The Lie-Santilli admissible hyperalgebras of type $\mathbf{D_n}$

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Abstract

The largest class of hyperstructures is the one which satisfy the weak properties, the H_v-structures. They were introduced in 1990 and they proved to have a lot of applications on several applied sciences. In this paper we present a construction of the hyperstructures used in the Lie-Santilli admissible theory on square matrices of type D_n.

Key words: hyperstructures, H_v-structures, hopes, weak hopes, admissible Lie-algebras

MSC2010: 20N20, 17B67, 17B70, 17D25.

1 Introduction

The largest class of hyperstructures are called H_v-structures. They were introduced in 1990 [15], and they satisfy the weak axioms, where the non-empty intersection replaces the equality. Some basic definitions are the following:

In a set H equipped with a hyperoperation (abbreviation *hyperoperation = hope)*

$$\cdot : H \times H \to P(H) - \{\varnothing\},$$

we abbreviate by

WASS the *weak associativity*: $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ and by

COW the *weak commutativity*: $xy \cap yx \neq \varnothing, \forall x, y \in H$.

The hyperstructure $(H, \cdot)$ is called an H_v-*semigroup* if it is *WASS*, it is called H_v-*group* if it is reproductive H_v-semigroup, i.e.,

$$xH = Hx = H, \forall x \in H.$$

Motivation. In the classical theory the quotient of a group with respect to an invariant subgroup is a group. F. Marty from 1934, states that, the quotient of a group with respect to any subgroup is a hypergroup. Finally, the quotient of a group with respect to any partition (or equivalently to any equivalence relation) is an H_v-group. This is the motivation to introduce the H_v-structures [15], [16].

In an H_v-semigroup the powers of an element $h \in H$ are defined as follows: $h^1 = \{h\}, h^2 = h \cdot h, ..., h^n = h \circ h \circ ... \circ h$, where $(\circ)$ denotes the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An H_v-semigroup $(H, \cdot)$ is called *cyclic of period s*, if there exists an element h, called *generator*, and a natural number s, the minimum one, such that $H = h^1 \cup h^2 ... \cup h^s$. Analogously the cyclicity for the infinite period is defined [16]. If there is an element h and a natural number s, the minimum one, such that $H = h^s$, then $(H, \cdot)$ is called *single-power cyclic of period s.*

$(R, +, \cdot)$ is called an H_v-*ring* if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y + z) \cap (xy + xz) \neq \varnothing, \ (x + y)z \cap (xz + yz) \neq \varnothing, \ \forall x, y, z \in R.$$

Let $(R, +, \cdot)$ be an H_v-ring, $(M, +)$ be a COW H_v-group and there exists an external hope

$$\cdot : R \times M \to P(M) : (a, x) \to ax$$

such that $\forall a, b \in R$ and $\forall x, y \in M$ we have

$$a(x + y) \cap (ax + ay) \neq \varnothing, \ (a + b)x \cap (ax + bx) \neq \varnothing, \ (ab)x \cap a(bx) \neq \varnothing,$$

then M is called an H_v-*module* over F.

For more definitions and applications on H_v-structures one can see [1], [2], [3], [5], [7], [8], [10], [12], [13].

The main tool to study hyperstructures is the fundamental relation. In 1970 M. Koscas defined in hypergroups the relation β and its transitive closure β^*. This relation connects the hyperstructures with the corresponding classical structures and is defined in H_v-groups as well. T. Vougiouklis [17] introduced the γ^* and ϵ^* relations, which are defined, in H_v-rings and H_v-vector spaces, respectively. He also named all these relations β^*, γ^* and ϵ^*, Fundamental Relations because they play very important role to the study of hyperstructures especially in the representation theory of them. For similar relations see [16], [17].

Definition 1.1. *The **fundamental relations** β^*, γ^* and ϵ^*, are defined, in H_v-groups, H_v-rings and H_v-vector space, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively.*

Specifying the above motivation we remark the following: Let $(G,\cdot)$ be a group and R be an equivalence relation (or a partition) in G, then $(G/R,\cdot)$ is an H_v-group, therefore we have the quotient $(G/R,\cdot)/\beta^*$ which is a group, the fundamental one. Remark that the classes of the fundamental group $(G/R,\cdot)/\beta^*$ are a union of some of the R-classes. Otherwise, the $(G/R,\cdot)/\beta^*$ has elements classes of G, where they form a partition which classes are larger than the classes of the original partition R.

The way to find the fundamental classes is given by the following:

Theorem 1.1. *Let $(\boldsymbol{H},\cdot)$ be an H_v-group and denote by $\boldsymbol{U}$ the set of all finite products of elements of H. We define the relation β in H by setting $x\beta y$ iff $\{x,y\} \subset \boldsymbol{u}$ where $\boldsymbol{u} \in \boldsymbol{U}$. Then β^* is the transitive closure of β.*

Theorem 1.2. *Let $(\boldsymbol{R},+,\cdot)$ be an H_v-ring. Denote by $\boldsymbol{U}$ the set of all finite polynomials of elements of $\boldsymbol{R}$. We define the relation γ in $\boldsymbol{R}$ as follows:*

$$x\gamma y \text{ iff } \{x,y\} \subset u \text{ where } u \in \boldsymbol{U}.$$

Then the relation γ^* is the transitive closure of the relation γ.

The fundamental relations are used for general definitions of hyperstructures. Thus, to define the general H_v-field we use the fundamental relation γ^*:

Definition 1.2. *[15], [16] The H_v-ring $(R, +, \cdot)$ is called H_v-field if R/γ^* is a field.*

Definition 1.3. *The H_v-semigroup $(H, \cdot)$ is called* ***h/v-group*** *if the quotient H/β^* is a group.*

Construction 1.1. *[19] Let $(H, \cdot)$ be an H_v-semigroup and $v \notin H$. We extend the $(\cdot)$ in $\underline{H} = H \cup \{v\}$ as follows:*

$$x \cdot v = v \cdot x = v, \forall x \in H, \text{ and } v \cdot v = H.$$

$(\underline{H}, \cdot)$ is an h/v-group, called ***attach****, where $(\underline{H}, \cdot)/\beta^* \cong Z_2$ and v is a single. The core of $(\underline{H}, \cdot)$ is H. Scalars and units of $(H, \cdot)$ are scalars and units (rep.) in $(\underline{H}, \cdot)$. If $(H, \cdot)$ is COW (resp. commutative) then $(\underline{H}, \cdot)$ is also COW (resp. commutative).*

The h/v-groups are a generalization of the H_v-groups because in h/v-groups the reproductivity is not necessarily valid. However, sometimes a kind of reproductivity of classes is valid. That means that if H is partitioned into equivalence classes $\sigma(x)$, then $x\sigma(y) = \sigma(xy) = \sigma(x)y, \forall x \in H$. This leads the quotient to be reproductive. In a similar way the *h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces* etc, are defined [20].

Construction 1.2. *Let $(H, \cdot)$ be an H_v-semigroup, $v \notin H$, and $(\underline{H}, \cdot)$ its attached h/v-group. Take an element $0 \notin H$ and define in $\underline{H}_o = H \cup \{v, 0\}$ two hopes:*

$$\textit{hyperaddition}$$

$$(+) : 0{+}0 = x{+}v = v{+}x = 0, 0{+}v = v{+}0 = x{+}y = v, 0{+}x = x{+}0 = v{+}v = H,$$
$$\forall x, y \in H$$

$$\textit{hypermultiplication}$$

$$(\cdot) : \text{ remains the same as in } \underline{H} \text{ moreover } 0 \cdot 0 = v \cdot x = x \cdot 0 = 0, \forall x \in \underline{H}$$

Then $(\underline{H}_o, +, \cdot)$ *is h/v-field with* $(\underline{H}_o, +, \cdot)/\gamma^* \cong Z_3$. $(+)$ *is associative,* $(\cdot)$ *is WASS and weak distributive with respect to* $(+)$. *0 is zero absorbing and single but not scalar in* $(+)$. $(\underline{H}_o, +, \cdot)$ *is called the* ***attached h/v-field*** *of the* H_v*-semigroup* $(H, \cdot)$.

Thus, it is an h/v-field, which is in fact an H_v-field introduced in 1990 [15].

Since the algebras are defined on vector spaces, we need before the definition of the H_v-Lie Algebra give the definition of the H_v-vector space.

Definition 1.4. *Let* $(F, +, \cdot)$ *be an* H_v*-field,* $(V, +)$ *be a COW* H_v*-group and there exists an external hope*

$$\cdot : F \times V \mapsto P(V) - \{\emptyset\} : (a, x) \mapsto ax$$

such that, for all $a, b \in F$ *and* $x, y \in V$ *we have* $a(x + y) \cap (ax + ay) \neq \emptyset, (a + b)x \cap (ax + bx) \neq \emptyset, (ab)x \cap a(bx) \neq \emptyset$, *then V is called an* H_v*-**vector space** over F. In the case of an* H_v*-ring instead of an* H_v*-field then the* H_v*-modulo is defined. In these cases the fundamental relation* ϵ^* *is the smallest equivalence relation such that the quotient* V/ϵ^* *is a vector space over the fundamental field* F/γ^*.

Theorem 1.3. *Let* $(\boldsymbol{M}, +)$ *be an* H_v*-module over the* H_v*-ring* $\boldsymbol{R}$. *Denote by* $\boldsymbol{U}$ *the set of all expressions consisting of finite hopes either on* $\boldsymbol{R}$ *and* $\boldsymbol{M}$ *or the external hope applied on finite sets of elements* $\boldsymbol{R}$ *and* $\boldsymbol{M}$. *We define the relation* ϵ *in* $\boldsymbol{M}$ *as follows:*

$$x \epsilon y \text{ iff } \{x, y\} \subset u \text{ where } u \in \boldsymbol{U}$$

Then the relation ϵ^* *is the transitive closure of the relation* ϵ

Definition 1.5. *[21] Let* $(L, +)$ *be an* H_v*-vector space over the* H_v*-field* $(F, +, \cdot)$, $\phi : F \to F/\gamma^*$ *the canonical map and* $\omega_F = \{x \in F : \phi(x) = 0\}$, *where 0 is the zero of the fundamental field* F/γ^*. *Similarly, let* ω_L *be the core of the canonical map* $\phi' : L \to L/\epsilon^*$ *and denote by the same symbol 0 the zero of* L/ϵ^*. *Consider the bracket (commutator) hope:*

$$[,] : L \times L \to P(L) : (x, y) \to [x, y]$$

then $\boldsymbol{L}$ *is an* H_v*-Lie algebra over F if the following axioms are satisfied:*

(L1) *The bracket hope is bilinear, i.e.*
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1, y] + \lambda_2[x_2, y]) \neq \varnothing$
$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1[x, y_1] + \lambda_2[x, y_2]) \neq \varnothing,$
$\forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in F$

(L2) $[x, x] \cap \omega_L \neq \varnothing, \quad \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \varnothing, \quad \forall x, y \in L$

2 Matrix Repesentations

The Lie-Santilli *isotopies* born to solve Hadronic Mechanics problems. Santilli proposed [11] a 'lifting' of the trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields*, used in physics or biology, correspond to H_v-structures called *e-hyperfields*.

Definition 2.1. *Let* $(H_o, +, \cdot)$ *be the attached* H_v*-field of the* H_v*-semigroup* $(H, \cdot)$*. If* $(H, \cdot)$ *has a left and right scalar unit e, then* $(H_o, +, \cdot)$ *is an e-hyperfield, the attached* H_v*-field of* $(H, \cdot)$*.*

Most of H_v-structures are used in Representation (abbreviate by rep) Theory. Reps of H_v-groups can be considered either by generalized permutations or by H_v-matrices [16], [18]. Reps by generalized permutations can be achieved by using translations. In the rep theory the singles are playing a crucial role. In representations of H_v-groups by H_v-matrices there are two difficulties: To find an H_v-ring and an appropriate set of H_v-matrices. Now, lets give some definitions.

The rep problem by H_v-matrices is the following:

H_v-matrix is called a matrix if has entries from an H_v-ring. The hyperproduct of H_v-matrices $A = (a_{ij})$ and $B = (b_{ij})$, of type $m \times n$ and $n \times r$, respectively, is a set of $m \times r$ H_v-matrices, defined in a usual manner:

$$A \cdot B = (a_{ij}) \cdot (b_{ij}) = \{C = (c_{ij}) | c_{ij} \in \oplus \sum a_{ik} \cdot b_{kj}\},$$

where $(\oplus)$ denotes the n-ary circle hope on the hyperaddition.

Definition 2.2. *Let* $(H,\cdot)$ *be an* H_v*-group,* $(R,+,\cdot)$ *an* H_v*-ring,* $M_R=\{(a_{ij})|a_{ij}\in R\}$*, then any map*

$$\boldsymbol{T}: H \to \boldsymbol{M}_R : h \to T(h) \text{ with } T(h_1h_2) \cap T(h_1)T(h_2) \neq \varnothing, \forall h_1, h_2 \in H,$$

is called H_v***-matrix rep****. If* $T(h_1h_2) \subset T(h_1)T(h_2)$*, then* $\boldsymbol{T}$ *is an* ***inclusion rep****, if* $T(h_1h_2) = T(h_1)T(h_2)$*, then* $\boldsymbol{T}$ *is a* ***good rep****.*

Hopes on any type of matrices can be defined, these are called helix hopes [22].

A large class of hopes is given as follows [14], [16]:

Let $(G,\cdot)$ be a groupoid then for every $P \subset G$, $P \neq \varnothing$, we define the following hopes called P-*hopes*: for all $x, y \in G$

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP),$$

$$\underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G,\underline{P})$,$(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called P-*hyperstructures*. The most usual case is if $(G,\cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G,\underline{P})$ is a semihypergroup but we do not know about $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$. In some cases, depending on the choice of P, the $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ can be associative or WASS. If more operations are defined in G, then for each operation several P-hopes can be defined.

Using several classes of H_v-structures one can face several representations. Some of those classes are as follows [4]:

Definition 2.3. *Let* $\mathbf{M} = \mathbf{M}_{m\times n}$ *be a module of* $m \times n$ *matrices over a ring* $\mathbf{R}$ *and* $\mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}$*. We define, a kind of, a P-hope* $\underline{P}$ *on* $\mathbf{M}$ *as follows*

$$\underline{P} : \mathbf{M} \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (A,B) \to A\underline{P}B = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

where P^t *denotes the transpose of the matrix* P*.*

The hope $\underline{P}$, which is a bilinear map, is a generalization of Rees' operation where, instead of one sandwich matrix, a set of sandwich matrices is used. The hope $\underline{P}$ is strong associative and the inclusion distributivity with respect to addition of matrices

$$A\underline{P}(B+C) \subseteq A\underline{P}B + A\underline{P}C \text{ for all } A, B, C \text{ in } \mathbf{M}$$

is valid. Therefore, $(\mathbf{M}, +, \underline{P})$ defines a multiplicative hyperring on non-square matrices. Multiplicative hyperring means that only the multiplication is a hope.

Definition 2.4. *Let* $\mathbf{M} = \mathbf{M}_{m\times n}$ *be a module of* $m \times n$ *matrices over* R *and let us take sets*

$$\mathbf{S} = \{S_k : k \in K\} \subseteq R, \quad \mathbf{Q} = \{Q_j : j \in J\} \subseteq \mathbf{M}, \quad \mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}.$$

Define three hopes as follows

$$\underline{S} : R \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (r, A) \to r\underline{S}A = \{(rs_k)A : k \in K\} \subseteq \mathbf{M}$$

$$\underline{Q}_+ : \mathbf{M} \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (A, B) \to A\underline{Q}_+B = \{A + Q_j + B : j \in J\} \subseteq \mathbf{M}$$

$$\underline{P} : \mathbf{M} \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (A, B) \to A\underline{P}B = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

Then $(\mathbf{M}, \underline{S}, \underline{Q}_+, \underline{P})$ *is a hyperalgebra over* $\boldsymbol{R}$ *called general matrix P-hyperalgebra.*

In a similar way a generalization of this hyperalgebra can be defined if one considers an H_v-ring or an H_v-field instead of a ring and using H_v-matrices.

3 Mathematical Realisation of type $\mathbf{D_n}$

The representation theory by matrices gives to researchers a flexible tool to see and handle algebraic structures. This is the reason to see Lie-Santilli's admissibility using matrices or hypermatrices to study the multivalued (hyper) case. Using the well known P-hyperoperations we extend the Lie-Santilli's admissibility into the hyperstructure case. We present the problem and we give the basic definitions on the topic which cover the four following cases:

Construction 3.1. *[12], [9] Suppose R, S be sets of square matrices (or hypermatrices). We can define the hyper-Lie bracket in the following ways:*

1. $[x,y]_{RS} = xRy - ySx$ *(General Case)*
2. $[x,y]_R = xRy - yx$
3. $[x,y]_S = xy - ySx$
4. $[x,y]_{RR} = xRy - yRx$

The question is when the anticommutativity and Jacobbi identity, for all square matrices (or hypermatrices) x, y, z,

$$[x,x]_{RS} \ni 0$$

$$[x,[y,z]_{RS}]_{RS} + [y,[z,x]_{RS}]_{RS} + [z,[x,y]_{RS}]_{RS} \ni 0$$

of a hyper-Lie algebra are satisfied [12].

We apply this generalization on the Lie algebras of the type D_n.

We deal with Lie-Algebra of type $D_n^{(1)}$, $n \geqslant 4$ which is a graded algebra, using the principal realisation used in Infinite Dimensional Kac Moody Lie Algebras [6].

Let

$$S = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

The set of matrices $M_{2n\times 2n}$ for which $MS + SM^t = 0$ constitutes the Lie-algebra, called D_n. So, the complex matrix obtained is the following:

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

Using the relation $MS + SM^t = 0$, we obtain the following limitations:

- $M_{22} = -M_{11}^t$

- $M_{12} = -M_{12}^t$
- $M_{21} = -M_{21}^t$

Denote by $E_{ij}(i, j = 1, ..., 2n)$ the $2n \times 2n$ complex matrix, which has 1 the ij-entry and 0 all other entries.

Then we have

- e_i the i^{th} element of Level 1, $i = 1, 2, ..., n-1$.

$$\begin{aligned} e_0 &= E_{2n-1,1} - E_{2n,2} \\ e_i &= E_{i,i+1} - E_{2n-i,2n-i+1} \\ e_n &= E_{n-1,n+1} - E_{n,n+2} \end{aligned}$$

- h_i the elements of Level 0, where $i = 1, 2, ..., n-1$.

$$\begin{aligned} h_0 &= E_{2n,2n} + E_{2n-1,2n-1} - E_{22} - E_{11} \\ h_i &= E_{2n-i,2n-i} + E_{i,i} - E_{2n-i+1,2n-i+1} - E_{i+1,i+1} \\ h_n &= E_{nn} + E_{n-1,n-1} - E_{n+2,n+2} - E_{n+1,n+1} \end{aligned}$$

- f_i the i^{th} element of Level $h-1$ basis, $i = 1, 2, ..., n-1$.

$$\begin{aligned} f_0 &= E_{1,2n-1} - E_{2,2n} \\ f_i &= E_{i+1,i} - E_{2n-i+1,2n-i} \\ f_n &= E_{n+1,n-1} - E_{n+2,n} \end{aligned}$$

These sets of elements are generators. Summarizing the elements of every basis, one can prove that the dimension of all odd levels including level-0 is $n+1$ and the dimension of even levels is n.

Proposition 3.1. *[13] A Lie algebra g of type D_n is a graded modh where the 1-principal $\mathbf{Z}/h\mathbf{Z}$-gradation is given by setting*

$$degE_{ij} = (\underline{j} - \underline{i})modh$$

where $h = 2(n-1)$, the number of the different levels and $\underline{k} = k$ if $k \leqslant n$ and $\underline{k} = k-1$ if $k > n$.

Denote that all the subscripts are mod $2n$ and the number of different Levels is $2n-2$. Therefore, based on the previous proposition the levels, for $n=4$, are:

Level-0

$$\begin{pmatrix} \mathbf{a_{11}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{a_{22}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{a_{33}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{a_{43}} & \mathbf{a_{44}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathbf{a_{44}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathbf{a_{43}} & -\mathbf{a_{33}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{22}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{11}} \end{pmatrix}$$

Level-1

$$\begin{pmatrix} 0 & \mathbf{a_{12}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{a_{23}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{a_{34}} & \mathbf{a_{35}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{35}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{34}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{23}} & 0 \\ \mathbf{a_{17}} & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{12}} \\ 0 & -\mathbf{a_{17}} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Level-2

$$\begin{pmatrix} 0 & 0 & \mathbf{a_{13}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{a_{24}} & \mathbf{a_{25}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{25}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{24}} & 0 \\ \mathbf{a_{61}} & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{13}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathbf{a_{61}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Level-3

$$\begin{pmatrix} 0 & 0 & 0 & \mathbf{a_{14}} & \mathbf{a_{14}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{a_{26}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{26}} & 0 \\ \mathbf{a_{41}} & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{14}} \\ \mathbf{a_{51}} & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{14}} \\ 0 & \mathbf{a_{62}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathbf{a_{62}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{a_{51}} & -\mathbf{a_{41}} & 0 & 0 & 0 \end{pmatrix}$$

Level-4

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \mathbf{a_{16}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{a_{31}} & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{16}} \\ 0 & \mathbf{a_{42}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{a_{52}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{a_{52}} & -\mathbf{a_{42}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{31}} & 0 & 0 \end{pmatrix}$$

Level-5

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{a_{17}} & 0 \\ \mathbf{a_{21}} & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{17}} \\ 0 & \mathbf{a_{32}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{a_{43}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{a_{53}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{a_{53}} & -\mathbf{a_{43}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{32}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{a_{21}} & 0 \end{pmatrix}$$

Remark 3.1. *The overall dimension of Lie-algebras of type D_n can be computed as follows: For $n-1$ odd levels we have $n+1$ elements of the bases, for $n-2$ even levels we have n elements of the bases and for the zero level we have $n+1$ elements of the base. Therefore:*

$$Dim(Level-0)+Dim(Level-1)+Dim(Level-2)+...+Dim(Level-(h-1)) =$$

$$= (n+1)+(n+1)+n+(n+1)+...+n+(n+1) =$$

$$= n(n+1)+n(n-2) =$$

$$= n(2n-1)$$

For our examples the Konstant's Cyclic Element E is the sum of *First* Level's Simple Base [6]:

$$E = e_0 + e_1 + ... + e_n$$

The element $F = f_0 + f_1 + ... + f_n$, which is the sum of the *Last* Level's Simple Base, can be used as well as Konstant's Cyclic Element.

The element E shifts every element of any level L to the next one $(L+1)$ [13]. The base of the first level as well as for every odd level, including zero has $n+1$ elements. On the other side, the base of every even level has n elements. So, this shifting is not a complete correspondance, since in odd levels there is one more element. Only the shifting from the last Level-(h-1) to Level-0 is an one-to-one correspondance, as both have the same $n+1$ dimension.

The element F shifts every element of any level L to the previous one $(L-1)$. These cyclic elements gets different element from the base and goes to different element of the next level or the previous level (depending on the Cyclic Element used), creating a correspondance between the elements of the Levels. This correspondance, though, is not a complete correspondance as we have extra elements. So, removing an element of every odd level and level zero, we will have a complete one-to-one correspondance.

In hyperstructure theory, this correspondance can be achieved by using the uniting elements procedure. In the uniting elements procedure we put together, in the same class, two or more elements. Therefore, we can unite the extra element of all odd levels with one of the other elements of the base of the same level.

An analogous construction to the A_n case [9] is described as follows:

Theorem 3.1. *Take a set P with only two elements from zero or first level, then we have the hyperstructure with P-hope on all elements of D_n. If P contains an element outside Level-0, then this hope shifts the elements analogously.*

Remark that, we can take P with more than two elements as well, but in this case the hope is greater, so the hyperstructure is not so usefull.

Moreover the general Construction 3.1 can be also applied in D_n case.

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