equations is mapped under isoduality into the negative value $L^d(t^d,r^d,v^d) = -L$ defined on isodual space $E^d(t^d,r^d,v^d)$ of Eq. (3.2). The *isodual Lagrange equations* are then given by

$$\frac{d^d}{d^d t^d} d \frac{\partial^d L^d(t^d,r^d,v^d)}{\partial^d v^{kd}} d - \frac{\partial^d L^d(t^d,r^d,v^d)}{\partial^d r^{kd}} d = 0\,. \tag{A.1}$$

All various aspects of the *isodual Lagrangian mechanics* can then be readily derived.

It is easy to see that Lagrange's equations change sign under isoduality and can therefore provide a *direct representation* (i.e. a representation without integrating factors) of the isodual Newton's equations,

$$\begin{aligned} &\frac{d^d}{d^d t^d} d \frac{\partial^d L^d(t^d,r^d,v^d)}{\partial^d v^{kd}} d - \frac{\partial^d L^d(t^d,r^d,v^d)}{\partial^d x^{kd}} d \\ &\quad = m_k^d \times^d d^d v_k^d /^d d^d t^d - F_k^{dSA}(t,r,v) = 0\,, \end{aligned} \tag{A.2}$$

where SA stands for *variational selfadjointness*, i.e. verification of the conditions to be derivable from a potential. The compatibility of the isodual Lagrangian mechanics with the primitive Newtonian results then follows.

A.2. *Isodual Hamiltonian mechanics*

The *isodual Hamiltonian* is evidently given by

$$H^d = p_k^d \times^d p^{dk} /^d (2m)^d + V^d(t^d,r^d,v^d) = -H\,. \tag{A.3}$$

It can be derived from (nondegenerate) isodual Lagrangians via a simple isoduality of the Legendre transforms and it is defined on the seven-dimensional carrier space (for one particle)

$$E^d(t^d,r^d,p^d) = E^d(t^d,R_t^d) \times E^d(r^d,\delta^d,R^d) \times E^d(p^d,\delta^d,R^d)\,. \tag{A.4}$$

The *isodual canonical action* is given by

$$A^{\circ d} = \int_{t_1}^{t_2} \left(p_k^d \times^d d^d r^{kd} - H^d \times^d d^d t^d\right)$$
$$= \int_{t_1}^{t_2} \left[R_\mu^{\circ d}(b^d) \times^d d^d b^{\mu d} - H^d \times^d d^d t^d\right], \tag{A.5}$$
$$R^\circ = \{p, 0\}, \quad b = \{x, p\}, \quad \mu = 1, 2, \ldots, 6.$$

Conventional variational techniques under simple isoduality then yield the *isodual Hamilton equations* which can be written in disjoint form

$$\frac{d^d x^{kd}}{d^d t^d} = \frac{\partial^d H^d(t^d, x^d, p^d)}{\partial^d p_k^d}, \qquad \frac{d^d p_k^d}{d^d t^d} = -\frac{\partial^d H^d(t^d, x^d, p^d)}{\partial^d x^{dk}}, \tag{A.6}$$

or in the unified notation

$$\omega_{\mu\nu}^d \times^d \frac{d^d b^{d\nu}}{d^d t^d} = \frac{\partial^d H^d(t^d, b^d)}{\partial^d b^{d\mu}}, \tag{A.7}$$

where $\omega_{\mu\nu}^d$ is the *isodual canonical symplectic tensor*

$$(\omega_{\mu\nu}^d) = (\partial^d R_\nu^{\circ d} /^d \partial^d b^{d\mu} - \partial^d R_\mu^{\circ d} /^d \partial^d b^{d\nu}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = -(\omega_{\mu\nu}). \tag{A.8}$$

Note that in matrix form the canonical symplectic tensor is mapped into the canonical Lie tensor, with intriguing geometric and algebraic implications studied elsewhere for brevity.

The isodual *Hamilton–Jacobi equations* are then given by

$$\partial^d A^{\circ d} /^d \partial^d t^d + H^d = 0,$$
$$\partial^d A^{\circ d} /^d \partial^d x_k^d - p_k^d = 0, \tag{A.9}$$
$$\partial^d A^{\circ d} /^d \partial^d p_k^d \equiv 0.$$

The *isodual Lie brackets* among two isodual functions A^d and B^d on $S^d(t^d, x^d, p^d)$ then become

$$[A^d, B^d]^d = \frac{\partial^d A^d}{\partial^d b^{d\mu}} d \times^d \omega^{d\mu\nu} \times^d \frac{\partial^d B^d}{\partial^d b^{d\nu}} d = -[A, B], \tag{A.10}$$

where

$$\omega^{d\mu\nu} = [(\omega_{\alpha\beta}^d)^{-}1]^{\mu\nu}, \tag{A.11}$$

is the *isodual Lie tensor*. The direct representation of the isodual Newton equations in the first-order form is self-evident.

In summary, all properties of the isodual theory at the Newtonian level carry over at the level of isodual Hamiltonian mechanics. In so doing, there is the emergence of a fundamental notion of these studies, the characterization of antimatter via *isodual space–time symmetries* nowadays called *Galilei-Santilli isodual symmetry* $G^d(3.1)$ for nonrelativistic treatments, the *Poincarè-Santilli isodual symmetry* $P^d(3.1)$ for relativistic treatments and the *Poincaré-Santilli isodual isosymmetry* for gravitational treatments.[6,13–15]

A.3. *Isodual naive quantization*

The isodual Hamiltonian mechanics and its underlying isodual symplectic geometry permit the identification of the novel *naive isodual quantization*

$$A^{od} \to -i^d \times^d \hbar^d \times^d Ln^d \psi^d(t^d, r^d)\,, \tag{A.12a}$$

$$\partial^d A^{od}/^d \partial^d t^d + H^d = 0 \to i^d \times^d \partial^d \psi^d /^d \partial^d t^d = H^d \times^d \psi^d = E^d \times^d \psi^d\,, \tag{A.12b}$$

$$\partial^d A^{od}/^d \partial^d x^{dk} - \hat{p}_k = 0 \to p^d_k \times^d \psi^d = -i^d \times^d \partial^d \psi^d\,, \tag{A.12c}$$

or more refined isodualities of symplectic quantization (see, e.g. Ref. 24 for the conventional case), which characterize a novel image of quantum mechanics for antiparticles, called *isodual quantum mechanics*, introduced in papers,[10] which is defined on the *isodual Hilbert space* $\mathcal{H}^d$ with isodual states $|\psi\rangle^d = -|\psi\rangle^\dagger$ and isodual inner product $\langle\psi|^d \times (-1) \times |\psi\rangle^d \times (-1)$ on the isodual complex field C^d with unit -1.

Note the compatibility of the classical and quantum isodual theories, e.g. the values of the energy remain negative after isodual quantization.

As one can see, isodual quantum mechanics originates from the invariance[10]

$$\langle\psi| \times (+1) \times |\psi\rangle \times (+1) \equiv \langle\psi|^d \times (-1) \times |\psi\rangle^d \times (-1)\,. \tag{A.13}$$

As a result, all physical laws holding for matter also hold for antimatter. The equivalence of charge conjugation and isoduality then follows (see Ref. 10 for details). Note that isoduality *replaces* charge conjugation at the operator level.

Note finally that, even though seemingly trivial, the above novel invariance of the Hilbert space has remained undetected throughout this century because it required *the prior discovery of new numbers with negative unit.*[4]

A.4. *Isodual Reinterpretation of Dirac, Majorana, Ahluwalia, and other Spinorial Representations*

Isoduality has permitted a novel interpretation of the *conventional* Dirac's equation[10] in which the negative-energy states are reinterpreted as belonging to the isodual images of conventional spaces, with explicit form

$$[\tilde{\gamma}^\mu \times (p_\mu - e \times A/c) + i \times m] \times \tilde{\Psi}(x) = 0\,, \tag{A.14a}$$

$$\tilde{\gamma}_k = \begin{pmatrix} 0 & \sigma^d_k \\ \sigma_k & 0 \end{pmatrix}, \qquad \tilde{\gamma}^4 = i \begin{pmatrix} I_s & 0 \\ 0 & I^d_s \end{pmatrix}, \tag{A.14b}$$

$$\{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2^d \times^d \eta^d_{\mu\nu}\,, \qquad \tilde{\Psi} = -\tilde{\gamma}_4 \times \Psi = i \times \begin{pmatrix} \Phi \\ \Phi^d \end{pmatrix}, \tag{A.14c}$$

which is defined on the following representation space with related total unit and symmetry[6]

$$M_{\text{Tot}} = \{M(x,\eta,R) \times S_{\text{spin}}\} \times \{M^d(x^d,\eta^d,R^d) \times^d S^d_{\text{spin}}\}, \tag{A.15a}$$

$$I_{\text{Tot}} = \{I_{\text{orb}} \times I_{\text{spin}}\} \times \{I^d_{\text{orb}} \times^d I^d_{\text{spin}}\}, \tag{A.15b}$$

$$S_{\text{Tot}} = \mathcal{P} \times \mathcal{P}^d = \{\text{SL}(2.C) \times T(3.1)\} \times \{\text{SL}^d(2.C^d) \times^d T^d(3.1)\}. \tag{A.15c}$$

A salient mathematical feature of the above structures is that they are all isoselfdual. As such, they are ideally suited to represent the superposition of particles and their antiparticles. A salient physical feature is the achievement of a representation of antiparticles fully valid at the level of *first quantization*, without any need for the second quantization exactly as it is the case for particles, and as expected by a theory of antimatter beginning at the *classical* level.

It should be noted that, as conventionally written, Dirac's equation *is not* isoselfdual because not sufficiently symmetric between (two-dimensional) states and their isoduals. It should also be recalled that the isodual theory was born via an inspection of Dirac's equation, the identification in γ_4 of an essential two-dimensional *negative unit*, and the reconstruction of the entire theory with respect to the latter new unit, resulting in the isodual mathematics of Sec. 2.

In summary, Dirac's[1] was forced to formulate the "hole theory" for antiparticles to eliminate unphysical behavior because he referred the *negative* energy states to the conventional *positive* unit, while their reformulation with respect to *negative* units yields fully physical results, thus avoiding the necessary use of second quantization, with consequential unbalance between particles and antiparticles.

In so doing, Dirac's historical representation $(1/2) \times (1/2)$ of the (spinorial covering of) the Poincaré symmetry $\mathcal{P}$ (which *is not* isoselfdual) is reinterpreted under isoduality as the representation $(1/2) \times (1/2)^d$ of $\mathcal{P} \times \mathcal{P}^d$ which is isoselfdual. Similarly, the conventional spinorial space $(1,2,0) + (0,1/2)$ is reinterpreted as the isoselfdual space $(1/2,0) + (1/2,0)^d$ which is also isoselfdual.

It is easy to see that the same isotopic reinterpretation applies for Majorana's spinorial representations[26(a)] (see also Refs. 26(b) and 26(c)) as well as Ahluwalia's broader spinorial representations[26(d)] (see also the subsequent paper[26(e)]) $(1/2,0)+(0,1/2)$ which are reinterpreted in the isoselfdual form $(1,2,0) + (1,2,0)^d$, thus extending their physical applicability to first quantization.

In the latter reinterpretation the representation $(1/2,0)$ is evidently referred to conventional spaces over conventional fields with unit $+1$, while the isodual representation $(1/2,0)^d$ is referred to the corresponding isodual spaces defined on isodual fields with unit -1. As a result, all quantities of the representation $(1/2,0)$ change sign under isoduality.

It should be finally indicated that Ahluwalia treatment of Majorana spinors has a deep connection with isoduality because the underlying Class II spinors have a *negative norm*[26(c)] precisely as it is the case for isoduality. As a result, the isodual reinterpretation under consideration here is quite natural and actually warranted for mathematical consistency (e.g. to have the topology characterized by a negative norm be compatible with the underlying fields unknown at the time of Refs. 26).

A.5. *Isodual discrete symmetries*

As it is well known (see, e.g. Wigner's historical contribution[27(a)]), the fundamental space–time symmetries used throughout this century in particle physics are the continuous Poincaré transforms plus the discrete transforms characterized by parity, time reversal and charge conjugation.

Note that in the above setting *antiparticles are treated in the same representation space and under the same symmetries as those of particles.*

The theory submitted in this paper and in Refs. 10 introduces a novel characterization in which *charge conjugation is replaced by isoduality.* This implies the introduction of the *isodual images of the continuous Poincaré transforms, as well as of parity and time inversions* (without any isodual image of charge conjugation).

The connected component of the Poincaré–Santilli isodual symmetry has been studied in Refs. 6. We here introduce the following *isodual space and time inversion* (formulated for simplicity for a scalar field)

$$\pi^d \times^d \phi^d(x^d) \times^d \pi^{d\dagger} = \phi^d(x'^d, x'^d = (-r^d, t^d) = (r, -t)\,, \tag{A.16a}$$

$$\tau^d \times^d \phi^d(x^d) \times^d \tau^{d\dagger} = \bar{\phi}^d(x''^d, x''^d = (r^d, -t^d) = (-r, t)\,, \tag{A.16b}$$

where r^d $(= -r)$ is the *isodual coordinate* on space $E^d(r^d\delta^d, R^d)$, and $t^d(= -t)$ is the *isodual time* on $E^d(t^d, 1, R^d_t)$.

As one can see, isodual space and time inversions formulated in their proper isodual spaces, *coincide* with the conventional space and time inversions formulated on conventional spaces.

Despite that, the isodual discrete symmetries are not trivial. In fact, all measurements are done in our space–time, thus implying the need to consider the *projection* of the isodual discrete symmetries into our space–time which are manifestly different than the conventional forms. In particular, they imply a sort of interchange, in the sense that the conventional *space* inversion $(r, t) \to (-r, t)$ emerges as belonging to the projection in our space–time of the isodual *time* inversion, and vice-versa.

In Ref. 10(a) we have introduced the *isodual photon* as a new photon emitted by antimatter which coincides with the conventional photon under all interactions, *except* gravitation. In particular, the isodual (conventional) photon is predicted by the isodual theory to experience a *repulsion* in a gravitational field due to matter (antimatter). In this paper we have introduced *two* additional means for a possible distinction between photons emitted by matter and those emitted by antimatter, given by the isodual space and time inversions, which are different than those of the ordinary photons irrespective of the applicable interactions.

The possible formulation of experiments based on the latter differences will be studies elsewhere. At this moment we merely mention that these two additional predicted differences are significant for the future theoretical and experimental resolution as to whether far away galaxies and quasars are made up of matter or of antimatter.

In closing, we point out that the notion of isodual parity has intriguing connections with the parity of antiparticles in the $(j,0)+(0,j)$ representation space originally studied by Bargmann, Wightman and Wigner[27(b)] and more recently studied by Ahluwalia, Johnson and Goldman.[27(c)] In fact, the latter parity results to be opposite to that of particles which is fully in line with isodual parity. The reformulation of the (non-iso-self-dual) space $(j,0)+(0,j)$ into the isoselfdual form $(j,0)+(j,0)^d$ would then permit: a topologically consistent treatment (indicated earlier); the isodual reinterpretation of parity; and the addition of the isodual time reversal, with intriguing interchanges in their projection into our space–time.

All in all, the isodual reinterpretation of current studies on antiparticles appears to be significant *per se*, as well as necessary preparatory grounds for the future experimental resolution of the correct theory of antiparticles valid at all levels of study, including classical, first quantization and second quantization theories, as well as under all interactions, including electromagnetic, weak, strong and gravitational interactions.[10(b)]

References

1. P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th ed. (Clarendon, Oxford, 1958).
2. R. L. Forward, in *Antiprotons Science and Technology*, eds. B. W. Augenstein, B. E. Bonner, F. E. Mills and M. M. Nieto (World Scientific, Singapore, 1988).
3. R. M. Santilli, *Hadronic J.* **8**, 25, 36 (1985).
4. R. M. Santilli, *Algebras, Groups and Geometries* **10**, 273 (1993).
5. R. M. Santilli, *Commun. Theor. Phys.* **3**, 153 (1993).
6. (a) R. M. Santilli, *J. Moscow Phys. Soc.* **3**, 255 (1993); (b) *Int. J. Mod. Phys.* **A10**, 4 (1998), in press.
7. (a) R. M. Santilli, *Hadronic J.* **17**, 257 (1994); (b) *Hadronic J.* **17**, 285 (1994).
8. R. M. Santilli, *Rendiconti Circolo Matematico di Palermo, Supplemento* **42**, 7 (1996).
9. R. M. Santilli, "Elements of hadronic mechanics," Vol. II, *Theoretical Foundations*, 2nd ed. (Ukraine Academy of Sciences, Kiev, 1995).
10. (a) R. M. Santilli, *Hyperfine Interactions* **109**, 63 (1997); (b) *Found. Phys. Lett.* **10**, 305 (1997).
11. R. M. Santilli, *Found. Phys.* **27**, 305 (1996).
12. J. P. Mills Jr., *Hadronic J.* **19**, 1 (1996).
13. J. V. Kadeisvili, *Algebras, Groups and Geometries* **9**, 283, 319 (1992); *Math. Meth. Appl. Sci.* **19**, 1349 (1996); *Santilli's Isotopies of Contemporary Algebras, Geometries and Relativities*, 2nd ed. (Ukraine Academy of Sciences, Kiev), in print.
14. (a) J. Lôhmus, E. Paal and L. Sorgsepp, *Nonassociative Algebras in Physics* (Hadronic Press, Palm Harbor, Florida, USA, 1994); (b) D. S. Sourlas and G. T. Tsagas, *Mathematical Foundations of the Lie-Santilli Theory* (Ukraine Academy of Sciences, Kiev, 1993).
15. M. Holzscheiter (editor), *Antimatter Gravity and Antihydrogen Spectroscopy* (Baltzez Science Publ., 1997).

16. E. C. G. Sudarshan and N. Mukunda, *Classical Mechanics: A Modern Perspective* (Wiley & Sons, New York, 1974).
17. W. Pauli, *Theory of Relativity* (Pergamon, London, 1958).
18. C. W. Misner, K. S. Thorne and A. Wheeler, *Gravitation* (Freeman, San Francisco, 1970).
19. P. Freud, *Ann. Math.* 40 (2), 417 (1939).
20. W. M. Fairbank and F. C. Witteborn, *Phys. Rev. Lett.* **19**, 1049 (1967).
21. R. E. Brown *et al.*, *Nucl. Instr. Meth. Phys. Res.* **B56**, 480 (1991).
22. M. M. Nieto and T. Goldman, *Phys. Rep.* **205**, 221 (1991); erratum **216**, 343 (1992).
23. R. M. Santilli, *Ann. Phys.* **83**, 108 (1974).
24. J. Sniatycki, *Geometric Quantization and Quantum Mechanics* (Springer-Verlag, New York, 1979).
25. D. Lovelock and H. Rund, *Tensors, Differential Forms and Variational Principles* (Wiley, New York, 1975).
26. R. M. Santilli, *Isotopic, Genotopic and Hyperstructural Methods in Theoretical Biology* (Ukraine Academy of Sciences, Kiev, 1996).
27. (a) E. Majorana, *Nuovo Cimento* 14, 171 (1937); (b) J. A. McLennan Jr., *Phys. Rev.* **106**, 821 (1957); (c) K. M. Case, *Phys. Rev.* **107**, 307 (1957); (d) D. V. Ahluwalia, *Int. J. Mod. Phys.* **A11**, 1855 (1996); (e) V. V. Dvoeglazov, *Hadronic J.* **20**, 435 (1997).
28. (a) E. P. Wigner, *Ann. Math.* 40, 149 (1939); (b) E. P. Wigner, in "Groups theoretical concepts and methods in elementary particle physics," lectures of the Instanbul Summer School of Theoretical Physics, ed. F. Gursey, (1962); (c) D. V. Ahluwalia, M. B. Johson and T. Goldman, *Phys. Lett.* **B316**, 102 (1993).