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ISOTOPIC GENERALIZATIONS OF GALILEI'S AND EINSTEIN'S RELATIVITIES

VOLUME I: MATHEMATICAL FOUNDATIONS

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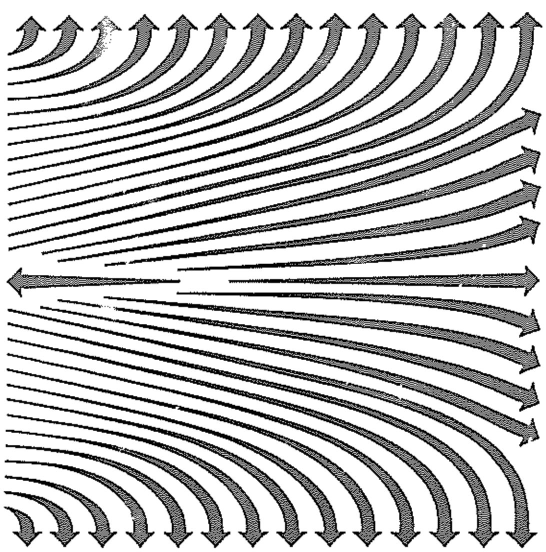

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my son-in-law ***ALLEN CRABTREE, jr***

because he personifies a gentle and
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ISOTOPIC GENERALIZATIONS OF GALILEI'S AND EINSTEIN'S RELATIVITIES

Volume I: Mathematical Foundations

TABLE OF CONTENTS

CHAPTER II: MATHEMATICAL FOUNDATIONS

PREFACE

OF VOLUMES I AND II

Physics is a science that will never admit *final theories* . No matter how autoritative, the generalization of fundamental physical theories is only a matter of time.

Physics is also a *quantitative science*, that is, requiring mathematically rigorous, quantitative formulations of predictions suitable for direct experimental verifications.

Finally, physics is a science with an absolute standard of values: the *experimental verification.* No matter how plausible a new theory is, it remains conjectural until verified in laboratories. By the same token, no matter how fundamental and authoritative an existing theory is, its validity remains conjectural for all physical conditions under which it has not been directly tested.

Along these lines, the sooner the scientific process is initiated with the submission of possible generalizations of existing theories and their critical examination by independent researchers, the better for the advancement of physical knowledge.

The author has spent his research life in studying possible classical and operator generalizations of *Galilei's relativity, Einstein's special relativity* and *Einsten's general relativity* (or *Einstein's gravitation* , for short). The studies were conducted along the following main lines:

1) Identifation of the *"physical conditions of unequivocal validity"* of conventional relativities;
2) Identification of broader physical conditions under which possible generalized relativities may be physically relevant;
3) Identification of the generalized mathematical tools needed for a quantitative representation of the broader physical conditions considered;
4) Construction of the generalized relativities, including the

identification of their mutual compatibility, implications and quantitative predictions; and, last but not least,

5) Formulation of specific experimental proposals for the verification or disproof of the new relativities.

In particular, this author has studied the above problem, in an evident preliminary way: a) for each of the Galilean, special and general profiles; b) for both classical and operator formulations; and c) in regard to the intrinsic compatibility of the emerging generalizations of Galilei's, the special and the general relativities, first, independently at the classical and operator level and, then, for the identification of a map from the classical into the operator formulations.

After an introductory chapter, Volume I is devoted to the review of the novel mathematical structures needed for a quantitative treatment of the broader physical conditions considered.

Volume II is devoted to: the construction of the classical generalized relativities; the study of their mutual compatibility; the identification of their most important implications; and the proposal of experiments for their verification or disproof.

The scope of these monographs is to identify the status of the studies in the field at this writing (Fall 1991), so that the interested researcher can appraise the new relativities, and participate in their mathematical-theoretical development or experimental verification.

The understanding of these volumes requires a mind open to the possibility that Galilei's relativity, Einstein's special relativity and Einstein's gravitation are not final theories, but only beautiful foundations for expected more general relativities for more complex physical conditions in the Universe.

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Fall 1991

CHAPTER I:

INTRODUCTION

I.1: INTERIOR AND EXTERIOR DYNAMICAL PROBLEMS.

The *"physical conditions of unequivocal applicability"* of contemporary relativities can be identified with the notion introduced by Galileo Galilei (1638) of *"massive point"*, subsequently implemented by Isaac Newton (1687) into the notion of *"massive point moving in empty space (vacuum) under action-a-distance, potential forces"*, and finally embraced in its entirety by Albert Einstein.

In fact, Einstein stated quite clearly in his limpid writings (see, e.g., Einstein (1905)) that his *special relativity* was conceived for

A) particles which can be well approximated as massive points;
B) when moving in vacuum conceived as a homogeneous and isotropic medium;
C) under action-at-a-distance potential interactions.

This physical field was subsequently implemented in *Galilei's relativity* (see, e.g., Levy-Leblond (1971) and quoted papers), and remains the central notion of a test particle in *Einstein's gravitation* (see, e.g., Pauli (1921) in its English edition (1958), and quoted historical sources).

The above physical conditions were well identified in the early treateses on relativities (see, e.g., the title of Chapter VI of Bergmann (1942) with a Foreword by Einstein). However, as a result of the successes of established relativities in various fields, they gradually disappeared in more recent treateses and papers, by acquiring the current assumption of their universal validity under all possible conditions of the Universe.

A primary purpose of this Introduction is the identification of physical conditions broader than A) and B) above under which conventional relativities are either inapplicable or have not been directly tested, and possible generalized relativities may have a physical value.

The best way to study this problem is by returning to the Founding Fathers of *analytic mechanics* and see that the soundedness of their description of the physical reality persists to this day.

In fact, Lagrange (1788), Hamilton (1834) Jacobi (1837) and others, based their analytic studies on the distinction between:

1) The *interior dynamical problem* , which is essentially the study of dynamics in the interior of the minimal surface $S°$ containing all matter of the celestial body considered, including any possible atmosphere; and

2) The *exterior dynamical problem* , which is essentially the study of dynamics in the empty space (vacuum) outside the above minimal surface $S°$, assumed as homogeneous and isotropic.

In fact, the original Lagrange's and Hamilton's equations were formulated with external terms precisely to represent the forces of the interior dynamics which were known to be outside the representational capabilities of the Lagrangian or Hamiltonian functions by their very originators. Similarly, Jacobi formulated his celebrated theorem for the original analytic equations with external terms, and not for the contemporary equations without external terms.

The distinction between the exterior and the interior problem gradually disappeared from the scientific scene as a result of an evolutionary process that does not appear to have been sufficiently studied by historians in the field until now. Without any claim of completeness, we can mention here:

a) The birth of *Lie's theory* (1893) which identified the algebraic structure of Hamilton's equations *without* external terms, and its vast impact in mathematical and physical studies;

b) The discovery of the special relativity by Lorentz (1904), Poincaré (1905), Einstein (1905) and others with its strictly Lie-Hamiltonian character, and the profound influence in the scientific thought which resulted from its experimental verification in the conditions originally conceived by Einstein;

c) The successes of the quantum mechanical description of the atomic structure with its strictly Lie-Hamiltonian character in

operator form (see, e.g., Pauli (*loc. cit*));
and numerous other factors.

Birkhoff (1927) identified a generalization of Hamilton's equations also derivable from a first-order variational principle, but its algebraic structure remained unknown. Also, he applied his equations to typical exterior problems, such as the stability of planetary trajectories.

Despite the general lack of recent interest, the historical motivations that led Lagrange, Hamilton, Jacobi and other Founders of analytic dynamics to formulate interior problems with external terms, were sound indeed. In fact, as we shall review shortly, the contact interactions of interior trajectories generally possess *nonlinear* as well as *nonlocal-integral forces* (see, e.g., Hofstandter (1970), Fujimura *et al.* (1970) and quoted papers).

Moreover, quantitative studies on the irreducibility of the interior to the exterior problem were initiated by H. Helmholtz (1887) who introduced the so-called conditions of *variational selfadjointness* (treated in detail in Santilli (1978e) as the integrability conditions for the existence of a first-order Lagrangian[1] or of the corresponding Hamiltonian representation. This provided rigorous mathematical tools to establish that the trajectories of interior problems, i.e., the external terms in Lagrange's and Hamilton's equations, violate the integrability conditions for the existence of a Lagrangian or a Hamiltonian in the frame of the experimenter.

As we shall elaborate throughout this monograph, we can therefore state that:

> *The "physical conditions of unequivocal applicability" of conventional relativities is provided by the exterior dynamical problem, while the different physical arena under which generalized relativities may have a physical value is that of the interior physical problem.*

A classical example of the dichotomy exterior vs interior problem is provided by Jupiter, whose exterior center-of-mass trajectory is manifestly stable and verifies all conventional symmetries and

[1] That is, via a Lagrangian L depending at most on the first-order derivative of the local coordinates, say, r, with respect to an independent variable t, $L = L(t, r, \dot{r})$. Unless the contrary is explicitly stated, the first-order character of all Lagrangians considered in these volumes will be tacitly assumed hereon. Some of the systems considered in these volumes will be representable in local approximation via second-order Lagrangians $L = L(t, r, \dot{r}, \ddot{r})$. The understanding is that the conventional Lie-Hamiltonian formulations are lost for the latter representations.

physical laws; nevertheless, the interior dynamics is manifestly unstable, nonconservative, as well as nonlagrangian-nonhamiltonian.

A conceptual guidance of the classical studies of this volume is therefore provided by Jupiter's structure. In fact, this volume can be essentially considered as attempting the identification of the generalizations of conventional Galilean, relativistic and gravitational treatments which can provide a classical, direct, representation of Jupiter's structure, as it appears to our experimental observations, and without hypothetical, seemingly inconsistent (see below), conservative reductions.

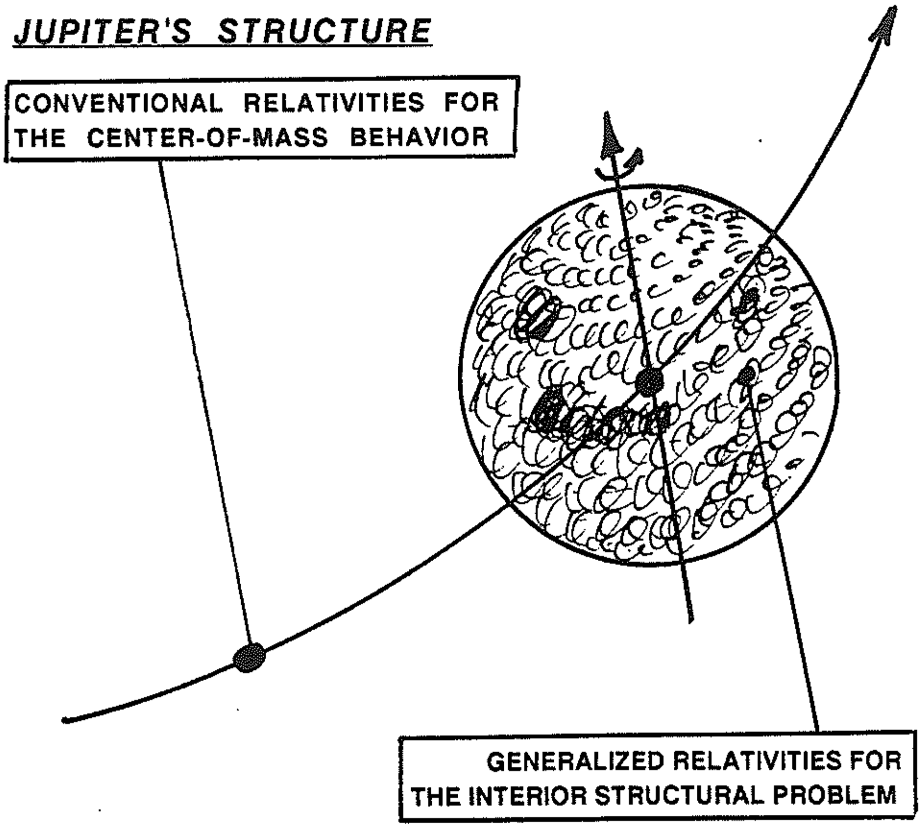

FIGURE I.1.1: The first experimental observation at the foundations of all contemporary relativities (Galilei's, Einstein's special and Einstein's general relativities) can be identified in the first visual observation of the Jovian system by Galileo Galilei with his telescope in 1609. The experimental observation at the foundations of the generalizations of contemporary symmetries and related relativities studied in these volumes can also be identified in the Jovian system, but this time in the observation of Jupiter itself, considered as an extended composite system. A dichotomy of far reaching implications emerges from this experimental evidence: the manifest local validity of the rotational, Galilei's and Poincare' symmetries in Jupiter's exterior motion in the Solar system, and the equally manifest breaking of the same symmetries in the interior structure, as established by nonconservative interior trajectories, vortices with monotonically varying angular momenta, etc. In these monographs we shall show that the Lie-isotopic generalization of the conventional formulation of Lie's theory permits the quantitative representation of this dichotomy at all levels, the Newtonian, the relativistic and the gravitational levels, as submitted in Santilli (1978a), (1982b), (1988a, b, c, d), and (1991a-d). As we shall see, the Lie-isotopic formulations permit the achievement of nontrivially generalized interior structures, in such a way to preserve the abstract symmetries and the axiomatic structure of the exterior physical laws, although realized in their most general possible, nonlinear and nonlocal form. In this way, we shall attempt an ultimate unity of mathematical and physical thought, whereby all distinctions between the exterior and the interior problem, or between conventional and generalized formulations cease to exist at the abstract, realization-free level.

In a subsequent monograph we hope to review the operator formulation of our results, and apply them to the problem of the hadronic structure. The primary objective which stimulated the Lie-isotopic studies (Santilli (1978a) and (1978b)) is that of attempting the identification of the hadronic constituents with ordinary (massive) physical particles modified in such a way by short range, nonlocal, internal forces, that they can be consistently produced free in the spontaneous decays, as occurring at the atomic and nuclear levels.

In different terms, the hope of these studies is that of investigating whether the historical identification of the atomic and nuclear constituents freely produced in the spontaneous decays, can also be extended, in due time, to the hadronic structure, of course, under suitably generalized formulations. For this purpose, it may be of some assistance to identify a conceivable operator counterpart of the classical lines of inquiry of this work.

Hadrons (see the reprints of the historical contributions in Lichtenberg and Rosen, editors (1980)) are currently conceived as being strictly Lagrangian or Hamiltonian, with a strictly local-differential and potential-Lagrangian structure. This essentially implies the treatment of the hadronic structure as an *exterior* dynamical problem.

In Santilli (1978b) we submitted the conjecture that the hadronic structure is an operator version of the classical structure of Jupiter, much along the historical open legacy of the ultimate nonlocality of strong interactions. In fact, in their exterior center-of-mass dynamics (e.g., in a particle accelerator) hadrons obey all conventional symmetries and relativities, as is the case for the motion of Jupiter in the Solar system. Nevertheless, the interior dynamics of hadrons could well be generalized, along Jupiter's interior structure.

After all, hadrons are the densest objects measured in laboratory until now. Moreover, under sufficient dynamical conditions on the wavelength of nearly free constituents (asymptotic freedom) and on the size of the hadrons (that is, the effective range of strong interactions), we have an evident wave overlapping producing the nonlocal nature of strong interactions.

Needless to say, the hadronic constituents can indeed have a point-like charge (such as the electron), as supported by current experimental evidence (Bloom *et al.* (1969)). However, "point-like wavepackets" do not exist in nature, thus resulting again in the nonlocal structure of hadrons which, rather than being new, is in actuality the open historical legacy of Fermi (1949), Bogoliubov (1960), and others.

It then follows that, while in the atomic structure we have very large mutual distances as compared to the size of the wavepackets of the constituents, in the hadronic structure we have instead mutual distances of the order of magnitude of the wavelength of the constituents themselves. This results in a quark motion which is typical of all interior problems because it is characterized by the motion of an extended wavepacket within the physical medium constituted by the wavepackets of all remaining constituents, called *hadronic medium* (Santilli (1978b)).

As a result, *the hadronic structure, could therefore be analytically equivalent to that of Jupiter in the sense that motion of a hadronic constituent inside a hadron could experience forces analytically equivalent to those, say, of a space-ship moving in Jupiter's atmosphere, resulting in both cases in the presence of short range, internal forces of nonlocal and nonhamiltonian type.*

In conclusion, a conceptual guidance for our operator studies can be given by the hadronic interior problem conceived precisely along the hystorical teaching by Lagrange, Hamiltpon and Jacobi recalled earlier for the interior problem.

I.2: INEQUIVALENCE OF INTERIOR AND EXTERIOR PROBLEMS

Consider a test particle in the gravitational field of a celestial object, say, Jupiter. When considering the exterior trajectory, motion occurs in empty space which is known to be homogeneous and isotropic to the best of our approximations. Under these conditions, the actual size and shape of the test particle do not affect its dynamical evolution. The particle can then be effectively assumed as being dimensionless, resulting in Galilei's concept of *"massive point"* indicated earlier.

In turn, this implies the exact validity of a local-differential geometry. Since points cannot collide, the only admissible forces are the conventional action-at-a-distance, potential forces called *variationally selfadjoint* (Helmholts (1887), Santilli (1978e)). The orbits are therefore necessarily stable, with consequential validity of conventional symmetries, such as the rotational symmetry O(3), the Galilei's symmetry G(3.1), or the Poincaré symmetry P(3.1).

We can therefore say that:

> *The exterior dynamical problem consists of motion of point-like particles in vacuum under potential (selfadjoint), and therefore Lagrangian-Hamiltonian forces, and can be effectively represented via the conventional Lagrange's and Hamilton's equations without external terms, their Lie algebra structure, and their local-differential geometries (e.g., symplectic or Riemannian).*

When the same test body penetrates Jupiter's atmosphere, thus passing to the interior dynamical problem, the physical framework is profoundly different. To begin, we now have motion within a physical medium which is evidently *inhomogeneous* (e.g., because the density of Jupiter tends to zero with the increase of the distance from its center), and *anisotropic* (e.g., because Jupiter's intrinsic angular momentum creates a preferred direction in the medium considered).

Also, the actual size and shape of the test particle cannot be ignored any longer, because they affect directly its dynamical evolution. As a result, we cannot any longer claim motion of a point-like particle in vacuum, but we have instead motion of an extended object within a generally inhomogeneous and anisotropic physical medium.

Moreover, the acting forces are given by the conventional potential (say, gravitational) forces which evidently remain unaffected by the interior dynamics, plus the contact forces, that is, forces caused by the actual contact of the extended object with the physical medium.

In particular, besides being nonlinear, these latter forces are known to be generally non-newtonian[2] as well as:

1) of *nonlocal type* , in the sense of requiring surface or volume integrals for their representation (Hofstadter (1970), or Fujimura *et al.* (1970));

2) of *nonlagrangian-nonhamiltonian type* , first, because the notion of potential has no physical meaning for contact interactions and,more deeply, because they are (variationally) *nonselfadjoint* (Helmholts (*loc. cit*), Santilli (*loc. cit*), i.e., they violate necessary and sufficient conditions for the existence of a Lagrangian or a Hamiltonian representation;
and, last but not least,

3) of *zero range* in the sense that they occur at the instant of mutual "contact" between the body considered and the medium, thus being *instantaneous* by conception at the classical level (with the understanding that, at the operator-particle level, they become of *short range* type owing to their technically different treatment).

Finally, as a result of the latter interactions, the orbits are manifestly unstable, e.g., because of the monotonic decay of the angular momentum, thus resulting in an apparent (see Chapter III) breaking of the rotational symmetry O(3). The local breaking of Galilei's symmetry G(3.1) is then consequential, as requested, e.g., by the inhomogeneous and anisotropic character of the interior media, the lack of applicability of canonical formulations, etc.

In fact, the insistence in the exact validity for the interior trajectories of the same symmetries of the exterior problem would directly imply excessive approximations, such as the acceptance of

[2] *Newtonian forces* F are traditionally assumed as depending on time t, the coordinates r and their derivatives $\dot{r}$, $F = F(t, r, \dot{r})$, but of being independent from the accelerations $\ddot{r}$. Forces of the type $F = F(t, r, \dot{r}, \ddot{r}, ...)$ are therefore referred to as being *non-newtonian.* While acceleration-dependent forces appear in the exterior problem only in particular cases, as well known (see, e.g., Sudarshan and Mukunda (1974)), they do play a crucial role in the interior problem, as we shall see in Chapter III. Acceleration-dependent forces have also been brought to the attention of the scientific community by other, quite intriguing aspects related to inertia (see Assis (1989) and (1990) and Graneau (1990)).

the perpetual motion in a physical environment, trivially, because of the necessary conservation of the angular momentum.

In conclusion:

> *The interior dynamical problem* consists of the motion of extended (and therefore deformable) test particles within generally inhomogeneous and anisotropic material media (with the understanding that the underlying space remains homogeneous and isotropic), and requires for an effective treatment the original Lagrange's and Hamilton's equations with external terms representing the contact forces precisely along the original conception of the Founders of analytic dynamics recalled in the preceding section; with suitably generalized algebraic and geometrical structures.

A primary task of these monographs is therefore that identifying the state of the art in the available generalized, analytic, algebraic and geometrical formulations for the quantitative treatment of the above interior systems, as a preparatory ground for their possible operator formulation.

I.3: IRREDUCIBILITY OF THE INTERIOR TO THE EXTERIOR PROBLEM

Physicists tend to react differently when exposed to novel physical conditions.

A first group confronts the novel conditions for the specific purpose of attempting a *generalization of established theories*, with the knowledge that, even if the primary objectives cannot be achieved, quantitative efforts always result in scientific advances, e.g., in the knowledge of new techniques.

A second group of physicists, instead, tends to *preserve as much as possible old knowledge* via all conceivable efforts in rendering the new physical conditions compatible with old theories.

The latter attitude has a clear scientific value, inasmuch as new theories should not be developed for physical conditions under which existing theories are sufficiently valid. However, there is a subtle line beyond which the latter attitude has no scientific value, e.g., when the insistence in old theories implies excessive approximations, and their applicability is not realizable on technical or experimental grounds.

Studies on interior dynamical problems have been systematically dismissed by the second group of physicists throughout this century on grounds of numerous (seemingly unpublished) objections. It is important for the analysis of these volumes to review the most important objections and show their excessively approximate character, or proved mathematical inconsistency, or rigorously established impossibility of experimental realization.

In particular, we shall point out that the conventional concepts of the exterior problem, when applied to the interior problem without sufficient care as a result of protracted use, generally lead to insidious inconsistencies as well as fundamental physical misrepresentations.

The first objection refers to the classical and local treatment of the interior problem and consists of the belief that the conventional Lagrange's and Hamilton's equations are sufficient to represent interior trajectories. The fact that this is indeed the case for numerous physical systems, is undeniable (see, e.g., the numerous examples of Santilli (1978e)). However, the lack of general applicability of the conventional analytic equations is rigorously established by the violation of the conditions of variational selfadjointness by the systems of our physical environment (the so-called *essentially nonselfadjoint* systems, see Santilli (*loc. cit.*)). Besides, the insistence in the general use of the conventional analytic equations would lead to evident, excessive approximations of physical reality.

In fact, nonlocal forces can be well approximated via power series in the velocities truncated (via suitable coefficients) at given powers. The point is that, to avoid excessive approximations, such powers must remain arbitrary, thus precluding the general existence of a direct Lagrangian or Hamiltonian representation in favor of suitably generalized representations of Birkhoffian type (Santilli (1982a)).

As an example, computerized guidance systems in contemporary rocketry require the use of up to *the tenth power in the velocity or more.* It is evident that, under these conditions, no Lagrangian or Hamiltonian exists in the frame of the observer. On the contrary, the existence of a direct Birkhoffian representation is ensured for the systems considered (Santilli *(loc. cit.)),* with numerous methodological possibilities, e.g., the consequential direct applicability of the optimal control theory in the frame of the experimenter.

A second objection, also of classical and local nature, is that the above nonlagrangian and nonhamiltonian systems can be transformed into suitable frames in which a conventional Lagrangian or

Hamiltonian exists. Under sufficient topological conditions (regularity, locality and analyticity) the *Lie-Koening Theorem* ensures that there always exists a tranformation under which a nonlagrangian or a nonhamiltonian system admits a conventional Lagrangian or Hamiltonian representation (see the geometric and analytic proofs of Santilli (1982a), Sect. 6.2). However, the transformation is necessarily *noncanonical* (evidently because the original system is nonhamiltonian by assumption), and generally *nonlinear in all variables.* Therefore, the admitted frames are strictly *noninertial* and, as such, *incompatible with the applicable relativity.* Besides, *the transformed frames are not realizable in experiments (e.g., because the needed transformations are hyperbolical or even transcendental*, see the examples in Santilli (1978a)), thus having a purely mathematical meaning. As a result, the objection here considered is not physically sound.

This is the reason for the insistence in various works by this author (Santilli (1978a, b, c, d, e.)) that *the methodological formulations of the interior problem must be directly applicable in the frame of the experimenter.* The same, evidently sound insistence will be kept throughout the analysis of these volumes. Only *after* the achievement of a direct representation of the classical physical reality in the frame of the experimenter, the use of the transformation theory may have a physical value.

We can therefore state that *a quantitative, classical treatment of interior trajectories requires:*

i) the treatment directly in the frame of the observer, to avoid mathematical noninertial frames nonrealizable in actual experiments and other inconsistencies;

ii) a generally nonlagrangian and nonhamiltonian theory, to represent the interior forces as they occur in the physical reality of the experimenter; and

iii) a generally nonlocal-integral theory; moreover, if locality is admitted in first approximation, the theory must remain non-Hamiltonian to avoid excessive approximations of the type of perpetual motion in a physical environment.

Still another attempt at reconciling old theories with the new physical conditions here considered, consists in the addition of a "nonlocal-integral potential" to a given Lagrangian or Hamiltonian.

This approach too is inconsistent on a number of counts. Mathematically, it implies the need of an integral topology, thus resulting in fundanental technical inconsistencies in the use of conventional topological symmetries, such as the Galilei's or Lorentz's symmetry.

Physically, the approach is manifestly inconsistent because, as stressed earlier, contact forces have no potential. When they are erroneously represented with a potential, this implies necessary deviations from physical trajectories.

A final objection is moved by physicists interested in preserving as much as possible old theories. It is the claim that the distinction between the exterior and the interior problem is "illusory" because, when the test body and the surrounding atmosphere are all reduced to their elementary constituents, one regains motion of point-like particles in vacuum, in which case all distinctions between the exterior and the interior problems cease to exist.

This latter objection itself has been proven to be "illusory" because intrinsically inconsistent and not technical realizable (Santilli (1985c)). Consider, again, a space-ship in Jupiter's atmosphere. Its trajectory is manifestly noncanonical and nonhamiltonian, as established by clear experimental evidence. On the contrary, the elementary constituents of the space-ship are evidently *assumed* to have unitary and Hamiltonian time evolutions. The following property can then be readily proved.

THEOREM I.3.1 ("NO-REDUCTION THEOREM I"; Santilli (1985c)): Under sufficient topological conditions, a classical noncanonical and nonhamiltonian interior system cannot be reduced to a finite collection of unitary and Hamiltonian particles; and, viceversa, a finite collection of unitary and Hamiltonian particles cannot produce a classical noncanonical and nonhamiltonian ensemble under the correspondence or other limits.

The proof is trivial. In fact, a macroscopic, monotonically unstable orbit simply cannot be decomposed into a finite number of stable elementary trajectories; and, viceversa, a collection of stable elementary trajectories simply cannot result into a classical, monotonically unstable ensemble. On the contrary, a classical, stable system can indeed be decomposed into a collection of unstable trajectories (Santilli (1978b)), as we shall review in Chapter III.

The symmetry counterpart of the above property is then predictably given by the following

THEOREM I.3.2 ("NO-REDUCTION THEOREM II"; loc. cit.): Under sufficient topological conditions, a classical Galilei- (or Lorentz-) noninvariant system cannot be reduced to a finite collection of Galilei- (or Lorentz-) invariant particles; and, viceversa, a finite collection of Galilei- (or Lorentz-) invariant particles cannot produce a Galilei- (or Lorentz-) noninvariant ensemble under the correspondence or other limits.

In fact, the local validity of the Galilei (or Lorentz) symmetry necessarily implies the stability of the constituents' orbits which, as such, cannot result into a nonconservative and, therefore, Galilei- (or Lorentz-) noninvariant ensemble. For the additional "No-Reduction Theorem III" we refer the reader to the quoted lietrature for brevity.

The above theorems essentially establish that classical systems such as a satellite during penetration in Jupiter's atmosphere with its continuously decaying angular momentum, is an experimental reality outside the field of applicability of conventional symmetries and relativities, and that the conceptual reduction of these systems to the elementary constituents, cannot be consistently realized.

Stated in different terms, the above theorems indicate that the contact, nonlocal and nonhamiltonian forces of the satellite in Jupiter's interior trajectory, by no means, are "illusory" and can be made to "disappear" in the reduction of the satellite to its elementary constituents.

In fact, we find exactly the same forces in the region of contact of the satellite with the atmosphere.

Furthermore, along the open historical legacy of the nonlocal nature of the strong interactions (Fermi (1949), Bogolioubov (1960)), and others, we expect to find again the nonlocal forces in the mutual overlapping of the wavepackets of the constituents in the hadronic as well as, to a lesser extent, in the nuclear structure (Santilli (1978b)). The fact that current particle theories cannot accomodate this historical legacy, is not a sufficient reason to conclude that strong interactions are necessarily local.

An objective of these monographs is therefore that of abandoning conceptual abstractions, and confronting the problem of the mathematical representation of interior trajectories as they appear in our physical reality, that is, with internal nonconservations, of course, in a way compatible with conventional settings in the exterior problem.

Equivalently, we shall attempt a reconciliation between the exact character of local relativities in the center-of-mass behavior (exterior problem), and the open historical legacy of the ultimate nonlocality of classical and operator structures (interior problem).

The necessity of a joint study of the problem at the Newtonian, as well as relativistic and gravitational levels can now be identified. In fact, the excellent results of current quark theories (see, e.g., Lichtenbergand Rosen, editors (1980)) indicate the possibility that, after all, the nonlocal and nonhamiltonian internal effects under consideration here could be small in the hadronic structure, and therefore ignorable at a first nonrelativistic and nongravitational treatment. This could be the case if the wavepackets of quarks could be experimentally proved to be very small as compared to the size of the hadrons.

However, when passing to the gravitational treatment, the physical evidence of the nonlocal structure of gravitation cannot be ignored. As an example, in a star undergoing gravitational collapse, we have not only the *total mutual penetration of the wavepackets of the constituents* (whatever their size), but also their *compression* in large numbers within the same very small region of space. The emerging, essentially nonlocal, and therefore nonselfadjoint structure of the interior gravitational problem is then beyond any scientific doubt.

At the same time, nonlocal and nonselfadjoint interior problems cannot be solely studied at the gravitational level, but require for consistency their study also at the preceding Newtonian and relativistic levels, as done in these volumes.

In conclusion, the historical distinction between the exterior and interior problems was established by the Founding Fathers of analytic mechanics on rather sound physical grounds and direct experimental evidence; it is confirmed by theoretical studies on their inequivalence, and all known objections are not technically consistent at this writing.

This establishes the open character of the central objective of these monographs: the construction of the space-time symmetries and relativities of interior nonlinear, nonlocal and nonhamiltonian problems at the Newtonian, relativistic and gravitational levels.

I.4: LIE-ISOTOPIC AND LIE-ADMISSIBLE FORMULATIONS

Contemporary theoretical physics is centered in the analytic, algebraic and geometric formulations underlying the Hamiltonian representation of conservative, exterior, dynamical systems, i.e., their representation via the familiar *Hamilton's equations without external terms*

$$\dot{r}^i = \frac{\partial H(r, p)}{\partial p_i}, \qquad \dot{p}_i = - \frac{\partial H(r, p)}{\partial r^i}, \tag{4.1}$$

where r represents the coordinates of the experimenter, p is the linear momentum and H = T + V is the *Hamiltonian* (the total energy).

The brackets of the theory, the familiar *Poisson brackets* among functions A(r,p) and B(r,p) in phase space

$$[A, B] = \frac{\partial A}{\partial r^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k}\frac{\partial A}{\partial p_k}, \tag{4.2}$$

verify the *Lie algebra axioms*

$$[A, B] + [B, A] = 0, \tag{4.3a}$$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \tag{4.3b}$$

The antisymmetry of the product, Eq. (4.3a), and the lack of external terms, render the theory particularly suited for the representation of conservative systems, for which the total energy is trivially conserved,

$$\dot{H} = [H, H] \equiv 0, \quad H = T + V. \tag{4.4}$$

Regrettably, the interior problem has been grossly ignored in this century, with only few contributions known to the author.

Some of the most comprehensive studies of nonconservative interior conditions are those conducted by Prigogine and his collaborators (see Prigogine (1962), (1968) and (1990) and quoted papers), primarily along statistical lines but with rather deep analytic and operator counterparts. Additional studies in interior dynamics have also been conducted by few other researchers (see, e.g., Edelen (1977), Ziman (1978) and quoted literature).

Somewhat stimulated by Prigogine's pioneering work, this author initiated his research (Santilli (1967), (1968) and (1969)) with a reinspection of the original *Hamilton's equations with external terms*

$$\dot{r}^i = = \frac{\partial H(r, p)}{\partial p_i}, \qquad \dot{p}_i = - \frac{\partial H(r, p)}{\partial r^i} + F_i(t, r, p), \tag{4.5}$$

and the indication that the brackets of the time evolution

$$A \times B = [A, B] + \frac{\partial A}{\partial p_i} F_i \tag{4.6}$$

violate the necessary conditions to constitute an algebra (the right scalar and distributive laws, see App. II.A), besides evidently violating the Lie algebra axioms (4.3). Nevertheless, when written in the form

$$(A, B) = [A, B] + \frac{\partial A}{\partial r^i} s^i{}_j \frac{\partial B}{\partial p_j}, \tag{4.7a}$$

$$s^i{}_j = \text{diag}\,(0, s), \quad s = F / (\partial H/\partial p), \tag{4.7b}$$

the brackets verify all the necessary conditions to constitute an algebra, and that algebra results to be a generalization of Lie algebras known as *Lie-admissible algebras* (Albert (1948)) with the following axiom in classical realization

$$(A,B,C) + (B,C,A) + (C,A,B) = (C,B,A) + (B,A,C) + (A,C,B), \tag{4.8}$$

where the quantity

$$(A,B,C) = (A, (B,C)) - ((A,B),C), \tag{4.9}$$

is called the *associator.*

While the primary emphasis of the conventional Hamiltonian-Lie formulations is on the conservation of total quantities, the primary emphasis of the above Lie-admissible formulation is on the characterization of the *time-rate-of-variation of physical quantities.* In fact, since product (4.7) is no longer antisymmetric, it cannot any longer represent the total conservation (4.4), but represents instead the variation of the energy H in time,

$$\dot{H} = (H,H) = \frac{\partial H}{\partial p_i} F_i \neq 0. \tag{4.10}$$

Therefore, *the Lie-admissible formulations are particularly suited for the representation of an individual, open-nonconservative interior trajectory,* say, a satellite during penetration in Jupiter's atmosphere which is considered as external.

The formulations were introduced in the memoir Santilli (1978a,c), and then developed in the monographs (Santilli (1978c) and (1981a)).

As a complement to these Lie-admissible studies, the author submitted an alternative formulation of the interior dynamics based on *Birkhoff's generalization of Hamilton's equations* (4.1) (Birkhoff (1927)), which can be written

$$\dot{a}^{\mu} = \Omega^{\mu\nu}(a) \frac{\partial H(a)}{\partial a^{\nu}}, \tag{4.11a}$$

$$a = (a^{\mu}) = (r, p), \quad \mu = 1, 2, ..., 2n, \tag{4.11b}$$

where *Birkhoff's tensor* $\Omega^{\mu\nu}$ is given by

$$\Omega^{\mu\nu} = (\| \Omega_{\alpha\beta} \|^{-1})^{\mu\nu} \tag{4.12a}$$

$$\Omega_{\mu\nu}(a) = \frac{\partial R_{\nu}(a)}{\partial a^{\mu}} - \frac{\partial R_{\mu}(a)}{\partial a^{\nu}}, \tag{4.12b}$$

and admit the Hamiltonian formulation as a particular case for R = (p, 0) (see Sect. II.8).

Unlike Hamilton's equations (4.1), Birkhoff's equations have been proved to be *directly universal* , i.e., capable of representing all possible nonlinear and nonhamiltonian interior trajectories in local and analytic treatment (universality), directly in the frame of the experimenter (direct universality) (Santilli (*loc. cit.*)).

The algebraic structure is characterized by the brackets

$$[A \hat{,} B] = \frac{\partial A}{\partial a^{\mu}} \Omega^{\mu\nu} \frac{\partial B}{\partial a^{\nu}}, \tag{4.13}$$

which result to be the most general possible classical, and regular realization of the Lie algebra axioms, i.e., to verify the axioms

$$[A \hat{,} B] + [B \hat{,} A] = 0, \tag{4.14a}$$

$$[A \hat{,} [B \hat{,} C]] + [B \hat{,} [C \hat{,} A]] + [C \hat{,} [A \hat{,} B]] = 0. \tag{4.14b}$$

The mapping from Hamilton's to Birkhoff's brackets within a fixed system of local variables

$$[A, B] \Rightarrow [A \hat{,} B], \tag{4.15}$$

is called (for certain historical reasons reviewed in Sect. II.1)) a *Lie-isotopy* and it implies, as we shall see, a corresponding generalization of the entire Lie's theory into the so-called *Lie-isotopic theory.*

Unlike the Lie-admissible formulations, the Lie-isotopic ones are based on brackets (4.13) which are totally antisymmetric as the conventional ones. As a result, *the Lie-isotopic formulations are particularly suited for the representation of the conservation of the total energy,*

$$\dot{H} = [H \hat{,} H] \equiv 0, \tag{4.16}$$

under a generalized internal structure represented precisely by the generalized Lie tensor $\Omega^{\mu\nu}$.

The Lie-isotopic formulations therefore possess a structure particularly suited for the representation of closed-isolated systems such as Jupiter's when considered as isolated from the rest of the Universe, which evidently verifies total conservation laws under a generalized nonhamiltonian internal structure.

The main lines of the Lie-isotopic theory (isotopic universal enveloping algebras, Lie-isotopic groups and Lie-isotopic algebras) were outlined in the memoir (Santilli (1978a)) as a particular case of the broader Lie-admissible formulations, and subsequently developed in the monographs (Santilli (1982a)) . A recent review for mathematicians has been provided by Aringazin *et al.* (1990).

An objective of the alternative, Lie-isotopic approach is to show that, in the transition from the exterior to the interior dynamics, there is no need to abandon the conventional analytic, algebraic and geometric structures of contemporary physics, but only the need to pass from their simplest possible (Hamiltonian) realizations to their most general possible (Birkhoffian) forms.

These monographs are primarily devoted to the Lie-isotopic formulation of interior dynamics. The reader should however be aware that our results can be reformulated in terms of the broader Lie-admissible approach. In fact, the central topics of these studies, the Lie-isotopic generalizations of the Galilei and Poincaré symmetries, are particular cases of still more general, Lie-admissible generalizations of the same symmetries (Santilli (1981a)).

I.5: BACKGROUND LITERATURE

The beginning of the analytic part of the Lie-isotopic studies of these monographs can be identified with Birkhoff's (1927) proposal of Eq.s (4.11). The algebraic and geometrical structures of the equations, however, had remained essentially unknown. Also, Birkhoff applied his equations to typical exterior problems, such as the stability of planetary orbits.

The *Lie-isotopic generalization of the conventional formulation of Lie's theory* was apparently submitted for the first time in Santilli (1978a, c), and continued with the monographs (Santilli (1978e) and (1982a)).

The author's contributions most relevant for these monographs are the identification of the transformation theory underlying the Lie-isotopic theory with a general theorem on the Lieisotopic symmetries (Santilli (1985a)), the study of the isotopies of contemporary algebras and geometries (Santilli (1988b) and (1991a, b)), and the isotopic liftings of: the rotational symmetry (Santilli (1985b)), the Galilei symmetry (Santilli (1982a) and (10988a)); the Lorentz symmetry (Santilli (1983a)); the Poincare' symmetry (Santilli (1988c)); and of Einstein's gravitation (1988d). Additional studied can be found in the ICTP preprints (Santilli (1991c-n)).

Independent contributions in *Lie-isotopy* are the following. The construction of specific examples of the Lie-isotopic generalizations of Galilei's relativity was done by by Jannussis *et al.* (1991).

Contributions on the Lie-isotopic generalizations of Einstein's special relativity known to the author are those by: de Sabbata and Gasperini (1982) on the maximal causal speed in the interior of hadronic matter; Aringazin (1989) on the "direct universality" of the Lie-isotopic relativity; Mignani (1992) on the application to the problem of the quasars'. redshift; Nishioka (1984a, b) and Jannussis (1985) on certain Lagrangian densities invariant under the Lie-isotopic Lorentz symmetry; and Cardone *et al.* (1992) on the phenopmenological interpretation of current experiments on the behaviour of the meanlife of unstable hadrons with speed.

Additional fundamental contributions of Lie-isotopic type outside relativity profiles are those by: Gasperini (1983a, b) on the first construction of a Lie-isotopic gauge theory (see also the isotopic degrees of freedom of gauge theories in Santilli (1979b); Nishioka (1983) and (1984b, c), Karajannis (1985a, b) and Karajannis and Jannussis (1986) on the development of certain aspects of the Lie-isotopic gauge theory; Karajannis's (1985) first Ph.D. Thesis on isotopic gauge theories;

and Mignani's (1984) first construction of the Lie-isotopic SU(3) symmetries.

The first isotopic generalizations of Einstein's gravitation on conventional Riemannian spaces were proposed by Gasperini (1984a, b, c) with a local Lie-isotopic Lorentz symmetry along Santilli's ((1978a) and (1983a)) proposal which were defined everywhere in space-time. Santilli (1988d, 1991a, b, c) restricted the validity of the theory to the interior gravitational problem only, and formulated the theory on suitable isotopies of the Riemannian space (see next chapter). A geometric Lie-isotopic treatment of torsion was provided by Rapoport-Campodonico (1991).

A first review article is that by Aringazin *et al.* (1990), following by a review monograph by the same authors (Aringazin *et al.* (1991)). Another review monograph of more mathematical orientation is that by Kadesivili (1992). This exhaust all physical contributions in classical Lie-isotopic theories known to the author (contribution on operator formulations have not been considered here to be quoted in a separate work).

Besides the papers (Santilli (1991a, b)), the only known contribution in a mathematical journal specifically devoted to Lie-isotopic algebras is the review for mathematicians by Aringazin *et al.* (1990).

Numerous theories possess an essential Lie-isotopic structure, although mostly unknown. A notable case is that of two of the last articles written by Dirac (1971, 1972) on a certain generalization of the Dirac's equation which has resulted to possess an essential invariance under the Lie-isotopic Poincare' symmetry, as shown by Santilli (1991d). Numerous other cases of theories with an unidentified Lie-isotopic structure exist in the literature, as the reader will be in a position to see following the study of these volume.

A further class of Lie-isotopic theories are those formulated within the context of the conventional Lie's theory, but which can be better treated within the context of the covering Lie-isotopic theory. The most important case is that of *Bogoslowski's special relativity* (Bogoslowki (1977), (1978) and (1984)) which achieved the first generalization of Einstein's special relativity for *anisotropic space-time*, and which is a particular case of our Lie-isotopic relativity for *inhomogenous and anisotropic physical media*, as we shall see. Edward (1963) and Strel'tsov (1990) suggested another class of generalized Lorentz transformations with an anisotropy in time. Also, Recami and Mignani (1972) introduced a yet different generalization capable of mapping space-time into time-like events, and viceversa which, as we shall see, is particularly significant for our analysis. Important work that appears to be directly related to ours can be found in the recent monograph by Logunov and Mestvirshvili (1989) and references quoted thereof, which focuses the attention on gravitational sources beginning at the relativistic level. Other generalizations of Lorentzian theories will be listed whenever needed.

A third class of studies worth a mention is that on the so-called *generalized Poisson brackets* which, as we shall see, are precisely of Lie-isotopic type (4.13). For a general Newtonian and relativistic presentation, the interested reader may consult Sudarshan and Mukunda (1974). The relativistic notion of generalized Poisson brackets was introduced by Dirac (1950), (1958) and (1964) and also constitutes a realization of the relativistic Lie-isotopic brackets. Further relativistic studies with generalized brackets can be found in Martin (1959), Hughes (1961), Hill (1967) and quoted papers. As we shall see, these latter studies too are particular cases of the relativistic Lie-isotopic theories.

The independent physical contributions in the broader Lie-admissible formulations (Santilli (1967), (1968) (1978a,b,c,d) and (1981a)) are more numerous, such as those by:

1) Kobussen (1979), who introduced a classical, Lie-admissible field theory;

2) Trostel (1982a, b), who proposed a geometrical extension of the conditions of variational selfadjointness;

3) Scoeber ((1981) and (1982)), who introduced a (non-Euclidean) non-Desarguesian geometry for open Lie-admissible systems;

4) Fronteau ((1979) and (1982)), who introduced the foundations of a Lie-admissible statistical mechanics for open systems along the original Jacobi's theorem recalled earlier (see also Fronteau *et al.* (1979));

5) Tellez-Arenas (1982) (see also Tellez-Arenas *et al.* (1979)), who worked out certain aspects of Fronteau's Lie-admissible statistics;

6) Ktorides et al. (1980), who worked out certain mathematical problems (a generalization of the Poincare'-Birkhoff-Witt theorem to flexible Lie-admissible algebras);

7) Mignani ((1982), (1984b), (1985), (1986) and (1989)), who made several important contributions, including the construction of a Lie-admissible generalization of the scattering theory for open systems; 8) Jannussis ((1985), (1986)) and Jannussis *et al.* ((1982), (1983), (1984), (1985), (1986), (1987), (1988), and (1991)), who made a large number of contributions on numerous aspects, including the proof of the direct universality of the Lie-admissible formulations for nonconservative classical and operator systems, the identification of the algebraic structure of the so-called q-algebras, of Caldirola's equation, etc.; 9) Eder

((1981) and (1982)), who developed the Lie-admissible notion of SU(2) spin proposed by Santilli (1978b) and (1981b)) for the case of flexible Lie-admissible algebras with applications to neutrons interferometric experiments (Rauch (1981) and (1982));

10) Animalu ((1982), (1986), (1987) and (1991a)), who worked out a number of developments, including an alternative formulation of Lie-isotopic and Lie-admissible relativities, a Lie-isotopic study of "Dirac's generalization of Dirac's equation", an application of Lie-isotopic theories to superconductivity (1991b), and others;

11) Nishioka ((1983), (1984), (1985), (1986), (1987) and (1988)), who made several important contributions, ranging from Lagrangian to geometrical structures, at the nonrelativistic, relativistic and gravitational levels, as well as in the classical and operator treatments;

12) Aringazin ((1990), (1991) and quoted papers) made several additional contributions;

13) Mijiatovic (1990), who studied several aspects of open systems;

14) Veljanoski *et al.* (1987), who studied certain aspects of open systems;

and others.

Worth a special mention is Gasperini (1985), who presented his Lie-isotopic gravitation as a particular case of the more general Lie-admissible generalization of Einstein's gravitation. Jannussis (1985), Gonzales-Diaz (1986) and others submitted a number of advances in Gasperini's Lie-admissible gravitation for open conditions, e.g., a test particle in the interior of Jupiter considered as external, or the possible Lie-admissible cosmological structure of an open Universe.

Adler (1978) identified a certain form of classical chromodynamics with a Lie-admissible structure of a particular type (called trace-admissible). but this line of studies was not continued. This is regrettable because contemporary string theories can also be shown to possess a Lie-admissible structure.

Unlike the Lie-isotopic algebras (which, rather oddly for mathematicians, have been solely developed in physics Journals), the mathematical literature in Lie-admissible algebras is rather vast (see, e.g, the mathematical bibliography on nonassociative algebras by Balzer *et al.* (1984)).

This completes the most significant classical contributions in Lie-isotopic and Lie-admissible theories known to this author at this time.

The mathematical content of the remaining part of this Volume I will essentially follow the presentation of Santilli (1988b) and (1991a, b). The physical contents of Volume II will essentially follow the most importart memoirs written by this author (Santilli (1978a), (1988a, c, d)).

CHAPTER II:

MATHEMATICAL FOUNDATIONS

II.1: STATEMENT OF THE PROBLEM

In this chapter we shall identify the analytic, algebraic and geometrical tools for the quantitative treatment of classical interior dynamical systems of N particles in their first-order, vector-field form.

From a mathematical viewpoint, these systems are given by the most general known regular, analytic (or C^∞), nonlinear, integro-differential, first-order systems of ordinary differential equations, which can be written as vector-fields in cotangent bundle (phase space) with local variables $a = (r, p)$

$$\dot{a} = (\dot{a}^\mu) = \begin{pmatrix} \dot{r}^{ia} \\ \\ \dot{p}_{ia} \end{pmatrix} = (\Gamma^\mu(t, a, \dot{a}, ..)) = \Gamma =$$

$$= \begin{pmatrix} p_{ia}/m_a \\ \\ F^{SA}_{\ ia}(r) + F^{NSA}_{\ ia}(t, r, p, \dot{p}, ..) + \int d\sigma\, \mathcal{F}^{NSA}_{\ ia}(t, r, p, \dot{p}, ..) \end{pmatrix}, \quad (1.1)$$

$$i = 1, 2, 3\ (x, y, z), \quad a = 1, 2, ..., N, \quad \mu = 1, 2, ... 6N.$$

where: r are the coordinates of the experimenter, the p's represent the linear momenta; the m's are the masses of the N particles all assumed to be non-null; SA and NSA stand for variational selfadjointness and nonselfadjointness, respectively (Santilli (1978e)); and σ reprersents a surface or volume.

More specifically, in this chapter we shall identify mathematical tools which can:

1) Represent forces that are generally nonlinear and nonlocal-integral in all variables, as well as nonlagrangian-nonhamiltonian (footnote[1], p. 5) and non-newtonian (footnote[2], p.10);

2) Permit the construction of generalized space-time symmetries representing conventional total conservation laws; and, last but not least,

3) Admit conventional representations as particular cases when performing the transition to the exterior problem, that is, when motion exits interior physical media and returns in vacuum.

The formulations verifying the above requirements were submitted by Santilli (1978a, c) under the name of *Lie-isotopic formulations*. This chapter is therefore devoted to the study of the analytic, algebraic and geometric branches of the Lie-isotopic formulations.

The fundamental mathematical (and physical) idea is the generalization of the conventional trivial unit I of current theories, I = diag. (1, 1, ...,1), into a quantity $\hat{1}$ which is nowhere null (i.e., everywhere invertible in the considered region of the local variables) and Hermitean (i.e., symmetric and real valued), but otherwise possesses the most general possible, nonlinear and nonlocal dependence on: time t, coordinates r; their derivatives of arbitrary order $\dot{r}$, $\ddot{r}$,.. (or p, $\dot{p}$,...); as well as any other needed quantity, such as the density $\mu = \mu(r)$ of the local medium considered, its local temperature $\tau(r)$, its index of refraction n = n(r) (if any) , etc.

$$1 = \text{diag.}\,(1, 1, \ldots\ldots, 1) \;\Rightarrow\; \hat{1} = \hat{1}(t, r, \dot{r}, \ddot{r}, \mu, \tau, n, \ldots). \qquad (1.2)$$

All our formulations, whether analytic, algebraic or geometric, are then generalized in such a way to admit the quantity $\hat{1}$ as their unit. In particular, systems (1.1) will be represented via a conventional Hamiltonian H = T + V characterizing the selfadjoint forces, and by embedding all nonhamiltonian forces in the generalized unit $\hat{1}$. The insensitivity of conventional formulations to the topology of their unit will then allow the representation of nonlocal-integral interactions.

The similarities as well as the differences with the original equations (4.5) should be noted. One of the most important teachings we have received from Hamilton is that the knowledge of only one function $H = T + V$ we call today the Hamiltonian is not sufficient to represent the physical reality, because we need, in general, $3N + 1$ functions, the Hamiltonian (representing all potential forces) and the external terms F_{ka} (representing the contact nonpotential forces).

This teaching is implemented in its entirety in our isotopic theories. In fact, as we shall see, the $6N \times 6N$-dimensional isounit $\hat{1}$ is diagonalizable and reducible to 3N independent quantities. The representation of physical systems via the isotopic theories therefore calls for the knowledge of $3N + 1$ quantities, the Hamiltonian (for the potential forces) and the 3N-independent elements of the isounit $\hat{1}$ (for the contact nonpotential forces), exactly along Hamilton's teaching.

The motivation for the transition from Hamilton's equations (4.5) to the isotopic theories is that the formers do not admit a consistent algebraic structures (Sect. I.4 and App. II.A[3]), while the latters not only admit a consistent algebra in the brackets of the time evolution, but in particular that algebra results to be Lie.

As we shall see, the assumption of $\hat{1}$ as the generalized unit of the theory has nontrivial mathematical implications, inasmuch as it implies the generalization of each and every notion used in contemporary mathematics, such as: fields, metric spaces, Lie algebras, symplectic geometry, affine geometry, Riemannian geometry, etc.

Physically, generalization (1.2) has equally far reaching implications, inasmuch as it requires a *necessary* generalization of conventional space-time symmetries and, consequently, of contemporary relativities.

The methods for the construction of the generalized formulations are known under the generic name of *isotopies* (Santilli, (*loc. cit.*)). In particular, the generalization $I \Rightarrow \hat{1}$, is called an *isotopic lifting* of the conventional unit I, and the generalized unit $\hat{1}$ is called an *isotopic unit*, or *isounit* for short.

The main idea of the isotopies is that of identifying the ultimate geometric properties and/or axioms of the theory considered, and realizing them in their most general possible way. This generally result in an infinite number of possible isotopies, and explains the reason for the use of the plurals in the title and throughout this volume.

As a first example, the primary properties of the conventional unit

[3] From hereon sections will be denoted with the chapter first, and then the section, while equations other than those of the section at hand, will be denoted with chapter, followed by section and equation number.

I are those of being nowhere singular, real valued and symmetric. The lifting $I \Rightarrow \hat{1}$ is then an isotopy. Later on, in physical applications we shall add the condition of positive-definiteness (which will be instrumental in proving the local isomorphism between the isotopic and conventional symmetries).

A similar situation occurs for fields, metric spaces, algebras, etc., as we shall see.

One aspect which should be brought to the reader's attention since these introductory words is that, owing to the deep inter-relation and mutual compatibility of the various mathematical structures used in dynamics, the isotopies of any one of them require the isotopies of all others.

For instance, the isotopy of an algebra soon requires that of the underlying field which, in turn, requires the isotopy of the space in which their modular action holds which, in turn, requires the isotopy of the applicable geometry, etc.

This is the reason why we shall start with the isotopies of fields, and then pass to those of linear spaces, metric spaces, algebras, geometries, etc.

A second important aspect of our analysis is the restriction of the isotopies to those admitting a well identified (left and right) isounit $\hat{1}$. As well known from a mathematical profile (see, e.g., Jacobson (1962)), the conventional Lie's theory is formulated with respect to the trivial unit I of current use in all its branches (universal enveloping associative algebra, Lie algebras, Lie group, representation theory, etc.). It is then evident that the selection, say, of an isotopy of the associative enveloping algebra which does not possess the unit[4] is bound to be inadequate for the quantitative treatment of interior systems of type (1.1).

The terms *Lie-isotopic theory* are specifically referred to the generalization of the various branches of the conventional Lie's theory when formulated with respect to, and under the condition of the existence of the most general possible isounit $\hat{1}$.

On physical grounds, the existence of the conventional unit I has fundamental implications, e.g., because a measurement theory cannot be consistently formulated without the existence of a left and right unit I of the universal enveloping associative algebra, classically and quantum mechanically. It is evident that no consistent generalization of current physical theories can be achieved without preserving, although in a generalized way, the fundamental notion of unit.

[4] See later on Sect. 4 of Chapter II for specific examples of isotopies of associative algebras without unit..

An intriguing point is that, despite the known interplay between Lie algebras, analytic mechanics and symplectic geometry, Lie-isotopic algebras readily permit the identification of their isounit (e.g., via the appearance of the isotopic element directly in the brackets of the algebra, as shown in Sect. II.5 and II.6) while the same identification was generally unknown until recently in the corresponding analytic and geometric treatments.

As an example, the Birkhoffian generalization of a Hamiltonian mechanics was formulated (Santili ((1978a) and (1982a)) as a classical realization of the Lie-isotopic algebras via product (I.4.13) which, as one can see, has no identification of the isounit of the theory. The two-forms of the corresponding symplectic geometry are equally studied in the literature without any identification of the related isounit (Sect. II.9).

In turn, the lack of identification of the isounit in the analytic and geometric structures is responsible for the restriction of the represented systems to those of nonlinear and nonhamiltonian type although in their local-differential approximation.

These latter aspects were resolved in Santilli (1988a, b, c, d) and (1991a, b). A central objective of this monograph is to identify analytic and geometric formulations that are true counterparts of the Lie-isotopic theory, that is, which admit a readily identifiable isounit in the structure of the brackets and of the two-forms.

In turn, such an identification is at the foundation of the classical nonlocal treatments of physical systems presented in this monograph.

A third and final guideline should be presented from these introductory words. The isotopies essentially represent the "degrees of freedom" of given mathematical axioms and, by central conditions, they produce no new abstract axiomatic structure.

As a matter of fact, this property is so universal that the most effective criterion for ascertaining the mathematical consistency of a given isotopy is that *the conventional and isotopic formulations must coincide, by construction, at the abstract, realization-free level.*

As a result, the reader should not expect the identification of new Lie algebras via the use of isotopies, trivially, because all Lie algebras (over a field of characteristic zero (see next section) are known from Cartan's classification. On the contrary, *the isotopies will merely produce infinitely many different realizations of known abstract Lie algebras.*[5]

[5] This is the case for Lie algebras over a field of characteristic zero (see footnote[7] in the next section) which have been fully classified. The situation for Lie algebras over a field of characteristic p is different inasmuch as their classification is far from being complete, as well known in mathematical circles. It is therefore possible that

Therefore, the Lie-isotopic generalizations of the Galilei and Poincaré symmetries treated in these volumes coincide, by conception and realization, with the conventional Galilei and Poincaré symmetries, respectively. More generally, our isotopies of Galilei's relativity, Einstein's special relativity and Einstein's gravitation for the interior problem will be such to coincide with the conventional exterior relativities at the level of abstract, realization-free formulations.

In short, the isotopies permit the achievement of a rather remarkable unity of mathematical and physical thought in which the fundamental space-time symmetries and physical laws, rather than being abandoned, are preserved in their entirety, and only realized in the most general possible nonlinear and nonlocal way.

The content of this chapter originated in a number of papers identified in the various sections. Its first comprehensive presentation appeared in Santilli (1988b), and was then expanded in (Santilli (1991a, b)). which are essentially followed in this review.

The mathematical literature on isotopies is rather limited indeed. While working on the original proposal (Santilli (1978a)) at the Department of Mathematics of Harvard University[6], this author conducted an extensive search in all Cantabridgean mathematical libraries. The *only* mathematical book that could be identified with the notion of isotopy was Bruck (1958), who points out that the notion dates back to the early stages of *set theory*, whereby *two sets were called isotopically related if they could be made to coincide via permutations*.

An extensive search in abstract algebras revealed that the notion had been applied to associative and (commutative) Jordan algebras (see the mathematical bibliography by Baltzer *et al.* (1984)), but this author could identify in the mathematical literature up to 1978 no application of the notion of isotopy to structures of direct physical relevance, such as Lie algebras, metric spaces, etc.

Still at this writing, to the author's best knowledge, no additional mathematical book has appeared with the notion of isotopy, and the only articles appeared in a mathematical Journal with the names "Lie-isotopic algebras" are those by Aringazin *et al.* (1990) and Santilli (1991a, b).

It appears appropriate to end this section with Bruck's (*loc. cit.*) remark:

"*The notion* [of isotopy] *is so natural to creep in unnoticed.*"

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the use of isotopic techniques may assist in identifying new Lie algebras.

[6] Under contracts from the US Department of Energy Numbers ER-78-S-02-4742.A000, AS02-78ER04742 and DE-AC02-8-OER10651 which are here gratefully acknowledged.

II.2: ISOFIELDS

Recall that a *field* (for a mathematical reference consult, e.g., Albert (1963) and a presentation for physicists is, e.g., that by Roman (1975)) is a set F of elements α, β, γ,... equipped with two (internal) operations, usually called *addition* $\alpha + \beta$ and *multiplication* or *product* $\alpha\beta$, such that

1) *Properties of addition:* For all $\alpha, \beta, \gamma \in F$, $\alpha + \beta = \beta + \alpha$, and $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$; for each element α there is an element 0, the unit for the addition here called *additive unit*, such that $\alpha + 0 = \alpha$ and an element $-\alpha$ such that $\alpha + (-\alpha) = 0$; and the set is not empty, i.e., there exist elements $\alpha \neq 0$;

2) *Properties of multiplication:* for all $\alpha, \beta, \gamma \in F$ we have $\alpha\beta = \beta\alpha$ and $\alpha(\beta\gamma) = (\alpha\beta)\gamma$; for all elements $\alpha \in F$ there exists an element I, the unit for the multiplication here called *multiplicative unit*, such that $\alpha I = I\alpha = \alpha$, and an element α^{-I} such that $\alpha\alpha^{-I} = \alpha^{-I}\alpha = I$; and the equations $\alpha x = \beta$, and $x\alpha = \beta$, for $\alpha \neq 0$, always admit solution;

3) *Distributive laws:* for all $\alpha, \beta, \gamma \in F$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$, and $(\beta + \gamma)\alpha = \beta\alpha + \gamma\alpha$.

Unless otherwise stated, all fields are assumed of characteristics zero[7] throughout the analysis of this volume, so as to avoid fields with an axiomatic structure different than that currently used in physics. The extension of the results of this chapter to fields of characteristic $p \neq 0$ is rather intriguing, but it will be left for brevity to the interested mathematician.

The sets of real numbers $\Re$, complex numbers C and quaternions Q constitute fields with respect to the conventional sum and multiplication. However, the octonions O do not constitute a field because of the loss of the associativity of the product.

DEFINITION II.2.1: Given a field F with elements α, β, γ..., sum $\alpha + \beta$, multiplication $\alpha\beta$, and respective units 0 and I, we define as an "isotope" $\hat{F}$ of F with respect to the multiplication the set F with the same sum $\alpha + \beta$ and related unit 0, but equipped with a new multiplication $\alpha * \beta$ and a new multiplicative unit $\hat{I}$, called

[7] Let F be a field with elements α, β,... If there exists a least positive integer p such that $p\alpha = 0$ for all $\alpha \in F$, then the field F is said to be of *characteristic* p. The fields of reals, complex numbers and quaternions evidently have *characteriztic zero*. Contrary to a general belief ib physics, the classification of simple Lie algebras is still incomplete. In fact, we have today the Cartan classification of *all* simple Lie algebras, but *only over a field of characteristic zero*, because that over a field of characteristic p is still incomplete at this writing.

"multiplicative isounit" which verifies all properties for $\hat{F}$ to be a field.

Thus, an isofield is a field by construction. The basic isofields of this analysis are the *real isofields* $\hat{\Re}$, also called *isoreals* , i.e., the infinitely possible isotopes $\hat{\Re}$ of the field of real numbers $\Re$, which can be symbolically written

$$\hat{\Re} = \{\hat{n} \mid \hat{n} = n\hat{1},\ n \in \Re,\ \hat{1} \neq 0\}, \tag{2.1}$$

and their elements $\hat{n}$ are called *isonumbers*. As per Definition II.2.1, the sum of two isonumbers is the conventional one,

$$\hat{n}_1 + \hat{n}_2 = (n_1 + n_2)\hat{1}. \tag{2.2}$$

To identify the appropriate isoproduct, recall that $\hat{1}$ must be the right and left isounit of $\hat{\Re}$. This is the case if one interprets $\hat{1}$ as the inverse of an element T, called *isotopic element*,

$$\hat{1} = T^{-1}, \tag{2.3}$$

and defines the isoproduct as

$$\hat{n}_1 * \hat{n}_2 \stackrel{\text{def}}{=} n_1 T n_2, \quad T \quad \text{fixed.} \tag{2.4}$$

Then,

$$\hat{1} * \hat{n} = \hat{n} * \hat{1} \equiv \hat{n}, \qquad \text{for all } \hat{n} \in \hat{\Re}, \tag{2.5}$$

as desired.

Note that the isotopic element T need not necessarily be an element of the original field $\Re$, because it can be, say, an integro-differential operator. As we shall see, this feature is of fundamental relevance for the applications of the isotopic theory.

Note also that the lifting $I \Rightarrow \hat{1}$ does not imply a change in the numbers used in a given theory. This can be seen in various ways, e.g., from the fact that the isoproduct of an isonumber $\hat{n}$ times a quantity Q coincides with the conventional product,

$$\hat{n} * Q \equiv nQ, \tag{2.6}$$

and in other ways we shall see in next chapters.

Note finally, from the complete arbitrariness of the isotopic element T in isoproduct (2.4), that the field of real number $\Re$ admits an infinite number of different isotopies.

Another field of basic physical relevance is the complex isofield $\hat{C}$,

$$\hat{C} = \{\hat{c} \mid \hat{c} = c\hat{1},\ c \in C,\ \hat{1} \neq 0\}, \tag{2.7}$$

which plays a fundamental role in the operator formulation of the classical isotopies of this volume. As such it will be considered elsewhere.

An important property of the notion of isofield is that of permitting the unification of all possible fields (of characteristics zero) into one single, abstract field, say $\mathcal{F}$. This unification can be expressed via the following

PROPOSITION II.2.1: The infinitely possible isotopies $\hat{\Re}$ of the field of real numbers $\Re$, called "isoreals", contain, as particular cases, all possible fields of characteristics zero.

PROOF: Let $\Re_0 = \Re 1$ be the field of real numbers with the ordinary unit 1. The field of complex numbers C is an isotope of $\Re$ because it can be written as the axiom-preserving tensorial product

$$C = \hat{\Re} = \Re 1 \times \Re \hat{1}_1,\ \ \hat{1}_1 = i, \tag{2.8}$$

(or, depending on the viewpoint at hand, as the direct sum $C \equiv \hat{R} = \Re 1 + \Re \hat{1}_1$), where i is the conventional imaginary unit. In this case the isounit is the tensorial procuct $\hat{1} = 1 \times \hat{1}_1$, while generic elements have the structure $n = \alpha \times \beta,\ \alpha, \beta \in \Re$. In turn, the field of quaternions Q is an isotope of C and, therefore, of $\Re$, because it can be written as the tensorial product

$$Q \equiv \hat{C} \equiv \hat{\Re}' = \Re 1 \times \Re \hat{1}_1 \times \Re \hat{1}_2 \times \Re \hat{1}_3, \tag{2.9}$$

where $\hat{1}_k = i,\ k = 1, 2, 3$. By keeping in mind that octonions violate the associativity law, structures (2.8) and (2.9) exhaust all possible conventional fields of characteristics zero (Albert (1963)), by therefore proving the proposition. Q.E.D.

Proposition II.2.1 provides the first illustration of rather general unifying capabilities of our isotopies. In fact, we shall see in the next section that all conventional metric and pseudo-metric spaces (such as the Minkowski, the Finslerian and the Riemannian spaces) are isotopes of the Euclidean space, with similar unifications holding for other mathematical structures.

It should stressed that *the class of infinitely possible isotopes $\hat{\Re}$ is substantially more general than the class of fields $\Re$, C and Q.* In fact, the isofield $\hat{\Re}$ are defined with respect to an arbitrary, integro-differential, n×n isounit $\hat{1}$.

As an illustration, the isofields $\hat{\Re}$ include the notion of *quaternionic isofield* $\hat{Q}$ with a structure evidently more general than that of the conventional field Q. Intriguingly, isofields $\hat{Q}$ do not appear to have been studied in the mathematical or physical literature until now

Finally, note that the isotopy $F \Rightarrow \hat{F}$ used in these volumes is solely referred to the multiplication, and not to the addition. Needless to say, a more general notion of isotopy including both sum and multiplication as well as internal and external operations is conceivable, but its study is left to the interested mathematician.

The notion of isofield was submitted by the author at the Clausthal Conference on *Differential Geometric Methods in Mathematical Physics* of 1980, and was subsequently elaborated in Santilli ((1980), (1981) and (1988b)) and Myung and Santilli (1982a).

II.3: ISOSPACES

A *linear space* V (see again Albert (1963) or Roman (1975)) is a set of elements a,b,c,.. over a field F of elements α, β, γ, ... and units 0 and I, equipped with the additions a+b, and a + b, and the multiplications ab, aa, and ab, such that, for all a,b,c $\in$ V and $\alpha, \beta, \gamma \in$ F: $a + b = b + a$; $a + (b + c) = (a + b) + c$; $\alpha(\beta a) = (\alpha\beta)a$; $\alpha(a + b) = \alpha a + \alpha b$; $(a + b)\alpha = \alpha a + \beta a$; for every a $\in$ V there exists an element $-a$ such that $a + (-a) = a - a = 0$; and the multiplicative unit I of F is the right and left unit of V, i.e., $Ia = aI = I$ for all a $\in$ V.

From the above definition one can see that we cannot construct an isotopy of a linear space without first introducing an isotopy of the field, because the multiplicative unit I of the space is that of the underlying field.

DEFINITION II.3.1: Given a linear space V over a field F, the "isotope" $\hat{V}$ of V with respect to the multiplication, or the "isolinear space", is the same set of elements a,b,c,.. $\in$ V defined *over the isofield $\hat{F}$ with multiplicative isounit $\hat{1}$ and is therefore equipped with a new multiplication a*b, which is such to verify all the axioms for a linear space, i.e.,*

$$\alpha*(\beta*a) = (\alpha*\beta)*a, \qquad \alpha*(a + b) = \alpha*a + \alpha*b, \tag{3.1a}$$

$$(\alpha + \beta)*a = \alpha*a + \beta*a, \ \alpha*(a + b) = \alpha*a + \alpha*b, \tag{3.1b}$$

$$\hat{1}*a = a*\hat{1} \equiv a, \tag{3.1c}$$

for all $a,b \in V$ and $\alpha, \beta \in F$.

Note the lifting of the field, but the elements of the vector space remain unchanged. This is a property of important physical consequence, inasmuch as it is at the foundation of the preservation of the conventional generators of Lie algebras under isotopies. In turn, this implies the preservation of conventional conservation laws under lifting.

The interested reader can prove as an exercise a number of properties of isolinear spaces. One which is particularly relevant for this analysis follows from the invariance of the elements a, b, c, ... of the space under isotopy and can be expressed as follows.

PROPOSITION II.3.1: The basis of a linear space V remains unchanged under isotopy.

The above property essentially anticipates the fact that, when studying later on the isotopies of Lie algebras, we shall expect no alteration of its basis, as originally proposed (Santilli (1978a)).

Linear spaces V are also called *vector spaces* in which case their elements a,b,c, are called *vectors.* The isotopes $\hat{V}$ are then called *isovector spaces* and their elements a,b,c *isovectors.*

A *metric space* hereon denoted M(x,g,F) is a (universal) set of elements x, y, z,.. over a field F equipped with a map (function) g: M x M ⇒ F, such that:

$$g(x,y) \geqq 0, \tag{3.2a}$$

$$g(x,y) = g(y,x) \text{ for all } x,y \in M; \qquad g(x,y) = 0 \text{ iff } x \neq 0 \text{ or } y \neq 0^{8}. \tag{3.2b}$$

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[8] This is due to realization (3.3) of the distance. If, instead, one assumes $g(x,y) = (x^i - y^i)g_{ij}(x^j - y^j)$, then $g(x,y) = 0$ iff $x = y$.

$$g(x,y) \leqq g(x,z) + g(y,z) \text{ for all } x,y,z \in M. \tag{3.2c}$$

A *pseudo-metric space*, hereon also denoted with M(x,g,F), occurs when the first condition (3.2a) is removed. Finally, recall that the field of metric spaces generally used in physics is that of the reals $\Re$.

Supppose that the space M(x,g,) is n-dimensional, and introduce the components $x = (x^i)$, $y = (y^i)$, $i = 1, 2, ..., n$. Then, the familiar way of realizing the map g(x,y) is that via a *metric* g of the form

$$g(x,y) = x^i g_{ij} y^j, \tag{3.3}$$

The axiom $g(x,y) > 0$ for metric spaces then implies that g is positive-definite, $g > 0$.

The best physical example of a metric space is the n-dimensional *Euclidean space* hereon denoted with the symbol $E(r,\delta,\Re)$, namely, the vector space E with local charts $r = (r^i)$ and realization of the metric

$$g(r_1,r_2) = r_1{}^i \delta_{ij} r_2{}^j, \tag{3.4}$$

where

$$\delta = (\delta_{ij}) = \text{diag.}\,(1, 1, ..., 1) \tag{3.5}$$

is the matrix of the *Kronecker delta* δ_{ij}.

A pseudo-metric space of primary physical relevance is the (3+1)-dimensional *Minkowski space* hereon denoted $M(x,\eta,\Re)$, namely, the vector space with charts

$$x = (x^\mu) = (x^i, x^4), \quad x^i \in \hat{E}(r,\delta,\Re), \quad x^4 = c_o t, \tag{3.6}$$

where $c_o \in \Re$ represents the speed of light in vacuum. The map is then indefinite,

$$\eta(x,y) = x^\mu \eta_{\mu\nu} y^\nu \gtreqless 0, \tag{3.7}$$

where η is the celebrated Minkowski (1913) metric, hereon assumed of the type

$$\eta = \text{diag.}\,(1, 1, 1, -1). \tag{3.8}$$

Further spaces also relevant in physics are the *Riemannian spaces* hereon denoted $R(x,g,\Re)$, which are the fundamental spaces of Chapter V.

The simplest possible way of constructing an infinite family of isotopes of $M(x,g,F)$ is by introducing n-dimensional, nowhere null and Hermitean isounits

$$\hat{1} = (\hat{1}_i^{\ j}) = (\hat{1}^j_{\ i}), \quad i, j, r, s, = 1, 2,..., n. \tag{3.9}$$

with isotopic elements

$$T = \hat{1}^{-1} = (T_i^{\ j}) = (T^j_{\ i}), \tag{3.10}$$

Then, we can introduce the *isomap*

$$\hat{g}(x,y) = (x^i\, \hat{g}_{ij}\, y^j)\, \hat{1}, \tag{3.11}$$

where the quantity

$$\hat{g} = Tg = (T_i^{\ k}\, g_{kj}), \tag{3.12}$$

shall be called hereon the *isometric*.

The *basis* $e = (e_i)$, $i = 1, 2, ...,n$ of an n-dimensional space $M(x,g,F)$ can be defined via the rules

$$g(e_i, e_j) = g_{ij}. \tag{3.13}$$

Then, under isotopy we have the rules

$$\hat{g}(e_i, e_j) = \hat{g}_{ij}, \tag{3.14}$$

which illustrate the preservation of the basis as per Proposition II.3.1.

The above isotopic generalizations can be expressed as follows.

DEFINITION II.3.2: The "isotopic liftings" of a given, n-dimensional, metric or pseudometric space $M(x,g,\Re)$ over the *reals $\mathfrak{R}$, or "isospaces" for short, are given by the infinitely possible isotopes $\hat{M}(x,\hat{g},\hat{F})$ characterized by: a) the same dimension n and the same local coordinates x of the original space; b) the isotopies of the original metric g into one of the infinitely possible nonsingular, Hermitean "isometric" $\hat{g} = Tg$ with isotopic element T depending on the local variables x, their derivatives $\dot{x}$, $\ddot{x}$, ... with respect to an independent parameter, as well as any needed additional quantity*

$$g \Rightarrow \hat{g} = Tg, \tag{3.15a}$$

$$T = T(x, \dot{x}, \ddot{x},...), \quad \det T \neq 0, \quad T^{\dagger} = T, \quad \det. g \neq 0, \quad g = g^{\dagger}, \tag{3.15b}$$

and c) the lifting the field $\mathfrak{R}$ into an isotope $\hat{\mathfrak{R}}$ whose isounit $\hat{1}$ is the inverse of the isotopic element T, i.e.,

$$\hat{\mathfrak{R}} = \mathfrak{R}\,\hat{1}, \quad \hat{1} = T^{-1} = \hat{\delta}^{-1}, \tag{3.16}$$

with composition now in $\hat{\mathfrak{R}}$

$$(x \,\hat{,}\, y) = (x, Ty)\,\hat{1} = (Tx, y)\,\hat{1} = \hat{1}\,(x, Ty) =$$

$$= (x^i\,\hat{g}_{ij}\,y^j)\,\hat{1} \in \hat{\mathfrak{R}}, \tag{3.17}$$

The liftings of the conventional n-dimensional Euclidean spaces $E(r,\delta,\mathfrak{R})$ over the reals $\mathfrak{R}$ into "Euclidean-isotopic spaces" or "isoeuclidean spaces", are given by the particular case

$$E(r,\delta,\mathfrak{R}) \Rightarrow \hat{E}(r,\hat{\delta},\hat{\mathfrak{R}}], \tag{3.18a}$$

$$\delta = I_{n\times n} \Rightarrow \hat{\delta} = T(r, \dot{r}, \ddot{r},...)\,\delta, \tag{3.18b}$$

$$\det. T \neq 0, \quad T = T^{\dagger}, \quad \det. \hat{\delta} \neq 0, \quad \hat{\delta}^{\dagger} = \hat{\delta}, \tag{3.18c}$$

$$\mathfrak{R} \Rightarrow \hat{\mathfrak{R}} = \mathfrak{R}\hat{1}, \quad \hat{1} = T^{-1} = \hat{\delta}^{-1} \tag{3,18d}$$

$$(r,r) = r^i\,\delta_{ij}\,r^j \Rightarrow (r \,\hat{,}\, r) = (r, \hat{\delta}r)\,\hat{1} = \tag{3.18e}$$

$$= (\hat{\delta}r, r)\,\hat{1} = \hat{1}\,(r, \hat{\delta}r) = [r^i\,\hat{\delta}_{ij}(r, \dot{r}, \ddot{r},..)\,r^j\,]\,\hat{1}, \tag{3.18f}$$

The liftings of the conventional Minkowski space $M(x,\eta,\Re)$ in (3+1)-space-time dimensions are given by the isotopes called "Minkowski-isotopic spaces" or "isominkowski spaces"

$$M(x,\eta,\Re) \;\Rightarrow\; \hat{M}(x,\hat{\eta},\hat{\Re}), \tag{3.19a}$$

$$\eta \;=\; \text{diag.}(1, 1, 1, -1) \;\Rightarrow\; \hat{\eta} \;=\; T(x, \dot{x}, \ddot{x},..)\, \eta, \tag{3.19b}$$

$$\det. T \neq 0, T = T^{\dagger}, \;\; \det \hat{\eta} \neq 0, \eta^{\dagger} = \hat{\eta}, \tag{3.19c}$$

$$\Re \;\Rightarrow\; \hat{\Re} \;=\; \Re \hat{1}, \;\; \hat{1} = T^{-1}, \tag{3.19d}$$

$$\begin{aligned}(x,x) \;=\; x^{\mu}\, \eta_{\mu\nu}\, x^{\nu} \;\Rightarrow\; (x \hat{,} x) \;=\; (x, Tx)\, \hat{1} \;=\; (Tx, y)\, \hat{1} \;=\\ = \hat{1}\,(x, Ty) \;=\; [x^{\mu}\, \hat{\eta}_{\mu\nu}(x, \dot{x}, \ddot{x},..)\, x^{\nu}]\, \hat{1},\end{aligned} \tag{3.19e}$$

Finally, the liftings of a given n-dimensional, Riemannian or pseudoriemannian space $R(x,g,\Re)$ over the reals $\Re$ into the infinitely possible "Riemannian-isotopic spaces" or "isoriemannian spaces" $\hat{R}(x,g,\hat{\Re})$ are given by the particular case

$$R(x,g,\Re) \;\Rightarrow\; \hat{R}(x,\hat{g},\hat{\Re}), \tag{3.20a}$$

$$g = g(x) \;\Rightarrow\; \hat{g} \;=\; T(x, \dot{x}, \ddot{x}, ...)\, g(x), \tag{3.20b}$$

$$\det. g \neq 0, g = g^{\dagger}, \;\; \det. T \neq 0, T = T^{\dagger}, \;\; \det. \hat{g} \neq 0, \hat{g} = \hat{g}^{\dagger}, \tag{3.20c}$$

$$\Re \;\Rightarrow\; \hat{\Re} \;=\; \Re \hat{1}, \quad \hat{1} \;=\; T^{-1}, \tag{3.20d}$$

$$\begin{aligned}(x, y) \;=\; x^{i}\, g_{ij}(x)\, x^{j} \;\Rightarrow\; (x \hat{,} x) \;=\; (x, Tx)\, \hat{1} \;=\; (Tx,x)\, \hat{1} \;=\\ =\; \hat{1}\,(x, Tx) \;=\; [x^{i}\, \hat{g}_{ij}(x, \dot{x}, \ddot{x}, ...)\, x^{j}\,]\, \hat{1}.\end{aligned} \tag{3.20e}$$

The general character of the concept of isotopy is illustrated by the following property of evident proof.

PROPOSITION II.3.2: All possible metric and pseudometric spaces in n-dimension $M(r,g,F)$ can be interpreted as isotopes of the Euclidean space in the same dimension $E(r,\delta,F)$

$$\hat{M}(r,g,\hat{F}): \qquad \hat{F} = F\hat{1}, \qquad \hat{1} = g^{-1}. \tag{3.21}$$

The reader should therefore keep in mind that there is no need to study the isotopies of all spaces, because those of the fundamental Euclidean space are sufficient, and inclusive of all others, as illustrated by the following

COROLLARY: II.3.2.a: The conventional Minkowski space $M(x,\eta,\Re)$ in (3+1) space-time dimensions over the reals $\Re$ can be interpreted as an isotope $\hat{M}(r,\eta,\hat{\Re})$ of the 4-dimensional Euclidean space $E(x,\delta,\Re)$ characterized by the isotopy of the metric

$$\delta = I_{4\times4} = \hat{\delta} = T\delta = \eta = \text{diag.}(1, 1, 1, -1), \tag{3.22}$$

under the redefinition of the fields

$$\Re \Rightarrow \hat{\Re} = \Re\hat{1}, \quad \hat{1} = T^{-1} = \eta^{-1} = \eta. \tag{3.23}$$

The reader should remember that the isotopy of the field is a feature needed for the mathematical consistency, which however does not affect the practical numbers of the theory owing to the property $\hat{N}*x \equiv Nx$, $\hat{N} \in \hat{\Re}$, $x \in M$. Also, as we shall see in Sect. II.8, the symmetries of $\hat{M}(x,\eta,\hat{\Re})$ and those of $M(x,\eta.\Re)$ coincide ebcause characterized by the metric η. Thus, the isotopic Minkowski space $\hat{M}x,\eta,\hat{\Re})$ and the conventional Minkowski space $M(x,\eta,\Re)$ space can be made to coincide for all practical purposes used in physics (see Chapter IV for details).

COROLLARY II.3.2.b: The conventional Riemannian spaces $R(x,g,\Re)$ in (3+1)-space-time dimensions over the reals $\Re$ is an isotope $\hat{R}(x,g,\hat{\Re})$ of the 4-dimensional Euclidean space $E(x,\delta,\Re)$ characterized by the lifting of the Euclidean metric δ into the Riemannian metric g

$$\delta = I_{4\times4} \Rightarrow T\delta = g, \tag{3.24}$$

and by the corresponding lifting of the field

$$\Re \Rightarrow \hat{\Re} = \Re\hat{1}, \quad \hat{1} = T^{-1} = g^{-1}. \tag{3.25}$$

We also have the following alternative interpretation of the Riemannian space.

COROLLARY II.3.2.c: The conventional Riemannian space $R(x,g,\mathfrak{R})$ in (3+1)-space-time dimensions over the reals $\mathfrak{R}$ can be interpreted as an isotope $\hat{R}(x,g,\hat{\mathfrak{R}})$ of the Minkowski space $M(x,\eta,\mathfrak{R})$ in the same dimension characterized by the isotopy of the Minkowski metric

$$\eta = \text{diag.}\,(1, 1, 1, -1) \;\Rightarrow\; T(x)\,\eta = g(x), \tag{3.26}$$

and of the field

$$\mathfrak{R} \Rightarrow \hat{\mathfrak{R}} = \mathfrak{R}\hat{1}, \quad \hat{1} = T^{-1}. \tag{3.27}$$

The notion of isotopy of a metric or pseudometric space is therefore first useful for *conventional* formulations. In fact, *the transition from relativistic to gravitational aspects is an isotopy*. This concept is at the foundations of our study of the *general*[9] *symmetries of conventional gravitational theories* which can be readily studied via the Lie-isotopic theory, as we shall see, but which is otherwise of rather difficult treatment via conventional techniques.

Notice also the *chain of isotopies* illustrated by the above Corollaries, also called *isotopies of isotopies*,

$$E(x,\delta,\mathfrak{R}) \;\Rightarrow\; M(x,\eta,\mathfrak{R}) \;\Rightarrow\; R(x,g,\mathfrak{R}). \tag{3.28}$$

Corollary II.3.2.c is useful to illustrate the insensitivity of the isotopies to the explicit functional dependence of the isounit. The reader can then begin to see the vastity of the isotopies of the Euclidean space, which encompass, not only the Minkowski and Riemannian space, but also all known metric and pseudometric spaces of the same dimension, such as Finslerian spaces, etc., as well as additional classes of infinitely possible, genuine isotopies of the Euclidean, Minkowski, Riemannian and other spaces.

[9] We here distinguish between the *local* symmetry of a gravitational theory, which is eveidently the conventional Poincaré symmetry, from the *general* symmetry of a separation in a Riemannian space under the full gravitational metric, which should not be confused with *global* aspects (e.g., of topological nature).

DEFINITION II.3.3: Given a metric or pseudo-metric space $M(x,g,\Re)$ with metric g, its "isodual" space $\hat{M}^d(x,\hat{g},F\hat{\Re})$ is the isotopic space $\hat{M}$ characterized by the isotopic element

$$T = - I = \text{diag.}\ (-1, -1, -1, ..., -1). \tag{3.29}$$

The isodual of the Euclidean space $E(x,\delta,\Re)$ is therefore the isotope $\hat{E}^d(x,\hat{\delta},\hat{\Re})$ where the isometric is given by

$$\hat{\delta} = - \delta. \tag{3.30}$$

As we shall see, the above spaces are useful for the construction of the isodual realization of given simple Lie groups with rather intriguing implications.

Similarly, the isodual of the Minkowski space $M(x,\eta,\Re)$ is the isospace $\hat{M}^d(x,\hat{\eta},\hat{\Re})$ where the isometric $\hat{\eta}$ is given by

$$\hat{\eta} = T\eta = -\eta = \text{diag.}\ (-1, -1, -1, +1). \tag{3.31}$$

Clearly, the notion of isoduality in Minkowski space allows the mapping of time-like into space-like vactors and viceversa. As such, isodual spaces are at the basis of the generalized Lorentz transformations $x \Rightarrow x'(x)$ introduced by Recami and Mignani (1972) for which

$$x'^{\mu}\ \eta_{\mu\nu}\, x'^{\,\nu} = -\ x^{\mu}\ \eta_{\mu\nu}\, x^{\nu}, \tag{3.32}$$

and they are important to identify certain properties of the isotopies of the Lorentz group (Chapter IV).

The notion of isospace was introduced by Santilli (1983a), with particular reference to the case of the isominkowski space, as a structure necessary for the mathematically consistent formulation of the isotopies of Einstein's special relativity. The notion was subsequently studied in more details for the general case in Santilli (1985a), and specialized to the case of isoeuclidean spaces in Santilli (1985b). The notion of isoriemannian space was introduced in Santilli (1988d) and then studied in more detail in Santilli (1991b).

II.4: ISOTRANSFORMATIONS

Let V and V′ be two linear spaces over the same field F. A *linear transformation* (Albert (1963) or Roman (1975)) is a map f: V ⇒ V′ which preserves both the sum and the multiplication, i.e., it is such that

$$f(a + b) = f(a) + f(b), \tag{4.1a}$$

$$f(\alpha a) = \alpha\, f(a), \tag{4.1b}$$

which can be equivalently written

$$f(\alpha a + \beta b) = \alpha\, f(a) + \beta\, f(b) \text{ for all } a,b \in V \text{ and } \alpha, \beta \in F. \tag{4.2}$$

DEFINITION II.4.1: An "isotopic transformation" is an isomap $\hat{f}$: $\hat{V}$ ⇒ $\hat{V}'$ between two isolinear vector spaces $\hat{V}$ and $\hat{V}'$ of the same dimension over the same isofield $\hat{F}$ which preserves the sum and isomultiplication, i.e., which is such that

$$\hat{f}(\alpha * a + \beta * b) = \hat{f}(\alpha) * \hat{f}(a) + \hat{f}(\beta) * \hat{f}(b)$$

$$\text{for all } a,b, \in V \text{ and } \alpha, \beta \in \Re. \tag{4.3}$$

In physical applications, the spaces V and V′ are usually assumed to coincide, V ≡ V′, in which case the *linear map* f is an *endomorphism* with realizations of the familiar right, modular-associative type

$$x' = Ax,\ x \in V,\ x' \in V'. \tag{4.4}$$

where: A is independent from the local variables; the product Ax is associative; and the notion of module will be treated in more details in the next section. A similar notion would evidently result for a left modular associative action x′ = xA.

The transformations are *nonlinear* when of the form

$$x' = A(x)\, x. \tag{4.5}$$

i.e., when A has an explicit dependence in the local coordinates x. If the x-dependence is of integral type, we shall see that the above transformations are *nonlocal*

Assume now that $\hat{V} \equiv \hat{V}'$. Then the isomap $\hat{f}$ can be realized with the isotransformations characterized by the right modular, associative-isotopic action

$$x' = A{*}x = ATx, \quad T = \text{fixed}. \tag{4.6}$$

where the action $A{*}a$ is still associative. A similar notion would result for a left, modular-isotopic action $x' = x{*}A = xTA$.

DEFINITION II.4.2: An "isotransformation" (4.6) is said to be "isolinear" and/or "isolocal" when the element A is conventionally linear and/or local, respectively, i.e., when all nonlinear and/or nonlocal terms are embedded in the isotopic element T.

A number of properties of isotransformations can be easily proved. At the level of abstract axioms, all distinctions between the ordinary multiplication ab and the isotopic one $a{*}b$ (transformations Ax and $A{*}x$) cease to exist, in which case linear and isolinear spaces (linear and isolinear transformations) coincide.

However, the isotopies are nontrivial, as illustrated by a number of properties. First, one can readily prove the following

PROPOSITION II.4.1: Conventional linear transformations f on an isolinear space $\hat{V}$ violate the conditions of isolinearity.

Explicitly stated, the lifting of the Euclidean spaces and of the Minkowski spaces into their corresponding isospaces requires the necessary abandonment, for mathematical consistency, of the Galilean and Lorentz transformations in favor of suitable isolinear and isolocal generalizations to be identified in the next chapter.

PROPOSITION II.4.2: A transformation $\hat{f}$ which is isolinear and isolocal in an isospace $\hat{V}$ is generally nonlinear and nonlocal in V.

In fact, when explicitly written out, isotransformations (4.6) become

$$x' = ATx = A\,T(t, x, \dot{x}, \ddot{x}, \ldots)\,x. \tag{4.7}$$

the nonlinearity and nonlocality of the transformations then becomes evidently dependent on the assumed explicity form of the isotopic element T.

Another simple but important property is the following

PROPOSITION II.4.3: Under sufficient topological conditions, nonlinear transformations on a linear vector V space can always be cast into an equivalent isolinear form on an isospace $\hat{V}$.

In different terms, given a map f in V which violates the conditions of linearity and/or of locality, there always exist an isotope $\hat{V}$ of V under which $\hat{f}$ is isolinear and/or isolocal. Explicitly, nonlinear transformations (4.5) can always be written

$$x' = A(x)\,x \equiv BT(x)x = B*x, \qquad (4.8)$$

i.e., for A = BT, with B linear.

The above property has important mathematical and physical implications. On mathematical grounds we learn that nonlinearity and nonlocality are mathematical characteristics without an essential axiomatic structure, because they can be made to disappear at the abstract level via isotopic liftings.

In turn, this feature is not a mere mathematical curiosity, but has a number of possible mathematical applications. As an example the isotopies of the current theory of linear equations may be of assistance in solving equivalent nonlinear systems.

On physical grounds, the first application of the notions presented in this section is that of rendering more manageable the formulation and treatment of nonlinear and nonlocal generalizations of Galilean or Lorentzian theories which, if treated conventionally, are of a notoriously difficult (if not impossible) treatment.

The physical implications are however deeper than this. Recall that the electromagnetic interactions are fully treatable with linear and local theories, such as the symmetry under the conventional Lorentz transformations.

One of the central open problems of contemporary theoretical physics (as well as of applied mathematics) is the still unanswered, historical legacy by Fermi (1949) and other Founders of contemporary physics on the nonlinearity and nonlocality of the strong interactions (Sect. I.1).

All attempts conducted until now in achieving a nonlinear and nonlocal extension of current theories via conventional techniques have met with rather serious problems of mathematical consistency and of physical effectiveness, as well known.

Because of their simplicity, our isotopies appear to have all the necessary ingredients for the achievement of a mathematically consistent and physically effective nonlinear and nonlocal generalization of the current theories for the electromagnetic interactions via the mere generalization of the trivial unity I into our isounit $\hat{1}$, and the consequential isotopic generalization of the various notions of field, spaces, transformations, etc.

The mathematical consistency of the isotopies is self-evident from their simplicity. Their physical effectiveness is due to the fact that, given a linear theory on a metric space, all its possible nonlinear and nonlocal generalizations are guaranteed by the mere isotopies of the underlying space, without any need of even considering the equations of motion in their explicit form, as we shall see.

The notion of isotransformation as used in this section was first considered in Santilli (1979) and then studied in Santilli (1983a) as another central tool for the construction of the isotopies of Einstein's special relativity. The notion was then studied in Santilli (1985a, b), (198a, b), (1991a). Additional studies are those by Myung and Santilli (1982a), and by Mignani, Myung and Santilli (1983).

II:5: ISOALGEBRAS

A (finite-dimensional) *linear algebra* U, or *algebra* for short (see Albert (1963) or Behnke *et al.* (1974)) is a linear vector space V over a field F equipped with a multiplication ab verifying the following axioms

$$\alpha(ab) = (\alpha a)b = a(\alpha), \ (ab)\beta = a(b\beta) = (a\beta)b, \qquad (5.1a)$$

$$a(b + c) = ab + ac, \ (a + b)c = ac + bc, \qquad (5.1b)$$

called *right and left scalar and distributive laws*, respectively, which must hold for all elements a,b,c $\in$ U, and α, $\beta \in$ F.

The reader should keep in mind that *the above axioms must be verified by all products to characterize an algebra* (see Appendix II.A for products commonly used in physics which do not characterize a consistent algebra).

Algebras play a fundamental role in physics, as well known, and their use is predictably enlarged by the isotopies. Among the existing large number of algebras, a true understanding of the formulations presented in this volume, as well as for attempting their operator image require a knowledge of the following primary algebras (see, e.g., Albert (1963) and Schafer (1966)):

1) *Associative algebras* A, characterized by the additional axiom (besides laws (5.1))

$$a(bc) = (ab)c, \tag{5.2}$$

for all a,b,c ∈ A, called the *associative law.* Algebras violating the above law are called *nonassociative.* All the following algebras are nonassociative:

2) *Lie algebras* L which are characterized by the additional axioms

$$ab + ba = 0, \tag{5.3a}$$

$$a(bc) + b(ca) + (c(ab) = 0. \tag{5.3b}$$

A familiar realization of the Lie product is given by

$$[a,b]_A = ab - ba, \tag{5.4}$$

with the classical counterpart being given by the familiar Poisson brackets among functions A, B in phase space $T^*E(r,\delta,\mathfrak{R}($ (or the cotangent bundle of Sect. II.9)

$$[A,B]_{\text{Poisson}} = \frac{\partial A}{\partial r^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k}\frac{\partial A}{\partial p_k}. \tag{5.5}$$

3) *Commutative Jordan algebras* J, characterized by the additional axioms

$$ab - ba = 0, \tag{5.6a}$$

$$(ab)a^2 = a(ba^2), \tag{5.6b}$$

A realization of the special commutative Jordan product is given by

$$\{a,b\}_A = ab + ba. \tag{5.7}$$

No realization of the commutative Jordan product in classical mechanics is known at this writing. As an example, the brackets

$$\{A,B\} = \frac{\partial A}{\partial r^k}\frac{\partial B}{\partial p_k} + \frac{\partial B}{\partial r^k}\frac{\partial A}{\partial p_k}, \qquad (5.8a)$$

evidently verify axiom (5.6a), but violate axiom (5.6b).

4) *General Lie-admissible algebras* U (Albert (1948), Santilli (1967) and (1978a)), which are characterized by a product ab verifying laws (5.1), which is such that the attached product $[a,b]_U = ab - ba$ is Lie. This implies, besides (5.1), the unique axiom

$$(a,b,c) + (b,c,a) + (c,a,b) = (c,b,a) + (b,a,c) + (a,c,b), \qquad (5.9)$$

where

$$(a,b,c) = a(bc) - (ab)c, \qquad (5.10)$$

is called the *associator.*

Note that *Lie algebras are a particular case of the Lie-admissible algebras.* In fact, given an algebra L with product $ab = [a,b]_A$, the attached algebra L^- has the product $[a,b]_U = 2\,[a,b]_A$ and, thus, L is Lie-admissible.

Therefore, the classification of the Lie-admissible algebras contains all the Lie algebras. Lie algebras therefore enter in the Lie-admissible algebras in a two-fold way.: first, in their classification and, second, as the attached antisymmetric algebras. Finally, *associative algebras are trivially Lie-admissible.*

The first abstract realization of the general Lie-admissible algebras was given by Santilli ((1978b), Sect. 4.14) and can be written

$$U: \quad (a,b)_A = arb - bsa, \qquad (5.11)$$
$$r,s \ \text{fixed} \in A, \quad r \neq s, \quad r,s \neq 0,$$

where ar, rb, etc., are associative. In fact, the antisymmetric product attached to U is a particular form of a Lie algebra (see below).

The first realization of U in classical mechanics was also identified by Santilli (1969) and (1978a) and it is given by the following product for functions A(r,p) and B(r,p) in $T^*E(r,\delta,\Re)$

$$U: \quad (A,B) = \frac{\partial A}{\partial r^k}\frac{\partial B}{\partial p_k}, \qquad (5.12)$$

namely, *the general, nonassociative Lie-admissible algebras are at the foundations of the structure of the conventional Poisson brackets*, which can be written

$$[A,B]_{Poisson} = [A,B]_U = (A,B) - (B,A), \tag{5.13}$$

5) *Flexible Lie-admissible algebras* U (Albert (1948), Santilli (1978a)), which are characterized by the axioms in addition to (5.1)

$$(a,b,a) = 0, \tag{5.14a}$$

$$(a,b,c) + (b,c,a) + (c,a,b) = 0, \tag{5.14b}$$

where condition (5.14a), called the *flexibility law*, is a simple generalization of the anticommutative law, as well as a weaker form of associativity. An abstract realization of the flexible Lie-admissible product is given by (Santilli (1978b))

$$(a, b)_U = \lambda\, ab - \mu\, ba, \quad \lambda, \mu \in F \tag{5.15}$$

where the products λa, ab, etc. are associative. No classical realization of flexible Lie-admissible algebras has been identified until now, to the best knowledge of this author. As an example, the brackets on $T^*E(r,\delta,\Re)$

$$(A, B) = \lambda \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k} - \mu \frac{\partial B}{\partial r^k} \frac{\partial A}{\partial p_k}, \tag{5.16}$$

are Lie-admissible, but violate the flexibility law.

6) *General Jordan-admissible algebras* U (Albert (1948), Santilli (1978a, b)), which are characterized by a product ab verifying laws (5.1), such that the attached symmetric product $\{a,b\}_U = ab + ba$ is Jordan, i.e., verifies the axiom

$$(a^2,b,a) + (a,b,a^2) + (b,a^2,a) + (a,a^2,b) = 0. \tag{5.17}$$

Again, *associative and Jordan algebras are trivially Jordan-admissible.* Also, Jordan algebras enter in the Jordan-admissible algebras in a two-fold way, in the classification of the latter, as well as the attached symmetric algebras.

It is important for the operator formulation of the isotopies of this volume to point out that the Lie-admissible product (5.11) is, jointly, Lie-admissible and Jordan-admissible (Santilli (1978b)), because the attached symmetric product characterizes a special commutative Jordan algebra (see below).

Finally, we should note that the classical Lie-admissible product (5.12) is only Lie-admissible and not jointly Jordan-admissible.

7) *Flexible Jordan-admissible algebras* U (Albert (1948), Santilli (1978a, b)), which, in addition to axioms (5.1), are characterized by the axioms

$$a(ba) = (ab)a, \tag{5.18a}$$

$$a^2(ba) + a^2(ab) = (a^2b)a + (a^2a)b. \tag{5.18b}$$

The flexible Lie-admissible product (5.15) is also a flexible Jordan-admissible product, but the classical product (5.16) is only Lie-admissible, and not flexible Lie-admissible nor Jordan-admissible.

We now pass to the study of the isotopies of the above notions.

*DEFINITION II.5.1 (Santilli (1978a)): An "isoalgebra", or simply an "isotope" $\hat{U}$ of an algebra U with elements a,b,c,... and product ab over a field F, is the same vector space U but defined over the isofield $\hat{F}$, equipped with a new product a*b, called "isotopic product", which is such to verify the original axioms of U.*

Thus, by definition, the isotopic lifting of an algebra does not alter the type of algebra considered.

It is important for this study (as well as for its operator formulation) to review the isotopies of the primary algebras listed above, beginning with the associative algebras.

Given an associative algebra A with product ab over a field F, its simplest possible *isotope* $\hat{A}$, hereon called *associative-isotopic* or *isoassociative algebra*, is given by

$$\hat{A}_1 : \quad a*b = \alpha ab, \qquad \alpha \in F\text{, fixed and} \neq 0, \tag{5.19}$$

and called a *scalar isotopy*. The preservation of the original associativity is trivial in this case.

A second less trivial isotopy is the fundamental one of the Lie-isotopic theory, and it is characterized by product (Santilli (*loc. cit.*))

$$\hat{A}_2: \quad a*b = aTb, \tag{5.20}$$

where T is an nonsingular (invertible) and Hermitean (real valued and symmetric) element not necessarily belonging to the original algebra A. The associativity of product (5.20) can also be readily proved.

Note the necessary condition, from Definition II.5.1, that the isoproduct and isounit in U and F coincide. This is the technical reason for the lifting of the universal enveloping associative algebras of a Lie algebra (Sect. II.6) into a form whose center coincides with the isounit of the underlying isofield.

The reader should keep in mind that the identity of the isoproduct and isounit for U and F occurs in the associative cases (5.19) and (5.20), but does not hold in general, e.g., for nonassociative algebras. This is due to the lack of general admittance of a unit, while such a unit is always well defined in the underlying field.

Only a third significant isotopy of an associative algebra is known, to the author's best knowledge. It is given by (Santilli (1981b))

$$\begin{aligned} \hat{A}_3: \quad & a*b = wawbw, \\ & w^2 = ww = w \neq 0, \end{aligned} \tag{5.21}$$

Additional isotopies are given by the combinations of the preceding ones, such as

$$\begin{aligned} \hat{A}_4: \quad & a*b = wawTwbw, \\ & w^2 = ww = w \neq 0, \end{aligned} \tag{5.22}$$

and

$$\begin{aligned} \hat{A}_5: \quad & a*b = \alpha wawTwbw, \\ \alpha \in F, \quad & w^2 = w, \quad a, w, T \neq 0. \end{aligned} \tag{5.23}$$

It is believed that the above isotopies (of which only the first three are independent) exhaust all possible isotopies of an associative algebra (over a field of characteristics zero, see footnote[6], page 31), although this property has not been rigorously proved to this writing.

The issue is not trivial, physically and mathematically. In fact, any new isotopy of an associative algebra implies a potentially new mechanics, while having intriguing mathematical implications (see later on Lemma II.5.1).

It should be finally indicated that this author has selected isotopy (5.20) over (5.21) because the former possesses a well defined isounit, while the latter does not, thus creating a host of problems of physical consistency in its possible use for an operator theory.

Nevertheless, the study of isoassociative algebras (5.21) remains intriguing indeed, although it has not yet been conducted in the mathematical and physical literatures, to our best knowledge.

We now pass to the study of the *isotopes* $\hat{L}$ of a Lie algebra L with product ab over a field F, which are the same vector space L but equipped with a *Lie-isotopic product* $a \times b$ over the isofield $\hat{F}$ which verifies the left and right scalar and distributive laws (5.1), and the axioms

$$a \times b + b \times a = 0, \tag{5.24a}$$

$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0, \tag{5.24b}$$

Namely, the abstract axioms of the Lie algebras remain the same by assumption.

The simplest possible *realization of the Lie-isotopic product* is that attached to isotopes $\hat{A}_1$, Eq. (5.19) (*loc. cit.*)

$$\hat{L}_1: \quad a \times b = [a,b]_{\hat{A}_1} = a*b - b*a = \alpha(ab - ba) = \alpha\,[a,b]_A, \quad \alpha \in F, \quad \alpha \neq 0 \tag{5.25}$$

and it is also called a *scalar Lie-isotopy*. It is generally the first lifting of Lie algebras one can encounter in the operator formulation of the theory.

The second independent realization of the Lie-isotopic algebras is that characterized by the isotope $\hat{A}_2$, also introduced in (*loc. cit.*)

$$\hat{L}_2: \quad a \times b = [a,b]_{\hat{A}_2} = a*b - b*a = aTb - bTa, \tag{5.26}$$

The third, independent isotopy is that attached to $\hat{A}_3$, and it was introduced in Santilli (1981b)

$$\hat{L}_3: \quad a \times b = [a,b]_{\hat{A}_3} = wawbw - wbwaw, \quad w^2 = ww \neq 0. \tag{5.27}$$

A fourth isotope is that attached to $\hat{A}_4$, i.e.,

$$\hat{L}_4: \quad a \times b = [a,b]_{\hat{A}_4} = waw T wbw \; - \; wbw T waw, \qquad (5.28)$$
$$w^2 = w, \quad w, T \neq 0.$$

A fifth and final (abstract) isotope is that characterized by $\hat{A}_5$, i.e.

$$\hat{L}_5 : a \times b = [a,b]_{\hat{A}_5} = \alpha \, [a,b]_{\hat{A}_4}. \qquad (5.29)$$

Again, it is believed that the above five isotopes exhaust all possible abstract Lie algebra isotopies, although this property has not been proved to date on rigorous grounds.

Note that the Lie algebra attached to the general Lie-admissible product (5.11) are not conventional, but isotopic. In fact, we can write

$$[a,b]_U = (a,b)_A - (b,a)_A = arb - bsa - bra + asb, \qquad (5.30a)$$

$$= aTb - bTa = a*b - b*a, \qquad (5.30b)$$

$$r \neq s, \quad r, s, T \neq 0, \quad T = r + s \neq 0.$$

As a matter of fact, this author first encountered the Lie-isotopic algebras by studying precisely the Lie content of the general Lie-admissible algebras (Santilli (1978a)).

The following property can be easily proved from properties of type (5.30).

LEMMA 5.1: An abstract Lie-isotopic algebra $\hat{L}$ attached to a general, nonassociative, Lie-admissible algebra U, $\hat{L} \approx U^-$, can always be isomorphically rewritten as the algebra attached to an isoassociative algebra $\hat{A}$, $\hat{L} \approx \hat{A}^-$, and viceversa, i.e.

$$\hat{L} \approx U^- \approx \hat{A}^-. \qquad (5.31)$$

The above property has the importance consequence that *the construction of the abstract Lie-isotopic theory does not necessarily require a nonassociative enveloping algebra because it can always be done via the use of an isoassociative enveloping algebra.* In turn, this focuses again the importance of knowing all possible isotopes of an associative algebra, e.g., from the viewpoint of the representation theory.

As an example, the studies by Eder ((1981) and 1982)) on a conceivable spin fluctuation of thermal neutrons caused by sufficiently intense external nuclear fields, are formulated via a flexible *nonassociative,* Lie-admissible generalization of the enveloping associative algebra of Pauli's matrices. As such, these studies can be identically reformulated via an *associative-isotopic* enveloping algebra. The consequential simplification of the structure is then expected to permit further physical advances.

Note also that *the construction of the abstract Lie-isotopic theory necessarily requires the isotopies of conventional associative envelopes.*

As typical for all abstract formulations of Lie's theory, the Lie-isotopies indicated above are in a form readily interpretable in terms of operators. As such, they put the foundations for the operator formulations of the generalized relativities of this volume, to be studied in a separeate work. Note in particular the identification of the inverse of the isounit $\hat{1}$ in the structure of product (5.30).

A primary objective of this monograph is that of identifying the *classical realizations of the Lie-isotopic product* in such a way to admit a ready identification of the isounit, as pointed out in Sect. II.1.. The latter problem will be the subject of subseqwuent sections of this chapter. At this point, we shall identify some classical realizations without the identification of their underlyiong isounit.

The most general possible, classical realization of Lie-isotopic algebras via functions A(a) and B(a) in $T^*E(r,\delta,\Re)$ with local chart

$$a = (a^\mu) = (r, p) = (r^i, p_i), \quad i, = 1, 2,..., n, \quad \mu = 1, 2,...., 2n, \tag{5.32}$$

is provided by the *Birkhoffian brackets* (Santilli (1978a), (1982a)) also called *generalized Poisson brackets* (see, e.g., Sudarshan and Mukunda (1974)),

$$(A \, , B)_{\text{Birkhoff}} = (A,B)_U = \frac{\partial A}{\partial a^\mu} \; \Omega^{\mu\nu}(a) \; \frac{\partial B}{\partial a^\nu}, \tag{5.33}$$

where $\Omega^{\mu\nu}$, called the *Lie-isotopic tensor,* is the contravariant form of (the exact, symplectic) *Birkhoff's tensor* (Santilli (1978a)) and (1982a))

$$\Omega^{\mu\nu} = (|\Omega_{\alpha\beta}|^{-1})^{\mu\nu}, \tag{5.34a}$$

$$\Omega_{\mu\nu} = \frac{\partial R_\nu(a)}{\partial a^\mu} - \frac{\partial R_\mu(a)}{\partial a^\nu}, \tag{5.34b}$$

where the R's are the so-called *Birkhoff's functions*, and the symplectic character of the covariant tensor (5.34b) ensures the Lie-isotopic character of brackets (5.33) (see the geometric, algebraic and analytic proofs in Santilli (1982a)).

Recall that, unlike the conventional, abstract, Lie brackets (5.4), the conventional Poisson brackets (5.5) characterize a Lie algebra attached to a *nonassociative* Lie-admissible algebra U, Eq.s (5.12). It is then evident that the covering Birkhoff's brackets (5.33) are also attached to a nonassociative Lie-admissible algebra, although of a more general type (see Santilli (*loc. cit.*) for details).

Numerous other classical Lie-isotopic brackets exist in the literature, the most notable being Dirac's generalized brackets for systems with subsidiary constraints (Dirac (1964)).

Note the lack of identification of the underlying generalized unit in Birkhoff's brackets (5.33), as well as in Dirac's brackets. This problem will be studied in Sect.s II.8 and II.9.

Realizations of the abstract isotopes $\hat{U}$ of the Lie-admissible algebras can be easily constructed via the above techniques. For instance, an isotope of the general Lie-admissible product (5.11) is given by

$$\hat{U}: \quad (a \,\hat{,}\, b) = wawrwbw - wbwswaw, \qquad w^2 = w, \quad w,r,s \neq 0, \quad r \neq s. \tag{5.35}$$

An isotope of the classical realization (5.12) is then given by

$$\hat{U}: \qquad (A \,\hat{,}\, B) = \frac{\partial A}{\partial a^\mu}\, S^{\mu\nu}(t,a)\, \frac{\partial B}{\partial a^\nu}, \tag{5.36}$$

where the tensor $S^{\mu\nu}$, called the *Lie-admissible tensor*, is restricted by the conditions of admitting Birkhoff's tensor as the attached antisymmetric tensor, i.e.,

$$S^{\mu\nu} - S^{\nu\mu} \equiv \Omega^{\mu\nu}, \tag{5.37}$$

(see Appendix II.A and, for a detailed study, Santilli (1981a)).

II.6: LIE-ISOTOPIC THEORY

To avoid an excessive length of this volume, in this section we can only outline the central ideas of the Lie-isotopic theory, with particular reference to those aspects needed for the physical objectives of the following chapters.

The theory was originally proposed by Santilli (1978a)). A first review appeared in Santilli (1982a), while recent reviews can be found in Aringazin *et al.* ((1990) and 1991)) andf in Kadeisvili (1992).

The literature on the conventional formulation of Lie's theory is so wast to discourage even a partial outline. A mathematical treatment of structural theorems on universal enveloping associative algebras and other aspects can be found in Jacobson (1962). A physical treatment of the theory can be found in Gilmore (1974). The reprersentation theory can be found, e.g., in Barut and Raczka (1977). Classical realizations of the theory are available in Sudarshan and Mukunda (1974).

In the following we shall first outline the Lie-isotopic theory in its abstract formulation, (i.e., a formulation admitting a direct interpretation via matrices), and then point out its classical realization (i.e., a realization in terms of functions in phase space).

To begin, let us recall that the conventional formulation of Lie's theory is based on the notion of *unit* I realized in its simplest possible form, e.g., via the unit value $1 \in \Re$ for the case of a scalar representation, or the trivial n-dimensional unit matrix I = Diag. (1,1,...,1) for the case of an n-dimensional matrix representation, and so on.

In this case, the *universal enveloping associative algebra* A (Jacobson (1962)) with elements a, b, c, ... over a field F (again assumed of characteristic zero) has the structure

$$A: \quad ab = \text{ass. product}, \qquad Ia = aI = a \quad \text{for all } a \in A. \tag{6.1}$$

The *Lie algebra* L (Jacobson (*loc., cit.*)) is then homomorphic to the antisymmetric algebra A^- attached to A characterized by the familiar commutator

$$L: \quad [a,b]_A = ab - ba\,. \tag{6.2}$$

Connected Lie groups G (Gilmore (1974)) can then be defined via power series expansions in A, according to the familiar form for one dimension

$$G: \quad g(w) = \exp_A^{iwx} = I + (iwx) / 1! + (iwx)(iwx) / 2! + \ldots \quad (6.3)$$
$$w \in F, \qquad x = x^{\dagger} \in A.$$

with well known generalizations to more than one dimension, as well as to discrete components such as the inversions (Gilmore (*loc. cit.*)).

As recalled in Sect. II.1, the central idea of the Lie-isotopic theory is to realize Lie's theory with respect to the most general possible unit $\hat{1}$ which, besides invertibility and Hermiticity, has no restriction on its functional dependence. As such, $\hat{1}$ can have a generally nonlinear dependence on all possible or otherwise needed quantities. For an operator interpretation of the theory (see below for its classical counterpart), such a dependence is on an independent parameter t, coordinates x, velocity $\dot{x}$ = dx/dt or momenta p), accelerations $\ddot{x}$ = d^2x/dt^2 (or $\dot{p}$), wavefunctions ψ, their conjugate $\psi^{\dagger}$, their derivatives $\partial\Psi$ = $\partial\psi/\partial x$ and $\partial\Psi^{\dagger}/\partial x$, etc.,

$$\hat{1} = \hat{1}(t, x, \dot{x}, \ddot{x}, \Psi, \Psi^{\dagger}, \partial\Psi, \partial\Psi^{\dagger}, \ldots) . \quad (6.4)$$

Furthermore, Lie's theory is known to be insensitive to the topology of its unit. As a result, the generalized unit $\hat{1}$ can be, not only nonlinear, but also nonlocal in all its variables.

The Lie-isotopic theory therefore has the necessary characteristics to admit, *ab initio*, nonlinear, nonlocal, nonhamiltonian and non-newtonian forces of systems (II.1.1), provided that they are all incorporated in the generalized unit (see the subsequent chapters for their analytic representation).

The reader should be aware that representations of nonlocal forces outside the unit of the theory would require a new topology precisely of nonlocal-integral type which, at any rate, does not appear to be available at the pure mathematical level at this time in a form usable for physical applications.

It is easy to see that the lifting $I \Rightarrow \hat{1}$ requires a necessary, corresponding, generalization of the entire structure of Lie's theory. In fact, for $\hat{1}$ to be the left and right unit, the universal enveloping algebra, say ξ, of Lie's theory must be generalized into the form, say, $\hat{\xi}$, which is the same vector space as ξ, but now equipped with the generalized product (II.5.20) , i.e.,

$$\hat{\xi}: \quad a*b = aTb, \quad (6.5)$$

where T is fixed, invertible and Hermitean.

As shown in Sect. II.5, the new product $a*b$ is still associative and, for this reason, $\hat{\xi}$ is called *associative-isotopic algebra*, or *isoassociative algebra* for short (Santilli (1978a)). Under the assumption $\hat{1} = T^{-1}$, $\hat{1}$ is indeed the correct right and left unit of the theory, i.e.,

$$\hat{1} = T^{-1}, \tag{6.6a}$$

$$\hat{1}*a = a*\hat{1} \equiv a, \quad \text{for all} \quad a \in \hat{A}, \tag{6.6b}$$

and is called the *isounit* (*loc. cit.*).

Owing to the isotopic character of the generalizations (often referred to as *liftings*), the structural theorems of conventional universal enveloping associative algebras ξ, such as the *Poincaré-Birkhoff*[10]*-Witt Theorem* for the infinite-dimensional basis (see Jacobson (*loc. cit.*)), admit consistent extensions to the *iso-associative envelope* $\hat{\xi}$, as shown since the original proposal (Santilli (*loc. cit.*)).

In particular, the ordered basis $X = (X_i)$, $i = 1, 2, \ldots, n$, of the original Lie algebra L is left unchanged by the isotopy, as anticipated in Sect. II.3. In fact, Proposition II.3.1. applies to L as a vector space, thus preserving the basis X.

In the transition to the underlying associative algebra we have evidently the same occurrence. However, when ξ is turned into an *isotopic envelope*, the original infinite-dimensional basis of ξ is lifted into the form characterized by the *isotopic Poincaré-Birkhoff-Witt Theorem*

$$\hat{\xi}: \; \hat{1}, \;\; X_i, \;\; X_i*X_j \; (i \leq j), \;\; X_i*X_j*X_k \; (i \leq j \leq k), \ldots.. \tag{6.7}$$

For brevity, we refer the interested reader to the reviews by Santilli (1982a) and Aringazin *et al.* (1990). Thus, $\hat{A}$ is indeed, a *bona fide, universal enveloping isoassociative algebra.*

Additional associative isotopies independent from form (6.5) are presented in Sect. II.5. Isotopy (6.5) is however the fundamental one of this analysis because it admits a right and left generalized unit.

The Lie algebras, say, $\hat{L}$ are now homomorphic to the antisymmetric algebra $\hat{\xi}^-$ attached to $\hat{\xi}$, $\hat{L} \approx \hat{\xi}^-$, with the new product (II.5.26), i.e.,

[10] It should be indicated that the name "Birkhoff" here refers to the author's former colleague at the Department of Mathematics of Harvard University, G. Birkhoff, son of G.D.Birkhoff, the discoverer of Birkhoff's equations.

$$\hat{L}: \quad [a,b]_{\hat{\xi}} = a*b - b*a = aTb - bTa, \tag{6.8}$$

which verifies the Lie-algebra axioms (II.5.24), while possessing a structure less trivial than the simplest possible Lie product "ab - ba" of current use. For this reason, the algebras $\hat{L}$ were called *Lie-isotopic algebras* (Santilli (1978a)).

The interplay between the Lie-isotopic algebra $\hat{L}$ and its isoenvelope $\hat{\xi}$ is intriguing. Consider an n-dimensional Lie algebra L with ordered basis X. In the conventional theory, the Poincaré-Birkhoff-Witt Theorem then characterizes the envelope ξ such that

$$\xi = \xi(L), \qquad [\xi(L)]^{-} \approx L. \tag{6.9}$$

The corresponding context of the covering Lie-isotopic theory is considerably broader. In fact, the isotopic Poincaré-Birkhoff-Witt Theorem now characterizes an isoenvelope $\hat{\xi}$ which, since it is constructed via the original basis of L, was denoted from its original formulation as $\hat{\xi} = \hat{\xi}(L)$ (and not as $\hat{\xi}(\hat{L})$). The novelty is that now, in general, we have

$$[\hat{\xi}(L)]^{-} \not\approx L, \qquad [\hat{\xi}(L)]^{-} \approx \hat{L}, \qquad \hat{L} \not\approx L. \tag{6.10}$$

More particularly, the original envelope $\xi(L)$ can characterize only one Lie algebra, the algebra L. On the contrary, it has been shown that *the infinite number of possible isoenvelopes $\hat{\xi}(L)$ for each given original algebra L can characterize in one, single, unified algorithm $\hat{\xi}(L)$ all possible Lie algebras $\hat{L}$ of the same dimension, with the sole possible exception of the exceptional Lie algebras*[11]. It is hoped that, in this, way the reader can begin to see the power of geometric unification of our isotopies.

Again, owing to the isotopic character of the lifting, conventional structural theorems of Lie algebras, such as the celebrated *Lie's First, Second and Third Theorems* (Gilmore (1974)), admit consistent Lie-isotopic generalizations identified since the original proposal. We refer the interested reader for brevity to the locally quoted reviews.

Most importantly, the conventional *structure constants* $C_{ij}{}^{k}$ of a Lie algebra are generalized under isotopy into the *structure*

[11] The exclusion of exceptional Lie algebras is due to the assumption of the Hermiticity of the isotopic element (see also Sect. II.8).

functions $\hat{C}_{ij}{}^{k}(t,x,\dot{x},\ddot{x},...)$ as requested from the Lie-isotopic Second Theorem with isocommutation relations for the generators

$$\hat{L}: \quad [X_i, X_j]_{\hat{\xi}} = X_i * X_j - X_j * X_i = \hat{C}_{ij}{}^{k}(r,x,\dot{x},\ddot{x},...)\, X_k. \qquad (6.11)$$

where the $\hat{C}$'s are restricted by certain integrability conditions originating from the *Lie-isotopic Third Theorem* (Santilli *(loc. cit.)*).

An important objective of this volume is to review the classification of all infinitely possible isotopes of given simple Lie algebras for the physically important cases O(3) (Chapter III) and O(3.1) (Chapter IV).

As we shall see, for all isotopes $\hat{L}$ of given, conventional, simple Lie algebras L with basis X_i and structure constants $C_{ij}{}^{k}$ emerging in the phhysical applications pof Volume II, rules (6.10) always admit a reformulation X'_i of the basis while keeping the isotopic element $T > 0$ unchanged, which recovers the conventional structure constants, i.e.,

$$\hat{L}: \quad [X'_i, X'_j]_{\hat{\xi}} = X'_i * X'_j - X'_j * X'_i = C_{ij}{}^{k}\, X_k \qquad (6.12)$$

In turn, this is evidently useful to prove the local isomorphism of the infinitely possible isotopes $\hat{L}$ with the original simple algebra L, $\hat{L} \approx$ L, by keeping in mind that, for an isotopic element of an arbitrary topology, $\hat{L} \not\approx$ L, as pointed out earlier.

More particularly, the conventional Cartan's classification of simple Lie algebras is intended to identify the nonisomorphic simple algebras of the same dimension, e.g.,

$$\text{Simple 3-dim. algebras: } O(3), \text{ and } O(2.1), \qquad (6.13a)$$

$$\text{Simple 6-dim. algebras: } O(4), \quad O(3.1), \quad \text{and} \quad O(2.2), \qquad (6.13b)$$

etc. (or algebras isomorphic to the above; see, e.g., Gilmore (*loc. cit.*)).

The covering Lie-isotopic theory allows instead the representation of all simple Lie algebras (6.13) with one, unique, abstract, simple, Lie-isotopic algebra of the same dimension,

$$\text{Simple 3-dim, isoalgebra } \hat{O}(3), \qquad (6.14a)$$

$$\text{Simple 6-dim. isoalgebra } \hat{O}(6), \qquad (6.14b)$$

The recovering of different, generally nonisomorphic, and compact or noncompact algebras is then reduced to the mere realization of the isounit $\hat{1}$.

The above results are expected from the capability indicated earlier of the universal enveloping isoassociative algebra $\xi(L)$ of representing all Lie algebra of the same dimension, with the sole possible exclusion of the exceptional Lie algebras. Unification (6.14a) will be studied in detail in Chpo. III of Vol. II, and unification (6.14b) in Chapter IV..

A technical understanding of the above unification is a necessary pre-requisite for the understanding of certain physical results of this monograph, such as the geometric unification of Einstein's special relativity in a Minkowski space, with Einsten's gravitation in a Riemannian space, as well as all their isotopic generalizations for the interior problem (Chapter IV), which is achieved via one unique, abstract notion, that of the Poincaré-isotopic symmetry, admitting of an infinite number of different realizations, whether in Minkowskian, or in Riemannian or in more general spaces.

The reader should keep in mind that, in physical applications, the generators have a direct physical meaning. The isotopic algebras with a direct physical meaning therefore remain structures (6.9), while reformulations (6.10) lose the directly physical meaning of the generators and, as such, they generally carry a sole mathematical meaning.

We also recall the *isodifferential rule* for the isocommutators

$$[A*B,C]_{\xi} = A*[B,C]_{\xi} + [A,C]_{\xi}*B, \tag{6.15a}$$

$$[A,B*C]_{\xi} = [A,B]_{\xi}*C + B*[A,C]_{\xi}, \tag{6.15b}$$

which is based on the fact that the *conventional* product AB of elements A and B of the isoassociative envelope has no mathematical or physical meaning in ξ, and must be replaced with the isotopic product $A*B$.

In particular, this implies that all conventional operations based on multiplications are now inapplicable to the isotopic theory. As an example, the insistence in the use of the *conventional square*

$$a^2 = aa, \tag{6.16}$$

such as the magnitude of the angular momentum

$$J^2 = \sum_{k=1,2,3} J_k J_k, \tag{6.17}$$

within the context of the isoenvelope $\hat{\xi}$ would imply the violation of isolinearity and numerous other inconsistencies (Sect.s II.3, II.4). The correct quantity is evidently given by the *isotopic square*

$$a^{\hat{2}} = a * a. \tag{6.18}$$

such as the isotopic magnitude of the angular momentum

$$J^{\hat{2}} = \sum_{k=1,2,3} J_k * J_k, \tag{6.19}$$

The lack of knowledge of these basic elements is reason for considerable confusions. In fact, readers not familiar with the Lie-isotopic theory tend to preserve the old notion of square, say, of the angular momentum under isotopy, by therefore resulting in a host of mathematical and physical inconsistencies of which they are generally unaware.

Additional isotopies of Lie algebras independent of (6.7) are presented in Sect. II.5, although structure (6.7) is the fundamental one for the analysis of this volume, as well as for its possible operator extensions.

Finally, connected Lie groups cannot be any longer defined via power series in ξ (which would violate the linearity condition), and must be defined in the new envelope $\hat{\xi}$ via infinite basis (6.7) with expressions of the following type called *isoexponentiation*

$$\hat{G} : \hat{g}(w) = \exp_{\hat{\xi}}{}^{iwX} = \hat{1} + (iwX) / 1! + (iwX)*(iwX) / 2! + \ldots =$$

$$= \hat{1}\,[\exp_{\xi}{}^{iwTX}] = [\exp_{\xi}{}^{iXTw}]\,\hat{1}, \tag{6.20}$$

with corresponding expressions for more than one dimension as well as for discrete components.

The elements $\hat{g}(w)$ cannot evidently verify the old group laws (Gilmore (*loc. cit.*)) but must verify instead the *isotopic group laws*

$$\hat{g}(w)*\hat{g}(w') = \hat{g}(w')*\hat{g}(w) = \hat{g}(w + w'), \tag{6.21a}$$

$$\hat{g}(w) * \hat{g}(-w) = \hat{g}(0) = \hat{1}, \tag{6.21b}$$

where the associativity of the isotopic group product $\hat{g}(w) * \hat{g}(w')$ is understood.

All Lie groups verifying the above laws were called *Lie-isotopic groups* (Santilli (1978a)). Again, we refer the interested reader to the locally quoted literature for the isotopic lifting of conventional theorems on Lie groups.

As an example the *isotopic lifting of the Baker-Campbell-Hausdorff Theorem* (Santilli (*loc. cit.*)) is given by

$$[e_{\hat{\xi}}^{X_i}] * [e_{\hat{\xi}}^{X_j}] = e_{\hat{\xi}}^{X_k}, \tag{6.22a}$$

$$X_k = X_i + X_j + [X_i, X_j]_{\hat{\xi}} / 2 + [(X_i - X_j), [X_i, X_j]_{\hat{\xi}}]_{\hat{\xi}} / 12 + \ldots\ldots \tag{6. 22b}$$

As now predictable from the preceding remarks on isoenvelopes and Lie-isotopic algebras, *the covering Lie-isotopic theory is expected to unify into one, single, abstract, n-dimensional, Lie-isotopic group $\hat{G}(n)$ all possible conventional Lie groups in the same dimensions G(n), with the sole possible exclusion of the exceptional groups.*

In fact, as we shall see in Chapter III, the abstract isotope $\hat{O}(3)$ can smoothly interconnect, as particular cases, the compact and noncompact Lie groups O(3) and O(2,1), with similar results for other dimensions as well as for inhomogeneous (non-simple) Lie groups (see Chapter IV)

With the terms *Lie-isotopic theory* we shall specifically refer to the collection of formulations based on: 1) the universal enveloping isoassociative algebras, 2) the Lie-isotopic algebras, and 3) the Lie-isotopic groups, including all related structural theorems.

Needless to say, an inspection of the quoted literature indicates that the theory is just at the beginning and so much remains to be done. Nevertheless, the main structural lines developed so far are amply sufficient for the primary objectuives of this monograph, i.e., the construction of Lie-isotopic generalizations of conventional relativities, as we shall see.

The Lie-isotopic generalization of the conventional formulation of Lie's theory was submitted along structural lines conceptually similar to those of the *Birkhoffian generalization of Hamiltonian mechanics* (Santilli (1982a)), i.e., under the condition that the generalized theory coincides with the conventional one at the abstract, realization-free level. In fact, *the isoenvelopes $\hat{\xi}$, the Lie-isotopic algebras $\hat{L}$ and the Lie-isotopic groups $\hat{G}$ coincide by construction with the original structures ξ, L and G, respectively, at the abstract, realization-free level.*

Note that, by central assumption, *the Lie-isotopies preserve the generators and parameters of the original group and generalize instead the structure of the group itself in an axiom preserving way.* These features are of central relevance for the characterization of closed non-Hamiltonian and non-Newtonian systems via Lie-isotopic symmetries, as we shall see in the subsequent chapters.

The representation of the time evolution is evidently of fundamental physical importance. In the conventional case it is given by a one-dimensional Lie group G_1 with the Hamiltonian H as generator and time t as parameter.

For the Lie-isotopic case, we have instead the more general structure in finite and infinitesimal forms for an arbitrary quantity Q(t)

$$\hat{G}_1 : \quad Q(t) = [e_{\xi}^{-itH}]*Q(0)*[e_{\xi}^{itH}] = \hat{1}[e_{\xi}^{-itTH}]*Q(0)*[e_{\xi}^{iHTt}]\hat{1}$$

$$= e_{\xi}^{-itHT}\, Q(0)\, e_{\xi}^{iTH}, \tag{6.23a}$$

$$i\,\frac{dQ(t)}{dt} = [G,H]_{\hat{A}} = Q*H - H*Q = QTH - HTQ =$$

$$= Q\, T(t,x,p,\dot{p},\psi,\psi^{\dagger},\partial\psi,\partial\psi^{\dagger}\ldots)\, H - H\, T(t, x,\dot{p},\ddot{p},\psi,\psi^{\dagger},\partial\psi,\partial\psi^{\dagger},\ldots)\, G. \tag{6.23b}$$

$$H = H^{\dagger}, \quad T = T^{\dagger}, \tag{6.24c}$$

characterizing the *Lie-isotopic generalization of Heisenberg's equations*, today called *isoeisenberg's representation*, originally submitted in Santilli (1978b), p. 752[12].

[12] Eq.s (6.23) have been proved to be *directly universal*, that is, admitting as particular cases (under sufficient topological conditions) all possible operator equations (universality), directly in the local quantities considered (direct universality) (see Santilli (1989) for details and references). As an example, certain nonlinear equations proposed by Weinberg (1989) are a particular case of Eq.s (6.23), as shown in detail by Jannussis *et al.* (1991). Note that the nonlinearity generally treated until

The corresponding, equivalent, *isoscherödinger's representation* is characterized by the right and left modular-isotopic eigenvalue equations

$$\begin{cases} i \dfrac{\partial}{\partial t} | \psi > \; = \; H * | \psi > \; = \; \hat{E} * | \psi > \; \equiv \; E | \psi >, & (6.24a) \\[2ex] - i < \psi | \dfrac{\partial}{\partial t} \; = \; < \psi | * H \; = \; < \psi | * \hat{E} \; = \; < \psi | E, & (6.24b) \end{cases}$$

proposed by Mignani (1982) and Myung and Santilli (1982a).

By inspection, one can see that Eq.s (6.23) or (6.24) represent a physical system with all possible potential forces, characterized by the conventional Hamiltonian $H = T(\dot{x}) + V(t, x, \dot{x})$ with potential forces

$$F^{SA} = \frac{d}{dt} \frac{\partial V}{\partial \dot{x}} - \frac{\partial V}{\partial x}, \qquad (6.25)$$

as well as an additional class of forces beyond the representational capability of the Hamiltonian, characterized precisely by the isotopic element T which, as one can see, *multiplies* the Hamiltonian from the right and from the left.

Eq.s (6.23) and (6.24) are evidently at the foundations of the operator formulation of the Lie-isotopic theory. In fact, as shown in Santilli (1978b) and subsequently developed by Myung and Santilli (1982a), Mignani, Myung and Santilli (1983), and Santilli (1989), the abstract formulation of the Lie-isotopic theory can be directly interpreted as representing a generalization of quantum mechanics on a suitable isotopic form of the Hilbert space, called *hadronic mechanics*.

In this case, the generators X are generally expressed via matrices or via local-differential operators, while the isounit $\hat{1}$ is generally represented by a nonlinear and nonlocal, integro-differential operator.

As an example, for the case of two particles with wavepackets $\psi(t,r)$ and $\phi(t,r)$ in conditions of deep mutual penetrations, the isotopic element can be expressed via the form first introduced by Animalu

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now, including that by Weinberg (*loc. cit.*), is the nonlinearity in the wavefunction ψ, while Eq.s (6.23) are nonlinear also in the derivatives of the wavefunction $\partial\psi$, the latter one being more relevant than the former for the study of short range, internal interactions caused by wave overlapping, as typical for all motions within physical media. Note also that conventional nonlinear models, including that by Weinberg (*loc. cit.*), are strictly local-differential, while Eq.s (6.23) are directly universal also for all possible nonlocal-integral equations, as illustrated later on via isounit (6.26).

(1991)

$$\hat{1} = e_A^{-itK\int dv\, \psi(t,r)\, \phi(t,r)}, \qquad K \in \mathfrak{R}, \tag{6.26}$$

One can therefore see in this way that for null wave-overlapping, the integral in the exponent of Eq. (6.26) is null, the isounit $\hat{1}$ assumes the conventional trivial value I, and all Lie-isotopic formulations recover the conventional formulations identically at both the quantum mechanical as well as Lie levels.

In turn, the emerging isotopic generalization of quantum mechanics under isounit (6.26) is useful for a more adequate, nonlocal treatment of systems such as: bound states of particles at very small mutual distances, as expected in the structure of hadrons and, to a lesser quantitative extent, in the structure of nuclei (but not in the structure of atoms); the Cooper pairs in superconductivity; Bose-Einstein correlations; etc.

Realization (6.26) is useful to provide the reader with a simple illustration of the needed type of nonlocality, as well as of the type of operator Lie-isotopic theory which is expected from the classical formulations of this analysis. The understanding is that the actual form of the isounit in specific models is rather complex.

We now pass to the *classical realization of the Lie-isotopic theory*, which is the central topic of the remaining parts of this volume. In this section we shall present only a few introductory notions. The topic will be studied in detail at the analytic level in Sect. II.7 and at the geometric level in Sect. II.9.

Introduce the conventional phase space $T^*E(r,\delta,\mathfrak{R})$ with local coordinates

$$a = (a^\mu) = (r, p) = (r_i, p_i), \quad i = 1,2,...,n, \quad \mu = 1, 2,..., 2n, \tag{6.27}$$

where we shall ignore for simplicity of notation any distinction between covariant and contravariant indeces in the r- and p-variables, but keep such a distinction for the a-variables.

As well known, the celebrated Lie's First, Second and Third Theorems provide a direct characterization of the conventional Hamilton's equations without external terms, as presented in their original derivation (Lie (1893)). Equivalently, we can say that Lie's Theorems provide a direct characterization of the familiar *Poisson brackets* among functions A(a) and B(a) on $T^*Er,\delta,\mathfrak{R})$

$$[A, B] = \frac{\partial A}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial B}{\partial a^\nu} = \frac{\partial A}{\partial r_k} \frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r_k} \frac{\partial A}{\partial p_k}, \qquad (6.28)$$

where $\omega^{\mu\nu}$ is the *contravariant, canonical, Lie tensor* with components

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{pmatrix}, \qquad (6.29)$$

(see the next sections for details and geometrical meaning).

A primary physical motivation for proposing the Lie-isotopic theory was to show that the Lie-isotopic First, Second and Third Theorems characterize a generalization of Hamilton's equations originally discovered by Birkhoff (1927) and called *Birkhoff's equations*, with the ensuing mechanics called *Birkhoffian mechanics* (see next section).

In fact, the Lie-isotopic First, Second and Third Theorems directly characterize the most general possible, regular realization of Lie brackets on $T^*E(r,\delta,\Re)$, given by *Birkhoff's brackets*

$$[A \hat{,} B] = \frac{\partial A(a)}{\partial a^\mu} \Omega^{\mu\nu}(a) \frac{\partial B(a)}{\partial a^\nu}, \qquad (6.30)$$

where $\Omega^{\mu\nu}$, called *contravariant Birkhoff's tensor,* verifies the conditions for brackets (6.30) to be Lie

$$\Omega^{\mu\nu} + \Omega^{\nu\mu} = 0, \qquad (6.31a)$$

$$\Omega^{\tau\rho} \frac{\partial \Omega^{\mu\nu}}{\partial a^\rho} + \Omega^{\mu\rho} \frac{\partial \Omega^{\nu\tau}}{\partial a^\rho} + \Omega^{\nu\rho} \frac{\partial \Omega^{\mu\tau}}{\partial a^\rho} = 0. \qquad (6.31b)$$

The reader should also be aware that, in a classical realization, the isotopic rules (6.15) no longer hold because the product of functions A and B in the phase space $T^*E(r,\delta,\Re)$ is conventional, AB, and the *isodifferential rules* for the classical isotopic brackets are given by

$$[A \hat{,} BC] = [A \hat{,} B]\, C + B\, [A \hat{,} C], \qquad (6.32a)$$

$$[AB \hat{,} C] = [A \hat{,} C]\, B + A\, [B \hat{,} C]. \qquad (6.32b)$$

A comprehensive treatment of Birkhoffian mechanics was presented in Santilli (1982a). However, such a formulation is basically insufficient for the needs of this monograph, and a further structural generalization is needed.

In essence, the abstract formulation of the Lie-isotopic theory, as reviewed in this section, is directly suited for the representation of nonlinear as well as nonlocal interactions, via their embedding in the isounit of the theory.

On the contrary, the classical realization of the Lie-isotopic theory as originally proposed and developed into the Birkhoffian mechanics could indeed represent all possible nonlinear and non-Hamiltonian systems (II.1.1), but only in their local-differential approximation.

The occurrence was dictated by the use of the conventional *symplectic geometry*, although in its most general possible exact realization precisely given by the Birkhoffian mechanics (see next sections). The inability to represent any form of nonlocal-integral interactions was then due to the strictly local-differential topology of the underlying geometry.

This created a rather unusual dichotomy whereby the operator formulations of the theory does indeed permit nonlocal interactions (Myung and Santilli (1982a), Mignani, Myung and Santilli (1983)), but their classical counterpart could only admit local interactions (Santilli 1982a)).

This problem was solved only lately via the submission (Santilli (1988a, b), (1991b)) of the so-called *symplectic-isotopic geometry* and *Birkhoffian-isotopic mechanics* as the true, classical, geometric and analytic counterparts, respectively, of the abstract Lie-isotopic theory reviewed earlier, in the sense of being capable of identifying the underlying isounit and therefore admitting of nonlocal-integral interactions.

The symplectic-isotopic geometry and the Birkhoffian-isotopic mechanics will be reviewed in the next sections. In this section, we shall merely present the main idea of the needed Lie-isotopic brackets.

Let us review the conventional Poisson brackets in the unified notation (6.27) on the conventional 2n-dimensional space $T^*E(r,\delta,\mathfrak{R})$. Its underlying unit is evidently given by the trivial unit I in 2n-dimension,

$$I = (I_\rho{}^\sigma) = I_{2n\times 2n} = \mathrm{diag.}(1, 1, ..., 1). \qquad (6.33)$$

We therefore note that, while the conventional way of writing the Poisson brackets in the disjoint r- and p-coordinates does not allow an identification of the underlying unit, this is not the case for the brackets written in the unified a-notation, because they can be written

$$[A\,,B] = \frac{\partial A(a)}{\partial a^\mu}\ \omega^{\mu\rho}\, I_\rho^{\ \nu}\ \frac{\partial B(a)}{\partial a^\nu}\,, \tag{6.34}$$

thus exhibiting the unit of the theory directly in their structure.

The *classical Lie-isotopic brackets* first submitted in Santilli ((1988a, b), (1991a)) are given by a direct generalization of brackets (6.34) of the form

$$[A \,\hat{,}\, B] = \frac{\partial A(a)}{\partial a^\mu}\ \omega^{\mu\rho}\, \hat{I}_\rho^{\ \nu}(t,a,\dot{a},...)\ \frac{\partial B(a)}{\partial a^\nu}\,, \tag{6.35}$$

that is, with the following realization of Birkhoff's tensor

$$\Omega^{\mu\nu} = \omega^{\mu\rho}\, \hat{I}_\rho^{\ \nu}(t,a,\dot{a},...), \tag{6.36}$$

evidently under integrability conditions (6.31) for brackets (6.35) to be Lie-isotopic.

Brackets (6.35) can be readily written in the disjoint r- and p-coordinates by assuming the diagonal form of the isounit

$$\hat{I} = \mathrm{diag.}\,(\hat{\delta}_{n\times n}, \hat{\delta}_{n\times n}), \qquad \hat{\delta} = \hat{\delta}^\dagger, \qquad \det \hat{\delta} \neq 0, \tag{6.37}$$

under which, by using Eq.s (6.29), we have

$$[A \,\hat{,}\, B] = \frac{\partial A}{\partial r_i}\ \hat{\delta}_{ij}\,(t,r,p,...)\ \frac{\partial B}{\partial p_j} - \frac{\partial B}{\partial r_i}\ \hat{\delta}_{ij}(t,r,p,...)\ \frac{\partial A}{\partial p_j}\,, \tag{6.38}$$

Again, one can see in this way that the unified notation (6.27) permits the direct identification of the isounit $\hat{I}$, while such isounit is not directly exhibited by the disjoint r- and p-formulation.

Classical Lie-isotopic brackets (6.35) do indeed permit the representation of nonlocal systems without any need of introducing a nonlocal topology. This is essentially due to the fact that the canonical structure $\omega^{\mu\nu}$ is preserved in its entirety in structure (6.35), while all nonlocal terms are factorized into the isounit $\hat{1}$, exactly as it occurs at the abstract formulation of the theory.

The only difference between the abstract and classical realizations is that in the abstract case, brackets (6.8) exhibit the presence of the *isotopic element*, while in the classical realization (6.35), the brackets exhibit the presence of the *isounit*.

This is a fully normal occurrence and it is due to the interchange between covariant and contravariant quantities in the transition from the abstract to the classical formulation of the enveloping algebra.

The classical realization of the Lie-isotopic time evolution (6.23a) is straightforward, and it is given by

$$\hat{G}_1: \quad Q'(a(t)) = \{ [e_{\xi}^{\,t\, \omega^{\mu\alpha}\hat{1}_{\alpha}{}^{\nu}\, (\partial H/\partial a^{\nu})\, (\partial/a^{\mu}})] \hat{1} \} * Q(a(0)), \qquad (6.39)$$

which constitute precisely a classical realization of the abstract Lie-isotopic transformation groups considered earlier, with a ready extention to more than one dimension.

The infinitesimal version of time evolution (6.39) is evidently given

$$\dot{Q} = [A \hat{,} H] = \frac{\partial Q}{\partial a^{\mu}} \, \omega^{\mu\alpha}\hat{1}_{\alpha}{}^{\nu} \, \frac{\partial H}{\partial a^{\nu}}, \qquad (6.40)$$

and characterizes the *Birkhoff-isotopic equations* submitted in Santilli (1988a, b), (1991a, b)) as shown in the next section.

The reader should be aware that, while the classical realizations of Lie algebras and groups in their conventional or isotopic realizations are rather simple, as shown above, the classical realizations of universal enveloping associative algebras are rather complex, whether conventional or isotopic, and they will not be treated here for brevity.

In particular, the most notable differences between the abstract isoenvelopes and their classical realizations is the appearance of the so-called *neutral elements* d_{ij} (see, e.g., Sudarshan and Mukunda (1974), p. 222). Conventional closure rules must therefore be generally written

$$L: \qquad [X_i , X_j] = C_{ij}{}^{k} X_k + d_{ij}, \qquad (6.41)$$

where the X's are vector-fields on $T^*E(r,\delta,\Re)$, the brackets are the conventional Poisson brackets, and the neutral elements d_{ij} are pure numbers.

In the transition to the Lie-isotopic brackets, the situation is predictably more complex, inasmuch as rules (6.41) are now lifted into the isotopic form

$$[X_i \hat{,} X_j] = \hat{C}_{ij}{}^k(t,a,\dot{a},..)\, X_k + \hat{d}_{ij}(t,a,\dot{a},..), \tag{6.42}$$

namely, not only the structure constants $C_{ij}{}^k$ are lifted into the structure functions $\hat{C}_{ij}{}^k$, but also the constant neutral elements d_{ij} are lifted into the *isoneutral elements* $\hat{d}_{ij}$ with a nontrivial dependence in the local variables.

Now, we shall see in the next chapters that the elimination of the neutral elements is rather simple at the level of abstract Lie algebras and groups, whether conventional or isotopic. Nevertheless their elimination is rather complex at the level of isoenvelopes in classical realizations.

This occurrence has direct implications in the identification of the classical realizations of *Casimir invariants* of the Lie-isotopic theory, called *isocasimir invariants*, but not in their abstract (or operator) counterpart. In fact, in the abstract case the isocasimir invariant can be globally identified in a rather simple way, while in the corresponding classical realization, the same isocasimir invariants are generally defined only locally in the neighborhood of a point of the local variables.

This occurrence can be best illustrated by inspecting the global identification of the isocasimir invariants of the Lorentz-isotopic group in Santilli (1983a), and only the local identification of their classical counterpart in Santilli (1988c) (see Chapter IV of Vol. II for the latter case).

The ultimate roots of this occurrence are due to the fact that the envelopes underlying the abstract Lie brackets "ab − ba" or their isotopic generalization "a∗b − b∗a" are *associative Lie-admissible* in the conventional or isotopic sense,

$$\xi : ab = \text{assoc.}, \qquad \xi^- : ab - ba, \tag{6.43a}$$

$$\hat{\xi} : a*b = \text{assoc.}, \qquad \hat{\xi}^- : a*b - b*a. \tag{6.43b}$$

On the contrary, the envelopes underlying the classical Lie brackets (6.34) or their isotopic generalizations (6.35) are *nonassociative Lie-admissible*, Eq.s (II.5.12),

$$U: \frac{\partial A}{\partial r_k}\frac{\partial B}{\partial p_k} = \text{nonassoc.}, \quad U^-: \frac{\partial A}{\partial r_k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r_k}\frac{\partial A}{\partial p_k}, \tag{6.44a}$$

$$\hat{U}: \frac{\partial A}{\partial r_i}\hat{\delta}_{ij}\frac{\partial B}{\partial p_j} = \text{nonassoc.}, \quad \hat{U}^-: \frac{\partial A}{\partial r_i}\hat{\delta}_{ij}\frac{\partial B}{\partial p_j} - \frac{\partial B}{\partial r_i}\hat{\delta}_{ij}\frac{\partial A}{\partial p_j}, \tag{6.44b}$$

We know nowadays how to generalize the Poincaré-Birkhoff-Witt theorem for isoassociative algebras, but their generalization for nonassociative algebras is known only for *flexible Lie-admissible algebras* (Santilli (1978a), Ktorides, Myung and Santilli (1980)), namely, for a type of algebra for which no classical realization is known at this writing (Sect. II.5).

It is evidently true that the classical Lie algebras and groups can be equivalently formulated via an associative envelope, Lemma II.5.1. In fact, the Lie-isotopic expansion (6.39) is precisely of conventionally associative type.

However, in such an associative reformulation of nonassociative envelopes, the neutral elements emerge. The difficulties in their elimination at this time therefore lie in our lack of knowledge of the infinite-dimensional basis for the nonassociative envelopes U and $\hat{U}$ above.

It should be stressed, however, that this is a merely mathematical aspect here left open for the interested mathematician, and implies no major drawback for the physical studies of the theory of the next chapters.

II.7: BIRKHOFF-ISOTOPIC MECHANICS

We shall now first review the elements of the *Birkhoffian generalization of Hamiltonian mechanics*, or *Birkhoffian mechanics*, as originally derived in (Santilli (1978a), (1982a), that is, via formulations on conventional spaces with the algebra structure being the Lie-isotopic theory, and the underlying geometry being the conventional symplectic geometry.

We shall also review the *direct universality of Birkhoffian mechanics for local-differential systems*, that is, its capability of representing all possible nonlinear and nonhamiltonian systems of ordinary local-differential equations verifying certain continuity and regularity conditions (universality) directly in the coordinate system of their experimental detection (direct universality).

We shall then reformulate the Birkhoffian mechanics in a form, called *Birkhoffian-isotopic mechanics*, which is formulated on suitable isospaces in such a way to exhibit the isounit of the theory directly in the analytic equations and, therefore, in the Lie-isotopic brackets. The geometric structure of the latter mechanics will be studied in Stc. II.9.

The primary reason for such a reformulation was indicated earlier, and it is due to the fact that the Birkhoffian mechancs can only represent local-differential systems because it is based on a geometry, the symplectic geometry, which is strictly local-differential in topological character. The Birkhoffian-isotopic mechanics, instead, permits the representation of nonlocal-integral systems under the condition that all the nonlocal terms are incorporated in the isounit of the theory, as permitted by the Lie-isotopic algebra.

In turn, the achievement of a mechanics capable of representing nonlocal interactions is necessary, not only for the classical representation of systems of type (II.1.1), but also for the operator formulation of the theory. In fact, the interactions of primary interest for the *interior* problem in both classical and particles physics are precisely of nonlocal-integral type.

As recalled in the Introduction, the studies of this section were initiated by Birkhoff (1927) who identified the central equations of the new mechanics. However, their algebraic and geometric structures remained unknown. Also, Birkhoff applied his equations to conventional, conservative, Hamiltonian systems, such as the problem of stability of planetary orbits.

Birkhoff's studies went essentially un-noticed for about fifty one years. Santilli (1978a) rediscovered the equations, by calling them "Birkhoff's equations", and identified: 1) their algebra structure as being that of the Lie-isotopic theory; 2) their geometric structure as being that of the conventional symplectic geometry in its most general possible exact and local formulation; and 3) the capability of the equations of representing all possible nonlinear and nonhamiltonian systems in local-differential approximation. The name of "Birkhoffian mechanics " was submitted in the quoted memoir for the first time.

A presentation of the foundations of the studies, Helmholtz's (1887) *conditions of variational selfadjointness*, were subsequently presented in the monograph Santilli (1978e), while a comprehensive presentation of Birkhoffian mechanics was provided in the subsequent monograph (1982a).

The *nonrelativistic Birkhoffian-isotopic mechanics* was introduced for the first time in Santilli (1988a), with the relativistic formulation being presented in Santilli (1988c). In this section we shall present the structure of the mechanics in isospaces of unspecified physical interpretation. The relativistic and nonrelativistic particularizations will therefore be studied in the subsequent chapters.

The reader should be aware that the Birkhoffian-isotopic mechanics provides the ultimate analytic foundations of the isotopies of Galilei's and Einstein's relativities studied in the subsequent chapters. No in depth knowledge of the isotopic relativities can therefore be reached without an in depth knowledge of their analytic structure. The rudimentary outline of this section is basically insufficient for this task, and a study of the original, locally quoted references is recommendable.

We should finally mention that the author presented in the same memoir of (1978a) a still more general mechanics possessing, this time, the broader Lie-admissible structure and symplectic-admissible geometry. This more general mechanics was subsequently studied in detail in the monograph Santilli (1981a). The rudiments of this latter mechanics are presented, for completeness, in Appendices II.A and II.B.

As anticipated in Sect. I.4, the primary physical motivation for this latter generalization is the following. Whether conventional or isotopic, Birkhoffian mechanics is an axiom preserving generalization of Hamiltonian mechanics. As such, its primary physical emphasis is in space-time symmetries and related first-integrals which represent total conservation laws. This renders Birkhoffian mechanics ideally suited for the characterization of *closed-isolated* interior systems, such as Jupiter when studied as a whole.

The more general Birkhoffian-admissible mechanics implies instead a generalization of the axiomatic structure of Hamiltonian mechanics into a form which represents instead the time-rate-of-variations of physical quantities. This renders the Birkhoffian-admissible mechanics particularly suited when studying open-nonconservative interior systems, such as a satellite during penetration in Jupiter's atmosphere considered as external.

Our notations will be the following. Manifolds over the reals $\Re$ of arbitrary physical interpretation will be indicated with the generic symbol $M(\Re)$. Specific physical interpretation of $M(\Re)$ (such as the

Euclidean space over the reals), will be generally indicated with different symbols (such as $E(\Re)$).

Generic local coordinates on an N-dimensional manifold $M(\Re)$ will be indicated with the symbol x, and their components with the symbol $x = (x^i)$, where for all Latin indices $i = 1, 2, \ldots N$. Coordinates of specific physical interpretation (e.g., the Carthesian coordinates on a Euclidean space) will be indicated with generally different symbols (e.g., $r = (r^i)$).

To begin, considered a 2n-dimensional manifold $M(\Re)$ with local coordinates $x = (x^i)$, $i = 1, 2, \ldots, 2n$, over the reals $\Re$. Let t be an independent variable and $\dot{x} = dx/dt$. Birkhoffian mechanics is based on the most general possible variational principle in $M(\Re)$ which is of linear and first-order character, i.e., depending linearly in the $\dot{x}$'s. Our basic analytic tool is then the *Pfaffian variational principle*

$$\delta \hat{A} = \delta \int_{t_1}^{t_2} (R_i(x)\,\dot{x}^i - B(t, x))_{|\tilde{E}}\, dt = 0, \quad i = i, 2, \ldots, 2n, \tag{7.1}$$

here written in its *semiautonomous* form, i.e., with the t-dependence restricted only to the function B called *Birkhoffian* because generally different than the Hamiltonian (i.e., total energy) $H = T + V$.

When computed along an actual path $\tilde{E}$ of the system, principle (7.1) characterizes the following equations

$$\Omega_{ij}(x)\,\dot{x}^j = \frac{\partial B(t,x)}{\partial x^i}, \qquad i, j = 1, 2, \ldots, 2n, \tag{7.2}$$

called *covariant, semiautonomous Birkhoff's equations*, and

$$\Omega_{ij}(x) = \frac{\partial Rj(x)}{\partial x^i} - \frac{\partial R_i(x)}{\partial x^j}, \tag{7.3}$$

is the *covariant Birkhoff's tensor* hereon restricted to be nowhere degenerate (i.e., $\det(\Omega_{ij}) \neq 0$ everywhere in the region considered).

The *contravariant, semiautonomous Birkhoff's equations* are given by

$$\dot{x}^i = \Omega^{ij}(x)\frac{\partial B(t,x)}{\partial x^j}, \tag{7.4}$$

where

$$\Omega^{ij}(x) = (\,|\,\Omega_{rs}(x)|^{-1}\,)^{ij}, \tag{7.5}$$

is the *contravariant Birkhoff's tensor.*

It is easy to see that the brackets of the algebraic structure of the theory among functions $A(x), B(x), \ldots$ on $M(\Re)$ are characterized by the contravariant tensor Ω^{ij}

$$[A \hat{,} B] = \frac{\partial A(x)}{\partial x^i} \Omega^{ij}(x) \frac{\partial B(x)}{\partial x^j}, \tag{7.6}$$

while the covariant tensor Ω_{ij} characterizes the two-form

$$\Omega = d\,\Theta = d\,(R_i dx^i)$$
$$= \tfrac{1}{2}\Big(\frac{\partial R_j}{\partial x^i} - \frac{\partial R_i}{\partial x^j}\Big)\, dx^i \wedge dx^j = \tfrac{1}{2}\Omega_{ij}\, dx^i \wedge dx^j. \tag{7.7}$$

As we shall review in Section II.9, the two-form (7.7) is the most general possible exact symplectic two-form in local coordinates. This provides the necessary and sufficient conditions for brackets (7.6) to be the most general possible classical, regular , unconstrained[13] Lie-isotopic product on $M(\Re)$ (see the proof in Santilli (1982a)).

Brackets (7.6) are called *Birkhoff's brackets* in our terminology or *generalized Poisson brackets* in other studies (e.g., Sudarshan and Mukunda (1974)). The *fundamental Birkhoff's brackets* are then given by

$$[x^i \hat{,} x^j] = \Omega^{ij}(x), \tag{7.8}$$

and they play an important role in the operator formulation of the theory.

Other fundamental equations are given by the *Birkhoffian Hamilton-Jacobi equations*

$$\left\{ \begin{array}{ll} \dfrac{\partial \hat{A}}{\partial t} + B(t,x) = 0, & (7.9a) \\[2ex] \dfrac{\partial \hat{A}}{\partial x^i} = R_i, & (7.9b) \end{array} \right.$$

[13] As we shall see in Chapter IV, in addition to the brackets of this section, we also have those defined on an hypersurface of constraints, as it is the case for Dirac's brackets (Dirac (1964)).

which are directly derivable from variational principle (7.1) (under the condition of nowhere degeneracy of Birkhoff's tensors) and, as such, are equivalent to Birkhoff's equations (see Santilli (1982a) for details).

As we shall see, Eq.s (7.9) play a predictable fundamental role for the construction of the operator formulation of the isotopic relativities, although in a reduced isotopic form discussed below.

In Hamiltonian mechanics, one usually assigns the Hamiltonian and then computes the equations of motion, when needed. In Birkhoffian mechanics, the situation is the opposite. In fact, one starts with an arbitrary nonlinear and nonhamiltonian system and then computes its Birkhoffian representations.

A main result can be formulated as follows.

THEOREM II.7.1 (Direct Universality of Birkhoffian Mechanics for Local First-Order Systems; Santilli (1982a), Theorem 4.5.1, p. 54): All local, analytic, regular, nonautonomous, finite-dimensional, first-order, ordinary differential equations on a 2n-dimensional manifold $M(\Re)$ with local coordinates $x = (x^i)$, $i = 1, 2, \ldots, 2n$, and derivatives $\dot{x} = dx/dt$ with respect to an independent variable t,

$$\dot{x}^i = \Gamma^i(t, x), \tag{7.10}$$

always admit, in a star-shaped neighborhood of a regular point of their variables, a representation in terms of Birkhoff's equations directly in the local variables at hand, i.e.,

$$\left[\frac{\partial R_j(t, x)}{\partial x^i} - \frac{\partial R_i(t, x)}{\partial x^j}\right] \Gamma^j(t, x) = \frac{\partial B(t, x)}{\partial x^i} + \frac{\partial R_i(t, x)}{\partial t}, \tag{7.11}$$

Namely, for each given vector-field $\Gamma(t, x)$ on $M(\Re)$ verifying the topological conditions of the theorem, one can always construct (n + 1) functions $R_i(t, x)$ and $B(t, x)$ which characterize Birkhoffian representation (7.11).

The reader should be warned that, as the representation emerges from the techniques of Birkhoffian mechanics, it is generally of the *nonautonomous* type (7.11), even when the equations of motion at hand are *autonomous*. Now, representation (7.11) is certainly correct from an analytic viewpoint, i.e., for the use of variational principle (7.1), the Hamilton-Jacobi theory, etc. However, structure (7.11) is not suitable for a generalization of conventional relativities because it violates the condition for the characterization of any algebra, let alone the Lie-isotopic algebras (see Appendix II.A for details).

This requires the reduction of nonautonomous representation (7.11) to the semiautonomous form (7.2) (with a consistent Lie-isotopic structure) via the use of the "degrees of freedom" of the theory which are considerably broader than those of the conventional Hamiltonian mechanics.

We limit here to the indication that the so-called *Birkhoffian gauge transformations*

$$\begin{cases} R_i(t, x) \Rightarrow R'_i(x) = R_i(t, x) + \dfrac{\partial G(t, x)}{\partial x^i}, & (7.12a) \\ \\ B(t,x) \Rightarrow B'(t,x) = B(t,a) - \dfrac{\partial G(t, x)}{\partial t}, & (7.12b) \end{cases}$$

leave unchanged the integrand of principle (7.1) as well as brackets (7.6) and two-form (7.7), within the fixed system of local coordinates of the vector-field. For other degrees of freedom, see the locally quoted references.

Birkhoffian mechanics is evidentlty a *covering* of the conventional Hamiltonian mechanics because:

1) The former mechanics is based on methods (the Lie-isotopic theory) structurally more general then those of the latter mechanics (Lie's theory in its simplest possible realization);

2) The former mechanics represents physical systems (local, but arbitrarily nonlinear and nonhamiltonian systems) which are structurally more general than those represented by the latter systems (local potential systems); and, last but not least

3) The former mechanics admits the latter as a particular case.

To illustrate the latter occurrence, we now introduce a physical realization of the preceding formulation. Let $E(r,\Re)$ be an n-dimensional Euclidean space with Carthesian coordinates $r = (r_i)$, and let $p = dr/dt = (p_i)$ be their tangent vectors (the ordinary linear momenta). Then, the 2n-dimensional manifold $M(\Re)$ can be interpreted as the cotangent bundle (the conventional phase space) $T^*E(r,\Re)$ with local coordinates $a = (a^{\mu}) = (r_i, p_i)$, $\mu = 1, 2, ..., 2n$, and $i = 1, 2, ...,n$, where, for simplicity of notation, we shall assume all upper and lower Latin indeces on coordinates and momenta to be equivalent, but preserve the distinction between the Greek upper (contravariant) and lower (covariant) indices on $T^*E(r,\Re)$.

It is then easy to see that the particular case of Birkhoffian mechanics characterized by

$$a = (a^{\mu}) = (r, p) = (r_i, p_j), \tag{7.13a}$$

$$R = R^{\circ} = (R^{\circ}_{\mu}) = (p, 0) = (p_i, 0_i), \tag{7.13b}$$

$$B = B(t, a) = B(t, r, p) = H(t, r, p) = H(t, a), \tag{7.13c}$$

$$\mu = 1, 2, ..., 2n, \quad i = 1, 2, ..., n,$$

reproduces the conventional Hamiltonian mechanics in its entirety.

In fact, under values (7.13) Pfaffian principle (7.1) reacquires its canonical form

$$\delta A = \delta \int_{t_1}^{t_2} [p_i \dot{r}^i - H(t, r, p)]_{|\hat{E}} dt =$$
$$= \delta \int_{t_1}^{t_2} [R^{\circ}_{\mu}(a)\, \dot{a}^{\mu} - H(t, a)]_{|\hat{E}} dt = 0, \tag{7.14}$$

the covariant tensor (7.3) assumes the canonical-symplectic value on $T^*E(r,\Re)$

$$(\omega_{\mu\nu}) = \frac{\partial R^{\circ}_{\nu}}{\partial x^{\mu}} - \frac{\partial R^{\circ}_{\mu}}{\partial x^{\nu}} = \begin{pmatrix} 0_{n\times n} & -I_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{7.15}$$

with canonical-Lie counterpart)

$$(\omega^{\mu\nu}) = (|\omega_{\alpha\beta}|^{-1}) = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{7.16}$$

Birkhoff's equations (7.2) then reduce to the covariant Hamilton's equations

$$\omega_{\mu\nu}\,\dot{a}^{\nu} = \frac{\partial H(t, a)}{\partial a^{\mu}}, \quad \mu = 1, 2, \ldots, 2n, \tag{7.17}$$

with contravariant form

$$\dot{a}^{\mu} = \omega^{\mu\nu}\frac{\partial H(t, a)}{\partial a^{\nu}}, \tag{7.18}$$

which, when written in the disjoint coordinates x = (r,p), assumes the familiar form

$$\dot{r}_i = \frac{\partial H(t, r, p)}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H(t, r, p)}{\partial r_i}, \tag{7.19}$$

Finally, theBirkhoffian Hamilton-Jacobi equations (7.9) assume the familiar canonical form

$$\left\{\begin{array}{ll} \dfrac{\partial A}{\partial t} + H(t, r, p) = 0, & (7.20a) \\ \dfrac{\partial A}{\partial r_i} = p_i, \qquad \dfrac{\partial A}{\partial p_i} = 0. & (7.20b) \end{array}\right.$$

Note that, while the canonical action A is independent from the variables p as expressed in Eq.s (7.20b), the Pfaffian action $\hat{A}$ is generally dependent on all a's and, thus also on the momenta, as expressed by Eq.s (7.9b).

This occurrence creates problems in the use of the general equations (7.9) for the construction of an operator image of Birkhoffian mechanics, owing to its excessive generality (e.g., because, after using conventional quantization techniques, it would imply "wavefunctions" ψ depending also on momenta, i.e., $\psi = \psi(t, r, p)$).

The first motivation for the reformulation of the above mechanics into a Birkhoffian-isotopic form is therefore of physical character, and it consists of the study of Pfaffian variational principles which, while being of a genuinely generalized nature, imply an action independent from the p-variables.

It is easy to see that the above objective is achieved by the followong particular form of the R-functions

$$\hat{R}^\circ = (\hat{R}^\circ{}_\mu(a)) = (P_i(r,p), 0_i) = (p_k\, T_i{}^k{}_i,\ 0_i) = (pT_1, 0)\,, \qquad (7.21)$$

where T_1 is an n×n symmetric, nonsingular and real-values matrix

$$T_1 = (T_{1ij}) = (T_{1\,ji}) = (T_1{}^i{}_j) = (T_{1j}{}^i). \qquad (7.22)$$

Realization (7.21) characterizes a phase space $T^*E_1(r,\mathfrak{R})$ for which principle (7.1) becomes

$$\delta\hat{A}^\circ = \delta\int_{t_1}^{t_2} [\,\hat{R}^\circ{}_\mu(a)\,\dot{a}^\mu - H(t,a)\,]_{|\hat{E}}\, dt$$

$$= \delta\int_{t_1}^{t_2} [\,p_i\, T_1{}^{ij}\,\dot{r}_j - H(t,r,p)]_{|\hat{E}}\, dt = 0, \qquad (7.23)$$

and can be interpreted as acting on a 2n-dimensional iso-phase-space $T^*\hat{E}_1(r,\hat{\mathfrak{R}})$ equipped with the isounit

$$\hat{1}_1 = \mathrm{diag.}\,(T_1{}^{-1}, T_1{}^{-1}). \qquad (7.24)$$

Eq.s (7.9) then become

$$\frac{\partial\hat{A}^\circ}{\partial t} + H(t,r,p) = 0, \qquad (7.25a)$$

$$\frac{\partial\hat{A}^\circ}{\partial r_i} = p_j T_1{}^j{}_i, \qquad \frac{\partial\hat{A}^\circ}{\partial p_i} = 0, \qquad (7.25b)$$

thus confirming the independence of the generalized action from the velocities, as desired.

Intriguingly, the mapping oof Eq.s (7.25) into an operator forms yields precisely the Sxchrödinger-isotopic equations (6.24), as studied by Santilli (1989), thus confirming the expectation that Birkhoffian mechanics admits, as operator image, the isotopic generalization of quantum mechanics.

The covariant Birkhoff's tensor characterized by Pfaffian principle (7.23) is calleed *covariant canonical-isotopic tensor* and is given by

$$(\Omega^\circ{}_{\mu\nu}) = \left(\frac{\partial\hat{R}_\nu}{\partial x^\mu} - \frac{\partial\hat{R}_\mu}{\partial x^\nu}\right) = \begin{matrix} 0_{n\times n} & (T_2)_{n\times n} \\ -(T_2)_{n\times n} & 0_{n\times n} \end{matrix} =$$

$$
= \begin{pmatrix} 0_{n\times n} & (T_{1\,ij} + p_k \dfrac{\partial T_1{}^k{}_i}{\partial p_j})_{n\times n} \\ -(T_{1\,ij} + p_k \dfrac{\partial T_1{}^k{}_i}{\partial p_j})_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{7.26}
$$

namely, has the factorized structure

$$
\hat{\Omega}^\circ = \omega \times \hat{T}_2, \tag{7.27a}
$$

$$
\hat{T}_2 = \text{diag.}\,(T_2, T_2), \tag{7.27b}
$$

$$
T_2 = (T_{1\,ij} + p_k \frac{\partial T_1{}^k{}_i}{\partial p_j})_{n\times n}\,, \tag{7.27c}
$$

with corresponding two-form

$$
\hat{\Omega}^\circ = \Omega^\circ_{\mu\nu}\, dx^\mu \wedge dx^\nu = (\,\omega_{\mu\alpha}\, \hat{T}_2{}^\alpha{}_\nu)\, da^\mu \wedge da^\nu, \tag{7.28}
$$

where the upper script " ° " in structure $\hat{\Omega}^\circ$ stands to indicate that the factorized structure ω is canonical. The analytic equations, called *covariant Hamilton-isotopic equations*, are given by

$$
\hat{\Omega}^\circ{}_{\mu\nu}\,(a)\, \dot{a}^\nu = \omega_{\mu\alpha}\, \hat{T}_2{}^\alpha{}_\nu(a)\, \dot{a}^\nu = \frac{\partial H(t, a)}{\partial a^\mu}\,. \tag{7.29}
$$

The *contravariant canonical-isotopic tensor* has the structure

$$
(\,\hat{\Omega}^{\circ\mu\nu}) = (\omega^{\mu\nu}) \times \hat{1}_2 = \hat{1}_2 \times (\omega^{\mu\nu}) = (\omega^{\mu\alpha}\, \hat{1}_{2\alpha}{}^\nu) =
$$

$$
= \begin{pmatrix} 0_{n\times n} & (I_2)_{n\times n} \\ -(I_2)_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{7.30a}
$$

$$
\hat{1}_2 = \hat{T}_2^{-1} = \text{diag.}\,(T_2{}^{-1}, T_2{}^{-1}) = \text{diag.}\,(I_2, I_2), \tag{7.30b}
$$

$$I_2 = (T_{1\,ij} + p_k \frac{\partial T_1{}^k{}_i}{\partial p_j})^{-1}, \qquad (7.30c)$$

and characterizes the brackets, called *Poisson-isotopic brackets*

$$[A \hat{,} B] = \frac{\partial A}{\partial a^\mu}\, \omega^{\mu\alpha}\, I_{2\alpha}{}^{\nu}(a, ..)\, \frac{\partial B}{\partial a^\nu} =$$

$$= \frac{\partial A}{\partial r_i}\, I_{2ij}(r,p)\, \frac{\partial B}{\partial p_j} - \frac{\partial B}{\partial r_i}\, I_{2ij}(r,p)\, \frac{\partial A}{\partial p_j}, \qquad (7.31)$$

with *contravariant Hamilton-isotopic equations*

$$\dot{a}^\mu = \hat{\Omega}^{\circ\mu\nu}(a)\, \frac{\partial H(t, a)}{\partial a^\nu} = \omega^{\mu\alpha}\, \hat{I}_{2\alpha}{}^{\nu}(a)\, \frac{\partial H(t, a)}{\partial a^\nu}, \qquad (7.32)$$

which can be written in the disjoint r- and p-coordinates

$$\dot{r}_i = I_{2\,ij}(r, p)\, \frac{\partial H(t, r, p)}{\partial p_j}, \qquad (7.33a)$$

$$\dot{p}_i = -\, I_{2\,ij}(r, p)\, \frac{\partial H(t, r, p)}{\partial r_i}, \qquad (7.33b)$$

A few comments are here in order. First, two-form (7.28) remains exact and symplectic, as the reader can verify (see Sect. II.9 for details). As a result, brackets (7.31) remain Lie-isotopic under factorization (7.27), provided that the elements I_2 are computed as in Eq.s (7.30c).

Second, we note that the particular version of the Birkhoffian mechanics characterized by analytic equations (7.30) or (7.32) on $T^*\hat{E}_2(r\hat{\mathfrak{R}})$ does indeed permit the representation of nonlocal-integral interactions, provided that they are all incorporated in the isotopic element $\hat{T}_2$ or, equivalently, in the isounit $\hat{I}_2$. In fact, the local-differential topology of Hamiltonian mechanics is preserved in its entirety in the factorized canonical forms, while the formulations are insensitive to the possible nonlocality of their units (see Sect. II.9 for more details).

Moreover, we note that, when the equations of motion represented by Eq.s (7.30) or (7.32) are written in their second-order form (see Santilli (1982a) for details), they characterize nonnewtonian forces[14].

As a result, analytic equations (7.30) or (7.32) on isospaces $T^*\hat{E}_2(r,\hat{\Re})$ do indeed characterize systems of type (I.2.1), that is, the most general possible, nonlinear, nonlocal, nonhamiltonian and nonnewtonian systems known at this writing.

The reader should finally be aware of the distinction between spaces $T^*\hat{E}_1(r,\hat{\Re})$ and $T^*\hat{E}_2(r,\hat{\Re})$. The former is characterized by a one-form, the integrand of Pfaffian principle (7.23), while the latter is characterized by a two-form, Eq.s (7.28). As a result, they have different isotopic elements, $\hat{T}_1$ and $\hat{T}_2$, and different isounits, $\hat{1}_1$ and $\hat{1}_2$, respectively. The isospace characterizing the Lie-isotopic algebra is evidently that of the analytic equations, $T^*\hat{E}_2(r,\hat{\Re})$.

The extension of the above results to a full isotopy of Birkhoffian mechanics, i.e., for structures (7.28) and (7.30) in which the factorized structures are Birkhoffian, rather than Hamiltonian, is straightforward (see also Sect. II.9 for its geometrical treatment).

We reach in this way the following

DEFINITION II.7.1: Let $T^*\hat{M}_2\,(r,\hat{\Re})$ be a 2n-dimensional iso-phase spaces with local coordinates $x = (r,\, p)$, isofield $\hat{\Re} = \Re\,\hat{1}_2$ and isounit

$$\hat{1}_2 = (\hat{1}_{2\alpha}{}^{\beta}) = (\hat{1}_2{}^{\beta}{}_{\alpha}) = (\hat{1}_2{}^{\alpha}{}_{\beta}) = (\hat{1}_{2\beta}{}^{\alpha}) = \text{diag.}\,(I_2,\, I_2) =$$

$$= (I_{2\,ij},\, I_{2\,ij}) = = (I_2{}^{ij},\, I_2{}^{ij}) = \hat{T}_2{}^{-1} = (T_2{}^{-1},\, T_2{}^{-1}) > 0, \quad (7.34a)$$

$$T_2 = (T_{1\,ij} + p_k \frac{\partial T_1{}^k{}_i(r,p)}{\partial p_j}), \qquad (7.34b)$$

with $T_1\,(r,p)$ being an $n{\times}n$ symmetric, nonsingular and real-value matrix. Then, the "Birkhoff-isotopic equations" are given in their covariant form by

$$\hat{\Omega}_{\mu\nu}\,(a)\,\dot{a}^{\nu} = [\hat{T}_{2\mu}{}^{\alpha}(a)\,\Omega_{\alpha\nu}(a)]\,\dot{a}^{\nu} = \frac{\partial B(t,\,a)}{\partial a^{\mu}}, \qquad (7.35)$$

[14] see footnote[2] in Sect. I.2.

with contravariant version

$$\dot{a}^{\mu} = \hat{\Omega}^{\mu\nu}(a) \frac{\partial H(t, a)}{\partial a^{\nu}} = [\Omega^{\mu\alpha}(a) \hat{1}_{2\alpha}{}^{\nu}(a)] \frac{\partial B(t, a)}{\partial a^{\nu}} , \qquad (7.36)$$

namely, they occur when the contravariant tensor $\hat{\Omega}^{ij}$ (covariant tensor Ω°_{ij}) is, first, Lie-isotopic (symplectic-isotopic see Sect. II.9), and, second, admits the factorization of the isounit (isotopic element) of the isospace $T^*\hat{E}_2$ $(r,\hat{\Re})$ in which it is defined

$$(\hat{\Omega}^{\mu\nu}) = (\hat{1}_{2\alpha}{}^{\nu}) \times (\Omega^{\mu\alpha}) , \qquad (7.37a)$$

$$(\hat{\Omega}_{\mu\nu}) = (\Omega_{\mu\alpha}) \times (\hat{T}_2{}^{\alpha}{}_{\nu}), \qquad (7.37b)$$

where Ω^{ij} (Ω_{ij}) are conventioinal, local-differential contravariant (covariant) Birkhoff's tensors. The "Birkhoff-isotopic mechanics" is the mechanics of the Birkhoff-isotopic equations.

It is easy to see that the Birkhoff-isotopic mechanics is *broader* then the conventional one, trivially, because of the preservation of the most general possible Birkhoff's tensors in its structure, plus the isotopic element. As a result, the Birkhoffian-isotopic mechanics, not only verifies Theorem II.7.1 of Direct Universality, but actually verifies it in an *extended* form inclusive of nonlocal integral terms.

In particular, Birkhoff-isotopic equations (7.32) are expected to be "directly universal" for systems (II.1.1), although the study of this property is not needed for these monographs, and is left to the reader interested in acquiring a technical knowelege of the field.

The Birkhoffian-isotopic mechanics is, however, excessively broad for our needs. In the subsequent chapters of this work we shall use its particularized form as per the following

DEFINITION II.7.2: The "Hamilton-isotopic mechanics" is the particular case of the Birkhoff-isotopic mechanics in which the general Birkhoff's tensor $\Omega_{\mu\nu}$ is replaced by the canonical one $\omega_{\mu\nu}$. The "Hamilton-isotopic equations" on $T^*\hat{E}_2$ $(r,\hat{\Re})$ are therefore given in their covariant form by Eq.s (7.30),and in their copntravariant form by Eq.s (7.32), with explicit form (7.33) in the r- and p-coordinates.

The simpler analytic equations (7.30) and (7.32) with underlying variational principle (7.23), Lie-isotopic brackets (7.30) and symplectic-isotopic two-forms (7.28) are fully sufficient for our analysis. In fact, the Hamiltonian H can represent the totality of potential forces, as in the conventional theory, while the isounit can represent the totality of nonlocal-integral forces that are admitted by the theory.

The problem of whether the simpler Hamilton-isotopic mechanics is sufficient to reach the directly universal for all possible systems (II.1.1) will be investigated at some subsequent time.

Let us provide some examples of the Hamilton-isotopic mechanics. The simplest conceivable case is that in which T_1 is a diagonal, positive-definite and constant matrix, e.g.,

$$T_1 = \text{diag.}\,(b_1^2, b_2^2, b_3^2) > 0. \tag{7.38}$$

Then, $T_1 \equiv T_2$, and the isounit of $T^*\hat{E}_2(r,\hat{\mathfrak{R}})$ is given by

$$\hat{1}_2 = \text{diag.}\,(b_1^{-2}, b_2^{-2}, b_3^{-2}; b_1^{-2}, b_2^{-2}, b_3^{-2}) > 0. \tag{7.39}$$

As we shall see in Chapter III, the above case can first represent a free particle with an extended shape, say, of oblate spheroidal type, see Eq.s (III.7.4), as well as the deformation of that shape due to external forces, see Eq.s (III.7.7).

A simple class of systems with nontrivial dependence of the isounit is given by

$$T'_1 = N\, e^{F(r)}\, T_1, \tag{7.40}$$

where T_1 is given by Eq. (7.39), F is an arbitrary function of r, and $N \in \mathfrak{R}$. In this case

$$\hat{1}'_2 = N\, e^{-F(r)}\, \hat{1}_2. \tag{7.41}$$

The represented systems are then, in general, nonlinearly damped systems, such as (III.7.10).

A simple illustration of nonlocal isounits is given by

$$\hat{1} = N\, e^{-f(r)\int_\sigma \mathcal{F}(r)} \tag{7.42}$$

which generally represents nonlinearly damped, extended systems with nonlocal corrections of the trajectory due to their shape σ, as in Eq.s (III.7.15).

Similar examples hold for the Birkhoff-isotopic mechanics, with structurally more general examples (see Sect. II.9 and the following chapters).

II.8: LIE-ISOTOPIC SYMMETRIES

We are now sufficiently equipped to introduce the notion of symmetry of systems (II.1.1) characterized by the Lie-isotopic theory on isomanifolds, called *isosymmetries*, which play a fundamental role for the construction of the isotopies of Galilei's, Einstein's special and Einstein's general relativities of Volume II.

In this section we shall consider two main topics. The first is the notion of isosymmetries as the largest possible nonlinear, nonlocal and noncanonical groups of isometries of given isometric spaces. The second is the notion of isosymmetries of given equations of motion on isomanifolds, with related lifting of Noether's theorem and conservation laws.

The notion of isotopic space-time symmetries was introduced in the original proposal of the Lie-isotopic theory (Santilli (1978a)), although it was formulated in conventional manifolds.

The formulation of isotopic space-time symmetries as isosymmetries, that is, as symmetries on isomanifolds, appeared in print, apparently for the first time, in Santilli (1983a) in conjunction with the first construction of the infinite family of isotopes $\hat{O}(3.1)$ of the Lorentz symmetry $O(3.1)$. In fact, the paper first constructed the infinite family of isotopies $\hat{M}$ of the Minkowski space M, then introduced the Fundamental Theorem on Isosymmetries (see below), and finally constructed the isotopies of $\hat{O}(3.1)$.

The theory was formalized in Santilli (1985a), which constitutes the main reference of this section, and applied to the lifting $\hat{O}(3)$ of the group of rotations in the adjoining paper (Santilli (1985b)).[15]

The second part of this section dealing with the isotopic symmetries of given equations of motions, was first introduced in the monograph Santilli (1982a) as part of the Birkhoffian generalization of Hamiltonian mechanics, including the isotopic generalization of Noether's theorem, and related conservation laws. As now familiar, the

[15] In actuality, the author wrote first, in 1982, the papers (Santilli (1985a and b)) on the background methods and then wrote the paper (1983a)) on the isotopies of the Lorentz symmetry. The preceding two papers appeared in print some two years after the latter, for reasons reported in detail in (Santilli (1985a)), p. 26.

theory is nonlinear and nonhamiltonian but local, owing to the use of conventional local-differential manifolds.

The operator counterpart was presented in Santilli (1983c) via an isotopy of Wigner's theorem on unitary symmeries (see also the memoirs Santilli (1989)).

The study of the classical profile was then resumed in Santilli (1988a) and (1991a) where the theory is reformulated as isosymmetries on isomanifolds, including the reformulation of Noether's theorem on an isospace, which constitutes the basis of the related review of this section.

To begin, consider a *pseudometric space* M (Sect. II.3), here defined as an n-dimensional topological space over the field F of real numbers $\Re$, complex numbers C or quaternions Q with local coordinates $x = (x^i)$, $y = (y^i)$, $i = 1, 2, ..., n$, equipped with a nonsingular, sesquilinear and Hermitean composition (x,y) characterizing the mapping

$$(x,y): \quad M \times M \Rightarrow M. \tag{8.1}$$

Let $e = (e^i)$ be the basis of M, and define the metric tensor via the familiar form

$$(e_i, e_j) = g_{ij}, \quad g = (g_{ij}) \tag{8.2}$$

The condition of nonsingularity is intended to ensure the existence of the inverse

$$g^{ij} = (|g_{rs}|^{-1})^{ij}, \tag{8.3}$$

everywhere in the region considered, which permits the customary raising and lowering of indeces

$$x_i = g_{ij}\, x^j, \qquad x^i = g^{ij}\, x_j, \tag{8.4}$$

The condition of sequilinearity

$$((x, \alpha y + \beta z) = \alpha(x,y) + \beta(x,z), \quad (\alpha x + \beta\psi, z) = \bar{\alpha}(x,z) + \bar{\beta}(y,z), \tag{8.5}$$

where the upper bar denotes conjugation in the field, permits the realization of the composition in the familiar form

$$(x,y) = x^{\dagger} g\, y = x^i g_{ij} y^j. \tag{8.6}$$

Finally, the condition of Hermiticity implies that

$$x^{\dagger} (gy) = (g^{\dagger} x^{\dagger})\, y = (g x^{\dagger})\, y \tag{8.7}$$

by characterizing abstract spaces hereon denote M(x,g,F).

The additional condition of positive-definiteness of the metric g implies that we have a *metric space* as per the definition of Sect. II.3.

A metric space of particular physical relevance is the three-dimensional *Euclidean space* E(r,δ,F), with local coordinates $r = (r^i)$ over the field F = ℜ, C, Q, with composition (Sect. II.3)

$$r^2 = r^i \delta_{ij} r^j \in \Re, \qquad \delta = \mathrm{diag.}\,(1, 1, 1). \tag{8.8}$$

A pseudometric space also relevant in physica is the (3+1)-dimensional *Minkowski space* M(x,η,ℜ) (Sect. II.3) with local coordinates $x = (r, x^4)$, $x^4 = c_o t$, where c_o represents the speed of light in vacuum, r ∈ E(r,δ,ℜ), and the composition is given by the familiar expression

$$x^2 = x^{\mu} \eta_{\mu\nu} x^{\nu}, \quad \eta = \mathrm{diag.}\,(1, 1, 1, -1). \tag{8.9}$$

Let us also recall the notion of *isometry* G(m) of a generic manifold M(x,g,F), here defined as *the largest possible m-dimensional Lie group G(m) of linear and local transformations x ⇒ x′ leaving invariant the composition for the separation* $x_1 - x_2$ *among two points* x_1, x_2 *of an n-dimensional manifold M(x,g,F), F = ℜ, C, Q,*

$$x'_1 - x'_2)^{\dagger} g\, (x'_1 - x'_2) \equiv (x_1 - x_2)^{\dagger} g\, (x_1 - x_2), \tag{8.10}$$

(see for details, e.g., Gilmore (1974) and quoted literature).

The connected component $G_o(m)$ of G(m) can be defined as an m-dimensional Lie transformation group on M(x,g,F)(, i.e., as a topological space $G_o(m)$ equipped with a binary associative composition φ characterizing the mapping

$$\varphi\!: \; G_o(m) \times G_o(m) \Rightarrow G_o(m), \tag{8.11}$$

for $G_o(m)$ to be a topological Lie group, and the additional mapping

$$f:\ G_\circ(m) \times M \ \Rightarrow\ M, \tag{8.12}$$

characterized by n analytic functions f(w; x) depending on m parameters w and the local coordinates x $\in$ M, which verify the conditions for $G_\circ(m)$ to be a Lie transformation group.

It is finally assumed that G(m) is a *linear* transformation group on M(x,g,F), i.e., the f-functions have the particular form

$$x' = f(w; x) = A(w)\, x, \tag{8.13}$$

under which the group conditions can be written

$$A(0) = I, \tag{8.14a}$$

$$A(w)\, A(w') = A(w')\, A(w) = A(w + w), \tag{8.14b}$$

$$A(w)\, A(-w) = I, \tag{8.14c}$$

where I is the trivial identity of Lie's theory and the composition is the associative one.

The isometry $G_\circ(m)$ can then be defined as the largest possible group of transformations (8.13) leaving invariant separation (8.10), i.e.

$$[(x_1 - x_2)^\dagger A^\dagger]\, g\, [A(x_1 - x_2)] \equiv (x_1 - x_2)^\dagger\, g\, (x_1 - x_2), \tag{8.15}$$

which can hold iff in F

$$A^\dagger g\, A = A\, g\, A^\dagger = gI, \tag{8.16}$$

and

$$\det A = \pm 1, \tag{8.17}$$

where the trivial unit is added for subsequent convenience.

Among the rather large number of methodological aspects needed for a comprehensive characterization of G(m), we now restrict our attention to the following.

I) The *universal enveloping associative algebra* $\xi(\mathbf{G}_\circ(m))$ of the Lie algebra $\mathbf{G}_\circ(m)$ recalled in Sect. II.6. For readiness in the comparison of the results under isotopy, let us recall that the basis of $\mathbf{G}_\circ(m)$

$$X = (X_k), \quad X_k^{\dagger} = -X_k, \quad k = i, 2, \ldots, m, \tag{8.18}$$

must be ordered, and that the envelope $\xi(G_\circ(m))$ is characterized by the infinite-dimensional basis

$$\xi : \quad I, \ X_r, \quad X_r X_s \quad (r \leqq s), \quad X_r X_s X_t, \ (r \leqq s \leqq t), \ldots\ldots \tag{8.19}$$

A generic element of $\xi(G(m))$ is then an arbitrary polynomial $P = P(X)$ in the X's. The *center* C of $\xi(G_\circ(m))$ is the set of all elements P which commuite with all components X_k of the basis, and can be characterized via the set of all possible scalar multiples of the fundamental unit I in F

$$C = \{\alpha I \mid \alpha \in F, \ I = \mathrm{diag.}(1, 1, \ldots, 1)\}, \tag{8.20}$$

, where the dimension of I is that of the basis (e.g., for the regular representation of $G_\circ(m)$, I is the $m \times m$ unit, etc.), and I is the right and left unit of Lie's theory

$$I X_r = X_r I = X_r, \ \forall X_r \in \xi. \tag{8.21}$$

II) The *connected Lie group* $G_\circ(m)$ of transformations on $M(x,g,F)$, which is characterized by exponentiations in $\xi(G(m))$ via the infinite basis (8.19). For the case of the right modular transformations (8.13), it can be written in the symbolic form

$$G_\circ(m) : \ A(w) = \prod e_{|\xi}^{X_k w_k}, \tag{8.22}$$

where the exponentiation is the conventional one (owing to the associativity of $\xi(G_\circ(m))$ and the trivial value of I, see below).

Exponentiation (8.22) can then be reduced to the desired form via the *Baker-Campbell-Hausdorff Theorem* (Gilmore (*loc. cit.*)). For the left modular action of G(m) on M(x,g,F)

$$x'^{\dagger} = x^{\dagger} A^{\dagger}(w), \tag{8.23}$$

we have the realization

$$G_\circ(m): \quad A^\dagger = (\prod e_{|\xi}^{X_k w_k})^\dagger, \tag{8.24}$$

where the skew-Hermiticity of the basis should be taken into account.

III) The *Lie algebra* $\mathbf{G}_\circ(m)$ of $G(m)$, which is homomorphic to the antisymmetric algebra $[\xi(\mathbf{G}_\circ(m))]^-$ attached to the envelope $\xi(\mathbf{G}_\circ(m))$, and it is characterized by the commutation rules

$$\mathbf{G}_\circ(m): [X_r, X_s]_{|\xi} = X_r X_s - X_s X_r = C_{rs}{}^t X_t, \tag{8.25}$$

where $X_r X_s$ is the trivial associative product in $\xi(\mathbf{G}_\circ(m))$, and the C's are the structure constants.

Finally, the discrete part $D(m)$ of $G(m)$ is characterized by the inversions

$$D(m): \quad x' = Px = -x. \tag{8.26}$$

As a specific example, the largest possible group of isometries $G(m)$ of a three-dimensional Euclidean space $E(r,\delta,F)$ is the *Euclidean group* (Gilmore (1974))

$$E(3) = O(3) \otimes T(3), \tag{8.27}$$

where $O(3)$ is the familiar *group of rotations*, and $T(3)$ is the *group of translations* (see Chapter III for details).

Similarly, the largest possible group of isometries of the (3+1)-dimensional Minkowski spoace $M(x,\eta,\Re)$ is the *Poincaré group* (*loc. cit.*)

$$P(3.1) = O(3.1) \otimes T(3.1), \tag{8.28}$$

where $O(3.1)$ is the *Lorentz group* and $T(3.1)$ is the *group of translations* in space-time (see Chapter IV for details).

We pass now to the study of the infinitely possible isotopies of each given group of isometry. For this purpose, the first needed notion is that of *isospaces* introduced in Sect. II.3, with particular reference to Definition II.3.2:

a) The infinitely possible isotopes $\hat{M}(x,\hat{g},\hat{F})$ of $M(x,g,F)$, called *isospaces*, which preserve the dimensionality and local coordinates of $M(x,g,F)$, and generalize instead the metric g and field F into the *isometrics* and *isofields*

$$\hat{g} = Tg, \quad T = T^{\dagger}, \ \det T \neq 0, \tag{8.29a}$$

$$\hat{F} = F\,\hat{1}, \quad \hat{1} = T^{-1}. \tag{8.29b}$$

respectively, with generic composition

$$(x \,\hat{,}\, x) = (x^{\dagger}\hat{g}\,x)\,\hat{1} = (x^{\dagger}\,Tg\,x)\,\hat{1} \in \hat{F}. \tag{8.30}$$

2) The infinitely possible isotopes $\hat{E}(r,\hat{\delta},\hat{\Re})$ of the Euclidean space $E(r,\delta,\Re)$, called *isoeuclidean spaces,* with

$$\hat{\delta} = T_{\delta}\delta \equiv T_{\delta}, \quad \hat{\Re} = \Re\hat{1}_{\delta}, \ \hat{1}_{\delta} = T_{\delta}^{-1} = \hat{\delta}^{-1}, \tag{8.31}$$

and composition

$$r^{\hat{2}} = (r^{t}\,\hat{\delta}\,r)\,\hat{1}_{\delta} \in \hat{\Re}. \tag{8.32}$$

and

3) The infinitely possible isotopes $\hat{M}(x,\hat{\eta},\hat{\Re})$ of the Minkowski space $M(x,\eta,\Re)$, called *isominkowski spaces,* with

$$\hat{\eta} = T_{\eta}\eta, \quad \hat{\Re} = \Re\,\hat{1}_{\eta}, \quad \hat{1}_{\eta} = T_{\eta}^{-1}, \tag{8.33}$$

and composition

$$x^{\hat{2}} = (\,x^{\mu}\,\hat{\eta}_{\mu\nu}\,x^{\nu}\,)\,\hat{1}_{\hat{\eta}} \in \hat{\Re}, \tag{8.34}$$

We introduce now the isotransformation theory of Sect. II.4 on isospaces $M(x,\hat{g},\hat{F})$, i.e., the right, modular isotopic transformations

$$x' = A * x \overset{\text{def}}{=} ATx. \tag{8.35}$$

where T is the isotopic element of the isospace.

The following important properies then follows.

PROPOSITION II.8.1: Given linear and local transformations on a metric or pseudometric space M(x,g,F)

$$x' = A(w)\, x, \quad w \in F, \tag{8.36}$$

their images for the infinitely possible isotopes $\hat{M}(x,\hat{g},\hat{F})$

$$x' = A(w)*x, \tag{8.37}$$

are "isolinear" and "isolocal" in the sense that they are linear and local at the abstract, coordinate-free level, but they are generally nonlinear and nonlocal when projected in the original space $M(x,g,F)$,

$$x' = A*x = A(w)\, T(x, \dot{x}, \ddot{x}, ...)\, x. \tag{8.38}$$

We study now the groups of *isometries of generic isospaces* $\hat{M}(x,\hat{g},\hat{F})$, namely, the largest possible, m-dimensional group of isolinear and isolocal transformations, hereon denoted $\hat{G}(m)$, leaving invariant the isoseparation $(x \,\hat{;}\, y)$ on $\hat{F}$.

It is evident that the old group of isometries G(m) cannot act consistently on $\hat{M}(x,\hat{g},\hat{F})$, e.g., because of the violation of the linearity condition and other problems. This renders *necessary* the lifting of Lie's theory, from the conventional formulation outlined earlier in this section, to the Lie-isotopic theory.

It is easy to see that the isotopes $\hat{G}(m)$ are constituted by union of two components, the isotopes $\hat{G}_o(m)$ of the connected Lie symmetry $G_o(m)$, and the isotopes $\hat{D}(m)$ of the discrete component D(m). It is also easy to see that $\hat{G}_o(m)$ remains connected, and $\hat{D}(m)$ remains discrete, because isotopic liftibngs do not alter the topological characters of the original symmetry.

We can therefore assume that $\hat{G}_o(m)$ is an (abstract) topological space equipped with the isomap

$$\hat{\varphi} : \hat{G}_o(m) \,\hat{\times}\, \hat{G}_o(m) \Rightarrow \hat{G}_o(m), \tag{8.39}$$

verifying the conditions for $\hat{G}_o(m)$ to be a Lie-isotopic group (Sect. II.6), and equipped with the additional isomap

$$\hat{f} : \hat{G}_o(m) \,\hat{\times}\, \hat{M} = \hat{M}, \tag{8.40}$$

characterized by analytic functions $\hat{f}(x,...; w)$ depending on the same parameters w and the same local variables x of the original isometry

G(m), as well as verifying the Lie-isotopic First, Second and Third Theorems mentioned earlier.

We finally impose that isomap (8.40) is isolineart and isolocal, i.e., of the left and right modular isotopic type

$$x'^{\dagger} = x^{\dagger} * \hat{A}(w) = x^{\dagger} T \hat{A}(w), \tag{8.41a}$$

$$x' = \hat{A}(w) * x = A(w) T x. \tag{8.41b}$$

This implies that the elements $\hat{A}(w)$ of $\hat{G}_{\circ}(m)$ verify the *Lie-isotopic group laws*

$$\hat{A}(0) = \hat{1} = T^{-1}, \tag{8.42a}$$

$$\hat{A}(w) * \hat{A}(w') = \hat{A}(w') * \hat{A}(w) = \hat{A}(w+w'), \tag{8.42b}$$

$$\hat{A}(w) * \hat{A}(-w) = \hat{1}, \tag{8.42c}$$

where the product $\hat{A}(w) * \hat{A}(w)$ is isoassociative (Sect. II.5), with similar laws for the conjugate elements.

DEFINITION II.8.1 (Santilli (1983a)): The group of isometries of a generic, n-dimensional isospace $\hat{M}(x,\hat{g},\hat{F})$, $\hat{F} = \hat{\mathfrak{R}}, \hat{C}, \hat{Q}$, herein called "isotopic-isometries", is the largest possible, m-dimensional, isolinear and isolocal, Lie-isotopic group $\hat{G}(m)$ of isotransformations (8.41) leaving invariant the isoseparation for the difference $z = x - y$ of two points $x, y \in \hat{M}(x,\hat{g},\hat{F})$,

$$(z' \hat{,} z') = (z * \hat{A} \hat{,} \hat{A} * z) = [(z^{\dagger} * \hat{A}^{\dagger}) \hat{g} (\hat{A} * z)] \hat{1} =$$

$$= [[(x^i - x^i) T_i^{\ j} A^{\dagger}{}_j{}^r] \hat{g}_{rs} [A^s{}_k T^k{}_l (x^l - y^l)]] \hat{1} \equiv$$

$$\equiv (x^r - y^s) \hat{g}_{rs} (x^s - y^s) \hat{1}. \tag{8.43}$$

For the construction of $\hat{G}_{\circ}(m)$ we evidently use the Lie-isotopic theory (Sect. II.6), with particular reference to:

I': The *universal enveloping isoassociative algebra* $\hat{\xi}(G_{\circ}(m)$ of $\hat{G}_{\circ}(m)$ which, by central assumption, is constructed via the same generators of the original isometry $G_{\circ}(m)$, i.e., the ordered basis (8.18).

The isotopy $\xi(G_\circ(m) \Rightarrow \xi(G(m))$ is characterized by the isotopic Poincaré-Birkhoff-Witt Theorem, with infinite-dimensional isobasis

$$\xi(\hat{G}_\circ(m)) \; : \; \hat{1} \; , \; X_k, \quad X_r * X_s \; (r \leqq s), \quad X_r * X_s * X_t \; (r \leqq s \leqq t), \ldots., \tag{8.44}$$

where $\hat{1} = T^{-1}$ is the fundamental isounit of the theory and the product is isoassociative, i.e.,

$$X_r * X_s \; = \; X_r T \, X_s, \tag{8.45a}$$

$$\hat{1} * P = P * \hat{1} \; \equiv \; P \qquad \forall \quad P \in \hat{\xi}, \tag{8.45b}$$

where P is a generic element of $\hat{\xi}$, i.e., a generic polynomial on the basis X. The *isocenter* $\hat{C}$ of the envelope is then characterized by all elements which isocommute with the basis X and all its possible polynomial forms, and it can be represented via all possible isoscalar multiples of $\hat{1}$ on $\hat{F}$

$$\hat{C} = \{ \; \hat{\alpha} * \hat{1} \mid \hat{\alpha} \; \in \; \hat{F} \; \}, \tag{8.46}$$

II') The *connected Lie-isotopic group* $\hat{G}_\circ(m)$ which can be characterized by power series expansions in the new envelope $\xi(G_\circ(m))$. For the case of one parameter w and one generator X, these generalized group structures are of type (II.5.8) and can be written for the m-dimensional case

$$\hat{G}(m) \; = \; \hat{A}(w) \; : \; \prod^{*}_{k=1,..,m} e_{|\xi}^{X_k w_k} \; = (\prod_{k \, = \, 1,\ldots,m} e_{|\xi}^{X_k * w_k} \,) \, \hat{1} =$$

$$\stackrel{\text{def}}{=} \; B(w; x,..) \, \hat{1}, \tag{8.47}$$

with composition characterized by the isotopic Baker-Campbell-Hausdorff Theorem. The conjugate expression is evidently given by

$$\hat{G}(m) : \hat{A}(w)^{\dagger} \; = \; \prod^{*} k{=}1,\ldots,m \; e_{|\xi}^{w_k X^{\dagger}_k} = \hat{1} \; (\prod k{=}1,\ldots,m \; e_{|\xi}^{w_k * X_k} =$$

$$\stackrel{\text{def}}{=} \; \hat{1} \, B^{\dagger}(w; x,..). \tag{8.48}$$

III') The *Lie-isotopic algebra* $\hat{G}_\circ(m)$ of $\hat{G}(m)$ charactdrized by the Lie-isotopic First, Second and Third Theorems with isocommutation rules

$$\hat{G}_\circ(m): \quad [X_r, X_s]_{|\hat{\xi}} = [X_r \hat{,} X_s] = X_r * X_s - X_s * X_r =$$

$$= \hat{C}_{rs}{}^{t}(x, \dot{x}, \ddot{x}, ...)\, X_t, \tag{8.49}$$

where the $\hat{C}$'s are the *structure functions* of $\hat{G}_\circ(m)$.

Suppose now that the original group $G_\circ(m)$ is an isometry of the original space M(x,g,F), i.e., it verifies conditions (8.15)-(8.17). It is then easy to see that all infinitely possible isotopes $\hat{G}_\circ(m)$ of G(m) as constructed above automatically leave invariant the new isocomposition

$$\{ (x-y)^{\dagger} * \hat{A}^{\dagger}\, \hat{g}\, \hat{A} * (x-y) \} \hat{1} \equiv [(x-y)^{\dagger}\, \hat{g}\, (x-y)] \hat{1}, \tag{8.50}$$

or, equivalently, they satisfy by construction the property

$$\hat{A}^{\dagger}\, \hat{g}\, \hat{A} = \hat{A}\, \hat{g}\, \hat{A}^{t} = g\, \hat{1}, \tag{8. 51}$$

with

$$\det. (\hat{A}\ T) = \pm 1, \tag{8.52}$$

without any need of additional conditions.

In fact, property (8.50) holds for the continuous part in view of the identities

$$e^{-w_k T X_k}\, Tg\, e^{X_k T w_k} \equiv Tg, \tag{8.53}$$

which hold iff the original invariance conditions

$$e^{-w_k X_k}\, g\, e^{X_k w_k}, \tag{8.54}$$

are verified, where the exponentiation in ξ has been omitted for simplicity.

In particular, if the original isometry is the orthogonal group O(n) of an n-dimensional Euclidean space E(r,δ,F), the isometric $\hat{\delta}$ coincides with the isotopic element T (Definition II.3.2), and expressions (8.53) reduce to an identity, as one can see in the one-dimensional case

$$e^{-w\hat{\delta}X}\,\hat{\delta}\,e^{X\hat{\delta}w} = \hat{\delta} + w(\hat{\delta}X\hat{\delta} - \hat{\delta}X\hat{\delta}) +$$

$$+ \tfrac{1}{2}w^2(\hat{\delta}X\hat{\delta}X\hat{\delta} - \hat{\delta}X\hat{\delta}X\hat{\delta}) + \ldots\ldots \equiv \hat{\delta}. \tag{8.55}$$

The isotopies $\hat{D}(m)$ of the discrete component $D(m)$ (8.26) are trivially given by the *isoinversions*

$$\hat{D}(m): \quad \hat{P}x = P\hat{1}*x = Px = -x, \tag{8.56}$$

where P is the original discrete generator.

The main results of of the aboive topiccan then be expressed as follows.

THEOREM II.8.1 (Fundamental Theorem on Isotopic Isometries; Santilli (1983a) and (1985a)) : Let G(m) be an m-dimensional Lie group of isometries of an n-dimensional metric or pseudometric space M(x,g,F) over the field of real numbers $\Re$*, complex numbers C or quaternions Q,*

$$G(m):\ x'^{\dagger} = x^{\dagger}A^{\dagger}(w), \quad x' = A(w)\,x, \tag{8.57a}$$

$$[(x-y)^{\dagger}\,A^{\dagger}(w)]\,g\,[A(w)\,(x-y)] \equiv (x-y)^{\dagger}\,g\,(x-y). \tag{8.57b}$$

$$A^{\dagger}g\,A = A\,g\,A^{\dagger} = gI, \tag{8.57c}$$

$$\det A = \pm 1. \tag{8.57d}$$

Then, the infinitely possible isotopes $\hat{G}(m)$ *of G(m) characterized by the same parameters and generators of G(m), and the infinitely possible, nowhere singular, Hermitean and sufficiently smooth isounits* $\hat{1} = T^{-1}$ *(isotopic elements T), leave invariant the isocomposition* $(x^{\dagger}Tgx)\,\hat{1}$ *of the isotopic spaces* $\hat{M}(x,\hat{g},\hat{F})$*,* $\hat{g} = Tg$*,* $\hat{F} = F\hat{1}$*,* $\hat{1} = T^{-1}$*,*

$$\hat{G}(m) = x'^{\dagger} = x^{\dagger}*\hat{A}(w) = x^{\dagger}T\,A^{\dagger}(w), \quad x' = A(w)*x = A(w)Tx, \tag{8.58a}$$

$$[(x-y)^{\dagger}*\hat{A}^{\dagger}]\ \hat{g}\ [\hat{A}*(x-y)] \equiv (x-y)^{\dagger}\ \hat{g}\ (x-y), \qquad (8.58b)$$

$$\hat{A}^{\dagger}\ \hat{g}\ A = \hat{A}\ \hat{g}\ \hat{A}^{\dagger} = \hat{g}\hat{1}, \qquad (8.58c)$$

$$\mathrm{Det}\ \hat{A} = (\det\hat{B})\ \hat{1} = \pm\hat{1}. \qquad (8.58d)$$

The following comments are now in order:

1) Each given isometry G(m) admits an infinite number of different isotopes $\hat{G}(m)$ characterized by infinitely possible, different isounits which, from a physical viewpoint, represent the infinitely possible interior physical media.

2) Each of the infinite isotopes can be explicitly computed, from expansions (8.47), via the sole knowledge of the old isometry G(m) and the isotopic element T.

3) Even though the mathematical formulation can be unified for all infinitely possible isotopes $\hat{G}(m)$, the explicit form of the isotransformations is different for different isounits $\hat{1}$.

4) As indicated earlier, the isotransformations are generally nonlinear and nonlocal, because of the dependence of T.

5) All isotopes $\hat{G}(m)$ are *coverings groups* of the original isometry G(m) under the sole condition that the old metric g is admitted as a particular case (or the isotopic element T admits the trivial unit I as a particular case).

6) All Lie algebras, including that of the isometries $G_o(m)$, admit the following *trivial isotopy* $X_r \Rightarrow \hat{X}_r = X_r\hat{1}$, under which

$$\hat{G}_o(m):\ [\hat{X}_r , \hat{X}_s]_{|\hat{\xi}} = \hat{X}_r*\hat{X}_s - \hat{X}_s*\hat{X}_r =$$

$$= [X_r , X_s]_{|\xi}\ \hat{1} = (C_{rs}{}^t\ X_t)\ \hat{1} = C_{rs}{}^t\ \hat{X}_t. \qquad (8.59)$$

The above isotopies are *excluded* from Theorem II.8.1 because they do not produce the invariance of the new isoseparation, as the reader is encouraged to verify.

7) The dimension m of the original isometries G(m) is preserved by all infinitely possible isotopic isometries $\hat{G}(m)$, as the reader is encouraged to verify. In particular, the condition for closure of $\hat{G}(m)$, Eq.s (8.49) are reducible to those for G(m).

8) The isotopic isometries $\hat{G}(m)$ are generally nonisomorphic to the original symmetry G(m). However, as we shall see in the subsequent

chapters, all infinitely possible isotopes $\hat{G}(m)$ can be restricted to be locally isomorphic to the original isometry $G(m)$ under the sole condition of positive- (or negative-) definiteness of the isotopic element T.

To understand the physical relevance of Theorem II.8.1, one should be aware that the isotopic generalizations of Galilei's relativity, of Einstein's special relativity and of Einstein's general relativity to be studied in the subsequent chapters are particular applications of the theorem.

The first, physically relevant particularization of Theorem II.8.1 is given by the following

COROLLARY II.8.1.a (loc. cit.)): Let O(3) be the simple, three-dimensional orthogonal group of isometries of the three-dimensional Euclidean space $E(r,\delta,\Re)$ over the reals $\Re$,

$$O(3):\quad r'^t = r^t R^t(\theta), \quad \dot{r} = R(\theta)\, r, \tag{8.60a}$$

$$r'^2 = r'^t \delta\, r = r R^t R\, r \equiv r^2 = r^t\, r, \tag{8.60b}$$

$$R^t R = R\, R^t = I, \tag{8.60c}$$

$$\det R = \pm 1. \tag{8.60d}$$

where the θ's are the Euler's angles. Then, the infinitely possible isotopic generalizations $\hat{O}(3)$ of O(3) characterized by the same parameters and generators of O(3), and a nowhere singular, Hermitean and sufficiently smooth isounits $\hat{1} = T^{-1}$ (isotopic elements $T\ (r,\ \dot{r},\ \ddot{r},\ldots)$), leave invariant the corresponding, infinitely possible isocompositions $(r^t \hat{\delta}\ r)\ \hat{1}$ of the isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{\Re})$ with $\hat{\delta} = T\delta = T$, $\hat{\Re} = \Re\hat{1}$, $\hat{1} = T^{-1}$,

$$\hat{G}(m):\quad r^{\hat{2}} = (r'^t\, \hat{\delta} r')\, \hat{1} = \{\, [\, (r * \hat{R}^t(\theta)\,]\, \hat{\delta}\, [\, \hat{R}(\theta) * r]\}\hat{1}$$

$$\equiv r^{\hat{2}} = (r^t \hat{\delta}\, r)\, \hat{1}, \tag{8.61a}$$

$$\hat{R}^t * \hat{R} = \hat{R} * \hat{R}^t = \hat{1} = \hat{\delta}^{-1}, \tag{8.61b}$$

$$\mathrm{Det}\,(\hat{R}T) = \det\,(\hat{R}\hat{\delta}) = \pm \hat{1}. \tag{8.61c}$$

As we shall see in Chapter III, the isotopes $\hat{O}(r)$ leave invariant all infinitely possible deformations of the sphere, while resulting to be locally isomorphic to O(3) for T > 0.

A further important case is given by the isotopies of the Lorentz isometry.

COROLLARY II.8.1.b (loc. cit.): Let O(3.1) be the simple Lorentz group of isometries of the conventional Minkowski space $M(x,\eta,\Re)$ over the reals $\Re$,

$$O(3.1):\quad x'^t = x^t \Lambda^t(w), \quad x' = \Lambda(w)\, x, \qquad (8.62a)$$

$$x'^2 = x'^t \Lambda^t \eta \Lambda x \equiv x^t \eta x, \quad \eta = \text{diag.}(1, 1, 1, -1), \qquad (8.62b)$$

$$\Lambda^t \eta \Lambda = \Lambda \eta \Lambda^t = \eta I, \qquad (8.62c)$$

$$(\det \Lambda) = \pm 1\, . \qquad (8.62d)$$

where the w's are the conventional six parameters of O(3.1). Then, the infinitely possible Lorentz-isotopic groups $\hat{O}(3.1)$ characterized by the same parameters and generators of the original group O(3.1), and by nowhere singular, Hermitean and sufficiently smooth isounits $\hat{I}$ (or isotopic elements $T(x, \dot{x}, \ddot{x}, ..)$), leave invariant the isoseparation $(xT\eta x)\hat{I}$ of the corresponding infinite class of Minkowski-isotopic spaces $\hat{M}(x,\hat{\eta},\hat{\Re})$ with $\hat{\eta} = T\eta$, $\hat{\Re} = \Re\hat{I}$, $\hat{I} = T^{-1}$,

$$\hat{O}(3.1):\quad x'^t = x^t{*}\hat{\Lambda}(w), \quad x' = \hat{\Lambda}(w){*}x, \qquad (8.63a)$$

$$x'^{\hat{2}} = (x'^t{*}\hat{\Lambda}^t \hat{\eta} \hat{\Lambda}{*}x)\, \hat{1} = (x^t \hat{\eta} x)\, \hat{1}, \qquad (8.63b)$$

$$\hat{\Lambda}^t \hat{\eta} \hat{\Lambda} = \hat{\Lambda} \hat{\eta} \hat{\Lambda}^t = \hat{1}\eta, \qquad (8.63c)$$

$$\det.(\hat{\Lambda} T) = \pm 1. \qquad (8.63d)$$

As we shall see in Chapter IV, the Lorentz-isotopic isometries provides a relativistic geometrization of inhomogeneous and anisotropic interior physical media. All isotopes $\hat{O}(3.1)$ result to be locally isomorphic to O(3.1) for all isotopic elements T > 0, and they constitute the basis for our isotopies of the special relativity.

As we shall see in Chapter V, the isometric $\hat{\eta}$ can be a conventional Riemannian metric g(x) (Corollary II.3.2.c). As a result, the Lorentz-isotopic group $\hat{O}(3.1)$ for T = g results to be the global group of isometries of Einstein's exterior gravitational theory, thus creating the possibility of constructing covering gravitational theories for the interior problem via the mere isotopies of isotopies

$$\eta \Rightarrow \hat{\eta} = T(x)\eta = g(x) \Rightarrow \hat{g} = T'(x, \dot{x}, \ddot{x}, ..) \, g(x). \tag{8.64}$$

A further case of physical relevance is the following.

COROLLARY II.8.1.c (Mignani (1984), Mignani and Santilli (1991)): Let SU(3) be the semisimple special unitary group of isometries of a two-dimensional Euclidean space E(x,η,C) over the complex field C

$$SU(3)\colon \; z'^{\dagger} = z^{\dagger} \, U^{\dagger}(w), \qquad z' = U(w) \, z, \tag{8.65a}$$

$$z'^{\dagger} U^{\dagger} \, \delta \, U z = z^{\dagger} \delta \, z, \tag{8.65b}$$

$$U^{\dagger} U = U U^{\dagger} = I_{2\times 2}, \tag{8.65c}$$

$$\det. U = +1, \tag{8.65d}$$

Then, the infinitely possible isotopes $S\hat{U}(3)$ of SU(3) characterized by the same parameters and generators of SU(3) and by nowhere degenerate, Hermitean and sufficiently smooth isounits $\hat{I}$ (or isotopic elements $T(z, z^{\dagger}, ...)$) leave invariant the isotopic separation $(z^{\dagger} T\delta z) \hat{I}$ of the isotopic spaces $\hat{E}(z,\hat{\delta},\hat{C})$ with $\hat{\delta} = T\delta$, $\hat{C} = C\hat{I}$, $\hat{I} = T^{-1}$,

$$S\hat{U}(3): \; z'^{\dagger} = z^{\dagger} * \hat{U}^{\dagger}, \qquad z' = \hat{U} * z, \tag{8.66a}$$

$$z'^{\dagger} \, \hat{\delta} \, z' = z^{\dagger} * \hat{U}^{\dagger} \hat{\delta} \, \hat{U} * z = z^{\dagger} \, \hat{\delta} \, z, \tag{8.66b}$$

$$\hat{U}^{\dagger} \hat{\delta} \, \hat{U} = \hat{U} \, \hat{\delta} \, \hat{U}^{t} = \hat{1}, \tag{8.66c}$$

$$\det. (\hat{U}T) = +1. \tag{8.66d}$$

The above corollary is instrumental in introducing the notion of *"isoquark"* (Sect. III.7) as an ordinary quark with an extended wavepacket under conventional local-potential as well as nonlocal-nonhamiltonian interactions. represented precisely by the isounit $\hat{1}$.

We pass now to the study of *isosymmetries* of given equations of motion (II.1.1) on an isospace. Since these equations are represented by the Birkhoff-isotopic equations (Sect. II.7), we can effectively restrict our analysis to the isosymmetries of the latter equations.

To avoid an excessive discontinuity over current views on symmetries and conservation laws, it appears recommendable to review first the symmetries of Birkhoff's equations on a conventional manifold, and then generalize them to our isospaces.

Let $E(r,\delta,\Re)$ be the 3N-dimensional Euclidean space of system (I.1.1) of N particles. Its cotangent bundle $T^*E(r,\delta,\Re)$ is the 6N-dimensional space with local coordinates

$$a = (a^\mu) = (r, p) = (r_{ia}, p_{ia}), \tag{8. 67}$$

$$\mu = 1, 2, ..., 6N, \quad i = 1, 2, 3, \quad a = 1, 2, ..., N.$$

The full representation space is then given by the (6N+1)-dimensional space $\Re_t \times T^*E(r,\delta,\Re)$, where $\Re_t$ represents (nonrelativistically) the time.

Suppose as first step that all nonlocal forces in system (II.1.1) are null (but the vector-field remains nonlinear and nonhamiltonian), and denote the correspondiong vector-field with $\Gamma^\circ = (\Gamma^{\circ\mu}(t,a))$. Then, the theorems of direct universality of Birkhoffian mechanics (Sect. II.7) ensures that, under the assumed topological conditions, a representation of the vector-field Γ° always exist in terms of Birkhoff's equations in the local coordinates considered, and we shall write

$$\left[\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right] \Gamma^{\circ\nu} = \frac{\partial B}{\partial a^\mu} \pm \frac{\partial R_\mu}{\partial t}, \tag{8.68}$$

The first concept needed for the understanding of the physical applications of the next chapters is the behavior of Birkhoff's equations under the most general possible transformations of the local variables.

Recall that Hamilton's equations preserve their form only under a special class of transformations, the canonical ones.

On the contrary, Birkhoff's equations are the most general equations which can be written in $T^*E(r,\delta,\mathfrak{R})$ with a Lie/symplectic structure. As such, they preserve their form under the most general possible transformations of the local variables.

A detailed treatment of this property is provided in Chapter 5.3 of Santilli (1982). Here let us illustrate the property by introducing the unified notation

$$b = (b^{\mu}) = (t,a), \quad \mu = 0, 1, ..., 6N. \qquad (8.69)$$

Then Birkhoff's equations (II.7.2) can be written in the unified form

$$\tilde{\Omega}_{\mu\nu}(b)\, db^{\nu} = 0, \qquad \mu = 0, 1, ..., 6N, \qquad (8.70)$$

where Birkhoff's tensor $\Omega_{\mu\nu}$ in $T^*E(r,\delta,\mathfrak{R})$ is now extended to the form in $\mathfrak{R}_t \times T^*E(r,\delta,\mathfrak{R})$

$$\tilde{\Omega}_{\mu\nu} = \frac{\partial \tilde{R}_{\nu}(b)}{\partial b^{\mu}} - \frac{\partial \tilde{R}_{\mu}(b)}{\partial b^{\nu}}, \qquad (8.71)$$

and $\tilde{R} = (-B, R)$ characterizes the one-form in $\mathfrak{R}_t \times T^*E(r,\delta,\mathfrak{R})$

$$\tilde{\mathfrak{R}}_{\mu}(b)\, db^{\mu} = R_{\mu}(a)\, da^{\mu} - B(t,a)\, dt, \qquad (8.72)$$

namely, it characterizes the complete integrand of the basic variational principle (II.7.1).

Eq.s (8.68) are represented by Eq.s (8.70) fopr $\mu = 1, 2, ..., 6N$, with the additional equation fo $\mu = 0$ being the identity

$$\left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t}\right) da^{\nu} =$$

$$= \left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t}\right) \Omega^{\nu a} \left(\frac{\partial B}{\partial a^{\alpha}} + \frac{\partial R_{\alpha}}{\partial t}\right) \equiv 0. \qquad (8.73)$$

What we have done here is performed the transition from the *symplectic geometry* in $T^*E(r,\delta,\mathfrak{R})$, to the so-called *contact geometry* in $\mathfrak{R}_t \times T^*E(r,\delta,\mathfrak{R})$ (see, e.g., Abraham and Marsden (1967) and Santilli (1982a) for a specific treatment of Birkhoff's equations in the contact geometry). Equivalently, we can say that Birkhoff's tensor in $T^*E(r,\delta,\Re)$ is of symplectic type while its extended version (8.71) is of contact type.

Once the contact character of tensor (8.71) is understood, one can readily see the invariance of Birkhoff's equations (8.70) under the local, but most general possible, smoothness and regularity preserving transformations in $\Re_t \times T^*E(r,\delta,\Re)$

$$b = (t, a) \Rightarrow b' = b'(b) = (t', a') = (t'(t, a), a'(t, a)), \qquad (8.74)$$

In fact, contact tensor (8.71) transforms as follows

$$\tilde{\Omega}_{\mu\nu}(b) \Rightarrow \tilde{\Omega}'_{\mu\nu}(b'(b)) = \frac{\partial b^{\mu}}{\partial b'^{\alpha}} \tilde{\Omega}_{\alpha\beta}(b(b')) \frac{\partial b^{\nu}}{\partial b'^{\beta}} , \qquad (8.75)$$

by evidently preserving its structure. The form invariant (but not the symmetry) of Birkhoff's equations then follows.

The implications of the above findings for the interior problem are the following. The space-time symmetries of contemporary relativities for the exterior problem are, first of all, canonical, and then symmetries of the system considered. In the transition to our treatment of the interior problem all smoothness and regularity preserving transformations are "canonical" and, therefore possible candidates for interior symmetries.

DEFINITION II.8.2 (Santilli (1978a), (1982a)): The local, but most general possible smoothness and regularity preserving transformations (8.74) on $\Re_t \times T^*E(r,\delta,\Re)$ constitute a "symmetry of Birkhoff's equations", when they leave invariant Birkhoff's tensor in its contact form, i.e., when Eq.s (8.75) imply the particular form

$$\tilde{\Omega}_{\mu\nu}(b) \Rightarrow \tilde{\Omega}'_{\mu\nu}(b') \equiv \tilde{\Omega}_{\mu\nu}(b'), \qquad (8.76)$$

or, alternatively, when the underlyng contact one-form (8.72) is invariant up to Birkhoffian gauge transformations, i.e.,

$$\hat{R}'_{\mu}(b')\, db'^{\mu} = [\, \hat{R}_{\mu}(b') + \frac{\partial \hat{G}(b')}{\partial b'^{\mu}} \,]\, db'^{\mu}, \qquad (8.77)$$

We now review the construction of *first integral* (i.e., conserved quantities) from a given symmetry of Birkhoff's equations.

THEOREM II.8.2. (Birkhoffian Noether's Theorem; loc. cit.): If Birkhoff's equations admit a symmetry under an r-dimensional, connected Lie group G_r of infinitesimal transformations

$$G_r:\ b \Rightarrow b' = b + \delta b = (b^\mu + w^i \alpha^\mu{}_i(b)) =$$

$$= \begin{matrix} t + w^i \rho_i(t, a) \\ \\ a^\mu + w^i \eta^\mu{}_i(t, a) \end{matrix} \qquad (8.78)$$

then there exist r first integrals $\mathcal{F}_i(b)$ of the equations of motion which are conserved along an actual path $\tilde{E}$

$$\frac{d}{dt} \mathcal{F}_i(b)|_{\tilde{E}} \equiv 0, \qquad (8.79)$$

namely, there exist r linear combinations of Birkhoff's equations which are exact differentials along $\tilde{E}$, i.e.,

$$\frac{d}{dt} \mathcal{F}_i(b) = \tilde{\Omega}_{\mu\nu}(b)\, b^\nu \alpha^\mu{}_i , \qquad (8.80)$$

given explicitly by

$$\mathcal{F}_i(b) = \tilde{R}_\mu a^\mu{}_i =$$

$$= R_\mu(t, a)\, \eta^\mu{}_i(t, a) - B(t, a)\, \rho(t, a) + G_i(t, a). \qquad (8.81)$$

Note that the "new time" t' in Birkhoffian mechanics is a function of the old time t as well as of the coordinates r and momenta p,

$$t' = t'(t, r, p). \qquad (8.82)$$

This property is important to understand the isotranslations in time of the new relativities presented in the next chapters.

Intriguingly, this property is typical of relativistic formulations but not of Hamiltonian mechanics. The Birkhoffian mechanics then achieves a form of symmetric behavior of time for both nonrelativistic and relativistic formulations.

Note that the symmetry G_r of Theorem II.8.2 is a *conventional Lie symmetry* defined on a *conventional space.*

Recall also that the nonautonomous Birkhoff's equations considered until now in this section do not admit a consistent algebraic structure (Appendix II.1). From now on we shall therefore restrict our attention to the semiautonomous case.

Finally, we shall consider the Birkhoff-isotopic representation of systems (II.1.1) introduced in Sect. II.7 on a on isospace $T^*\hat{E}_2(r,\hat{\delta},\hat{\mathfrak{R}})$, i.e.,

$$\hat{T}_{2\mu}{}^{\alpha}(a)\,\Omega_{\mu\alpha}(a)\,\Gamma^{\nu}(t,a) \;=\; \frac{\partial B(t,a)}{\partial a^{\mu}}, \tag{8.83}$$

where $\Omega_{\mu\nu}$ is the symplectic tensor and $\hat{T}_2$ the isotopic element of $T^*\hat{E}_2(r,\hat{\delta},\hat{\mathfrak{R}})$.

It is easy to see that, owing to the appearance of the isotopic element $\hat{T}_2$ directly in the analytic equations, structure (8.83) does indeed allow the representation of the nonlocal forces of systems (II.1.1), in addition to the nonlinear and nonhamiltonian forces represented with structure (8.69).

Our objective here is, not only that of reaching isosymmetries on isospaces, but also that of studying their most general known nonlinear, nonlocal and nonhamiltonian form.

DEFINITION II.8.3 (Santilli (1982a), (1991a)): An r-dimensional symmetry of Birkhoff-isotopic equations (8.83) is an "isosymmetry" $\hat{G}_r$ when it is defined on isospaces $T^*\hat{E}_2(r,\hat{\delta},\hat{\mathfrak{R}})$ and admits infinitesimal transformations of the Lie-isotopic type

$$a'^{\mu} \;=\; a^{\mu} + w^{i}\,\Omega^{\mu\alpha}(a)\,\hat{1}_{2\alpha}{}^{\nu}\,\frac{\partial X_i}{\partial a^{\nu}}\,, \tag{8.84}$$

where $\hat{1}_2 = \hat{T}_2^{-1}$ is the basic isounit of the isospace, the w's are the parameter and the X's are the generators of $\hat{G}_r$, with isocommutation rules

$$[X_r \hat{,} X_s] = \frac{\partial X_r}{\partial a^\mu} \Omega^{\mu\alpha} \hat{1}_{2\alpha}{}^{\nu} \frac{\partial X_s}{\partial a^\nu} = \hat{C}_{rs}(a)\, X_s, \tag{8.85}$$

It is easy to see that a *necessary* condition for transformations a ⇒ a′ to be a symmetry of the Birkhoff-isotopic equations is that they have a Lie-isotopic structure. This renders *necessary* the use of the Lie-isotopic theory for the study of isosymmetries and their first integrals.

THEOREM II.8.3 (Integrability Conditions for the Existence of an Isosymmetry, loc. cit.): Necessary and sufficient conditions for a smoothness and regularity preserving transformation (8.84) to be an isosymmetry of the Birkhoff-isotopic equations (8.83) is that they leave the Birkhoffian invariant, i.e.,

$$B'(a') = B(a) + w_i\, [X_i \hat{,} B] \equiv B(a), \tag{8.86}$$

which can hold iff the Birkhoffian B isocommutes with all generators X_i, i.e.,

$$[X_i \hat{,} B] \equiv 0, \quad i = 1, 2, \ldots, r. \tag{8.87}$$

It is easy to see that the isosymmetries here considered are not only nonlinear, but also nonlocal, as desired, owing to the appearance of the isounit $\hat{1}_2$ directly in their infinitesimal structure.

Therefore, the above framework does indeed provide the methodological foundations for studying the most general possible nonlinear and nonlocal generalizations of conventional space-time symmetries for the interior problem.

Until now we have essentially outlined the isotopic generalizations of analytic mechanics and Lie's theory. Nevertheless, no in depth study of the problem considered can be achieved without the identification of compatible isotopies of the fundamental geometries of contemporary relativities, the symplectic, affine and Riemannian geometries, which are studied in the remaining parts of this chapter.

II.9: ISOSYMPLECTIC GEOMETRY

We now pass to the review of the new geometries needed for the characterization of the most general possible class of systems (II.1.1) beginning with the isotopies of the symplectic geometries, and then passing to those of the affine and Riemannian geometries.

The new geometries have been introduced, apparently for the first time, in Santilli (1988a, d) and then studied in more details in Santilli (1991b), under the names of *isosymplectic, isoaffine and isoriemannian geometries* . As we shall see, these geometries are centrally dependent on a generalization of the conventional differential calculus submitted under the name of *isodifferential calculus*.

The literature in the conventional symplectic geometry is rather vast indeed. A list of references can be found in Santilli (1982a), p.77. In the following, we shall review only the most essential aspects needed for our analysis following Abraham and Marsden (1967). The literature in the calculus of exterior forms is equally vast. We here follow Lovelock and Rund (1975) of which we adopt the notation for clarity in the comparison of the results.

All quantities considered are assumed to verify the needed continuity conditions, e.g., of being of Class C^{∞}, which shall be hereon omitted for brevity. Similarly, all neighborhoods of given points are assumed to be star-shaped, or have a similar topology also ignored hereon for brevity.

Let $M(\Re)$ be an n-dimensional (abstract) manifold over the reals $\Re$ and let $T^*M(\Re)$ be its cotangent bundle. We shall denote with $T^*M_1(\Re)$ the manifold $T^*M(\Re)$ equipped with the *canonical one-form* θ defined by (see, e.g., Abraham and Marsden (*loc. cit.*))

$$\theta \ : \ T^*M_1(\Re) \Rightarrow T^*(T^*M_1(\Re)), \quad \theta \in \Lambda_1(T^*M_1(\Re)). \qquad (9.1)$$

The *fundamental symplectic form* is then given by the two-form

$$\omega = d\theta, \qquad (9.2)$$

which is nowhere degenerated, exact and therefore closed, i.e., such that $d\omega \equiv 0$. The manifold $T^*M(\Re)$, when equipped with the symplectic two-form ω becomes an *exact symplectic manifold* $T^*M_2(\Re)$ in canonical realization. The *symplectic geometry* is the geometry of symplectic manifolds as characterized by exterior forms, Lie's derivative, etc.

Let H be a function on $T^*M_2(\Re)$ called the *Hamiltonian* A vector-field X on $T^*M_2(\Re)$ is called a *Hamiltonian vector-field* when it verifies the condition

$$X \rfloor \omega = - dH. \tag{9.3}$$

The above equation provides a global, coordinate-free characterization of the conventional Hamilton's equations (those without external terms) for the case of *autonomous systems*, i.e., systems without an explicit dependent in the independent variable (time t).

Finally, we recall that the *Lie derivative* of a vector-field Y with respect to the vector field X on $T^*M_2(\Re)$ can be defined by

$$L_X Y = [X,Y], \tag{9.4}$$

where [X,Y] is the canonical commutator.

The case of *nonautonomous systems* (those with an explicit dependence on time) requires the further extension to the *contact geometry* (see, e.g., Abraham and Marsden (1967), Santilli (1982a)), and it will not be considered here for brevity because it does not affect the Lie content of the geometry of primary interest for this study.

The Birkhoffian generalization of the above canonical geometry is straightforward, and was worked out in Santilli (1978a) and (1982a).

Introduce in the same cotangent bundle $T^*M_1(\Re)$ the most general possible one-form Θ, called the *Birkhoffian* or *Pfaffian one-form*,

$$\Theta: T^*M_1(\Re) \Rightarrow T^*(T^*M_1(\Re)), \qquad \Theta \in \Lambda_1(T^*M_1(\Re)). \tag{9.5}$$

The *Birkhoffian two-form* is then given by

$$\Omega = d\Theta, \tag{9.6}$$

under the conditions of being nowhere degenerate. Ω is exact by construction and therefore closed, that is, symplectic. The manifold $T^*M(\Re)$, when equipped with the two-form Ω, becomes an *exact, Birkhoffian, symplectic manifold* $T^*M_2(\Re)$.

Let B be another function on $T^*M_2(\Re)$ called the *Birkhoffian* Then, a non-Hamiltonian vector-field $\hat{X}$ on $T^*M_2(\Re)$ is called a *Birkhoffian* *vector-field* when it verifies the property

$$\hat{X} \rfloor \Omega = - dB. \tag{9.7}$$

which provides a global, coordinate-free characterization of Birkhoff's equations for autonomous systems.

Similarly, we recall that the *Lie-isotopic derivative* of a vector-field $\hat{Y}$ with respect to a *nonhamiltonian* vector field $\hat{X}$ (Santilli (1982a), p.88) can be written

$$\hat{L}_{\hat{X}}\hat{Y} = [\hat{X} \hat{,} \hat{Y}], \tag{9.8}$$

where the brackets are Birkhoffian (see below).

The realization of the above global structures in local coordinates is straightforward. Interpret the space $M(\Re)$ as an Euclidean space $E(r,\Re)$ with local coordinates $r = (r_i)$, i= i, 2, ..., n. Then, the cotangent bundle T^*M becomes $T^*E(r,\Re)$ with local coordinates $(r,p) = (r_i, p_i)$, where $p = dr/dt$ represents the tangent vectors, and all Latin indeces are assumed to be contravariant for simplicity of notation. The canonical one-form (9.1) then admits the local realization

$$\theta = p_i \, dr_i. \tag{9.9}$$

The Hamiltonian two-form (9.2) admits the realization

$$\omega = d\theta = dp_i \wedge dr_i, \tag{9.10}$$

from which one can easily verify that $d\omega \equiv 0$. A vector-field can then be written

$$X = A_i(r,p) \, \partial / \partial r_i + B_i(r,p) \, \partial / \partial p_i, \tag{9.11a}$$

$$A_i \, dr_i + B_i \, dp_i = -dH, \tag{9.111b}$$

which can hold iff Hamilton's equations are verified, i.e.,

$$\frac{dr_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = - \frac{\partial H}{dr_i}, \tag{9.12}$$

Finally, Lie's derivative (9.4) admits the simple realization

$$L_X Y = [X,Y] = \frac{\partial X}{\partial r_i}\frac{\partial Y}{\partial p_i} - \frac{\partial Y}{\partial r_i}\frac{\partial X}{\partial p_i}, \tag{9.13}$$

where one recognizes in the commutator the familiar Poisson brackets (Sect. II.7).

The realization of the Birkhoffian generalization of the above structures requires the introduction of the unified notation introduced in Sect. II.9, Eq.s (7.13), i.e.,

$$a = (a^\mu) = (r, p) = (r_i, p_i), \quad \mu = 1, 2, ..., 2n, \quad i = 1, 2, ..., n, \tag{9.14}$$

where we preserve the distinction between contravariant and covariant Greek indeces. In the a-chart, the canonical one-form can be written

$$\theta = R^\circ{}_\mu \, da^\mu \equiv p_i \, dr_i, \quad R^\circ = (p, 0), \tag{9.15}$$

and the the Hamiltonian two-form (9.10) becomes

$$\omega = d\theta = \tfrac{1}{2}\omega_{\mu\nu} \, da^\mu \wedge da^\nu \equiv dp_i \wedge dr_i, \tag{9.16}$$

where $\omega_{\mu\nu}$ is the *covariant, canonical, symplectic tensor* (II.7.15), i.e.,

$$(\omega_{\mu\nu}) = \left(\frac{\partial R^\circ{}_\nu}{\partial a^\mu} - \frac{\partial R^\circ{}_\mu}{\partial a^\nu}\right) = \begin{pmatrix} 0_{n\times n} & -I_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{9.17}$$

A vector-field can then be written

$$X = X_\mu(a) \, \partial / \partial a^\mu. \tag{9.18}$$

The conditions for a Hamiltonian vector-field become

$$\omega_{\mu\nu} \, X^\mu \, da^\mu = - dH, \tag{9.19}$$

and can hold iff

$$X = X_\mu \frac{\partial}{\partial a^\mu} = \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu} \frac{\partial}{\partial a^\mu}, \tag{9.20}$$

where

$$\omega^{\mu\nu} = \left(|\omega_{\alpha\beta}|^{-1} \right)^{\mu\nu}, \tag{9.21}$$

namely, iff Hamilton's equations (9.12) hold, which in our unified notation can be written as in Eq.s (7.18), i.e.,

$$\dot{a}^\mu = \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu}. \tag{9.22}$$

Finally, Lie's derivative becomes in unified notation

$$L_X Y = [X,Y] = \frac{\partial X}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial Y}{\partial a^\nu}, \tag{9.23}$$

The transition to the Birkhoffian realization is now straightforward. In fact, it merely requires the transition from the canonical quantities $R^\circ(a) = (p, 0)$ to arbitrary quantities $R(a)$ on $T^*E_1(r,\mathfrak{R})$ under which the Birkhoffian one-form (9.5) assumes the realization

$$\Theta = R_\mu(a)\, da^\mu, \tag{9.24}$$

while the Birkhoffian two-form (9.6) becomes

$$\Omega = d\Theta = \tfrac{1}{2}\, \Omega_{\mu\nu}(a)\, da^\mu \wedge da^\nu. \tag{9.25}$$

where $\Omega^{\mu\nu}$ is the *(covariant) symplectic Birkhoff's tensor* (7.3), i.e.

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}, \tag{9.26}$$

A Birkhoffian vector-field $\hat{X}$ can no longer be decomposed in the simple form (9.11), but can be written

$$\hat{X} = X^{\mu}\ \partial / \partial a_{\mu}. \qquad (9.27)$$

The conditions for a vector-field $\hat{X}$ to be Birkhoffian, Eq.s (9.7), then become

$$\hat{X} \rfloor \Omega = \Omega_{\mu\nu}\ \hat{X}^{\nu}\, da^{\mu} = -\, dB, \qquad (9.28)$$

and they hold iff

$$\hat{X} = \hat{X}^{\mu} \frac{\partial}{\partial a^{\mu}} = \Omega^{\mu\nu} \frac{\partial B}{\partial a^{\nu}} \frac{\partial}{\partial a^{\mu}}, \qquad (9.29)$$

where

$$\Omega^{\mu\nu} = (\,|\,\Omega_{\alpha\beta}|^{-1})^{\mu\nu}, \qquad (9.30)$$

which can hold iff the autonomous Birkhoff's equations (Birkhoff (1927)), Eq.s (7.4), hold, i.e.,

$$\dot{a}^{\mu} = \hat{X}^{\mu} = \Omega^{\mu\nu}(a) \frac{\partial B(a)}{\partial a^{\nu}}. \qquad (9.31)$$

Similarly, the Lie-isotopic derivative (9.8) assumes the realization

$$\hat{L}_{\hat{X}}\hat{Y} = [\hat{X} \,\hat{,}\, \hat{Y}] = \frac{\partial \hat{X}}{\partial a^{\mu}}\ \Omega^{\mu\nu}(a) \frac{\partial \hat{Y}}{\partial a^{\nu}}\ , \qquad (9.32)$$

For additional aspects, the reader may consult Santilli (1982a), the appendices of Ch. 4.

Note that an arbitrary vector-field $\hat{X}$ is not Hamiltonian in a given local chart. This illustrates the relevance of the following

THEOREM II.9.1 (DIRECT UNIVERSALITY OF THE SYMPLECTIC GEOMETRY FOR LOCAL NEWTONIAN SYSTEMS, Santilli (loc. cit.)): An arbitrary, local-differential, analytic, semiautonomous and regular vector-field $\hat{X}$ on a given chart on $T^*M_2\ (r,\Re)$ always admits a direct representation as a Birkhoffian vector-field, i.e., a representation directly in the chart considered.

The physical implications are the following. When considering conservative-potential systems of the exterior dynamical problem (Sect. I.1), the vector-fields are evidently Hamiltonian in the frame of the experimenter. However, when considering the nonconservative systems of the interior dynamical problem, the vector-fields are generally nonhamiltonian in the frame of the experimenter (which lead Lagrange and Hamilton to formulate their historical equations with external terms).

Now, under sufficient topological conditions, the *Lie-Koening theorem* ensures that a nonhamiltonian vector-field can always be transformed into a Hamiltonian form under a suitable change of coordinates (see the analytic and geometric proofs of Santilli (1982a)).

However, since the original vector-field is nonhamiltonian the tranformations must necessarily be noncanonical and nonlinear, thus creating evident physical problems, e.g., conventional relativities become inapplicable because turned into noninertial formulations.

This creates the problem of the "direct representation" of the physical systems considered, that is, their representation, first, in the frame of the experimenter, as per Theorem II.9.1.

Intriguingly, the identification of the Lie-Koening transformation $a \Rightarrow a'$ of a nonhamiltonian systems $\hat{X}(a)$ into a mathematical frame a' in which it is Hamiltonian, $\hat{X}(a(a')) = X(a')$, directly implies the Birkhoffian representation of Theorem II.9.1 in the a-frame of the observer. In fact, Birkhoff's equations (9.31) in the a-frame can be characterized precisley via a *noncanonical* transformation $a' \Rightarrow a$ of Hamilton's equations (9.22) in the a'-frame, i.e.

$$\omega_{\mu\nu}\dot{a}'^{\nu} - \frac{\partial H(a')}{\partial a'^{\mu}} = \frac{\partial a^{\rho}}{\partial a'^{\mu}} [\Omega_{\rho\sigma}(a) - \frac{\partial B(a)}{\partial a^{\rho}}] = 0, \qquad (9.33a)$$

$$H(a'(a)) = B(a), \qquad (9.33b)$$

(see Santilli (*loc. cit.*), p.130 for details).

We are now sufficiently equipped to study the *symplectic-isotopic geometry* or *isosymplectic geometry* for short (Santilli (1988a,b), (1991b)). To begin, let us recall that the geometry outlined above is strictly local-differential. In particular, the vector-fields cannot incorporate nonlocal-integral terms without the construction of a suitable, rather complex revision of the geometry via an appropriate nonlocal-integral topology.

We now want to generalize the symplectic geometry into a nonlocal-integral form which is mathematically simple and physically effective, while permitting the direct representation of vector-fields with nonlocal-integral components.

For this purpose, let us first rewrite the canonical realization of the symplectic geometry in the following way. Consider again the original, abstract cotangent bundle $T^*M(\Re)$, and let

$$I^\circ = (I_{n\times n}) = \text{diag.}(1, 1, ..., 1) \equiv T^{\circ -1} \qquad (9.34)$$

be its unit. Then, the canonical one form (9.1) can be identically written in terms of the factorization

$$\theta \equiv \hat{\theta}^\circ = \theta \times T^\circ : T^*\hat{M}_1{}^\circ \Rightarrow T^*(T^*\hat{M}_1{}^\circ), \qquad (9.35)$$

while the canonical two-form (9.2) becomes

$$\omega \equiv \hat{\omega}^\circ = d\hat{\theta}^\circ = (d\theta) \times T + \theta\, dT \equiv \omega \times T^\circ \qquad (9.36)$$

This implies that, in the realization $T^*E(r,\Re)$ of $T^*M(\Re)$ with local chart $a = (r, p)$, we can exhibit the isotopic element, this time, given by the trivial identity T°, directly in the canonical-symplectic tensor

$$\hat{\omega}^\circ{}_{\mu\nu} = T^\circ{}_\mu{}^\alpha \omega_{\alpha\nu} , \qquad (9.37)$$

. Then, its contravariant version, the unit I°, is exhibited in the Lie-tensor of the theory,

$$\hat{\omega}^{\circ\mu\nu} = \omega^{\mu\alpha}\, I^\circ{}_\alpha{}^\nu . \qquad (9.38)$$

The main idea of the isosymplectic geometry is that of reaching a generalization of two-form (9.38) in which the trivial isotopic element T° is replaced by the most general possible, nonlinear and nonlocal isotopic element $T(t, a, \dot{a},..)$, i.e.,

$$\hat{\omega}_{\mu\nu} = \omega_{\mu\alpha}\, T^\alpha{}_\nu(t, a, \dot{a}, ...), \qquad (9.39)$$

under the conditions of characterizing an exact and therefore closed two-form.

In this way, the conventional, local-differential, topological structure of the symplectic geometry is preserved in its entirety in the canonical two-form ω, while all nonlocal-integral terms are incorporated in the isotopic element T.

The corresponding algebraic tensor is then of the type

$$\hat{\omega}^{\mu\nu} = \omega^{\mu\alpha}\, \hat{1}_{\alpha}{}^{\nu}(t, a, \dot{a}, \ldots), \tag{9.40}$$

namely, it is precisely of the Lie-isotopic type with the explicit identification of the isounit directly in the structure of the Lie product, as desired for this study.

The topological consistency of the geometry then follows from that of the underlying Lie-isotopic algebra discussed earlier.

For clarity as well as for ready comparison of the results, we shall follow the presentation of the conventional exterior calculus by Lovelock and Rund (1975), by preserving their notation in terms of a generic 2n-dimensional bundle $T^*M(\Re)$ with generic local chart $x = (x^i)$, $i = 1, 2, \ldots, 2n$. We shall return to our a-coordinates later on for specific physical interpretations.

To begin, let us submit the manifold $M(\Re)$ to one of the infinitely possible isotopic liftings into n-dimensional isospaces $\hat{M}(\hat{\Re})$ over the isofields $\hat{\Re}$, and let $T^*\hat{M}(\hat{\Re})$ be its *"isocotangent bundle"*, that is, the conventional bundle only referred to isospace $\hat{M}$. Introduce one of the infinitely possible, symmetric, nonsingular and real-valued isounits of $\hat{\Re}$ in the original charts x

$$\hat{1} = \hat{1}(t, x, \dot{x}, \ldots) = (\hat{1}^i{}_j) = (\hat{1}_j{}^i) = (\hat{1}^j{}_i) = (\hat{1}_i{}^j) = T^{-1} \tag{9.41a}$$

$$T = T(t, x, \dot{x}, \ldots) = (T_i{}^j) = (T^j{}_i) = (T_j{}^i) = (T^i{}_j). \tag{9.41b}$$

The fundamental tool for the construction of the isosymplectic geometry is an isotopic generalization of the conventional differential calculus introduced, apparently for the first time in Santilli (1988a) under the name of *isodifferential calculus*, and then treated in more details in Santilli (1988b) and (1991b).

For mathematical consistency, conventional linear transformations on $T^*M(\Re)$, e.g.,

$$x' = Ax, \quad \text{or} \quad x'^i = A^i{}_j\, x^j, \tag{9.42}$$

must be necessarily generalized on $T^*\hat{M}(\hat{\Re})$ into those of the isotopic type (e.g., to preserve isolinearity, see Sect. II.4)

$$x' = A*x, \quad \text{or} \quad x' = A^i{}_r\, T^r{}_s\, x^s. \tag{9.43}$$

In the conventional case, the differentials dx and dx′ of the two coordinate systems are related by the familiar expressions

$$dx' = A\, dx, \quad \text{or} \quad dx'^i = A^i{}_j\, dx^j, \tag{9.44}$$

with the realization, say, for the coordinate transformations $x \Rightarrow x' = x'(x)$

$$dx' = \frac{\partial x'}{\partial x}\, dx, \quad \text{or} \quad d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j}\, dx^j. \tag{9.45}$$

However, the same notion of differentials dx and dx′ becomes inconsistent in the isocontangent bundle $T^*\hat{M}(\hat{\mathfrak{R}})$. We therefore introduce the generalized notion of *isodifferentials* $\hat{d}x$ and $\hat{d}\bar{x}$ which holds when interconnected by the isotopic laws

$$\hat{d}\bar{x} = A*\hat{d}x, \quad \text{or} \quad \hat{d}\bar{x}^i = A^i{}_r\, T^r{}_s\, \hat{d}x^s, \tag{9.46}$$

with the particular realization, say, for the case of the isotransformations $x \Rightarrow \bar{x}(x)$

$$\hat{d}\bar{x} = \frac{\partial \bar{x}}{\partial x} * \hat{d}x, \quad \text{or} \quad \hat{d}\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^r}\, T^r{}_s\, \hat{d}x^s. \tag{9.47}$$

The full geometrical meaning of the above isotransformations and of the isodifferential $\hat{d}x$, will be evident later on in this chapter when studying the notions of isoparallel transport and isogeodesics. At this moment we shall simply assume the notions and derive their consequences.

Let $\phi(x)$ be an *isoscalar function* on $T^*\hat{M}(\hat{\mathfrak{R}})$. Then its isodifferential is given by

$$\hat{d}\phi = \frac{\partial \phi}{\partial x} * \hat{d}x, \quad \text{or} \quad \hat{d}\phi(x) = \frac{\partial \phi}{\partial x^r}\, T^r{}_s\, \hat{d}x^s. \tag{9.48}$$

where the partial derivative is the conventional one.

Similarly, let $X = (X^i)$ be a contravariant *isovector-field* on $T^*\hat{M}(\hat{\mathfrak{R}})$, that is, an ordinary vector-field although defined on an isospace. Then its isodifferential is given by

$$\hat{\partial} X = \frac{\partial X}{\partial x} * \hat{\partial} x, \quad \text{or} \quad \hat{\partial} X^i = \frac{\partial X^i}{\partial x^r} T^r{}_s \hat{\partial} x^s, \qquad (9.49)$$

Thus, an isovector-field on $T^*\hat{M}(\hat{\Re})$ transforms according to the isotopic laws

$$\overline{X}(\overline{x}) = \frac{\partial \overline{X}}{\partial x} * X(x), \quad \text{or} \quad \overline{X}^i(\overline{x}) = \frac{\partial \overline{X}^i}{\partial x^r} T^r{}_s(x)\, X^s(x). \qquad (9.50)$$

Note that, while for conventional transformations (9.42) on $T^*M(x,\Re)$ we have $\partial x'/\partial x = A$, we now have for isotransformations (9.43)

$$\frac{\partial \overline{x}^i}{\partial x^j} = A^i{}_r T^r{}_j + A^i{}_r \frac{\partial T^r{}_s}{\partial x^j} x^s. \qquad (9.51)$$

By using the above results and the usual chain rule for partial differentiation, one easily gets from law (9.51)

$$\frac{\partial \overline{X}^j}{\partial \overline{x}^k} = \frac{\partial^2 \overline{x}^j}{\partial x^s \partial x^i} \frac{\partial x^s}{\partial \overline{x}^k} T^i{}_r X^r + \frac{\partial \overline{x}^j}{\partial x^i} \frac{\partial x^s}{\partial \overline{x}^k} \mathrm{Tir} \frac{\partial X^r}{\partial x^s} +$$

$$+ \frac{\partial \overline{x}^j}{\partial x^i} \frac{\partial x^s}{\partial \overline{x}^k} \frac{\partial T^i{}_r}{\partial x^s} X^r. \qquad (9.52)$$

One can see in this way that, in addition to the isotopy of the conventional two terms of this expression (see Eq.s (3.5), p. 67, Lovelock and Rund, (*loc., cit.*)), we have an additional third term. Note that the quantity $\partial \overline{X}_j / \partial \overline{x}^k$ is not a mixed tensor of rank (1.1), exactly as it happens in the conventional case.

From the preceding results one can then compute the isodifferential of a contravariant isovector-field

$$\hat{\partial}\, \overline{X}^j = \frac{\partial \overline{X}_j}{\partial \overline{x}^k} T^k{}_r \hat{\partial} \overline{x}^r =$$

$$= \frac{\partial^2 \overline{x}^j}{\partial x^s\,\partial x^i} T^i{}_r X^r\, \hat{\partial} x^s + \frac{\partial \overline{x}^j}{\partial x^i} T^i{}_r \frac{\partial X^r}{\partial x^s} \hat{\partial} x^s + \frac{\partial \overline{x}^j}{\partial x^i} \frac{\partial T^i{}_r}{\partial x^s} X^r\, \hat{\partial} x^s \tag{9.53}$$

A *contravariant isotensor* X^{ij} of rank two on $\hat{M}(\hat{\Re})$ is evidently characterized by the transformation laws

$$\overline{X}^{(2)}(\overline{x}) = \frac{\hat{\partial} \overline{x}}{\hat{\partial} x} * \frac{\hat{\partial} \overline{x}}{\hat{\partial} x} * X^{(2)}(x), \quad \overline{X}^{ij}(\overline{x}) = \frac{\partial \overline{x}^i}{\partial x^r} T^r{}_p \frac{\partial \overline{x}^j}{\partial x^s} T^s{}_q X^{pq}(x), \tag{9.54}$$

Similar extensions to higher orders, as well as to contravariant isotensors of rank (0.s) and to generic tensors of rank (r.s) are left as an exercise for the interested reader.

In all preceding expressions (9.42)-(9.54) we have shown both, the abstract forms and their realization in local coordinates, to illustrate that the notion of isotransformations and isodifferentials do constitute isotopies, in the sense that all distinctions between conventional and isotopic notions cease to exist at the abstract, realization-free level.

We are now equipped to outline the isosymplectic geometry, beginning with the introduction of *one-isoforms* on $T^*\hat{M}_1(\hat{\Re})$ as the quantities

$$\hat{\Phi}_1 = A * \hat{\partial} x = A_i\, T^i{}_j\, \hat{\partial} x^j. \tag{9.55}$$

We shall now study the algebraic operations of isodifferentials and one-isoforms. The sum of two one-isoforms is the conventional sum. In fact, given two one-isoforms $\Phi_1{}^1 = A * \hat{\partial} x$ and $\Phi_1{}^2 = B * \hat{\partial} x$, their sum is given by

$$\Phi_1{}^1 + \Phi_1{}^2 = (A + B) * \hat{\partial} x. \tag{9.56}$$

The isoproduct of one-isoform $\hat{\Phi}_1 = A * \hat{\partial} x$ with an isonumber $\hat{n} \in \hat{\Re}$ is the conventional product,

$$\hat{n} * \tilde{\Phi}_1 \equiv n\, \Phi_1. \tag{9.57}$$

For the product of two or more one-isoforms $\Phi_1{}^k = A^k * \hat{\partial} x$, $k = 1, 2, 3, \ldots$ we introduce the *isoexterior*, or *isowedge product* denoted with the symbol $\hat{\wedge}$, which verifies the same axioms of the conventional exterior product, that is, distributive laws and anticommutativity, i.e.

$$(\hat{\Phi}_1{}^1 + \hat{\Phi}_1{}^2) \hat{\wedge} \hat{\Phi}_1{}^3 = \hat{\Phi}_1{}^1 \hat{\wedge} \hat{\Phi}_1{}^3 + \hat{\Phi}_1{}^2 \hat{\wedge} \hat{\Phi}_1{}^3, \quad (9.58a)$$

$$\hat{\Phi}_1{}^1 \hat{\wedge} (\hat{\Phi}_1{}^2 + \hat{\Phi}_1{}^3) = \hat{\Phi}_1{}^1 \hat{\wedge} \hat{\Phi}_1{}^2 + \hat{\Phi}_1{}^1 \hat{\wedge} \hat{\Phi}_1{}^3, \quad (9.58b)$$

$$\hat{\Phi}_1{}^1 \wedge \hat{\Phi}_1{}^2 = - \hat{\Phi}_1{}^2 \hat{\wedge} \hat{\Phi}_1{}^1. \quad (9.58c)$$

The product of two one-isoforms $\hat{\Phi}_1{}^1 = A * \hat{d}x$ and $\hat{\Phi}_1{}^2 = B * \hat{d}x$ shall be called a *two-isoform* on $T^*\hat{M}(x,\hat{\mathfrak{R}})$, and can be written

$$\hat{\Phi}_2 = \hat{\Phi}_1{}^1 \hat{\wedge} \hat{\Phi}_1{}^2 = A_i T^i{}_r B_j T^j{}_s \hat{d}x^r \hat{\wedge} \hat{d}x^s =$$

$$= \tfrac{1}{2} (A_i T^i{}_r B_j T^j{}_s - A_i T^i{}_s B_j T^j{}_r) \hat{d}x^r \hat{\wedge} \hat{d}x^s =$$

$$= \tfrac{1}{2} A_i B_j (T^i{}_r T^j{}_s - T^i{}_s T^j{}_r) \hat{d}x^r \hat{\wedge} \hat{d}x^s, \quad (9.59)$$

thus showing the first deviations from the conventional exterior calculus (compare with Lovelock and Rund (*loc. cit.*), p. 132).

For the case of the isoexterior product of the one-isoforms we have the *three-isoforms*

$$\hat{\Phi}_3 = \hat{\Phi}_1{}^1 \wedge \hat{\Phi}_1{}^2 \hat{\wedge} \hat{\Phi}_1{}^3 = \quad (9.60)$$

$$= A^1{}_{i_1} A^2{}_{i_2} A^3{}_{i_3} \delta^{j_1 j_2 j_3}{}_{k_1 k_2 k_3} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} \hat{d}x^{k_1} \wedge \hat{d}x^{k_2} \hat{\wedge} \hat{d}x^{k_3},$$

where (see Lovelock and Rund (*loc. cit.*), Santilli (1982a) and others)

$$\delta^{i_1 i_2}{}_{j_1 j_2} = \det. \begin{bmatrix} \delta^{i_1}{}_{j_1} & \delta^{i_1}{}_{j_2} \\ \delta^{i_2}{}_{j_1} & \delta^{i_2}{}_{j_2} \end{bmatrix}, \quad (9.61a)$$

$$\delta^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \det. \begin{bmatrix} \delta^{i_1}{}_{j_1} & \delta^{i_1}{}_{j_2} & \delta^{i_1}{}_{j_3} \\ \delta^{i_2}{}_{j_1} & \delta^{i_2}{}_{j_2} & \delta^{i_2}{}_{j_3} \\ \delta^{i_3}{}_{j_1} & \delta^{i_3}{}_{j_2} & \delta^{i_3}{}_{j_3} \end{bmatrix}, \quad (9.61b)$$

etc. The extension to *n-isoforms* on $T^*\hat{M}(\hat{\Re})$ is left to the interested reader.

Given n one-isoforms $\hat{\Phi}_1{}^k = A^k \hat{*} \hat{\partial} x$, k = 1, 2, n, they are said to be *isolinearly dependent* when

$$\hat{\Phi}_1{}^1 \hat{\wedge} \ldots\ldots \hat{\wedge} \hat{\Phi}_1{}^n \equiv 0. \qquad (9.62)$$

Note that given n one-isoforms linearly dependent on $M(x,\Re)$, their isotopic images are not necessarily dependent.

Evidently, in an n-dimensional isomanifold $\hat{M}(\hat{\Re})$ there exist a maximum of n lineraly independent one-isoforms as in the conventional case, with basis $\hat{\partial}x^1, \ldots, \hat{\partial}x^n$. The space $\hat{M}(\hat{\Re})$ equipped with iso-oneforms is the cotangent space $T^*\hat{M}_1(\hat{\Re})$ at a given point.

Similarly, two-isoforms are elements of an isomanifold here denoted $T^*\hat{M}_2(\hat{\Re})$ of $\frac{1}{2}n(n-1)$-dimension with basis $\hat{\partial}x^i \hat{\wedge} \hat{\partial}x^j$, $i < j$, as in the conventional case. A similar situation occurs for p-isoforms

$$\Phi_p = A_{i_1 i_2 \ldots i_p} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} \times\times\times T^{i_p}{}_{j_p} \hat{\partial}x^{j_1} \hat{\wedge} \hat{\partial}x^{j_2} \hat{\wedge} \ldots \hat{\wedge} \hat{\partial}x^{j_p}, \qquad (9.63)$$

and related isomanifolds $T^*\hat{M}_p(\hat{\Re})$.

As an incidental note we point out without treatment the *Grassmann-isotopic algebra* $\hat{G}$, or *isograssmann algebra*, which is given by the direct sum

$$\hat{G} = \sum_{k = o,1,2,\ldots,n} T^*\hat{M}_k(\hat{\Re}). \qquad (9.64)$$

The necessary and sufficient conditions for a two-isoform (9.59) to be identically null are that

$$\delta^{i_1 i_2}{}_{j_1 j_2} A^1{}_{k_1} A^2{}_{k_2} T^{k_1}{}_{i_1} T^{k_2}{}_{i_2} =$$

$$= A^1{}_{k_1} A^2{}_{k_2} (T^{k_1}{}_{i_1} T^{k_2}{}_{i_2} - T^{k_1}{}_{i_2} T^{k_2}{}_{i_1}) \equiv 0. \qquad (9.65)$$

A similar situation occurs for p-isoforms.

We now study the *isodifferential calculus of p-isoforms*. Let $\hat{\Phi}_1 = A \hat{*} \hat{d}x$ be a one-isoform. We define as the *isoexterior derivative* of $\hat{\Phi}_1$ (also called *isoexterior differential*) and denoted with $\hat{d}\hat{\Phi}_1$, the two-isoform

$$\hat{\Phi}_2 = \hat{d}\hat{\Phi}_1 = \frac{\partial (A_{i_1} T^{i_1}{}_{j_1})}{\partial x^{i_2}} T^{i_2}{}_{j_2} \, \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} = \tag{9.66}$$

$$= \Big(\frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \Big) \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} =$$

$$= \tfrac{1}{2} \delta^{j_1 j_2}{}_{k_1 k_2} \Big(\frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \Big) \hat{d}x^{k_1} \hat{\wedge} \hat{d}x^{k_2}$$

from which one can see that $\hat{d}\hat{\Phi}_1$ is no longer the curl of the vector field A_{i_1}, but something more general, although admitting the conventional formulation as a particular case for $\hat{1} = I$.

The *isoexterior derivative of a two-isoform*

$$\hat{\Phi}_2 = A_{i_1 i_2} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} \, \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2}, \tag{9.67}$$

is given by the three-isoform

$$\hat{\Phi}_3 = \hat{d}\hat{\Phi}_2 = \Big(\frac{\partial A_{i_1 i_2}}{\partial x^{i_3}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + A_{i_1 i_2} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_3}} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} +$$

$$+ A_{i_1 i_2} T^{i_1}{}_{j_1} \frac{\partial T^{i_2}{}_{j_2}}{\partial x^{i_3}} T^{i_3}{}_{j_3} \Big) \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} \wedge \hat{d}x^{j_3}. \tag{9.68}$$

It is easy to see that the isoexterior derivative of the isoexterior product of a p-isoform $\hat{\Phi}_p$ and a q-isoform $\hat{\Phi}_q$ is given by

$$\hat{d}(\hat{\Phi}_p \hat{\wedge} \hat{\Phi}_q) = (\hat{d}\hat{\Phi}_p) \hat{\wedge} \hat{\Phi}_q + (-1)^p \hat{\Phi}_p \hat{\wedge} (\hat{d}\hat{\Phi}_q). \tag{9.69}$$

A p-isoform $\hat{\Phi}_p$ shall be called *isoexact* when there exists a (p-1) form $\hat{\Phi}_{p-1}$ such that

$$\hat{\Phi}_p = \hat{\partial}\, \hat{\Phi}_{p-1}. \tag{9.70}$$

Similarly, a p-isoform $\hat{\Phi}_p$ shall be called *isoclosed* when

$$\hat{\partial}\, \hat{\Phi}_p \equiv 0. \tag{9.71}$$

The most significant result of this section can be expressed as follows.

LEMMA II.9.1 (Isotopic Poincaré Lemma; Santilli (1988a,b), (1991b): Under sufficient regularity and continuity conditions, the Poincaré Lemma admits an infinite nunber of isotopic images, i.e., given an exact p-form $\Phi_p = d\Phi_{p-1}$, there exists an infinite number of isotopies of Φ_{p-1} into isoforms $\hat{\Phi}_{p-1}$

$$\Phi_{p-1} \Rightarrow \hat{\Phi}_{p-1}, \tag{9.72}$$

with consequential isotopies of the p-form

$$\Phi_p = d(\Phi_{p-1}) \Rightarrow \hat{\Phi}_p = \hat{\partial}(\hat{\Phi}_{p-1}), \tag{9.73}$$

for which the isoexterior derivative of the isoexact p-isoforms are identically null,

$$\hat{\partial}(\hat{\partial}\, \Phi_{p-1}) \equiv 0. \tag{9.74}$$

PROOF: Consider an isoexact two-isoform

$$\hat{\Phi}_2 = \hat{\partial}\, \hat{\Phi}_1 = \hat{\partial}(A_i\, T^i{}_j \hat{\partial} x^j). \tag{9.75}$$

Then, under the necessary regularity and continuity conditions, its isoexterior derivative

$$\hat{\partial}\, \hat{\Phi}_2\ (\hat{\partial} \hat{\Phi}_1) =$$

$$= \Big(\frac{\partial^2 A_{i_1}}{\partial x^{i_2} \partial x^{i_3}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + \frac{\partial A_{i_1}}{\partial x^{i_2}} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_3}} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} +$$

$$+ \frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} \frac{\partial T^{i_2}{}_{j_2}}{\partial x^{i_3}} T^{i_3}{}_{j_3} \Big) \, \partial x^{j_1} \hat{\wedge} \partial x^{j_2} \wedge \partial x^{j_3}, \qquad (9.76)$$

is identically null for all infinitely possible isotopic elements, as the reader can verify via simple but tedious calculations based on the antisymmetrization of all indeces. An iteration of the procedure then proves the lemma at any (finite) order p. QED.

In short, the existence of consistent isotopies of the Poincare' Lemma proves the consistency of the isotopic generalization of the conventional and exterior calculus under consideration here.

The mathematical relevance of Lemma II.9.1 is provided by the fact that the abstract, realization-free axioms

$$\Phi_2 = d\Phi_1, \quad d\Phi_2 \equiv 0, \qquad (9.77a)$$

$$\Phi_3 = d\Phi_2, \quad d\Phi_3 \equiv 0, \quad \text{etc.} \qquad (9.77b)$$

admit the conventional realization based on an ordinary manifold, as well as an infinite number of additional realizations for each given original form which can be readily identified via our isomanifolds. The latter realizations are generally inequivalent owing to the generally different isotopic elements or isounits.

The conventional Poincaré Lemma constitutes a geometric foundation of Galilei's, Einstein's special and Einstein's general relativities for the exterior problem in vacuum. As we shall see in Volume II, the Isotopic Poincaré Lemma constitutes a geometric foundation of the isotopic coverings of the above relativities for the interior dynamical problem within physical media.

Note that, for each given, conventional realization of axioms (9.77), there exist an infinite number of isotopies which are all geometrically equivalent, but physically inequivalent, because they characterize different integro-differential systems (II.1.1) with inequivalent solutions.

We shall now consider some cases of exact isoclosed isoforms. Consider a one-isoform $\hat{\Phi}_1$ on $T^*\hat{M}_1(\hat{\mathfrak{R}})$. Then, $d\hat{\Phi}_1 \equiv 0$, iff

$$\tfrac{1}{2}\delta^{j_1 j_2}{}_{k_1 k_2} \left(\frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \right) \equiv 0, \tag{9.78}$$

namely, the isoclosure of a one-isoform does not imply that the conventional curl of the vector A is null.

Similarly, given a exact two-isoform $\hat{\Phi}_2 = \hat{d}\hat{\Phi}_1$, the property $\hat{d}\hat{\Phi}_2 \equiv 0$ holds iff

$$\delta^{j_1 j_2 j_3}{}_{k_1 k_2 k_3} \left(\frac{\partial^2 A_{i_1}}{\partial x^{i_2} \partial x^{i_3}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + \frac{\partial A_{i_1}}{\partial x^{i_2}} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_3}} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + \frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} \frac{\partial T^{i_2}{}_{j_2}}{\partial x^{i_3}} T^{i_3}{}_{j_3} \right) \equiv 0. \tag{9.79}$$

We are now equipped to identify the desired geometry. Let us review the interplay between exact symplectic two-forms and Lie-isotopic algebras (see Santilli (1982a) for details). Recall that a conventional two-form on an even, 2n-dimensional manifold $T^*M_2(\Re)$ with covariant-geometric tensor $\Omega_{i_1 i_2}$

$$\Phi_2 = \tfrac{1}{2}\Omega_{i_1 i_2}\, dx^{i_1} \wedge dx^{i_2}, \tag{9.80}$$

characterizes the algebra brackets among functions A(x) and B(x) on $T^*M_2(\Re)$

$$[A \,\hat{,}\, B] = \frac{\partial A}{\partial x^{i_1}} \Omega^{i_1 i_2} \frac{\partial B}{\partial x^{i_2}}, \tag{9.81}$$

where the contravariant-algebraic tensor $\Omega^{i_1 i_2}$ is given by the familiar rule

$$\Omega^{i_1 i_2} = (|\Omega_{j_1 j_2}|^{-1})^{i_1 i_2}. \tag{9.82}$$

Now, the integrability conditions for two-form (9.80) to be an exact symplectic two-form are given by

$$\Omega_{i_1 i_2} + \Omega_{i_2 i_1} \equiv 0, \tag{9.83a}$$

$$\frac{\partial \Omega_{i_1 i_2}}{\partial x^{i_3}} + \frac{\partial \hat{\Omega}_{i_2 i_3}}{\partial x^{i_1}} + \frac{\partial \Omega_{i_3 i_1}}{\partial x^{i_2}} \equiv 0, \qquad (9.83b)$$

The above conditions are equivalent to the integrability conditions

$$\Omega^{i_1 i_2} + \Omega^{i_2 i_1} \equiv 0, \qquad (9.84a)$$

$$\Omega^{i_1 k} \frac{\partial \Omega^{i_2 i_3}}{\partial x^k} + \Omega^{i_2 k} \frac{\partial \Omega^{i_3 i_1}}{\partial x^k} + \Omega^{i_3 k} \frac{\partial \Omega^{i_1 i_2}}{\partial x^k} \equiv 0, \quad (9.84b)$$

for generalized brackets (9.81)) to be Lie-isotopic, i.e., verify the Lie algebra axioms in their most general possible, classical, regular realization on $T^*M_2(\Re)$

$$[A \hat{,} B] + [B \hat{,} A] \equiv 0, \qquad (9.85a)$$

$$[[A \hat{,} B] \hat{,} C] + [[B \hat{,} C] \hat{,} A] + [[C \hat{,} A] \hat{,} B] \equiv 0. \qquad (9.85b)$$

Thus, the exact character of the two-form $\Phi_2 = d\Phi_1$ implies its closure $d\Phi_2 \equiv 0$ (Poincaré Lemma), which, in turn, guarantees that the underlying brackets are Lie-isotopic, with the canonical case being a trivial particular case (see the analytic, algebraic, and geometric proofs of Santilli (1982), Sect. 4.1.5.

Lemma II.9.1 establishes that all the above results on the conventional exterior calculus persist under isotopies. Our objective is than that of using the isotopies for the identification of the isounit of the Lie-isotopic algebra directly in the structure of the brackets.

DEFINITION II.9.1 (Santilli (1988a,b), (1991b)): Under sufficient continuity and regularity conditions, an "exact symplectic-isotopic manifold", or "isosymplectic manifold" for short, is a 2n-dimensional isomanifold $T^*\hat{M}_2(x,\hat{\Re})$ equipped with an exact and nowhere degenerate two-isoform

$$\Phi_2 = \tfrac{1}{2}\, \Omega_{i_1 i_2}(t, x, \dot{x}, \ldots)\ \hat{d}x^{i_1} \hat{\wedge} \hat{d}x^{i_2} = \qquad (9.86)$$

$$= \hat{d}\Phi_1 = \frac{\partial (A_{i_1} T^{i_1}{}_{j_1})}{\partial x^{i_2}} T^{i_2}{}_{j_2}\ \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} =$$

$$= \left(\frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \right) dx^{j_1} \wedge dx^{j_2} =$$

$$= \tfrac{1}{2} \delta^{j_1 j_2}{}_{k_1 k_2} \left(\frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \right) dx^{k_1} \wedge dx^{k_2}$$

$$A_i = A_i(x), \qquad T = T(t, x, \dot{x}, ...)$$

which is such to admit the factorization

$$\hat{\Omega}_{i_1 i_2} = \Omega_{i_1 k}(x) \times T_2{}^{k}{}_{i_2}(t, x, \dot{x}, ...), \qquad \hat{T}_2 > 0, \qquad (9.87)$$

where T_2 *is the nowhere singular, symmetric and real-valued isotopic element of* $T^*\hat{M}_2(\hat{x}, \hat{\Re})$*, and*

$$\Omega_{i_1 i_2} = \frac{\partial A_{i_2}}{\partial x^{i_1}} - \frac{\partial A_{i_1}}{\partial x^{i_2}}, \qquad (9.88)$$

is Birkhoff's tensor (II.7.3), with corresponding Lie-isotopic brackets

$$[A \hat{,} B] = \frac{\partial A}{\partial x^{i_1}} \hat{1}_2{}^{i_1}{}_{k}(t, x, \dot{x}, ...)\, \Omega^{k i_2}(x) \frac{\partial B}{\partial x^{i_2}} \qquad (9.89a)$$

$$\hat{1}_2 = \hat{T}_2^{-1}, \qquad (\Omega^{i_1 i_2}) = (\|\Omega_{k_1 k_2}\|^{-1}), \qquad (9.89b)$$

where $\hat{1}_2$ *is the isounit of the universal enveloping associative algebra of the Lie-isotopic algebra with brackets (9.89) on* $T^*\hat{M}_2(\hat{x}, \hat{\Re})$*. The "symplectic-isotopic geometry" or "isosymplectic geometry" for short, is the geometry of the isosymplectic manifolds.*

As an illustration, we shall now work-out an explicit model of isosymplectic manifolds. For physical applications it is sufficient to consider the *canonical isosymplectic manifolds*, i.e., the isomanifolds of Definition II.9.1 where Birkhoff's tensor Ω is replaced by the simpler canonical tensor ω.

Let us consider again the physical realization of the abstract

$T^*\hat{M}_2(x,\Re)$ manifold as the cotangent bundle $T^*\hat{E}_2(r,\Re)$ with local coordinates

$$a = (a^\mu) = (r, p) = (r_i, p_i), \quad \mu = 1, 2, ..., 2n, \quad i = 1, 2, ..., n, \qquad (9.90)$$

where r represents the Cartesian coordinates and p the linear momenta.

Then, we can introduce the *canonical one-isoform* on $T^*\hat{E}_1(r,\hat{\Re})$ of the particular type

$$\hat{\Phi}^\circ_1 = R^\circ_\mu T_1{}^\mu{}_\nu \, \hat{\partial} x^\nu , \qquad (9.91a)$$

$$R^\circ = (p, 0), \qquad (9.91b)$$

$$T_1 = \text{diag.}\,(b^2{}_1, ..., b^2{}_{2n}) > 0, \quad b_k > 0 \qquad (9.91c)$$

$$T_1{}^i{}_j = \delta^i{}_j \; b^2{}_j \quad \text{(no sum)}, \qquad (9.91d)$$

Its isoexterior derivative on $T^*\hat{E}_2(r,\hat{\Re})$ is given by

$$\hat{\Phi}^\circ_2 = \hat{\partial}\hat{\Phi}^\circ_1 = \tfrac{1}{2}[\omega_{\mu_1\mu_2} b^2_{\mu_1} b^2_{\mu_2} + $$

$$+ (R^\circ_{\mu_2} \frac{\partial b^2_{\mu_2}}{\partial a^{\mu_1}} b^2_{\mu_1} - R^\circ_{\mu_1} \frac{\partial b^2_{\mu_1}}{\partial a^{\mu_2}} b^2_{\mu_2})] \, \hat{\partial} a^{\mu_1} \hat{\wedge} \hat{\partial} a^{\mu_2} \qquad (9.92)$$

and it always admits the factorization

$$\hat{\Phi}_2 = \hat{\partial}\, \hat{\Phi}_1 = \tfrac{1}{2}\, \omega_{\mu_1 \nu} \times \hat{T}_2{}^\nu{}_{\mu_2}(t, x, \dot{x}, ...) \, \hat{\partial} a^{\mu_1} \hat{\wedge} \hat{\partial} a^{\mu_2} \qquad (9.93)$$

with

$$T_2{}^{\mu_1\mu_2} = b^{2\mu_1} b^{2\mu_2} + \omega^{\mu_1\alpha} \omega^{\mu_2\beta} (R^\circ_\beta \frac{\partial b^2_\beta}{\partial a^\alpha} b^2_\alpha - R^\circ_\alpha \frac{\partial b^2_\alpha}{\partial a^\beta} b^2_\beta) \qquad (9.94)$$

The isomanifold $T^*\hat{E}_2(r,\hat{\Re})$ equipped with two-isoform (9.93) is isosymplectic when T_2 coincides with its isotopic element.

Under these conditions, the generalized brackets characterized by structure (9.93)

$$[A \hat{,} B] = \frac{\partial A}{\partial a^{\mu_1}} \omega^{\mu_1 \nu} \times \hat{1}_{2\nu}{}^{\mu_2}(t, x, \dot{x}, ...) \frac{\partial B}{\partial a^{\mu_2}}, \tag{9.95a}$$

$$\hat{1}_2 = \hat{T}_2^{-1}, \tag{9.95b}$$

are indeed Lie-isotopic and exhibit the isounit $\hat{1}_2$ of $T^*\hat{E}_2(r,\hat{\mathfrak{R}})$ directly in their structure, as desired.

An example is given when

$$T_2 = \text{diag.}\,(b^2{}_1, ..., b^2{}_{2n}) > 0, \quad b_K > 0, \tag{9.96a}$$

$$\frac{\partial b_i{}^2}{\partial x^j} = \text{Cost.}, \qquad i.j, = 1, 2, ..., 2n. \tag{9.96b}$$

The interested reader can then work out an endless number of examples of isosymplectic manifolds of both Birkhoff-isotopic and Hamilton-isotopic type. Additional examples will be provided in the physical applications of the subsequent chapters.

We close this section with a comparative analysis of the isosymplectic geometry of this section and the Birkhoffian-isotopic mechanics of the preceding section. In Sect. II.7 wee did introduce symplectic-isotopic two-forms, Definition II.7.1, however, they were not symplectic *isoforms.* In fact, the one-forms on $T^*\hat{E}_1(r,\hat{\mathfrak{R}})$ of this section

$$\hat{\Phi}_1 = R_\mu(a)\, T^\mu{}_\nu(t, a, \dot{a}, ...)\, \hat{\partial} a^\nu, \tag{9.97}$$

formally coincides with those of Sect. II.7 in a fixed local chart in which $dx \equiv \hat{\partial}x$.

However, forms (9.97) were characterized in Sect. II.7 via the ordinary calculus of differential forms. In fact, the main geometrical structure of Definition II.7.1 is the *conventional exterior derivative of an exact conventional two-form,*

$$\Phi_2 = d(\Phi_1). \tag{9.98}$$

Since the Poincaré Lemma does indeed apply to the exact two-form $\hat{\Phi}_2$, we have

$$d\Phi_2 \equiv 0, \tag{9.99}$$

and the isotopy of Definition II.7.1 follows.

In this section we have brought the notion of isotopy to its most general possible level, by introducing the isodifferential calculus of isoforms, with isoexterior derivatives $\hat{d}$, and then computed the two-isoforms

$$\hat{\Phi}_2 = \hat{d}(\hat{\Phi}_1). \tag{9.100}$$

The Isotopic Poincaré Lemma then ensures that

$$\hat{d}\hat{\Phi}_2 \equiv 0. \tag{9.101}$$

The infinite isotopies of Definition II.9.1 then follow.

What we have gained in the process is a further enlargement of the geometric structure which is needed for the study of the possible *direct universality of the isosymplectic geometry for the most general known class of nonlinear and nonlocal vector-fields (II.1.1)* , which we hope to study at some later time.

In addition, we have learned how to reinterpret any exact, noncanonical symplectic structure as an isocanonical structure, directly in the local chart considered, which is a necessary condition for the identification of the isounit of the related Lie-isotopic algebra.

II.10: ISOAFFINE GEOMETRY.

We shall now proceed with our study of the geometrical characterization of systems (II.1.1) by reviewing the *affine-isotopic geometry* or *isoaffine geometry* for short, introduced, apparently for the first time, in Santilli (1988d) and then studied in more detail in Santilli (1991b).

The new geometry essentially permits a generalization of the current local-differential character of the affine geometry into a nonlocal-integral form capable of treating systems of type (II.1.1), with consequential generalization of the notions of curvature, parallel transport, geodesic, etc.

The literature in the conventional affine geometry is predictably vast. Among the earliest references, the presentation by Schrödinger (1950) still has considerable value. In this section we shall follow the treatise by Lovelock and Rund (1975) of which we preserve the notation mostly unchanged for clarity in the comparison of the results.

For a full understanding of this and of the following sections, the reader is expected to have a prior knowledge of the following notions introduced in the preceding sections: *isofields* $\hat{\mathfrak{R}}$, *isovector spaces* $\hat{V}$ and *isometric spaces* $\hat{M}$, with particular reference to the *isoeuclidean space* $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ and the *isominkowski space* $\hat{M}(x,\hat{\eta},\hat{\mathfrak{R}})$.

The study of the implications of isotopies for differentiable manifolds was initiated in the preceding section on the isosymplectic geometry, by introducing the notion of isodifferentials $\hat{d}x$, and then using it for the constructions of the elements of the isoexterior calculus.

In this section we shall enter deeper into this study and identify the implications of isodifferentials for the notions of connections, curvature, etc.

Let $M(x,\mathfrak{R})$ be an n-dimensional *affine space* (Lovelock and Rund (1975)) here referred as a differentiable manifold with local coordinates $x = (x^i)$, $i = 1, 2, ..., n$, over the reals $\mathfrak{R}$. We shall denote: the conventional scalars on $M(x,\mathfrak{R})$ with $\phi(x)$; contravariant and covariant vectors with $X^j(x)$ and $X_j(x)$, respectively; and mixed tensors of rank (r.s)

$$X^{(r.s)} = X^{j_1 j_2 .. j_r}{}_{k_1 k_2 .. k_s}(x). \tag{10.1}$$

Unless otherwise stated, all tensors considered on $M(x,\mathfrak{R})$ will be assumed hereon to be local-differential and to verify all needed continuity and regularity conditions.

DEFINITION II.2.1 (Santilli (1988d), (1991b): The infinite class of isotopic liftings $\hat{M}(x,\hat{\mathfrak{R}})$ of an affine space $M(x,\mathfrak{R})$, called "affine-isotopic spaces" or "isoaffine spaces" for short, are characterized by the same local coordinates x and the same local-differential tensors $X^{(r.s)}$ of $M(x,\mathfrak{R})$ but now defined with respect to the isotopic liftings of the field

$$M(x,\mathfrak{R}) \Rightarrow \hat{M}(x,\hat{\mathfrak{R}}) \; : \; \hat{\mathfrak{R}} = \mathfrak{R}\hat{1}, \tag{10.2}$$

for all infinitely possible isounits $\hat{1}$ in $n{\times}n$ dimension which are *nowhere singular and Hermitean, but otherwise possess an arbitrary, generally nonlinear and nonlocal dependence on an independent parameter s, the variables x, their derivatives with respect to s of arbitrary order, and any other quantity needed for physical applications, such as density μ, temperature τ, index of refraction n,...),*

$$\hat{1} = \hat{1}(s, x, \dot{x}, \ddot{x}, \mu, \tau, n, ...). \tag{10.3}$$

In this and in the next two sections we shall study isoaffine spaces for arbitrary isounits $\hat{1}$. Nevertheless, it may be recommendable to keep in mind the intended use of the theory, that of attempting a more general formulation of the *interior* gravitational problem, which is capable of recovering identically the conventional gravitational theories for the exterior problem (Chapter V).

As a result, the reader should keep in mind that:

1) The isounits $\hat{1}$ are defined in a well identified region, in the interior of the minimal surface S° encompassing all matter, including its boundary (e.g., the interior of Jupiter);

2) The isounits $\hat{1}$ shall represent the nonlinear, nonlocal, nonlagrangian and nonnewtonian forces expected in the interior gravitational problem, as well as the generally inhomogeneous and anisotropic character of interior physical media; and,

3) All possible isounits $\hat{1}$ shall admit as a particular case the trivial units I = diag. (1, 1, ..., 1) of the affine geometry in the same dimension and recover it everywhere in the exterior of S°, $\hat{1}_{>S^\circ} \equiv I$, so as to permit the recovering in their entirety of the conventional gravitational theories for the exterior problem.

As done in the preceding sections, the isounit $\hat{1}$ will be assumed to be nonsingular, real-valued and symmetrical

$$\hat{1}^\dagger \equiv \hat{1}, \qquad \hat{1} = (\hat{1}^i{}_j) = (\hat{1}_j{}^i), \qquad \hat{1} \neq 0, \tag{10.4}$$

The *isotopic element* T = T(s, x, $\dot{x}$,...) of the theory can then be written

$$\hat{1} = T^{-1}, \qquad T = (T_i{}^j) = (T^j{}_i). \tag{10.5}$$

A first salient feature of the liftings $M(x,\mathfrak{R}) \Rightarrow \hat{M}(x,\hat{\mathfrak{R}})$ is that the conventional *linear transformations*, i.e., the linear, right, modular, associative transformations on $M(x,\mathfrak{R})$

$$x' = A\,x, \tag{10.6}$$

must now be necessarily generalized into the *isolinear transformations (or isotransformations)* on $\hat{M}(x,\hat{\mathfrak{R}})$, i.e., the *isolinear, right, modular, associatiove isotopic transformations* studied in Sect. II.4,

$$\bar{x} = A * x \overset{\text{def}}{=} ATx, \tag{10.7}$$

where T is fixed.

In turn, the lifting $Ax \Rightarrow A*x$ has a number of consequences. First, it permits the treatment of nonlocal-integral structures which would be otherwise precluded by the conventional theory of affine spaces.

This is readily done via the embedding of all nonlocal-integral terms in the isotopic element of the theory. The insensitivity of the affine geometry to the topology of its unit then ensures the achievement of a mathematically consistent structure.

Secondly, isotransformations (10.7) are called *isolinear* and *isolocal* (Sect. II.4) in the sense that they verify all abstract linearity and locality conditions on $\hat{M}(x,\hat{\mathfrak{R}})$. Nevertheless, *they are generally nonlinear and nonlocal when written in the original space $M(x,\mathfrak{R})$*, i.e.

$$\bar{x} = A * x = A\,T(x, \dot{x}, \ddot{x}, ..)\,x \tag{10.8}$$

The liftings $Ax \Rightarrow A*x$ imply that all conventional contractions of indeces are now lifted via the insertion of the isotopic element, i..e.,

$$A^i_{\ j}\, x^j \quad \Rightarrow \quad A^i_{\ j}\, T^j_{\ k}\, x^k. \tag{10.9}$$

Let us also recall that the use of conventional transformations (10.6) on the isotopic spaces $\hat{M}(x,\hat{\mathfrak{R}})$ would violate the condition of (iso) linearity. This illustrates the *necessity* of the liftings $Ax \Rightarrow A*x$.

Finally, we assume the reader is familiar with the fact that all distinctions between conventional transformations (10.6) and their isotopic forms (10.7) cease to exist, by construction, at the abstract, realization-free level. Thus, by their very conception, *isoaffine spaces are a more general realization of the mathematical axioms of the conventional spaces*, that is, the spaces $M(\mathfrak{R})$ and $\hat{M}(\hat{\mathfrak{R}})$ are locally isomorphic.

This ultimate geometric equivalence ensures the mathematical consistency of our liftings. As a matter of fact, the equivalence can be used to verify the consistencies of individual treatments.

Despite this geometrical equivalence, the physical implications of our isotopies are rather deep, as we shall see.

Recall in the conventional case that, given two contravariant vectors x_1 and x_2 on $M(x,\mathfrak{R})$, their difference Δx is a contravariant vector iff the transformation is *linear*. Similarly, Δx is a contravariant vector on $\hat{M}(x,\hat{\mathfrak{R}})$ iff the transformation is *isolinear*. We reach in this way the following simple result (see also Propositions II.3.1).

PROPOSITION II.10.1 (loc. cit.): For any given (sufficiently smooth and regular) nonlinear and nonlocal transformation on $M(x,\mathfrak{R})$, there always exists an isounit $\hat{1}$ under which the transformation becomes isolinear and isolocal on $\hat{M}(x,\hat{\mathfrak{R}})$($\hat{\mathfrak{R}} = \mathfrak{R}\hat{1}$. Similarly, for any given coordinate difference Δx of two contravariant vectors on $M(x,\mathfrak{R})$ which does not transform contravariantly, there always exists an isotope $\hat{M}(x,\hat{\mathfrak{R}})$ of $M(x,\mathfrak{R})$ under which Δx transforms isocovariantly.

The *left, modular isotransformations* are evidently defined by

$$\bar{x} = A{*}x = ATx, \tag{10.10a}$$

$$\bar{x}^t = x^t * A^t = x^t\, T\, A^t, \tag{10.10b}$$

where t denotes conventional transpose. The inverse, right-modular transformations are given by the isotopic rule

$$x = A^{-\hat{1}} * \bar{x} = A^{-\hat{1}}\, \bar{T}\, x, \tag{10.11}$$

where $A^{-\hat{1}}$ is the *isoinverse*, i.e., it verifies the isotopic rules

$$A^{-\hat{1}} * A = A * A^{-\hat{1}} = \hat{1}, \tag{10.12}$$

and, from here on, when considering the isotopy in the new coordinate system, we shall put

$$\bar{T} = \bar{T}(\bar{x}, \bar{x}, ...) \equiv T(x, \dot{x}, ...). \tag{10.13}$$

Note the preservation of the isotopic element for the left and inverse isotransformations. This preservation is ensured by the assumed Hermiticity of the element T and it is at the very foundations of the *Lie-isotopic theory* (Santilli (1978a) (1982a)) with basic product

$$[A \,\hat{,}\, B] = A{*}B - B{*}A = ATB - BTA. \qquad (10.14)$$

$\hat{M}(x,\hat{\Re})$ is then the correct *isomodule* for the isorepresentations of the Lie-isotopic algebra characterized by product (10.14) (App. II-D).

If the Hermiticity of T is relaxed, the right isotopic element becomes different than the left one

$$\bar{x}' = A > x \stackrel{\text{def}}{=} A\,T\,x, \qquad \bar{x}'^{t} = x^{t} < A^{t} \stackrel{\text{def}}{=} x^{t}T^{\dagger}A^{t}, \qquad (10.15a)$$

$$T \neq T^{\dagger}, \qquad (10.15b)$$

This signals the necessary emergence of the covering *Lie-admissible theory* (Santilli (1967), (1978a) (1981a)) with basic product

$$(A\,,B) = A < B - B > A = AT^{\dagger}B - BTA, \qquad (10.16)$$

verifying the axioms of the covering Lie-admissible algebra. In this case the generalized affine space is the correct *isobimodule* of the Lie-admissible algebra (Appendix II.D).

Note that in this case we have *two different isounits*,

$$I^{>} = T^{-1}, \quad {}^{<}I = T^{\dagger -1}, \qquad (10.17)$$

and *two different isofields*

$$\Re^{>} = \Re I^{>}\,, \quad {}^{<}\Re = {}^{<}I\,\Re, \qquad (10.18)$$

or, equivalently, we have one single quantity ${}^{<}\Re^{>}$, representing both the right- or left-modular-isotopic action depending on the assumed conjugation.

We shall reserve the name of *affine-admissible spaces* , or *genoaffine spaces* [16] for short, and the symbold ${}^{<}M^{>}(x, {}^{<}\Re^{>})$ for the

[16] In Santilli (1978a) we introduced the following two main lines of research:

A) The *isotopies* conceived as *axiom-preserving generalizations of a given mathematical structure* .This first notion has resulted in the Lie-isotopic formulations for the treatment of nonhamiltonian systems as closed-isolated, which are the main mathematical tools of these monographs. And emerging structure.

The isoaffine spaces are necessary for the study of interior gravitation as a whole, i.e., in closed-conservative conditions (see App. II.C for details). In fact, the antisymmetry of the Lie-isotopic product (10.14) ensures the conservation of the total energy,

$$dH/dt = [H \hat{,} H] = HTH - HTH \equiv 0, \tag{10.19}$$

and similar conservations follow for the other total quantities under a generalized internal structure evidently represented by the isotopic element T. As a result, isospaces $\hat{M}(x,\hat{R})$ are the fundamental ones of the analysis of this paper for interior gravitation.

The genoaffine spaces instead imply the necessary study of gravitation in open-nonconservative conditions. In fact, owing to the lack of antisymmetry of the Lie-admissible product (10.16), we now have time-rate-of-variations of the energy H of the considered interior particle (Sect. I.4)

B) The *genotopies* conceived as *axiom-inducing alterations of given mathematical structures* . This second notion has resulted into the still more general Lie-admissible formulations for the treatment of nonhamiltonian systems as open-nonconservative, as presented in the appendices.

The two notions were illustrated as follows. Let L be a Lie algebra with (ordered) basis $\{ X_i \}$, $i = 1, 2, ..., n$, and trivial product $[X_i, X_j] = X_iX_j - X_jX_i$, where X_iX_j is the conventional associative product. Then, the isotopies $\hat{L}$ of L are given by the now familiar mappings of the original Lie product while keeping the basis unchanged

$$L: \ [X_i X_j] = X_iX_j - X_jX_i \quad \Rightarrow \quad \hat{L}: \ [X_i \hat{,} X_j] = X_i\, T\, X_j - X_j\, T\, X_i, \qquad T = T^\dagger$$

which *prerserve the Lie algebra axioms by central condition* On the contraries, the genotopies U of L are given by mappings of the original Lie product while keeping the basis unchanged, which now *violate the original Lie algebra axioms in favor of more general covering axioms also by central condition* The realizations of U suggested in Santilli (1978b) is that of form (10.16), i.e.

$$L: \ [X_i,X_j] = X_iX_j - X_jX_i \ \Rightarrow \ {}^{<}L^{>}: \ (X_i,X_j) = X_i < X_j \ - \ X_j > X_i = X_iT^\dagger X_j - X_jTX_i$$

which verify the axioms of the covering Lie-admissiblealgebras. From hereon, the prefix "iso" shall therefore denote the preservation of the original axioms, while the prefix "geno" shall denote the alterations of the original axioms in favor of covering axioms. Note that the original axioms are not lost under genotopies, but preserved in their entirety, although as particular cases of covering axioms. In fact, the Lie algebras are not lost in the transition to the Lie-admissible algebras, but are preserved in their entirety because contained in the classification of the covering Lie-admissible algebras.

$$dH/dt = (H\,,H) \neq 0, \tag{10.20}$$

while the remaining system is considered to be external. The affine-admissible space therefore are the fundamental ones on the still more general, Lie-admissible approach, which will not be studied in detail in this volume for brevity, but only outlined in the Appendices..

We now study the *affine-isotopic geometry*, or *isoaffine geometry*, i.e., the isotopic liftings of the conventional geometry characterized by isotransformations (10.8).

Recall from Sect. II.9 that, in the conventional case, the differentials of the two coordinates x and x′ are given by the familiar forms dx′ and dx with interconnecting rule

$$dx' = A\,dx, \qquad dx'^{\,i} = A^i{}_j\,dx^j. \tag{10.21}$$

But the same interconnection does not hold for the differentials $d\bar{x}$ and dx because of property (10.7), i.e., by central assumption of isotopy, $d\bar{x} \neq A\,dx$.

Following Sect. II.9, we therefore introduce the generalized notion of *isodifferentials* $\hat{d}\bar{x}$ and $\hat{d}x$ when interconnected by the isotopic law

$$\hat{d}\bar{x} = A * dx, \qquad \hat{d}\bar{x}^i = A^i{}_j\,T^j{}_k\,\hat{d}x^k. \tag{10.22}$$

Similarly, we recall from Sect. II.9 the *isodifferential of an isoscalar* $\phi(x)$ on $\hat{M}(x,\hat{\mathfrak{R}})$

$$\hat{d}\phi(x) = \frac{\partial\phi}{\partial x} * \hat{d}x = \frac{\partial\phi}{\partial x^i}\,T^i{}_j\,\hat{d}x^j \tag{10.23}$$

where the partial derivative is the conventional one, as well as the *isodifferential of a contravariant isovector* $X = (X^i(x))$ on $\hat{M}(x,\hat{\mathfrak{R}})$

$$\hat{d}X = \frac{\partial X}{\partial x} * \hat{d}x, \qquad \hat{d}X^i = \frac{\partial X^i}{\partial x^j}\,T^j{}_k\,\hat{d}x^k, \tag{10.24}$$

The above quantity then allows the introduction of the *isotransformation laws of the contravariant isovector*

$$\bar{X}(\bar{x}) = \frac{\partial\bar{x}}{\partial x} * X(x), \qquad \bar{X}^i = \frac{\partial\bar{x}^i}{\partial x^j}\,T^j{}_k\,X^k(x), \tag{10.25}$$

Recall also that, while in the conventional (linear) case $x' = Ax$, $\partial x'/\partial x = A$, we now have on $\hat{M}(x,\hat{\mathfrak{R}})$

$$\frac{\partial \bar{x}^i}{\partial x^j} = A^i{}_k T^k{}_j + A^i{}_k \frac{\partial T^k{}_r}{\partial x^j} x^r, \tag{10.26}$$

Similarly, we have the *isotransformations of a contravariant isotensor* X^{ij} of rank two on $\hat{M}(x,\hat{\mathfrak{R}})$

$$\bar{X}^{(2)}(\bar{x}) = \frac{\partial \bar{x}}{\partial x} * \frac{\partial \bar{x}}{\partial x} * X^{(2)}(x), \quad \bar{X}^{ij}(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^r} T^r{}_p \frac{\partial \bar{x}^j}{\partial x^s} T^s{}_q X^{pq}(x), \tag{10.27}$$

with similar extension to higher orders, as well as contravariant isotensors of rank (0.s) and generic tensors of rank (r.s).

The reader should also recall from Sect. II.9 the identity of the above isoquantities with the conventional quantities.

From the preceding results we have the *isodifferential of a contravariant isovector-field*

$$\hat{\partial} \bar{X}^j = \frac{\partial \bar{X}^j}{\partial \bar{x}^k} T^k{}_r \, \hat{\partial} \bar{x}^r =$$

$$= \frac{\partial^2 \bar{x}^j}{\partial x^s \partial x^i} T^i{}_r X^r \hat{\partial} x^s + \frac{\partial \bar{x}^j}{\partial x^i} T^i{}_r \frac{\partial X^r}{\partial x^s} \hat{\partial} x^s + \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial T^i{}_r}{\partial x^s} X^r \hat{\partial} x^s \tag{10.28}$$

We now introduce the *isocovariant (or isoabsolute) differential* $\hat{D}X^r$

$$\hat{D}X^j = \hat{\partial}X^j + \hat{P}^j(x, X, \hat{\partial}x), \tag{10.29}$$

under the condition that it preserves the original axioms (Lovelock and Rund (*loc.cit.*), p.68), i.e.,

1) $\hat{D}(X^j + Y^j) = \hat{D}X^j + \hat{D}Y^j$, which can hold iff $\hat{P}^j$ is isolinear in X^r;

2) $\hat{D}X^j$ is isolinear in $\hat{\partial}x^s$; and

3) $\hat{D}X_j$ transforms as a contravariant isovector.

By again using Lovelock-Rund's symbols with a "hat" to denote isotopy, we can write

$$\hat{D}X^j = \partial X^j + \hat{\Gamma}_{h\,k}^{\;\;j} T^h{}_r X^r T^k{}_s \partial x^s , \qquad (10.30)$$

where the $\hat{\Gamma}$'s are here called the component of an *isoaffine connection.*

By lifting the conventional procedure, one can readily see that the necessary and sufficient conditions for the n^3 quantities $\hat{\Gamma}_m{}^s{}_n$ to be the coefficients of an isoaffine connection are given by

$$\bar{\Gamma}_{m\,p}^{\;\;j} T^m{}_r \frac{\partial \bar{x}^r}{\partial x^s} T^s{}_t X^t T^p{}_q \frac{\partial \bar{x}^q}{\partial x^w} T^w{}_z \partial x^z =$$

$$= \frac{\partial \bar{x}^j}{\partial x^r} T^r{}_s \hat{\Gamma}_m{}^s{}_n T^m{}_p X^p T^n{}_q \partial x^q - \frac{\partial^2 \bar{x}^j}{\partial x^s \partial x^i} T i^r X^r \partial x^s +$$

$$+ \frac{\partial x^j}{\partial x^i} T^h{}_r \frac{\partial X^r}{\partial x^s} (T^s{}_t \partial x^t - \partial x^s) - \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial T^i{}_r}{\partial x^s} X^r \partial x^s. \qquad (10.31)$$

As in the conventional case, the $\hat{\Gamma}$'s do not constitute a tensor of rank (1.2). The extra terms in conditions (10.31), therefore, do not affect the consistency of the isoaffine geometry, but constitute the *desired generalization.*

An important particular case occurs when T is a constant, which is the case when the characteristic isotopic functions representing the interior physical medium are averaged into a constant (see next chapter). In this case the isotopy of the conventional terms persists, but the additional terms are null. Finally, note that all conventional notions and properties are admitted as a trivial particular case by the isoaffine geometry whenever T = I.

The extension of the above results to the *isocontravariant derivatives* is evidently given by

$$\hat{D}X_j = \partial X_j - \hat{\Gamma}_j{}^s{}_n T_s{}^r X_r T^n{}_p \partial x^p. \qquad (10.32)$$

As a result, ***the isocovariant derivative of a scalar coincides with the isodifferential,*** as in the conventional case, i.e.,

$$\hat{D}\phi = \hat{D}(X^i X_i) = \partial\phi. \qquad (10.33)$$

We shall say that the isoaffine connection is *symmetric* if the following property is verified

$$\hat{\Gamma}_m{}^s{}_n = \hat{\Gamma}_n{}^s{}_m. \tag{10.34}$$

The following property can be trivially proved (but carries important physical consequences).

PROPOSITION II.10.2 (loc. cit.): The isotopic image $\hat{\Gamma}_h{}^j{}_k$ of a conventional, symmetric, affine connection $\Gamma_h{}^j{}_k = \Gamma_k{}^j{}_h$ is not necessarily symmetry.

The isotopic liftings of all remaining properties of covariant derivatives, as well as the extension to the isocovariant differential of tensors, will be left for brevity to the interested reader.

It is easy to see that the isocovariant (isoabsolute) differential preserves the basic axioms of the conventional differential, i.e. (Lovelock and Rund ((*loc. cit.*), p.74):

AXIOM 1: The isocovariant differential of a constant is identically null; that of a scalar coincides with the isodifferential; and that of a tensor of rank (r.s) is a tensor of the same rank.

AXIOM 2: The isocovariant differential of the sum of two tensors of the same rank is the sum of the isoabsolute differential of the individual tensors. And

AXIOM 3: The isocovariant differential of the product of two tensors of the same rank verifies the conventional chain rule of differentiation.

By following again the pattern of the conventional formulation, and as a natural generalization of the isocovariant differential, we introduce the *isocovariant derivative* of a contravariant vector field X^p

$$X^j{}_{\hat{|}k} \overset{\text{def}}{=} \frac{\partial X^j}{\partial x^k} + \hat{\Gamma}_h{}^j{}_k \, T^h{}_r \, X^r, \tag{10.35}$$

under which the isocovariant differential can be written

$$\hat{D}X^j = X^j{}_{\lceil k} T^k{}_s \hat{d}x^s. \tag{10.36}$$

It is an instructive exercise for the interested reader to prove that the isocovariant derivatives (10.35)) constitute the components of a (1.1) isotensor.

It is also easy to verify that the isocovariant derivatives preserve the axioms of the conventional covariant derivatives (Lovelock and Rund (*loc. cit.*), p. 77):

AXIOM 1': The isocovariant derivative of a constant is identically null; that of a scalar is equal to the conventional partial derivative; and that of an isotensor of rank (r.s) is an isotensor of rank (r.s+1).

AXIOM 2': The isocovariant derivative of the sum of two tensors of the same rank is the sum of the isocovariantderivatives of the individual tensors. And

AXIOM 3': The isocovariant derivative of the product of two isotensors of the same rank is that of the usual chain rule of partial derivatives.

Axioms 1, 2, 3 and 1', 2', 3' imply the most important result of this section, which can be expressed via the following

PROPOSITION II.10.3 Santilli (loc. cit.): Under sufficient continuity conditions, all infinitely possible isotopic liftings of an affine geometry coincide with the same geometry at the abstract, coordinate-free level.

In actuality, the capability of our isotopies of preserving the basic axioms is such that, the isotopic liftings can be used as a test of geometric consistency of a *conventional* theory.

In fact, *if a given property is not preserved under isotopy, the definition of the property itself is geometrically incomplete.* As we shall see in the next section, this is precisely the case of the historical Einstein's tensor.

We now pass to the study of a central notion of our analysis, the generalized curvature, herein called *isocurvature* ,and the generalized torsion, herein called *isotorsion* , which are inherent in the isoaffine geometry prior to any introduction of an isometric (to be done in the next section).

For this purpose, let us study the lack of commutativity of the isocovariant derivatives on isoaffine spaces $\hat{M}(x,\hat{\mathfrak{R}})$ with respect to an arbitrary, not necessarily symmetric, isoconnection $\hat{\Gamma}_h{}^j{}_k$. Via a simple isotopy of the corresponding equations (see Lovelock–Rund (*loc. cit.*), pp 82–83), and by noting that

$$X^j{}_{\uparrow h \uparrow k} = \frac{\partial}{\partial x^k}(X^j{}_{\uparrow h}) + \hat{\Gamma}^{2}{}_p{}^j{}_k\, T^p{}_q\, (X^q{}_{\uparrow h}) - \hat{\Gamma}^{2}{}_h{}^p{}_k\, T_p{}^q (X^j{}_{\uparrow q}), \tag{10.37}$$

we get the expression

$$\begin{aligned} X^j{}_{\uparrow h \uparrow k} - X^j{}_{\uparrow k \uparrow h} &= \left(\frac{\partial \hat{\Gamma}^{2}{}_l{}^j{}_h}{\partial x^k} - \frac{\partial \hat{\Gamma}^{2}{}_l{}^j{}_k}{\partial x^h}\right) + \\ &+ (\hat{\Gamma}^{2}{}_m{}^l{}_k\, T^m{}_r\, \hat{\Gamma}^{2}{}_l{}^r{}_h - \Gamma^{2}{}_m{}^l{}_h\, T^m{}_r\, \hat{\Gamma}^{2}{}_l{}^r{}_k)\, T^l{}_s\, X^s - \\ &- (\Gamma^{2}{}_h{}^l{}_k - \hat{\Gamma}^{2}{}_k{}^l{}_h) T_l{}^r\, X^j{}_{\uparrow r} - \\ &- \left(\hat{\Gamma}^{2}{}_l{}^j{}_h \frac{\partial T^l{}_r}{\partial x^k} - \hat{\Gamma}^{2}{}_l{}^j{}_k \frac{\partial T^l{}_r}{\partial x^h}\right) X^r, \end{aligned} \tag{10.38}$$

DEFINITION II.10.2 (loc. cit.): The "isocurvature tensor" of a vector field X_r on an n-dimensional isoaffine space $\hat{M}(x,\hat{\mathfrak{R}})$ is given by the isotensor of rank (1.3)

$$\begin{aligned} \hat{K}_l{}^j{}_{hk} &= \frac{\partial \hat{\Gamma}_l{}^j{}_h}{\partial x^k} - \frac{\partial \hat{\Gamma}_l{}^j{}_k}{\partial x^h} + \\ &+ \hat{\Gamma}_m{}^j{}_k\, T^m{}_r\, \hat{\Gamma}^{2}{}_l{}^r{}_h - \hat{\Gamma}_m{}^j{}_h\, T^m{}_r\, \Gamma^{2}{}_l{}^r{}_k + \\ &+ \hat{\Gamma}_r{}^j{}_h \frac{\partial T^r{}_s}{\partial x^k} \hat{1}^s{}_l - \hat{\Gamma}_r{}^j{}_k \frac{\partial T^r{}_s}{\partial x^h} \hat{1}^s{}_l ; \end{aligned} \tag{10.39}$$

while the "isotorsion" is given by

$$\hat{\tau}_{h\ k}^{\ l} = \hat{\Gamma}_{h\ k}^{\ l} - \hat{\Gamma}_{k\ h}^{\ l}; \tag{10.40}$$

Expression (10.38) can then be written

$$X^{j}{}_{\restriction h \restriction k} - X^{j}{}_{\restriction k \restriction h} = \hat{K}_{l\ hk}^{\ j}\, T^{l}{}_{s}\, X^{s} - \hat{\tau}_{h\ k}^{\ l}\, T^{s}{}_{l}\, X^{j}{}_{\restriction s}. \tag{10.41}$$

Comparison with the corresponding conventional expression (Eq.s (6.9), p. 83, Lovelock and Rund (*loc. cit.*)) is instructive to understand the *modification of the curvature as well as of the torsion caused by our geometrization of interior physical media*. As we shall see, this modification is the desired feature to avoid excessive approximations, such as the admission of the perpetual motion within a physical environment, which is inherent in conventional gravitational theories.

The extension of the results to a (0.2)-rank tensor is tedious but trivial, yielding the expression

$$X^{jl}{}_{\restriction h \restriction k} - X^{jl}{}_{\restriction k \restriction h} = \hat{K}_{r\ hk}^{\ j}\, T^{r}{}_{s}\, X^{sl} + \hat{K}_{r\ hk}^{\ l}\, T^{r}{}_{s}\, X^{js} -$$

$$- \hat{\tau}_{h\ k}^{\ r}\, T_{r}{}^{s}\, X^{jl}{}_{\restriction s} \quad . \tag{10.42}$$

Similarly, for contravariant isovectors and isotensors we have

$$X_{j \restriction h \restriction k} - X_{j \restriction k \restriction h} = -\hat{K}_{j\ hk}^{\ r}\, T_{r}{}^{s}\, X_{s} - \hat{\tau}_{h\ k}^{\ r}\, T_{r}{}^{s}\, X_{j \restriction s} \tag{10.43a}$$

$$X_{jl \restriction h \restriction k} - X_{jl \restriction k \restriction h} = -\hat{K}_{j\ hk}^{\ r}\, T_{r}{}^{s} X_{sl} - \hat{K}_{l\ hk}^{\ r}\, T_{r}{}^{s}\, X_{js} -$$

$$- \hat{\tau}_{h\ k}^{\ r}\, T^{s}{}_{k}\, X_{jl \restriction s}. \tag{10.43b}$$

Relations (10.42) and (10.43) will be at times referred to as the *isoricci identities.*

We now pass to the study of the properties of the isocurvature tensor. The following first property is an easy derivation of definition (10.39).

PROPERTY 1:

$$\hat{K}_l{}^j{}_{hk} = -\hat{K}_l{}^j{}_{kh}. \tag{10.44}$$

The second property requires some algebra, which can be derived via a simple isotopy of the conventional derivation (Lovelock and Rund (*loc. cit.*) pp. 91–92).

PROPERTY 2:

$$\hat{K}_l{}^j{}_{hk} + \hat{K}_h{}^j{}_{lh} + \hat{K}_k{}^j{}_{lh} = \hat{\tau}_l{}^j{}_{h\hat{|}k} + \hat{\tau}_h{}^j{}_{kl\hat{|}l} + \hat{\tau}_k{}^j{}_{l\hat{|}h} +$$

$$+ \hat{\tau}_l{}^j{}_r \, T^r{}_s \, \hat{\tau}_h{}^s{}_k + \hat{\tau}_h{}^j{}_r \, T^r{}_s \, \hat{\tau}_k{}^s{}_l + \hat{\tau}_k{}^j{}_r \, T^r{}_s \, \hat{\tau}_l{}^s{}_h +$$

$$+ \hat{\tau}^j{}_{rh} \frac{\partial T^r{}_s}{\partial x^k} \hat{\Gamma}^s{}_l + \hat{\tau}^j{}_{rk} \frac{\partial T^r{}_s}{\partial x^l} \hat{\Gamma}^s{}_h + \hat{\tau}^j{}_{rl} \frac{\partial T^r{}_s}{\partial x^h} \hat{\Gamma}^s{}_k. \tag{10.45}$$

where, again, the reader should note the isotopies of the conventional terms, plus two new terms which are important physical applications indicated earlier in which the interior characteristic functions are averaged into constants.

Note that, for a symmetric isoconnection, the isotorsion is null and the above property reduces to the familiar form

$$\hat{K}_l{}^j{}_{hk} + \hat{K}_h{}^j{}_{kl} + K_k{}^j{}_{lh} = 0. \tag{10.46}$$

The third property also requires some tedious but simple algebra given by an isotopy of the conventional derivation (Lovelock and Rund (*loc. cit.*), pp.92–93), which results in

PROPERTY 3:

$$(\hat{K}_j{}^l{}_{h\hat{|}p} + \hat{K}_j{}^l{}_{kp\hat{|}h} + \hat{K}_j{}^l{}_{ph\hat{|}k}) \, Y_l =$$

$$= (\hat{S}_h{}^r{}_k \, T_r{}^s \, K_j{}^l{}_{sp} + \hat{S}_k{}^r{}_p \, T_r{}^s \, \hat{K}_j{}^l{}_{sh} + \hat{S}_k{}^r{}_h \, T_r{}^s \, \hat{K}_j{}^l{}_{sk}) \, Y_l +$$

$$+ (\hat{K}_j{}^r{}_{hk} \, T_r{}^l{}_{\hat{|}p} + \hat{K}_j{}^r{}_{kp} \, T_r{}^l{}_{\hat{|}h} + \hat{K}_j{}^r{}_{ph} \, T_r{}^l{}_{|k}) \, Y_l +$$

$$+ (\hat{S}_{h}{}^{r}{}_{k} T_{r}{}^{l}{}_{\lceil p} + \hat{S}_{k}{}^{r}{}_{p} T_{r}{}^{l}{}_{\lceil h} + \hat{S}_{p}{}^{r}{}_{h} T_{r}{}^{l}{}_{\lceil k}) TY_{j \lceil l}, \tag{10.47}$$

hereon called *isobianchi identity*, which can be written in a number of equivalent forms here left to the interested reader (see an alternative expression in the next section).

Again, as it was the case for property (10.45), the isobianchi identities (10.47) for the case of a symmetric isoconnection reduces to[17]

$$\hat{K}_{j}{}^{l}{}_{hk \lceil p} + \hat{K}_{j}{}^{l}{}_{kp \lceil k} + \hat{K}_{j}{}^{l}{}_{ph \lceil k} \equiv 0. \tag{10.48}$$

This completes the identification of all primary properties of an isocurvature tensor prior to the introduction of the isometric. Other properties, such as the *Freud identity* (Freud (1939), will be studied in the next section because they require the isometric for their proper definition.

II.11: ISORIEMANNIAN GEOMETRY

In this section we shall review the foundations of the *Riemannian-isotopic geometry*, or *isoriemannian geometry* for short, i.e., the most general possible, *nonlinear and nonlocal* geometry which can be constructed with a *symmetric* connection on the modular, affine-isotopic spaces $\hat{M}(x,\hat{\mathfrak{R}})$. The new geometry was introduced, apparently for the first time, in Santilli (1988d) and then developed in more details in Santilli (1991b).

As now predictable, the study should be intended as preparatory for the construction of the more general *Riemannian-admissible geometry*, or *genoriemannian geometry*[18] for short, namely, the yet more general, nonlinear and nonlocal geometry which can be constructed with a *nonsymmetric* connection on the bimodular, genoaffine spaces $^{<}M^{>}(x, {}^{<}\mathfrak{R}^{>})$ (see Sect. 10 and App. II.C).

To begin, let us perform the transition from the n-dimensional isoaffine spaces $\hat{M}(x,\hat{\mathfrak{R}})$ of the preceding section, to the corresponding isospaces $\hat{M}(x,\hat{g},\hat{\mathfrak{R}})$ equipped with a (sufficiently smooth, real valued

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[17] The generalization of the properties studied in this section to the case of genoaffine manifolds is rather intriguing, but it will not be done at this time for brevity.

[18] For the meaning of the prefix "geno" see footnote [16], p. 136.

and nowhere singular) *symmetric isotensor* $\hat{g}_{ij}$ *of rank (0.2)* on $\hat{M}(x,\hat{\mathfrak{R}})$, hereon called *isometric*, with a dependence on: an independent parameter s, the local coordinates x, their derivatives with respect to the parameter s of arbitrary order, as well as any additional quantity needed for specific physical applications, such as the density μ of the interior physical medium considered, its temperature τ, its possible index of refraction n, etc. ,

$$\hat{g}_{ij} = g_{ij}(s, x, \dot{x}, \ddot{x}, \mu, \tau, n, ...), \quad \hat{g}_{ij} = \hat{g}_{ji}, \quad \det \hat{g} \neq 0 \qquad (11.1)$$

It is easy to see that isospaces $\hat{M}(x,\hat{g},\hat{\mathfrak{R}})$ are a direct extension to an arbitrary dimension n of the *isoeuclidean spaces* $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ in three-dimensions used for the construction of the *Galilei-isotopic symmetries* $\hat{G}_{\hat{\delta}}(3.1)$ of Chapter III, as well as of the *isominkowski spaces* $\hat{M}^{III}(x,\hat{g},\hat{\mathfrak{R}})$ in (3.1)-dimension used for the construction of the *Poincaré-isotopic symmetries* $\hat{P}_{\hat{g}}(3.1)$ of Chapter IV. In this section we shall continue our study of the general n-dimensional case, by keeping in mind that, from a physical viewpoint, we are primarily interested in the isoeuclidean and isominkowski subcases.

Among the infinite class of possible isoeuclidean spaces $\hat{E}(x,\hat{\delta},\hat{\mathfrak{R}})$, we now restrict our attention to the following subclass.

DEFINITION II.11.1 (Santilli (1988d), (1991b): The "isotopic liftings" $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ of a conventional Riemannian space $R(x,g,\mathfrak{R})$ in n-dimension (see, e.g., Lovelock-Rund (1975)) called "Riemannian-isotopic spaces" or "isoriemannian spaces" for short, are the n-dimensional isoaffine spaces $\hat{M}(x,\hat{\mathfrak{R}})$ equipped with a (sufficiently smooth, nowhere singular, real valued and symmetric) isometric $\hat{g} = Tg$ characterizing, first, the isofield $\hat{\mathfrak{R}}$ via the rules

$$\hat{g} = \hat{g}(s, x, \dot{x}, ...) = T(s, x, \dot{x}, ...)\, g(x)\,, \qquad (11.2a)$$

$$\hat{g} \in \hat{R}(x,\hat{g},\hat{\mathfrak{R}}), \qquad g \in R(x,g,\mathfrak{R}), \qquad (11.2b)$$

$$\hat{\mathfrak{R}} = \mathfrak{R}\,\hat{1}, \qquad \hat{1} = T^{-1}, \qquad (11.2c)$$

and then a symmetric isoaffine connection, hereon called "isochristoffel's symbols of the first kind"

$$\hat{\Gamma}^1{}_{hlk} = \tfrac{1}{2}\left(\frac{\partial \hat{g}_{kl}}{\partial x^h} + \frac{\partial \hat{g}_{lh}}{\partial x^k} - \frac{\partial \hat{g}_{hk}}{\partial x^l}\right) = \hat{\Gamma}^1{}_{klh} \qquad (11.3)$$

as well as the "isochristoffel's symbols of the second kind"

$$\hat{\Gamma}^{2\ i}_{h\ k} = \hat{g}^{ij}\,\hat{\Gamma}^{1}_{hjk} = \hat{\Gamma}^{2\ i}_{k\ h} \qquad (11.4)$$

where the capability for an isometric of raising and lowering the indices is understood (as in any affine space), and

$$(\hat{g}^{ij}) = (|(\hat{g}_{rs})^{-1}|^{ij}) = (g^{ij}(x))\ \hat{1}(s, x, \dot{x}, \ddot{x}, \ldots). \qquad (11.5)$$

The "Riemannian-isotopic geometry", or "isoriemannian geometry" for short, is the geometry of isospaces $\hat{R}(x,\hat{g},\hat{\Re})$.

THE GEOMETRIES OF GRAVITATION

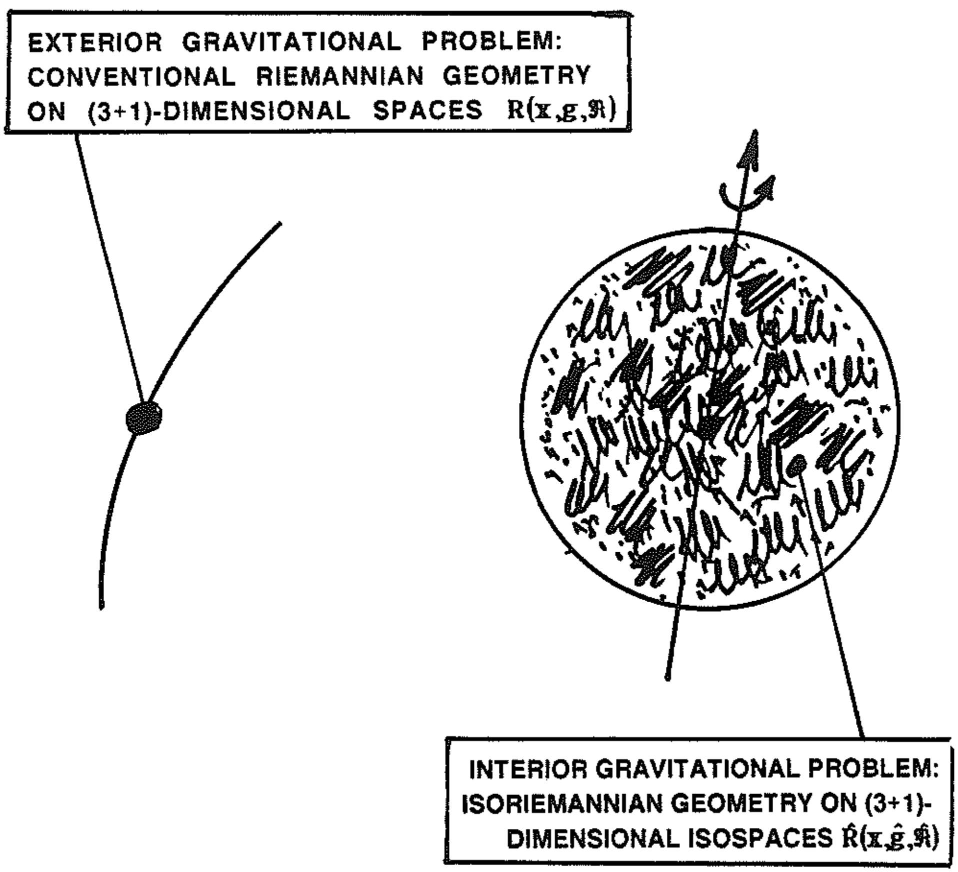

FIGURE II.11.1: A schematic view of the central objectives of the isoriemannian geometry: the geometrization of interior gravitational problems via a direct representation, not only of gravity, but also of the inhomogenuity, anisotropy and nonlocality of interior physical systems, conceived in such a way to admit the conventional exterior treatment as a particular case, as well as to coincide with the same at the abstract, realization-free level, as submitted in Santilli (1988d), (1991b). Let $R(x,g,\Re)$ be a conventional (3+1)-dimensional Riemannian space of the exterior gravitational problem in vacuum with metric $g = g(x)$. The representation of the physical media of the interior gravitation is represented via the alteration of $g(x)$, called *mutation* (Santilli (1978b)) of the type $g(x) \Rightarrow \hat{g}(s, x, \dot{x}, \ddot{x}, \mu, \tau, n, ...)$, under the conditions that the new metric $\hat{g}$ is also nowhere degenerate and symmetric. The inhomogenuity is directly represented, say, via an explicit dependence of $\hat{g}$ from the locally varying density $\mu = \mu(x)$, temperature $\tau = \tau(x)$, index of refraction $n = n(x)$, etc.; the anisotropy is directly represented, e.g., by factorizing in the Finslerian fashion a preferred direction $n{\times}x$ in the medium, such as that of its intrinsic angular momentum; and the nonlocal interactions can be directly represented via integral realizations of $\hat{g}$. The preservation of the original geometric axioms of the Riemannian geometry is permitted by

1) the decomposition of the isometric

$$\hat{g} = (\hat{g}^{ij}) = \hat{g}(s, x, \dot{x}, \ddot{x}, ...) = T(s, x, \dot{x}, ...)\, g(x), \qquad \text{(a)}$$

where $g(x)$ is the original Riemannian metric and all inhomogeneous, anisotropic and nonlocal terms are embedded in the isotopic element T;

2) the assumption of T^{-1} as the generalized unit $\hat{1}$ of the theory, with $\hat{g}^{-1} = (\hat{g}^{ij}) = (g^{ij})\,\hat{1}$, and

3) the definition of the geometry with respect to the isofield $\hat{\Re} = \Re\,\hat{1}$, resulting in the isoriemannian spaces

$$\hat{R}(x,\hat{g},\hat{\Re}), \qquad \hat{g} = Tg, \qquad \hat{\Re} = \Re\,\hat{1}, \qquad \hat{1} = T^{-1}. \qquad \text{(b)}$$

The above structure implies the important geometric property that all possible isoriemannian liftings $\hat{R}(x,\hat{g},\hat{\Re})$ of a given Riemannian space $R(x,g,\Re)$ are locally isomorphic to the latter, $\hat{R}(x,\hat{g},\hat{\Re}) \approx R(x,g,\Re)$, even though the formers are inhomogeneous, anisotropic and nonlocal-inntegral, while the latter is homogeneous, isotropic and local-differential. The preservation of the original geometric axioms can be also seen from the transformation theory of isoriemannian spaces which must be necessarily of isotopic type. The isospaces $\hat{R}(x,\hat{g},\hat{\Re})$ then remain nonlinear as the original one, but become isolocal (Sect.

III.4), by therefore avoiding the need for an integral topology, because all the nonlocal-integral terms are embedded in its isounbit $\hat{1}$. Note that *each given gravitational theory (each given Riemannian metrix g) can be subjected to an infinite number of isotopic liftings (isoriemannian images $\hat{g} = Tg$)*. This is a *necessary condition* for the new geometry because, for each given total gravitational mass m, there exist an infinite variety of astrophysical bodies with different interior characteristics all admitting the same mass m. This is the reason why we use the plural "isotopies" or "isoriemannian spaces" in these monographs. Note that the Riemannian geometry provides general laws for the gravitational metric g(x), but cannot identify its numerical value at a given point in space-time, which must be identified via experimental measures, e.g., on the gravitational mass m. Along exactly the same lines, the isoriemannian geometry provides general laws for the metric g and the isotopic element T, but their values must be identified via experiments, not only on the total mass (to identify the metric g), but also on the size, density, temperature, orientation of the intrinsic angular momentum, and other characteristics of the body (for the isotopic element T). Finally, the compatibility with the exterior problem is achieved by the conditions that all isotopic elements T (isounits $\hat{1}$) considered in these volumes recover the conventional trivial identity I everywhere outside the surface S° encompassing all matter of the uinterior problem, in which case isoriemannian spaces recover the conventional spaces identically

$$T_{\,|\,>S^\circ} \equiv I, \qquad \hat{\mathfrak{R}}(x,\hat{g},\hat{\mathfrak{R}}) \equiv R(x,g,\mathfrak{R}), \qquad \text{(c)}$$

thus establishing the covering character of the isoriemannian over the riemannian geometry. The above isoriemannian geometrization of the interior gravitational problem constitutes the ultimate geometric achievement of all the studies presented in these two volumes and related references, inasmuch it permit an axiom-preserving generalization of Einstein's gravitation for interior problems (Chapter V), which includes as a particular case on tangent planes the isotopies of the special relativity (Chapter IV) and, under group contraction, the isotopies of Galilei's relativity (Chapter III).

In order to avoid insidious topological problems, the reader should be aware that both metrics g(x) and $\hat{g}(s, x, \dot{x}, \ddot{x},..)$ are nonlinear, but the nonlocal-integral terms, jointly with all dependence other than that on x, must be all embedded in the isotopic part T of the isometric $\hat{g}$, and cannot be admitted in the original Riemannian metric g(x). This implies the embedding of all nonlocal terms in the isounit $\hat{1} = \hat{1}(s,x,\dot{x},\ddot{x},...)$, thus ensuring the topological consistency of the theory.

For simplicity, but without loss of generality, we shall consider hereon the case in which the isounit is independent from the parameter s, but depends on all other quantities identified above, $\hat{1} = \hat{1}(x, \dot{x}, \ddot{x}, \mu, \tau, n, ...)$.

The physical applications will be considered in more detail in Volume II. In this section we shall study the isoriemannian geometry *per se*, i.e., from a mathematical viewpoint, and without any boundary condition on the isotopic element of type (c) of Fig. II.11.1.

As indicated in Definition II.11.1, the introduction of a metric on an affine space implies the capability of raising and lowering the indeces. The same property evidently persists under isotopy.

Given a contravariant isovector X^i on $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$, we can define its covariant form via the familiar rule

$$X_i = \hat{g}_{ij} X^j. \tag{11.6}$$

Similar conventional rules apply for the lowering of the indeces of all other quantities.

It is easy to see that the inverse of g_{ij}, Eq.s (11.5), is a bona-fide contravariant isotensor of rank (2.0). Given a covariant isovector X_i on $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$, its contravariant form is then defined by

$$X^i = \hat{g}^{ij} X_j. \tag{11.7}$$

Rules (11.6) and (11.7) can then be used to raise or lower the indices of an arbitrary isotensor of rank (r.s).

From the definition of the isochristoffel's symbols of the first kind, Eq.s (11.3), we have

$$\frac{\partial \hat{g}_{hl}}{\partial x^k} = \hat{\Gamma}^1{}_{hlk} + \hat{\Gamma}^1{}_{lhk}, \tag{11.8}$$

and

$$\hat{g}_{hl\,\hat{|}\,k} = \frac{\partial g_{hl}}{\partial x^k} - \hat{\Gamma}^1{}_{hlk} - \hat{\Gamma}^1{}_{lhk}. \tag{11.9}$$

Thus,

$$\hat{g}_{hl\,\hat{|}\,k} \equiv 0, \qquad \hat{g}^{hl}{}_{\hat{|}\,k} \equiv 0, \tag{11.10}$$

We reach in this way the following

LEMMA II.11.1 (Isoricci Lemma; Santilli (loc. cit.)) : All (sufficiently smooth, and regular) isotopic liftings of the Riemannian geometry preserve the vanishing character of the covariant derivative of the isometrics.

In different terms, the familiar property of the Riemannian geometry

$$g_{ij|k} = 0 \tag{11.11}$$

is a true geometric axiom because it is invariant under all infinitely possible isotopies. As we shall see, this property is not shared by all quantities of current use in gravitation.

The isotransformation law of the isometric $\hat{g}$ is given by

$$\hat{g}_{ij}(x, \dot{x},..) = \frac{\partial \bar{x}^p}{\partial x^i} \bar{T}_p{}^r(\bar{x}, \dot{\bar{x}},..) \, \bar{g}_{rs}(\bar{x}, \dot{\bar{x}},..)) \, \bar{T}^s{}_q(\bar{x}, \dot{\bar{x}},...) \frac{\partial \bar{x}^q}{\partial x^j} \tag{11.12}$$

where the isotopic elements $\bar{T}^r{}_p$ in the r.h.s. are again computed in the new coordinate system as in Eq.s (10.32).

By repeating the conventional procedure (see Lovelock and Rund (*loc. cit.*)), pp. 78-70) under isotopy, we obtain the following expression for the *isochristoffel's symbol of the first kind*

$$\hat{\Gamma}^1{}_{hlk} = \tfrac{1}{2} \left(\frac{\partial \hat{g}_{kl}}{\partial x^h} + \frac{\partial \hat{g}_{lh}}{\partial x^k} - \frac{\partial \hat{g}_{hk}}{\partial x^l} \right) = \tag{11.13}$$

$$= \hat{g}_{jp} \, T^j{}_r \frac{\partial^2 x^r}{\partial x^h \partial x^k} T^p{}_s \frac{\partial x^s}{\partial x^l} + \frac{\partial \bar{g}_{jp}}{\partial x^m} T^j{}_r \, T^p{}_s \Big(\frac{\partial \bar{x}^r}{\partial x^h} \frac{\partial \bar{x}^s}{\partial x^k} \frac{\partial \bar{x}^m}{\partial x^l} +$$

$$+ \frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial \bar{x}^s}{\partial x^l} \frac{\partial \bar{x}^m}{\partial x^h} - \frac{\partial \bar{x}^r}{\partial x^l} \frac{\partial \bar{x}^s}{\partial x^h} \frac{\partial \bar{x}^m}{\partial x^k} \Big) +$$

$$+ \tfrac{1}{2}\hat{g}_{jp}\, T^p{}_s \Big[\frac{\partial T^j{}_r}{\partial x^l}\Big(\frac{\partial \bar{x}^r}{\partial x^h} \frac{\partial \bar{x}^s}{\partial x^k} + \frac{\partial \bar{x}^s}{\partial x^h} \frac{\partial \bar{x}^r}{\partial x^k}\Big) +$$

$$+ \frac{\partial T^j{}_r}{\partial x^h}\Big(\frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial \bar{x}^s}{\partial x^l} + \frac{\partial \bar{x}^s}{\partial x^k} \frac{\partial x^r}{\partial x^l}\Big) - \frac{\partial T^j{}_r}{\partial x^k}\Big(\frac{\partial \bar{x}^r}{\partial x^l} \frac{\partial \bar{x}^s}{\partial x^h} + \frac{\partial \bar{x}^s}{\partial x^l} \frac{\partial \bar{x}^r}{\partial x^h}\Big),$$

with a number of alternative formulations and simplifications, e.g., for diagonal isotopic elements T, which are left to the interested reader for brevity.

The contravariant isometric $\hat{g}^{ij}$ evidently verifies the isotransformation laws

$$\hat{g}^{ij}(x, \dot{x},..) = \frac{\partial x^i}{\partial \bar{x}^r}\, \bar{T}^r{}_p(\bar{x}, \dot{\bar{x}},..)\, \bar{g}^{pq}(\bar{x}, \dot{\bar{x}},..)\, \bar{T}^s{}_q(\bar{x}, \dot{\bar{x}},..)\, \frac{\partial x^j}{\partial \bar{x}^s}. \tag{11.14}$$

In order to proceed with our study, we need the following

DEFINITION II.11.2 (Santilli (loc. cit.)): Given an isoriemannian space $\hat{R}(x,\hat{g},\hat{\Re})$ in n dimension, the "isocurvature tensor" is given by

$$\hat{R}_l{}^j{}_{hk} = \frac{\partial \hat{\Gamma}^2{}_l{}^j{}_h}{\partial x^k} - \frac{\partial \hat{\Gamma}^2{}_l{}^j{}_k}{\partial x^h} +$$

$$+ \hat{\Gamma}^2{}_m{}^j{}_k\, T^m{}_r\, \hat{\Gamma}^2{}_l{}^r{}_h - \Gamma^2{}_m{}^j{}_h\, T^m{}_r\, \Gamma^2{}_l{}^r{}_k +$$

$$+ \hat{\Gamma}^2{}_r{}^j{}_h\, \frac{\partial T^r{}_s}{\partial x^k}\, \hat{1}^s{}_l - \hat{\Gamma}^2{}_r{}^j{}_k\, \frac{\partial T^r{}_s}{\partial x^h}\, \hat{1}^s{}_l \tag{11.15}$$

and can be rewritten

$$\hat{R}_l{}^j{}_{hk} = \tfrac{1}{2}\hat{g}^{jp}\Big(\frac{\partial \hat{g}_{ph}}{\partial x^k \partial x^l} - \frac{\partial \hat{g}_{pk}}{\partial x^h \partial x^l} + \frac{\partial \hat{g}_{lk}}{\partial x^h \partial x^j} - \frac{\partial \hat{g}_{lh}}{\partial x^k \partial x^j}\Big) +$$

$$+ \hat{g}^{jp}\big(\hat{\Gamma}^1{}_{prh}\, T^r{}_s\, \Gamma^2{}_l{}^s{}_h - \Gamma^1{}_{prk}\, T^r{}_s\, \Gamma^2{}_l{}^s{}_k \big) +$$

$$+ \hat{\Gamma}^{2\,j}_{r\,h} \frac{\partial T^r{}_s}{\partial x^k} \hat{1}^s{}_l \quad - \quad \hat{\Gamma}^{2\,j}_{r\,k} \frac{\partial T^r{}_s}{\partial x^h} \hat{1}^s{}_l, \tag{11.16}$$

the "isoricci tensor" is given by

$$\hat{R}_{lh} = \hat{R}_{l\,hj}^{\;j} = g^{ij} \hat{R}_{lihj}; \tag{11.17}$$

the "isoeinstein's tensor" is given by

$$\hat{G}_i^{\;j} = \hat{R}_i^{\;j} - \tfrac{1}{2} \delta_i^{\;j} \hat{R}; \tag{11.18}$$

and the "completed isoeinstein's tensor" is given by

$$\hat{S}_i^{\;j} = \hat{R}_i^{\;j} - \tfrac{1}{2} \delta_i^{\;j} \hat{R} - \tfrac{1}{2} \delta_i^{\;j} \hat{\theta}; \tag{11.19}$$

where $\hat{R}$ is the "isocurvature isoscalar"

$$\hat{R} = \hat{R}^i{}_i = \hat{g}^{ij} \hat{R}_{ij}, \tag{11.20}$$

and $\hat{\theta}$ is the "isotopic isoscalar"

$$\hat{\theta} = \hat{g}^{jh} \hat{g}^{lk} (\hat{\Gamma}^1{}_{rjk} T^r{}_s \hat{\Gamma}^{2\,s}_{l\,h} - \hat{\Gamma}^1{}_{rjh} T^r{}_s \hat{\Gamma}^{2\,s}_{l\,k}) =$$

$$= \hat{\Gamma}^1{}_{rjk} T^r{}_s \hat{\Gamma}^{2\,s}_{l\,h} (\hat{g}^{jh} \hat{g}^{lk} - \hat{g}^{jk} \hat{g}^{lh}). \tag{11.21}$$

We are now equipped to study the isotopies of the various properties of the Riemannian geometry as available in texbooks in gravitation. From definition (11.16) we readily obtain (Santilli (*loc. cit.*))

PROPERTY 1: Antisymmetry of the last two indices of the isocurvature tensor:

$$\hat{R}_l{}^j{}_{hk} = - \hat{R}_l{}^j{}_{kh}. \tag{11.22}$$

The specialization of properties (II.10.45) to the case at hand easily implies the following

PROPERTY 2: Vanishing of the totally antisymmetric part of the isocurvature tensor

$$\hat{R}_{l\ hk}^{\ j} + \hat{R}_{h\ kl}^{\ j} + \hat{R}_{k\ lh}^{\ j} = 0, \tag{11.23}$$

or, equivalently,

$$\hat{R}_{lmhk} + \hat{R}_{hmkl} + \hat{R}_{kmlh} = 0. \tag{11.24}$$

The use of property (11.22) and Lemma II.10.1 then yields

PROPERTY 3: Antisymmetry in the first two indices of the isocurvature tensor

$$\hat{R}_{jlhk} = \hat{R}_{ljhk}, \tag{11.25}$$

or, equivalently,

$$\hat{R}_{ljhk} = \hat{R}_{hklj}. \tag{11.26}$$

From Definition (11.15) and the use of the Isoricci Lemma, after tedious but simple calculations we have the following

PROPERTY 4: Isobianchi identity

$$\hat{R}_{l\ hk\hat{|}p}^{\ j} + \hat{R}_{l\ ph\hat{|}k}^{\ j} + \hat{R}_{l\ kp\hat{|}h}^{\ j} = \hat{S}_{l\ hkp}^{\ j}, \tag{11.27}$$

where

$$\begin{aligned}\hat{S}_{l\ hkp}^{\ j} = {} & \Gamma^{2\ j}_{\ r\ h}\left(T^{r}{}_{s|k}\,\hat{\Gamma}^{2}{}_{l}{}^{s}{}_{p} - T^{r}{}_{s\hat{|}p}\,\hat{\Gamma}^{2}{}_{l}{}^{s}{}_{k}\right) + \\ & + \hat{\Gamma}^{1\ j}_{\ r\ p}\left(T^{r}{}_{s|h}\,\hat{\Gamma}^{2}{}_{l}{}^{s}{}_{k} - T^{r}{}_{s|k}\,\hat{\Gamma}^{2}{}_{l}{}^{s}{}_{h}\right) + \\ & + \hat{\Gamma}^{1\ j}_{\ r\ k}\left(T^{r}{}_{s\hat{|}p}\,\hat{\Gamma}^{2}{}_{l}{}^{s}{}_{h} - T^{r}{}_{s|h}\,\hat{\Gamma}_{l}{}^{s}{}_{p}\right) +\end{aligned}$$

$$+ \hat{\Gamma}^{2\,j}_{r\,h}(\hat{Q}_{k}{}^{r}{}_{l\lceil p} - \hat{Q}_{p}{}^{r}{}_{l\lceil k}) + \hat{\Gamma}^{2\,j}_{r\,p}(\hat{Q}_{h}{}^{r}{}_{l\lceil k} - \hat{Q}_{k}{}^{r}{}_{l\lceil h}) +$$

$$+ \hat{\Gamma}^{2\,j}_{r\,k}(\hat{Q}_{p}{}^{r}{}_{l\lceil h} - \hat{Q}_{h}{}^{r}{}_{r\lceil p}), \tag{11.28}$$

and

$$\hat{Q}_{k}{}^{r}{}_{l\lceil p} = (\frac{\partial T^{r}{}_{s}}{\partial x^{k}}\, \hat{1}^{s}{}_{l})_{\lceil p}. \tag{11.29}$$

For isotopic liftings independent from the local coordinates (but dependent on the velocities and other variables, as it is generally the case, see Chapter V), or for the characteristic functions of the interior physical medium averaged into constants, isodifferential property (11.27) assumes the simpler form

$$\hat{R}_{l}{}^{j}{}_{hk\lceil p} + \hat{R}_{l}{}^{j}{}_{ph\lceil k} + \hat{R}_{l}{}^{j}{}_{kp\,|\,h} \equiv 0. \tag{11.30}$$

The isobianchi identity can also be equivalently written in the general case

$$\hat{R}_{ljhk\lceil p} + \hat{R}_{ljph\lceil k} + \hat{R}_{ljkp\,|\,h} = \hat{S}_{ljhkp}, \tag{11.31}$$

where the $\hat{S}$-term is that defined by Eq.s (11.28), with the reduced form for the isotopies not dependent on the local coordinates or constant

$$\hat{R}_{ljhk\lceil p} + \hat{R}_{ljph\lceil k} + \hat{R}_{ljkp\lceil h} \equiv 0. \tag{11.32}$$

At this point, the reader is suggested to verify that the above properties (in their conventional forms) are all the properties generally presented in contemporary textbooks and papers in gravitation.

The above properties, however, do not exhaust all independent properties of the Riemannian geometry owing to the existence of an additional property discovered by Freud (1939), and subsequently studied in details by Pauli (1958), but which remained thereafter ignored. This additional property, today called the *Freud identity*, was recently brought back to the attention of the physics community by Yilmaz (1990), and also treated by Carmeli *et al.* (1990). The property reached a final maturity of mathematical formulation in the recent article by Rund (1991), subsequently reviewed in Santilli (1991b). This completes all references on Freud's identity known to this author.

In this work, we shall follow Rund's paper (*loc. cit.*). It is easy to see that a step-by-step isotopy of Rund's analysis leads to the following

PROPERTY 5: Isofreud identity

$$\hat{U}'^{k}{}_{j} + \hat{G}'^{k}{}_{j} = \frac{\partial \hat{V}'^{kl}}{\partial x^{l}}, \tag{11.33}$$

where (see Rund (loc. cit.), p. 269)

$$\hat{V}'^{kl}{}_{j} = \tfrac{1}{2}\hat{g}^{\frac{1}{2}}\{\hat{g}^{rs}(\delta^{k}{}_{j}\hat{\Gamma}2_{r}{}^{l}{}_{s} - \delta^{l}{}_{j}\hat{\Gamma}2_{r}{}^{k}{}_{s}) + \\ + (\delta^{l}{}_{j}\hat{g}^{kr} - \delta^{k}{}_{j}\hat{g}^{lr})\Gamma2_{r}{}^{s}{}_{s} + \hat{g}^{lr}\hat{\Gamma}2_{j}{}^{k}{}_{r} - \hat{g}^{kr}\hat{\Gamma}2_{j}{}^{l}{}_{r}\}, \tag{11.34a}$$

$$\hat{U}'^{k}{}_{j} = \tfrac{1}{2}\hat{g}^{\frac{1}{2}}\left(\frac{\partial \hat{G}}{\partial \hat{g}^{lm}{}_{\lceil k}}\hat{g}^{lm}{}_{\lceil j} - \delta^{k}{}_{j}\hat{G}\right), \tag{11.34b}$$

$$\hat{G} = \hat{g}^{jk}(\hat{\Gamma}2_{j}{}^{p}{}_{s}\, T^{q}{}_{p}\,\Gamma2_{k}{}^{s}{}_{q} - \Gamma2_{j}{}^{p}{}_{k}\, T^{q}{}_{p}\,\Gamma2_{q}{}^{s}{}_{s}), \tag{11.34c}$$

$$\hat{G}'^{k}{}_{j} = \hat{g}^{\frac{1}{2}}\hat{G}^{k}{}_{j}. \tag{11.34d}$$

A major result of Rund's analysis is that *the conventional Freud identity holds for all symmetric and nonsingular metric on a conventional Riemannian space of arbitrary (finite) dimension and signature.* The same property evidently persists under isotopies. Thus, *Property 5 is automatically satisfied for all symmetric and nonsingular isometrics on isoriemannian spaces of arbitrary (finite) dimension and signature.*

The physical implications of the Freud identity will be considered in Chapter V of Vol. II.

We are now in a position to identify the most salient consequences of the isoriemannian geometry. First, it is an instructive exercise for the interested reader to prove the following important property

LEMMA II.11.2 (Santilli (loc. cit.)): Einstein's tensor

$$G^i_j = R^i_j - \tfrac{1}{2}\delta^i_j R \qquad (11.35)$$

does not preserve under isotopies the vanishing value of its covariant divergence (contracted Bianchi identity)

$$G^i_{j|i} = R^i_{j|i} - \tfrac{1}{2}\delta^i_j R_{|i} \equiv 0, \qquad (11.36)$$

that is, the isoeinsteinian tensor (11.18) violates property (11.36),

$$\hat{G}^i_{j\hat{|}i} = \hat{R}^i_{j\hat{|}i} - \tfrac{1}{2}\delta^i_j \hat{R}_{\hat{|}i} \neq 0, \qquad (11.37)$$

Therefore, Einstein's tensor does not possess an axiomatically complete structure, and the contracted Bianchi identity does not constitute an axiom of the Riemannian geometry.

This was at first a rather unexpected occurrence. But, under a deeper analysis, it emerged to be relevant for a rigorous understanding of the problematic aspects of Einstein's exterior gravitation, including quantization (see Chapter V).

It is interesting to note that *the Freud identity is a true geometric axiom of the Riemannian geometry in the sense that it persists under isotopies, while the contracted Bianchi identity is not, evidently because it is not preserved by isotopies.*

The following important additional property can also be proved via tedious but simple calculations from isodifferential property (11.27).

LEMMA II.11.3 (loc. cit.): The completed isoeinsteinian tensor (11.19) does possess an identically null isocovariant isodivergence, i.e.,

$$\hat{S}^i_{j\hat{|}j} = (\hat{R}^i_j - \tfrac{1}{2}\delta^i_j \hat{R} - \tfrac{1}{2}\delta^i_j \hat{\theta})_{\hat{|}i} \equiv 0. \qquad (11.38)$$

hereon referred to as the "completed and contracted isobianchi identity".

By reinspecting the above findings, we can say that Einstein's tensor G^i_j is not "axiomatically complete" because it does not characterize an axiom which is invariant under all infinitely possible isotopies. However, if Einstein's tensor is "completed" by adding a suitable tensor with null covariant divergence, then it is turned into a true axiomatic form invariant under isotopy.

Let us first identify the implications of the above findings for the *conventional* theory of gravitation, and then study their isotopies.

For this purpose, we introduce the following

DEFINITION II.11.3: The "completed Einstein's tensor" on $R(x,g,\Re)$ is given by the expression

$$S^i_j = R^i_j - \tfrac{1}{2}\delta^i_j R - \tfrac{1}{2}\delta^i_j \theta, \tag{11.39}$$

where R^i_j is the conventional Ricci tensor, R is the conventional curvature scalar and θ is given by the isotopic quantity $\hat{\theta}$, Eq. (11..21), for $T = I$, i.e.,

$$\theta = g^{jh} g^{lk} (\Gamma^1_{rjk} \Gamma^{2r}_{l\ h} - \Gamma^1_{rjh} \Gamma^{2r}_{l\ k})$$

$$= \Gamma^1_{rjk} \Gamma^{2r}_{l\ h} (g^{jh} g^{lk} - g^{jk} g^{lh}). \tag{11.40}$$

But, the covariant derivatives of the θ-quantity are identically null (from the conventional Ricci lemma). We therefore have the following

COROLLARY II.11..2.1 : Einstein's tensor can be axiomatically completed by substracting the term $\tfrac{1}{2}\delta^i_j\,\theta$ with null covariant derivatives as per Definiton II.11.3, while preserving the null value of the covariant divergence, i.e.,

$$(S^i_{j\lceil i})_{T=I} = (R^i_j - \tfrac{1}{2}\delta^i_j R - \tfrac{1}{2}\delta^i_j\theta)_{|i} \equiv (R^i_j - \tfrac{1}{2}\delta^i_j R)_{|i} \equiv 0. \tag{11.41}$$

which is called hereon the "completed and contracted Bianchi identity".

The axiomatic structure which can be subjected to a consistent lifting is therefore the generalized tensor (11.39), and not Einstein's tensor.

It should be recalled that our *"completed Einstein's tensor"* has no relationship to the *"modified Einstein's tensor"* with the cosmological constant Λ, i.e., the familiar form (see, e.g., Pauli (1958))

$$'G^i_j = R^i_j - \tfrac{1}{2}\delta^i_j R + \delta^i_j\Lambda. \tag{11.42}$$

for numerous reasons. First, Λ is a constant in quantity (11.42), while θ is a scalar function in Eq.s (11.39). Secondly, tensor (11.42) leads to a static universe, as well known, while this is not the case for our completed tensor, as we shall see. Third, the modified tensor (11.42) also does not possess sufficient generality to constitute a geometric axiom invariant under isotopies.

At this point, it is important to identify the implications for the gravitational equations prior to the addition of gravitational sources (to be done shortly in this section).

A repetition of the analysis by Lovelock and Rund ((*loc. cit.*), p. 313 and the Theorem of p. 321) for the completed Einstein's tensor leads to the following

THEOREM II.11.1: In a (conventional) four-dimensional Riemannian space $R(x,g,\Re)$ the most general possible, axiomatically complete Euler-Lagrange equations

$$\tilde{E}^{ij} = 0, \tag{11.43}$$

verifying the properties

$$\tilde{E}^{ij} = \tilde{E}^{ji}, \qquad \tilde{E}^{ij}{}_{|j} = 0, \tag{11.44a}$$

$$\tilde{E}^{ij} = \tilde{E}^{ij}(g_{ij}, g_{ij,k}, g_{ij,kl}), \; g_{ij,k} = \partial g_{ij} / \partial x^k, \text{etc.} \tag{11.44b}$$

(where the latter property also expresses the lack of source), are characterized by the variational principle

$$\delta\tilde{A} = \delta\int L(g_{ij}, g_{ij,k}, g_{ij,kl})$$
$$= \delta\int \Delta^{\frac{1}{2}}[\lambda(R + \theta) - 2\Lambda] = 0, \qquad \Delta^{\frac{1}{2}} = (\det g)^{\frac{1}{2}} \tag{11.45}$$

and read

$$\tilde{E}^{ij} = \Delta^{\frac{1}{2}}\{\lambda[R^{ij} - \tfrac{1}{2}g^{ij}(R + \theta)] + \Lambda g^{ij}\} = 0, \tag{11.46}$$

where R is the curvature scalar and θ is quantity (11.40).

The reader will recognize in the above theorem the cosmological constant Λ, as well as its differentiation from our θ-quantity. The reader will also see the difference of the gravitational equations (11.50) with the corresponding Einstenian form.

The isotopies of the above property can be readily done, via the methods illustrated earlier, thus reaching the following

THEOREM II.11..2 (loc. cit): In a four-dimensional isoriemannian space $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$, the most general possible Euler-Lagrange equations

$$\hat{E}^{ij} = 0, \tag{11.47}$$

verifying the properties

$$\hat{E}^{ij} = \hat{E}^{ji}\ , \quad \hat{E}^{ij}{}_{\upharpoonright j} = 0, \tag{11.48a}$$

$$\hat{E}^{ij} = \hat{E}^{ij}(\hat{g}_{ij}, \hat{g}_{ij\,,\,k}, \hat{g}_{ij\,kl}), \quad \hat{g}_{ij\,,\,k} = \partial \hat{g}_{ij} / \partial x^k, \text{etc.} \tag{11.48b}$$

where the latter property denotes absence of sources, are characterized by the variational principle

$$\delta \hat{A} = \delta \int \hat{L}(\hat{g}_{ij}, \hat{g}_{ij\,,\,k}, \hat{g}_{ij\,,\,kl})\, d^4x =$$

$$= \delta \int \hat{\Delta}^{\frac{1}{2}} [\, \lambda\, (\hat{R} + \hat{\theta}\,) - 2\Lambda]\, d^4x = 0, \quad \hat{\Delta}^{\frac{1}{2}} = (\hat{g})^{\frac{1}{2}} \tag{11.49}$$

and read

$$\hat{E}^{ij} = \hat{\Delta}^{\frac{1}{2}} \{\, \lambda\, [\hat{R}^{ij} - \tfrac{1}{2} \hat{g}^{ij} (\hat{R} + \hat{\theta}\,)] + \Lambda\, \hat{g}^{ij} \}, \tag{11.50}$$

where $\hat{R}$ is the isocurvature isoscalar (11..20) and $\hat{\theta}$ is the isotopic isoscalar (11..21).

This completes our analysis of the conventional and isotopic Riemannian geometry without sources.

To study the case with sources, let us first return to the conventional exterior gravitational problem on a Rimannian manifold. In this case, the conventional Einstein's equations for the exterior problem in vacuum, e.g., for an astrophysical body with null total charge and magnetic monents, assume the familiar form

$$G^{ij} = R^{ij} - \tfrac{1}{2} g^{ij} R = 0, \tag{11.51}$$

expressing the ultimate conception of Einstein's gravitation as the reduction of the exterior gravitational field to pure geometry without source.

The lack of souces in Einstein's equations (11.51) is the reason of considerable problematic aspects which do not appear to have been resolved at this time. These issues will be considered in Chapter V of Volume II. It is nevertheless important to recall the following problematic aspects which have a direct connection with the mathematical study of this section.

As shown in details by Santilli (1974), even when the total charge and magnetic moments are null, any gravitational mass possesses in its exterior problem in vacuum a nonnull source of electromagnetic nature $T^{ij}{}_{elm}$ which is so large, to be able to account in first approximation for the entire gravitational mass,

$$m_{grav.} \approx \int dv \, T^{\circ\circ}{}_{elm} , \qquad (11.52)$$

This occurrence implies the following modification of Eq.s (11.51) in vacuum

$$G^{ij} = R^{ij} - \tfrac{1}{2} g^{ij} R = 8\pi \, T^{ij}{}_{elm} , \qquad (11.53)$$

which is necessary from the well known electromagnetic origin of the masses of all elementary particles, as proved in Santilli (1974) for the case of the π° particle (which has precisely a null total electromagnetic phenomenology).

In a series of articles initiated in 1958 (see the bibliography), Yilmaz has advocated a different generalization of Eq.s (11.51) into the following form inclusive of the stress-energy tensor

$$G^{ij} = R^{ij} - \tfrac{1}{2} g^{ij} R = 8\pi \, t^{ij}{}_{stress} \qquad (11.54)$$

Moreover, in a recent article, Yilmaz (1990a) suggests the possibility that a modified form of the Freud identity (defined for quantities without the metric as a factor) implies a modification of Einstein's equations (11.51) of the type

$$G^{ij} = R^{ij} - \tfrac{1}{2} g^{ij} R = 8\pi \, (T^{ij} + t^{ij}{}_{stress}). \qquad (11.55)$$

Finally, we recently suggested (Santilli (1991b)) that the equations for the completed Einstein's tensor with the most general possible sources

$$S^{ij} = R^{ij} - \tfrac{1}{2} g^{ij} R - \tfrac{1}{2} g^{ij} \theta = (8\pi (T^{ij} + t^{ij}_{stress}) \qquad (11.56)$$

can be recast in the form

$$G^{ij} = R^{ij} - \tfrac{1}{2} g^{ij} R = 8\pi (T^{ij}_{elm} + t^{ij}_{stress}). \qquad (11.57)$$

where

$$T^{ij}_{elm} = T^{ij} + g^{ij} \theta / 16\pi \qquad (11.58)$$

is a traceless tensor, to qualify for an electromagnetic source.

We therefore submitted the hypothesis (Santilli (1974) and (1991b)) on the origin of the gravitational field as being entirely generated by the electromagnetic field of the individual charged constituents of a massive body, plus a comparatively smaller contribution from the short-range, weak and strong interactions in the interior of hadrons , i.e.,

$$M^{ij}_{matter} = T^{ij}_{elm} + t^{ij}_{s.r.\ int.} \qquad (11.59a)$$

$$m_{grav} = \int dv\, T^{\circ\circ}_{elm} + \int dv\, t^{\circ\circ}_{s.r.int}. \qquad (11.59b)$$

This lead in a natural way to the hypothesis that the physical origin of *Yilmaz's stress-energy tensor in vacuum is due precisely to the short range, weak and strong interactions in the structure of matter*

$$t^{ij}_{\mathcal{S}.r.int.} \equiv t^{ij}_{stress} \qquad (11.60)$$

The assumption of Eq.s (11.57) as exterior gravitational equations in vacuum apparently resolves at least some of the problematic aspects of Einstein's equations (11.51). In fact, Eq.s (11.57):

a) avoid the incompatibility of Eq.s (11.51) with Maxwell's electrodynamics, with specific reference to the electromagnetic origin of matter (Santilli (1974));

b) permit a theory on the origin of the gravitational field in which the now vexing problem of the "unification" of the gravitational and electromagnetic field is replaced with their "identification" in the sense of Eq.s (11.59) (Santilli (loc. cit.)); and

c) admitting rather intriguing features, such as a reduction to easily solvable field equations, an apparently unambiguous quantization, and others (Yilmaz (1979), (1980), (1982), (1984), (1990)).

Following the recent appearance of Rund's paper (1991), the role of the Freud identity in the characterization of possible sources in vacuum is under re-examination (Yilmaz, private communication) and no final conclusion can be drawn at this writing.

We are now equipped to study the most general possible interior gravitation on an isoriemannian manifold with sources, which can be expressed via the following

THEOREM II.11..5 (GENERAL THEOREM FOR ISOINTERIOR GRAVITATION, Santilli (1991b)): In a four-dimensional isoriemannian space $\hat{R}(x,\hat{g},\hat{\Re})$, the most general possible Euler-Lagrange equations

$$\hat{E}^{ij} = 0, \tag{11.61}$$

verifying the properties: 1) symmetric condition on the Euler-Lagrange tensor

$$\hat{E}^{ij} = \hat{E}^{ji}, \tag{11.62}$$

2) contracted isobianchi identity

$$\hat{E}^{il}{}_{\hat{|}j} = 0, \tag{11.63}$$

and 3) isofreud identity

$$\partial\, (\, \hat{g}^{\frac{1}{2}}\, \hat{U}^{ij}\,)\, /\, \partial x^{j} = 0, \tag{11.64}$$

are characterized by the variational principle

$$\begin{aligned} \delta\hat{A} &= \delta\int \hat{L}(\hat{g}_{ij}, \hat{g}_{ij,k}, \hat{g}_{ij,kl}, \tau_{ij}, t_{ij})\, d^4x \\ &= \delta\int \hat{g}^{\frac{1}{2}}\, [\, \lambda\, (\, \hat{R} + \hat{\theta}\,) + 2\Lambda + \rho(\hat{\tau} + \hat{t})\,]\, d^4x \\ &= \delta\int \hat{g}^{\frac{1}{2}}\, [\, \lambda\hat{R} + 2\Lambda + \rho(\hat{\Upsilon} + \hat{t})\,]\, d^4x = 0\, , \end{aligned} \tag{11.65}$$

where λ, Λ, and ρ are constants, and

$$\hat{\Upsilon} = \hat{\tau} + \lambda\hat{\theta}/\rho, \tag{11.66}$$

is the source tensor. For the case $\lambda = \rho = 1$ and $\Lambda = 0$, the Euler-Lagrange equations are given by

$$\hat{E}^{ij} = \hat{R}^{ij} - \tfrac{1}{2}\hat{g}^{ij}\hat{R} - \tfrac{1}{2}\hat{g}^{ij}\hat{\theta} - \hat{T}^{ij} - \hat{t}^{ij} = 0, \qquad (11.67)$$

or, equivalently,

$$\hat{G}^{ij} = \hat{R}^{ij} - \tfrac{1}{2}\hat{g}^{ij}R = \hat{T}^{ij} + \hat{t}^{ij}, \qquad (11.68)$$

where $\hat{T}^{ij}$ can be traceless and, thus, can represent the electromagnetic field originating from each charged constitent of matter, and $\hat{t}^{ij}$ is the stress-energy tensor.

Throughout the analysis of these sections we have often considered interior trajectories of "nonlagrangian" type. It is important to understand that this term is referred to the lack of analytic representations in terms of a *first-order Lagrangian*, i.e., a Lagrangian L depending at most on the first order derivatives of the variables, $L = L(s, x, \dot{x})$[19]. In this case the Euler-Lagrange equations are of second-order.

The theory of Lagrangians of order higher than the first (with Euler-Lagrange equations of order higher than the second), even though quite intriguing, implies a rather deep revision of the analytic mechanics, e.g., for the construction of the corresponding "Hamiltonian" formulation.

A first way to understand the nonlagrangian character of the geometry considered in the sense indicated above, is by recalling that the "Lagrangian" equivalent of the Birkhoffian mechanics is preciselty of the second order (Santilli (1982a)).

The generally nonlagrangian character of the geometry under consideration is then made clear by the following

COROLLARY II.11.5.1: The Lagrangians of Theorem II.11..5 are first-order in the metric tensor, $L = L(g_{ij}, \hat{g}_{ij,k}, \hat{g}_{ij,kl})$, but generally of the second- or higher-order in the coordinate derivatives, $L = L(s, \dot{x}, \ddot{x}, \ldots)$.

Euler-Lagrange equations of order higher than the second are avoided in the isoriemannian geometry because all derivative terms are embedded in the isometric, while Euler-Lagrange equations (11.67) are computed precisely with respect to such isometric, and not with

[19] See footnote[1], p. 5.

respect to the local variables and their derivatives, as in the conventional case.

The analysis of this section is complemented with that of the next section on the notions of isoparrallel transport and isogeodesics.

We close this section with a few complementary aspects. As well known, a most important system of local coordinates is that introduced by Riemann (1868) with the name of *normal coordinates*, say,

$$x^i \Rightarrow y^{\circ i}(x), \tag{11.69}$$

under which the (3+1)-dimensional Riemannian space $R(x,g,\Re)$ can be considered locally flat, i.e., in the neighborhood of the point $P^\circ = (y^\circ)$ all coefficients of the connection $\Gamma^2{}_\rho{}^\mu{}_\sigma$ are identically null[20],

$$\Gamma^2{}_\rho{}^\mu{}_\sigma(y^\circ) \equiv 0. \tag{11.70}$$

Moreover, it has been proved in the literature that a system of normal coordinates always exists for all affine spaces with a *symmetric* connection. We can therefore introduce the following

DEFINITION II.11.4 (Santilli (1988d), (1991b): The "isonormal coordinates" of an isoriemannian (3+1)-dimensional space $\hat{R}(x,\hat{g},\hat{\Re})$ are the coordinates $y^{\circ\, i}(x)$ such that, in the neighborhood of a point $y^{\circ\, i}$ all isoconnections coefficients are identically null

$$\hat{\Gamma}^2{}_\rho{}^\mu{}_\sigma y^\circ) \equiv 0, \qquad \rho, \mu, \sigma = 1, 2, 3, 4. \tag{11.71}$$

Normal coordinates have a fundamental physical meaning in conventional gravitational theories, because they allow the identification of the *local Euclidean or Lorentz frames.*

In the transition to an isotopic formulation, we encounter another difference with the conventional setting with fundamental physical implications investigated in Chapter V of Volume II.

LEMMA II.11..6 (loc. sit.): The metric holding in the neighborhood of a point of the isonormal coordinates of an isoriemannian space is isoeuclidean or isominkowskian.

PROOF: Suppose that the transformations $x \Rightarrow y^\circ(x)$ are such to

20 According to our notation, we now return to the use of Greek indeces because we are dealing, specifically, with the physical space-time.

eliminate the space-dependence of the transformed isoconnection coefficients. Then, Eq.s (11.71) hold, but the local metric remains generally dependent on the derivatives $\dot{y}$, $\ddot{y}$, and other quantities, thus being of isoeuclidean or isominkowskian type. Q.E.D.

Stated differently, in the conventional case, the connection coefficients can only depend on the local coordinates, $\Gamma^{2\ i}_{r\ s} = \Gamma^{2\ i}_{r\ s}(x)$. The recovering of the Euclidean metric δ or of the Minkowskian metric η under local coordinates then follows.

In the isotopic case, the isoconnection coefficients depend on the local coordinates x as well as all possible (or otherwise needed) derivatives and other quantities, $\Gamma^{2\ i}_{r\ s} = \Gamma^{2\ i}_{r\ s}(x, \dot{x}, \ddot{x}, \mu, \tau, n, ..)$. Their transformation under isonormal coordinates then eliminates the coordinate dependence of the metric, but generally leaves the dependence on the remaining variables, and we shall write

$$\bar{g}^{\mu\nu} = \frac{\partial y^{i\mu}}{\partial x^{\alpha}} T^{\alpha}_{\ \rho}(x,\dot{x},\ddot{x},..)\, g^{\rho\sigma}(x,\dot{x},\ddot{x},..)\, T^{\beta}_{\ \sigma}(x\dot{x}\ddot{x},..) \frac{\partial y^{\nu}}{\partial x^{\beta}} = \\ = \bar{g}^{\mu\nu}(\dot{y}, \ddot{y}, ..). \tag{11.72}$$

Needless to say, coordinate transformations of an isoriemannian manifold

$$x^{\mu} \Rightarrow w^{\mu}(x, \dot{x}, \ddot{x}, ..), \tag{11.73}$$

admitting the Euclidean or Minkowskian metric may indeed exist, but they are generally nonlinear and nonlocal. In fact, for the case in which the 3-dimensional Riemannian geometry generalizes the Euclidean setting with metric δ = diag. (1, 1, 1), transformation (11.73) via rule (11.72) would imply

$$x^{i}\, \hat{g}_{ij}(x, \dot{x}, \ddot{x}, ..)\, x^{j} = w^{m}\, \delta_{mn}\, w^{n}, \qquad i, j, m, n = 1, 2, 3 \tag{11.74}$$

with similar results for the case of the Minkowski metric (see Chapter V). Needless to say, the latter coordinates are considerably more difficult to identify than the isonormal coordinates, although their existence is not excluded here.

The central point remains that reduction (11.74) *is not* necessarily implied by the isotopic conditions (11.72). The local isotopic metric (11.72) then persists as the geometrically natural case.

As now familiar, we have initially considered a conventional gravitational theory on and arbitrary (finite-dimensional) space R(x,g,$\mathfrak{R}$) which, as well known, has *null torsion*, and have reached an infinite family of isotopies all of which also have a *null isotorsion* on $\hat{R}(x,\hat{g},\hat{\Re})$. In fact, the original symmetric connection $\Gamma^2{}_h{}^s{}_k$ has been lifted into an infinite family of isoconnections which are also symmetric

$$\tau_h{}^s{}_k = \Gamma^2{}_h{}^s{}_k - \Gamma^2{}_k{}^s{}_h = 0 \quad \Rightarrow \quad \hat{\tau}_h{}^s{}_k = \hat{\Gamma}^2{}_h{}^s{}_k - \hat{\Gamma}^2{}_k{}^s{}_h \equiv 0. \qquad (11.75)$$

However, the null value of torsion occurs at the level of our *geometrical isospaces* $\hat{R}(x,\hat{g},\hat{\Re})$ which are not the *physical space* of the experimenter, the latter remaining the conventional space-time (see Chapter IV for details).

The physical issue whether or not the isotopies of Einstein's gravitation have a non-null torsion must therefore be inspected in the physical space and not in our geometrical isospace.

This can be easily done, e.g., by projecting the isocovariant derivative of an isovector on $\hat{R}(x,\hat{g},\hat{\Re})$ as a conventional covariant derivative in the ordinary space $R(x,g,\Re)$, i.e.,

$$X^i{}_{\hat{|}k} = \frac{\partial X^i}{\partial x^k} + \hat{\Gamma}^2{}_h{}^i{}_k \, T^h{}_r \, X^r$$

$$\equiv X^i{}_{|k} = \frac{\partial X^i}{\partial x^k} + \hat{\Gamma}^2{}_r{}^i{}_k \, X^r, \qquad (11.76a)$$

$$\hat{\Gamma}^2{}_r{}^i{}_k = \hat{\Gamma}^2{}_h{}^i{}_k \, T^h{}_r. \qquad (11.76b)$$

It is then evident that, starting with a symmetric isoconnection $\hat{\Gamma}_h{}^i{}_k$ on $\hat{R}(x,\hat{g},\hat{\Re})$, the corresponding connection $\hat{\Gamma}_r{}^i{}_k$ on $R(x,g,\Re)$ is no longer necessarily symmetric, and we have proved the following

LEMMA II.11.5 (Santilli (loc. cit.): The isotopic liftings $\Gamma^2{}_h{}^i{}_k \Rightarrow \hat{\Gamma}^2{}_h{}^i{}_k$ of a symmetric connection $\Gamma^2{}_h{}^i{}_k$ on a Riemannian space $R(x,g,\Re)$ into an infinite familiy of isotopic connections $\hat{\Gamma}^2{}_h{}^i{}_k$ on isoriemannian spaces $\hat{R}(x,\hat{g},\hat{\Re})$ of the same dimension, imply that the isospace always possesses a null isotorsion, but, when the isotopies are projected into the original space, a non-null torsion generally occurs.

At this point the advances in torsion made by Gssperini (1984a, b, c), Rapoport-Campodonico ((1991) and others become applicable to our *interior* gravitation. We regret the inability to review these studies and reformulate them in terms of our null isotorsion.

Let us recall that any nonlinear and nonlocal theory can always be identically written in an isolinear and isolocal form (Sect. II.4). By reversing the proof of Lemma II.11.5, it is then easy to prove the following

COROLLARY II.11.5.1 (Santilli (loc. cit.)): Under sufficient continuity and regularity conditions, any theory on a conventional affine space $R(x,\Re)$ with non-null torsion, can always be written in an identical form on a suitable isoaffine space $\hat{R}(x,\hat{\Re})$ of the same dimension with an identically null isotorsion.

Let us recall that the reasons which render Einstein's exterior gravitation so effective for the characterization of the *stability* of the planetary orbits and other exterior features are exactly due to the null value of its torsion. The same reasons are then at the origin for the *inability* of the theory to represent the *instability* of the interior orbits.

In turn, these results necessarily lead to *the need for two different, but compatible gravitational theories: one for the exterior gravitational problem with null torsion, and one for the interior gravitational problem with null isotorsion but non-null torsion.*

II.12: ISOPARALLELISMS AND ISOGEODESICS.

In the contemporary treatment of the Riemannian geometry, the notions of *parallelism* and *geodesic* play a fundamenmtal role for the geometric characterization of the trajectories of the applicable relativity which, as well as known, are geodesic whether in a curved or a flat space.

In this monograph we have studied the isoriemannian geometry which characterizes more general notions of parallelism and geodesic, apparently introduced for the first time in Santilli (1988d) under the names of *isoparallelism* and *isogeodesic*, and then studied in more details in Santilli (1991b). Independent reviews can be found in Aringazin *et al.* (1991) and Kadeisvili (1992).

Predictably, the latter notions play a fundamental role in the isotopies of conventional relativities studied in the next volume, because they show that the admitted trajectories in the interior dynamical problem are still geodesics, although referred to a more general space, whether flat or curved.

In fact, the generalized notions were originally derived precisely via axiom-preserving isotopies of the conventional notions. In particular, one should expect that our generalized notions coincide with the conventional notions at the abstract, realization-free level, as it has been the case for *all* our isotopies.

On physical grounds, the *deviations* of the generalized from the conventional notions are expected to represent the transition from motion in vacuum (exterior problem), to motion within physical media (interior problem).

Let $R(x,g,\Re)$ be a conventional n-dimensional Riemannian space. Under sufficient smoothness and regularity conditions hereon assumed, a vector field X^i on $R(x,g,\Re)$ is said to be parallel along a curve C if it satisfies the differential equation along C (see Lovelock and Rund (1975))

$$DX^i = X^i{}_{|s}\, dx^s = \left(\frac{\partial X^i}{\partial x^s} + \Gamma 2^{\,i}_{r\,s} X^r\right) dx^s = 0, \qquad (12.1)$$

where $\Gamma 2^i_{rs}$ is a symmetric connection. Then, by recalling the notions of isodifferentials of Sect. II.11, we have the following

DEFINITION II.12.1 Santilli (1988d), (1991b): An isovector field X^i on an n-dimensional isoriemannian space $\hat{R}(x,\hat{g},\hat{\Re})$ is said to be "isoparallel" along a curve C on $\hat{R}(x,\hat{g},\hat{\Re})$, iff it verifies the isotopic equations along C

$$\hat{D}X^i = X^i{}_{\hat{|}r}\, T^r{}_s(x,\dot{x},..)\, \hat{\partial}x^s =$$

$$= \left[\frac{\partial X^i}{\partial x^s} + \hat{\Gamma} 2^{\,i}_{r\,s}\, T^r{}_t(x,\dot{x},..)\, X^t\right] T^s{}_p(x,\dot{x},..)\,]\, \hat{\partial}x^p = 0, \qquad (12.2)$$

where $\hat{\Gamma} 2^{\,i}_{r\,s}$ is the symmetric isoconnection (II.11.4) and the T's are the isotopic elements.

The identity of axioms (12.1) and (12.2] at the abstract level is evident, again, because of the loss of all distinction between the right, modular, associative product, say Xx, and its isotopic generalization $X*x = XTx$.

To understand the physical differences between the above two definitions, let us introduce an independent (invariant) parameter s, such that the isovector field $\dot{x} = \hat{\partial}x/\hat{\partial}s$ is tangent to C, and let $X^i = X^i(s)$. Consider the curve C at a point P(1) for $s = s_1$ and let $X^i(1)$ be the corresponding value of the isovector field X^i at P(1).

Consider now the transition from P(1) to P(2), i.e., from s_1 to $s_1 + \hat{\partial}s$. The corresponding transported value of the isovector field $X^i(2) = X^i(1) + \hat{\partial}X^\mu$ is said to occur under an *isoparallel displacement* on $\hat{\mathfrak{R}}(x,\hat{g},\hat{\mathfrak{R}})$ in accordance with Definition II.12.1, iff

$$\hat{\partial}X^i = \frac{\partial X^i}{\partial x^r} T^r{}_s \hat{\partial}x^s = -\hat{\Gamma}^2{}_r{}^i{}_s T^r{}_p X^p T^s{}_q \hat{\partial}x^q. \tag{12.3}$$

The iteration of the process up to a finite displacement is equivalent to the solution of the differential equation

$$\frac{\hat{\partial}X^i}{\hat{\partial}s} = \frac{\partial X^i}{\partial x^r} T^r{}_s \frac{\hat{\partial}x^s}{\hat{\partial}s} = -\Gamma^2{}_r{}^i{}_s T^r{}_p X^p T^s{}_q \frac{\hat{\partial}x^q}{\hat{\partial}s}, \tag{12.4}$$

By integrating the above expression in the finite interval (s_1, s_2), we reach the following

LEMMA II.12.1 (Santilli (loc. cit.)): The isoparallel transport of an isovector field $X^i(s)$ on an n-dimensional isoriemannian manifold $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ from the point s_1 to a point s_2 on a curve C verifies the isotopic laws

$$\hat{X}^i(2) = \hat{X}^i(1) - \int_1^2 \hat{\Gamma}^2{}_r{}^i{}_s(x, \dot{x}, ..) T^r{}_p(x, \dot{x}, ..) X^p(x) T^s{}_q(x, \dot{x}, ..) \dot{x}^q \hat{\partial}s \tag{12.5}$$

where

$$\hat{X}^i(2) - \hat{X}^i(1) = \int_1^2 \hat{\partial}X^i = \int_1^2 \frac{\partial X^i}{\partial x^p} T^p{}_q \frac{\hat{\partial}x^q}{\hat{\partial}s} \hat{\partial}s. \tag{12.6}$$

The physical implications are pointed out by the fact that the isotransported isovector does not start at the value $X^i(1)$, but at the *modified value* $\hat{X}^i(1)$ characterized by Eq.s (12.6). Additional evident modifications are characterized by the isotopic connection $\hat{\Gamma}^2{}_r{}^i{}_s$ and the two isotopic elements T of the r.h.s. of Eq.s (12.5).

These departures from the conventional definition can be better understdood in a *flat isospace*, via the following evident

COROLLARY II.12.1.1 (loc. cit.): In a flat isospace, such as the isominkowski space $\hat{M}(x,\hat{\eta},\hat{\Re})$ in (3.1)-space-time dimensions, or the isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{\Re})$ in 3-dimensional, the conventional notion of parallelism no longer holds, in favor of the following flat isoparallelism

$$\hat{\Gamma}^{2\ i}_{\ r\ s} \equiv 0, \tag{12.7a}$$

$$\hat{x}^i(2) - \hat{x}^i(1) = \int_1^2 \hat{\partial} x^i = \int_1 \frac{2\partial x^i}{\partial x^p} T^p{}_q \frac{\hat{\partial} x^q}{\hat{\partial} s} \hat{\partial} s. \tag{12.7b}$$

Consider, as an illustration, a straight line C in conventional Euclidean space $\Re_t \times E(r,\delta,\Re)$, with only two space-components. Then a vector $\vec{R}(1)$ at $s = t_1$ is transported in a parallel way to $\vec{R}(2)$ at $s = t_2$ by keeping unchanged the characteristic angles with the reference axis, i.e.,

$$R^k(2) - R^k(1) = \int_1^2 \Big(\frac{\partial R^k(r)}{\partial x^1} dx^1 + \frac{\partial R^k(r)}{\partial x^2} dx^2\Big). \tag{12.8}$$

Under isotopy, the situation is no longer that trivial. In fact, assume the simple diagonal isotopy (Chapters III and IV)

$$T = \text{diag.}\,(b_1^2(\dot{r}), b_2^2(\dot{r})) > 0. \tag{12.9}$$

Then Eq.s (B.8) are lifted into the form

$$\hat{R}^k(2) - \hat{R}^k(1) = \int_1^2 \Big(\frac{\partial R^k(r)}{\partial r^1} b_1^2(\dot{r})\, \hat{\partial} r^1 + \frac{\partial R_k(r)}{\partial r^2} b_2^2(\dot{r})\, \hat{\partial} r^2 \tag{12.10}$$

In figurative terms, a given straight and rigid arrow in 3-space is, first, twisted under isotopy, and then transported in an isoparallel way, that is, in such a way that the isotopic (rather than the conventional) characteristic angles with the reference axis are preserved (see also the example of isorotation of Chapter III). For additional comments, see later on in thhis section Fig. II.12.1.

The *irreducibility* of the notion of isoparallel transport to the conventional notion can be illustrated even in the case of null curvature. In fact, consider for simplicity the isominkowski-space $M(x,\hat{\eta},\hat{\Re})$ with local coordinates $x = (x^\mu)$, $\mu = 1, 2, 3, 4$, and constant diagonal isotopy $\hat{\eta} = T\eta$, $T = \text{diag.}\,(b_1^2, b_2^2, b_3^2, b_4^2) > 0$, and introduce the redefinitions

$$\hat{x}^{\mu} = b_{\mu}{}^2 x^{\mu} \quad \text{(no sum)}, \quad X^{\mu}(x(\hat{x})) = \hat{X}^{\mu}(\hat{x}). \tag{12.11}$$

Then Eq.s (12.7b) become

$$\int_1 [\frac{2\partial X^{\mu}(x)}{\partial x^{\alpha}} b_{\alpha}{}^2 \partial x^{\alpha} \equiv \int_1 \frac{2\partial \hat{X}^{\mu}(\hat{x})}{\partial \hat{x}^{\alpha}} b_{\alpha}{}^2 \, \partial \hat{x}^{\alpha}, \tag{12.12}$$

namely, the isotopy persists even under the simplest possible constant isotopy (12.11), thus confirming the achievement of a novel geometrical notion.

By submitting the conventional treatment (Sect. 3.7 of Lovelock and Rund (*loc. cit.*)) to isotopies, we now identify the *integrability conditions for the existence of isoparallelism.* By performing partial derivatives of Eq.s (12.7) with respect to x^t and then interchanging symbols, we obtain

$$\frac{\partial X^{i}}{\partial x^s \, \partial x^t} = - \frac{\partial \hat{\Gamma}^{2}{}_{r\,s}{}^{i}}{\partial x^t} T^r{}_p X^p + \Gamma_{r\,s}{}^{i} T^r{}_q \Gamma^2{}_m{}^q{}_t T^m{}_n X^n +$$

$$+ \hat{\Gamma}^{2\,i}_{r\,s} \frac{\partial T^r{}_p}{\partial x^t} X^p = - \frac{\partial X^i}{\partial x^t \, \partial x^s} = \frac{\partial \hat{\Gamma}^{2}{}_{r\,t}{}^{i}}{\partial x^s} T^r{}_p X^p +$$

$$+ \hat{\Gamma}^{2}{}_{r\,t}{}^{i} T^r{}_p \Gamma^2{}_m{}^p{}_s T^m{}_n X^n + \hat{\Gamma}^{2}{}_{r\,t}{}^{i} \frac{\partial T^r{}_p}{\partial x^s} X^p \tag{12.13}$$

from which we obtain the following

LEMMA II.12.2 (loc. cit.): Necessary and sufficient conditions for the existence of an isoparallel transport of an isovector X^i on an n-dimensional isoriemannian space $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ are that all the following equations are identically verified

$$\hat{R}_{l}{}^{i}{}_{hk} \, T^l{}_s \, X^s = 0, \tag{12.14}$$

where $\hat{R}_{r\,pq}^{\;i}$ is the isocurvature tensor (Sect. II.11), i.e.,

$$\hat{R}_{l\,hk}^{\;i} = \frac{\partial\hat{\Gamma}2^{i}_{l\,h}}{\partial x^{k}} - \frac{\partial\hat{\Gamma}2^{i}_{l\,k}}{\partial x^{h}} +$$

$$+ \hat{\Gamma}2^{\;i}_{m\,k}\, T^{m}_{\;r}\, \hat{\Gamma}2^{r}_{l\,h} - \Gamma 2^{\;i}_{m\,h}\, T^{m}_{\;r}\, \Gamma 2^{r}_{l\,k} +$$

$$+ \hat{\Gamma}2^{\;i}_{r\,h} \frac{\partial T^{r}_{\;s}}{\partial x^{k}} \hat{1}^{s}_{\;l} - \hat{\Gamma}2^{\;i}_{r\,k} \frac{\partial T^{r}_{\;s}}{\partial x^{h}} \hat{1}^{s}_{\;l}. \tag{12.15}$$

The re-emergence of the isocurvature tensor as part of the integrability conditions of isoparallel transport, can then be considered as a confirmation of the achievement and consistency of isoparallelism as a novel geometrical notion.

We now pass to the study of the *isogeodesics*. Let s be an invariant parameter and consider the tangent $\dot{x}^{i} = \hat{\partial}x^{i}/\hat{\partial}s$ of the curve C on an n-dimensional isoriemannian space $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$. Its absolute isodifferential is given by

$$\hat{D}\dot{x}^{i} = \hat{\partial}\dot{x}^{i} + \hat{\Gamma}2^{\;i}_{r\,s}\, T^{r}_{\;p}\, \dot{x}^{p}\, T^{s}_{\;q} \hat{\partial}x^{q}. \tag{12.16}$$

In accordance with Definition II.12.1, $\hat{D}X^{i}$ remains isoparallel along C iff

$$\hat{D}\dot{x}^{i} = 0. \tag{12.17}$$

We can therefore introduce the following

DEFINITION II.12..2 (Santilli (loc. cit.)): The "isogeodesics" of an n-dimensional isoriemannian manifold $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ are the solutions of the differential equations

$$\frac{\hat{\partial}^{2}x^{i}}{\hat{\partial}s^{2}} + \hat{\Gamma}2^{\;i}_{r\,s}(x,\dot{x},\ddot{x}..)\, T^{r}_{\;p}(x,\dot{x},\ddot{x}..) \frac{\hat{\partial}x^{p}}{ds}\, T^{s}_{\;q}(x,\dot{x},\ddot{x}.) \frac{\hat{\partial}x^{q}}{ds} = 0 \tag{12.18}$$

By recalling that $\hat{\partial}s = ds$, it is easy to see that *the isogeodesics of flat isospaces remain the straight line (i.e., linear functions of s), while* *those of curved isospaces remain curves.*

It is a simple but instructive exercise to prove the following

LEMMA II.12.3 (loc. cit.): The isogeodesics of an n-dimensional isoriemannian manifold $\hat{R}(x,\hat{g},\hat{\Re})$ are the curves verifying the variational principle . . . .

$$\delta\int \hat{d}s \ = \ \delta\int (\hat{g}_{ij}(x, \dot{x}, \ddot{x}..)\, \hat{d}x^i\, \hat{d}x^j)^{\frac{1}{2}} \ = \ 0. \tag{12.19}$$

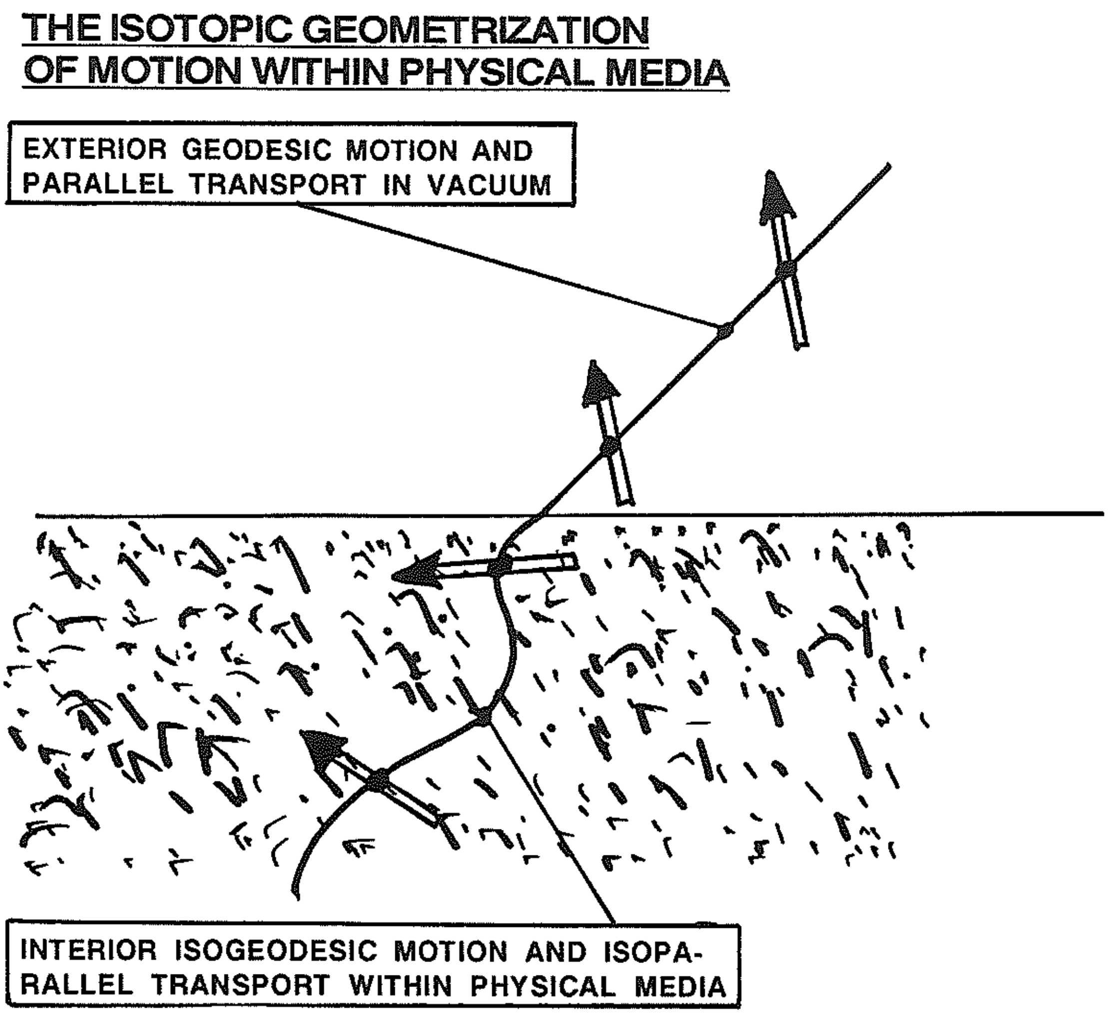

FIGURE II.12.1: A schematic view of the preservation of parallel transport and geodesic motion of the trajectories in the transition from the exterior to the interior problem submitted in Santilli (1988d) and (1991b). By assuming both problems to be flat and without potential forces for simplicity, in the upper portion of the figure we depict a rocket whose exterior geodesic in vacuum is a straight line, and whose parallel transport is such to preserve the angle with the direction of motion, as well known. In the lower portion of the figure, we see the same rocket, but now moving within a generally inhomogeneous and anisotropic, physical medium. The first difference with the exterior case is that, in general, the trajectory of the center of mass is no longer a straight line even in the absence of curvature, as shown by clear physical evidence (say, a rocket felling in our atmosphere "sideway"). The second difference is that the angle between the orientation of the rocket is not preserved, but varies locally, depending on the physical conditions at hand (shape of rocket, its density, the density of the atmosphere, etc.). Despite these differences, the motion of the rocket keeps verifying the axioms of geodesic motion and parallel transport, although in a more general realization called *isogeodesic* (Def. II.12.2) and *isoparellel* (Def. II.12.1). In different terms, the isotopic theories allow to prove that the exterior and interior motions of this figure are geometrically equivalent under the sole condition that the action-at-a-distance, potential forces are the same for both cases. In fact, the generally curved character of the geodesic is clearly shown by the solution of the variational principle (12.19), while the lack of preservation of the angle with the local direction of motion is clearly shown by the solutions of Eq.s (12.10). The equivalence follows from the fact that, owing to their isotopic character, the structures in isospace coincide with the conventional, corresponding structuresin empty space at the abstract, realization-free level. The notions outlined in this figure are fundamental for the understanding of the isotopic relativities per se, as presented in Vol. II, as well as of their geometric equivalence with the conventional relativities. In fact, the upper portion of the figure represents a system characterized by the Galilei or Lorentz boosts, while, the lower portion represents motion characterized by the isogalilean or isolorentz boosts (see Chap.s III and IV, respectively). The important geometric result that permits these advances is that no alteration of structural axioms has occurred in the transition from the exterior to the interior relativity, thus permitting an ultimate geometric unity between conventional and isotopic relativities.

By recally that the isominkowski spaces are locally isomorphic to the conventional ones, $\hat{R}(x,\hat{g},\hat{\mathfrak{R}}) \approx R(x,g,\mathfrak{R})$, the abstract identity of the above isotopic variational principle with the conventional one confirms the achievement and consistency of isogeodesic as a novel geometrical notion.

The notions of isoparallel transport and isogeodesic have a fundamental role in our geometrization of physical media. In fact they are the geometrical counterpart of our preservation of the exact Galilei and Poincaré symmetries under Lie-isotopies.

An important application of the isogeodesics and isoparallelism can be found in the isosymmetries. Consider the rotational symmetry O(3), that is, the symmetry of an ideal (rigid) sphere represented by the trivial metric δ = diag. (1, 1, 1).. It is well known that the trajectories under the modular action of O(3) on the sphere, the circles, are geodesic.

Consider now the isorotational symmetry $\hat{O}(3)$ characterized by the isometric $\delta = T\hat{\delta}$ = diag. (b_1^2, b_2^2, b_3^2), which leaves invariant all possible ellispoidical deformations of the sphere (See Chap. III for details). But $T > 0$. It is then possible to prove that all possible isosymmetries $\hat{O}(3)$ of the class considered are locally isomorphic to the conventional symmetry O(3).

The understanding of the theories presented in these volumes requires the understanding that the preservation of the rotational symmetry for the ellipsoidical deformations of the sphere is geometrically made possible by the preservation of the geodesic character. In fact, the interested reader can readily see that the trajectories under the modular-isotopic action of $\hat{O}(3)$ on the ellipsoids, the ellipses, are isogeodesic.

A similar situation occurs for the full Galilei's and Poincaré's symmetries. In fact, equations of motion which appear to violate these symmetries because of the presence of contact interactions, can be proved instead to verify them exactly at the higher isotopic level.

The restoration of the exact space-time symmetries at the isotopic level when believed to be broken at the conventional level is geometrically permitted precisely by the results of this section, the preservation of the geodesic character of the group action under isotopy.

We hope the above comments are sufficient to illustrate the importance of the central line of these volumes, the isotopic geometrization of interior physical media.

This completes the presentation of the essential methodological aspects of the Lie-isotopic formulations which are needed to study the isotopies of Galilei's and Einstein's special, and Einstein's general relativities for the closed-isolated treatment of interior dynamical problems.. A few lines for the more general Lie-admissible formulations for the open-nonconservative treatment of interior systems are given in the appendices.

APPENDICES

APPENDIX II.A: LIE-ADMISSIBLE STRUCTURE OF HAMILTON'S EQUATIONS WITH EXTERNAL TERMS

No in depth knowledge of the topic of these monographs can be achieved without a study of the analytic, algebraic and geometrical structures underlying the equations originally conceived by Hamilton (1834) for interior dynamical systems, those with external terms

$$\begin{cases} \dot{r}_{ka} = \dfrac{\partial H t, r, p)}{\partial p_{ka}} = p_{ka}/m_a, & \text{(A.1a)} \\ \dot{p}_{ka} = -\dfrac{\partial H(t, r, p)}{\partial r_{ka}} + F_{ka}, & \text{(A.1b)} \end{cases}$$

$$H = p_{ka}\, p_{ka} / 2m_a + V(r), \qquad \text{(A.1c)}$$

$$F_{ka} = F^{NSA}{}_{ka}(t, r, p, \dot{p},..) + \int_\sigma d\sigma \mathcal{F}^{NSA}{}_{ka} t, r, p, \dot{p},..), \qquad \text{(A.1d)}$$

$$k = 1, 2, 3\ (= x, y, z), \quad a = 1, 2, \ldots, N.$$

As one can see, the "direct universality" of the equations for the representation of all possible systems (II.1.1) in the coordinates of the experimenter is direct and immediate, because the Hamiltonian H represents all local and potential forces, while the external terms F_{ka} represents all remaining nonlinear, nonlocal and nonhamiltonian forces.

However, in so doing, the Hamiltonian H is necessarily nonconserved (Sect. I.1.4) and, for this reason, the equations characterize open nonconservative systems.

As we shall see momentarily, an algebraically similar situation occurs for the most general possible nonautonomous Birkhoff's equation (II.7.11) in $T^*E(r,\delta,\Re)$ with local coordinates $a = (a^\mu) = (r, p) = (r_k, p_k)$,

$$\dot{a}^\mu = \Omega^{\mu\nu}(t,a) \left[\frac{\partial B(t, a)}{\partial a^\nu} + \frac{\partial R_\nu(t, a)}{\partial t} \right], \quad \mu, \nu = 1, 2, ..., N, \tag{A.2a}$$

$$\Omega^{\mu\nu} = (| \Omega_{\alpha\beta}|^{-1})^{\mu\nu}, \tag{A.2b}$$

$$\Omega_{\mu\nu} = \frac{\partial R_\nu(t, a)}{\partial a^\mu} - \frac{\partial R_\mu(t,a)}{\partial a^\nu}, \tag{A.2c}$$

The algebraic structure of Eq.s (A.1) was identified, apparently for the first time in Santilli ((1967), (1968), (1969)). The studies were then continued in Santilli (1978a). A comprehensive presentation can be found in Santilli (1981a), including the identification of an underlying geometric structure and the extension of the results to Eq.s (A.2).

In this appendix we shall outline the algebraic properties of Eq.s (A.1) and (A.2). We shall also point out in more details the reasons why the restriction of the studies of interior trajectories solely to Lie-isotopic treatements is insufficient, and the need for the complementary Lie-admissible formulation is necessary. Additional properties will be outlined in the subsequent appendices.

To begin, the conventional Poisson brackets [A,B] of Hamilton's equations without external terms are generalized for Eq.s (A.1) in a form, say A×B, explicitly given by

$$AxB = [A,B] + \frac{\partial A}{\partial p_{pa}} F_{ka}. \tag{A.3}$$

PROPOSITION A.1. : Brackets (A.3) of Hamilton's equations with external terms violate the conditions to characterize any algebra.

PROOF. Brackets (A.3) violate the right scalar and right distributive laws (II.5.1), i.e.,

$$\alpha x(BxC) = Ax(\alpha xB) = (\alpha xA)xB, \qquad (A.4a)$$

$$(AxB)x\alpha \neq Ax(Bx\alpha) \neq (Ax\alpha)xB, \qquad (A.4b)$$

and

$$(A + B)xC = AxC + BxC, \qquad (A.5a)$$

$$Ax(B + C) \neq AxB + AxC, \qquad (A.5b)$$

As a result, brackets (A.3) do not characterize an algebra as commonly intended in contemporary mathematics (Jacobson (1962). QED

In different terms, *in the transition from the contemporary Hamilton's equations to their original form with external terms, we have the loss, not only of the Lie algebras, but more precisely of all algebras.*

Exactly the same situation occurs for the quantum mechanical treatment of nonconservative forces via nonhermitean Hamiltonians $H_d \neq H_d^\dagger$ (Santilli (1978b)). In fact, under these conditions, the conventional Heisenberg's brackets among operators A, B, ... on a Hilbert space $\mathcal{H}$, $[A,B] = AB - BA$, over a complex field C are generalized into a form, say $A\tilde{\times}B$, which is evidently defined by the new equations of motion

$$i\,\dot{A} = A\tilde{\times}H_d = AH_d^\dagger - H_dA\,, \qquad \hbar = 1, \qquad (A.6)$$

Again, *the nonconservative Heisenberg's brackets $A\tilde{\times}H$, not only lose the Lie algebra character of conventional quantum mechanics, but do not characterize any consistent algebra, because they violate the right scalar and right distributive laws,* as the reader can verify.

This is not a mere mathematical occurrence, because it carries rather deep physical implications. For instance, the notion of *angular momentum* can be consistently defined in conventional (classical and quantum) Hamiltonian mechanics, and treated via its underlying Lie symmetry O(3).

In the transition to Hamilton's equations with external terms (A.1) and their operator counterpart (A.6), we have lost *all* Lie algebras, let alone that of the rotational symmetry. This has the direct consequence that, even though the use of angular momenta is often kept for Eq.s (A.6) according to a rather widespread use in papers and books (particularly those in nuclear physics), the reality is that the notion has lost all necessary background for its definition, let alone its quantitative treatment.

In fact, it would be fundamentally inconsistent to use one product $A\tilde{\times}H$ for the time evolution, and a *different product*, say, [A,H] for the characterization of physical quantities such as the angular momentum.

This is due to the well known, ancient rule of dynamics whereby *the product of the algebra characterizing a given theory, whether classically or operationally, must coincide with that of the time evolution law.*

To put it explicitly, a statement to the effect that, say, a particle described by Eq.s (6) has spin one, is mathematically inconsistent, because of the loss of any algebra, and physically undefined, because the spin of particles in open nonconservative conditions is ultimately unknown to this wrtiting.

Numerous other inconsistencies of Eq.s (A.6) will be pointed out when studying, specifically, the operator formulation of the theory.

Exactly the same situation occurs for the nonautonomous Birkhoff's equations (A.2). In fact, Birkhoff's brackets $[A\,\hat{,}\,B]$ for the autonomous case (Sect. II.8),

$$[A\,\hat{,}\,B] = \frac{\partial A}{\partial a^{\mu}}\ \Omega^{\mu\nu}(a)\ \frac{\partial B}{\partial a^{\nu}}, \tag{A.7}$$

have to be generalized for Eq.s (A.2) in the form

$$A \odot B = [A\,\hat{,}\,B] + \frac{\partial A}{\partial a^{\mu}}\ \Omega^{\mu\nu}\ \frac{\partial R_{\nu}}{\partial t} \tag{A.8}$$

which again violate the right scalar and distributive laws.

Equivalently, one can say that for, the case of time-dependent R-functions, Birkhoff's equations can be expressed with the $(2N+1)\times(2N+1)$ contact tensor (Sect. II.9) which, being odd-dimensional, do not admit a consistent contravariant (Lie) counterpart.

The reader should therefore be aware that *the isotopies of conventional relativities to be studied in the next chapters are inapplicable to the nonautonomous Birkhoff's equations, because of the loss of a consistent algebraic structure, let alone the loss of their Lie-isotopic character.*

The above occurrences evidently creates the problem of identifying the relativities which are directly applicable to open, nonconservative, nonautonomous, interior systems, such as oscillator with a time-dependent applied force, etc.

In turn, the above relativities cannot be identified without first reformulating Eq.s (A.1) and (A.2) in an analytically *identical* way (to avoid the alteration of the equations of motion) which is however admitting of a consistent algebraic structure.

This problem signals the birth of the Lie-admissible algebras in physics. In fact, on one side, the consistent brackets for Eq.s (A.1), say, (A,B), cannot be antisymmetric, to permit the representation of the time-rate-of-variation of the energy

$$\dot{H} = (H,H) = \frac{\partial H}{\partial p_{ka}} F_{ka} = v_{ka} F_{ka} \neq 0, \qquad (A.9)$$

while, on the other side, Lie algebras cannot be evidently abandoned, because they must be admitted as a particular case for null nonselfadjoint forces, i.e.

$$(A,B)_{|F_{ka}=0} \equiv [A,B]. \qquad (A.10)$$

This problem was originally studied in Santilli ((1967),(1968), (1969)) and then reinspected in Santilli (1978a), where it was pointed out that conditions (A.9) and (A.10) identify the so-called *general Lie-admissible algebras*.

Recall from Sect. II.5, an algebra $\mathbf{U}$ with (abstract) elements a,b,c,..and (generally nonassociative, abstract) product ab over a field F is called a *Lie-admissible algebra* (Albert (1948)), when the attached algebra $\mathbf{U}^-$, which is the same vector space as $\mathbf{U}$, but equipped with the product

$$\mathbf{U}^- : \ [a,b]_{\mathbf{U}} = ab - ba, \qquad (A.11)$$

is Lie.

The most general possible algebras of the type considered are called *general Lie-admissible algebras U* (Santilli (1978a)) when they verify no condition other than the Lie-admissibility law (A.12)

$$(a, b, c) + (b, c, a) + (c, a, b) = (c, b, a) + (b, a, c) + (a, c, b). \qquad (A.12)$$

The first classical realization of the Lie-admissible algebras in physics was introduced in Santilli (1978a, c) and then worked out in more details in Santilli (1981a) Let A, B. ...be (nonsingular, sufficiently smooth) functions in $\Re_t \times T^*E(r,\delta,\Re)$. Then the brackets

$$(A,B) = \frac{\partial A}{\partial a^\mu} S^{\mu\nu}(t,a) \frac{\partial B}{\partial a^\nu}, \qquad (A.13)$$

over the reals $\Re$ characterize a Lie-admissible algebra **U** when the attached antisymmetric brackets

$$\mathbf{U}^- : \quad [A,B]_{\mathbf{U}} = (A,B) - (B,A) \qquad (A.14)$$

are Lie, or, equivalently, when the attached antisymmetric tensor

$$S^{\mu\nu} - S^{\nu\mu} = \Omega^{\mu\nu} \qquad (A.15)$$

is Birkhoffian.

Now, brackets (A.3) can be written in an algebraically consistent way by introducing the tensor in $\Re_t \times T^*E(r\delta,\Re)$

$$S^{\mu\nu}(t,a) = \omega^{\mu\nu} + s^{\mu\nu}(t,a), \qquad (A.16)$$

where $\omega^{\mu\nu}$ is the (totally antisymmetric) canonical Lie tensor (II.7.16), and $s^{\mu\nu}$ is the totally symmetric tensor

$$s = (s^{\mu\nu}) = \text{diag.}\,(0, s), \quad s = F/(\partial H/\partial p) \qquad (A.17)$$

The brackets (A,B), when written in form (A.13) with the S-tensor given by *symmetric* form (A.16), first of all, verify both right and left scalar and distributive laws, and, secondly, they characterize a Lie-admissible algebra because the attached brackets are Lie

$$(A,B) - (B,A) = 2\,[A,B], \quad S^{\mu\nu} - S^{\nu\mu} = 2\omega^{\mu\nu}. \qquad (A.18)$$

The historical "true" equations by Hamilton, when rewritten in term of tensor (A.16)

$$\dot a^\mu = S^{\mu\nu} \frac{\partial H(t,a)}{\partial a^\nu} = (a^\mu, H), \qquad (A.19)$$

were called *Hamilton-admissible equations* (Santilli (1978a))., and are more expliocitly given by

$$\dot{r}_{ka} = \partial H/\partial p_{ka}, \tag{A.20a}$$

$$\dot{p}_{ka} = -\partial H/\partial r_{ka} + s_{kajb}\,\partial H/\partial p_{jb} = -\,\partial H/\partial R_{ka} + F_{ka}. \tag{A.20b}$$

In particular, the brackets (A,B) preserve the correct time-rate-of-variation of the Hamiltonian

$$\dot{H} = (H,H) = v_{ka}F_{ka}, \tag{A.21}$$

by construction.

The regaining of a consistent mathematical structure carries rather intriguing mathematical and physical implications.

As an example, *Eq.s (A.1) do not admit a consistent exponentiation into a finite group* .On the contrary, when written in their equivalent Lie-admissible form (A.19), they can be easily exponentiated into the form

$$a' = \{ e|_A^{\,t^o S^{\mu\nu}\partial_\nu H)\,(\partial_\mu)} \}\, a, \tag{A.22}$$

In particular, *the above structure leaves invariant the equations of motion.* In fact, from a general property of vector-fields on manifolds, we have

$$\Gamma'(t,a') =$$

$$\{ e|_A^{\,t^o S^{\mu\nu}(\partial_\nu H)\,(\partial_\mu)} \}\, \Gamma(t,a) = \Gamma(t,a'), \tag{A.23}$$

For this reason, structures of type (A.22) constitute an intriguing generalization of the notion of Lie-isotopic symmetry (Sect. II.9) known as a *Lie-admissible symmetry* (Santilli (*loc. cit.*)).

The physical differences with the conventional approach are, however, rather deep. In fact, *the conventional Lie and Lie-isotopic symmetries represent the conservation of the energy and other quantities.* In the more general case under consideration here, we can say that *the broader Lie-admissible symmetry characterized by* *the Hamiltonian as generator represent the time-rate-of-variation of the energy*

$$\dot{H} = H(t,a) - \{ e|_A^{t^o S^{\mu\nu} (\partial_\nu H) (\partial_\mu)} \} H(t,a) = v_{ka} F_{ka}. \qquad \text{(A.24)}$$

Moreov

er, exponentiation (A.19) admits the following explicit form

$$\{ e|_A^{t^o S^{\mu\nu} (\partial_\nu H) (\partial_\mu)} \} A =$$

$$A + t^o(A,H)/1! + t^{o2}((A,H),H)/2! + \ldots.. \qquad \text{(A.25)}$$

namely, *symmetries (A.19) admit non-Lie, Lie-admissible algebras in the neighborhood of the identity.* This signals the possibility of generalizing the entire Lie's and Lie-isotopic theories in a yet more general Lie-admissible theory (Sanbtilli (1978a), (1981a)).

The covering character of the Lie-admissible formulations over the Lie-isotopic and Lie formulations is then evident.

By recalling that the symmetry characterized by the Hamiltonian as generator is the time component of the Galilei and of the Galilei-isotopic relativities, symmetry (A.23) can then be considered as the time component of conceivable, still more general relativities, tentatively called *Lie-admissible relativities* (*loc. cit.*) for open nonconservative systems, in which the form-invariance characterizes, this time, the time-rate-of-variation of the Galilean quantities. The understanding is that the studies on Lie-admissibility are considerably less advanced than the corresponding Lie-isotopic theories, and so much remains to be done.

The identification of the algebraic structure of the nonautonomous Birkhoff's equations (A.2) is now easy (*loc. cit.*). Introduce the generalized tensor

$$\hat{S}^{\mu\nu}(t,a) = \Omega^{\mu\nu}(a) + T^{\mu\nu}(t,a), \qquad \text{(A.26)}$$

where $\Omega^{\mu\nu}$ is the (totally antisymmetric) Birkhoff's tensor (A.2b), and $T^{\mu\nu}$ is given by the totally symmetric form

$$\tau = (\tau^{\mu\nu}) = \text{diag.}(0,\sigma), \quad \sigma = (\partial_t R) / (\partial_\mu B). \tag{A.27}$$

Then, the generalized brackets

$$(A \hat{,} B) = \frac{\partial A}{\partial a^\mu} \hat{S}^{\mu\nu}(t,a) \frac{\partial B}{\partial a^\nu}, \tag{A.28}$$

are algebraically consistent and Lie-admissible, as one can see. This results in the generalized equations

$$\dot{a}^\mu = \hat{S}(t, a) \frac{\partial B(t, a)}{\partial a^\nu}, = (\Omega^{\mu\nu}(a) + \tau^{\mu\nu}(t, a)] \frac{\partial B(t, a)}{\partial a^\nu}, \tag{A.29}$$

called *Birkhoff-admissible equations* (Santilli (1978a), (1982a)) and which eviodently constitute a covering of both Birkhoff's and Birkhoff-isotopic equations.

In particular, the transition from brackets (A.13) to (A.28) (A,B) $\Rightarrow$ (A$\hat{,}$ B), is an example of *Lie-admissible isotopies* (Sect. II.5).

For further studies, we refer the interested reader to Santilli (1981a), where one can see the elements for a further generalization of Birkhoffian mechanics into a covering discipline, tentatively called *Birkhoffian-admissible mechanics.*

The operator counterpart of Hamilton-admissible equations (A.16) was achieved in Santilli (1978b). Here we shall briefly outline it, because the operator Lie-admissible equations possesses considerable guidance value in the study of the broader Lie-admissible formulations.

The most salient physical difference in the transition from closed-isolated-stable systems to open-nonconservative-unstable systems is the appearance of *irreversibility*, i.e., the lack of invariance of physical processes under time reversal. As an example, the trajectory of Jupiter in the Solar system is manifestly reversible, while the trajectory of a satellite penetrating Jupiter's atmosphere is manifestly irreversible. Corresponding similar situations occur at the particle level.

Consider then the *forward direction in time*, and denote it with the symbol " > ". Let ξ be the conventional enveloping operator algebra of quantum mechanics with operators A, B, ... and trivial associative product AB on a Hilbert space $\mathcal{H}$ over the field of complex numbers C.

Introduce the isotope $\xi^>$ of ξ (Sect. II.5) describing the motion forward in time

$$\xi^{>}: \quad A>B \stackrel{\text{def}}{=} AT^{>}B, \qquad \text{(A.30)}$$

which is characterized by a nowhere null and sufficiently smooth, but *nonhermitean* operator $T^{>}$, with *isounit for motion forward in time*

$$I^{>} = (T^{>})^{-1}, \qquad \text{(A.31a)}$$

$$I^{>}>A = A>I^{>} = A, \qquad \forall A \in \xi^{>}, \qquad \text{(A.31b)}$$

Introduce now the isotope $^{<}\xi$ for motion backward in time, denoted with the symbol " < ",

$$^{<}\xi: \quad A<B \stackrel{\text{def}}{=} A^{<}TB, \qquad \text{(A.32)}$$

characterized by a different isotopic element $^{<}T \neq T^{>}$, with *isounit for motion backward in time*

$$^{<}I = (^{<}T)^{-1}, \qquad \text{(A.33a)}$$

$$^{<}I<A = A<^{<}T, \qquad \text{(A.33b)}$$

Finally, assume that the forward description via envelope $\xi^{>}$ is the time reversal of the backward one $^{<}\xi$, i.e.,

$$I^{>} = (^{<}I)^{\dagger}. \qquad \text{(A.34)}$$

LEMMA II.A.1: The axiomatic structure of irreversibility from the algebraic viewpont can be expressed via isoassociative algebras with two different isounits $I^{>} = (^{<}I)^{\dagger} \neq {}^{<}I$, and related isofields, one for the motion forward in time $I^{>}$ and the other for the motion backward in time $^{<}I$.

It is an instructive exercise for the reader interested in learning the techniques of these volumes to prove that structures (A.30)-(A.34) are indeed invariant under isotopy and, thus possess an axiomatic character.

Lemma II.A.1 is of particular guidance value in studying abstract problems, i.e., the identification of the generalization of the Riemannian geometry needed for the Lie-admissible formyulations (Appendix II.C).

Under envelopes $\xi^>$ and $^<\xi$, the time evolution is given in infinitesimal form by

$$i\dot{A} = (A \hat{,} B) = A{<}H - H{>}A = A\,{}^<TH - HT^>A, \qquad \hbar = 1, \quad \text{(A.35)}$$

with finite version

$$A(t) = {}^<I\,\{e|_\xi{}^{-it<H}\} < A(0) > \{e|_\xi{}^{iH>t}\}\,I^>, \qquad \text{(A.36)}$$

which were proposed, apparently for the first time, in Santilli (1978b), p. 746.

It is easy to see that Eq.s (A.35) are Lie-admissible. In fact, their attached antisymmetric brackets are precisely the brackets of the Lie-isotopic time evolution in operator form (Sect. II.6)

$$i\dot{A} = [A \hat{,} B] = ATB - BTA, \qquad \text{(A.37a)}$$

$$T = {}^<T + T^>. \qquad \text{(A.37b)}$$

This shows again, this time at the operator level, the complementarity of the Lie-isotopic and Lie-admissible formulations.

In particular, structure (A.36) is an operator realization of the Lie-admissible groups (A.22).

It should be stressed that, *by no means Eq.s (A.35) alter the physical content of conventional nonconservative systems (A.6). In fact, Eq.s (A.35) merely provide the identical reformulation of the systems but, this time, in an algebraically consistent form.*

In fact, the nonhermitean Hamiltonians H_d of current use in physics are generally the sum of a Hermitean term H and a dissipative nonhermitean term

$$H_d = H + H_{diss.} \qquad \text{(A.38)}$$

The desired, algebraically consistent, but physically identical reformulation of systems (A.6) is then given by (Santilli (*loc. cit.*))

$$H_d{}^\dagger = {}^<TH, \qquad H_d = HT^>, \qquad \text{(A.39a)}$$

$$i\dot{A} = AH_d^{\dagger} - H_d A \equiv A<H - H>A. \qquad (A.39b)$$

where the *Hermitean* operator H evidently represents the *nonconserved* energy.

Eq.s (A.35) were proved to be "directly universal" for the representation of the most general known local and nonlocal, Hamiltonian and nonhamiltonian and continuous or discrete operator systems (Jannussis *et al.* (1984), (1985), (1986), (1987), Santilli (1989)).

The similarities of the above operator formulation with the corresponding, classical, Birkhoff's and Birkhoff-admissible formulations, are remarkable, thus illustrating the applicability of the complementary Lie-isotopic and Lie-admissible formulations at both the classical and operator level.

We can therefore close this appendix with the view expressed in the Preface that, by no means, the isotopic relativities prersented in the subsequent chapters can be considered as the final relativities, because physics is a discipline that will never admit final theories.

APPENDIX II.B: SYMPLECTIC-ADMISSIBLE GEOMETRY

As stressed thoughout this analysis, physical theories in general, and relativities in particular, are a simbiotic expression of analytic, algebraic and geometric formulations.

The analytic and algebraic structures of the Birkhoff-admissible equations (A. 29) have been indicated in the preceding appendix. It may therefore be of some value for the interested reader to outline their geometric structure identified, apparently for the first time, in Santilli (1978a) and then developed in Santilli (1981a) under the name of *symplectic-admissible geometry* or *genosymplectic*[21] *geometry* for short.

Recall that the direct geometric structure underlying Birkhoff's brackets (Sect. II.7) in $T^*E(r,\delta,\Re)$ with the now usual unified notation $a = (a^\mu) = (r, p), \mu = 1, 2, ..., 2n,$

$$A \hat{,} B] = \frac{\partial A}{} \; \Omega^{\mu}(a) \; \frac{\partial B}{}, \qquad (B.1)$$

[21] The meaning of the prefix "geno" has been pointed out in footnote[16], p. 136.

$$\partial a^{\mu} \qquad \partial a^{\nu}$$

is the symplectic geometry also on $T^*E(r,\delta,\mathfrak{R})$ characterized by the exact, symplectic, Birkhoffian two-forms

$$\Omega = \tfrac{1}{2} \Omega_{\mu\nu}(a)\, da^{\mu} \wedge da^{\nu}, \tag{B.2}$$

where the algebraic-contravariant and geometric-covariant tensors are interconnected by the familiar rule

$$\Omega^{\mu\nu} = (\,|\Omega_{\alpha\beta}|^{-1})^{\mu\nu}. \tag{B.3}$$

In the transition to the Birkhoff-isotopic brackets on isospaces $T^*\hat{E}_2 r,\hat{\delta},\hat{\mathfrak{R}})$ with isounit $\hat{1}_2$ (Sct. II.8),

$$[A \hat{,} B] = \frac{\partial A}{\partial a^{\mu}}\, \Omega^{\mu\alpha}(a)\, \hat{1}_{2\alpha}{}^{\nu}(a)\, \frac{\partial B}{\partial a^{\nu}}, \tag{B.4}$$

we have the transition to the symplectic-isotopic geometry (Sect. II.9) characterized by the isoexact, isosymplectic two-isoform

$$\hat{\Omega} = \tfrac{1}{2} \hat{1}_{\mu}{}^{\alpha}(a)\, \Omega_{\alpha\nu}(a)\, \hat{d}a^{\mu} \hat{\wedge} \hat{d}a^{\nu} \quad , \tag{B.5}$$

where, again, the algebraic and geometric tensors are interconnected by the rule

$$(\Omega^{\mu\alpha}\, \hat{1}_{2\alpha}{}^{\nu}) = (\hat{1}_{2\alpha}{}^{\rho}\, \Omega_{\rho\beta})^{-1}. \tag{B.6}$$

The problem of the geometry underlying the Birkhoff-adnmissible brackets (B.28), i.e.,

$$(A \hat{,} B) = \frac{\partial A}{\partial a^{\mu}}\, \hat{S}^{\mu\nu}(t,a)\, \frac{\partial B}{\partial a^{\nu}} \ , \tag{B.7a}$$

$$\hat{S}^{\mu\nu} = \Omega^{\mu\nu}\ T^{\mu\nu}, \tag{B.7b}$$

$$\Omega^{\mu\nu} = -\ \Omega^{\nu\mu}, \tag{B.7c}$$

$$T^{\mu\nu} = T^{\nu\mu}, \tag{B.7d}$$

was the central geometrical problem studied in Santilli (1981a), and resolved via the introduction of a geometry more general than the symplectic and the symplectic-isotopic ones, called *symplectic-admissible geometry*, or *genosymplectic geometry*.

We cannot possibly review these studies in detail here. nevertheless an outline of the central ideas may be of some value for the interested reader.

The first point to realize is that *the symplectic geometry and related exterior calculus, whether in their conventional or isotopic formulations, are intrinsically unable to characterize the Lie-admissible algebras.*

This is due to the fact that the calculus of exterior forms is essentially *antisymmetric* in the indices, while the Lie-admissible tensors $\hat{S}^{\mu\nu}$ are not, and the same occurs for the covariant counterpart

$$\hat{S}_{\mu\nu}(t, a) \; = \; (\| \hat{S}^{\alpha\beta}\|^{-1})_{\mu\nu} \neq \pm \hat{S}_{\nu\mu} \qquad \text{(B.8)}$$

. In fact, the construction of a conventional exterior two-form with the above tensor implies the reduction

$$\hat{S}_{\mu\nu}\, da^{\mu} \wedge da^{\nu} \; \equiv \; \tfrac{1}{2}\, \hat{\Omega}_{\mu\nu}\, da^{\mu} \wedge da^{\nu}, \qquad \text{(B.9)}$$

namely, the symplectic geometry automatically eliminates the symmetric component of the S-tensor, thus characterizing only its Lie content.

The main idea of the genosymplectic geometry is that of generalizing the conventional exterion calculus, say, of two differentials

$$da^{\mu} \wedge da^{\nu} \; = \; - \; da^{\nu} \wedge da^{\mu}, \qquad \text{(B.10)}$$

into a more general calculus of differentials da^{μ} and da^{ν}, called *exterior-admissible calculus* , or *genoexterior calculus* ,which is based on a product, say $\odot$ which is neither totally symmetric nor totally antisymmetric, but such that its antisymmetric component is the conventional exterior one,

$$da^{\mu} \odot da^{\nu} \; = \; da^{\mu} \wedge da^{\nu} \; + \; da^{\nu} \times da^{\nu}, \qquad \text{(B.11a)}$$

$$da^{\mu} \wedge da^{\nu} = - da^{\nu} \wedge da^{\mu}, \tag{B.11b}$$

$$\hat{d}a^{\mu} \times \hat{d}a^{\nu} = \hat{d}a^{\nu} \times \hat{d}a^{\mu}, \tag{B.11c}$$

This allows the introduction of the *exterior-admissible forms* , or *genoexterior forms*, via the sequence

$$\hat{S}_o = \phi(a), \tag{B.12a}$$

$$\hat{S}_1 = \hat{S}_{\mu}\, da^{\mu}, \tag{B.12b}$$

$$\hat{S}_2 = \hat{S}_{\mu\nu}\, da^{\mu} \odot da^{\nu}, \tag{B.12c}$$

....

The *exact exterior-admissible forms* , or *exact genoexterior forms* , are then given by

$$\hat{S}_1 = d\hat{S}_o = \frac{\partial \phi}{\partial a^{\mu}}\, da^{\mu}, \tag{B.13a}$$

$$\hat{S}_2 = d\hat{S}_1 = \frac{\partial A_{\nu}}{\partial a^{\mu}}\, da^{\mu} \odot da^{\nu}, \tag{B.13b}$$

....

The calculus of genoexterior forms can indeed characterize the Lie-admissible algebras in full, because they characterize, not only the antisymmetric component of the Lie-admissible algebras, but also their symmetric part, via the two-forms

$$\hat{S}_2 = \hat{S}_{\mu\nu}(t,a)\, da^{\mu} \odot da^{\nu} =$$

$$= \hat{\Omega}_{\mu\nu}(a)\, da^{\mu} \wedge da^{\nu} + \hat{\tau}_{\mu\nu}(t, a)\, da^{\mu} \times da^{\nu}, \tag{B.14}$$

Structures (B.14) were called in Santilli (*loc. cit.*) *symplectic-admissible two-forms* , or *genosymplectic teo-forms* , because their antisymmeyric component is symplectic, in a way fully parallel to the property whereby the antisymmetric part of the Lie-admissible algebras is Lie. Spaces $T^*E(r,\delta,\Re)$ when equipped with two-form (B.16) were called *symplectic-admissible manifolds* , or *genosymplectic manifolds* , and the related geometry *symplectic-admissible geometry.*

As incidental comments, note that the dependence on time appears only in the symmetric part, as needed for consistency in the symplectic component. Also, under inversion (B.8), we generally have

$$(\tilde{\Omega}_{\mu\nu}) \neq (\Omega^{\alpha\beta})^{-1}, \quad (\tilde{\tau}_{\mu\nu}) \neq (\tau^{\alpha\beta})^{-1}, \tag{B.15}$$

which is a rather intriguing feature of the generalized geometry here considered, whereby the symplectic content of a Lie-admissible tensor is more generalk then the symplecticv counterpart of the antisymmetric component of a Lie-admissible tenmsor (see Santilli (*loc. cit.*) for details).

The most salient departure of the exterior-admissible calculus from the exterior calculus in its conventional or isotopic formulation (Sect. II.9) is that the Poincare' Lemma no longer holds, i.e., for exact symplectic-admissible two-forms

$$\hat{S}_2 = d\hat{S}_1, \tag{B.16a}$$

$$\hat{d}\hat{S}_2 = d(d\hat{S}_1) \neq 0. \tag{B.16b}$$

In actuality, within the contest of the exterior-admnissible calculus, the Poincarte' Lemma is generalized into a rather intriguing geometric struture which evidently admits the conventional Lemma as a particular case when all symmetric components are null.

The geometric understanding of the Lie-isotopic algebras requires the understanding that *the validity of the Poincaré Lemma within the context of the symplectc-isotopic geometry is a necessary condition for the representation of the conservation of the total energy under nonhamiltonian internal forces,* as studied in the next chapters.

By the same token, the geometric understanding of the Lie-admissible algebras requires the understanding that *the lack of validity of the Poincaré Lemma within the context of the symplectic-admissible geometry is a necessary condition for the representation of the nonconservation of the energy of an interior dynamical system.*

APPENDIX II.C: RIEMANNIAN-ADMISSIBLE GEOMETRY

There is little doubt that future historians will consider our contemporary studies in gravitations as being in their first infancy.

Among a rather large number of problems that remain to be solved in gravitation, a further open problem is the representation of the dichotomy constituted by the clearly (time-) reversible exterior dynamics with a clearly irreversible interior behavior.

This is majestically illustrated, e.g., by Jupiter (Figure I.1) whose center-of-mass trajectory in the solar system is clearly reversible, while the interior dynamics is manifestly irreversible.

It is at this point that the dual use of the Lie-isotopic and Lie-admissible formulations becomes useful. In fact, the *Lie-isotopic formulations are ultimately reversible in their structure,* because they provide a global treatment of nonhamiltonian systems via *Hermitian* isounits. By comparison, *the Lie-admissible formulations are intrinsically irreversible.*

We are referring here to formulations that are structurally reversible or irreversible, rather than the achievement of reversibility or irreversibility via the selection of suitable Hamiltonians. In fact, Lie-isotopic formulations are irreversible irrespective of the selected Hamiltonian.

The dual representation of reversible center-of-mass-trajectories *versus* irreversible interior dynamics, is then permitted by the complementarity of the Lie-isotopic and Lie-admissible formulations via inter-relations of type (II.A.37).

Note the necessity of the Lie-isotopic formulations for this complementarity. In fact, reversible, conventionally Lie formulations for the global-exterior description are not compatible with irreversible, Lie-admissible, interior descriptions because their attached Lie algebra is of Lie-isotopic character, as clearly expressed by Eq.s (II.A.37).

A first axiomatic characterization of irreversibility was provided in Appendix II.A, via different isounits for motion forward $I^{>}$ and backward ${}^{<}I$ in time. A further axiomatic approach to irreversibvility will be provided in Appendix II.D via the notions of inequivalent right and left isorepresentations.

In this appendix we would like to merely indicate a conceivable generalization of the Riemannian geometry submitted by Santilli (1988d), (1991b) under the name of *Riemannian-admissible geometry* , or *genoriemannian geometry* , which could provide an irreversible description of interior gravitation in a way compatible with the reversible description provided by the *Riemannian-isotopic geometry* (or *isoriemannian geometry*) of the exterior problem (Sect. II.11).

In Sect. II.10 we introduced the notion of *affine-admissible manifolds* (or *isoaffine manifolds*) as the manifolds $^{<}M^{>}(x, {}^{<}\Re^{>})$ which possess the same dimension, local coordinates and continuity propertties of a conventional affine manifold $M(x,\Re)$, but are defined over an isofield $^{<}\Re^{>}$ with two different isounits $I^{>}$ and $^{<}I$ for the modular-isotopic action to the right and to the left, respectively

$$x'^{>} = A>x = AT^{>}x, \quad I^{>} = (T^{>})^{-1}, \tag{C.1a}$$

$$^{<}x' = x<A = x^{<}TA, \quad ^{<}I = (^{<}T)^{-1}, \tag{C.1b}$$

$$I^{>} = (^{<}I)^{\dagger}. \tag{C.1c}$$

DEFINITION II.C.1 (Santilli (1989), (1991g)): A "Riemannian-admissible manifold", or "genoriemannian manifold" is an isoaffine manifold (Derfinition II.10.1) equipped with inequivalent isometrics and isomodular actions to the right (forward in time) and to the left (backward in time), here donoted with $^{<}R^{>}(x, {}^{<}g^{>}, {}^{<}\Re^{>})$ *, namely, characterized by the "isometrics for motions forward and backward in time"*

$$g^{>} = T^{>}(x, \dot{x}, \ddot{x}, ..) \, g(x), \tag{C.2a}$$

$$^{<}g = {}^{<}T(x, \dot{x}, \ddot{x}, ..) \, g(x), \tag{C.2b}$$

$$T^{>} = (^{<}T)^{-1}, \tag{C.2c}$$

and equipped with two nonequivalent, nonsymmetric, isoaffine connections, one for the modular-isotopic action to the right (forward) and the other to the left (backward), called "Christoffel-admissible symbols of the first kind for motions forward and backward in time",

$$\Gamma^{>1}{}_{hlk} = \tfrac{1}{2}\left(\frac{\partial g^{>}{}_{kl}}{\partial x^h} + \frac{\partial g^{>}{}_{lh}}{\partial x^k} - \frac{\partial g^{>}{}_{hk}}{\partial x^l}\right) \neq \Gamma^{>1}{}_{klh} \qquad \text{(C.3a)}$$

$$^{<}\Gamma^{1}{}_{hlk} = \tfrac{1}{2}\left(\frac{\partial\, ^{<}g_{kl}}{\partial x^h} + \frac{\partial\, ^{<}g_{lh}}{\partial x^k} - \frac{\partial\, ^{<}g_{hk}}{\partial x^l}\right) \neq\ ^{<}\Gamma^{1}{}_{klh} \qquad \text{(C.3b)}$$

with corresponding "Christoffel-admissible symbols of the second kind"

$$\Gamma^{>2}{}_{h\ k}^{\ i} = g^{>ij}\, \Gamma^{>1}{}_{hjk} = \Gamma^{>2}{}_{k\ h}^{\ i} \qquad \text{(C.4a)}$$

$$^{<}\Gamma^{2}{}_{h\ k}^{\ i} =\ ^{<}g^{ij}\ ^{<}\Gamma^{1}{}_{hjk} =\ ^{<}\Gamma^{2}{}_{k\ h}^{\ i} \qquad \text{(C.4b)}$$

where the capability for an isometric of raising and lowering the indeces is understood (as in any affine space), and

$$g^{>ij} = |\,(g^{>}{}_{rs})^{-1}|^{ij}. \qquad \text{(C.5a)}$$

$$^{<}g^{ij} = |\,(^{<}g_{rs})^{-1}|^{ij}. \qquad \text{(C.5b)}$$

The "Riemannian-admissible geometry", or "genoriemannian geometry" for short, is the geometry of spaces $^{<}\hat{R}^{>}(x, {}^{<}g^{>}, {}^{<}\mathfrak{R}^{>})$.

The construction of the Riemannian-admissible geometry can be done via the appropriate generalization of the Riemannian-isotopic geometry presented in Sect. II.11, with particular reference to the isoconnections which, besides being different for the right and left modular-isotopic action, are now necessarily *nonsymmetric.*

Comparison of the above setting with that of Proposition II.A.1 and II.D.1 then yields the following

PROPOSITION II.C.1 (loc. cit.): An axiomatization of irreversibility in interior gravitation is provided by inequivalent modular-

isotopic actions to the right (forward in time) and to the left (backward in time) with necessarily nonsymmetric genoaffine connections.

Regrettably, we cannot study the Riemannian-admissible geometry in the necessary details to avoid a prohibitive length of this volume. It is however hoped that geometers in the field will indeed develop this new geometry for, in the final analysis, it is so general to encompass and include as particular cases all generalized geometries presented in this monograph.

The first generalization of Einstein's gravitation with a Lie-admissible structure was achieved by Gasperini (1983) in the full spirit of the formulations: to represent interior, nonconservative, irreversible trajectories, as well as a covering of his Lie-isotopic lifting of Einstein's gravitation (Gasperini (1984a,b,c)). Nevertheless, Gasperini formulated both, the Lie-isotopic and Lie-admissible studies in conventional Riemannian spaces and, as such, they need a reinspection for the proper formulation in suitable isospaces (see Chapter V).

Additional important gravitational studies of Lie-admissible type were conducted by Jannussis (1986), Gonzalez-Diaz (1986), Nishioka ((1985), (1987)), and others.

APPENDIX D: ISOREPRESENTATIONS AND GENOREPRESENTATIONS

A deep understanding of the Lie-isotopic and Lie-admissible algebras cannot be reached without an understanding of the structure of their representation theories. In turn, the latters have well known, profound implications in physics, inasmuch as they characterize the notion of particle reviewed in the next appendix.

The Lie-isotopic and Lie-admissible formulations imply the following sequence of generalizations of the representation theory:

A) REPRESENTATION THEORY OF LIE ALGEBRAS. As well known, it is characterized by one-sided, left or right, modular representations, generally called *"representations"*;

B) REPRESENTATION THEORY OF LIE-ISOTOPIC ALGEBRAS. It is characterized by one sided, left or right modular-isotopic representations, called *"isorep-resentations"* [25] apparently jintroduced for the first time in Santilli (1979); and

C) REPRESENTATION THEORY OF LIE-ADMISSIBLE ALGEBRAS. It is characterized by two-sided, left and right, modular-isotopic representations, called *two -sided isobirepresentations* , or *genore-presentations* [22] for short, also introduced introduced in Santilli (*loc. cit.*);

To outline the main ideas, consider a nonassociative algebra U over a field F. The right and left multiplications in U (Albert (1963), Schafer (1966)) are given by the following linear transformations of U onto itself as a vector space

$$R_x : a \Rightarrow ax, \quad \text{or} \quad aR_x = a_x, \tag{D.1a}$$

$$L_x : a \Rightarrow xa, \quad \text{or} \quad aL_x = xa, \tag{D.1b}$$

for all a, x e U, and verify the following general properties

$$(a\alpha)R_x = (a\alpha)x = a(\alpha x), \quad \text{or} \quad \alpha R_x = R_{\alpha x}, \tag{D.2a}$$

$$aR_{(x+y)} = a(x + y) = aR_x + aR_y = a(R_x + R_y)$$

$$\text{or } R_{(x+y)} = R_x + R_y \tag{D.2b}$$

with evident similar properties for the left multiplications L_x.

When the algebra is associative, we have the additional properties

$$a(xy) = (ax)y, \text{ or } aR_{xy} = aR_xR_y, \text{ or } R_{xy} = R_xR_y, \tag{D.3a}$$

$$(xy)a = x(ya), \text{ or } aL_{xy} = aL_xL_y, \text{ or } L_{xy} = L_xL_y. \tag{D.3b}$$

The above properties imply that the mapping $a \Rightarrow R_a$ ($a \Rightarrow L_a$) is a homomorphism (antihomomorphism) of A into the associative algebra V(A) of all linear transformations in A. Thus, they provide a *right representation* $a \Rightarrow R_a$ or a *left representation* $a \Rightarrow L_a$, respectively,

[22] By recalling the meaning of the prefixes *"iso"* and *"geno"* (footnote[16], p. 136), the terms *"isorepresentations"* and *"genorepresentations"* stand to indicate the *"preservation"* and *"alteration"*, respectively, of the axiomatic structure of Lie's representations.

of A, also called left or right $Hom^A{}_F(V_T)$, for T = R, L.

If the algebra A contains the identity I, we have a *one-to-one (or faithful) representation* because $R_a = R_b$ implies $I R_a = I R_b$ which can hold iff a = b. When the space L is the algebra A itself, we have the so-called *adjoint (or regular) representation*

In the case of nonassociative algebras, the mapping $a \Rightarrow R_a$ is no longer a homomorphism, and this illustrates the reason for the study of the representation theory of Lie algebras via that of the underlying universal enveloping associative algebra, as done in the mathematical literature (see, e.g., Jacobson (1962)), but generally not in the physical literature.

Consider now an isoassociative algebra $\hat{A}$ over an isofield $\hat{F}$ with isounit $\hat{1}$ and isoassociative product a*b. Introduce the *right and left isomultiplications*

$$\hat{R}_x: a \Rightarrow a*x, \qquad \text{or} \qquad a*\hat{R}_x = a*x, \tag{D.4a}$$

$$\hat{L}_x: a \Rightarrow x*a, \qquad \text{or} \qquad a*\hat{L}_x = x*a, \tag{D.4b}$$

for all $a \in A$. It is then easy to see that properties (D.2) are lifted into the forms

$$\alpha*\hat{R}_x = \hat{R}_{\alpha*x}, \quad \hat{R}_{(x+y)} = \hat{R}_x + \hat{R}_y, \tag{D.5a}$$

$$\hat{R}_{x*y} = \hat{R}_x*\hat{R}_y, \quad \hat{1}*\hat{R}_a = \hat{1}*\hat{R}_b \quad \Rightarrow \quad a = b, \tag{D.5b}$$

with similar properties for the left isomultiplications.

It is easy to see that the mapping $a \Rightarrow R_a$ characterizes a *right, faithful, isorepresentation* of $\hat{A}$ in the isoassociative algebra $\hat{V}(\hat{A})$ of isolinear transformations of $A\mathfrak{R}$, and denoted $Hom^{\hat{A}}{}_F(\hat{V}_{\hat{R}})$, with similar results holding for the left isorepresentations.

The nontriviality of the isotopy is made clear by the following

LEMMA II.D.1 (Santilli (1991b): Isorepresentations of isoassociative algebras $\hat{A}$ over an isofield $\hat{F}$ are isolinear and isolocal in $\hat{V}$ but generally nonlinear and nonlocal in V.

Namely, *the transition from Lie algebras to Lie-isotopic algebras generally implies the transition from linear and local to nonlinear and nonlocal representations.*

A *module* of an algebra U over a field F, also called *U-module*, (Schafer (1966)) is a linear vector space V over F together with a mapping $U \times V \Rightarrow V$ denoted with the symbol $(a,v) \Rightarrow av$ which verifies the distributive and scalar rules

$$a(v + t) = av + at, \qquad (a + b)v = av + bv, \tag{D.6a}$$

$$\alpha(a,v) = (\alpha a,v) = (a,\alpha v), \tag{D.6b}$$

as well as all the axioms of U, for all $a,b \in U$, $v,t \in V$, and $a \in F$.

The mappings $a \Rightarrow R_V = av$ and $a \Rightarrow L_V = va$ clearly show that the space V is a left and right U-module.

The above notion of module implies only one action, e.g., that to the right. In order to reach a two-sided action, consider an algebra U over a field F. Let V be a vector space over F. Introduce the direct sum $S = U \oplus V$ in such a way that S is an algebra verifying the same axioms of U while V is a two-sided ideal of S. This can be done as follows (see, e.g., Schafer (1966)):

1) retain the product of U;

2) introduce a left and a right composition av and va, for all elements $a \in U$ and $v \in V$ which verify all axioms of U (including the and right and left scalar and distributive laws); and

3) to complete the requirement that V is an ideal of S, assume $vt = tv = 0$ for all elements of V.

When all the above properties are verified, V is called a *two-sided, left and right module*, or a *bimodule* of U, and the algebra S is called a *split null extension* of U (Schafer (*loc. cit*).

Bimodules clearly provide a generalized, left and right representation theory of all algebras, whether associative or nonassociative.

It is important to understand why bimodules are *not* needed for the representation theory of Lie algebras and of Lie-isotopic algebras, but they become essential for the covering Lie-admissible algebras.

A *bimodule V of a Lie algebra* L or *Lie-bimodule* (Santilli (1979a)) is characterized by left and right compositions av and va, $a \in L$, $v \in V$, verifying the properties

$$av = -va, \tag{D.7a}$$

$$v(ab) = (va)b - (vb)a, \tag{D.7b}$$

which can be identically expressed via the left and right multiplications

$$L_a = - R_a, \tag{D.8a}$$

$$R_{ab} = R_a R_b - R_b R_a, \tag{D.8b}$$

The mappings $a \Rightarrow R_a$ and $a \Rightarrow L_a$ then provide *a left and right representation,* or a *birepresentation,* of the Lie algebra L over the bimodule V as a $Hom^L{}_F(V_R,V_L)$.

However, owing to property (D.8a), the left representation is trivially equivalenty to the right representation, $R_a = - L_a$. This is the reason why only one-sided representations of Lie algebras are significant in physics.

The notions of isomodules and isobimodules, apparently introduced for the first time in Santilli (1979a), can then be defined via the one sided and two-sided isotopic liftings, respectively.

A *Lie-isobimodule* (Santilli (*loc. cit.*))is therefore an isovector space $\hat{V}$ with left and right isocompositions $a*v$ and $v*a$ verifying the distributive and scalar laws, and the rules

$$a*v = - v*a, \tag{D.9a}$$

$$v*(a*b) = (v*a)*b - (v*b)*a, \tag{D.9b}$$

or, equivalently in terms of isomultiplications

$$\hat{R}_a = - \hat{L}_a, \tag{D.10a}$$

$$\hat{R}_{a*b} = \hat{R}_a * \hat{R}_b - \hat{R}_b * \hat{R}_a, \tag{D.10b}$$

which characterizes an *isobirepresentation of a Lie-isotopic algebra* $\hat{L}$ as $Hom^L{}_F(\hat{V}_{\hat{R}},\hat{V}_{\hat{L}})$.

However, the left and right isorepresentations are again equivalent because of the property $\hat{R}_a = - \hat{L}_a$. Thus, *only one-sided isorepresentations are needed for the physical applications of Lie-isotopic algebras.*

The notion of isobirepresentations on bimodules becomes *necessary* when passing to the study of the covering Lie-admissible algebras U (Santilli (*loc. cit.*)). In fact, in this case, the action to the right is no longer equivalent to the action to the left, thus resulting in a much richer structure. In this case a *Lie-admissible bimodule* V has the right and left isotopic compositions a>v and v<a, such that the attached composition $a \odot v = a>v - v<a$ verifies the conditions

$$a \odot v = - v \odot a, \tag{D.11a}$$

$$v \odot (a \odot b) = (v \odot a) \odot b - (v \odot b) \odot a \tag{D.11b}$$

X

which can be equivalently expressed via the right and left multiplications

$$\hat{R}_{a>b-b<a} + \hat{L}_{a>b-b<a} = [(\hat{R}_a - \hat{L}_a),(\hat{R}_b - \hat{L}_b)], \tag{D.12}$$

and they characterize a *left and right isobirepresentation (genorepresentation) of a general Lie-admissible algebra* U as $\mathrm{Hom}^{U_F}(\hat{V}_{\hat{R}}, \hat{V}_{\hat{L}})$.

Similar structures for commutative Jordan and Jordan-admissible algebras and for other algebras (see also Santilli (*loc. cit.*)), but their study is not considered here for brevity.

By recalling Propositions B.1 and C.1 the following property is evident.

PROPOSITION II.D.1 (Santilli (1991b): An axiomatization of irreversibility from the viewpoint of the representation theory is provided by generepresentations of Lie-admissible algebras, that is, by modular-isotopic representations with inequivalent actioms to the right and to the left on bimodular vector spaces.

The reader should note the rather remarkable unity of mathematical and physical thought provided by Propositions II.B.1, II.C.1 and II.D.1.

APPENDIX E: ISOPARTICLES AND GENOPARTICLES

The sequence of representations, isorepresentations and genorepresentations of the preceding appendix implies the characterization of the following sequential physical notions:

A') "PARTICLES", which are characterized by conventional representations of Lie algebras, and consist of the Galilean or Einsteinian notion of massive point moving in a stable orbit in vacuum under action-at-a-distance, local-potential interactions;

B') "ISOPARTICLES"[23], which are given by the more general notion of particle characterized by isorepresentations of Lie-isotopic algebras, and consist of extended-deformable particles in stable orbit[24] under the most general known, linear and nonlinear, local and nonlocal, potential and nonpotential interactions; and

C') "GENOPARTICLES"[23], which constitute the most general possible particles, characterized by genorepresentations of Lie-admissible algebras, and constitute extended-deformable particles under the most general dynamical conditions conceivable at this writing, that is, in open-nonconservative-unstable orbits while moving within a physical medium under linear and nonlinear, local and nonlocal, and Hamiltonian and nonhamiltonian external forces.

From the content of Appendix D, we can say that

The Galilean or Einsteinian particle is a linear, local, one-sided, conventionally modular representation of a Lie algebra.

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[23] By keeping in mind the meaning of the prefixes *"iso"* and *"geno"* (footnote[16], p. 136), the terms *"isoparticles"* and *"genoparticles"* stand to indicate the *"preservation"* and *"alteration"*, respectively, of the axiomatic structure of the Galilean or Einsteinian particles.

[24] Recall that the Lie-isotopic algebras preserve the antisymmetry of the product of Lie algebras. As such, they characterize conserved quantities which, when representing physical entities, imply stable orbits. The effective treatment of a particle in an unstable (say, decaying) orbit with all algorithms at hand representing physical quantities (e.g., the Hamiltonian H represents the energy of the particle, p represents the linear momentum, etc.), requires the use of the Lie-admissible formulations. These aspects have profound implications for the hadronic structure, which we hope to review in a possible operator sequel of these papers. In fact, they imply that *the hadronic constituents are "isoparticles" only when in stable orbits, otherwise they are "genoparticles"* (Santilli (1988) and (1989)). Needless to say, the two formulations are interchangeable, in the sense that Lie-admissible formulations can also represent stable orbits, but then the algorithms at hand must necessarily lose their physical meaning (e.g., $H = \{ \alpha \exp (\beta \dot{r}^2) \}$). This illustrate the insidious possibility of misrepresenting physical results whenever one relaxes the condition that all algorithms at hand must have a direct physical meaning.

The Lie-isotopic theory outlined in the main text implies a nontrivial generalization of the preceding notion. In fact,

The notion of isoparticle is a nonlinear, nonlocal, one-sided, isomodular representation (isorepresentation) of a Lie-isotopic algebra.

The Lie-admissible formulations outlined in these appendices imply the following further generalization

The notion of genoparticle is a nonlinear, nonlocal, two-sided, isobimodular representation (genorepresentation) of a Lie-admissible algebra.

On physical grounds, the implications are rather deep. Recall that for Einstein's special relativity a particle is a massive point which, as such, is a perennial and immutable geometric concept. Moreover, the orbits of Einstein's particles are necessarily stable, as trivially requested by the excact character of its rotational subsymmetry.

As indicated in the main text, the Lie-isotopic theory can instead represent the actual shape of the particle considered, as well as all its infinitely possible deformations. Thus, an isoparticle can have an infinite number of different intrinsic characteristics, depending on the infinite number of different interior conditions, and as permitted by the infinite number of isotopes of the Galilei or Poincaré symmetry. However, isoparticles should always be restricted to stable orbits, to avoid possible, insidious misinterpretations of the algorithms at hand.

The more general Lie-admissible theory outlined in these appendices implies further physical generalizations. In fact, besides representing the actual shape of the particle considered and all its possible deformations, genoparticle are in unstable orbits, and possess an intrinsically irreversible time evolution.

Now, the Galilean or Einsteinian notion of particle is unquestionably exact for the physical conditions of their original *exterior* conception, say, for the motion of our Earth in the solar system or of an electron in an atomic cloud. The lack of exact applicability of the same notion in *interior* conditions is evident following the mathematucal studies of this volume.

In fact, the insistence, say, for the characterization of a spaceship during re-entry in Earth's atmosphere or of a proton in the core of a star via the Galilean or Einsteinian notion of particle, would imply that the spaceship penetrates and moves inside Earth's atmosphere with a conserved angular momentum, or that a proton freely orbits inside the core of a star undergoing gravitational collapse with a conserved angular momentum.

The use instead of the covering notion of genoparticles offers clear possibilities for advances, both classically and operationally. The understanding is that we are referring to one of the most complex and, by far, unexplored notions of contemporary mathematics, as expectedly needed to represent some of the most complex physical conditions in the Universe.

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INDEX

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About the Author

Ruggero Maria Santilli obtained his PhD Degree at the Istitute of Theoretical Physics of the University of Turin, Italy, in 1966. He then moved with his family to the USA where he held several faculty positions at various universities, including: the Center for Theoretical Physics of the University of Miami, Coral Gables, Florida (1967–1968); Department of Physics of Boston University, Boston, MA (1968–1974); Center for Theoretical Physics, M.I.T., Cambridge, MA (1975-1977); Lyman Laboratory of Physics, Harvard University (1977–1978); and Department of Mathematics, Harvard University, Cambridge, MA (1978-1981). Santilli is the founder of The Institute for Basic Research, Cambridge, MA, and Palm Harbor, FL, of which he is president and full professor of theoretical physics (1981–). He also is the founder and editor in chief of three Journals, one in pure mathematics *Algebras, Groups and Geometries* (nine years of publication) and two in physics, the *Hadronic Journal* and the *Hadronic Journal Supplement* (fifteen years of publication). Santilli has organized twelve international workshops and conferences in the Lie-admissible theory, Lie-isotopic theory, and hadronic mechanics. As research associate, co-investigator or principal investigator, he has been the recipient of several reseaerch grants from U.S. Governmental Agencies, including: AFOSR, NASA, NSF, ERDA and DOE. In addition, Santilli is the author of seven research monographs in theoretical physics, and over one hundred papers published in various Journals, mostly devoted to mathematical, theoretical and experimental studies of nonlinerar, nonlocal and nonhamiltonian systems in classical and quantum mechanics. As a result of this intense academic activity, Santilli received several honors, including Gold Medals from the Cities of Orl éans, France, and Campobasso, Italy. He was recently listed by the Estonian Academy of Sciences among the most illustrious scientists of all times.

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