Hyperring Theory and Applications

Bijan Davvaz
and
Violeta Leoreanu-Fotea

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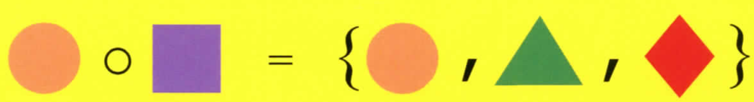

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Preface

Hyperstructures represent a natural extension of classical algebraic structures and they were introduced in 1934 by the French mathematician F. Marty [79]. Since then, hundreds of papers and several books have been written on this topic. A recent book on hyperstructures [27] points out on their applications in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. The books published till now on hyperstructures deal especially with hypergroups. That is why we think that a book on hyperring theory is a necessity, especially for the new researchers on this topic, for Ph.D. students who need a material on hyperrings. Hyperrings have been also the starting point for the study of hypermodules, which were considered by P. Corsini, C.G. Massouros, G.G. Massouros and others.

The presented book is composed by eight chapters. At the beginning, we recall some notions and basic results on ring theory, that we shall extend to the context of hyperrings. The second chapter is about algebraic hyperstructures, their history and some basic results especially on some important classes of hypergroups. The following chapters are about several types of hyperrings. We can consider several definitions for a hyperring, by replacing at least one of the two operations by hyperoperations.

A well known type of a hyperring, called the Krasner hyperring, is obtained by considering the addition as a hyperoperation, such that the structure $(R, +)$ is a canonical hypergroup, a structure which is analyzed in the second chapter. Then we consider the multiplication as a hyperoperation and we obtain the so-called multiplicative hyperrings, presented in the fourth chapter. Finally, if both addition and multiplication are hyperoperations, then we obtain general hyperrings, studied in the fifth chapter.

Speaking about hyperrings, we must recall several names of the mathematicians who have introduced and given an important development to this notion. After M. Krasner, who gave his name of an important class of hyperrings, we have to mention D. Stratigopoulos, who wrote one of the first Ph.D. thesis on hyperrings, namely on Artinian hyperrings, J. Mittas who studied in depth the Krasner hyperrings, T. Vougiouklis and M. De Salvo who considered general hyperrings, R. Procesi and R. Rota, who introduced the multiplicative hyperrings. T. Vougiouklis and S. Spartalis analyzed H_v-rings, C.G. Massouros studied especially hyperfields and G.G. Massouros mentioned some interesting of applications of hyperrings to automata. Other mathematicians gave a contribution to this theory and they are mentioned in each chapter together with several of their main results on this topic. Another hyperstructure that can be obtained from a ring, by replacing the associative law by a weak associative law is analyzed in the sixth chapter. They are called H_v-rings and were introduced by Vougiouklis. In the seventh chapter, we consider commutative rings that can be obtained from hyperrings. This is a new connection between rings and hyperrings. For each type of hyperrings, we analyze the quotient structure, the fundamental relation and polynomials. In the last chapter, we present an outline of applications of hyperstructures in chemistry and physics.

There are many new research directions on which the presented research can be continuated. Our book contains only a background on hyperrings, that can be useful to everybody who wishes to do research on this topic.

We thank to Prof. Piergiulio Corsini, Prof. Thomas Vougiouklis, Prof. Jan Chvalina and Lecturer Dr. Šárka Hošková for their suggestions and help in providing us some necessary material for this book.

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The Authors

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Chapter 1

Introduction to rings

1.1 Introduction

Any book on Abstract Algebra will contain the definition of a ring. It will define a ring as a set endowed with two operations, called addition and multiplication, satisfying a collection of axioms. These axioms require addition to satisfy the axioms of an abelian group, while multiplication is associative and the two operations are connected by the distributive laws. A ring is therefore a setting for generalizing integer arithmetic. Familiar examples of rings such as the real numbers, the complex numbers, the rational numbers, the integers, the even integers, 2×2 real matrices, the integers modulo m for a fixed integer m, will almost certainly be given in an Abstract Algebra book, as well as many beautiful theorems on rings. What will be probably missing are the reasons for which these particular axioms have been singled out for a such intensive study. What motivates this abstract definition of a ring?

Richard Dedekind introduced the concept of a ring. The word *ring* has the origin in the German word Zahlring (=number ring). The term ring (Zahlring) was coined by David Hilbert in [53]. The first abstract definition of a ring was given by Fraenkel (of set-theory fame) in a 1914 paper entitled "On zero divisors and the decomposition of rings" [50]. The Fraenkel's definition meant to encompass both commutative and noncommutative rings. He considered integers modulo m, matrices, p-adic integers, and hypercomplex number systems as examples of rings. The Fraenkel's aim was to do for rings what Steinitz had just (1910) done for fields, namely to give an abstract and comprehensive theory of commutative and noncommutative rings. Of course he was not successful (he did admit that the task here is not so "easy" as in the case of fields). The undertaking to subsume the structure of both commutative and noncommutative rings under one theory was too ambitious. Among the main concepts introduced in Fraenkel's paper are "zero divisors" and "regular elements". Fraenkel considered only rings which are not integral domains (i.e. rings with zero divisors) and discussed divisibility for such rings. A great part of the paper deals with the decomposition of rings in direct products of "simple" rings (not the usual notion of simplicity), see [9]. Fraenkel's definition of a ring is almost the definition that we use nowadays. He defined a ring as "a system" with two abstract operations, that he named addition and multiplication. The system is a group with respect to one of the operations (addition). The second operation (multiplication) is associative and it is distributive over the first. The two axioms give the closure of the system under the operations, and there is the requirement of an identity in the definition of the ring. The commutativity of the addition does not appear as an axiom but it is proved! Similarly, other elementary properties of a ring such as $a \cdot 0 = 0$; $a(-b) = (-a)b = -(ab)$, and $(-a)(-b) = ab$ are proved. There are two "extraneous" axioms, dealing with "regular" elements in the ring, which derive from an otherwise modern definition, given by Sono in a 1917 paper entitled "On congruences" [117]. Sono's is a very modern and abstract paper, which discusses about cosets, quotient rings, maximal and minimal ideals, simple rings, the isomorphism theorems, and composition series (see [9]). Although Fraenkel's and Sono's papers were not in the mainstream of contemporary ring theoretic studies, they were significant since the rings are studied as independent, abstract objects, not just as rings of polynomials, as rings of algebraic integers, or as rings (algebras) of hypercomplex numbers.

Ideals were firstly proposed by Dedekind in 1876 in the third edition of his book. They are a generalization of the concept of an ideal number developed by Ernst Kummer. Later, the concept was expanded by David Hilbert and especially by Emmy Noether. The ideal concept generalizes some important properties of integers like "even number" or "multiple of 3" in an appropriate way. An ideal can be used to construct a quotient ring in a similar way as a normal subgroup can be used to construct a factor group in group theory. The order ideal concept in order theory is derived from the ideal notion in ring theory.

Prime numbers were generalized to prime ideals by Dedekind in 1871. A prime ideal is an ideal which contains the product of two elements only if it contains one of the two elements. For example all integers which are divisible by a fixed prime p form a prime ideal of the ring of integers. This trend which consists in looking at ideals rather than at the elements marks an important stage in the development of the ring theory. The decomposition of an integer into the product of powers of primes has an analogue in rings where prime integers are replaced by prime ideals. On the other hand, powers of prime integers are not replaced by powers of prime ideals but rather by "primary ideals". Primary ideals were introduced by Lasker in 1905, in the context of polynomial rings. (Lasker was World Chess Champion from 1894 to 1921.) Lasker proved the existence of a decomposition of an ideal into primary ideals but the uniqueness of the decomposition was proven by Macaulay only in 1915.

In ring theory, one can study prime ideals instead of prime numbers, one can define coprime ideals as a generalization of coprime numbers, and one can prove a generalized Chinese remainder theorem for ideals. In the Dedekind domains, which form an important class of rings in number theory, one can even obtain a version of the arithmetic fundamental theorem: in these rings, every nonzero ideal can be uniquely written as a product of prime ideals.

Despite of the abstract definition of a ring, the study of rings of polynomials, rings of algebraic integers, and rings of hypercomplex numbers represents an important topic of algebra. In the 1920s, two master algebraists Noether and Artin have transformed these subjects into powerful, abstract theories. The Noether's two seminal papers of 1921 and 1927 extended and abstracted the decomposition theories of polynomial rings on the one hand and of the rings of integers of algebraic number fields and algebraic function fields on the other hand, to abstract commutative rings which satisfy the ascending chain condition, that we call now Noetherian rings. More exactly, Noether showed in the 1921 paper, entitled "Ideal theory in rings", that the results of Hilbert, Lasker, and Macauley on primary decomposition in polynomial rings hold for any (abstract) ring which satisfy the ascending chain condition. Thus, results which seemed inextricably connected with the properties of polynomial rings were obtained from a single axiom!

Commutative ring theory has its origins in algebraic number theory, algebraic geometry, and invariant theory, and has in turn been applied mainly to these subjects. In 1921, Emmy Noether gave the first axiomatic system of the commutative ring theory in her fundamental paper Ideal Theory in Rings.

In her 1927 paper, "Abstract development of ideal theory in algebraic number fields and function fields", she discussed about the Dedekind and Dedekind-Weber results on the decomposition of ideals as unique products of prime ideals in rings of integers of algebraic number fields and function fields in the setting of abstract rings respectively. In particular, she characterized abstract commutative rings in which every nonzero ideal is a unique product of prime ideals.

A ring is called commutative if its multiplication is commutative. Commutative rings resemble familiar number systems, and various definitions for commutative rings are designed to recover properties known from the integers. In the commutative ring theory, numbers are often replaced by ideals, and the definition of prime ideal tries to capture the essence of prime numbers. Integral domains generalize another property of the integers and can be used as the proper realm to study divisibility. Principal ideal domains are integral domains in which every ideal can be generated by a single element, which is another property shared by the integers. Euclidean domains are integral domains in which the Euclidean algorithm can be carried out. Important examples of commutative rings can be constructed as rings of polynomials and their factor rings.

Noncommutative rings resemble rings of matrices in many respects. Following the model of algebraic geometry, attempts have been made recently to define noncommutative geometry based on noncommutative rings. Noncommutative rings and associative algebras (rings that are also vector spaces) are often studied via their categories of modules.

Nowadays, ring theory is a fertile meeting ground for group theory (group rings), representation theory (modules), functional analysis (operator algebras), Lie theory (enveloping algebras), algebraic geometry (finitely generated algebras, differential operators, invariant theory), arithmetic (orders, Brauer groups), universal algebra (varieties of rings), and homological algebra (cohomology of rings, projective modules, Grothendieck and higher K-groups).

1.2 The abstract definition of a ring and some examples

In this section, the definition of a ring and numerous examples are given. A ring is a two-operational system and these operations are usually called *addition* and *multiplication*.

Definition 1.2.1. A nonempty set R is said to be a *ring* if in R there are defined two binary operations, denoted by $+$ and $\cdot$ respectively, such that for all a, b, c in R:

(1) $a + b = b + a$,

(2) $(a + b) + c = a + (b + c)$,

(3) there is an element 0 in R such that $a + 0 = a$,

(4) there exists an element $-a$ in R such that $a + (-a) = 0$,

(5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,

(6) $\cdot$ is distributive with respect to $+$, i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot y + y \cdot z$.

Axioms 1 through 4 merely state that R is an *abelian group* under the operation $+$. The additive identity of a ring is called the *zero element*. If $a \in R$ and $n \in \mathbb{Z}$, then na has its usual meaning for additive groups.

If in addition:

(7) $a \cdot b = b \cdot a$ for all a, b in R,

then R is said to be a *commutative ring*. If R contains an element 1 such that

(8) $1 \cdot a = a \cdot 1 = a$ for all a in R,

then R is said to be a *ring with unit element*.

If R is a system with unit satisfying all the axioms of a ring expect possibly $a + b = b + a$ for all $a, b \in R$, then one can show that R is a ring.

For any two elements a, b of a ring R, we shall denote $a + (-b)$ by $a - b$ and for convenience sake we shall usually write ab instead of $a \cdot b$.

Before going on to work out some properties of rings, we pause to examine some examples. Motivated by these examples we shall define various special types of rings which are of importance.

Example 1.2.2. (Some examples of commutative rings)

(1) Each of the number sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ forms a ring with respect to ordinary addition and multiplication.

(2) For every $m \in \mathbb{Z}$, $\{ma \mid a \in \mathbb{Z}\}$ forms a ring with respect to ordinary addition and multiplication.

(3) The set $\mathbb{Z}_n$ is a ring with respect to addition and multiplication modulo n.

(4) We say that a ring R is a *Boolean ring* (after the English Mathematician George Boole) if $x^2 = x$ for all $x \in R$. A Boolean ring is commutative. Let X be a set and A, B be subsets of X. The symmetric difference between two subsets A and B, denoted by $A \triangle B$, is the set of all x such that either $x \in A$ or $x \in B$ but not both. The set $\mathcal{P}(X)$ of all subsets of a set X is a ring. The addition is the symmetric difference $\triangle$ and the multiplication is the set operation intersection $\cap$. Its zero element is the empty set, and its unit element is the set X. This is an example of a Boolean ring.

(5) Let $\mathbb{Z}[i]$ denote the set of all complex numbers of the form $a+bi$ where a and b are integers. Under the usual addition and multiplication of complex numbers, $\mathbb{Z}[i]$ forms a ring called the *ring of Gaussian integers*.

(6) The set of all continuous real-valued functions defined on the interval $[a, b]$ forms a ring, the operations are addition and multiplication of functions.

(7) A *polynomial* is a formal expression of the form

$$p(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} + a_nx^n,$$

where $a_0, \dots, a_n \in \mathbb{R}$ and x is a variable. Polynomials can be added and multiplied as usual. With these operations the set $\mathbb{R}[x]$ of all polynomials is a ring. In fact, given any commutative ring R, one can construct the ring $R[x]$ of polynomials over R in a similar way.
We define now the ring of polynomials in the n-variables $x_1, \dots, x_n$ over R, $R[x_1, \dots, x_n]$, as follows: let $R_1 = [x_1]$, $R_2 = R_1[x_2]$,..., $R_n = R_{n-1}[x_n]$. R_n is called the *ring of polynomials in* $x_1, \dots, x_n$ *over* R.

(8) Let R be a commutative ring with unit element and denoted by $R[[x]]$ the set of all formal power series over the ring R. Then $R[[x]]$ is a ring with addition and multiplication defined by

$$\begin{aligned} \sum_{i=0}^{\infty} a_ix^i + \sum_{i=0}^{\infty} b_ix^i &= \sum_{i=0}^{\infty} (a_i + b_i)x^i, \\ \sum_{i=0}^{\infty} a_ix^i \cdot \sum_{i=0}^{\infty} b_ix^i &= \sum_{i=0}^{\infty} c_ix^i, \end{aligned}$$

where $c_n = \sum_{i=0}^{n} a_ib_{n-i}$. The ring $R[[x]]$ is called the *ring of power series.*

(9) Let R be a commutative ring with unit. A nonempty subset S of R is called a *multiplicative subset* if $0 \notin S$ and $s_1, s_2 \in S$ implies $s_1s_2 \in S$. Let $R \times S$ be the set of all ordered pairs (r, s) where $r \in R$ and $s \in S$. In $R \times S$ we define now a relation as follows: $(r_1, s_1) \sim (r_2, s_2)$ if and only if there exists $s \in S$ such that $s(r_1s_2 - s_1r_2) = 0$. The relation $\sim$ is an equivalence relation on $R \times S$. Let $[r, s]$ be the equivalence class of (r, s) in $R \times S$, and let $S^{-1}R$ be the set of all such equivalence classes

$[r, s]$ where $r \in R$ and $s \in S$. The quotient set $S^{-1}R$ is a commutative ring with unit under addition and multiplication defined by

$$\begin{aligned} [r_1, s_1] + [r_2, s_2] &= [r_1 s_2 + r_2 s_1, s_1 s_2], \\ [r_1, s_1] \cdot [r_2, s_2] &= [r_1 r_2, s_1 s_2], \end{aligned}$$

for all $r_1, r_2 \in R$ and $s_1, s_2 \in S$. $S^{-1}R$ is usually called the *ring of fractions* of R. In the special case in which R is the ring of integers, the $S^{-1}R$ so constructed is, of course, the ring of rational numbers.

Example 1.2.3. (Some examples of noncommutative rings)

(1) One of the smallest noncommutative rings is the Klein 4-ring $(R, +, \cdot)$, where $(R, +)$ is the Klein 4-group $\{0, a, b, c\}$ with 0 the neutral element and the binary operation $\cdot$ given by the following table:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	b	0	b
c	0	c	0	c

(2) The set $M_n(\mathbb{R})$ of all $n \times n$ matrices with entries from $\mathbb{R}$ forms a ring with respect to the usual addition and multiplication of matrices. In fact, given an arbitrary ring R, one can consider the ring $M_n(R)$ of $n \times n$ matrices with entries from R.

(3) If G is an abelian group, then $End(G)$, the set of endomorphisms of G, forms a ring, the operations in this ring are the addition and composition of endomorphisms.

(4) Let Ω consist of all complex valued functions f of real variable x such that

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

Ω is an additive abelian group with respect to the ordinary addition. We consider the binary operation $*$ called *convolution*,

$$h = f * g,$$

where $(f * g)(x)$ is defined by the equation

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

It can be shown that if f and g are in Ω, then h is also in Ω (it follows from Fubini's theorem in analysis). The remaining axioms are easy to verify and we conclude that Ω is a ring with respect to the ordinary addition $+$ and convolution $*$. This ring lacks a unit element.

(5) Let G be a group and R a ring. Firstly, we define the set $R[G]$ to be one of the following:

- The set of all formal R-linear combinations of elements of G.
- The set of all functions $f : G \longrightarrow R$ with $f(g) = 0$ for all but finitely many g in G.

No matter which definition is used, we can write the elements of $R[G]$ in the form $\sum_{g \in G} a_g g$, with all but finitely many of the a_g being 0, and the addition on $R[G]$ is the addition of formal linear combinations or addition of functions, respectively. The multiplication of elements of $R[G]$ is defined by setting

$$(\sum_{g \in G} a_g g)(\sum_{h \in G} b_h h) = \sum_{g,h \in G} (a_g b_h) gh$$

If R has a unit element, this is the unique bilinear multiplication for which $(1g)(1h) = (1gh)$. In this case, G is commonly identified with the set of elements $1g$ of $R[G]$. The identity element of G then serves as the 1 in $R[G]$. It is not difficult to verify that $R[G]$ is a ring. This ring is called the *group ring* of G over R.
Note that: If R and G are both commutative (i.e., R is commutative and G is an abelian group), then $R[G]$ is commutative.

(6) This last example is often called the *ring of real quaternions.* This ring was firstly described by the Irish mathematician Hamilton. Initially

it was extensively used in the study of mechanics; today its primary interest is that of an important example, although still it plays key roles in geometry and number theory.

Let Q be the set of all symbols $a_0+a_1i+a_2j+a_3k$, where all the numbers a_0, a_1, a_2 and a_3 are real numbers. Define the equality between two elements of Q as follows: $a_0+a_1i+a_2j+a_3k = b_0+b_1i+b_2j+b_3k$ if and only if $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$. We define the addition and multiplication on Q by

$$\begin{aligned}
&(a_0 + a_1i + a_2j + a_3k) + (b_0 + b_1i + b_2j + b_3k) \\
&\quad = (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k, \\
&(a_0 + a_1i + a_2j + a_3k) \cdot (b_0 + b_1i + b_2j + b_3k) \\
&\quad = (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)i \\
&\quad +(a_0b_2 + a_2b_0 + a_3b_1 - a_1b_3)j + (a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1)k.
\end{aligned}$$

It is easy to see that Q is a noncommutative ring in which $0 = 0 + 0i + 0j + 0k$ and $1 = 1 + 0i + 0j + 0k$ are the zero and unit elements respectively. Note that the set $\{1, -1, i, -i, j, -j, k, -k\}$ forms a nonabelian group of order 8 under this product.

(7) (Differential operator rings). Consider the homogeneous linear differential equation $a_n(x)D^ny + \dots + a_1(x)Dy + a_0(x)y = 0$, where the solution $y(x)$ is a polynomial with complex coefficients, and also the terms $a_i(x)$ belong to $\mathbb{C}[x]$. The equation can be written in compact form as $L(y)=0$, where L is the differential operator $a_n(x)D^n + \dots + a_1(x)D + a_0(x)$, with $D = \frac{d}{dx}$. Thus the differential operator can be thought as a polynomial in the two indeterminates x and D, but in this case the indeterminates do not commute, since $D(xy(x)) = y(x) + xD(y(x))$, yielding the identity $Dx = 1 + xD$. The repeated use of this identity makes possible to write the composition of two differential operators in the standard form $a_0(x) + a_1(x)D + \dots + a_n(x)D^n$, and we denote the resulting ring by $\mathbb{C}[x][D]$.

We wish to be able to compute in rings in the same manner in which we compute with real numbers, keeping in mind always that there are different. It may happen that $ab \neq ba$, or a does not divide b. To this end we mention some preliminary results, which assert that certain something we should like to be true in rings are indeed true.

Preliminary results 1.2.4. Let R be a ring. Then

(1) Since a ring is an abelian group under $+$, there are certain things we know from the group theory background, for instance, $-(-a) = a$ and $-(a+b) = -a-b$ for all a, b in R and so on,

(2) $0a = a0 = 0$ for all a in R,

(3) $(-a)b = a(-b) = -(ab)$ for all a, b in R,

(4) $(-a)(-b) = ab$ for all a, b in R,

(5) $(na)b = a(nb) = n(ab)$ for all $n \in \mathbb{Z}$ and a, b in R,

(6) $\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{m} b_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j$ for all a_i, b_j in R.

Moreover, if R has a unit element 1, then

(7) $(-1)a = -a$ for all $a \in R$,

(8) $(-1)(-1) = 1$.

1.3 Some special classes of rings

In dealing with an arbitrary ring R there may exist nonzero elements a and b in R, such that their product is zero. Such elements are called *zero-divisors*.

Definition 1.3.1. A nonzero element a is called a *zero-divisor* if there exists a nonzero element $b \in R$ such that either $ab = 0$ or $ba = 0$.

Example 1.3.2. As examples of such rings, we have

(1) In the ring $\mathbb{Z}_6$ we have $2 \cdot 3 = 0$ and so 2 and 3 are zero-divisors. More generally, if n is not prime then $\mathbb{Z}_n$ contains zero-divisors.

(2) Consider the ring R of all order pairs of real numbers (a, b). If (a, b) and (c, d) are two elements in R, we define the addition and multiplication in R by the equalities: $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b) \cdot (c, d) = (ac, bd)$. Then R is a ring. The zero element is $(0, 0)$ and the ring has zero-divisors.

Definition 1.3.3. A commutative ring is an *integral domain* if it has no zero-divisors.

The ring of integers, is an example of an integral domain. It is easy to verify that a ring R has no zero-divisors if and only if the right and left cancellation laws hold in R.

Definition 1.3.4. If the nonzero elements of a ring R form a multiplicative group, i.e., R has unit element and every element except the zero element has an inverse, then we shall call the ring a skew field or a *division ring.*

Definition 1.3.5. A *field* is a commutative division ring.

The inverse of an element a under multiplication will be denoted by a^{-1}.

Example 1.3.6.

(1) If p is prime, then $\mathbb{Z}_p$ is a field.

(2) $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are examples of fields whereas $\mathbb{Z}$ is not.

(3) In Example 1.2.2 (9), let R be an integral domain and $S = R\backslash\{0\}$. Then $S^{-1}R = F$ is a field. F is usually called the *field of fractions.*

(4) Consider the set $\{a + bx \mid a, b \in \mathbb{Z}_2\}$ with x a "indeterminate". We use the arithmetic addition modulo 2 and multiplication using the "rule" $x^2 = x + 1$. Then we obtain a field with 4 elements: $\{0, 1, x, 1 + x\}$.

(5) Consider the set $\{a + bx + cx^2 \mid a, b, c \in \mathbb{Z}_2\}$, where we now use the rule $x^3 = 1 + x$. This gives a field with 8 elements: $\{0, 1, x, 1 + x, x^2, 1 + x^2, x + x^2, 1 + x + x^2\}$.

(6) Consider the set $\{a + bx \mid a, b \in \mathbb{Z}_3\}$ with arithmetic modulo 3 and the "rule" $x^2 = -1$ (so it is similar as the multiplication in $\mathbb{C}$). Then we get a field with 9 elements: $\{0, 1, 2, x, 1 + x, 2 + x, 2x, 1 + 2x, 2 + 2x\}$.

More generally, using "tricks" like the above ones, we can construct a finite field with p^k elements for any prime p and positive integer k. This is denoted by $GF(p^k)$ and it is called the *Galois Field* named after the French mathematician Evariste Galois.

The ring of quaternions is a division ring which is not a field. Many other examples of noncommutative rings exist, for instance see the following example.

Example 1.3.7. Consider the set $M = \left\{ \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} \mid a, b \in \mathbb{C} \right\}$, where $\overline{a}, \overline{b}$ are conjugates of a, b. M is a ring with unit under matrix addition and multiplication. If $A = \begin{bmatrix} x+iy & u+iv \\ -u+iv & x-iy \end{bmatrix}$ is a nonzero matrix in M, then

$$A^{-1} = \begin{bmatrix} \dfrac{x-iy}{x^2+y^2+u^2+v^2} & -\dfrac{u+iv}{x^2+y^2+u^2+v^2} \\ \dfrac{u-iv}{x^2+y^2+u^2+v^2} & \dfrac{x+iy}{x^2+y^2+u^2+v^2} \end{bmatrix}.$$

Hence M is a division ring, but is not commutative, since

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \neq \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Clearly, every field is an integral domain, but, in general, an integral domain is not a field. For example, the ring of integers is an integral domain, but not all nonzero elements have inverse under multiplication. However, for finite domains of integrity, we have the following theorem.

Theorem 1.3.8. *Any finite ring without zero-divisors is a division ring.*

Proof. Let $R = \{x_1, x_2, ..., x_n\}$ be a finite ring without zero-divisors and suppose that $a(\neq 0) \in R$. Then $ax_1, ax_2, ..., ax_n$ are all n distinct elements lying in R, as cancellation laws hold in R. Since $a \in R$, there exists $x_i \in R$ such that $a = ax_i$. Then we have $a(x_i a - a) = a^2 - a^2 = 0$, and so $x_i a = a$. Now, for every $b \in R$ we have $ab = (ax_i)b = a(x_i b)$, hence $b = x_i b$ and further $ba = b(x_i a) = (bx_i)a$ which implies that $b = bx_i$. Hence x_i is the unit element for R and we denote it by 1. Now, $1 \in R$, so there exists $c \in R$

such that $1 = ac$. Also $a(ca - 1) = (ac)a - a = a - a = 0$, and so $ca = 1$. Consequently, R is a division ring. ∎

Corollary 1.3.9. *A finite integral domain is a field.*

By a famous theorem of Wedderburn, "every finite division ring is a field". Therefore we can say that "any finite ring without zero-divisors is a field.

Definition 1.3.10. Let R be a ring. Then D is said to be of *finite characteristic* if there exists a positive integer n such that $na = 0$ for all $a \in R$. If no such of n exists, R is said to be of *characteristic* 0. If R is of finite characteristic, then we define the *characteristic* of R as the smallest positive integer n such that $na = 0$ for all $a \in R$.

The characteristic of $\mathbb{Z}_n$ is equal to n, whereas $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are of characteristic 0.

Basic results 1.3.11.

(1) Any finite field is of finite characteristic. However, an integral domain may be infinite and with a finite characteristic.

(2) The characteristic of an integral domain with unit element is either zero or a prime number.

(3) If D is an integral domain and if $na = 0$ for some $a \neq 0$ in D and some integer $n \neq 0$, then D is of finite characteristic. Note that, it is not true for an arbitrary ring; it is enough to consider the ring $\mathbb{Z}_2 \times \mathbb{Z}$.

(4) Let R be a ring with unit element. Then the characteristic of R is equal to n if and only if n is the least positive integer such that $n \cdot 1 = 1$.

1.4 Subrings, ideals and quotient rings

In the study of groups, subgroups play a crucial role. Subrings, the analogous notion in ring theory, play a much less important role than their counterparts in group theory. Nevertheless, subrings are important.

Definition 1.4.1. Let R be a ring and S be a nonempty subset of R, which is closed under the addition and multiplication in R. If S is itself a ring under these operations then S is called a *subring* of R; more formally, S is a subring of R if the following conditions hold:

$$a, b \in S \text{ implies that } a - b \in S \text{ and } a \cdot b \in S.$$

Example 1.4.2.

(1) For each positive integer n, the set $n\mathbb{Z} = \{0, \pm n, \pm 2n, \pm 3n, ...\}$ is a subring of $\mathbb{Z}$.

(2) $\mathbb{Z}$ is a subring of the ring of real numbers and also a subring of the ring of polynomials $\mathbb{Z}[X]$.

(3) The ring of Gaussian integers is a subring of the complex numbers.

(4) The set A of all 2×2 matrices of the type $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$, where a, b and c are integers, is a subring of the ring $M_2(\mathbb{Z})$.

(5) The polynomial ring $R[x]$ is a subring of $R[[x]]$.

(6) If R is any ring, then the *center* of R is the set $Z(R)=\{x \in R \mid xy=yx, \forall y \in R\}$. Clearly, the center of R is a subring of R.

Let R be a ring and S be a proper subring of it. Then there exists the following five cases:

- R and S have a common unit element.
- R has a unit element but S does not.
- R and S both have their own nonzero unities but these are distinct.
- R has no unit element but S has a unit element.
- Neither R nor S have unit element.

Example 1.4.3.

(1) The ring $\mathbb{Q}$ and its subring $\mathbb{Z}$ have the common unit element 1.

(2) The subring S of even integers of the ring $\mathbb{Z}$ has no unit element. Actually, the only subring with unit of $\mathbb{Z}$ is $\mathbb{Z}$.

(3) Let S be the subring of all pairs (a, b) of the ring $\mathbb{Z} \times \mathbb{Z}$ for which the operations $+$ and $\cdot$ are defined component by component. Then S and $\mathbb{Z} \times \mathbb{Z}$ have the unities $(1, 0)$ and $(1, 1)$, respectively.

(4) Let S be the subring of all pairs $(a, 0)$ of the ring $R=\{(a, 2b) \mid a, b\in\mathbb{Z}\}$ (operations are defined component by component). Now S has the unit element $(1, 0)$ but R has no unit element.

(5) Neither the ring $\{(2a, 2b) \mid a, b \in \mathbb{Z}\}$ (operations are defined component by component) nor its subring consisting of the pairs $(2a, 0)$ have unit element.

In group theory, normal subgroups play a special role, they permit us to construct quotient groups. Now, we introduce the analogous concept for rings.

Definition 1.4.4. A nonempty subset I of a ring R is said to be an *ideal* of R if

(1) I is a subgroup of R under addition,

(2) for every $a \in I$ and $r \in R$, both ar and ra are in I.

Clearly, each ideal is a subring. For any ring R, $\{0\}$ and R are ideals of R. The ideal $\{0\}$ is called the *trivial ideal.* An ideal I of R such that $I \neq 0$ and $I \neq R$ is called a *proper ideal.* Observe that if R has a unit element and I is an ideal of R, then $I = R$ if and only if $1 \in I$. Consequently, a nonzero ideal I of R is proper if and only if I contains no invertible elements of R. It is easy to see that the intersection of any family of ideals of R is also an ideal.

Example 1.4.5.

(1) For any positive integer n, the set $n\mathbb{Z}$ is an ideal of $\mathbb{Z}$. In fact, every ideal of $\mathbb{Z}$ has this form, for suitable n.

(2) Let I be the set of all polynomials over $\mathbb{R}$ with zero constant term. Then I is an ideal of $\mathbb{R}[x]$.

(3) Let R be the ring of all real-valued functions of a real variable. The subset S of all differentiable functions is a subring of R but not an ideal of R.

(4) Let $f \in \mathbb{Q}[x]$. Then the set $\{fg \mid g \in \mathbb{Q}[x]\}$ is an ideal of $\mathbb{Q}[x]$. In fact, every ideal, though not every subring, of $\mathbb{Q}[x]$ has this form.

(5) Let

$$R = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{bmatrix} \mid a, b, c, d, e, f \in \mathbb{Z} \right\}$$

then R is a ring under matrix addition and multiplication. The set

$$R = \left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \mid x, y \in \mathbb{Z} \right\}$$

is an ideal of R.

(6) Let R be a ring and let $M_n(R)$ be the ring of matrices over R. If I is an ideal of R then the set $M_n(I)$ of all matrices with entries in I is an ideal of $M_n(R)$. Conversely, every ideal of $M_n(R)$ is of this type.

(7) Let m be a positive integer such that m is not a square in $\mathbb{Z}$. If $R = \{a + \sqrt{m}b \mid a, b \in \mathbb{Z}\}$, then R is a ring under the operations of sum and product of real numbers. If p is an odd prime number, consider the set $I_p = \{a + \sqrt{m}b \mid p|a \text{ and } p|b\}$, where $a + \sqrt{m}b \in R$. Then I_p is an ideal of R.

(8) For ideals I_1, I_2 of a ring R define $I_1 + I_2$ to be the set $\{a + b \mid a \in I_1, b \in I_2\}$ and $I_1 I_2$ to be the set $\left\{ \sum_{i=1}^{n} a_i b_i \mid n \in \mathbb{Z}^+, a_i \in I_1, b_i \in I_2 \right\}$. Then $I_1 + I_2$ and $I_1 I_2$ are ideals of R.

(9) Let R be an arbitrary ring and let $a_1, a_2, \ldots a_m \in R$. Then the set of all elements of the form

$$\sum_{i=1}^{m} z_i a_i + \sum_{i=1}^{m} s_i a_i + \sum_{i=1}^{m} a_i t_i + \sum_{i=1}^{m} \left(\sum_{i=1}^{n_i} u_{i,k} a_i v_{i,k} \right),$$

where $m, z_i, n_i \in \mathbb{Z}$, $s_i, t_i, u_{i,k}, v_{i,k} \in R$, is an ideal. In fact it is the smallest ideal of R which contains $a_1, a_2, ...a_m$. Hence it is called the *ideal generated* by $a_1, a_2, ...a_m$.
If R is commutative and has a unit element, the above set reduces to the set $\{a_1r_1 + a_2r_2 + ... + a_mr_m \mid r_i \in R\}$. We denote this ideal briefly by $\langle a_1, a_2, ...a_m\rangle$. If $m = 1$ the ideal $\langle a_1\rangle$ is called the *principal ideal* generated by a_1. In particular, $\langle 1\rangle = R$.

(10) The subset E of $\mathbb{Z}[x]$ composed by all polynomials with even constant term is an ideal of $\mathbb{Z}[x]$. In fact $E = \langle x, 2\rangle$ and it is not principal.

(11) Let X be a nonempty set and $\mathcal{P}(X)$ denotes the ring of power set of X. Then a nonempty subset I of $\mathcal{P}(X)$ is an ideal of $\mathcal{P}(X)$ if and only if $\mathcal{P}(A \cup B) \subseteq I$ for all $A, B \in I$.

(12) Let R be a commutative ring and let A be an arbitrary subset of R. Then the *annihilator* of A, $Ann(A) = \{r \in R \mid ra = 0 \text{ for all } a \in A\}$ is an ideal.

Lemma 1.4.6. *Let R be a commutative ring with unit element whose only ideals are the trivial ideal and R. Then R is a field.*

Proof. In order to prove this lemma, for any nonzero element $a \in R$ we must find an element $b \in R$ such that $ab = 1$. The set $Ra = \{xa \mid x \in R\}$ is an ideal of R. By our assumptions on R, $Ra = \{0\}$ or $Ra = R$. Since $0 \neq a = 1 \cdot a \in Ra$, $Ra \neq \{0\}$, and so $Ra = R$. Since $1 \in R$, there exists $b \in R$ such that $1 = ba$. ■

Definition 1.4.7. (Quotient ring). Let R be a ring and let I be an ideal of R. In order to define the quotient ring, we consider firstly an equivalence relation on R. We say that the elements $a, b \in R$ are equivalent, and we write $a \sim b$, if and only if $a - b \in I$. If a is an element of R, we denote the corresponding equivalence class by $[a]$. The *quotient ring* of modulo I is the set $R/I = \{[a] \mid a \in R\}$, with a ring structure defined as follows. If $[a], [b]$ are equivalence classes in R/I, then

$$[a] + [b] = [a + b] \quad \text{and} \quad [a] \cdot [b] = [ab].$$

Since I is closed under addition and multiplication, it follows that the ring structure in R/I is well defined. Clearly, $a + I = [a]$.

Example 1.4.8. Let us present some quotient rings.

(1) $\mathbb{Z}/6\mathbb{Z} = \{6\mathbb{Z}, 1+6\mathbb{Z}, 2+6\mathbb{Z}, 3+6\mathbb{Z}, 4+6\mathbb{Z}, 5+6\mathbb{Z}\}$.

(2) We consider the ring of polynomials $\mathbb{R}[x]$ with real coefficients and $\langle x^2+1 \rangle$ generated by x^2+1. Then

$$\mathbb{R}[x]/\langle x^2+1 \rangle = \{ax + b + \langle x^2+1 \rangle \mid a, b \in \mathbb{R}\}.$$

(3) If $R = \mathbb{Z}[x,y]$ and $I = \langle x^2, y^2+1 \rangle$, then every element of R/I has the form $a + bx + cy + dxy + I$, where $a, b, c, d \in \mathbb{Z}$.

1.5 Ring homomorphisms and isomorphisms

In this section, we consider one of the most fundamental notions of ring theory-"homomorphism". The homomorphism term comes from the Greek words "homo", which means like and "morphe", which means form. In our presentation about rings we see that one way to discover information about a ring is to examine its interaction with other rings using homomorphisms.

Just as a group homomorphism preserves the group operation, a ring homomorphism preserves the ring operations.

Definition 1.5.1. A mapping φ from the ring R into the ring R' is said to be a *(ring) homomorphism* if

(1) $\varphi(a+b) = \varphi(a) + \varphi(b)$,

(2) $\varphi(ab) = \varphi(a)\varphi(b)$,

for all $a, b \in R$.

If φ is a ring homomorphism from R to R', then $\varphi(0) = 0$ and $\varphi(-a) = -\varphi(a)$ for every $a \in R$.

A ring homomorphism $\varphi : R \longrightarrow R'$ is called an *epimorphism* if φ is onto. It is called a *monomorphism* if it is one to one, and an *isomorphism* if it is both one to one and onto. A homomorphism φ of a ring R into itself is called an *endomorphism*. An endomorphism is called an *automorphism* if it is an isomorphism. The rings R and R' are said to be *isomorphic* if there exists an isomorphism between them, in this case, we write $R \cong R'$.

Before going on we examine these concepts for certain examples.

Example 1.5.2.

(1) For any positive integer n, the mapping $k \to k \bmod n$ is a ring homomorphism from $\mathbb{Z}$ onto $\mathbb{Z}_n$.

(2) Let I be an ideal of a ring R. We define $\varphi : R \longrightarrow R/I$ by $\varphi(a) = a+I$ for all $a \in R$. Then φ is an epimorphism. This map is called a *natural homomorphism*.

(3) Let $\mathbb{Z}(\sqrt{2})$ be the set of real numbers of the form $m + n\sqrt{2}$ where m, n are integers; $\mathbb{Z}(\sqrt{2})$ forms a ring under the usual addition and multiplication of real numbers. We define $\varphi : \mathbb{Z}(\sqrt{2}) \longrightarrow \mathbb{Z}(\sqrt{2})$ by $\varphi(m + n\sqrt{2}) = m - n\sqrt{2}$. Then φ is an automorphism.

Preliminary results 1.5.3. Let φ be a homomorphism from the ring R to the ring R'. Let S be a subring of R, I an ideal of R and J an ideal of R'.

(1) $\varphi(S) = \{\varphi(a) \mid a \in S\}$ is a subring of R'.

(2) If φ is onto, then $\varphi(I)$ is an ideal of R'.

(3) $\varphi^{-1}(J) = \{r \in R \mid \varphi(r) \in J\}$ is an ideal of R.

(4) If R is commutative then $\varphi(R)$ is commutative.

(5) If R has a unit element 1 and φ is onto, then $\varphi(1)$ is the unit element of R'.

(6) If φ is an isomorphism from R to R', then φ^{-1} is an isomorphism from R' to R.

Now, we introduce an important ideal that is intimately related to the image of a homomorphism.

Definition 1.5.4. If φ is a ring homomorphism of R into R', then the *kernel* of φ is defined by $\{x \in R \mid \varphi(x) = 0\}$.

Corollary 1.5.5. *If φ is a ring homomorphism from R to R', then $ker\varphi$ is an ideal of R.*

Theorem 1.5.6. *A ring homomorphism φ from R to R' is one to one if and only if $ker\varphi = \{0\}$.*

We are in a position to establish an important connection between homomorphisms and quotient rings. Many authors prefer to call the next theorem the Fundamental theorem of ring isomorphism.

Theorem 1.5.7. (First isomorphism theorem). *Let $\varphi : R \longrightarrow R'$ be a homomorphism from R to R'. Then $R/ker\varphi \cong \varphi(R)$; in fact, the mapping $\psi : R/ker\varphi \longrightarrow \varphi(R)$ defined by $\psi(a+ker\varphi) = \varphi(a)$ defines an isomorphism from $R/ker\varphi$ onto $\varphi(R)$. Moreover there is a one to one correspondence between the set of ideals of R' and the set of ideals of R which contain $ker\varphi$. This correspondence can be achieved by associating with an ideal J in R', the ideal I in R defined by $I = \{x \in R \mid \varphi(x) \in J\}$. With I so defined, R/I is isomorphic to R'/J.*

We go on to the next isomorphism theorem.

Theorem 1.5.8. (Second isomorphism theorem). *Let I and J be two ideals of a ring R. Then $(I + J)/I \cong J/(I \cap J)$.*

Finally, we come to the last of the isomorphism theorem that we wish to state.

Theorem 1.5.9. (Third isomorphism theorem). *Let I and J be two ideals of a ring R such that $J \subseteq I$. Then $R/I \cong (R/J)/(I/J)$.*

Example 1.5.10.

(1) $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$.

(2) Let $R = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$. We define $\psi : R \longrightarrow \mathbb{C}$ by $\psi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right) = a + bi$. Then ψ is an isomorphism and so R is isomorphic to the field of complex numbers.

(3) Let R be the ring of all real valued continuous functions defined on the closed unit interval. Then $I = \{f \in R \mid f(\frac{1}{2}) = 0\}$ is an ideal of R. One can shows that R/I is isomorphic to the real field.

Lemma 1.5.11. *Let R be a ring with unit element 1. The mapping $\varphi : \mathbb{Z} \longrightarrow R$ given by $\varphi(n) = n1$ is a ring homomorphism.*

Corollary 1.5.12. *If R is a ring with unit element and the characteristic of R is $n > 0$, then R contains a subring isomorphic to $\mathbb{Z}_n$. If the characteristic of R is 0, then R contains a subring isomorphic to $\mathbb{Z}$.*

Proof. The set $S = \{n1 \mid n \in \mathbb{Z}\}$ is a subring of R. Lemma 1.5.11 shows that the mapping φ from $\mathbb{Z}$ onto S given by $\varphi(n) = n1$ is a homomorphism, and by the first isomorphism theorem, we have $\mathbb{Z}/ker\varphi \cong S$. But, clearly $ker\varphi = n\mathbb{Z}$. So $S \cong \mathbb{Z}_n$ if $n > 0$, whereas $S \cong \mathbb{Z}/ < 0 > \cong \mathbb{Z}$ if $n = 0$. ■

Corollary 1.5.13. *If F is a field of characteristic p, then F contains a subfield isomorphic to $\mathbb{Z}_p$. If F is a field of characteristic 0, then F contains a subfield isomorphic to $\mathbb{Q}$.*

Proof. By Corollary 1.5.12, F contains a subring isomorphic to $\mathbb{Z}_p$ if F has characteristic p and F has a subring S isomorphic to $\mathbb{Z}$ if F has characteristic 0. In the latter case, let $K = \{ab^{-1} \mid a, b \in S, b \neq 0\}$. Then K is isomorphic to $\mathbb{Q}$. ■

1.6 Maximal ideals and prime ideals

In this section, we define some special ideals of a ring and we give some important results about them. Firstly, we begin with the definition of maximal ideal of a ring.

Definition 1.6.1. A proper ideal M of R is said to be a *maximal ideal* of R if whenever U is an ideal of R and $M \subseteq U \subseteq R$ then $U = M$ or $U = R$.

Example 1.6.2. Examples of maximal ideals.

(1) In a division ring, $< 0 >$ is a maximal ideal.

(2) In the ring of even integers, $< 4 >$ is a maximal ideal.

(3) In the ring of integers, an ideal $n\mathbb{Z}$ is maximal if and only if n is a prime number.

(4) The ideal $\langle x^2 + 1 \rangle$ is maximal in $\mathbb{R}[x]$.

(5) Let R be the ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. The set $M = \{f \in R \mid f(0) = 0\}$ is a maximal ideal of R.

Zorn's lemma is a form of the axiom of choice which is technically very useful for proving existence theorems. For instance, from Zorn's lemma it follows directly that every ring has a maximal ideal.

Theorem 1.6.3. *If R is a commutative ring with unit element and M is an ideal of R, then M is a maximal ideal of R if and only if R/M is a field.*

Proof. Suppose that M is a maximal ideal and let $a \in R$ but $a \notin M$. It suffices to show that $a + M$ has a multiplicative inverse. Consider

$$U = \{ar + b \mid r \in R, b \in M\}.$$

This is an ideal of R that contains M properly. Since M is maximal, we have $U = R$. Thus $1 \in U$, so there exist $c \in R$ and $d \in M$ such that $1 = ac + d$. Then $1 + M = ac + d + M = ac + M = (a + M)(c + M)$.

Now, suppose that R/M is a field and U is an ideal of R that contains M properly. Let $a \in U$ but $a \notin M$. Then $a + M$ is a nonzero element of R/M and so there there exists an element $b+M$ such that $(a+M)(b+M) = 1+M$. Since $a \in U$, we have $ab \in U$. Also, we have $1 - ab \in M \subseteq U$. So $1 = (1 - ab) + ab \in U$ which implies that $U = R$. ∎

The motivation for the definition of a prime ideal comes from the integers.

Definition 1.6.4. An ideal P in a ring R is said to be *prime* if $P \neq R$ and for any ideals A, B in R

$$AB \subseteq P \implies A \subseteq P \text{ or } B \subseteq P.$$

The definition of a prime ideal excludes the ideal R for both historical and technical reasons. The following corollary is a very useful characterization of prime ideals.

Corollary 1.6.5. *Let R be a commutative ring. An ideal P of R is prime if $P \neq R$ and for any $a, b \in R$*

$$ab \in P \implies a \in P \text{ or } b \in P.$$

Example 1.6.6. Examples of prime ideals.

(1) A positive integer n is a prime number if and only if the ideal $n\mathbb{Z}$ is a prime ideal in $\mathbb{Z}$.

(2) In the ring $\mathbb{Z}[x]$ of all polynomials with integer coefficients, the ideal generated by 2 and x is a prime ideal.

(3) The prime ideals of $\mathbb{Z} \times \mathbb{Z}$ are $\{0\} \times \mathbb{Z}$, $\mathbb{Z} \times \{0\}$, $p\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times q\mathbb{Z}$, where p and q are primes.

(4) If R denotes the ring $\mathbb{C}[x, y]$ of polynomials in two variables with complex coefficients, then the ideal generated by the polynomial $y^2 - x^3 - x - 1$ is a prime ideal. Also the ideals $< 0 > \subsetneq < y - x - 1 > \subsetneq < x - 2, y - 3 >$ are all prime.

(5) In $\mathbb{Z}[x, y, z]$, the ideals $< x > \subsetneq < x, y > \subsetneq < x, y, z >$ are all prime, but none is maximal.

Theorem 1.6.7. *If R is a commutative ring with unit element and P is an ideal of R, then P is a prime ideal of R if and only if R/P is an integral domain.*

Proof. Suppose that R/P is an integral domain and $ab \in P$. Then $(a + P)(b + P) = ab + P = P$. So either $a + P = P$ or $b + P = P$; that is either $a \in P$ or $b \in P$. Hence P is prime.

Now, suppose that P is prime and $(a+P)(b+P) = 0 + P = P$. Then $ab \in P$ and therefore $a \in P$ or $b \in P$. Thus one of $a+P$ or $b+P$ is zero. ∎

Theorem 1.6.8. *Let R be a commutative ring with unit element. Each maximal ideal of R is a prime ideal.*

Proof. Suppose that M is maximal in R but not prime, so there exist $a, b \in R$ such that $a \notin M$, $b \notin M$ but $ab \in M$. Then each of the ideals $M + < a >$ and $M + < b >$ contains M properly. By maximality we obtain $M + < a > = R = M + < b >$. Therefore $R^2 = (M + < a >)(M + < b >) \subseteq M^2 + < a > M + M < b > + < a >< b > \subseteq M \subsetneq R$. This is a contradiction. ∎

Definition 1.6.9. The *radical* of an ideal I in a commutative ring R, denoted by $Rad(I)$ is defined as

$$Rad(I) = \{r \in R \mid r^n \in I \text{ for some positive integer } n\}.$$

Intuitively, one can think that the radical of I is obtained by taking all the possible roots of elements of I. $Rad(I)$ turns out to be an ideal itself, containing I. The above definition is equivalent to: The radical of an ideal I in a commutative ring R is

$$Rad(I) = \bigcap_{\substack{P \in Spec(R) \\ I \subseteq P}} P,$$

where $Spec(R)$ is the set of all prime ideals of R.

Lemma 1.6.10. *If $J, I_1, ..., I_n$ are ideals in a commutative ring R, then*

(1) $Rad(Rad(J)) = Rad(J)$,

(2) $Rad(I_1, ..., I_n) = Rad\left(\bigcap_{i=1}^{n} I_i\right) = \bigcap_{i=1}^{n} Rad(I_i)$.

Example 1.6.11. In the ring of integers

(1) $Rad(12\mathbb{Z}) = 2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$,

(2) let $n = p_1^{k_1}...p_r^{k_r}$, where p_i's are distinct prime numbers. Then we have $Rad(n\mathbb{Z}) = \langle p_1, ..., p_r \rangle$.

The concept of a maximal ideal in a commutative ring leads immediately to the very important notion of a Jacobson radical of that ring.

Definition 1.6.12. Let R be a commutative ring. We define the Jacobson radical of R, denoted by $Jac(R)$, as the intersection of all the maximal ideals of R.

We can provide a characterization for the Jacobson radical of a commutative ring.

Lemma 1.6.13. (Nakayama's lemma). *Let R be a commutative ring, and let $r \in R$. Then $r \in Jac(R)$ if and only if for every $a \in R$, the element $1 - ra$ is an invertible element of R.*

1.7 Noetherian rings

Noetherian rings are named after Emmy Noether who made many contributions to algebra. Towards the end of this chapter, we shall establish some basic facts about Noetherian rings.

Definition 1.7.1. Let R be a ring. Then R is said to be a *Noetherian ring* when it satisfies the equivalent conditions:

(1) R satisfies the *ascending chain condition* for ideals, i.e., whenever

$$I_1 \subseteq I_2 \subseteq ... \subseteq I_i \subseteq I_{i+1} \subseteq ...$$

is an ascending chain of ideals of R, then there exists $k \in \mathbb{N}$ such that $I_k = I_{k+i}$ for all $i \in \mathbb{N}$;

(2) every nonempty set of ideals of R has a maximal member with respect to the inclusion.

Principal ideal domains are integral domains in which every ideal can be generated by a single element.

Lemma 1.7.2. *Every principal ideal domain is a Noetherian ring.*

Example 1.7.3.

(1) $\mathbb{Z}$ is a Noetherian ring.

(2) If F is a field then $F[x]$ is a Noetherian ring.

(3) If R is a division ring then the ring $M_n(R)$ of all $n \times n$ matrices over R is a Noetherian ring.

Basic results 1.7.4.

(1) Let R be a ring and I be an ideal of R. Then R is a Noetherian ring if and only if I and R/I are Noetherian rings.

(2) A commutative ring R is Noetherian if and only if every ideal of R is finitely generated.

(3) (Hilbert's basis theorem). If R is a commutative Noetherian ring with unit element, then so is $R[x]$.

By the following example, we show that a subring of a Noetherian ring is not necessary Noetherian.

Example 1.7.5. Let $S = \left\{ \sum\limits_{i=0}^{n} a_i x^i \mid a_i \in \mathbb{Q},\ n \in \mathbb{N},\ a_0 \in \mathbb{Z} \right\}$. Then S is a subring of $\mathbb{Q}[x]$. The strictly ascending chain $< x > \subset < x/2 > \subset < x/4 > \ldots$ of ideals of S does not stop.

Definition 1.7.6. Let R be a ring. Then R is said to be an *Artinian ring* when it satisfies the equivalent conditions:

(1) R satisfies the *descending chain condition* for ideals, i.e., whenever

$$J_1 \supseteq J_2 \supseteq \ldots \supseteq J_i \supseteq J_{i+1} \supseteq \ldots$$

is an descending chain of ideals of R, then there exists $k \in \mathbb{N}$ such that $J_k = J_{k+i}$ for all $i \in \mathbb{N}$;

(2) every nonempty set of ideals of R has a minimal member with respect to the inclusion.

Example 1.7.7. Consider the set $\mathbb{Z}(p^\infty)$ of all rational numbers between 0 and 1 of the form m/p^n, where p is a fixed prime number, m is an arbitrary positive integer and n runs through all nonnegative integers. Then $\mathbb{Z}(p^\infty)$ is an abelian group under addition modulo 1. $\mathbb{Z}(p^\infty)$ can be endowed with a ring structure by defining $ab = 0$ for all $a, b \in \mathbb{Z}(p^\infty)$. Then $\mathbb{Z}(p^\infty)$ is an Artinian ring.

The Hilbert's basis theorem fails to hold for Artinian rings. Also, a subring of an Artinian ring is not necessary Artinian.

By *a chain of prime ideals* of a ring R we mean a finite strictly increasing sequence $P_0 \subset P_1 \subset ... \subset P_n$; the *length* of the chain is n. We define *dimension* of R as the supremum of the lengths of all chains of prime ideals in R. Theorem 1.7.8 is a well known theorem in commutative ring theory.

Theorem 1.7.8. *Let R be a commutative ring with unit element. Then R is an Artinian ring if and only if it is a Noetherian ring with zero dimension.*

Chapter 2

Algebraic hyperstructures

2.1 What algebraic hyperstructures are?

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. More exactly, if H is a nonempty set and $\mathcal{P}^*(H)$ is the set of all nonempty subsets of H, then we consider maps of the following type:

$$f_i : H \times H \longrightarrow \mathcal{P}^*(H),$$

where $i \in \{1, 2, ..., n\}$ and n is a positive integer. The maps f_i are called (*binary*) *hyperoperations*. For all x, y of H, $f_i(x, y)$ is called the (*binary*) *hyperproduct* of x and y. An algebraic system $(H, f_1, ..., f_n)$ is called a (*binary*) *hyperstructure*. Usually, $n = 1$ or $n = 2$.

Under certain conditions, imposed to the maps f_i, we obtain the so-called semihypergroups, hypergroups, hyperrings or hyperfields. Sometimes, external hyperoperations are considered, which are maps of the following type:

$$h : R \times H \longrightarrow \mathcal{P}^*(H),$$

where $R \neq H$. Usually, R is endowed with a ring or a hyperring structure. An example of a hyperstructure, endowed both with an internal hyperoperation and an external hyperoperation is the so-called hypermodule.

A binary structure (H, f) endowed with only one internal hyperoperation is called a *hypergroupoid*. Hypergroups play an important role among hypergroupoids. Several kinds of hypergroups have been intensively studied, such as: regular hypergroups, reversible regular hypergroups, canonical hypergroups, cogroups, cyclic hypergroups, associativity hypergroups. The situations that occur in hyperstructure theory, particularly in hypergroup theory, are often extremely diversified and complex with respect to the classical ones. For instance, there are homomorphisms of various types between hypergroups and there are several kinds of subhypergroups, such as: closed, invertible, ultraclosed, conjugable.

One of the first books, dedicated especially to hypergroups, is "Prolegomena of Hypergroup Theory", written by P. Corsini in 1993 [23]. Another book on "Hyperstructures and Their Representations", by T. Vougiouklis, was published one year later [133]. On the other hand, algebraic hyperstructure theory has a multiplicity of applications to other disciplines: geometry, graphs and hypergraphs, binary relations, lattices, groups, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, C-algebras, artificial intelligence, probabilities and so on. A recent book on these topics is "Applications of Hyperstructure Theory", by P.Corsini and V. Leoreanu, published by Kluwer Academic Publishers in 2003 [27]. Finally, we mention here another important book for the applications in Geometry and for the clearness of the exposition, written by W. Prenowitz and J. Jantosciak [104]. Some particular generalizations of hyperstructures have been also considered and we mention here three of them. H.S. Wall [143] introduced hyperoperations, for which for all x, y of H, the hyperproduct $f(x, y)$ contains not necessarily distinct elements $a_1, ..., a_k$. In other words, each element a_i can occur in $f(x, y)$ with a certain multiplicity, which means that a_i can occur one or two or more times in $f(x, y)$. Moreover, a set of conditions of regularity are considered to be satisfied. Such hypergroups, called *Wall-hypergroups*, have applications in physics, especially in atomic physics, in harmonic analysis.

A second kind of generalization consists in considering n-ary hyperoperations, instead of binary hyperoperations, where $n \geq 3$. In other words, we consider maps of the following type:

$$f : H \times ... \times H \longrightarrow \mathcal{P}^*(H).$$

This study was introduced by B. Davvaz and T. Vougiouklis [42] and studied then by them and other mathematicians in different contexts.

In the third generalization, for all x, y of H the image $f(x, y)$ is a fuzzy set on H, instead of a subset of H. This generalization has been considered especially by Iranian mathematicians, and we mention here B. Davvaz, M.M. Zahedi, R. Ameri, R.A. Borzooei, but also by T. Kehagias.

Algebraic hypertructures represent a field of algebra of major attraction and productive of many significant results in algebra. There are a lot of topics about hyperstructures, which can be in depth analyzed and there also are open problems and new connections to other fields that can be explored more in the future.

2.2 A historical development of algebraic hyperstructures

The hypergroup notion was introduced in 1934 by a French mathematician F. Marty [79], at the 8th Congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions, non commutative groups.

Around the 40's, the general aspects of the theory, the connections with groups and various applications in geometry were studied in France by F. Marty, M. Krasner, M. Kuntzmann, R. Croisot, in U.S.A. by M. Dresher, O. Ore, W. Prenowitz, H.S. Wall, J.E. Eaton, H. Campaigne, L. Griffiths, in Russia by A. Dietzman, A. Vikhrov, in Italy by G. Zappa, in Japan by Y. Utumi.

Over the following two decades, other interesting results on hyperstructures were obtained, for instance, in Italy, A. Orsatti studied semiregular hypergroups, in Czechoslovakia, K. Drbohlav studied hypergroups of two sided classes, in Romania, M. Benado studied hyperlattices.

The theory knew an important progress starting with the 70's, when its research area enlarged. In France, M. Krasner, M. Koskas and Y. Sureau investigated the theory of subhypergroups and the relations defined on hyperstructures; in Greece, J. Mittas, and his students M. Konstantinidou, K. Serafimidis, S. Ioulidis and C.N. Yatras studied the canonical hypergroups, the hyperrings, the hyperlattices, Ch. Massouros obtained important results about hyperfields and other hyperstructures. G. Massouros, together with J. Mittas studied applications of hyperstructures to Automata. D. Stratigopoulos continued some of Krasner ideas, studying in depth noncommutative hyperrings and hypermodules. T. Vougiouklis, L. Konguetsof and later S. Spartalis, A. Dramalidis analyzed especially the cyclic hypergroups, the P-hyperstructures and the representation of the H_v-structures.

Significant contributions to the study of regular hypergroups, complete hypergroups, of the heart and of the hypergroup homomorphisms in general or with applications in Combinatorics and Geometry were brought by the Italian mathematician P. Corsini and his group of research, among whom we mention M.de Salvo, R. Migliorato, F. de Maria, G. Romeo, P. Bonansinga.

Also around 70's, some connections between hyperstructures and ordered systems, particularly lattices, were established by T. Nakano and J.C. Varlet. Around the 80's and 90's, associativity semyhypergroups were analyzed in the context of semigroup theory by T. Kepka and then by J. Jezec, P. Nemec and K. Drbohlav, and in Finland by M. Niemenmaa.

In U.S.A., R. Roth used canonical hypergroups in solving some problems of character theory of finite groups, while S. Comer studied the connections among hypergroups, combinatorics and the relation theory. J. Jantosciak continued the study of join spaces, introduced by W. Prenowitz, he considered a generalization of them for the noncommutative case and studied correspondences between homomorphisms and the associated relations.

In America, hyperstructures have been studied both in U.S.A. (at Charleston, South Carolina – The Citadel, New York-Brooklyn College, CUNY, Cleveland, Ohio – John Carroll University) and in Canada (at Université de Montréal).

A big role in spreading this theory is played by the Congresses on Algebraic Hyperstructures and their Applications.

The first three Congresses were organized by P. Corsini in Italy. The contribution of P. Corsini in the development of Hyperstructure Theory has been decisive. He has delivered lectures about hyperstructures and their applications in several countries, several times, for instance in Romania, Thailand, Iran, China, Montenegro, making known this theory. After his visits in these countries, hyperstructures have had a substantial development.

Coming back to the Congresses on Algebraic Hyperstructures, the first two were organized in Taormina, Sicily, in 1978 and 1983, with the names: "Sistemi Binari e loro Applicazioni" and "Ipergruppi, Strutture Multivoche e Algebrizzazione di Strutture d'Incidenza". The third Congress, called "Ipergruppi, altre Strutture Multivoche e loro Applicazioni" was organized in Udine in 1985.

The fourth congress, organized by T. Vougiouklis in Xanthi in 1990, used already the name of Algebraic Congress on Hyperstructures and their Applications, also known as AHA Congress. After 1990, AHA Congresses have been organized every three years. Beginning with the 90's Hyperstructure Theory represents a constant concern also for the Romanian mathematicians, the decisive moment being the fifth AHA Congress, organized in 1993 at the University "Al.I.Cuza" of Iasi by M. Stefanescu. This domain of the modern algebra is a topic of a great interest also for the Romanian researches, who have published a lot of papers on hyperstructures in national or international journals, have given communications in conferences and congresses, have written Ph.D. theses in this field.

The sixth AHA Congress was organized in 1996 at the Agriculture University of Prague by T. Kepka and P. Nemec, the seventh was organized in 1999 by R. Migliorato in Taormina, Sicily, then the eighth was organized in 2002 by T. Vougiouklis in Samothraki, Greece. All these congresses were organized in Europe. Nowadays, one works successfully on Hyperstructures in the following countries of Europe:

- in Greece, at Thessaloniki (Aristotle University), at Alexandropoulis (Democritus University of Thrace), at Patras (Patras University), Orestiada (Democritus University of Thrace), at Athens;
- in Italy at Udine University, at Messina University, at Rome (Universita' "La Sapienza"), at Pescara (D'Annunzio University), at Teramo (Universita' di Teramo), Palermo University;
- in Romania, at Iasi ("Al.I. Cuza" University), Cluj ("Babes-Bolyai" University), Constanta ("Ovidius" University);
- in Czech Republic, at Praha (Charles University, Agriculture University), at Brno (Brno University of Technology, Military Academy of Brno, Masaryk University), Olomouc (Palacky University);

- in Montenegro, at Podgorica University.

Let us continue with the following AHA Congresses.

The ninth congress on hyperstructures, organized in 2005 by R. Ameri in Babolsar, Iran, was the first of this kind in Asia. In the past millenniums, Iran gave fundamental contributions to Mathematics and in particular, to Algebra (for instance Khwarizmi, Kashi, Khayyam and recently Zadeh), many scientists have well understood the importance of hyperstructures, on the theoretical point of view and for the applications to a wide variety of scientific sectors.

Nowadays, hyperstructures are cultivated in many universities and research centers in Iran, among which we mention Yazd University, Shahid Bahonar University of Kerman, Mazandaran University, Kashan University, Ferdowsi University of Mashhad, Tehran University, Tarbiat Modarres University, Zahedan (Sistan and Baluchestan University), Semnan University, Islasmic Azad University of Kerman, Shahid Beheshti University of Tehran, Center for Theoretical Physics and Mathematics of Tehran, Zanjan (Institute for Advanced Studies in Basic Sciences). Iranian mathematicians have especially studied hyperstructures in connections with Fuzzy Sets and Rough Sets.

Another Asian country where hyperstructures have had success is Thailand. In Chulalornkorn University of Bangkok, important results have been obtained by Y. Kemprasit and her students Y. Punkla, S. Chaoprakhoi, N. Triphop, C. Namnak on the connections among hyperstructures, semigroups and rings.

There are other Asia centers for researches in hyperstructures. We mention here India (University of Calcutta, Aditanar College of Arts and Sciences, Tiruchendur, Tamil Nadu), Korea (Chiungju National University, Chiungju National University of Education, Gyeongsang National University, Jinju), Japan (Hitotsubashi University of Tokyo), Sultanate of Oman (Education College for Teachers), China (Northwest University of Xian, Yunnan University of Kunming).

Hypertructures have been also cultivated in Germany, Netherlands, Belgium, Macedonia, Serbia, Slovakia, Spain, Uzbekistan, Australia. The tenth AHA Congress was held in in Brno, Czech Republic in the autumn of 2008. It was organized by Šárka. Hošková, at the Military Academy of Brno.

More than 700 papers and some books have been written till now on hyperstructures. Many of them are dedicated to the applications of hyperstructures in other topics. We shall mention here some of the fields connected with hyperstructures and only some names of mathematicians who have worked in each topic:

- *Geometry* (W. Prenowitz, J. Jantosciak, and later G. Tallini),
- *Codes* (G. Tallini),
- *Cryptography and Probability* (L. Berardi, F. Eugeni, S. Innamorati, A. Maturo),
- *Automata* (G. Massouros, J. Chvalina, L. Chvalinova),
- *Artificial Intelligence* (G. Ligozat),
- *Median Algebras, Relation Algebras*, C-algebras (S. Comer),
- *Boolean Algebras* (A.R. Ashrafi, M. Konstantinidou),
- *Categories* (M. Scafati, M.M. Zahedi, C. Pelea, R. Bayon, N. Ligeros, S.N. Hosseini, B. Davvaz, M.R. Khosharadi-Zadeh),
- *Topology* (J. Mittas, M. Konstantinidou, M.M. Zahedi, R. Ameri, S. Hošková),
- *Binary Relations* (J. Chvalina, I.G. Rosenberg, P. Corsini, V. Leoreanu, B. Davvaz, S. Spartalis, I. Chajda, S. Hošková, I. Cristea, M. De Salvo, G. Lo Faro),
- *Graphs and Hypergraphs* (P. Corsini, I.G. Rosenberg, V. Leoreanu, M. Gionfriddo, A. Iranmanesh, M.R. Khosharadi-Zadeh),
- *Lattices and Hyperlattices* (J.C. Varlet, T. Nakano, J. Mittas, A. Kehagias, M. Konstantinidou, K. Serafimidis, V. Leoreanu, I.G. Rosenberg, B. Davvaz, G. Calugareanu, G. Radu, A.R. Ashrafi),

- *Fuzzy Sets and Rough Sets* (P. Corsini, M.M. Zahedi, B. Davvaz, R. Ameri, R.A. Borzooei, V. Leoreanu, I. Cristea, A. Kehagias, A. Hasankhani, I. Tofan, C. Volf, G.A. Moghani, H. Hedayati),
- *Intuitionistic Fuzzy Hyperalgebras* (B. Davvaz, R.A. Borzooei, Y.B. Jun, W.A. Dudek, L. Torkzadeh),
- *Generalized Dynamical Systems* (M.R. Molaei) and so on.

Another topic which has aroused the interest of several mathematicians, is that one of H_v-structures, introduced by T. Vougiouklis and studied then also by B. Davvaz, M.R. Darafsheh, M. Ghadiri, R. Migliorato, S. Spartalis, A. Dramalidis, A. Iranmanesh, M.N. Iradmusa, A. Madanshekaf. H_v-structures are a special kind of hyperstructures, for which the weak associativity holds.

Recently, n-ary hyperstructures, introduced by B. Davvaz and T. Vougiouklis, represent an intensively studied field of research.

Therefore, there are good reasons to hope that Hyperstructure Theory will be one of the more successful fields of research in algebra.

2.3 The hypergroup of Marty

Now, it is time to present the hypergroup notion, introduced by Marty [79].

Definition 2.3.1. Let H be a nonempty set and $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ be a hyperoperation. The couple $(H, \circ)$ is called a *hypergroupoid.*

For any two nonempty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b, \quad A \circ x = A \circ \{x\} \quad \text{and} \quad x \circ B = \{x\} \circ B.$$

Definition 2.3.2. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all a, b, c of H we have $(a \circ b) \circ c = a \circ (b \circ c)$, which means that

$$\bigcup_{u\in a\circ b} u \circ c = \bigcup_{v\in b\circ c} a \circ v.$$

Definition 2.3.3. A hypergroupoid $(H, \circ)$ is called a *quasihypergroup* if for all a of H we have $a \circ H = H \circ a = H$.

The above condition is also called the *reproduction axiom.*

Definition 2.3.4. A hypergroupoid $(H, \circ)$ which is both a semihypergroup and a quasihypergroup is called a *hypergroup.*

Now, we look at some examples of hypergroups.

Example 2.3.5.

(1) If H is a nonempty set and for all x, y of H, we define $x \circ y = H$, then $(H, \circ)$ is a hypergroup, called the *total hypergroup.*

(2) Let $(S, \cdot)$ be a semigroup and let P be a nonempty subset of S. For all x, y of S, we define $x \circ y = xPy$. Then $(S, \circ)$ is a semihypergroup. If $(S, \cdot)$ is a group, then $(S, \circ)$ is a hypergroup, called a *P-hypergroup.*

(3) If G is a group and for all x, y of G, $< x, y >$ denotes the subgroup generated by x and y, then we define $x \circ y =< x, y >$. We obtain that $(G, \circ)$ is a hypergroup.

(4) If $(G, \cdot)$ is a group, H is a normal subgroup of G and for all x, y of G, we define $x \circ y = xyH$, then $(G, \circ)$ is a hypergroup.

(5) Let $(G, \cdot)$ be a group and let H be a non-normal subgroup of it. If we denote $G/H = \{xH \mid x \in G\}$, then $(G/H, \circ)$ is a hypergroup, where for all xH, yH of G/H, we have $xH \circ yH = \{zH \mid z \in xHy\}$.

(6) If $(G, +)$ is an abelian group, ρ is an equivalence relation in G, which has classes $\overline{x} = \{x, -x\}$, then for all $\overline{x}, \overline{y}$ of G/ρ, we define $\overline{x} \circ \overline{y} = \{\overline{x+y}, \overline{x-y}\}$. We obtain that $(G/\rho, \circ)$ is a hypergroup.

(7) Let D be an integral domain and let F be its field of fractions. If we denote by U the group of the invertible elements of D, then we define the following hyperoperation on F/U: for all $\overline{x}, \overline{y}$ of F/U, we have $\overline{x} \circ \overline{y} = \{\overline{z} \mid \exists (u, v) \in U^2$ such that $z = ux + vy\}$. We obtain that $(F/U, \circ)$ is a hypergroup.

(8) [113] Let R be a binary relation on a nonempty set H. An element x of H is called an *outer element* of R if there exists $h \in H$ such that $(h, x) \notin R^2$ and an *inner element* of R otherwise. We define the following hyperoperation on H: for all x, y of H, we have $x \circ y = \{z \in H \mid (x, z) \in R \text{ or } (y, z) \in R\}$. The hypergroupoid $(H, \circ)$ is a hypergroup if and only if the following conditions hold: R has full domain and full range, $R \subsetneq R^2$ and $(a, x) \in R^2$ implies $(a, x) \in R$, whenever x is an outer element of R.

(9) [23] Let $\Gamma = (H, \{A_i\}_{i\in I})$ be a hypergraph, i.e., $A_i \in \mathcal{P}^*(H)$ for all $i \in I$ and $\bigcup\limits_{i\in I} A_i = H$. Set $E(x) = \bigcup\limits_{x\in A_i} A_i$. We define the following hyperoperation on H: For all x, y of H, we have $x \circ y = E(x) \cup E(y)$. The hypergroupoid $(H, \circ)$ is a hypergroup if and only if for all x, y of H, we have $x \circ x \circ x \setminus x \circ x \subseteq y \circ y \circ y$.

(10) Let $(L, \wedge, \vee)$ be a lattice with a minimum element 0. If for all $a \in L$, $F(a)$ denotes the principal filter generated from a, then we obtain a hypergroup $(L, \circ)$, where for all a, b of L, we have $a \circ b = F(a \wedge b)$.

(11) [17] Let $(L, \wedge, \vee)$ be a modular lattice. If for all x, y of L, we define $x \circ y = \{z \in L \mid z \vee x = x \vee y = y \vee z\}$, then $(L, \circ)$ is a hypergroup.

(12) [130] Let $(L, \wedge, \vee)$ be a distributive lattice. If for all x, y of L, we define $x \circ y = \{z \in L \mid x \wedge y \leq z \leq x \vee y\}$, then $(L, \circ)$ is a hypergroup.

(13) [24] Let H be a nonempty set and $\mu : H \longrightarrow [0, 1]$ be a function. If for all x, y of H we define $x \circ y = \{z \in L \mid \mu(x) \wedge \mu(y) \leq \mu(z) \leq \mu(x) \vee \mu(y)\}$, then $(H, \circ)$ is a hypergroup.

(14) [25] Let H be a nonempty set and R be an equivalence relation in H, such that for all x of H, the equivalence class $R(x)$ of x has at least three elements. For any subset A of H, $\overline{R}(A)$ denotes the set $\bigcup\limits_{R(x)\cap A\neq\emptyset} R(x)$, while $\underline{R}(A)$ denotes the set $\bigcup\limits_{R(x)\subseteq A} R(x)$. The couple $(\underline{R}(A), \overline{R}(A))$ is called a *rough set*. If for all x, y of H, we define $x \circ y = \overline{R}(\{x, y\}) \setminus \underline{R}(\{x, y\})$, then $(H, \circ)$ is a hypergroup.

(15) [24] Let H be a nonempty set and μ, λ be two functions from H to $[0,1]$. For all x, y of H we define $x \circ y = \{u \in H \mid \mu(x) \wedge \lambda(x) \wedge \mu(y) \wedge \lambda(y) \leq \mu(u) \wedge \lambda(u)$ and $\mu(u) \vee \lambda(u) \leq \mu(x) \vee \lambda(x) \vee \mu(y) \vee \lambda(y)\}$. The hyperstructure $(H, \circ)$ is a commutative hypergroup.

(16) Define the following hyperoperation on the real set $\mathbb{R}$: for all $x \in R$, $x \circ x = x$ and for all different real elements x, y, $x \circ y$ is the open interval between x and y. Then $(R, \circ)$ is a hypergroup.

Some of the above examples are join spaces, which constitute an important class of hypergroups. Join spaces have been introduced by W. Prenowitz [103] and used by him and J. Jantosciak [104] to rebuild several branches of geometry.

Remark 2.3.6. A hypergroup for which the hyperproduct of any two elements has exactly one element is a group. Indeed, let $(H, \circ)$ be a hypergroup, such that for all x, y of H, we have $|x \circ y| = 1$. Then $(H, \circ)$ is a semigroup, such that for all a, b in H, there exist x and y for which we have $a = b \circ x$ and $a = y \circ b$. It follows that $(H, \circ)$ is a group.

And now, some words about subhypergroups.

Definition 2.3.7. A nonempty subset K of a semihypergroup $(H, \cdot)$ is called a *subsemihypergroup* if it is a semihypergroup.

In other words, a nonempty subset K of a semihypergroup $(H, \circ)$ is a subsemihypergroup if $K \circ K \subseteq K$.

Definition 2.3.8. A nonempty subset K of a hypergroup $(H, \circ)$ is called a *subhypergroup* if it is a hypergroup.

Hence, a nonempty subset K of a hypergroup $(H, \circ)$ is a subhypergroup if for all a of K we have $a \circ K = K \circ a = K$.

There are several kinds of subhypergroups. In what follows, we introduce closed, invertible, ultraclosed and conjugable subhypergroups and some connections among them.

Among the mathematicians who studied this topic, we mention F. Marty, M. Dresher, O. Ore, M. Krasner who analyzed closed and invertible subhypergroups. M. Koskas considered another type of subhypergroups, which are complete parts and that we present in Paragraph 2.5. Later Y. Sureau has studied ultraclosed, invertible and conjugable subhypergroups. Corsini has obtained important results about ultraclosed and complete parts. Also, Leoreanu has studied and obtained other interesting results on subhypergroups.

Let us present now the definition of these types of subhypergroups. Let $(H, \circ)$ be a hypergroup and $(K, \circ)$ be a subhypergroup of it.

Definition 2.3.9. We say that K is:

- *closed on the left* (*on the right*) if for all k_1, k_2 of K and x of H, from $k_1 \in x \circ k_2$ ($k_1 \in k_2 \circ x$, respectively), it follows that $x \in K$;
- *invertible on the left* (*on the right*) if for all x, y of H, from $x \in K \circ y$ ($x \in y \circ K$), it follows that $y \in K \circ x$ ($y \in x \circ K$, respectively);
- *ultraclosed on the left* (*on the right*) if for all x of H, we have $K \circ x \cap (H \backslash K) \circ x = \emptyset$ ($x \circ K \cap x \circ (H \backslash K) = \emptyset$);
- *conjugable on the right* if it is closed on the right and for all $x \in H$, there exists $x' \in H$ such that $x' \circ x \subseteq K$.

We say that K is *closed* (*invertible, ultraclosed, conjugable*) if it is closed (invertible, ultraclosed, conjugable respectively) on the left and on the right.

Example 2.3.10.

(1) Let $(A, \circ)$ be a hypergroup, $H = A \cup T$, where T is a set with at least three elements and $A \cap T = \emptyset$. We define the hyperoperation $\otimes$ on H, as follows:
    if $(x, y) \in A^2$, then $x \otimes y = x \circ y$;
    if $(x, t) \in A \times T$, then $x \otimes t = t \otimes x = t$;
    if $(t_1, t_2) \in T \times T$, then $t_1 \otimes t_2 = t_2 \otimes t_1 = (A \cup (T \setminus \{t_1, t_2\})$.
    Then $(H, \otimes)$ is a hypergroup and $(A, \otimes)$ is an ultraclosed, non-conjugable subhypergroup of H.

(2) Let $(A, \circ)$ be a total hypergroup, with at least two elements and let $T = \{t_i\}_{i \in \mathbb{N}}$ such that $A \cap T = \emptyset$ and $t_i \neq t_j$ for $i \neq j$. We define the hyperoperation $\otimes$ on $H = A \cup T$ as follows:

if $(x, y) \in A^2$, then $x \otimes y = A$;
if $(x, t) \in A \times T$, then $x \otimes t = t \otimes x = (A \setminus \{x\}) \cup T$;
if $(t_i, t_j) \in T \times T$, then $t_i \otimes t_j = t_j \otimes t_i = A \cup \{t_{i+j}\}$.

Then $(H, \otimes)$ is a hypergroup and $(A, \otimes)$ is a non-closed subhypergroup of H.

(3) Let us consider the group $(\mathbb{Z}, +)$ and the subgroups $S_i = 2^i\mathbb{Z}$, where $i \in \mathbb{N}$. For any $x \in \mathbb{Z} \setminus \{0\}$, there exists a unique integer $n(x)$, such that $x \in S_{n(x)} \setminus S_{n(x)+1}$. Define the following commutative hyperoperation on $\mathbb{Z} \setminus \{0\}$:
if $n(x) < n(y)$, then $x \circ y = x + S_{n(y)}$;
if $n(x) = n(y)$, then $x \circ y = S_{n(x)} \setminus \{0\}$;
if $n(x) > n(y)$, then $x \circ y = y + S_{n(x)}$.

Notice that if $n(x) < n(y)$, then $n(x + y) = n(x)$. Then $(\mathbb{Z} \setminus \{0\}, \circ)$ is a hypergroup and for all $i \in \mathbb{N}$, $(S_i \setminus \{0\}, \circ)$ is an invertible subhypergroup of $\mathbb{Z} \setminus \{0\}$.

Other examples can be found in [23].

Lemma 2.3.11. *A subhypergroup K is invertible on the right if and only if $\{x \circ K\}_{x \in H}$ is a partition of H.*

Proof. If K is invertible on the right and $z \in x \circ K \cap y \circ K$, then $x, y \in z \circ K$, whence $x \circ K \subseteq z \circ K$ and $y \circ K \subseteq z \circ K$. It follows that $x \circ K = z \circ K = y \circ K$. Conversely, if $\{x \circ K\}_{x \in H}$ is a partition of H and $x \in y \circ K$, then $x \circ K \subseteq y \circ K$, whence $x \circ K = y \circ K$ and so we have $x \in y \circ K = x \circ K$. Hence, for all x of H we have $x \in x \circ K$. From here, we obtain that $y \in y \circ K = x \circ K$. ■

The following theorems present some connections among the above types of subhypergroups.

If A and B are subsets of H such that we have $H = A \cup B$ and $A \cap B = \emptyset$, then we denote $H = A \oplus B$.

Theorem 2.3.12. *If a subypergroup K of a hypergroup $(H, \circ)$ is ultraclosed, then it is invertible.*

Proof. First we check that K is closed. For $x \in K$, we have $K \cap x \circ (H \setminus K) = \emptyset$ and from $H = x \circ K \cup x \circ (H \setminus K)$, we obtain $x \circ (H \setminus K) = H \setminus K$, which means that $K \circ (H \setminus K) = H \setminus K$. Similarly, we obtain $(H \setminus K) \circ K = H \setminus K$, hence K is closed. Now,we show that $\{x \circ K\}_{x \in H}$ is a partition of H. Let $y \in x \circ K \cap z \circ K$. It follows that $y \circ K \subseteq x \circ K$ and $y \circ (H \setminus K) \subseteq x \circ K \circ (H \setminus K) = x \circ (H \setminus K)$. From $H = x \circ K \oplus x \circ (H \setminus K) = y \circ K \oplus y \circ (H \setminus K)$, we obtain $x \circ K = y \circ K$. Similarly, we have $z \circ K = y \circ K$. Hence $\{x \circ K\}_{x \in H}$ is a partition of H, and according to the above lemma, it follows that K is invertible on the right. Similarly, we can show that K is invertible on the left. ∎

Theorem 2.3.13. *If a subhypergroup K of a hypergroup $(H, \circ)$ is invertible, then it is closed.*

Proof. Let $k_1, k_2 \in K$. If $k_1 \in x \circ k_2 \subseteq x \circ K$, then $x \in k_1 \circ K \subseteq K$. Similarly, from $k_1 \in k_2 \circ x$, we obtain $x \in K$. ∎

We denote the set $\{e \in H \mid \exists x \in H,$ such that $x \in x \circ e \cup e \circ x\}$ by I_p and we call it the *set of partial identities* of H.

Theorem 2.3.14. *A suhypergroup K of a hypergroup $(H, \circ)$ is ultraclosed if and only if K is closed and $I_p \subseteq K$.*

Proof. Suppose that K is closed and $I_p \subseteq K$. First, we show that K is invertible on the left. Suppose there are x, y of H such that $x \in K \circ y$ and $y \notin K \circ x$. Hence $y \in (H \setminus K) \circ x$, whence $x \in K \circ (H \setminus K) \circ x = (H \setminus K) \circ x$, since K is closed. We obtain that $I_p \cap (H \setminus K) \neq \emptyset$, which is a contradiction. Hence K is invertible on the left. Now, we check that K is ultraclosed on the left. Suppose there are a and x in H such that $a \in K \circ x \cap (H \setminus K) \circ x$. It follows that $x \in K \circ a$, since K is invertible on the left. We obtain $a \in (H \setminus K) \circ x \subseteq (H \setminus K) \circ K \circ a = (H \setminus K) \circ a$, since K is closed. This means that $I_p \cap (H \setminus K) \neq \emptyset$, which is a contradiction. Therefore K is ultraclosed on the left and similarly it is closed on the right.

Conversely, suppose that K is ultraclosed. According to Theorems 2.3.12 and 2.3.13, K is closed. Now, suppose that $I_p \cap (H \setminus K) \neq \emptyset$, which means that there is $e \in H \setminus K$ and there is $x \in H$, such that $x \in e \circ x$, for instance. We obtain $x \in (H \setminus K) \circ x$, whence $K \circ x \subseteq (H \setminus K) \circ x$, which contradicts that K is ultraclosed. Hence $I_p \subseteq K$. ∎

Theorem 2.3.15. *If a subypergroup K of a hypergroup $(H, \circ)$ is conjugable, then it is ultraclosed.*

Proof. Let $x \in H$. Denote $B = x \circ K \cap x \circ (H \setminus K)$. Since K is conjugable it follows that K is closed and there exists $x' \in H$, such that $x' \circ x \subseteq K$. We obtain

$$\begin{aligned} x' \circ B &= x' \circ (x \circ K \cap x \circ (H \setminus K)) \\ &\subseteq K \cap x' \circ x \circ (H \setminus K) \\ &\subseteq K \cap K \circ (H \setminus K) \\ &\subseteq K \cap (H \setminus K) = \emptyset. \end{aligned}$$

Hence $B = \emptyset$, which means that K is ultraclosed on the right. Similarly, we check that K is ultraclosed on the left. ■

Finally, we give some ideas about hypergroup homomorphisms. Several types of homomorphisms have been considered since the first papers on hypergroups (for instance, by M. Dresher, O. Ore, M. Krasner, J. Kuntzmann) and later by M. Koskas. However, the first explicit construction of hypergroup homomorphisms was given by P. Corsini [19]. A unified theory of various types of homomorphisms was given by J. Jantosciak [60]. Some other types of homomorphisms and connections among them were studied by V. Leoreanu [73]. There are more than 10 types of hypergroup homomorphisms. A detailed presentation of all these homomorphisms, connections between them and various examples can be found in [23].

We present here some more important types of homomorphisms.

Definition 2.3.16. Let $(H_1, \circ)$ and $(H_2, *)$ be two hypergroupoids. A map $f : H_1 \longrightarrow H_2$, is called

- a *homomorphism* if for all x, y of H, we have $f(x \circ y) \subseteq f(x) \circ f(y)$;
- a *good homomorphism* if for all x, y of H, we have $f(x \circ y) = f(x) * f(y)$;
- a *very good homomorphism* if it is good and for all x, y of H, we have $f(x/y) = f(x)/f(y)$ and $f(x \backslash y) = f(x) \backslash f(y)$, where $x/y = \{z \in H \mid x \in z \circ y\}$ and $x \backslash y = \{u \in H \mid y \in x \circ u\}$;
- an *isomorphism* if it is a homomorphism, and its inverse f^{-1} is a homomorphism, too.

Let us explain why the above definition is used for a hypergroup homomorphism. Define the following ternary relation on a hypergroupoid $(H, \circ)$:

$$(x, y, z) \in R \ \text{ if } \ z \in x \circ y.$$

Hence, we can associate a model (H, R) with any hypergroupoid $(H, \circ)$. According to the model theory, a homomorphism between two such models (H_1, R_1) and (H_2, R_2), is a map $f : H_1 \longrightarrow H_2$, such that if $(x, y, z) \in R_1$, then $(f(x), f(y), f(z)) \in R_2$. In other words, f is a *model homomorphism* if $z \in x \circ y$ implies $f(z) \in f(x) \circ f(y)$, which means that f is a model homomorphism if $f(x \circ y) \subseteq f(x) * f(y)$ for all x, y of H.

Example 2.3.17. [19] First, we consider the following hyperoperation defined on an abelian totally ordered group $(G, \cdot, \leq)$ as follows: $x \circ y = \{x + y, |x - y|\}$. We denote this hypergroup by $\mathcal{H}(G)$ and we call it the *sd-hypergroup*.

(1) The map

$$f : \mathcal{H}(\mathbb{Z}) \longrightarrow \mathcal{H}(\mathbb{Z}), \text{ defined by } f(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

is a homomorphism, but it is not a good homomorphism.

(2) Consider the group $\mathbb{Z} \times \mathbb{Z}$ ordered lexicographically. The map

$$f : \mathcal{H}(\mathbb{Z} \times \mathbb{Z}) \longrightarrow \mathcal{H}(\mathbb{Z} \times \mathbb{Z}), \text{ defined by } f(m, n) = |(2m - n, m - 2n)|$$

is a very good homomorphism.

Theorem 2.3.18. *A bijective homomorphism of semihypergroups is an isomorphism if and only if it is good.*

Proof. If $f : H_1 \longrightarrow H_2$ is an isomorphism of semihypergroups, then for all x, y of H_1, we have

$$\begin{aligned} f(f^{-1}(f(x) * f(y))) &= f(x) * f(y) \\ &= f(f^{-1}(f(x)) \circ f^{-1}(f(y))) \\ &= f(x \circ y). \end{aligned}$$

Conversely, suppose that f is good. For all $x'{=}f(x)$, $y'{=}f(y)$ of H_2, we have

$$\begin{aligned} f^{-1}(x' * y') &= f^{-1}(f(x) * f(y)) \\ &= f^{-1}(f(x \circ y)) \\ &= x \circ y \\ &= f^{-1}(x') \circ f^{-1}(y'). \ \blacksquare \end{aligned}$$

Definition 2.3.19. Let $(H_1, \circ)$ and $(H_2, *)$ be two hypergroupoids. A map $f : H_1 \longrightarrow H_2$ is called

- a *2-homomorphism* if for all x, y of H, we have
$f^{-1}(f(x) * f(y)) = f^{-1}(f(x \circ y))$;
- an *almost strong homomorphism* if for all x, y of H, we have
$f^{-1}(f(x) * f(y)) = f^{-1}(f(x)) \circ f^{-1}(f(y))$.

Theorem 2.3.20. *If $f : H_1 \longrightarrow H_2$ is a very good homomorphism, then f is both a 2-homomorphism and an almost strong homomorphism.*

Proof. Since f is a homomorphism, for all x, y of H_1, we have $f^{-1}(f(x)) \circ f^{-1}(f(y)) \subseteq f^{-1}(f(x) * f(y))$. Indeed, for all a, b of H_1, such that $f(a) = f(x)$ and $f(b) = f(y)$, we have $f(a \circ b) \subseteq f(a) * f(b) = f(x) * f(y)$. Hence $a \circ b \subseteq f^{-1}(f(a) * f(b))$, for all $a \in f^{-1}(f(x))$ and $b \in f^{-1}(f(y))$, which means that$f^{-1}(f(x)) \circ f^{-1}(f(y)) \subseteq f^{-1}(f(x) * f(y))$. Now, let $z \in f^{-1}(f(x) * f(y))$. We have $f(z) \in f(x) * f(y)$, whence $f(y) \in f(x) \backslash f(z) = f(x \backslash z)$. It follows that there exists $y' \in x \backslash z$, such that $f(y) = f(y')$. We obtain $z \in x \circ y' \subseteq f^{-1}(f(x)) \circ f^{-1}(f(y))$, and so $f^{-1}(f(x) * f(y)) \subseteq f^{-1}(f(x)) \circ f^{-1}(f(y))$. ■

Definition 2.3.21. Let $(H_1, \circ)$ and $(H_2, *)$ be two semihypergroups. A homomorphism $f : H_1 \longrightarrow H_2$ is called *strong on the left* if for all x, y of H, the following implication holds:

$$f(z) \in f(x) * f(y) \implies \exists x' \in H \text{ such that } f(x) = f(x'), \ z \in x' \circ y.$$

Similarly, we define a strong on the right homomorphism. A strong homomorphism is a strong on the left and right homomorphism.

Theorem 2.3.22. *Any strong homomorphism is almost strong.*

Proof. Let f be a strong homomorphism. Let $z \in f^{-1}(f(x) * f(y))$, which means that $f(z) \in f(x) * f(y)$. Hence there exists $x' \in H$ such that $f(x) = f(x')$, $z \in x' \circ y$, whence $z \in f^{-1}(f(x)) \circ y \subseteq f^{-1}(f(x)) \circ f^{-1}(f(y))$. Therefore, f is an almost strong homomorphism. ■

2.4 Join spaces, canonical hypergroups and polygroups

There are many classes of hypergroups, which have aroused a major interest. We mention here some of them: regular hypergroups, regular reversible hypergroups, canonical hypergroups, join spaces, polygroups, complete hypergroups, cambiste hypergroups, cogroups, associativity hypergroups, cyclic hypergroups, P-hypergroups, 1-hypergroups and others. In this section, we shall shortly present three classes of hypergroups: join spaces, canonical hypergroups, and polygroups.

Join spaces were introduced by W. Prenowitz and then applied by him and J. Jantosciak both in Euclidian and in non Euclidian geometry. Using this notion, several branches of non Euclidian geometry were rebuilt: descriptive geometry, projective geometry and spherical geometry. Then, several important examples of join spaces have been constructed in connection with binary relations, graphs, lattices, fuzzy sets, rough sets.

In order to define a join space, we need the following notation: If a, b are elements of a hypergroupoid $(H, \circ)$, then we denote $a/b = \{x \in H \mid a \in x \circ b\}$. Moreover, by A/B we intend the set $\bigcup\limits_{a\in A, b\in B} a/b$.

Definition 2.4.1. A commutative hypergroup $(H, \circ)$ is called a *join space* if the following condition holds for all elements a, b, c, d of H:

$$a/b \cap c/d \neq \emptyset \implies a \circ d \cap b \circ c \neq \emptyset.$$

Definition 2.4.2. A join space $(H, \circ)$ is called *geometric* if there exists $x \in H$ such that $x \circ x = \{x\} = x/x$.

Some important examples of join spaces were presented in the above paragraph (see Examples 2.3.5. (11), (12), (13), (14), (16)). We give here some other examples (see [23]).

Example 2.4.3.

(1) Let $(L, \wedge, \vee)$ be a distributive lattice. If for all a, b of L we define

$$a \circ b = \{x \in L \mid x = (a \wedge b) \vee (a \wedge x) \vee (b \wedge x)\},$$

then $(L, \circ)$ is a non geometrical join space in which every element is an identity. The above hyperoperation can be considered in a more general context, that one of a median semi-lattice. A *median semi-lattice* is a meet semi-lattice $(S, \wedge)$, such that the following conditions hold:

• Every principal ideal is a distributive lattice;

• Any three elements of S have an upper bound whenever each pair of them has an upper bound.

(2) Let V be a vector space over an ordered field F. If for all a, b of V we define

$$a \circ b = \{\lambda a + \mu b \mid \lambda > 0, \mu > 0,\ \lambda + \mu = 1\},$$

then $(V, \circ)$ is a join space, called an *affine join space* over F.

(3) Let $G = (V, E)$ be a connected simple graph. We say that a subset A of V is convex if for all different elements a, b of A, we have that A contains all points on all geodetics from a to b. Denote by (a, b) the least convex set containing $\{a, b\}$. A convex set P is called *prime* if $V \setminus P$ is convex. Finally, G is called a *strong prime convex intersection graph* if:

• For any convex set A and any point x, which does not belong to A, there exists a prime convex set P, such that $A \subseteq P$, $x \in V \setminus P$;

• For any (a, b), (c, d) such that $(a, b) \cap (c, d) = \emptyset$, there exists a convex prime set P such that $(a, b) \subseteq P$ and $(c, d) \subseteq V \setminus P$.

If G satisfies the above two conditions and for all different elements a, b of V we define $a \circ b = (a, b)$ and $a \circ a = a$, then $(V, \circ)$ is a join space.

(4) Denote by $)a, b($ an open real interval. We define the following hyperoperation on the Cartesian plane $\mathbb{R}^2$: for all different elements $(x_1, x_2), (y_1, y_2)$ of $\mathbb{R}^2$, we have $(x_1, x_2) \circ (y_1, y_2) = \{(z_1, z_2) \mid z_1 \in)x_1, x_2($ and $z_2 \in)x_2, y_2(\}$ and for all element (x_1, x_2) of $\mathbb{R}^2$, we have $(x_1, x_2) \circ (x_1, x_2) = (x_1, x_2)$. Then $(\mathbb{R}^2, \circ)$ is a geometric join space, not provided with identity elements.

(5) Let $G = (V, E)$ be a connected simple graph. We define the following hyperoperation on V: for all different elements x, y of V, we have $x \circ x = x$ and $x \circ y$ is the set of all points $z \in V$, which belong to some paths $\gamma : x - y$. Then $(V, \circ)$ is a non-geometric join space in which every element is an identity.

If N is a closed subhypergroup of a join space H and $\{x, y\} \subseteq H$, then we define the following binary relation: xJ_Ny if $x \circ N \cap y \circ N \neq \emptyset$.

Theorem 2.4.4. *J_N is an equivalence relation on H and the equivalence class of an element a is $J_N(a) = (a \circ N)/N$. In particular, $J_N(a) = N$ for all $a \in N$.*

Proof. Clearly, J_N is reflexive and symmetric. Now, suppose that $a \circ N \cap b \circ N \neq \emptyset$ and $b \circ N \cap c \circ N \neq \emptyset$. It follows that $b \in (a \circ N)/N \cap (c \circ N)/N$ and since $(H, \circ)$ is a join space, we obtain $a \circ N \cap c \circ N \neq \emptyset$, which means that aJ_Nc. Hence, J_N is also transitive, and so it is an equivalence relation on H. We check now that for all $a \in H$ we have $J_N(a) = (a \circ N)/N$. If $d \in J_N(a)$, then $d \circ N \cap a \circ N \neq \emptyset$, hence there exist $v \in a \circ N$ and $m \in N$ such that $v \in d \circ m$, whence it follows that $d \in v/m \subseteq (a \circ N)/N$. We obtain $J_N(a) \subseteq (a \circ N)/N$. Now, let $y \in (a \circ N)/N$. Then there exist $u \in a \circ N$ and $m \in N$, such that $u \in y \circ m$, whence $y \circ N \cap a \circ N \neq \emptyset$, which means that $y \in J_N(a)$ and so, $(a \circ N)/N \subseteq J_N(a)$. Clearly, if $a \in N$, then $J_N(a) = N$, since N is closed. ■

Canonical hypergroups are a particular case of join spaces. The structure of canonical hypergroups was individualized for the first time by M. Krasner as the additive structure of hyperfields. In 1970, J. Mittas was the first who studied them independently from the other operations. In 1973, P. Corsini analyzed the sd-hypergroups, which are a particular type of canonical hypergroups and in 1975 Roth used canonical hypergroups in the character theory of finite groups. W. Prenowitz and J. Jantosciak emphasized the role of canonical hypergroups in geometry, while J.R. McMullen and J.F. Price underlined the role of a generalization of canonical hypergroups in harmonic analysis and particle physics. Some connected hyperstructures with canonical hypergroups were introduced and analyzed by P. Corsini, P. Bonansinga, K. Serafimidis, M. Kostantinidou, J. Mittas, De Salvo. We mention here some of them: strongly canonical, i.p.s. hypergroups, quasi-canonical hypergroups (also called polygroups), feebly (quasi)canonical hypergroups.

Let us see now what a canonical hypergroup is.

Definition 2.4.5. We say that a hypergroup $(H, \circ)$ is *canonical* if

(1) it is commutative,

(2) it has a scalar identity (also called scalar unit), which means that

$$\exists e \in H,\ \forall x \in H,\ x \circ e = e \circ x = x,$$

(3) every element has a unique inverse, which means that for all $x \in H$, there exists a unique $x^{-1} \in H$, such that $e \in x \circ x^{-1} \cap x^{-1} \circ x$,

(4) it is reversible, which means that if $x \in y \circ z$, then there exist the inverses y^{-1} of y and z^{-1} of z, such that $z \in y^{-1} \circ x$ and $y \in x \circ z^{-1}$.

Clearly, the identity of a canonical hypergroup is unique. Indeed, if e is a scalar identity and e' is an identity of a canonical hypergroup $(H, \circ)$, then we have $e \in e \circ e' = \{e'\}$.

Some interesting examples of a canonical hypergroup is the following ones (see [23]).

Example 2.4.6.

(1) Let $C(n) = \{e_0, e_1, ..., e_{k(n)}\}$, where $k(n) = n/2$ if n is an even natural number and $k(n) = (n-1)/2$ if n is an odd natural number. For all e_s, e_t of $C(n)$, define $e_s \circ e_t = \{e_p, e_v\}$,where $p = \min\{s+t, n-(s+t)\}$, $v = |s-t|$. Then $(C(n), \circ)$ is a canonical hypergroup.

(2) Let (S, T) be a projective geometry, i.e., a system involving a set S of elements called *points* and a set T of sets of points called *lines*, which satisfies the following postulates:

• Any lines contains at least three points;

• Two distinct points a, b are contained in a unique line, that we shall denote by $L(a, b)$;

• If a, b, c, d are distinct points and $L(a, b) \cap L(c, d) \neq \emptyset$, then $L(a, c) \cap L(b, d) \neq \emptyset$.

Let e be an element which does not belong to S and let $S' = S \cup \{e\}$. We define the following hyperoperation on S':

• For all different points a, b of S, we consider $a \circ b = L(a, b) \setminus \{a, b\}$;

• If $a \in S$ and any line contains exactly three points, let $a \circ a=\{e\}$, otherwise $a \circ a=\{a, e\}$;

• For all $a \in S'$, we have $e \circ a = a \circ e = a$.

Then $(S', \circ)$ is a canonical hypergroup.

In what follows, we present some basic results of canonical hypergroups.

Theorem 2.4.7. *If $(H, \circ)$ is a canonical hypergroup, then the following implication holds for all x, y, z, t of H:*

$$x \circ y \cap z \circ t \neq \emptyset \implies x \circ z^{-1} \cap t \circ y^{-1} \neq \emptyset.$$

Proof. Let $u \in x \circ y \cap z \circ t$. Since H is reversible, we obtain $u^{-1} \in z^{-1} \circ t^{-1}$, whence $u \circ u^{-1} \subseteq x \circ y \circ z^{-1} \circ t^{-1}$. If e is an identity of H, we obtain $e \in (x \circ z^{-1}) \circ (t \circ y^{-1})^{-1}$. Hence there exists an element $v \in x \circ z^{-1} \cap t \circ y^{-1}$. ■

Theorem 2.4.8. *A commutative hypergroup is canonical if and only if it is a join space with a scalar identity.*

Proof. Suppose that $(H, \circ)$ is a canonical hypergroup. For all a, b of H we have $a/b = a \circ b^{-1}$. Then the implication $\Longrightarrow$ follows by the above theorem.

Conversely, let us check that the inverse of an element is unique. Let e be the scalar identity. If $e \in a \circ b \cap a \circ c$, then $a \in e/b \cap e/c$, whence it follows that $e \circ c \cap e \circ b \neq \emptyset$, hence $b = c = a^{-1}$. Let us check now the reversibility of H. We have $a \in b \circ c$ if and only if $b \in a/c$. From $e \in b \circ b^{-1}$ we obtain $b \in e/b^{-1}$, hence $a \circ b^{-1} \cap e \circ c \neq \emptyset$, which means that $c \in a \circ b^{-1}$. Therefore, H is canonical. ■

Theorem 2.4.9. *If $(H, \circ)$ is a join space and N is a closed subhypergroup of H, then the quotient $(H/J_N, \otimes)$ is a canonical hypergroup, where for all $\overline{a}, \overline{b}$ of H/J_N, we have $\overline{a} \otimes \overline{b} = \{\overline{c} \mid c \in a \circ b\}$.*

Proof. First, we check that the hyperoperation $\otimes$ is well defined. In other words, we have to check that if $a_1 J_N a_2$ and $x \in H$, then for all $z \in a_1 \circ x$, there exists $w \in a_2 \circ x$, such that $z J_N w$. Indeed, from $a_1 J_N a_2$, it follows there exist m, n of N and v of H, such that $v \in a_1 \circ m \cap a_2 \circ n$. If $z \in a_1 \circ x$, then we have $a_1 \in z/x \cap v/m$, hence $z \circ N \cap v \circ x \neq \emptyset$, whence $z \circ N \cap N \circ a_2 \circ x \neq \emptyset$. It follows that there exists $w \in a_2 \circ x$, such that

zJ_Nw. Therefore the hyperoperation $\otimes$ is well defined. Since $(H, \circ)$ is a join space, it follows that $(H/J_N, \otimes)$ is a join space, too. Moreover, notice that N is a scalar identity for $(H/J_N, \otimes)$, and according to the above theorem, we obtain that $(H/J_N, \otimes)$ is a canonical hypergroup. ∎

Definition 2.4.10. A subhypergroup K of a canonical hypergroup $(H, \circ)$ is called a *canonical subhypergroup* of H, if it is a canonical hypergroup with respect to the hyperoperation $\circ$ of H.

Theorem 2.4.11. *Let K be a commutative hypergroup, such that the following conditions hold for all x, y, z of K:*

(1) $(x/y) \circ z = (x \circ z)/y$,

(2) $x \in (y/y) \circ x$,

(3) $x/(y/z) \subseteq (x \circ z)/y$.

Then there exists a canonical hypergroup H such that K is a noncanonical subhypergroup of H.

Proof. Let K' be a set, such that $K \cap K' = \emptyset$ and there exists a bijective map $f : K \longrightarrow K'$. For all x of K, we denote $f(x) = x'$. If $T \subseteq K$, then we shall write T' instead of $f(T)$. For all x, y of K, we define $x' \otimes y' = f(x \circ y)$. Then $(K', \otimes)$ and $(K, \circ)$ are isomorphic hypergroups. Let e be an element, such that $e \notin K \cup K'$. We define a canonical hypergroup structure on the set $H = K \cup K' \cup \{e\}$, such that K and K' are noncanonical subhypergroups of H. For all x, y of H, we denote the hyperproduct of x and y in H by $(x \circ y)_H$. Similarly, for all x, y in K and all x', y' in K', we denote $(x \circ y)_K$ and $(x' \circ y')_{K'}$ respectively. Moreover, for all x, y in H (in K, in K') we denote the set $\{z \in H\ (z \in K,\ z \in K') \mid x \in z \circ y\}$ by $(x/y)_H$ ($(x/y)_K$, $(x/y)_{K'}$ respectively). For all different elements x, y of K, we consider $(x \circ y')_H = (y' \circ x)_H = (x/y)_K \cup (y'/x')_{K'}$ and $(x \circ x')_H = (x' \circ x)_H = (x/x)_K \cup (x'/x')_{K'} \cup \{e\}$. We have $(x \circ y)_H = (x \circ y)_K$, $(x' \circ y')_H = x' \otimes y'$. It follows that e is a scalar identity in H and for all x of K, x' is the inverse of x. In what follows, we consider the steps:

(1) For all x, y of K we have $(x/y')_K = (x/y')_H$.
Indeed, $(x/y')_H = \{h \in H \mid x \in h \circ y'\} = \{k \in K \mid x \in k \circ y'\} \cup \{k' \in K' \mid x \in k' \circ y'\}$ and we use then that $K \cap K' = \emptyset$.

(2) For all x, y of K we have $x \circ y = (x/y')_K$.
Indeed, $(x/y')_K = \{k \in K \mid x \in k \circ y'\} = \{k \in K \mid x \in k/y\} = x \circ y$.
From (1), (2) and according to the definition, we obtain

(3) For all x, y of K we have $(x \circ y)_H = (x/y')_H$.

Now, let us check that $(H, \circ)$ is a quasihypergroup. Let x and y be elements of H. If x and y are both in K or both in K', then there exists $z \in H$, such that $x \in y \circ z$, since K and K' are hypergroups. Suppose now that x is arbitrary of K and y' is arbitrary in K'. Let $v' \in y' \circ x'$. Then $y' \in (v'/x')_{K'} \subseteq v' \circ x$, whence $K' \subseteq H \circ x$. Moreover, if $w \in x \circ y$, then $x \in w/y \subseteq w \circ y'$ and so, $x \in H \circ y'$.

In order to check that $(H, \circ)$ is a canonical hypergroup, we still have to verify: the associativity of $\circ$ and the fact that H is reversible.

Let $(x, y', z') \in K \times K' \times K'$.

(a) Suppose $x \neq y$.

(i) Let $x' \notin y' \circ z'$. We have $x \circ (y' \circ z') = (\bigcup\limits_{h' \in y' \circ z'} h'/x') \cup (\bigcup\limits_{h \in y \circ z} x/h)$.
Denote by B the right side of the above equality. On the other hand, $(x \circ y') \circ z' = ((y'/x')_{K'} \circ z') \cup ((x/y)_K \circ z')$. But $((y'/x')_{K'} \circ z') = ((y'/x')_{K'} \circ z')_{K'} = ((y' \circ z')/x')_{K'}$. Moreover, $((x/y)_K \circ z')_H = ((x/y)_K/z)_K \cup (z'/(x/y)'_{K'})_{K'}$. For all x, y of K we have $x \in h \circ y$ if and only if $x' \in h' \circ y'$ and then $(x/y)'_{K'} = (x'/y')_{K'}$. Notice that if H is a hypergroup with identity, such that each element has a unique inverse, then is reversible on the right if and only if for all x, y of H, we have $x \circ y' = x/y$. On the other hand for all x, y, z of a commutative hypergroup H, we have $(x/y)/z = x/(y \circ z)$. Hence, we obtain $((x/y) \circ z')_H = (x/(y \circ z))_K \cup (z'/(x'/y')_{K'})_{K'}$. According to condition (3) of the statement, the last term is contained in $((z' \circ y')/x')_{K'}$.

(ii) If $x' \in y' \circ z'$ then $x \circ (y' \circ z') = B \cup \{e\}$. We have the equivalences $x' \in y' \circ z' \iff x \in y \circ z \iff z \in x/y$ and so $e \in (x \circ y') \circ z'$.

(b) Now, suppose that $x = y$.

(i) If $x' \notin x' \circ z'$, then we have: $(x \circ x') \circ z' = ((x/x)_K \circ z')_H \cup ((x'/x')_{K'} \circ z') \cup (e \circ z')$. Moreover, $x \circ (x' \circ z') = (\bigcup\limits_{h' \in x' \circ z'} (h'/x')_{K'}) \cup (\bigcup\limits_{h \in x \circ z} (x/h)_K)$, but

$((x'/x')_{K'} \circ z')_H = ((x'/x')_{K'} \circ z')_{K'} = ((x' \circ z')/x')_{K'}$. On the other hand,

$$\begin{aligned}((x/x)_K \circ z')_H &= ((x/x)_K/z)_K \cup (z'/(x'/x')_{K'})_{K'} \\ &= (x/(x \circ z))_K \cup (z'/(x'/x')_{K'})_{K'}.\end{aligned}$$

According to condition (3) of the statement, we have $(z'/(x'/x')_{K'})_{K'} \subseteq ((x' \circ z')/x')_{K'}$, so we have to show only that $z' \in x \circ (x' \circ z')$. By conditions (1) and (2) of the statement, we have $z' \in ((x'/x')_{K'} \circ z')_{K'} = ((x' \circ z')/x')_{K'} \subseteq x \circ (x' \circ z')$.

(ii) If $x' \in x' \circ z'$, then we have $e \in x \circ (x' \circ z')$. Since $x \in x \circ z$, it follows that $z \in x/x$ and so $e \in (x \circ x') \circ z'$.

Now, we consider $(x, y', z) \in K \times K' \times K$.

(c) Suppose $x \neq y \neq z$.

(i) If $x' \notin y' \circ z$, then we obtain $x \circ (y' \circ z) = ((y'/z')_{K'} \circ x)_H \cup ((z/y)_K \circ x)_K$ and $(x \circ y') \circ z = ((y'/x')_{K'} \circ z)_H \cup ((x/y)_K \circ z)_H$ and according to condition (1), we have $((z/y)_K \circ x)_K = ((z \circ x)/y)_K = ((x/y)_K \circ z)_K = ((x/y)_K \circ z)_H$. Moreover, $((y'/x')_{K'}/z)_H = ((y'/x')_{K'}/z')_{K'} \cup (z/((y'/x')'_K)_K$. Since in any commutative hypergroup for all x, y, z we have $(x/y)/z = x/(y \circ z)$, it follows that $((y'/x')_{K'}/z')_{K'} = (y'/(x' \circ z'))_{K'}$. So we obtain $((y'/x')_{K'} \circ z)_{K'} = (y'/(x' \circ z'))_{K'} \cup ((z/(y/x)_K)_K$. By condition (3) of the statement, it follows that $((z/(y/x)_K)_K \subseteq ((z \circ x)/y)_K$. On the other hand, $((y'/z')_{K'} \circ x)_H = ((y'/z')_{K'}/x')_{K'} \cup ((x/(y/z)_K)_K$, and by condition (3), we obtain $((x/(y/z)_K)_K \subseteq ((x \circ z)/y)_K$ and $((y'/z')_{K'}/x')_{K'} = (y'/(x' \circ z'))_K = ((y'/x')_{K'}/z')_{K'}$.

(ii) Suppose now that $x' \in y' \circ z$. Then we have $e \in x \circ (y' \circ z)$, but we also have $z \in x'/y'$, whence $z' \in x/y$ and so we obtain $e \in (x \circ y') \circ z$.

(d) If $x = y \neq z$, then we distinguish the following cases:

(i) Suppose that $x' \notin x' \circ z$. Then we have

$$\begin{aligned}x \circ (x' \circ z) &= (x \circ (x'/z')_{K'})_H \cup (x \circ (z/x)_K)_H, \\ (x \circ x') \circ z &= ((x/x)_K \circ z)_H \cup ((x'/x')_{K'} \circ z)_H \cup \{z\}.\end{aligned}$$

According to condition (1), we have

$$(x \circ (z/x)_K)_H = (x \circ (z/x)_K)_K = ((x/x)_K \circ z)_K = ((x/x)_K \circ z)_H.$$

Moreover

$$\begin{aligned}(x \circ (x'/z')_{K'})_H &= ((x'/z')_{K'}/x')_{K'} \cup ((x/(x'/z')'_K)_K \\ &= ((x'/z')_{K'}/x')_{K'} \cup ((x/(x/z)_K)_K.\end{aligned}$$

On the other hand,

$$\begin{aligned}((x'/x')_{K'} \circ z)_H &= ((x'/x')_{K'}/z')_{K'} \cup ((z/(x'/x')_K)_K \\ &= ((x'/x')_{K'}/z')_{K'} \cup (z/(x/x)_K)_K.\end{aligned}$$

We also have $((x'/x')_{K'}/z')_{K'} = (x'/(x' \circ z'))_{K'} = ((x'/z')_{K'}/x')_{K'}$. Notice that both of the terms $(x \circ x') \circ z$ and $x \circ (x' \circ z)$ are contained in $((x \circ z)/x)$ which is a common term both in $((x/(x/z)_K)_K$ and $((z/(x/x)_K)_K)$. We still have to show that $z \in x \circ (x' \circ z)$, but we obtain this in a similar way, as in the cases (b) and (c).

(ii) Suppose now that $x' \in x' \circ z$. Then $e \in x \circ (x' \circ z)$. We also have $z \in x'/x'$, whence $z' \in x/x$. It follows that $e \in (x \circ x') \circ z$.

(e) Finally, if $x = z$, then we obtain $(x \circ y') \circ x = (y' \circ x) \circ x = x \circ (y' \circ x)$.

To show that H is reversible, it is sufficient to consider the following cases:

(i) If $(x, k, y) \in K^3$, then $x \in k \circ y$ implies $y \in x/k \subseteq x \circ k'$.

(ii) If $(x, k', y) \in K \times K' \times K$, then $x \in k' \circ y$ implies $y \in x/k' = x \circ k$ and $x \in k' \circ y$ implies $k' \in x/y \subseteq x \circ y'$. ■

Example 2.4.12.

(1) We can consider the following hypergroup $(K, \circ)$:

$$K = \{x, y\} \text{ and } x \circ x = y, \text{ while } x \circ y = y \circ x = y \circ y = \{x, y\}.$$

The conditions (1), (2) and (3) of the above theorem are satisfied, so we can construct a canonical hypergroup H, such that K is a noncanonical subhypergroup of it.

(2) If K is a nonempty set and for all x, y of K we define $x \circ y = \{x, y\}$, then the conditions (1), (2) and (3) of the above theorem are satisfied, and so there exists a canonical hypergroup H, such that K is a noncanonical subhypergroup of it.

Quasicanonical hypergroups were introduced by P. Corsini and later, they were studied by P. Bonansinga and C. Massouros. They satisfy all the conditions of canonical hypergroups, except the commutativity. Later, S. Comer introduced this class of hypergroups independently, using the name of polygroups. He emphasized the importance of polygroups, by analyzing them in connections to graphs, relations, Boolean and cylindric algebras. Another connection between polygroups and artificial intelligence was considered and analyzed by G. Ligozat. Some of these results are exposed in [27]. The double cosets hypergroups are particular quasicanonical hypergroups and they were analyzed by K. Drbohlav, D.K. Harrison and S. Comer.

Similar as for canonical hypergroups, the subhypergroups of a quasicanonical hypergroup are not necessarily quasicanonical.

Theorem 2.4.13. *Let H be a quasicanonical hypergroup (polygroup) with n elements and S be a subhypergroup of H, which is not quasicanonical. If n is odd, then consider $k = (n-1)/2$, while if n in even, then consider $k = (n-2)/2$. Then S has at maximum k elements.*

Proof. Let $n = 2k+1$ and suppose S has $k+i$ elements, where $i > 0$. Then the cardinality of $H \setminus S$ is $|H \setminus S| = 2k+1-k-i \leq k$. On the other hand, for all x of S, we have that $x^{-1} \in H \setminus S$ and if e is the identity, then $e \in H \setminus S$. From here, we obtain $|H \setminus S| \geq k+i+1 \geq k+2$, which is absurd. Let us consider now $n = 2k+2$. Suppose again that S has $k+i$ elements, where $i > 0$. Similarly as above, we obtain that $|H \setminus S| \leq k+2$, which is absurd. ■

Now, we present some constructions of polygroups.

Example 2.4.14. Classical polygroup constructions.

(1) *Double coset algebra.* Suppose H is a subgroup of a group G. Define a system

$$G//H = < \{HgH \mid g \in G\}, *, H, ^{-I} >$$

where $(HgH)^{-I}=Hg^{-1}H$ and $(Hg_1H)*(Hg_2H)=\{Hg_1hg_2H \mid h\in H\}$. The algebra of double cosets $G//H$ is a polygroup introduced in (Dresher and Ore [47]). In [8], Ben-Yaacov studied polygroups and the blow-up procedure. He obtain a structure theorem for coreless polygroups as a double quotient space $G//H$, and a polygroup chunk theorem. A polygroup is called *chromatic* if it is isomorphic to the color algebra of some color scheme. Chromatic polygroups have the nice property that they are exactly the polygroups which have a faithful representation as a regular polygroup of generalized permutations.

(2) *Prenowitz algebras.* Suppose that G is a projective geometry with a set P of points and for $p \neq q$, suppose that $\overline{pq}$ denoted the set of all points on the unique line through p and q. Choose an object $I \notin P$ and form the system

$$P_G = < P \cup \{I\}, \cdot, I,^{-1} >$$

where $x^{-1} = x$ and $I \cdot x = x \cdot I = x$ for all $x \in P \cup \{I\}$ and for $p, q \in P$,

$$p \cdot q = \begin{cases} \overline{pq} \setminus \{p, q\} & \text{if } p \neq q \\ \{p, I\} & \text{if } p = q. \end{cases}$$

P_G is a polygroup (Prenowitz [103]).

(3) *Extensions of polygroups by polygroups.* In [15], extensions of polygroups by polygroups were introduced in the following way. Suppose that $\mathcal{A} =< A, \cdot, e,^{-1} >$ and $\mathcal{B} =< B, \cdot, e,^{-1} >$ are two polygroups whose elements were renamed so that $A \cap B = \{e\}$. A new system $\mathcal{A}[\mathcal{B}] =< M, *, e,^I >$ called the extension of $\mathcal{A}$ by $\mathcal{B}$ is formed in the following way: Set $M = A \cup B$ and let $e^I = e$, $x^I = x^{-1}$, $e * x = x * e = x$ for all $x \in M$, and for all $x, y \in M \setminus \{e\}$

$$x * y = \begin{cases} x \cdot y & \text{if } x, y \in A \\ x & \text{if } x \in B,\ y \in A \\ y & \text{if } x \in A,\ y \in B \\ x \cdot y & \text{if } x, y \in B,\ y \neq x^{-1} \\ x \cdot y \cup A & \text{if } x, y \in B,\ y = x^{-1}. \end{cases}$$

In this case $\mathcal{A}[\mathcal{B}]$ is a polygroup which is called the extension of $\mathcal{A}$ by $\mathcal{B}$.

(4) *Conjugacy class polygroups.* In dealing with a symmetry group two symmetric operations belong to the same class if they present the same map with respect to (possibly) different coordinate systems where one coordinate system is converted into the other by a member of the group. In the language of group theory this means the elements a, b in a symmetric group G belong to the same class if there exists a $g \in G$ such that $a = gbg^{-1}$, i.e., a and b are conjugate. The collection of all conjugacy classes of a group G is denoted by $\overline{G}$ and the system $< \overline{G}, *, \{e\}, ^{-1} >$ is a polygroup where e is the identity of G and the product $A * B$ of conjugacy classes A and B consists of all conjugacy classes contained in the elementwise product AB. This hypergroup was studied by Campaigne [10] and by Dietzman [45].
Now, we illustrate some constructions using Dihedral group D_4. This group is generated by a counter-clockwise rotation r of 90^o and a horizontal reflection h. The group consists of the following 8 symmetries:

$$\{1 = r^0, r, r^2 = s, r^3 = t, h, hr = d, hr^2 = v, hr^3 = f\}.$$

Dihedral groups occur frequently in art and nature. Many of the decorative designs used on floor coverings, pottery, and buildings have one of Dihedral groups as a group of symmetry. In the case of D_4 there are five conjugacy classes: $\{1\}, \{s\}, \{r, t\}, \{d, f\}$ and $\{h, v\}$. Let us denote these classes by $C_1, ..., C_5$ respectively. Then the polygroup $\overline{D_4}$ is

$*$	C_1	C_2	C_3	C_4	C_5
C_1	C_1	C_2	C_3	C_4	C_5
C_2	C_2	C_1	C_3	C_4	C_5
C_3	C_3	C_3	C_1, C_2	C_5	C_4
C_4	C_4	C_4	C_5	C_1, C_2	C_3
C_5	C_5	C_5	C_4	C_3	C_1, C_2

As a sample of how to calculate the table entries, consider $C_3 \cdot C_3$. To determine this product, compute the elementwise product of the conjugacy classes $\{r, t\}\{r, t\} = \{s, 1\} = C_1 \cup C_2$. Thus $C_3 \cdot C_3$ consists of the two conjugacy classes C_1, C_2.

(5) *Character polygroups.* Closely related to the conjugacy classes of a finite group are its characters. Let $\hat{G} = \{\chi_1, \chi_2, ..., \chi_k\}$ be the collection

of irreducible characters of a finite group G where χ_1 is the trivial character. The character polygroup $\hat{G}$ of G is the system $< \hat{G}, *, \chi_1, ^{-1} >$ where the product $\chi_i * \chi_j$ is the set of irreducible components in the elementwise product $\chi_i\chi_j$. The system $\hat{G}$ was investigated by R. Roth [112] who considered a duality between $\hat{G}$ and $\overline{G}$.
Before calculating $\hat{D}_4$ we need to know the five irreducible characters of Dihedral group D_4. These are given by the following character table. (since characters are constant on conjugacy classes it is usual to list only the conjugacy classes across the top of the table.)

	C_1	C_2	C_3	C_4	C_5
χ_1 :	1	1	1	1	1
χ_2 :	1	1	-1	1	-1
χ_3 :	1	1	-1	-1	1
χ_4 :	1	1	1	-1	-1
χ_5 :	2	-2	0	0	0

We illustrate the calculation of the polygroup product of two characters by considering $\chi_5 * \chi_5$. The pointwise product of χ_5 with itself yields the following (non-irreducible) character:

	C_1	C_2	C_3	C_4	C_5
$\chi_5\chi_5$:	4	4	0	0	0

This character can be written as a sum of irreducible characters in exactly one way: $\chi_5\chi_5 = \chi_1 + \chi_2 + \chi_3 + \chi_4$. This is indicated by the entry in the lower right hand corner of the polygroup table for $\hat{D}_4$. In general the polygroup product of two characters $\chi_i * \chi_j$ tell which are the irreducible characters in the product $\chi_i\chi_j$, but not the multiplicity. Using i in the place of the character χ_i the polygroup $\hat{D}_4$ is

	1	2	3	4	5
1	1	2	3	4	5
2	2	1	4	3	5
3	3	4	1	2	5
4	4	3	2	1	5
5	5	5	5	5	1, 2, 3, 4.

2.5 Complete hypergroups

Regular and strongly regular relations

By using a certain type of equivalence relations, we can connect semihypergroups to semigroups and hypergroups to groups. These equivalence relations are called strong regular relations. More exactly, starting with a (semi)hypergroup and using a strong regular relation, we can construct a (semi)group structure on the quotient set. A natural question arises: Do they also exist regular relations? The answer is positive, regular relations provide us new (semi)hypergroup structures on the quotient sets.

Let us define these notions. First, we do some notations.

Let $(H, \circ)$ be a semihypergroup and R be an equivalence relation on H. If A and B are nonempty subsets of H, then

$$\begin{aligned} A\overline{R}B &\text{ means that } \forall a \in A, \exists b \in B \text{ such that } aRb \text{ and} \\ &\qquad \forall b' \in B, \exists a' \in A \text{ such that } a'Rb'; \\ A\overline{\overline{R}}B &\text{ means that } \forall a \in A, \forall b \in B, \text{ we have } aRb. \end{aligned}$$

Definition 2.5.1. The equivalence relation R is called

(1) *regular on the right* (*on the left*) if for all x of H, from aRb, it follows that $(a \circ x)\overline{R}(b \circ x)$ $((x \circ a)\overline{R}(x \circ b)$ respectively);

(2) *strongly regular on the right* (*on the left*) if for all x of H, from aRb, it follows that $(a \circ x)\overline{\overline{R}}(b \circ x)$ $((x \circ a)\overline{\overline{R}}(x \circ b)$ respectively);

(3) R is called *regular* (*strongly regular*) if it is regular (strongly regular) on the right and on the left.

Theorem 2.5.2. *Let $(H, \circ)$ be a semihypergroup and R be an equivalence relation on H.*

(1) *If R is regular, then H/R is a semihypergroup, with respect to the following hyperoperation: $\overline{x} \otimes \overline{y} = \{\overline{z} \mid z \in x \circ y\}$;*

(2) *If the above hyperoperation is well defined on H/R, then R is regular.*

Proof. (1) First, we check that the hyperoperation $\otimes$ is well defined on H/R. Consider $\overline{x} = \overline{x_1}$ and $\overline{y} = \overline{y_1}$. We check that $\overline{x} \otimes \overline{y} = \overline{x_1} \otimes \overline{y_1}$. We have xRx_1 and yRy_1. Since R is regular, it follows that $(x \circ y)\overline{\overline{R}}(x_1 \circ y)$, $(x_1 \circ y)\overline{\overline{R}}(x_1 \circ y_1)$ whence $(x \circ y)\overline{\overline{R}}(x_1 \circ y_1)$. Hence for all $z \in x \circ y$, there exists $z_1 \in x_1 \circ y_1$ such that zRz_1, which means that $\overline{z} = \overline{z_1}$. It follows that $\overline{x} \otimes \overline{y} \subseteq \overline{x_1} \otimes \overline{y_1}$ and similarly we obtain the converse inclusion. Now, we check the associativity of $\otimes$. Let $\overline{x}, \overline{y}, \overline{z}$ be arbitrary elements in H/R and $\overline{u} \in (\overline{x} \otimes \overline{y}) \otimes \overline{z}$. This means that there exists $\overline{v} \in \overline{x} \otimes \overline{y}$ such that $\overline{u} \in \overline{v} \otimes \overline{z}$. In other words, there exist $v_1 \in x \circ y$ and $u_1 \in v \circ z$, such that vRv_1 and uRu_1. Since R is regular, it follows that there exists $u_3 \in v_1 \circ z \subseteq x \circ (y \circ z)$ such that u_1Ru_3. From here, we obtain that there exists $u_4 \in y \circ z$ such that $u_3 \in x \circ u_4$. We have $\overline{u} = \overline{u_1} = \overline{u_3} \in \overline{x} \otimes \overline{u_4} \subseteq \overline{x} \otimes (\overline{y} \otimes \overline{z})$. It follows that $(\overline{x} \otimes \overline{y}) \otimes \overline{z} \subseteq \overline{x} \otimes (\overline{y} \otimes \overline{z})$. Similarly, we obtain the converse inclusion.

(2) Let aRb and x be an arbitrary element of H. If $u \in a \circ x$, then $\overline{u} \in \overline{a} \otimes \overline{x} = \overline{b} \otimes \overline{x} = \{\overline{v} \mid v \in b \circ x\}$. Hence, there exists $v \in b \circ x$ such that uRv, whence $(a \circ x)\overline{R}(b \circ x)$. Similarly we obtain that R is regular on the left. ∎

Corollary 2.5.3. *If $(H, \circ)$ is a hypergroup and R is an equivalence relation on H, then R is regular if and only if $(H/R, \otimes)$ is a hypergroup.*

Proof. If H is a hypergroup, then for all x of H we have $H \circ x = x \circ H = H$, whence we obtain $H/R \otimes \overline{x} = \overline{x} \otimes H/R = H/R$. According to the above theorem, it follows that $(H/R, \otimes)$ is a hypergroup. ∎

Notice that if R is regular on a (semi)hypergroup H, then the canonical projection $\pi : H \longrightarrow H/R$ is a good epimorphism. Indeed, for all x, y of H and $\overline{z} \in \pi(x \circ y)$, there exists $z' \in x \circ y$ such that $\overline{z} = \overline{z'}$. We have $\overline{z} = \overline{z'} \in \overline{x} \otimes \overline{y} = \pi(x) \otimes \pi(y)$. Conversely, if $\overline{z} \in \pi(x) \otimes \pi(y) = \overline{x} \otimes \overline{y}$, then there exists $z_1 \in x \circ y$ such that $\overline{z} = \overline{z_1} \in \pi(x \circ y)$.

Theorem 2.5.4. *If $(H, \circ)$ and $(K, *)$ are semihypergroups and $f : H \longrightarrow K$ is a good homomorphism, then the equivalence ρ^f associated with f is regular and $\varphi : f(H) \longrightarrow H/\rho^f$, defined by $\varphi(f(x)) = \overline{x}$, is an isomorphism.*

Proof. Let $h_1 \rho^f h_2$ and a be an arbitrary element of H. If $u \in h_1 \circ a$, then

$$f(u) \in f(h_1 \circ a) = f(h_1) * f(a) = f(h_2) * f(a) = f(h_2 \circ a).$$

Then there exists $v \in h_2 \circ a$ such that $f(u) = f(v)$, which means that $u\rho^f v$. Hence, ρ^f is regular on the right. Similarly, it can be shown that ρ^f is regular on the left. On the other hand, for all $f(x), f(y)$ of $f(H)$, we have

$$\varphi(f(x) * f(y)) = \varphi(f(x \circ y)) = \{\overline{z} \mid z \in x \circ y\} = \overline{x} \otimes \overline{y} = \varphi(f(x)) \otimes \varphi(f(y)).$$

Moreover, if $\varphi(f(x)) = \varphi(f(y))$ then $x\rho^f y$, so φ is injective and clearly, it is also surjective. Finally, for all $\overline{x}, \overline{y}$ of H/ρ^f we have

$$\begin{aligned} \varphi^{-1}(\overline{x} \otimes \overline{y}) &= \varphi^{-1}(\{\overline{z} \mid z \in x \circ y\}) = \{f(z) \mid z \in x \circ y\} \\ &= f(x \circ y) = f(x) * f(y) = \varphi^{-1}(\overline{x}) * \varphi^{-1}(\overline{y}). \end{aligned}$$

Therefore φ is an isomorphism. ■

Theorem 2.5.5. *Let $(H, \circ)$ be a semihypergroup and R be an equivalence relation on H.*

(1) *If R is strongly regular, then H/R is a semigroup, with respect to the following operation: $\overline{x} \otimes \overline{y} = \{\overline{z} \mid z \in x \circ y\}$;*

(2) *If the above operation is well defined on H/R, then R is strongly regular.*

Proof. (1) For all x, y of H, we have $(x \circ y)\overline{\overline{R}}(x \circ y)$. Hence, $\overline{x} \otimes \overline{y} = \{\overline{z} \mid z \in x \circ y\} = \{\overline{z}\}$, which means that $\overline{x} \otimes \overline{y}$ has exactly an element. Therefore, $(H/R, \otimes)$ is a semigroup.

(2) If aRb and x is an arbitrary element of H, we check that $(a \circ x)\overline{\overline{R}}(b \circ x)$. Indeed, for all $u \in a \circ x$ and all $v \in b \circ x$ we have $\overline{u} = \overline{a} \otimes \overline{x} = \overline{b} \otimes \overline{x} = \overline{v}$, which means that uRv. Hence, R is strongly regular on the right and similarly, it can be shown that it is strongly regular on the left. ■

Corollary 2.5.6. *If $(H, \circ)$ is a hypergroup and R is an equivalence relation on H, then R is strongly regular if and only if $(H/R, \otimes)$ is a group.*

Proof. It follows from the above theorem and Remark 2.3.6. ■

Theorem 2.5.7. *If $(H, \circ)$ is a semihypergroup and $(S, *)$ is a semigroup and $f : H \longrightarrow S$ is a homomorphism, then the equivalence ρ^f associated with f is strongly regular.*

Proof. Let $a\rho^f b$, $x \in H$ and $u \in a \circ x$. It follows that

$$f(u) = f(a) * f(x) = f(b) * f(x) = f(b \circ x).$$

Hence for all $v \in b \circ x$, we have $f(u) = f(v)$, which means that $u\rho^f v$. Hence ρ^f is strongly regular on the right and similarly, it is strongly regular on the left. ■

The fundamental relation

This relation has an important role in the study of semihypergroups and especially of hypergroups.

Definition 2.5.8. Let $(H, \circ)$ be a semihypergroup and n be a nonzero natural number. We say that

$$x\beta_n y \text{ if there exists } a_1, a_2, ..., a_n \text{ in } H, \text{ such that } \{x, y\} \subseteq \prod_{i=1}^{n} a_i.$$

Let $\beta = \bigcup_{n \geq 1} \beta_n$. Clearly, the relation β is reflexive and symmetric. Denote by β^* the transitive closure of β.

Theorem 2.5.9. *β^* is the smallest strongly regular relation on H.*

Proof. We show that:

(1) β^* is a strongly regular relation on H;

(2) If R is a strongly regular relation on H, then $\beta^* \subseteq R$.

(1) Let $a\beta^* b$ and x be an arbitrary element of H. It follows that there exist $x_0 = a, x_1, ..., x_n = b$ such that for all $i \in \{0, 1, ..., n-1\}$ we have $x_i \beta x_{i+1}$. Let $u_1 \in a \circ x$ and $u_2 \in b \circ x$. We check that $u_1 \beta^* u_2$. From $x_i \beta x_{i+1}$ it follows that there exists a hyperproduct P_i, such that $\{x_i, x_{i+1}\} \subseteq P_i$ and so $x_i \circ x \subseteq P_i \circ x$ and $x_{i+1} \circ x \subseteq P_i \circ x$, which means that $x_i \circ x \overline{\overline{\beta}} x_{i+1} \circ x$. Hence for all $i \in \{0, 1, ..., n-1\}$ and for all $s_i \in x_i \circ x$ we have $s_i \beta s_{i+1}$. If we consider $s_0 = u_1$ and $s_n = u_2$, then we obtain $u_1 \beta^* u_2$. Then β^* is strongly regular on the right and similarly, it is strongly regular on the left.

(2) We have $\beta_1 = \{(x, x) \mid x \in H\} \subseteq R$, since R is reflexive. Suppose

that $\beta_{n-1} \subseteq R$ and show that $\beta_n \subseteq R$. If $a\beta_n b$, then there exist $x_1, ..., x_n$ in H, such that $\{a, b\} \subseteq \prod_{i=1}^{n} x_i$. Hence, there exists u, v in $\prod_{i=1}^{n-1} x_i$, such that $a \in u \circ x_n$ and $b \in v \circ x_n$. We have $u\beta_{n-1}v$ and according to the hypothesis, we obtain uRv. Since R is strongly regular, it follows that aRb. Hence $\beta_n \subseteq R$. By induction, it follows that $\beta \subseteq R$, whence $\beta^* \subseteq R$. ∎

Hence the relation β^* is the smallest equivalence relation on H, such that the quotient H/β^* is a group.

Definition 2.5.10. β^* is called the *fundamental relation* on H and H/β^* is called the *fundamental group.*

Complete parts

Complete parts were introduced and studied for the first time by M. Koskas [69]. Later, this topic was analyzed by P. Corsini [22] and Y. Sureau [128] mostly in the general theory of hypergroups. M. De Salvo studied complete parts from a combinatorial point of view. A generalization of them, called n-complete parts, was introduced by R. Migliorato. Other hypergroupists gave a contribution to the study of complete parts and of the heart of a hypergroup. Among them, V. Leoreanu analyzed the structure of the heart of a hypergroup in her Ph.D. Thesis.

We present now the definitions.

Definition 2.5.11. Let $(H, \circ)$ is a semihypergroup and A be a nonempty subset of H. We say that A is a *complete part* of H if for any nonzero natural number n and for all $a_1, ..., a_n$ of H, the following implication holds:

$$A \cap \prod_{i=1}^{n} a_i \neq \emptyset \implies \prod_{i=1}^{n} a_i \subseteq A.$$

Theorem 2.5.12. *If $(H, \circ)$ is a semihypergroup and R is a strongly regular relation on H, then for all z of H, the equivalence class of z is a complete part of H.*

Proof. Let $a_1, ..., a_n$ be elements of H, such that

$$\overline{z} \cap \prod_{i=1}^{n} a_i \neq \emptyset.$$

Then there exists $y \in \prod_{i=1}^{n} a_i$, such that yRz. The homomorphism $\pi : H \longrightarrow H/R$ is good and H/R is a semigroup. It follows that $\pi(y) = \pi(z) = \pi\left(\prod_{i=1}^{n} a_i\right) = \prod_{i=1}^{n} \pi(a_i)$. This means that $\prod_{i=1}^{n} a_i \subseteq \overline{z}$. ∎

If A is a subset of H, denote by $C(A)$ the complete closure of A, which is the smallest complete part of H, that contains A.

Denote $K_1(A) = A$ and for all $n \geq 1$ denote

$$K_{n+1}(A)=\left\{x\in H \mid \exists p \in \mathbb{N}^*, \exists(h_1, ..., h_p) \in H^p : x \in \prod_{i=1}^{p} h_i, K_n(A) \cap \prod_{i=1}^{p} h_i \neq \emptyset\right\}.$$

Let $K(A) = \bigcup_{n\geq 1} K_n(A)$.

Theorem 2.5.13. *We have $C(A) = K(A)$.*

Proof. Notice that $K(A)$ is a complete part of H. Indeed, if we suppose $K(A) \cap \prod_{i=1}^{p} x_i \neq \emptyset$, then there exists $n \geq 1$, such that $K_n(A) \cap \prod_{i=1}^{p} x_i \neq \emptyset$, which means that $\prod_{i=1}^{p} x_i \subseteq K_{n+1}(A)$.

Now, if $A \subseteq B$ and B is a complete part of H, then we show that $K(A) \subseteq B$. We have $K_1(A) \subseteq B$ and suppose $K_n(A) \subseteq B$. We check that $K_{n+1}(A) \subseteq B$. Let $z \in K_{n+1}(A)$, which means that there exists a hyperproduct $\prod_{i=1}^{p} x_i$, such that $z \in \prod_{i=1}^{p} x_i$ and $K_n(A) \cap \prod_{i=1}^{p} x_i \neq \emptyset$. Hence $B \cap \prod_{i=1}^{p} x_i \neq \emptyset$, whence $\prod_{i=1}^{p} x_i \subseteq B$. We obtain $z \in B$. Therefore, $C(A) = K(A)$. ∎

Theorem 2.5.14. *If x is an arbitrary element of a hypergroup $(H, \circ)$, then*

(1) *For all $n \geq 2$ we have $K_n(K_2(x)) = K_{n+1}(x)$.*

(2) *The next equivalence holds:* $x \in K_n(y) \iff y \in K_n(x)$.

Proof. (1) We check the equality by induction. We have

$$K_2(K_2(x)) = \left\{ z \in H \mid \exists q \in \mathbb{N}^*, \exists (a_1, ..., a_q) \in H^q : z \in \prod_{i=1}^{q} a_i, K_2(x) \cap \prod_{i=1}^{q} a_i \neq \emptyset \right\}$$
$$= K_3(x).$$

Suppose that $K_{n-1}(K_2(x)) = K_n(x)$. Then

$$K_n(K_2(x))$$
$$= \left\{ z \in H \mid \exists q \in \mathbb{N}^*, \exists (a_1, ..., a_q) \in H^q : z \in \prod_{i=1}^{q} a_i, K_{n-1}(K_2(x)) \cap \prod_{i=1}^{q} a_i \neq \emptyset \right\}$$
$$= K_{n+1}(x).$$

(2) We check the equivalence by induction. For $n = 2$, we have

$$x \in K_2(y)$$
$$= \left\{ z \in H \mid \exists q \in \mathbb{N}^*, \exists (a_1, ..., a_q) \in H^q : z \in \prod_{i=1}^{q} a_i, K_1(y) \cap \prod_{i=1}^{q} a_i \neq \emptyset \right\}.$$

Hence $\{y, x\} \subseteq \prod_{i=1}^{q} a_i$, whence $y \in K_2(x)$.

Suppose that the following equivalence holds:

$$x \in K_{n-1}(y) \iff y \in K_{n-1}(x)$$

and we check $x \in K_n(y) \iff y \in K_n(x)$. If $x \in K_n(y)$ then there exists $\prod_{i=1}^{p} a_i$ with $x \in \prod_{i=1}^{p} a_i$ and there exists $v \in \prod_{i=1}^{p} a_i \cap K_{n-1}(y)$. It follows that $v \in K_2(x)$ and $y \in K_{n-1}(v)$. Hence $y \in K_{n-1}(K_2(x)) = K_n(x)$. Similarly, we obtain the converse implication. ■

Corollary 2.5.15. *The binary relation defined as follows:*

$$xKy \iff \exists n \geq 1,\ x \in K_n(y)$$

is an equivalence relation.

Theorem 2.5.16. *The equivalence relations K and β^* coincide.*

Proof. If $x\beta y$, then x and y belong to the same hyperproduct and so, $x \in K_2(y) \subseteq K(y)$. Hence $\beta \subseteq K$, whence $\beta^* \subseteq K$. Now, if we have xKy and $x \neq y$, then there exists $n \geq 1$, such that $xK_{n+1}y$, which means that there exists a hyperproduct P_1, such that $x \in P_1$ and $P_1 \cap K_n(y) \neq \emptyset$. Let $x_1 \in P_1 \cap K_n(y)$. Hence $x\beta x_1$. From $x_1 \in K_n(y)$ it follows that there exists a hyperproduct P_2, such that $x_1 \in P_2$ and $P_2 \cap K_{n-1}(y) \neq \emptyset$. Let $x_2 \in P_2 \cap K_{n-1}(y)$. Hence $x_1\beta x_2$ and $x_2 \in K_{n-1}(y)$. After a finite number of steps, we obtain that there exist x_{n-1}, x_n such that $x_{n-1}\beta x_n$ and $x_n \in K_{n-(n-1)}(y) = \{y\}$. Therefore $x\beta^* y$. ∎

Theorem 2.5.17. *If B is a nonempty subset of H, then we have*

$$C(B) = \bigcup_{b\in B} C(b).$$

Proof. Clearly, for all $b \in B$, we have $C(b) \subseteq C(B)$. On the other hand, $C(B) = \bigcup\limits_{n\geq 1} K_n(B)$. We shall prove by induction. For $n = 1$, we have $K_1(B) = B = \bigcup\limits_{b\in B} K_1(b)$. Suppose $K_n(B) \subseteq \bigcup\limits_{b\in B} K_n(b)$. If $z \in K_{n+1}(B)$ then there exists a hyperproduct P, such that $z \in P$ and $K_n(B) \cap P \neq \emptyset$, whence there exists $b \in B$ such that $K_n(b) \cap P \neq \emptyset$. Hence $z \in K_{n+1}(b)$. We obtain $K_{n+1}(B) \subseteq \bigcup\limits_{b\in B} K_{n+1}(b)$. Therefore, $C(B) = \bigcup\limits_{b\in B} C(b)$. ∎

The heart of a hypergroup

The notion of a heart of a hypergroup is directly connected to the fundamental relation on that hypergroup.

Definition 2.5.18. Let $(H, \circ)$ is a hypergroup and consider the canonical projection $\varphi_H : H \longrightarrow H/\beta^*$. The *heart* of H is the set $\omega_H = \{x \in H \mid \varphi_H(x) = 1\}$, where 1 is the identity of the group $(H/\beta^*, \otimes)$.

We shall prove that the heart of a hypergroup $(H, \circ)$ is the smallest complete part subhypergroup of H. First, we show that

Theorem 2.5.19. *ω_H is a complete part of H.*

Proof. We consider the following steps:

(1) For all nonempty subset A of H, we prove that $\omega_H \circ A = A \circ \omega_H = \varphi_H^{-1}(\varphi_H(A))$. If $y \in \varphi_H^{-1}(\varphi_H(A))$ then there exists $a \in A$ such that $\varphi_H(y) = \varphi_H(a)$. On the other hand, there exists $u \in H$ such that $y \in a \circ u$. It follows that $\varphi_H(y) = \varphi_H(a) \otimes \varphi_H(u)$, whence $\varphi_H(u) = 1$. Hence $u \in \omega_H$ and so, $y \in A \circ \omega_H$. Therefore $\varphi_H^{-1}(\varphi_H(A)) \subseteq A \circ \omega_H$. Conversely, if $z \in A \circ \omega_H$ then $\varphi_H(z) \in \varphi_H(A) \otimes \varphi_H(\omega_H) = \varphi_H(A)$, whence $z \in \varphi_H^{-1}(\varphi_H(A))$.

(2) For all nonempty subset A of H, we prove that $\omega_H \circ A = A \circ \omega_H = C(A)$. Indeed, we have $x \in \varphi_H^{-1}(\varphi_H(A))$ if and only if there exists $a \in A$ such that $x\beta^* a$, which means $x \in K(a) = C(a)$. Hence $\varphi_H^{-1}(\varphi_H(A)) = \bigcup_{a \in A} C(a) = C(A)$. From here we obtain:

(3) A nonempty subset A of H is complete if and only if $\omega_H \circ A = A$.

(4) We have $\omega_H \circ \omega_H = \omega_H$, which means that ω_H is a complete part subhypergroup of H. ■

Notice that if A and B are nonempty subsets of H, such that one of them is a complete part, then $A \circ B$ is a complete part, since $\omega_H \circ A \circ B = A \circ B$.

Theorem 2.5.20. *Any complete part subhypergroup K of $(H, \circ)$ is invertible.*

Proof. Let $y \in K \circ x$. We obtain $\varphi_H(y) = \varphi_H(k) \otimes \varphi_H(x)$ for some $k \in K$. Since $\varphi_H(K)$ is a subgroup of $(H/\beta^*, \otimes)$, it follows that $\varphi_H(x) = (\varphi_H(k))^{-1} \otimes \varphi_H(y) = \varphi_H(K) \otimes \varphi_H(y) = \varphi_H(K \circ y)$. Hence $x \in \varphi_H^{-1}(\varphi_H(K \circ y)) = C(K \circ y) = K \circ y$. Therefore, K is left invertible. Similarly, it can be shown that K is right invertible. ■

Denote the class of all complete parts subhypergroups of H by $CPS(H)$. We obtain the following:

Theorem 2.5.21. $\omega_H = \bigcap_{K \in CPS(H)} K.$

Proof. Since $\omega_H \in CPS(H)$, it follows that $\bigcap_{K \in CPS(H)} K \subseteq \omega_H$. Now, we show that for all $K \in CPS(H)$ we have $\omega_H \subseteq K$. Let $x \in \omega_H$. Since

$K \circ \omega_H = K$, it follows that there are a, b in K, such that $b \in a \circ x \subseteq K \circ x$. Since K is invertible, it follows that $x \in K \circ b = K$. Hence $\omega_H \subseteq \bigcap_{K \in CPS(H)} K$. ■

Theorem 2.5.22. *Any complete part subhypergroup K of $(H, \circ)$ is ultraclosed.*

Proof. First of all, K is closed since it is invertible. On the other hand, for all $x \in K$, there exists $e \in H$ such that $x \in x \circ e$. From here, we obtain that $e \in \omega_H$. Moreover, there exists $x' \in H$ such that $e \in x' \circ x$. Since ω_H is a complete part, it follows that $x' \circ x \subseteq \omega_H$. Denote $B = x \circ K \cap x \circ (H \setminus K)$. We obtain

$$x' \circ B \subseteq x' \circ (x \circ K \cap x \circ (H \setminus K)) \subseteq \omega_H \circ K \cap \omega_H \circ (H \setminus K) \subseteq K \cap K \circ (H \setminus K) = \emptyset.$$

since K is closed. Hence $B = \emptyset$, which means that K is ultraclosed on the right. Similarly, it is ultraclosed on the left. ■

Using the hypergroupoid associated with a binary relation, defined by I.G. Rosenberg, we characterize the class of all semihypergroups for which the relation is transitive. (see [74]).

To each binary relation R on a nonempty set H, a partial hypergroupoid $(H, \circ_R)$ is associated [113], as follows:

$$\forall (x, z) \in H^2, \ x \circ_R x = \{y \in H \mid (x, y) \in R\}, \quad x \circ_R z = x \circ_R x \cup z \circ_R z.$$

From here, we obtain $x \circ_R x \circ_R x = \bigcup_{a \in x \circ_R x} a \circ_R a \cup x \circ_R x$. By a direct check, it follows:

Theorem 2.5.23. *Let R be a relation on H, such that $R \subseteq R^2$. Then R is transitive if and only if for all $x \in H$, we have $x \circ_R x \circ_R x = x \circ_R x$.*

Theorem 2.5.24. *Let $(H, \circ)$ be a semihypergroup. The relation β is transitive if and only if for all x of H, we have $C(x) = K_2(x)$.*

Proof. First, denote by $Pr(H)$ the set of all hyperproducts of H. According to the above theorem, the relation β is transitive if and only if for all $x \in H$, $x \circ_\beta x \circ_\beta x = x \circ_\beta x$. We have

$$x \circ_\beta x \circ_\beta x = \{t \in H \mid t \in K_2(x), \ a \in K_2(x)\} \cup K_2(x) = K_3(x) \cup K_2(x).$$

Hence β is transitive if and only if for all $x \in H$, we have $K_3(x) \cup K_2(x) = K_2(x)$, which means that for all $x \in H$, we have $K_3(x) \subseteq K_2(x)$. We show that for every positive integer n, $K_{n+1} \subseteq K_n(x)$. Indeed, if we suppose that $K_s(x) \subseteq K_{s-1}(x)$ where $s \in \mathbb{Z}^+$, then

$$\begin{aligned} K_{s+1}(x) &= \bigcup\{P_0 \in Pr(H) \mid P_0 \cap K_s(x) \neq \emptyset\} \\ &\subseteq \bigcup\{P_0 \in Pr(H) \mid P_0 \cap K_{s-1}(x) \neq \emptyset\} = K_s(x). \end{aligned}$$

Since $C(x) = \bigcup_{i \in \mathbb{Z}^+} K_i(x)$, it follows that β is transitive if and only if for all x of H, we have $C(x) = K_2(x)$. In other words, β is transitive if and only if for all x of H, $K_2(x)$ is a complete part of H. ■

Theorem 2.5.25. *If $(H, \circ)$ is a hypergroup, then for all x of ω_H, we have $K_2(x) = \omega_H$.*

Proof. Clearly, for all x of ω_H, we have $K_2(x) \subseteq \omega_H$. Now, it is sufficient to show that $K_2(x)$ is a complete part of H. Suppose $v \in K_2(x) \cap P$, where $P \in Pr(H)$. We check that $P \subseteq K_2(x)$. There exists $P_0 \in Pr(H)$ such that $x \in P_0$, $v \in P_0 \cap P$. On the other hand, there exists $e \in \omega_H$, such that $P \subseteq P \circ e$. Moreover, there are a, b in ω_H, such that $x \in v \circ a$, $e \in b \circ x$. We have

$$\begin{aligned} &P \subseteq P \circ e \subseteq P \circ b \circ x \subseteq P \circ b \circ v \circ a \subseteq P \circ b \circ P_0 \circ a = P_2 \in P_r(H) \\ &x \in v \circ a \subseteq P \circ a \subseteq P \circ e \circ a \subseteq\subseteq P \circ b \circ x \circ a \subseteq P \circ b \circ P_0 \circ a = P_2. \end{aligned}$$

Hence $P \subseteq K_2(x)$.

Corollary 2.5.26. *If $(H, \circ)$ is a hypergroup, then the relation β is an equivalence relation on H.*

Proof. We check that for all $x \in H$, we have $K_3(x) \subseteq K_2(x)$. Let $y \in K_3(x)$. Then there exists $z \in H$ and there exist P, Q in $Pr(H)$, such that $\{y, z\} \subseteq P$, $\{z, x\} \subseteq Q$. Hence $\varphi_H(y) = \varphi_H(z) = \varphi_H(x)$. There exist $u, e \in \omega_H$ such that $y \in z \circ u$, $x \in x \circ e$. By the above theorem, we obtain $u \in K_2(e)$ and so there exists $T \in Pr(H)$ such that $\{u, t\} \subseteq T$. We have $y \in Q \circ u \subseteq Q \circ T$ and $x \in x \circ e \subseteq Q \circ T$. Hence $y \in K_2(x)$, since x, y belong to the same hyperproduct. Therefore, β is transitive, so it is an equivalence relation. ■

Definition 2.5.27. Let $(H, \circ)$ and $(H', *)$ be hypergroups and let $f : H \to H'$ be a homomorphism. The *kernel* of f is the set $K(f) = \{x \in H \mid f(x) \in \omega_{H'}\}$.

Theorem 2.5.28. *The kernel $K(f)$ of a hypergroup homomorphism $f : H \to H'$ is a complete part subhypergroup of H. Moreover, it has the property: for all x in H, we have $x \circ K(f) = K(f) \circ x$, which means that $K(f)$ is a normal subhypergroup of $(H, \circ)$.*

Proof. First, we show that if K' is a closed subhypergroup of $(H', *)$, then $f^{-1}(K')$ is a closed subhypergroup of $(H, \circ)$. Indeed, if $x \in f^{-1}(K')$ then from $t \in x \circ f^{-1}(K')$ it follows that $f(t) \in f(x) * f(f^{-1}(K')) \subseteq f(x) * K' \subseteq K'$, whence $t \in f^{-1}(K')$, hence $x \circ f^{-1}(K') \subseteq f^{-1}(K')$. Now, if $u \in f^{-1}(K')$, then there exists v in H, such that $u \in x \circ v$, whence $f(u) \in f(x) * f(v)$ and since K' is closed, we obtain $v \in f^{-1}(K')$. Hence, $u \in x \circ f^{-1}(K')$, and so we have $f^{-1}(K') \subseteq x \circ f^{-1}(K')$. Similarly we show that for all x in H, we have $f^{-1}(K') = f^{-1}(K') \circ x$. Therefore, $f^{-1}(K')$ is a subhypergroup of $(H, \circ)$. Clearly, it is a closed subhypergroup. It follows that $f^{-1}(\omega_{H'}) = K(f)$ is a closed subhypergroup of $(H, \circ)$. Now, we show that $\omega_H \subseteq K(f)$. If x and e are elements of H, such that $x \in x \circ e$, then $e \in \omega_H$. The element e is called a *partial identity* for x. Let u be an arbitrary element of ω_H. Then we have $u \beta e$, which means that u and e belong to the same hyperproduct in H. Then $f(u)$ and $f(e)$ belong to the same hyperproduct in H'. But $f(e)$ is a partial identity for $f(x)$, so $f(e) \in \omega_{H'}$. Hence $f(u) \in \omega_{H'}$, whence $u \in f^{-1}(\omega_{H'}) = K(f)$. In other words, we have $\omega_H \subseteq K(f)$ and so $K(f)$ is a complete part subhypergroup of H. Finally, we check that for all x in H, we have $K(f) \circ x \subseteq x \circ K(f)$. If $y \in K(f) \circ x$, then $f(y) \in \omega_{H'} * f(x)$, whence $\varphi_{H'}(f(y)) = \varphi_{H'}(f(x))$. Let z be such that $y \in x \circ z$. We obtain $\varphi_{H'}(f(y)) = \varphi_{H'}(f(x)) \otimes \varphi_{H'}(f(z))$, whence $\varphi_{H'}(f(z)) = 1$ which means that $f(z) \in \omega_{H'}$ and so $z \in K(f)$. Hence $y \in x \circ K(f)$. Similarly, it follows the converse inclusion. Therefore, $K(f)$ is a normal subhypergroup of H. ∎

In the end of this section we present another important class of hypergroups: complete hypergroups. We present some interesting properties of this class of hypergroups, for instance we show that any complete hypergroup has at least an identity and any element has an inverse. In other words, any complete hypergroup is regular.

If $(H, \circ)$ is a semihypergroup and A is a nonempty subset of H, then we denote the complete closure of A by $C(A)$.

Theorem 2.5.29. *Let $(H, \circ)$ be a semihypergroup. The following conditions are equivalent:*

(1) *for all $x, y \in H$ and for all $a \in x \circ y$, $C(a) = x \circ y$,*

(2) *for all $x, y \in H$, $C(x \circ y) = x \circ y$,*

Proof. (1$\Longrightarrow$2): We have $C(x \circ y) = \bigcup_{a \in x \circ y} C(a) = x \circ y$.

(2$\Longrightarrow$1): From $a \in x \circ y$, we obtain $C(a) \subseteq C(x \circ y) = x \circ y$. This means that $C(a) \cap x \circ y \neq \emptyset$, whence $x \circ y \subseteq C(a)$. Therefore, $C(a) = x \circ y$. ∎

Definition 2.5.30. A semihypergroup is *complete* if it satisfies one of the above equivalent conditions. A hypergroup is *complete* if it is a complete semihypergroup.

Corollary 2.5.31. *If $(H, \circ)$ is a complete semihypergroup and $\bar{x}$ is the equivalence class of x with respect to the equivalence relation β^*, then either there exist $a, b \in H$ such that $\beta^*(x) = a \circ b$ or $\beta^*(x) = \{x\}$.*

Theorem 2.5.32. *If $(H, \circ)$ is a complete hypergroup, then*

(1) *$\omega_H = \{e \in H : \forall x \in H, x \in x \circ e \cap e \circ x\}$, which means that ω_H is the set of two-sided identities of H.*

(2) *H is regular (i.e. H has at least an identity and any element has an inverse) and reversible.*

Proof. (1) If $u \in \omega_H$, then for all $a \in H$, we have $a \in C(a) = a \circ \omega_H = a \circ u$. Similarly we have $a \in u \circ a$, which means that u is a two-sided identity of H. Conversely, any two-sided identity u of H is an element of ω_H, since $\varphi(u) = 1$.

(2) Let a, a', a'' be elements of H and e be a two-sided identity, such that $e \in a' \circ a \cap a \circ a''$. Then, $a' \circ a = \omega_H = a \circ a''$ and $a \circ a' \subseteq a \circ a' \circ a \circ a'' \subseteq a \circ \omega_H \circ a'' = \omega_H \circ a \circ a'' = \omega_H$, hence $a \circ a' = \omega_H$, so a' is an inverse of a.

Moreover, if $a \in b \circ c$, then $\omega_H = a' \circ a \subseteq a' \circ b \circ c$, so for any inverse c' of c, we have $c' \in \omega_H \circ c' \subseteq a' \circ b \circ c \circ c' = a' \circ b \circ \omega_H = a' \circ b$. Similarly, from here we obtain $b' \in c \circ a'$, and so $b' \circ a \subseteq c \circ a' \circ a = C(c)$, whence $c \in C(c) = b' \circ a$. In a similar way, we obtain $b \in a \circ c'$. ■

Definition 2.5.33. A hypergroup $(H, \circ)$ is called *flat* if for all subhypergroup K of H, we have $\omega_K = \omega_H \cap K$.

Theorem 2.5.34. *Any complete hypergroup is flat.*

Proof. Let H be a complete hypergroup and let K be a subhypergroup H. We have $\omega_H \cap K = \{e \in K : \ \forall a \in H, \ x \in e \circ x \cap x \circ e\} \subseteq \omega_K$.

Moreover, $y \in C_K(x) \Rightarrow y\beta_K x \Rightarrow y\beta_H x \Rightarrow y \in C_H(x)$, which means that $C_K(x) \subseteq C_H(x)$. Clearly, $\omega_H \cap K \neq \emptyset$. If $x \in \omega_H \cap K \subseteq \omega_K$, then $C_K(x) = \omega_K, C_H(x) = \omega_H$. Hence $\omega_K \subseteq \omega_H$ whence $\omega_K \subseteq \omega_H \cap K$. Hence, $\omega_K = \omega_H \cap K$. ■

Corollary 2.5.35. *If K is a subhypergroup of a complete hypergroup $(H, \circ)$, then $\omega_K = \omega_H$.*

Proof. Set $x \in \omega_H \cap K$. We have $\omega_H = C(x \circ x) = x \circ x \subseteq \omega_H \cap K$, whence $\omega_H \subseteq \omega_H \cap K$, then we apply the above theorem. Hence, $\omega_K = \omega_H$. ■

Theorem 2.5.36. *Let H, H' be complete hypergroups and $f : H \to H'$ be a good homomorphism. Then we have $f(\omega_H) = \omega'_H$.*

Proof. Let $x \in \omega_H$. Then $x \circ x = \omega_H$, whence $f(x) \circ f(x) = f(\omega_H)$. On the other hand, $f(x)$ is an identity of H', since x is an identity of H, which means that $f(x) \in \omega_{H'}$. Hence, $\omega'_H = f(x) \circ f(x) = f(\omega_H)$. ■

Chapter 3

The hyperring of Krasner

The more general structure that satisfies the ring-like axioms is the hyperring in the general sense: $(R, +, \cdot)$ is a hyperring if $+$ and $\cdot$ are two hyperoperations such that $(R, +)$ is a hypergroup and $\cdot$ is an associative hyperoperation, which is distributive with respect to $+$. There are different notions of hyperrings. If only the addition $+$ is a hyperoperation and the multiplication $\cdot$ is a usual operation, then we say that R is an additive hyperring. A special case of this type is the hyperring introduced by Krasner [70]. Also, Krasner introduced a new class of hyperrings and hyperfields: the quotient hyperrings and hyperfields. In a long list of papers, Mittas studied firstly the hyperfield's additive part, i.e., the canonical hypergroup, as he has named it, and then, the hyperfield itself.

Throughout this chapter, by a hyperring we mean a Krasner hyperring.

3.1 Definition and constructions of Krasner hyperrings

We give first the definition of a Krasner hyperring and a Krasner hyperfield.

Definition 3.1.1. A *Krasner hyperring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup, i.e.,

(i) for every $x, y, z \in R$, $x + (y + z) = (x + y) + z$,

(ii) for every $x, y \in R$, $x + y = y + x$,

iii) there exists $0 \in R$ such that $0 + x = \{x\}$ for every $x \in R$,

(iv) for every $x \in R$ there exists a unique element $x' \in R$ such that $0 \in x + x'$;
(We shall write $-x$ for x' and we call it the *opposite* of x.)

(v) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$;

(2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$.

(3) The multiplication is distributive with respect to the hyperoperation $+$.

The following elementary facts follow easily from the axioms: $-(-x) = x$ and $-(x+y) = -x-y$, where $-A = \{-a \mid a \in A\}$. Also, for all $a, b, c, d \in R$ we have $(a + b) \cdot (c + d) \subseteq a \cdot c + b \cdot c + a \cdot d + b \cdot d$. In Definition 3.1.1, for simplicity of notations we write sometimes xy instead of $x \cdot y$ and in (iii), $0 + x = x$ instead of $0 + x = \{x\}$.

A Krasner hyperring $(R, +, \cdot)$ is called *commutative* (*with unit element*) if $(R, \cdot)$ is a commutative semigroup (with unit element).

Definition 3.1.2.

(1) A Krasner hyperring is called a *Krasner hyperfield*, if $(R \setminus \{0\}, \cdot)$ is a group.

(2) A Krasner hyperring R is called a *hyperdomain* if R is a commutative hyperring with unit element and $ab = 0$ implies that $a = 0$ or $b = 0$ for all $a, b \in R$.

Some students of Krasner, namely Jean Mittas and D. Stratigopoulos have studied hyperrings and hyperfields. Other names can be also quoted in this topic with interesting contributions: P. Corsini, B. Davvaz, C. Massouros, A. Nakassis, T. Vougiouklis, T. Konguetsof, A. Dramalidis, S. Spartalis, G. Pinotsis, Y. Kemprasit, M. Stefanescu, V. Leoreanu, R. Ameri and many others. We pick up from their papers some constructions of hyperrings.

Example 3.1.3.

(1) Let $R = \{0, 1, 2\}$ be a set with the hyperoperation $+$ and the binary operation $\cdot$ defined as follow:

$+$	0	1	2
0	0	1	2
1	1	1	R
2	2	R	2

and

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

Then $(R, +, \cdot)$ is a hyperring.

(2) The first construction of a hyperring appeared in Krasner's paper [70] and it is the following one: Consider $(F, +, \cdot)$ a field, G a subgroup of $(F^*, \cdot)$ and take $F/G = \{aG \mid a \in F\}$ with the hyperaddition and the multiplication given by

$$aG \oplus bG = \{cG \mid c \in aG + bG\},$$
$$aG \odot bG = abG.$$

Then $(F/G, \oplus, \odot)$ is a hyperring. If $(F, +, \cdot)$ is a unitary ring and G is a subgroup of the monoid $(F^*, \cdot)$ such that $xG = Gx$, for all $x \in F$, then $(F/G, \oplus, \odot)$ is a hyperring with identity.

(3) Let $(A, +, \cdot)$ be a ring and N a normal subgroup of its multiplicative semigroup. Then the multiplicative classes $\overline{x} = xN$ $(x \in A)$ form a partition of R, and let $\overline{A} = A/N$ be the set of these classes. If for all $\overline{x}, \overline{y} \in \overline{A}$, we define

$$\overline{x} \oplus \overline{y} = \{\overline{z} \mid z \in \overline{x} + \overline{y}\}, \quad \text{and} \quad \overline{x} * \overline{y} = \overline{x \cdot y},$$

then the obtained structure is a hyperring.

(4) Let R be a commutative ring with identity. We set $\overline{R} = \{\overline{x} = \{x, -x\} \mid x \in R\}$. Then $\overline{R}$ becomes a hyperring with respect to the hyperoperation $\overline{x} \oplus \overline{y} = \{\overline{x + y}, \ \overline{x - y}\}$ and multiplication $\overline{x} \otimes \overline{y} = \overline{x \cdot y}$.

(5) (Inspired by Lyndon [76, 77]) Let $(G, \cdot)$ be a group, $H = G \cup \{0, u, v\}$, where 0 is an absorbing element under multiplication and u, v are distinct orthogonal idempotents, with

$$\begin{aligned} &a \cdot 0 = 0 \cdot a = 0, \\ &u^2 = u, \\ &v^2 = v, \\ &uv = vu = 0, \\ &ug = gu = u, \ gv = vg = v \text{ for all } g \in G. \end{aligned}$$

If we define the hypersum on H by

$$\begin{aligned} &a + 0 = \{a\} \text{ for all } a \neq 0, \\ &a + a = \{a, 0\} \text{ for all } a \in H, \\ &a + b = H \backslash \{a, b, 0\} \text{ for all } a, b \in H \setminus \{0\} \text{ and } a \neq b, \end{aligned}$$

then $(H, +, \cdot)$ is a hyperring.

(6) [2] If $(L, \wedge, \vee)$ is a relatively complemented distributive lattice, then we define a structure of hyperring on L, by taking:

$$\begin{aligned} &a \oplus b := \{c \in L \mid a \wedge b = a \wedge c = b \wedge c\}, \\ &a \cdot b := a \vee b \text{ for all } a, b \in L. \end{aligned}$$

(7) [93, 125] Let $(G, \cdot)$ be a finite group with m elements, $m > 3$, and define a hyperaddition and a multiplication on $H = G \cup \{0\}$, by:

$$\begin{aligned} &a + 0 = 0 + a = \{a\} \text{ for all } a \in H, \\ &a + a = \{a, 0\} \text{ for all } a \in G, \\ &a + b = b + a = H \backslash \{a, b\} \text{ for all } a, b \in G, a \neq b, \\ &a \odot 0 = 0 \text{ for all } a \in H, \\ &a \odot b = a \cdot b \text{ for all } a, b \in G. \end{aligned}$$

Then $(H, +, \odot)$ is a hyperring.

(8) [125] If $(H, \leq, +)$ is a totally ordered group, then

$$\begin{aligned} &x \oplus x = \{t \in H \mid t \leq x\} \text{ for all } x \in H, \\ &x \oplus y = \{\max\{x, y\}\} \text{ for all } x, y \in H, \ x \neq y, \end{aligned}$$

defines a structure of canonical hypergroup on H. If $(H, +, \cdot)$ is a totally ordered ring (for example $\mathbb{R}$) then $(H, \oplus, \cdot)$ is a hyperring.

(9) [81] Let $(H,+,\cdot)$ be a hyperfield. If we define a new hyperoperation on H, as follows:

$$\begin{aligned} &a \oplus b = (a+b) \cup \{a,b\}, \text{ if } a \neq -b,\ a,b \in H, \\ &a \oplus (-a) = H \text{ for all } a \in H \setminus \{0\}, \\ &a \oplus 0 = 0 \oplus a = a \text{ for all } a \in H, \end{aligned}$$

then $(H,\oplus,\cdot)$ is a new hyperfield. If $(H,+,\cdot)$ is a field then $a \oplus b = \{a,b,a+b\}$, for $a \neq b,\ a,b \in H^*$.

(10) Let R be a hyperring and let S be a multiplicatively closed subset of R such that $0 \notin S$. The relation $\sim$ is defined on $R \times S$ as follows: $(a,s) \sim (b,t)$ if and only if there exists $u \in S$ such that $uta = usb$. This is an equivalence relation on the set $R \times S$. The equivalence class of (a,s) is denoted by a/s and we let $S^{-1}R$ be the quotient set. On $S^{-1}R$, the hyperoperation $\oplus$ is defined by

$$\frac{a}{s} \oplus \frac{b}{t} = \{\frac{c}{st} \mid c \in ta+sb\} = \frac{ta+sb}{st}$$

and the multiplication defined in the standard way. One can easily verify all conditions of Definition 3.1.1. We prove only condition (v). Suppose that $\frac{z}{r} \in \frac{x}{s} \oplus \frac{y}{t} = \frac{xt+ys}{st}$. Then there exists $v \in xt+ys$ such that $\frac{z}{r} = \frac{v}{st}$. Hence there exists $u \in S$ such that $uzst = urv$. Thus $uzst \in ur(xt+ys) = urxt + urys$, and so $urxt \in uzst - urys$ and $urys \in -urxt + uzst$. Therefore

$$\frac{x}{s} = \frac{urtx}{urts} \in \frac{uzst - urys}{urts} = \frac{uzst}{urts} - \frac{urys}{urts} = \frac{z}{r} - \frac{y}{t}$$

and

$$\frac{y}{t} \in \frac{-x}{s} \oplus \frac{z}{r}.$$

Therefore $S^{-1}R$ is a hyperring.

3.2 Hyperideals, quotient hyperrings and homomorphisms

Definition 3.2.1. Let $(R, +, \cdot)$ be a hyperring and A be a nonempty subset of R. Then A is said to be a *subhyperring* of R if $(A, +, \cdot)$ is itself a hyperring.

The subhyperring A of R is *normal* in R if and only if $x + A - x \subseteq A$ for all $x \in R$.

Definition 3.2.2. A subhyperring A of a hyperring R is a *left (right) hyperideal* of R if $r \cdot a \in A \quad (a \cdot r \in A)$ for all $r \in R,\ a \in A$. A is called a *hyperideal* if A is both a left and a right hyperideal.

It would be useful to have some criterions for deciding whether a given subset of a hyperring is a left (right) hyperideal or not. This is the purpose of the next lemma.

Lemma 3.2.3. *A nonempty subset A of a hyperring R is a left (right) hyperideal if and only if*

(1) *$a, b \in A$ implies $a - b \subseteq A$,*

(2) *$a \in A$, $r \in R$ imply $r \cdot a \in A \quad (a \cdot r \in A)$.*

Definition 3.2.4. Let A and B be nonempty subsets of a hyperring R.

- The sum $A + B$ is defined by

$$A + B = \{x \mid x \in a + b \text{ for some } a \in A,\ b \in B\}.$$

- The product AB is defined by

$$AB = \left\{ x \mid x \in \sum_{i=1}^{n} a_i b_i, a_i \in A, b_i \in B, n \in \mathbb{Z}^+ \right\}.$$

If A and B are hyperideals of R, then $A + B$ and AB are also hyperideals of R.

The following two corollaries are obtained directly from definitions.

Corollary 3.2.5. *Let A be a normal hyperideal of R. Then*

(1) $(A+x)+(A+y)=A+x+y$ *for all* $x,y\in R$,

(2) $A+x=A+y$ *for all* $y\in A+x$.

Corollary 3.2.6. *Let A and B be hyperideals of a hyperring R with B normal in R. Then*

(1) $A\cap B$ *is a normal hyperideal of* A,

(2) B *is a normal hyperideal of* $A+B$.

In what follows, we present the isomorphism theorems in the context of hyperings.

Definition 3.2.7. If A is a normal hyperideal of a hyperring R, then we define the relation

$$x\equiv y(modA) \text{ if and only if } x-y\cap A\neq\emptyset.$$

This relation is denoted by xA^*y.

Lemma 3.2.8. *The relation A^* is an equivalence relation.*

Proof. (1) Since $0\in x-x\cap A$ for all $x\in R$, it follows xA^*x, i.e., A^* is reflexive. (2) Suppose that xA^*y then there exists $z\in x-y\cap A$ which implies $-z\in y-x$ and $-z\in A$, which means that yA^*x, and so A^* is symmetric. (3) Let xA^*y and yA^*z where $x,y,z\in R$. Then there exist $a\in x-y\cap A$ and $b\in y-z\cap A$. So $x\in a+y$ and $-z\in -y+b$. Hence $x\in y+a$, $-z\in b-y$ which imply that $x-z\subseteq y+a+b-y$. Since $a+b\subseteq A$ and A is normal, it follows $y+a+b-y\subseteq A$. Therefore $x-z\cap A\neq\emptyset$, hence xA^*z, and so A^* is transitive. ∎

Let $A^*(x)$ be the equivalence class of the element $x\in R$.

Lemma 3.2.9. *If A is a normal hyperideal of R, then $A+x=A^*(x)$ for all $x\in R$.*

Proof. Suppose $y\in A+x$ then there exists $a\in A$ such that $y\in a+x$, which implies $a\in y-x$ and so $y-x\cap A\neq\emptyset$. Thus $A+x\subseteq A^*(x)$. Similarly we have $A^*(x)\subseteq A+x$. ∎

Lemma 3.2.10. *Let A be a normal hyperideal of R. Then for all $x, y \in R$ and for all $z \in x + y$, we have $A + x + y = A + z$.*

Proof. Suppose that $z \in x + y$. It is clear that $A + z \subseteq A + x + y$. Now, let $a \in A + x + y$, by condition (v) of Definition 3.1.1, we get $y \in -(A + x) + a$ or $y \in A - x + a$, and so $x + y \subseteq x + A - x + a$. Since A is normal, we obtain $x + y \subseteq A + a$. Therefore for every $z \in x + y$, we have $z \in A + a$ which implies that $a \in A + z$. ∎

The next two corollaries are fundamental.

Corollary 3.2.11. *For all $x, y \in R$, we have $A^*(A^*(x) + A^*(y)) = A^*(x) + A^*(y)$.*

Proof. The proof follows easily from Lemma 3.2.10. ∎

Corollary 3.2.12. *For all $x, y \in R$, we have $A^*(A^*(x \cdot y)) = A^*(x \cdot y)$.*

Proof. Clearly, we have $A^*(x \cdot y) \subseteq A^*(A^*(x \cdot y))$. Now let $a \in A^*(A^*(x \cdot y))$. Then there exists $b \in A^*(x \cdot y)$ with $a \in A^*(b)$. So aA^*b and $bA^*x \cdot y$ which imply $aA^*x \cdot y$. Hence $a \in A^*(x \cdot y)$. ∎

Proposition 3.2.13. *Let R be a hyperring and A be a normal hyperideal of R. We define the hyperoperation $\oplus$ and the multiplication $\odot$ on the set of all classes $[R : A^*] = \{A^*(x) \mid x \in R\}$, as follows:*

$$\begin{aligned} A^*(x) \oplus A^*(y) &= \{A^*(z) \mid z \in A^*(x) + A^*(y)\}, \\ A^*(x) \odot A^*(y) &= A^*(x \cdot y). \end{aligned}$$

Then $[R : A^*]$ is a hyperring.

Proof. This follows from Corollary 3.2.11, Corollary 3.2.12 and Definition 3.1.1. ∎

Definition 3.2.14. Let R_1 and R_2 be hyperrings. A mapping φ from R_1 into R_2 is said to be a *good (strong) homomorphism* if for all $a, b \in R_1$,

$$\varphi(a + b) = \varphi(a) + \varphi(b), \quad \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b) \text{ and } \varphi(0) = 0.$$

Clearly, a good homomorphism φ is an *isomorphism* if φ is one to one and onto. We write $R_1 \cong R_2$ if R_1 is *isomorphic* to R_2.

Because R_1 is a hyperring, $0 \in a - a$ for all $a \in R_1$, then we have $\varphi(0) \in \varphi(a) + \varphi(-a)$ or $0 \in \varphi(a) + \varphi(-a)$ which implies that $\varphi(-a) \in$

$-\varphi(a)+0$, therefore $\varphi(-a) = -\varphi(a)$ for all $a \in R_1$. Moreover, if φ is a good homomorhism from R_1 into R_2, then the *kernel* of φ is the set $ker\varphi = \{x \in R_1 \mid \varphi(x) = 0\}$. It is trivial that $ker\varphi$ is a hyperideal of R_1, but in general it is not normal in R_1.

Corollary 3.2.15. *Let φ be a good homomorphism from R_1 into R_2. Then φ is one to one if and only if $ker\varphi = \{0\}$.*

Proof. Let $y, z \in R_1$ be such that $\varphi(y) = \varphi(z)$. Then $\varphi(y) - \varphi(y) = \varphi(z) - \varphi(y)$. It follows that $\varphi(0) \in \varphi(z-y)$, and so there exists $x \in z - y$ such that $0 = \varphi(0) = \varphi(x)$. Thus, if $ker\varphi = \{0\}$, $x = 0$, whence $y = z$.

Now, let $x \in ker\varphi$. Then $\varphi(x) = 0 = \varphi(0)$. Thus, if φ is one to one, we conclude that $x = 0$. ■

We present now the first isomorphism theorem.

Theorem 3.2.16. (First isomorphism theorem). *If φ is a good homomorphism from R_1 into R_2 with the kernel K, such that K is a normal hyperideal of R_1, then $[R_1 : K^*] \cong Im\varphi$.*

Proof. We define $\rho : [R_1 : K^*] \longrightarrow Im\varphi$ by setting $\rho(K^*(x)) = \varphi(x)$ for all $x \in R_1$. We prove firstly that ρ is well-defined. Suppose that xK^*y. So there exists $z \in x - y \cap K$. Consequently, $\varphi(z) = 0$ and $\varphi(z) \in \varphi(x) - \varphi(y)$. Thus $\varphi(x) = \varphi(y)$. Clearly ρ is onto. To show that ρ is one to one, suppose that $\varphi(x) = \varphi(y)$. Then $0 \in \varphi(x-y)$, and so there exists $z \in x - y$ with $z \in ker\varphi$. Therefore $x - y \cap K \neq \emptyset$ which implies that $K^*(x) = K^*(y)$, and so ρ is one to one. Moreover,

$$\begin{aligned}
\rho(K^*(x) \oplus K^*(y)) &= \rho(\{K^*(z) \mid z \in K^*(x) + K^*(y)\}) \\
&= \{\varphi(z) \mid z \in K^*(x) + K^*(y)\}) \\
&= \varphi(K^*(x)) + \varphi(K^*(y)) = \varphi(x) + \varphi(y) \\
&= \rho(K^*(x)) + \rho(K^*(y)), \\
\rho(K^*(x) \odot K^*(y)) &= \rho(K^*(x \cdot y)) = \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) \\
&= \rho(K^*(x)) \cdot \rho(K^*(y)),
\end{aligned}$$

and $\rho(K^*(0)) = \varphi(0) = 0$. Therefore ρ is an isomorphism. ■

Using Lemma 3.2.9, we can put $[R : A^*]=\{A + x \mid x \in R\}$ and then we have

Corollary 3.2.17. *If A is a normal hyperideal of R, then*

$$(A+x)\oplus(A+y)=\{A+z \mid z\in x+y\}.$$

Proof. Using Corollary 3.2.5 and Lemma 3.2.10 we have

$$\begin{aligned}(A+x)\oplus(A+y) &= \{A+c \mid c\in A+x+A+y\}\\ &= \{A+c \mid c\in A+x+y\}\\ &= \{A+c \mid c\in A+z,\ z\in x+y\}\\ &= \{A+z \mid z\in x+y\}. \blacksquare\end{aligned}$$

We are now in a position to state and prove the second and third isomorphism theorems in hyperrings.

Theorem 3.2.18. (Second isomorphism theorem). *If A and B are hyperideals of a hyperring R, and B is normal in R, then $[A:(A\cap B)^*]\cong[A+B:B^*]$.*

Proof. Clearly, B is a normal hyperideal of $A+B$; Consequently, we can consider $[A+B:B^*]$. Define $\rho: A\longrightarrow[A+B:B^*]$ by $\rho(a)=B+a$. ρ is a strong homomorphism. For all $B+y\in[A+B:B^*]$, $y\in A+B$, there are $a\in A$, $b\in B$, such that $y\in a+b$. Thus, by Lemma 3.2.10, $B+y=B+a+b=B+a=\rho(a)$. This shows that ρ is also onto. If we establish that $ker\rho=A\cap B$, then we shall obtain that $[A:(A\cap B)^*]\cong[A+B:B^*]$, since $A\cap B$ is a normal hyperideal of A. For any $a\in A$, we have $a\in ker\rho \Longleftrightarrow \rho(a)=B \Longleftrightarrow B+a=B \Longleftrightarrow a\in B \Longleftrightarrow a\in A\cap B$ (since $a\in A$). This yields $ker\rho=A\cap B$, whence it follows the thesis. $\blacksquare$

Theorem 3.2.19. (Third isomorphism theorem). *If A and B are normal hyperideals of a hyperring R such that $A\subseteq B$, then $[B:A^*]$ is a normal hyperideal of $[R:A^*]$ and $[[R:A^*]:[B:A^*]]\cong[R:B^*]$.*

Proof. We leave to reader to verify that $[B:A^*]$ is a normal hyperideal of $[R:A^*]$. Moreover, $\rho:[R:A^*]\longrightarrow[R:B^*]$ defined by $\rho(A+x)=B+x$ is a strong homomorphism from $[R:A^*]$ onto $[R:B^*]$ such that $ker\rho=[B:A^*]$. $\blacksquare$

Definition 3.2.20. Let R be a commutative ring with unit element, A be a subhyperring of R and I be a hyperideal of A. Then I is said to be *R-regular* if $I\cap U(R)\neq\emptyset$, where $U(R)$ is the set of all invertible elements in R.

At the end of this section we provide a general method for simplifying the study of hyperrings by decomposing a hyperring into more manageable hyperrings. The motivation comes from a venerable result of number theory often called the Chinese Remainder Theorem. Given $m_1, ..., m_t, n_1, ..., n_t$ in $\mathbb{N}$ where $m_1, ..., m_t$ are pairwise relatively prime, one can find some $n \in \mathbb{N}$ such that $n \equiv n_i(mod m_i)$ for $1 \leq i \leq t$. In other words, the Chinese Remainder Theorem says that if $m_1, ..., m_t$ are pairwise relatively prime then the canonical homomorphism $\mathbb{Z} \longrightarrow \prod_{i=1}^{t} \mathbb{Z}/m_i\mathbb{Z}$ is onto. A domain A is said to satisfy the Chinese Remainder Theorem when, for a finite number of integral ideals $A_1, ..., A_n$ of A and elements $x_i \in A$, $1 \leq i \leq n$ such that $x_i \equiv x_j(mod A_i + A_j)$, $1 \leq i, j \leq n$, there exists a solution of the system of congruences $x \equiv x_i(mod A_i)$, $1 \leq i \leq n$. It is well known that a domain A satisfies the Chinese Remainder Theorem if and only if A is a Prüfer domain. Now, we will consider the Chinese Remainder Theorem for hyperrings.

Definition 3.2.21. Let A be a subhyperring of a hyperring R. We say that the Chinese Remainder Theorem holds for A with respect to R if the following condition is valid:

For all normal hyperideals $A_1, ..., A_n$ of A, such that at most two are not R-regular and for all elements $x_1, ..., x_n \in A$, the system of congruences $x \equiv x_i(mod A_i)$ admits a solution $x \in A$ if and only if $x_i \equiv x_j(mod A_i + A_j)$ for all $i \neq j$.

Theorem 3.2.22. *Let R be a hyperring and let A be a subhyperring of R. If*

(1) $(\forall a, b, c, d \in R)\ \{a, b\} \subseteq c - d \implies ab \in c - d$,

(2) $L + (M \cap N) = (L + M) \cap (L + N)$ *for all hyperideals* L, M, N *of* A *if at least one of them is* R*-regular,*

then A satisfies the Chinese Remainder Theorem with respect to R.

Proof. We shall prove this theorem by mathematical induction. First, suppose that n=2. Since $x_1 \equiv x_2(mod A_1 + A_2)$, then $(x_1 - x_2) \cap (A_1 + A_2) \neq \emptyset$

and so there exists $z \in (x_1 - x_2) \cap (A_1 + A_2)$ which implies that $z \in (x_1 - x_2) \cap (a_1 + a_2)$ for some $a \in A_1$ and $a_2 \in A_2$. Now, we have

$$\begin{aligned} z \in a_1 + a_2 &\Longrightarrow a_2 \in -a_1 + z \\ &\Longrightarrow a_2 \in -a_1 + x_1 - x_2 \\ &\Longrightarrow a_2 + x_2 \subseteq -a_1 + x_1 - x_2 + x_2 \\ &\Longrightarrow a_2 + x_2 \subseteq x_1 + (-x_2 - a_1 + x_2) \\ &\Longrightarrow a_2 + x_2 \subseteq x_1 + A_1. \end{aligned}$$

Therefore if we consider $t_0 \in a_2 + x_2$ then $t_0 \in x_1 + A_1$ and so there exists $a_1' \in A_1$ such that $t_0 \in x_1 + a_1'$. Now, we have $a_2 \in t_0 - x_2$ and $a_1' \in -x_1 + t_0$. Hence $(t_0 - x_2) \cap A_2 \neq \emptyset$ and $(t_0 - x_1) \cap A_1 \neq \emptyset$. Therefore $t_0 \equiv x_2(modA_2)$ and $t_0 \equiv x_1(modA_1)$, i.e., t_0 is a solution of the system.

Now, let $x \equiv x_i(modA_i)$ for $i=1,\dots,k+1$ be such that $x_i \equiv x_j(modA_i + A_j)$ $i \neq j$ and $1 \leq i, j \leq k+1$. Suppose that there exists $y \in A$ such that $y \equiv x_i(modA_i)$ for $1 \leq i \leq k$. We consider the following system of congruences:

$$\begin{aligned} x &\equiv y(mod \bigcap_{i=1}^{k} A_i) \\ x &\equiv x_{k+1}(modA_{k+1}). \end{aligned}$$

It is easy to see that any solution x of this system satisfies also the system

$$x \equiv x_i(modA_i) \quad 1 \leq i \leq k.$$

In order to solve the first system of two congruences it suffices to show that

$$y \equiv x_{k+1}(mod((\bigcap_{i=1}^{k} A_i) + A_{k+1})).$$

By the condition (2) we have

$$\left(\bigcap_{i=1}^{k} A_i\right) + A_{k+1} = \bigcap_{i=1}^{k}(A_i + A_{k+1}).$$

Since $y \equiv x_i(modA_i)$ for $1 \leq i \leq k$, there exists $z_i \in (y - x_i) \cap A_i$ for $1 \leq i \leq k$. Since $x_{k+1} \equiv x_i(modA_{k+1} + A_i)$ for $1 \leq i \leq k$, there exists

$t_i \in (x_{k+1} - x_i) \cap (A_{k+1} + A_i)$. Hence $-x_i \in -x_{k+1} + t_i$ and $t_i \in A_{k+1} + A_i$. Also, we have

$$y \in z_i + x_i \subseteq z_i + (x_{k+1} - t_i) = (z_i - t_i) + x_{k+1} \subseteq (A_i + A_{k+1}) + x_{k+1}.$$

Therefore there exists $c_i \in A_i + A_{k+1}$ such that $y \in c_i + x_{k+1}$ for $1 \leq i \leq k$, and so $c_i \in y - x_{k+1}$. Now, we set $c = c_1 ... c_k$ and we obtain

$$c \in \bigcap_{i=1}^{k} (A_i + A_{k+1}).$$

Moreover, using the condition (1) we have $c \in y - x_{k+1}$ and hence

$$c \in (y - x_{k+1}) \cap \left(\bigcap_{i=1}^{k} A_i + A_{k+1} \right)$$

which implies that

$$y \equiv x_{k+1} \left(mod \left(A_{k+1} + \bigcap_{i=1}^{k} A_i \right) \right)$$

which completes the proof of the theorem. ∎

3.3 Special hyperideals

Definition 3.3.1. Let X be a subset of a hyperring R. Let $\{A_i \mid i \in J\}$ be the family of all hyperideals in R which contain X. Then $\bigcap_{i \in J} A_i$ is called the *hyperideal generated by* X. This hyperideal is denoted by $< X >$. If $X = \{x_1, x_2, ..., x_n\}$, then the hyperideal $< X >$ is denoted $< x_1, x_2, ..., x_n >$.

Theorem 3.3.2. *Let R be a hyperring, $a \in R$ and $X \subset R$. Then*

(1) *The principal hyperideal $< a >$ is equal to the set*

$$\{t \mid t \in ra + as + na + k(a - a) + \sum_{i=1}^{m} r_i a s_i, \; r, s, r_i, s_i \in R, m \in \mathbb{Z}^+ \; and \; n, k \in \mathbb{Z}\}.$$

(2) *If R has a unit element, then*

$$< a >= \{t \mid t \in k(a-a) + \sum_{i=1}^{m} r_i a s_i,\ r_i, s_i \in R, m, k \in \mathbb{Z}^+\}.$$

(3) *If a is in the center of R, then*

$$< a >= \{t \mid t \in ra + na + k(a-a),\ r \in R, n \in \mathbb{Z}^+\},$$

where the center of R is the set $\{x \in R \mid xy = yx \text{ for all } y \in R\}$.

(4) *$Ra = \{ra \mid r \in R\}$ is a left hyperideal in R and $aR = \{ar \mid r \in R\}$ is a right hyperideal in R. If R has a unit element then $a \in aR \cap Ra$.*

(5) *If R has a unit element and a is in the center of R then $Ra = \langle a \rangle = aR$.*

(6) *If R has a unit element and X is included in the center of R, then*

$$< X >= \{t \mid t \in \sum_{i=1}^{m} r_i x_i,\ r_i \in R, x_i \in X,\ m \in \mathbb{Z}^+\}.$$

Proof. The proof is straightforward. ∎

Definition 3.3.3. A proper hyperideal M of R is a *maximal hyperideal* of R if the only hyperideals of R that contain M are M itself and R.

Proposition 3.3.4. *Let R be a commutative hyperring with a unit element and let I be a proper hyperideal of R. Then there exists a maximal hyperideal of R containing I.*

Definition 3.3.5. A proper hyperideal P of a hyperring R is called *prime* if for every pair of hyperideals A and B of R

$$AB \subseteq P \text{ implies } A \subseteq P \text{ or } B \subseteq P.$$

Lemma 3.3.6. *Let R be a commutative hyperring. A hyperideal P of R is prime if $P \neq R$ and for every $a, b \in R$*

$$ab \in P \text{ implies } a \in P \text{ or } b \in P.$$

Proposition 3.3.7. *If R is a commutative hyperring with a unit element, then each maximal hyperideal M of a hyperring R is a prime hyperideal.*

Proof. Let $x, y \in R$ be elements such that $xy \in M$ and $x \notin M$. The set $M + xR = \cup\{a + b \mid a \in M, b \in xR\}$ is a hyperideal of R containing both M and xR. Therefore we have $x \in M + xR$ and $x \notin M$. Hence we conclude that $1 \in M + xR$, and so $1 \in a + xr$ for some $a \in M$ and $r \in R$. Therefore we obtain $y \in y(a + xr) = ya + yxr \subseteq M + M \subseteq M$. Hence $y \in M$. ∎

Proposition 3.3.8. *Let R be a commutative hyperring with a unit element.*

(1) *Let $M \neq R$ be a hyperideal of R. Then M is maximal if and only if R/M is a hyperfield.*

(2) *Let $P \neq R$ be a hyperideal of R. Then P is prime if and only if R/P is a hyperdomain.*

Proof. The proofs are similar to the proofs of Theorems 1.6.3 and 1.6.7. ∎

For a hyperring R we define the Jacobson radical $J(R)$ of R as the intersection of all maximal hyperideals of R. If R does not have any maximal hyperideal, then we define $J(R) = R$.

Proposition 3.3.9. *Let R be a commutative hyperring with a unit element and I be a hyperideal of R. Then $I \subseteq J(R)$ if and only if every element of $1 + I$ is invertible.*

Proof. Let $I \subseteq J(R)$ and suppose that there exists $x \in 1 + I$ such that x is not invertible. Clearly $x \in 1 + a$ for some $a \in I$. Since x is not invertible, there exists a maximal hyperideal M such that $x \in M$. But $x \in 1 + a$ implies that $1 \in x - a \subseteq M$, which is a contradiction. Hence every element of $1 + I$ is invertible.

Conversely, suppose that any element of $1+I$ is invertible but $I \not\subseteq J(R)$. Thus $I \not\subseteq M$ for some maximal hyperideal M of R. Then there exists $a \in I \setminus M$. Therefore $< M, a >= R$. So $1 \in m + ra$ for some $r \in R$ and $m \in M$, hence $m \in 1 - ra \subseteq 1 + I$. Thus m is invertible, which is a contradiction with maximality of M. ∎

Corollary 3.3.10. *Let R be a commutative hyperring with a unit element and $U(R)$ be the set of all invertible elements in R. Then an element $a \in R$ belongs to $J(R)$ if and only if $1 - ba \subseteq U(R)$ for all $b \in R$.*

Definition 3.3.11. Let R be a hyperring. A finite chain of n normal hyperideals $A_0, A_1, ..., A_{n-1}$ of R

$$R = A_0 \supset A_1 \supset ... \supset A_n = 0$$

is called a *composition series of length* n for R provided that $[A_{i-1} : A_i^*]$ is *simple* ($i = 1, ..., n$), i.e., if each term in the chain is maximal in its predecessor.

We shall consider a generalization of the Jordan-Holder theorem for hyperrings.

Theorem 3.3.12. (Jordan-Holder Theorem). *If a hyperring R has a composition series, then any two composition series for R are equivalent.*

Proof. If R has a composition series, then denote by $c(R)$ the minimum length of such series for R. We shall prove by induction on $c(R)$. Clearly, if $c(R) = 1$, there is nothing to prove. So assume that $c(R) = n > 1$ and that any hyperring with a composition series of smaller length than n has all of its composition series equivalent. Let

$$R = A_0 \supset A_1 \supset ... \supset A_n = 0 \tag{I}$$

be a composition series of length n for R and

$$R = B_0 \supset B_1 \supset ... \supset B_m = 0 \tag{II}$$

be a second composition series for R. If $A_1 = B_1$, then by induction hypotheses and since $c(A_1) \leq n-1$, if follows that the two series are equivalent. So we may assume that $A_1 \neq B_1$. Then since A_1 is a maximal hyperideal of R, we have $A_1 + B_1 = R$, so

$$[R : A_1^*] = [A_1 + B_1 : A_1^*] \cong [B_1 : (A_1 \cap B_1)^*],$$

and

$$[R : B_1^*] = [A_1 + B_1 : B_1^*] \cong [A_1 : (A_1 \cap B_1)^*].$$

Thus $A_1 \cap B_1$ is maximal in both A_1 and B_1. Now, $A_1 \cap B_1$ has a composition series

$$A_1 \cap B_1 = C_0 \supset C_1 \supset ... \supset C_k = 0.$$

So

$$A_1 \supset C_0 \supset C_1 \supset \ldots \supset C_k = 0 \quad \text{and} \quad B_1 \supset C_0 \supset C_1 \supset \ldots \supset C_k = 0,$$

are composition series for A_1 and B_1. Since $c(A_1) < n$, every two composition series for A_1 are equivalent, so the two series

$$R = A_0 \supset A_1 \supset \ldots \supset A_n = 0 \quad \text{and} \quad R = A_0 \supset A_1 \supset C_0 \supset \ldots \supset C_k = 0,$$

are equivalent. In particular, $k < n - 1$, so clearly $c(B_1) < n$. Thus by our induction hypothesis, every two composition series for B_1 are equivalent. Thus the two series

$$R = B_0 \supset B_1 \supset \ldots \supset B_m = 0 \quad \text{and} \quad R = B_0 \supset B_1 \supset C_0 \supset \ldots \supset C_k = 0,$$

are equivalent. Since

$$[R : A_1^*] \cong [B_1 : C_0^*] \text{ and } [R : B_1^*] \cong [A_1 : C_0^*];$$

thus the series (I) and (II) are equivalent. ■

The mapping $\varphi : R \longrightarrow S^{-1}R$ given by $\varphi(a) = a/1$ is a good homomorphism. If I is a hyperideal of R, then $\varphi(I) = S^{-1}I = \{\frac{i}{s} \in S^{-1}R \mid i \in I, s \in S\}$ is also a hyperideal of $S^{-1}R$. $S^{-1}I$ is called the *extension* of I in $S^{-1}R$. Note that $r/s \in S^{-1}I$ need not imply that $r \in I$, since it is possible to have $a/s = r/s$ with $a \in I$ and $r \notin I$.

The proofs of the following theorems are straightforward and they are left to the reader.

Theorem 3.3.13. *Let S be a multiplicative subset of a hyperring R. If I and J are ideals in R, then*

(1) $S^{-1}(I + J) = S^{-1}I + S^{-1}J$,

(2) $S^{-1}(IJ) = (S^{-1}I)(S^{-1}J)$,

(3) $S^{-1}(I \cap J) = S^{-1}I \cap S^{-1}J$.

Theorem 3.3.14. *Let I be a hyperideal of a hyperring R. Then $S \cap I \neq \emptyset$ if and only if $S^{-1}I = S^{-1}R$.*

Theorem 3.3.15. *Let $\varphi : R \longrightarrow S^{-1}R$ be the natural homomorphism and let I be a hyperideal in R. Then*

(1) *$I \subseteq \varphi^{-1}(S^{-1}I)$.*

(2) *If $I = \varphi^{-1}(J)$ for some hyperideal J in $S^{-1}R$, then $S^{-1}I = J$. In other words every hyperideal in $S^{-1}R$ is of the form $S^{-1}I$ for some hyperideal I in R.*

(3) *If P is a prime hyperideal in R and $S \cap P = \emptyset$, then $S^{-1}P$ is a prime hyperideal in $S^{-1}R$ and $\varphi^{-1}(S^{-1}P) = P$.* ■

Theorem 3.3.16. *Let S be a multiplicative subset of a hyperring R. Then there is a one to one correspondence between the set $\mathcal{U}$ of prime hyperideals of R, which are disjoint with S and the set $\mathcal{V}$ of prime hyperideals of $S^{-1}R$, given by $P \longmapsto S^{-1}P$.*

Let R be a hyperring and P a prime hyperideal of R. Then $S = R \backslash P$ is a multiplicative subset of R. The hyperring of fractions $S^{-1}R$ is called the *localization* of R at P and is denoted by R_P. If I is a hyperideal of R, then the hyperideal $S^{-1}I$ in R_P is denoted by I_P.

Theorem 3.3.17. *Let P be a prime hyperideal in a hyperring R.*

(1) *There is a one to one correspondence between the set of prime hyperideals of R which are contained in P and the set of prime hyperideals of R_P, given by $Q \longmapsto Q_P$.*

(2) *The hyperideal P_P in R_P is the unique maximal hyperideal of R_P.*

Let R be a hyperring and A be a subhyperring of R. For a multiplicatively closed subset S of A we can form the large hyperring of quotients $A_{[S]} = \{x \in R \mid (\exists s \in S)\ xs \in A\}$. In fact $A_{[S]}$ is a subhyperring of R and $A \subseteq A_{[S]}$. For a hyperideal I of A its large extension is defined as the set $[I]A_{[S]} = \{x \in R \mid (\exists s \in S)\ xs \in I\}$ which is a hyperideal of $A_{[S]}$. If $S = A \backslash P$, for a given prime hyperideal P of A, then we shall write $A_{[P]}$ instead of $A_{[A\backslash P]}$. It is evident that the following equality holds: $([P]A_{[P]}) \cap A = P$.

3.4 Hypervaluations

In this section, all hyperrings are commutative and we define a hypervaluation on a commutative hyperring. For this, as in the classical case we need a mapping from R onto an ordered group G. If the element $\infty \notin G$, then we define $a \cdot \infty = \infty \cdot a = \infty \cdot \infty = \infty > a$ for all $a \in G$. Then some interesting results concerning this concept are proved.

We say that an arbitrary group G is partially ordered by $\leq$ if $(G, \leq)$ is a poset in which $a < b$ implies $ga < gb$ and $ag < bg$ for all $g \in G$. We see that if $a < a'$ and $b < b'$ then $ab < a'b'$. Consequently, G has a submonoid $P = \{g \in G \mid 1 \leq g\}$ called the positive cone. The following properties are straightforward, writing P^{-1} for $\{a^{-1} \mid a \in P\}$:

(1) $P \cap P^{-1} = \{1\}$,

(2) If $\leq$ is a total order then $P \cup P^{-1} = G$.

Proposition 3.4.1. *If G is a totally ordered group, then $G_\infty = G \cup \{\infty\}$ is a hyperring with the hyperoperation $\oplus$ having the following properties:*

(1) $a < b \implies a \oplus b = \{a\}$ *for all* $a, b \in G_\infty$,

(2) $a \oplus a = \{g \in G_\infty \mid a \leq g\}$,

and the multiplication $a \odot b = a \vee b$ for all $a, b \in G_\infty$.

Proof. This proposition is due to Nakano (see [91]). Nakano showed in [91] that this structure is an m-ring. Notice that in m-ring, we don't have the conditions (iii) and (iv) in Definition 3.1.1, and instead of condition (v), we have $a \in b \oplus c \implies b \in a \oplus c$. Therefore it is enough to show that the conditions (iii), (iv) and (v) hold. If we set $0 = \infty$, then $a \oplus 0 = a$. Since $0 \in a \oplus a = \{g \in G_\infty \mid a \leq g\}$, then $a = -a$. Hence clearly the condition (v) of Definition 3.1.1 holds. ■

The symbol ∞ will be usually adjoined to an ordered group G in such a way that $a \cdot \infty = \infty \cdot a = \infty \cdot \infty = \infty > a$ for all $a \in G$. As above, we denote $G_\infty = G \cup \{\infty\}$.

Definition 3.4.2. Let R be a hyperring. By a *hypervaluation* on R we mean a map μ from R onto G_∞, where G is a totally ordered abelian group G, such that the following conditions are satisfied:

(1) $\mu(0) = \infty$,

(2) $\mu(xy) = \mu(x) \cdot \mu(y)$ for all $x, y \in R$,

(3) $\mu(-x) = \mu(x)$ for all $x \in R$,

(4) $z \in x + y \implies \mu(z) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y, z \in R$.

Lemma 3.4.3. *In* (4) *we have* $\mu(z) = \min\{\mu(x), \mu(y)\}$ *whenever* $\mu(x) \neq \mu(y)$.

Proof. Suppose that $z \in x + y$, $\mu(z) > \min\{\mu(x), \mu(y)\}$ and $\mu(x) < \mu(y)$, then $\mu(z) > \mu(x)$. Since $z \in x + y$, we have $x \in z - y$ and so

$$\mu(x) \geq \min\{\mu(z), \mu(-y)\} = \min\{\mu(z), \mu(y)\} > \mu(x)$$

which is a contradiction. ■

If $\mu : R \longrightarrow G_\infty = G \cup \{\infty\}$ is a hypervaluation, we say that (R, μ, G) is a hypervaluated hyperring.

Proposition 3.4.4. *Let* (R, μ, G) *be a hypervalued hyperring. We set*

$$R^\mu = \{x \in R \mid \mu(x) \geq 1\} \quad \text{and} \quad P^\mu = \{x \in R \mid \mu(x) > 1\}.$$

Then

(1) R^μ *is a subhyperring of* R,

(2) P^μ *is a prime hyperideal of* R^μ,

(3) *The set* $\mu^{-1}(\infty)$ *is a prime hyperideal of* R *contained in* P^μ.

Proof. (1) Let $x, y \in R^\mu$. We must show that $x - y \subseteq R^\mu$ and $xy \in R^\mu$. For every $z \in x - y$ we have $\mu(z) \geq \min\{\mu(x), \mu(-y)\} = \min\{\mu(x), \mu(y)\} \geq 1$ and so $z \in R^\mu$. Hence $x - y \subseteq R^\mu$. Also, we have $\mu(xy) = \mu(x)\mu(y) \geq 1$ thus $xy \in R^\mu$.

(2) Clearly, P^μ is a hyperideal of R^μ. Therefore we show that P^μ is a prime hyperideal of R^μ. Let $x, y \in R^\mu$ be elements such that $xy \in P^\mu$ and $x \notin P^\mu$. Since $x \in R^\mu$ and $x \notin P^\mu$, then $\mu(x) = 1$. From $xy \in P^\mu$, we have $\mu(xy) > 1$ and so $1 < \mu(xy) = \mu(x)\mu(y) = \mu(y)$ which implies that $y \in P^\mu$.

(3) Suppose that $x, y \in \mu^{-1}(\infty)$ and $r \in R$. Then $\mu(x) = \mu(y) = \infty$. For every $z \in x - y$ we have $\mu(z) \geq \min\{\mu(x), \mu(y)\} = \infty$, and so $\mu(z) = \infty$

or $z \in \mu^{-1}(\infty)$ which implies that $x - y \subseteq \mu^{-1}(\infty)$. Also, we have $\mu(rx) = \mu(r)\mu(x) = \mu(r)\infty = \infty$ which implies that $rx \in \mu^{-1}(\infty)$. Therefore $\mu^{-1}(\infty)$ is a hyperideal of R. Now, let $x, y \in R$ be elements such that $xy \in \mu^{-1}(\infty)$ and $x \notin \mu^{-1}(\infty)$. Then $\mu(xy) = \infty$ and $\mu(x) \neq \infty$. Since $\mu(xy) = \mu(x)\mu(y) = \infty$, we get $\mu(y) = \infty$ or $y \in \mu^{-1}(\infty)$. Thus $\mu^{-1}(\infty)$ is a prime hyperideal of R. Clearly we have $\mu^{-1}(\infty) \subseteq P^{\mu}$. ■

We say that μ is a nontrivial hypervaluation if $\mu(R) \neq \{\infty\}$.

The class of hypervalued hyperrings has many properties which are similar to the corresponding properties of the class of Manis valuation rings. In what follows we present some of these common properties.

Proposition 3.4.5. *Let R be a hyperring, let A be a subhyperring of R and let P be a proper prime hyperideal of A. Then the following statements are equivalent:*

(1) $(\forall x \in R \setminus A)(\exists y \in P)\ xy \in A \setminus P$,

(2) *there exists a hypervaluation $\mu : R \longrightarrow G_\infty$ such that $R^\mu = A$ and $P^\mu = P$.*

Proof. ($2 \Longrightarrow 1$) If an element x belong to the set $R \setminus A$, then $\mu(x) < 1$ and therefore $\mu(x) \neq \infty$; i.e., $\mu(x) \in G$. Thus we have $\mu(x)^{-1} = \mu(y)$ for some $y \in R$. Further, we get $1 < \mu(x)^{-1} = \mu(y)$; i.e., $y \in P^\mu = P$. Finally $\mu(xy) = \mu(x)\mu(y) = 1$ and therefore $xy \in R^\mu \setminus P^\mu$.

($1 \Longrightarrow 2$) Let us notice that (1) implies the following property:

$$(\forall x, y \in R)\ xy \in P \implies x \in P \text{ or } y \in P.$$

Now, for each $x \in R$ we set $(P : x)_R = \{z \in R \mid xz \in P\}$, and then we define the equivalence relation $\sim$ by

$$x \sim y \text{ if and only if } (P : x)_R = (P : y)_R.$$

We denote the equivalence class of an element $x \in R$ by $\mu(x)$. We define the multiplication on the set $R/\sim$ by $\mu(x) \cdot \mu(y) = \mu(xy)$. One can prove that the set $G = \{\mu(x) \mid x \in R\} \setminus \{\mu(0)\}$ is a totally ordered group with respect to the above defined multiplication where the ordering is given by

$$\mu(x) \leq \mu(y) \text{ if and only if } (P : x)_R \subseteq (P : y)_R.$$

Moreover, we take $\mu(0) = \infty$. It can be proved that μ is a hypervaluation. Indeed, let $z \in x+y$ and let $\mu(x) < \mu(y)$. Firstly, suppose that $\mu(z) < \mu(x)$; i.e., $zu \notin P$ and $ux \in P$ for some $u \in R$. We show that $yu \notin P$. Suppose that $yu \in P$. Then

$$zu \in (x+y)u = xu + yu \in P + P \subseteq P;$$

i.e., $zu \in P$, which is a contradiction. From the assumption $\mu(x) < \mu(y)$ we conclude that $yt \in P$ and $xt \notin P$ for some $t \in R$. Thus we obtain $xt \cdot yu = xu \cdot yt \in P$ and so we have $xt \in P$ or $yu \in P$, which is a contradiction. In the same way we obtain a contradiction in the case $\mu(x) < \mu(z)$. Thus we have proved that $\mu(z) = \mu(x)$ for all $z \in x+y$, if $\mu(x) \neq \mu(y)$.

If $\mu(x) = \mu(y)$, we must show that $\mu(z) \geq \mu(x)$ for any $z \in x+y$, i.e., $\{u \in R \mid ux \in P\} \subseteq \{u \in R \mid uz \in P\}$. Let $u \in R$ and $ux \in P$. Since $\mu(x) = \mu(y)$ we have $uy \in P$ and $uz \in ux + uy \subseteq P + P \subseteq P$. Hence $(P:x)_R \subseteq (P:z)_R$, i.e., $\mu(x) \leq \mu(z)$. The rest of the proof may be done easily and it is left to the reader. ■

In Proposition 3.4.5, the pair (A, P) is called a *hypervaluation pair* of R.

Proposition 3.4.6. *Let R be a hyperring and let $\mu : R \longrightarrow G_\infty$ be a nontrivial hypervaluation on R. Then we have*

(1) $\mu^{-1}(\infty) = \{x \in R \mid (\forall y \in R \backslash R^\mu)\ xy \in R^\mu\}$,

(2) *If P is a prime hyperideal of R^μ such that $P \subseteq P^\mu$ and $P \not\subseteq \mu^{-1}(\infty)$, then $\mu^{-1}(\infty) \subseteq P$.*

Proof. (1) Let $x \in R$ be an element such that $\mu(x) = \infty$ and $y \in R \setminus R^\mu$. Then we have $\mu(xy) = \mu(x)\mu(y) = \infty$, and hence $xy \in R^\mu$. Now, let us suppose that for an element $x \in R$ and for every $y \in R \backslash R^\mu$ we have $xy \in R^\mu$. Then $\mu(x) = \infty$. In fact, if $\mu(x) < 1$, we can take $y = x \in R \backslash R^\mu$ and deduce that $\mu(xy) < 1$, i.e., $xy \notin R^\mu$. So we consider the case $1 \leq \mu(x) < \infty$. If $\mu(x) > 1$, we take $y \in R$ such that $\mu(y) = \mu(x)^{-1} < 1$ and $\mu(y^2) < \mu(y)$. Therefore, it follows that $xy^2 \notin R^\mu$ and $y^2 \in R \backslash R^\mu$, which is a contradiction. Finally, if $\mu(x) = 1$, take any $y \in R \setminus R^\mu$, and hence $\mu(xy) < \mu(x) = 1$. Then it follows that $xy \notin R^\mu$ and $y \in R \setminus R^\mu$, a contradiction. Thus we can conclude that $\mu(x) = \infty$.

(2) Let P be a prime hyperideal of R^μ and let $P \subseteq P^\mu$ be such that $P \nsubseteq \mu^{-1}(\infty)$. Take $p \in P$ with $\mu(p) < \infty$ and let $n \in \mu^{-1}(\infty)$. Then $\mu(p)^{-1} = \mu(x)$ for some $x \in R \setminus R^\mu$. It is immediate that $nx \in P^\mu$. Further, we have

$$xp \in R^\mu \setminus P^\mu \subseteq R^\mu \setminus P \implies xp \cdot n = xn \cdot p \in P^\mu P \subseteq P \implies n \in P,$$

since $xp \notin P$. Therefore, we obtain that $\mu^{-1}(\infty) \subseteq P$. ■

Proposition 3.4.7. *Let R be a hyperring and let $\mu : R \longrightarrow G_\infty$ and $\lambda : R \longrightarrow H_\infty$ be nontrivial hypervaluations on R. Then $R^\mu = R^\lambda$ and $P^\mu = P^\lambda$ if and only if $\lambda = f \circ \mu$ for some order preserving isomorphism $f : G_\infty \longrightarrow H_\infty$.*

Proof. Let us suppose that $R^\mu = R^\lambda$ and $P^\mu = P^\lambda$. We set $f(\mu(x)) = \lambda(x)$ for every $\mu(x) \neq \infty$ and $f(\infty_G) = \infty_H$. This definition is correct, since $\infty_G \neq \mu(x) = \mu(y)$ implies $\lambda(x) = \lambda(y) \neq \infty_H$. In fact, $\mu(y)^{-1} \in G$, and hence $\mu(y)^{-1} = \mu(z)$ for some $z \in R$. Thus we have

$$1 = \mu(x)\mu(y)^{-1} = \mu(x)\mu(z) \implies xz \in R^\mu \setminus P^\mu = R^\lambda \setminus P^\lambda \implies \lambda(xz) = 1.$$

Analogously, from $1 = \mu(x)\mu(y)^{-1}$ and $\mu(y)\mu(z) = \mu(yz)$, it follows that $\lambda(x)\lambda(y) \neq \infty_H$. Finally from Proposition 3.4.6 it follows that $\mu^{-1}(\infty_G) = \lambda^{-1}(\infty_H)$. It is easy to see that f is a homomorphism. In fact, if $\mu(x) < \mu(y)$ then $\lambda(x) < \lambda(y)$. Otherwise, we obtain $\lambda(y) \leq \lambda(x) < \infty$, and so $\lambda(y)^{-1} = \lambda(z)$ for some $z \in R$. Furthermore, $1 \leq \lambda(x)\lambda(y)^{-1} = \mu(x)\mu(z)$ implies that $xz \in R^\lambda = R^\mu$, and hence $1 \leq \mu(xz)$. But $\lambda(z) = \lambda(y)^{-1} \neq \infty$ implies that $\mu(z) \neq \infty$. Therefore, $yz \in R^\lambda \setminus P^\lambda$, and so $yz \in R^\mu \setminus P^\mu$; i.e., $1 = \mu(yz)$. From $1 \leq \mu(xz)$ it follows that $\mu(y) = \mu(z)^{-1} \leq \mu(x)$. Thus, we obtain $\mu(y) \leq \mu(x)$, which is a contradiction.

It remains to prove that the following property holds:

$$(\forall a, b \in G_\infty)\ f(a + b) = f(a) \oplus f(b).$$

Since $a + \infty_G = a$ in G and $a_1 + \infty_H = a_1$ in H, it suffices to consider the case $a, b \in G$. Let $a, b \in G$ and $x, y \in R$ be such that $a = \mu(x)$ and $b = \mu(y)$. Then $f(a) + f(b) = f(\mu(x)) + f(\mu(y))$. Therefore, if $\mu(x) < \mu(y)$ we have $\lambda(x) < \lambda(y)$, and hence

$$f(\mu(x) + \mu(y)) = f(\mu(x)) = \lambda(x) = f(\mu(x)) + f(\mu(y)).$$

If $\mu(x) = \mu(y)$ then $\lambda(x) = \lambda(y)$, and so the set $f(\mu(x) + \mu(y))$ is equals

$$\{f(\mu(z)) \mid \mu(x) \leq \mu(z)\} = \{\lambda(z) \mid \lambda(x) \leq \lambda(z)\} = \lambda(x) + \lambda(y),$$

as required. ■

3.5 The existence of non-quotient hyperrings

The existence of non-quotient hyperrings and hyperfields plays a very important and quite determinative role in the independence, self sufficiency and further development of the theory of hyperrings and hyperfields. The existence of such hyperrings and hyperfields was proved in [80, 81, 82]. Moreover the construction methods used there endowed this theory with new interesting classes of hyperrings and hyperfields. The results of this paragraph were obtained by C.G. Massouros, G.G. Massouros and A. Nakassis [80, 81, 82, 83, 93]

Let $(R, +, \cdot)$ be a ring and G a subset of R. G is called a *multiplicative subgroup* of R if and only if $(G, \cdot)$ is a group. Moreover, if G is such that $R = RG$ and $rG = Gr$ for all r in R, then G is called a *normal subgroup* of R. We remark that only rings with an identity element admit normal subgroups. As we have already mentioned, a normal subgroup G of R induces an equivalence relation P in R and a partition of R in equivalence classes which inherits a hyperring structure from R. Hyperrings obtained via this construction are called *quotient hyperrings* and are denoted by R/G or by R/P. The following results answer to the questions:

(1) Are all hyperrings embeddable in quotient hyperrings?

(2) Are all hyperrings generated by a set of orthogonal idempotents, embeddable into quotient hyperrings?

(3) Are all primitive hyperrings embeddable into quotient hyperrings?

(4) Are all hyperfields embeddable in quotient hyperrings?

Actually, as we have already mentioned, one can generalize the notion of a quotient hyperring as follows: Assume that P is an equivalence relation in R and for each r in R, $P(r)$ is the equivalence class to which r belongs.

Assume that for all a and b in R, $P(a)P(b)$ is a subset of $P(ab)$. Let R/P be the set of all equivalence classes in R and for each subset X of R let the P-closure $P(X)$ of X be the set of all equivalence classes that intersect X. Clearly the multiplication in R induces an associative multiplication in R/P provided that the product of any two classes $P(a)$ and $P(b)$ is defined to be the P-closure of the set in R. Similarly, R's addition induces a commutative hyperoperation $\oplus$ in R/P provided that one defines $P(a) \oplus P(b)$ to be the P-closure of the set $P(a) + P(b)$ in R. If P is such that $(R/P, \oplus, \cdot)$ is a hyperring (as a rule it is not), then R/P is called a *partition hyperring*. One can verify that $(R/P, \oplus, \cdot)$ is a hyperring if and only if P satisfies the following conditions:

(i) $P(0)$ is a bilateral ideal of R such that for every a in R, $a + P(0)$ is a subset of $P(a)$,

(ii) for every a in R, $P(-a) = -P(a)$,

(iii) P is such that $\oplus$ is associative and the multiplication in R/P is left and right distributive over $\oplus$.

Clearly, condition (iii) is a restatement of the problem and it would be interesting to derive conditions on P that ensure that $(R/P, \oplus, \cdot)$ is a hyperring. Under this light, the Krasner's original construction can be seen as a proof of the fact that if P is induced by a normal subgroup G, then R/P inherits a hyperring structure from R. In the next section, we shall show that there are hyperrings that are not embeddable in partition hyperrings and that there are partition hyperrings that are not embeddable in quotient hyperrings. It turns out though, that the class of partition hyperfields and the class of quotient hyperfields are one and the same.

Massouros introduced a hyperring which is not isomorphic to a quotient hyperring because it contains more than one right unit (a quotient hyperring R/G has a single unit, the G's image). Nevertheless, a quotient hyperring R/G can have subhyperrings that do not contain the G's image. Thus, such a subhyperring H could contain more than one unit, either on the left or on the right (evidently, not both). Indeed, the Massouros' idea construct was to consider a ring A with an identity element 1, and to define a ring $R = AxA$, in which the addition is defined component by component and the multiplication via the following rule: $(a,b)\cdot(c,d) = (a(c+d), b(c+d))$. If we set now $G = \{(1,0),(-1,0)\}$, then G is not a normal subgroup of R (it fails to satisfy $rG = Gr$ for all r in R). Nevertheless, G induces an equivalence relation P in R such that R/P inherits a hyperring structure from R. We observe that $rG = -rG$ in R/P and R/P has more than one right unit (all rG with $r = (a,b)$ and $a+b=1$).

As we have already mentioned, the existence of multiple units from the right shows that the hyperring in question is not isomorphic either to quotient hyperrings or to quotient subhyperrings that contain the image of the normal group which induces the hyperring structure. But, there exist quotient subhyperrings that do not contain a unit element and the above construction is embeddable in a quotient hyperring. Indeed, assume that for every semigroup S and every ring A, $A[S]$ is the semigroup ring from S over A, i.e., the set of all mappings from S to A that have finite support. This set can be endowed with a ring structure where the algebra of a semigroup over a field is defined. Indeed, for any two such functions f and g it suffices to define

$(f+g)(s) = f(s) + g(s)$ for every s in S, and

$(fg)(r) = \sum f(s)g(t)$ where $r \in S$ and (s,t) ranges over all pairs such that $st = r$.

We observe that for every subsemigroup T of S, the elements of $A[T]$ can be identified with the elements of $A[S]$ whose support is a subset of T, i.e., that $A[T]$ can be isomorphically embedded in $A[S]$. Let X be a left zero semigroup (which means that $xy = x$ for all x and y in X) of at least two elements, let X^e be the smallest semigroup with an identity element e, that contains X, and assume that A has an identity element 1, such that $1+1$ is not zero. We observe that A is isomorphic to $A[\{e\}]$ and can be isomorphically mapped into $A[X^e]$ (identifying each a in A with the function that maps X to $\{0\}$ and e to a); therefore, $F = \{-1,1\}$ is a normal subgroup of $A[X^e]$. If Y is any two-element subset of X, then we have:

- $A[Y]$ is isomorphic to Massouros' ring and isomorphic to a subring of $A[X^e]$ (Y is a subsemigroup of X^e),
- F induces a partition of $A[Y]$ such that $A[Y]/F$ is isomorphic to a subhyperring of $A[X^e]/F$,

- $A[Y]/F$ is isomorphic to Massouros' hyperring.

We remark that F is not embeddable in $A[Y]$. But, if g maps Y onto $\{0, 1\}$, then F introduces in $A[Y]$ the same partition as $\{-g, g\}$ in $A[Y]$ and this group is isomorphic to the group that Massouros used in order to partition $A[Y]$. In what follows we propose to use the following symbols and terminology:

(1) H^* will represent a hyperring whose elements are $0^*, a^*, b^*, \ldots$.

(2) R/P will represent a partition hyperring (it is assumed that P is such that R/P inherits a hyperring structure from R).

(3) If H' is a subhyperrring of R/P, we denote its elements by $0', a', b', \ldots$. If H^* and H' are isomorphic, then we assume that the images of $0^*, a^*, b^*, \ldots$ are $0', a', b', \ldots$, respectively. We notice that a' can also be seen as a subset of R since it is a P equivalence class.

(4) An equivalence P is said to be *induced by a group* G, if and only if the classes of P are of the form rG and G is a multiplicative subgroup of R.

The next two propositions link the cardinality of $a^* \oplus b^*$ to the cardinality of b' as a subset of R (clearly, it is assumed that H^* is embeddable in a partition hyperring, R/P). They are a blueprint for constructing counterexamples and for proving non-embeddability in partition hyperrlngs. The second proposition is, after all, a "counting lemma" and therefore it provides a natural method for constructing counterexamples.

Proposition 3.5.1. *If P is an equivalence relation that induces a hyperring structure in R/P, then $I = P(0)$ is an ideal of R. Furthermore, $a + I$ is a subset of $P(a)$ for every a in R and P induces a partition $P^* = P/I$ over $R^* = R/I$. Finally, R/P and R^*/P^* are isomorphic hyperrings.*

Corollary 3.5.2. *If a hyperring H^* is embeddable in a partition hyperring R/P, then we can assume without loss of generality that $P(0) = \{0\}$.*

Proposition 3.5.3. *Assume that a hyperring H^* is embeddable in a partition hyperring R/P for which $P(0) = \{0\}$ and assume that there are two elements a^* and b^* in H^*, such that for every $c^* \in a^* \oplus b^*$, $c^* \oplus (-c^*)$ and*

$b^* \oplus (-b^*)$ have only 0^* in common. Then the cardinality of b' cannot exceed the cardinality of $a^* \oplus b^*$. (Clearly, b' is the image of b^* and when we speak about its cardinality, we view b' as a subset of R).

Proof. Indeed, if a is an element in a' and if b_1 and b_2 are distinct elements of b', then $a + b_1$ and $a + b_2$ belong to different equivalence classes in R/P. Therefore, there is an injection from b' into $a' \oplus b'$ in R/P, and as a result there is an injection from b' into $a^* \oplus b^*$. This injection is not always a surjection because the element a of a' is arbitrary, but fixed. ■

In the case when P is induced by a group G, we obtain that $aG + bG = (a + bG)G$. Therefore, the mapping we described in the above proof is onto.

Corollary 3.5.4. *Under the assumptions in Proposition 2.5.3, if P is induced by a group, then b' and $a^* \oplus b^*$ have the same cardinality.*

Now, we shall see how the above propositions can be used in the construction of counterexamples. Proposition 3.5.3 and its corollary can be used in the construction of hyperrings that are not embeddable in quotient hyperrings.

Proposition 3.5.5. *There are partition hyperrings that are not embeddable in quotient hyperrings.*

Proof. Clearly, commutative hypergroups can be seen as hyperrings, where every product is zero. Let $\mathbb{Q}$ be the set of all rational numbers, and let L be the set of all irreducible fractions of the form k/m with $m = 1$ or $m = 2$. Clearly, L is an additive subgroup of $\mathbb{Q}$ and, if it is equipped with the type of multiplication mentioned above (all products zero), then L is a ring. Let P be defined as follows: i/k and j/m are equivalent if and only if either $k = m = 2$ or $i + j = 0$. Then L/P is a partition hyperring H^* whose elements are $0^* = \{0\}$, $x^* = \{(2i + 1)/2 \mid i = 0, -1, 1, -2, 2, -3, 3, ...\}$, and $d^*(i) = \{-i, i\}$ for $i = 1, 2, ...$

One can prove that L/P is not embeddable in a quotient hyperring R/G by the way of contradiction. Indeed, Corollary 3.5.4 can be used then to prove that for each i, $d'(i)$ has two elements and their sum is zero. Furthermore, we can prove by induction that these two elements can be chosen in such a way that $d^*(i) = \{-id, id\}$ for $i = 1, 2, ...$ If L/P were embeddable into R/G, then x' would be of the form $x' = xG$ for every x in

x'. It if were also true that $x + x = 0$, then for every y in x', $y + y = 0$. Furthermore, since for every i, $d^*(i)$ is in $x^* \oplus x^*$, we can prove that the above d is of the form $x_1 + y_1$ with x_1 and y_1 are in x'. But, $d + d$ is not zero while, under our assumptions, $(x_1 + y_1) + (x_1 + y_1)$ is. Therefore, if x is in x', $x + x$ cannot be zero.

Furthermore, we observe that since $x' + x' = \{0', d'(1), d'(2), ...\}$, $x + x$ must belong to some $d'(i)$. This i cannot be even, because if i were equal to $2m$, then both $x + md$ and $x - md$ would belong to x' ($x^* \oplus d^*(m) = x^*$). But then, either $x + md$ or $x - md$ must satisfy $t + t = 0$, in contradiction to what we just proved. Thus, there is an m such that $x + x$ is in $d'(2m+1)$. If $x + x = (2m+1)d$, then it suffices to take $z = x - md$ in order to obtain an element z in x' such that $z + z = d$. If on the other hand $x + x = -(2m+1)d$, one can achieve the same result by taking $z = (m+1)d - x$. Thus we can always assume that the element x we chose satisfies $x + x = d$.

It ensues then that $x' = \{-x, x, -3x, 3x, -5x, 5x, ...\}$. Indeed, if y is in x', then either $y + y = (2m+1)d$ or $y + y = -(2m+1)d$ for a well choose m. In the first instance, it suffices to consider t, $t = y - (2m+1)x$. In the second, $t = y + (2m+1)x$. In both cases $t + t = 0$, and we can prove that t cannot belong either to x' or to any $d'(i)$, $i = 1, 2, ...$. Hence $t = 0$ and therefore, $x' = \{-x, x, -3x, 3x, -5x, 5x, ...\}$.

Finally, we observe that if $x' = xG = (3x)G$, then $x = (3x)g$ for some g in G. But, since $x' = xG$, xg must be of the form $(2k+1)x$. Thus, $x = 3(2k+1)x$ which implies that $(3k+1)d = (6k+2)x = 0$. But, $3k+1$ is not zero, and if its absolute value is n, then $(3k+1)d$ belongs to $d'(n)$. Therefore, the above line of reasoning produces a contradiction and this implies that H^* is not embeddable in a quotient hyperring. ■

While the above construction shows quite conclusively that there are hyperrings that are not embeddable in quotient hyperrings, it leaves open the possibility that all hyperrings generated by a set of multiplicative idempotents may be embeddable in quotient hyperrings.

Proposition 3.5.6. *Let T^* be a multiplicative group and let H^* be the disjoint union of $\{0^*, u^*, v^*\}$ and T^*. Then H^* can be endowed with a hyperring structure if one defines an hyperaddition $\oplus$ and a multiplication as follows:*

(1) *for every* a^* *in* H^*, $a^* \oplus 0^* = 0^* \oplus a^* = \{a^*\}$;

(2) *for every* a^* *other than zero,* $a^* \oplus a^* = \{0^*, a^*\}$,

(3) *for all distinct* a^* *and* b^*, $a^* \oplus b^* = H^* \setminus \{a^*, b^*, 0^*\}$, *provided that neither* a^* *nor* b^* *is* 0^*.

(4) *for every* a^* *in* H^*, $a^*0^* = 0^*a^* = 0^*$,

(5) $u^*u^* = u^*$, $v^*v^* = v^*$ *and* $u^*v^* = v^*u^* = 0^*$,

(6) *for every* t^* *in* T^*, $u^*t^* = t^*u^* = u^*$ *and* $v^*t^* = t^*v^* = v^*$,

(7) *the multiplication of* H^* *and of* T^* *are identical over* T^*.

The proof of this proposition is quite straightforward, albeit long. The important part is that we can prove the following proposition using Proposition 3.5.6.

Proposition 3.5.7. *If* H^* *is embeddable in a partition hyperring* R/P, *then the following assertion hold:*

(1) $u' \cup 0'$ *and* $v' \cup 0'$ *are finite fields, viewed as subsets of* R,

(2) u' *and* v' *are isomorphic to subgroups of* T^*,

(3) *the isomorphic images* $f(u')$ *and* $f(v')$ *of* u' *and* v' *are normal subgroups of* T^*,

(4) *there are homomorphisms from* u' *onto* $T^*/f(v')$ *and from* v' *onto* $T^*/f(u')$.

Before we delineate a proof of the proposition, let us observe that one, among many, way of constructing hyperrings not embeddable in quotient hyperrings is to let T^* have a prime number element q such that $q+1$ is not a power of 2, e.g., $q = 5$. Indeed, if Proposition 3.5.7 holds then either u' or v' would be isomorphic to T^* and we would have a finite field of $q+1$ elements. Since the latter is impossible, we have produced a class of hyperrings that are not embeddable in partition hyperrings. A fortiori, they cannot be embedded in quotient hyperrings.

Proof. If H^* is embeddable in a partition hyperring R/P, then we could use Proposition 3.5.3 to prove that the images u' and v' of u^* and v^* respectively, are finite subsets of R having at most q elements. Moreover, since u^* and v^* are multiplicative idempotents, it follows that u' and v' are multiplicative semigroups of R. We can construct a semigroup homomorphism f that maps $u'xv'$ onto T^* by defining f as follows: $f(u,v) = t^*$ if and only if $u+v$ is a member of t' (as we have already indicated, t' is the isomorphic image of t^* when H^* is mapped into a subhyperring of R/P). Since $u^*v^* = v^*u^* = 0^*$ and $0'$ has a single element, R's zero, for any two pairs (u_1, v_1) and (u_2, v_2) from $u'xv'$, $(u_1+v_1)(u_2+v_2)$ is equal to $u_1u_2+v_1v_2$. Therefore, $f(u_1u_2, v_1v_2) = f(u_1,v_1)f(u_2,v_2)$ and f is a semigroup homomorphism from $u'xv'$ onto T^*.

Now, let (u_0, v_0) be a multiplicative idempotent in $u'xv'$ (such an idempotent exists because $u'xv'$ is finite). Then, if $u = u_0$, then we obtain a homomorphism $f(u_0, v)$ that maps v' into a subset $f(v')$ of T^*. Similarly, if $v = v_0$, then we obtain a homomorphism $f(u, v_0)$, that maps u' into a subset $f(u')$ of T^*. It is elementary then to check that $f(u_0, v)$ and $f(u, v_0)$ are injections and all finite semigroups of a group are groups. It ensues that u', v' and $u'xv'$ are multiplicative groups and that f is a group homomorphism from $u'xv'$ onto T^*. It follows also that (u_0, v_0) is the unity of $u'xv'$ and that $\{u_0\}xv'$ and $u'x\{v_0\}$ are normal subgroups of $u'xv'$. Then u' is isomorphic to $u'xv'/\{u_0\}xv'$ and u' is homomorphic to $T^*/f(v')$. Similarly, v' is homomorphic to $T^*/f(u')$. It is easy now to check that $u' \cup \{0\}$ and $v' \cup \{0\}$ are finite fields (u' and v' are finite groups, $u' + u' = u' \cup \{0\}$, and $v' + v' = v' \cup \{0\}$. We notice also that if the cardinality of T^* is a prime number q, then either u' or v' is isomorphic to T^*. Indeed, if $f(u')$ is not T^*, it must be equal to $\{e^*\}$, where e^* is the identity of T^*. But then, the finite group v' is isomorphic to $f(v')$, which is a subgroup of T^*, and homomorphic to $T^*/\{e^*\}$ and the only way this can happen is if v' has q elements.■

A similar construction can be used in order to show that there exist hyperfields that are not embeddable into partition hyperrings. Given that each hyperfield is an irreducible and faithful module over itself, it follows that there are primitive rings that are not embeddable into partition hyperfields. Indeed, let T^* be any finite group of $m > 3$ elements and define a hyperfield structure over $H^* = T^* \cup \{0*\}$, as follows:

(1) $a^*0^* = 0^*a^* = 0^*$, for every a^* in H^*,

(2) $a^* \oplus 0^* = 0^* \oplus a^* = \{a^*\}$, for every a^* in H^*,

(3) $a^* \oplus a^* = \{a^*, 0^*\}$, for every a^* in T^*,

(4) $a^* \oplus b^* = b^* \oplus a^* = T^* \setminus \{a^*, b^*\}$ for every a^* and b^* in T^*, provided that a^* and b^* are distinct.

Structures that satisfy such properties are not very uncommon. Indeed, it suffices to consider the field of complex numbers $\mathbb{C}$ and the multiplicative group $\mathbb{R}^*$ of all nonzero reals, in order to obtain a quotient hyperfield $\mathbb{C}/\mathbb{R}^*$ that has properties (3) and (4). We can prove the following result:

Proposition 3.5.8. *If the above constructed H^* is embeddable in a partition hyperring R/P, and if H' is the isomorphic image of H^*, then the following statements hold:*

(1) *The isomorphism maps the unit e^* of T^* onto a finite multiplicative subgroup of R that will be called e' in what follows,*

(2) *If H_1 is the subset of R that corresponds to H', then e' and P induce the same partition on H_1,*

(3) *$e' \cup \{0\}$ is a field of $m-1$ elements while H_1 is a field of $m(m-2)+1 = (m-1)^2$ elements.*

Notice that if (1)-(3) hold, then we can choose T^* in such a way that H^* cannot be embeddable in a partition hyperring. Indeed, all finite fields are commutative, their cardinality is a power of a prime, and the multiplicative group of their nonzero elements is cyclic. One can choose then either m or the structure of T^* in such a way that H^* is not embeddable into a partition hyperring.

Proof. By Proposition 3.5.3, all nonzero elements of H' correspond to finite subsets of R having $m-2$ elements or less. Let H_1 be the union of all these subsets. We start by observing that e' is a finite set multiplicatively closed, without divisors of zero ($e'e' = e'$). Furthermore, we have $e' \oplus e' = \{e', 0'\}$ and so if x and y are distinct elements of e', then $x - y$ is in e'. It follows that for every a in e', the mappings $x \longmapsto ax$ and $x \longmapsto xa$ are injections from e' into e'. Therefore, e' is a group. The same reasoning can be used for $H_1 \setminus \{0\}$. Since $H' \setminus \{0\}$ is isomorphic to T^*, $H_1 \setminus \{0\}$ has no divisors of zero and is multiplicatively closed. If x and y are distinct elements of H_1, $x - y$ belongs also to H_1 and we deduce that for every a in $H_1 \setminus \{0\}$ the mappings $x \longmapsto ax$ and $x \longmapsto xa$ are injections. Since H_1 is finite it follows that $H_1 \setminus \{0\}$ is a group, e' is a subgroup of $H_1 \setminus \{0\}$, and these two groups share the same identity element e. Let x' be any nonzero element of H'. Since H' is a hyperfield, there is a y' such that $y'x' = x'y' = e'$ in H'. It ensues that if y is any element of y', then yx' and $x'y$ are subsets of e' in H_1. The inverse x of y in $H_1 \setminus \{0\}$, is clearly an element of x'. By multiplying by x we obtain that x' is a subset of xe' and $e'x$ in H_1. On the other hand, since $x'e' = e'x' = x'$ in H', it follows that xe' and $e'x$ must be subsets of x' in H_1. Hence $x' = xe = ex$ for some x in x'. But, if this property is true for one x in x', it is true for every x in x' (it suffices to remark that e' is a group). Since P is induced by e' over H_1, each class in H_1 has exactly $m-2$ elements (Proposition 3.5.3) except of $0'$ that contains only 0. Therefore, H_1 is a field of $m(m-2)+1$ elements while $e' \cup \{0\}$ is a field ($e^* \oplus e^* = \{e^*, 0^*\}$) of $(m-2)+1$ elements. ∎

Non quotient hyperfields

We shall construct a class of hyperfields which contains hyperfields that are not isomorphic to quotient ones. The idea there is to take a group G and to introduce a hyperfield structure over $H = G \cup \{0\}$ as follows:

(1) For every h in H, $h \oplus 0 = 0 \oplus h = \{h\}$,

(2) $g_l \oplus g_2 = \{g_l, g_2\}$ for every two distinct elements of G,

(3) $g \oplus g = H \setminus \{g\}$ for every g in G.

One can prove that G can be chosen in such a way that H is not embeddable in a quotient hyperfield. Indeed, we have:

Proposition 3.5.9. *If G is not trivial and $gg = e$ for every g in G, then H is not isomorphic to a quotient hyperfield.*

Proof. This can be proven by the way of contraposition. If H was isomorphic to a quotient hyperfield F/Q, then

(1) Since $gg = e$ in G, Q contains all squares in F other than zero,

(2) Since for each element g in G, $g \oplus g = H \setminus \{g\}$, it follows that $Q = -Q$ and $Q + Q = F \setminus Q$.

But, if all squares of F and their opposites are in Q, then we obtain a contradiction. If the characteristic of F is not two, then each element of F is the difference of two squares. If the characteristic of F is two, then the sum of two squares is a square. Otherwise, $Q + Q$ cannot be equal to $F \setminus Q$. ∎

Finally, one can prove that if a Cartesian product of hyperrings is embeddable in a quotient hyperring, then every term of the product which is a hyperfield must be isomorphic to a quotient hyperfield. Thus, one can produce hyperrings that are not embeddable in quotient hyperrings.

Therefore, the structures and counterexamples that appeared in this section, proved that the theory of hyperrings is not a straightforward extension of ring theory.

3.6 Semigroups admitting hyperring structure

We say that a semigroup $(S, \cdot)$ *admits a hyperring* (*ring*) *structure* if there is a hyperoperation (operation) $+$ on $S^0 = S \cup \{0\}$, such that $(S^0, +, \cdot)$ is a hyperring (ring). In [66], Kemprasit and Punkla gave a necessary and sufficient condition for a set X so that some transformation semigroups on X admit a hyperring structure.

Example 3.6.1. Let G be a group and define a hyperoperation $+$ on G^0 by

$$\begin{aligned}
&x + 0 = 0 + x = \{x\} \ \text{ for all } x \in G^0,\\
&x + x = G^0 \backslash \{x\} \ \text{ for all } x \in G^0 \backslash \{0\},\\
&x + y = \{x, y\} \ \text{ for all } x, y \in G^0 \backslash \{0\} \text{ with } x \neq y.
\end{aligned}$$

We can easily verify $(G^0, +, \cdot)$ is a hyperring. From this fact, we conclude that every group admits a hyperring structure.

Let $\mathcal{SR}$ and $\mathcal{SHR}$ denotes the class of all semigroups admitting a ring structure and the class of all semigroups admitting a hyperring structure, respectively. Then $\mathcal{SHR}$ contains $\mathcal{SR}$ as a subclass. Semigroups belonging to the class $\mathcal{SR}$ have long been studied, for example see [64, 65]. Kemprasit and Punkla characterized some standard transformation semigroups belonging to $\mathcal{SHR}$.

Let X be an arbitrary set and let

$$\begin{array}{ll} P_X & = \text{the partial transformation semigroup on } X, \\ T_X & = \text{the full transformation semigroup on } X, \\ I_X & = \text{the one to one partial transformation semigroup on } X, \\ G_X & = \text{the symmetric group on } X, \\ M_X & = \text{the semigroup of all one to one transformations of } X, \\ E_X & = \text{the semigroup of all onto transformations of } X. \end{array}$$

Observe that

$$\begin{array}{l} G_X \subseteq M_X \subseteq I_X \subseteq P_X, \\ G_X \subseteq M_X \subseteq T_X \subseteq P_X, \\ G_X \subseteq E_X \subseteq T_X \subseteq P_X. \end{array}$$

The following facts are known. If X contains more than two elements, then the center of G_X is $\{1_X\}$, where 1_X is the identity map on X. For $\alpha \in P_X$; $\alpha^2 = \alpha$ if and only if $Im\alpha \subseteq Dom\alpha$ and $x\alpha = x$ for all $x \in Im\alpha$ where $Dom\alpha$ and $Im\alpha$ denote the domain and the image of α, respectively. For convenience, the following notation will be used.

For distinct $a_1, a_2, ..., a_n \in X$, let $(a_1, a_2, ..., a_n)$ be the element of G_X defined by

$$(a_1, a_2, ..., a_n)(x) = \begin{cases} a_{i+1} & \text{if } x = a_i \text{ for some } i \in \{1, 2, ..., n-1\}, \\ a_1 & \text{if } x = a_n, \\ x & \text{otherwise} \end{cases}$$

For $A \subseteq X$, $A \neq \emptyset$ and $x \in X$, let $A_x \in P_X$ be defined by $DomA_x = A$ and $ImA_x = \{x\}$.

Let X be an arbitrary set and $|X|$ denote the cardinality of X. Firstly, we notice that if $|X| = 0$, then all of the previous transformation semigroups contain exactly one element. If $|X| = 1$, $P_X = I_X \cong (\mathbb{Z}_2, \cdot)$ and $T_X = M_X = E_X = G_X$ which contains exactly one element. Hence if $|X| \leq 1$, all of these transformation semigroups belong to $\mathcal{SR}$ ($\subseteq \mathcal{SHR}$).

Theorem 3.6.2. *Let S be P_X or I_X. Then $S \in \mathcal{SHR}$ if and only if $|X| \leq 1$.*

Proof. Assume that $S \in \mathcal{SHR}$. Then there exists a hyperoperation $+$ on S^0 such that $(S^0, +, \cdot)$ is a hyperring. To prove that $|X| \leq 1$, suppose to the contrary. Let $a, b \in X$ be such that $a \neq b$. The element $\{a\}_a$ of S maps a in b. Similarly, we can define other elements of S, such that $\{a\}_b$ and $\{b\}_a$. Since $\emptyset \neq \{a\}_a + \{a\}_b \subseteq S$, there exists an element $\alpha \in S$ such that $\alpha \in \{a\}_a + \{a\}_b$. Thus $\{a\}_a \alpha \in \{a\}_a(\{a\}_a + \{a\}_b)$. But $\{a\}_a(\{a\}_a + \{a\}_b) = \{a\}_a + 0 = \{a\}_a$, so $\{a\}_a \alpha = \{a\}_a$. This implies that $a \in Dom\alpha$ and $\alpha a = a$. Consequently, we have $\alpha\{a\}_a = \{a\}_a$. Hence $\{a\}_a \in (\{a\}_a + \{a\}_b)\{a\}_a$. Since $(\{a\}_a + \{a\}_b)\{a\}_a = \{a\}_a + \{a\}_b$,, it follows that

$$0 = \{b\}_a\{a\}_a \in \{b\}_a(\{a\}_a + \{a\}_b) = 0 + \{a\}_a = \{a\}_a.$$

This is a contradiction. Hence $|X| \leq 1$. The converse follows from what we have mentioned above. ■

Theorem 3.6.3. *$T_X \in \mathcal{SHR}$ if and only if $|X| \leq 1$.*

Proof. Assume that $T_X \in \mathcal{SHR}$ and suppose that $|X| > 1$. Then there exists a hyperoperation $+$ on T_X^0 such that $(T_X^0, +, \cdot)$ is a hyperring. Let us note that for $\alpha, \beta \in T_X^0$, $\beta\alpha = 0$ implies that $\alpha = 0$ or $\beta = 0$. Let a and b be distinct elements of X and define $\alpha, \beta : X \longrightarrow X$ by

$$\alpha(x) = \begin{cases} a & \text{if } x \in \{a, b\} \\ x & \text{otherwise} \end{cases}$$

$$\beta(x) = \begin{cases} b & \text{if } x \in \{a, b\} \\ x & \text{otherwise.} \end{cases}$$

Then $\alpha, \beta \in T_X$, $\alpha^2 = \alpha$ and $\beta^2 = \beta$. It is easy to see that $\alpha\beta = \alpha$. Thus

$$\alpha - \alpha = \alpha^2 - \alpha\beta = \alpha(\alpha - \beta).$$

Since $0 \in \alpha - \alpha$ and $\alpha \neq 0$, we have $0 \in \alpha - \beta$. Consequently, we have $\alpha = \beta$ which is a contradiction. As it was mentioned previously, the converse holds. ■

Theorem 3.6.4. *Let S be M_X or E_X. Then $S \in \mathcal{SHR}$ if and only if X is finite.*

Proof. If X is finite, then $S = G_X$, so $S \in \mathcal{SHR}$ since every group is in $\mathcal{SHR}$.

Conversely, assume that $S \in \mathcal{SHR}$. In order to show that X is finite, suppose to the contrary that X is infinite. Let $+$ be a hyperoperation on S^0 such that $(S^0, +, \cdot)$ is a hyperring. Again, let us note that for $\alpha, \beta \in S^0$, $\alpha\beta = 0$ implies $\alpha = 0$ or $\beta = 0$. Since for every $\alpha \in G_X$,

$$(-1_X)\alpha = -(1_X\alpha) = -(\alpha 1_X) = \alpha(-1_X),$$

it follows that -1_X is in the center of G_X. Since X is infinite, the center of G_X is $\{1_X\}$. Therefore $-1_X = 1_X$. Consequently, we have $-\alpha = \alpha$ for all $\alpha \in S$, so $0 \in \alpha + \alpha$ for all $\alpha \in S$. Next, let a and b be distinct elements of X. Since X is infinite, $|X| = |X \setminus \{a, b\}|$ so there exists a one to one map γ from X onto $X \setminus \{a, b\}$ and a map λ from $X \setminus \{a, b\}$ onto X. Define $\mu : X \longrightarrow X$ by $\mu(a) = \mu(b)$ and $\mu(x) = \lambda(x)$ for all $x \in X \setminus \{a, b\}$. Then $\gamma \in M_X$ and $\mu \in E_X$. Denote the transposition map of a and b by (a, b). Then we have $(a, b)\gamma = \gamma$ and $\mu(a, b) = \mu$.

Case 1: $S{=}M_X$. Since $0 \in \gamma{+}\gamma$ and $\gamma{+}\gamma{=}(a, b)\gamma{+}1_X\gamma{=}((a, b){+}1_X)\gamma$, we have $0 \in ((a, b) + 1_X)\gamma$. But $\gamma \neq 0$, so $0 \in (a, b) + 1_X$. Then (a, b) is an inverse of 1_X in $(S^0, +)$. This is a contradiction since 1_X is the unique inverse of 1_X in $(S^0, +)$.

Case 2: $S{=}E_X$. Since $0 \in \mu{+}\mu$ and $\mu{+}\mu{=}\mu(a, b){+}\mu 1_X{=}\mu((a, b){+}1_X)$, we have $0 \in \mu((a, b) + 1_X)$. But $\mu \neq 0$, so $0 \in (a, b) + 1_X$. This implies that (a, b) is an inverse of 1_X in $(S^0, +)$ which is a contradiction because 1_X is the unique inverse of 1_X in $(S^0, +)$.

Therefore the theorem is completely proved. ■

The following two corollaries are obtained from the fact that $\mathcal{SR} \subsetneq \mathcal{SHR}$, Theorem 3.6.2 and Theorem 3.6.3, respectively and the paragraph before Theorem 3.6.2.

Corollary 3.6.5. *Let S be P_X or I_X. Then $S \in \mathcal{SR}$ if and only if $|X| \leq 1$.*

Corollary 3.6.6. *$T_X \in \mathcal{SR}$ if and only if $|X| \leq 1$.*

We know that $G_X \in \mathcal{SHR}$ for any cardinality of X and if X is finite and $|X| \geq 3$, $G_X \notin \mathcal{SR}$. The following theorem shows that the condition $n \leq 2$ is necessary and sufficient for G_X to belong to $\mathcal{SR}$.

Theorem 3.6.7. *$G_X \in \mathcal{SR}$ if and only if $|X| \leq 2$.*

Proof. Assume that $G_X \in \mathcal{SR}$. Let $+$ be an operation on G_X^0 such that $(G_X^0, +, \cdot)$ is a ring. In order to show that $|X|{\leq}2$, suppose to the contrary that $|X|{>}2$. Let a, b and c be distinct elements of X. Then $(a,b,c){+}(a,c){=}\alpha$ for some $\alpha \in G_X^0$, where (a,b,c) is a cycle of length 3.

Case 1: $\alpha = 0$. Then $(a,b,c) + (a,c) = 0$, so

$$\begin{aligned}0 &= ((a,b,c) + (a,c))(a,c) = (a,b,c)(a,c) + 1_X = (b,c) + 1_X\\ 0 &= (a,c)((a,b,c) + (a,c)) = (a,c)(a,b,c) + 1_X = (a,b) + 1_X.\end{aligned}$$

These imply that $(a,b) + 1_X = 0 = (b,c) + 1_X$ which is a contradiction since $(a,b) \neq (b,c)$.

Case 2: $\alpha{\neq}0$. Then $(b,c){+}1_X{=}((a,b,c){+}(a,c))(a,c){=}\alpha(a,c)$, and so

$$1_X + (b,c) = (b,c)((b,c) + 1_X) = (b,c)\alpha(a,c).$$

Then we have $\alpha(a,c) = (b,c)\alpha(a,c)$. Since α and (a,c) are in the group G_X, it follows that $(b,c) = 1_X$, a contradiction.

Conversely, assume that $|X| \leq 2$. If $|X| \leq 1$, then $G_X \in \mathcal{SR}$. If $|X| = 2$, it is clear that $G_X^0 \cong (\mathbb{Z}_3, \cdot)$, so $G_X \in \mathcal{SR}$. ■

We use Theorem 3.6.4 and Theorem 3.6.7 to obtain the following theorem.

Theorem 3.6.8. *Let S be M_X or E_X. Then $S \in \mathcal{SR}$ if and only if $|X| \leq 2$.*

Proof. Let $S \in \mathcal{SR}$. Then $S \in \mathcal{SHR}$, so by Theorem 3.6.4, X is finite. Thus $S = G_X$. Hence $|X| \leq 2$ by Theorem 3.6.7.

Conversely, if $|X| \leq 2$, then $S = G_X$, so $S \in \mathcal{SR}$ by Theorem 3.6.7. ■

Now, we characterize multiplicative interval semigroups on $\mathbb{R}$ admitting hyperring structure.

Proposition 3.6.9.[65] *A subset S of $\mathbb{R}$ is a multiplicative interval semigroup on $\mathbb{R}$ if and only if S is one of the following types:*

(1) $\mathbb{R}$,
(2) $\{0\}$,
(3) $\{1\}$,
(4) $(0,\infty)$,
(5) $[0,\infty)$,
(6) (a,∞) *where* $a \geq 1$,
(7) $[a,\infty)$ *where* $a \geq 1$,
(8) $(0,b)$ *where* $0 < b \leq 1$,
(9) $[0,b]$ *where* $0 < b \leq 1$,
(10) $(0,b]$ *where* $0 < b \leq 1$,
(11) $[0,b)$ *where* $0 < b \leq 1$,
(12) (a,b) *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$,
(13) $[a,b]$ *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$,
(14) $(a,b]$ *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$,
(15) $[a,b)$ *where* $-1 < a < 0 < a^2 < b \leq 1$.

Lemma 3.6.10. *If S is a multiplicative interval semigroup on $\mathbb{R}$ such that $S \subseteq [0,\infty)$, then $S \in \mathcal{SHR}$.*

Proof. By assumption, S is one of types (2)–(11) in Proposition 3.6.9. Clearly, $S \in \mathcal{SHR}$ if S is of type (2) or (3).

Case 1: S is one of types (4), (5) and (8)–(11). Then S^0 can be considered as $S^0 = [0,\infty)$ or $S^0 = [0,b)$ or $[0,b]$ for some $b > 0$ with $b \leq 1$. Define the hyperoperation $\oplus$ on S^0 by

$$\begin{array}{rcll} x \oplus x &=& [0,x] & \text{for all } \ x \in S^0 \ \text{ and} \\ x \oplus y &=& \{\max\{x,y\}\} & \text{for all distinct } \ x,y \in S^0. \end{array}$$

Then, $(S^0,\oplus)$ is a canonical hypergroup. In order to show that $(S^0,\oplus,\cdot)$ is a hyperring where $\cdot$ is the multiplication on S^0, set $x,y,z \in S^0$. We have

$$y \oplus z = \begin{cases} \{z\} & \text{if } y < z, \\ \{y\} & \text{if } z < y, \\ [0,y] & \text{if } y = z. \end{cases}$$

But $x \geq 0$, so

$$x(y \oplus z) = \begin{cases} \{xz\} & \text{if } y < z, \\ \{xy\} & \text{if } z < y, \\ [0, xy] & \text{if y = z.} \end{cases}$$

Since $x \geq 0$, we obtain $x(y \oplus z) = xy \oplus xz$.

Case 2: S is of types (6) or (7). Then $S = (a, \infty)$ or $[a, \infty)$ for some $a \geq 1$, so S^0 can be considered as $S \cup \{0\}$ where 0 is the zero real number. Define the hyperoperation $\oplus$ on S^0 by

$$\begin{array}{rcll} x \oplus x & = & 0 \oplus x = \{x\} & \text{for all} \quad x \in S^0, \\ x \oplus x & = & [x, \infty) \cup \{0\} & \text{for all} \quad x \in S \quad \text{and,} \\ x \oplus y & = & \{min\{x, y\}\} & \text{for all distinct} \quad x, y \in S. \end{array}$$

Then, $(S^0, \oplus)$ is a canonical hypergroup. We claim that $(S^0, \oplus, \cdot)$ is a hyperring where $\cdot$ is a multiplication on S^0. Set $x, y, z \in S^0$. If $x = 0$, then $x(y \oplus z) = \{0\} = xy \oplus xz$. Assume that $x \neq 0$. Then $x \geq a \geq 1$. We have

$$y \oplus z = \begin{cases} \{z\} & \text{if } y = 0 \text{ or } 0 < z < y, \\ \{y\} & \text{if } z = 0 \text{ or } 0 < y < z, \\ [y, \infty) \cup \{0\} & \text{if } 0 < y = z. \end{cases}$$

The fact that $x > 0$ implies

$$x(y \oplus z) = \begin{cases} \{xz\} & \text{if } y = 0 \text{ or } 0 < z < y, \\ \{xy\} & \text{if } z = 0 \text{ or } 0 < y < z, \\ [xy, \infty) \cup \{0\} & \text{if } 0 < y = z. \end{cases}$$

Since $x > 0$, then we have that $x(y \oplus z) = xy \oplus xz$.

Hence the lemma is completely proved. ∎

Lemma 3.6.11. *Let S be a multiplicative interval semigroup on $\mathbb{R}$ such that $S \not\subseteq [0, \infty)$. Then $S \in \mathcal{SHR}$ if and only if for every $x \in S$, $-x \in S$.*

Proof. By assumption, S is one of types (1) and (12) – (15). Then $S^0 = S$. Firstly, assume that for every $x \in S$, $-x \in S$. Define the hyperoperation $\oplus$ on S by

$$\begin{array}{ll} x \oplus x = \{x\} & \text{for all } x \in S, \\ x \oplus y = y \oplus x = \{x\} & \text{for all } x, y \in S \text{ with } |y| < |x| \text{ and} \\ x \oplus (-x) = [-|x|, |x|] & \text{for all } x \in S. \end{array}$$

Then, $(S, \oplus)$ is a canonical hypergroup. Let x, y, $z \in S$. Then

$$y \oplus z = \begin{cases} \{y\} & \text{if } y = z, \\ \{z\} & \text{if } |y| < |z|, \\ \{y\} & \text{if } |z| < |y|, \\ [-|y|, |y|] & \text{if } z = -y. \end{cases}$$

If $x \geq 0$, then $x[-|y|, |y|] = [-x|y|, x|y|] = [-|xy|, |xy|]$, and if $x < 0$, then $x[-|y|, |y|] = [x|y|, -x|y|] = [-|xy|, |xy|]$. Therefore we have

$$x(y \oplus z) = \begin{cases} \{xy\} & \text{if } y = z, \\ \{xz\} & \text{if } |y| < |z|, \\ \{xy\} & \text{if } |z| < |y|, \\ [-|xy|, |xy|] & \text{if } z = -y. \end{cases}$$

So we have $x(y \oplus z) = xy \oplus xz$. Hence $(S, \oplus, \cdot)$ is a hyperring where $\cdot$ is the multiplication on S. Therefore $S \in \mathcal{SHR}$.

For the converse, assume that there exists $c \in S$ such that $-c \notin S$. Then S is one of types (12)–(15). In order to show that $S \notin \mathcal{SHR}$, suppose to the contrary that $S \in \mathcal{SHR}$. Then there exists a hyperoperation $\oplus$ on S such that $(S, \oplus, \cdot)$ is a hyperring where $\cdot$ is the multiplication on S. Set $K = \{x \in S \mid -x \in S\}$. Then $c \in S \setminus K$ and there exists $e > 0$ such that $K = [-e, e]$ or $K = (-e, e)$. If $x \in S \setminus K$, then $-x \notin S$ and $0 \in x \oplus y$ for some $y \in S$, so

$$0 \in x(x \oplus y) = x^2 \oplus xy \text{ and } 0 \in (x \oplus y)y = xy \oplus y^2$$

which implies that $x^2 = y^2$ and hence $y = x$. This proves that

$$0 \in x \oplus x \quad \text{for all} \quad x \in S \setminus K.$$

Hence, $0 \in xy \oplus xy \quad \text{for all} \quad x \in S \setminus K \quad \text{and} \quad y \in S.$

We claim that

$$\begin{array}{lll} \text{for every} & x \in K \setminus \{0\}, & 0 \notin x \oplus (-x) \quad \text{and} \\ \text{for every} & y \in x \oplus (-x), & -y \in x \oplus (-x). \end{array}$$

In order to prove this, let $x \in K \setminus \{0\}$. We have $0 \in cx \oplus cx$. If $0 \in x \oplus (-x)$, then $0 \in cx \oplus (-cx)$ which implies that $cx = -cx$, a contradiction. Hence $0 \notin x \oplus (-x)$. Next, let $y \in x \oplus (-x)$. Then

$$-xy \in (-x^2) \oplus x^2 = x(x \oplus (-x)).$$

It follows that $-xy = xt$ for some $t \in x \oplus (-x)$, so $-y = t \in x \oplus (-x)$.

Case 1: $K = [-e, e]$. Hence, there exists $y \in K$ such that $y > 0$ and $y \in e \oplus (-e)$. Then

$$0 < \frac{y}{e} \leq 1 \quad \text{and} \quad cy \in ce \oplus (-ce).$$

Since $(S, \oplus)$ is reversible, then $-ce \in ce \oplus cy$. Since $0 < y/e \leq 1$,

$$\begin{array}{lll} 0 < \dfrac{cy}{e} \leq c & \text{if} \quad c > 0 & \text{and} \\ 0 > \dfrac{cy}{e} \geq c & \text{if} \quad c < 0. & \end{array}$$

It follows that $cy/e \in S$ since S is an interval on $\mathbb{R}$ and $0, c \in S$. Hence

$$-ce \in ce \oplus (\frac{cy}{e})e = (c \oplus \frac{cy}{e})e$$

which implies that $-ce = te$ for some $t \in c \oplus cy/e$. Consequently, $-c = t \in S$, a contradiction.

Case 2: $K = (-e, e)$. Then $e \notin K$. Let $d \in K$ be such that $d > 0$. Then $0 < d < e$. So there exists $z \in K$ such that $z > 0$ and $z \in d \oplus (-d)$. Thus $0 < z < e$.

Subcase i: $c \geq e$. Since $c \in S$, $e \in S \setminus K$, so $-e \notin S$. Since $z \in d \oplus (-d)$, $ez \in ed \oplus (-ed)$. Since $(S, \oplus)$ is reversible, $-ed \in ed \oplus ez$. If $z \leq d$, then $0 < ez/d \leq e$, so $ez/d \in S$ and hence

$$-ed \in ed \oplus \left(\frac{ez}{d}\right) d = \left(e \oplus \frac{ez}{d}\right) d$$

whence $-e \in e \oplus ez/d \subseteq S$, a contradiction.

Next, assume that $z > d$. Then $0 < ed/z < e$. Thus $ed/z \in K$. From that $ez \in ed \oplus (-ed)$,

$$e^2 d \in \frac{ed}{z}(ed \oplus (-ed)) = \frac{e^2d^2}{z} \oplus (-\frac{e^2d^2}{z}) = ed\left(\frac{ed}{z} \oplus (-\frac{ed}{z})\right)$$

which implies that $e \in (ed/z) \oplus (-ed/z)$. So $-e \in (ed/z) \oplus (-ed/z) \subseteq S$, a contradiction.

Subcase ii: $c \leq -e$. Then $-e \in S \setminus K$, and so $e \notin S$. Since $z \in d \oplus (-d)$, $(-e)z \in (-e)d \oplus (-e)(-d)$. Since $(S, \oplus)$ is reversible, then

$$ed = (-e)(-d) \in (-e)d \oplus (-e)z = (-ed) \oplus (-ez).$$

First, assume that $z \leq d$. Then $0 > -ez/d \geq -e$, so $-ez/d \in S$. Hence

$$ed \in (-ed) \oplus \left(-\frac{ez}{d}\right) d = \left((-e) \oplus \left(-\frac{ez}{d}\right)\right) d$$

whence $e \in (-e) \oplus (-ez/d) \subseteq S$, a contradiction.

Next, assume that $z > d$. Then $0 > -ed/z > -e$. Thus $-ed/z \in K$. Since $(-e)z \in (-e)d \oplus (-e)(-d)$, we have

$$e^2 d \in -\frac{ed}{z}((-ed) \oplus ed) = \frac{e^2d^2}{z} \oplus \left(-\frac{e^2d^2}{z}\right) = (-e)d\left(\left(-\frac{ed}{z}\right) \oplus \frac{ed}{z}\right).$$

Consequently, we have that $-e \in (ed/z) \oplus (-ed/z)$.

So $e \in (ed/z) \oplus (-ed/z) \subseteq S$, a contradiction.

Hence the lemma is completely proved. ■

We note that $(\mathbb{R}, \cdot) \in \mathcal{SR} \subseteq \mathcal{SHR}$. The hyperoperation $\oplus$ defined for the case $S = \mathbb{R}$ makes $(\mathbb{R}, \oplus, \cdot)$ is a hyperring which is not a ring.

Now we are ready to state our main result which follows directly from Lemma 3.6.10 and Lemma 3.6.11.

Theorem 3.6.12. *Let S be a multiplicative interval semigroup on $\mathbb{R}$. Then $S \in \mathcal{SHR}$ if and only if either $S \subseteq [0, \infty)$ or for every $x \in S$, $-x \in S$.*

By Proposition 3.6.9, Theorem 3.6.12 is equivalent to

Theorem 3.6.13. *A multiplicative interval semigroup S on $\mathbb{R}$ belongs to $\mathcal{SHR}$ if and only if S is one of the following types:*

(1) $\mathbb{R}$,
(2) $\{0\}$,
(3) $\{1\}$,
(4) $(0, \infty)$,
(5) $[0, \infty)$,
(6) (a, ∞) *where* $a \geq 1$,
(7) $[a, \infty)$ *where* $a \geq 1$,
(8) $(0, b)$ *where* $0 < b \leq 1$,
(9) $[0, b]$ *where* $0 < b \leq 1$,
(10) $(0, b]$ *where* $0 < b \leq 1$,
(11) $[0, b)$ *where* $0 < b \leq 1$,
(12) $(-c, c)$ *where* $0 < c^2 \leq c \leq 1$,
(13) $[-c, c]$ *where* $0 < c^2 \leq c \leq 1$.

Remark 3.6.14. It is easy to show that there are exactly 6 types of additive interval semigroups on $\mathbb{R}$ as follows: $\mathbb{R}$, $\{0\}$, (a, ∞) where $a \geq 0$, $[a, \infty)$ where $a \geq 0$, $(-\infty, b)$ where $b \leq 0$, $(-\infty, b]$ where $b \leq 0$.

Let S be an additive interval semigroup on $\mathbb{R}$ and $S \neq \{0\}$. Then S has no zero. Thus S^0 can be considered as $S \cup \{-\infty\}$ where $x + (-\infty) = -\infty + x = -\infty$ for all $x \in S \cup \{-\infty\}$. Hence under the usual order on $\mathbb{R}$ and defining $x \geq -\infty$ for all $x \in S \cup \{-\infty\}$, we have that $S \cup \{-\infty\}$ is a totally ordered set having $-\infty$ as its minimum element. If S is of type (1), (5) or (6), by following the proof of Case 1 of Lemma 3.6.10, we have $S \in \mathcal{SHR}$. If S is type (3) or (4), we have $S \in \mathcal{SHR}$ by following the proof of Case 2 of Lemma 3.6.10.

Therefore every additive interval semigroup on $\mathbb{R}$ belongs to the class $\mathcal{SHR}$.

In the continuation of this section, we let x' be the opposite of x in any hyperring $(A, +, \cdot)$.

Let V be a vector space over a division ring R and n a positive integer. We denote by $L_R(V)$ and $G_R(V)$ the semigroup of all linear transformations $\alpha : V \to V$ with respect to the composition and the unit group of $L_R(V)$, respectively. Then

$$G_R(V) = \{\alpha : V \to V \mid \alpha \text{ is an isomorphism}\}.$$

Let $M_n(R)$ and $G_n(R)$ denote respectively the full $n \times n$ matrix semigroup over R and the matrix group of all invertible $n \times n$ matrices over R, that is, $G_n(R)$ is the unit group of $M_n(R)$. It is known that if $\dim_R V = n$, then

$$L_R(V) \cong M_n(R) \text{ and } G_R(V) \cong G_n(R).$$

Since every group is in $\mathcal{SHR}$, we have that $G_R^0(V) \in \mathcal{SHR}$.

We know that $L_R(V) \in \mathcal{SR} \subseteq \mathcal{SHR}$ with the usual addition $+$. Then in the ring $(L_R(V), +, \cdot)$, $\alpha' = -\alpha$ for all $\alpha \in L_R(V)$.

If k is any cardinal number such that $0 \leq k \leq \dim_R V$, then

$$S = G_R(V) \cup \{\alpha \in I_R(V) \mid \dim_R \operatorname{Im} \alpha \leq k\}$$

is clearly a subsemigroup of $L_R(V)$ containing $G_R(V)$.

We denote the center of R by $Z(R)$. It is clear that for $a \in Z(R)$, $a1_V \in L_R(V)$ where 1_V is the identity map on V and $a1_V$ is defined in the usual sense, that is, $(a1_V)(v) = a(1_V)(v) = av$ for all $v \in V$.

Lemma 3.6.15. *Assume that B is a basis of V. Let $\alpha \in L_R(V)$ be such that $\alpha\beta = \beta\alpha$ for all $\beta \in G_R(V)$ and for every $v \in B$, $\alpha(v) = a_v v$ for some $a_v \in R$. Then there exists $a \in Z(R)$ such that $\alpha = a1_V$.*

Proof. It is trivial if $B = \emptyset$. Assume that $B \neq \emptyset$. In order to show that $a_v \in Z(R)$ for every $v \in B$, let $u \in B$ and $b \in R \setminus \{0\}$. Define $\beta \in L_R(V)$ by

$$\beta(v) = \begin{cases} bu & \text{if } v = u, \\ v & \text{if } v \in B \setminus \{u\}. \end{cases}$$

Since $(B \setminus \{u\}) \cup \{bu\}$ is a basis of V, $\beta \in G_R(V)$. Then $\alpha\beta = \beta\alpha$ by hypothesis. Thus $(\beta\alpha)(u) = \beta(a_u u) = (a_u b)u$ and $(\alpha\beta)(u) = \alpha(bu) = (ba_u)u$ which implies that $a_u b = ba_u$. Hence $a_u \in Z(R)$.

Next, we shall show that $a_v = a_{v'}$ for all $v, v' \in B$. If $|B| = 1$ or $a_v = 0$ for all $v \in B$, there is nothing to prove. Assume that $|B| > 1$ and $a_w \neq 0$

for some $w \in B$. Let $z \in B \setminus \{w\}$. Define $\gamma \in L_R(V)$ by

$$\gamma(v) = \begin{cases} z & \text{if } v = w, \\ a_w w & \text{if } v = z, \\ v & \text{if } v \in B \setminus \{w, z\}. \end{cases}$$

Since $a_w \neq 0$, $(B \setminus \{w\}) \cup \{a_w w\}$ is a basis of V, so $\gamma \in G_R(V)$. By hypothesis, $\alpha\gamma = \gamma\alpha$. Then $(\gamma\alpha)(w) = \gamma(a_w w) = a_w z$ and $(\alpha\gamma)(w) = \alpha(z) = a_z z$. These imply that $a_z = a_w$. Hence the lemma is proved. ∎

Lemma 3.6.16. *Let $\alpha \in L_R(V)$ and assume that $\alpha\beta = \beta\alpha$ for all $\beta \in G_R(V)$. Then there exists $a \in Z(R)$ such that $\alpha = a1_V$.*

Proof. Let B be a basis of V. In order to show that for every $v \in B$

$$\alpha(v) = a_v v \quad \text{for some } a_v \in R,$$

suppose to the contrary that it is not true. Then there exists $u \in B$ such that $\alpha(u) \neq bu$ for all $b \in R$. Then $\alpha(u) \neq u$ and $\{u, \alpha(u)\}$ is linearly independent. Let B' be a basis of V containing $\{u, \alpha(u)\}$. Consequently, $\{u + \alpha(u)\} \cup (B' \setminus \{\alpha(u)\})$ is a basis of V. Define $\beta \in L_R(V)$ by

$$\beta(v) = \begin{cases} u + \alpha(u) & \text{if } v = \alpha(u), \\ v & \text{if } v \in B' \setminus \{\alpha(u)\}. \end{cases}$$

Then $\beta \in G_R(V)$ since $\{u + \alpha(u)\} \cup (B' \setminus \{\alpha(u)\})$ is a basis of V. By hypothesis, $\alpha\beta = \beta\alpha$. But $(\beta\alpha)(u) = u + \alpha(u)$ and $(\alpha\beta)(u) = \alpha(u)$, so we have $u = 0$, a contradiction. Now, by Lemma 3.6.15, $\alpha = a1_V$ for some $a \in Z(R)$. ∎

Theorem 3.6.17. *Let S be a subsemigroup of $L_R(V)$ containing $G_R(V)$. Assume that $\oplus$ is a hyperoperation on S^0 such that $(S^0, \oplus, \cdot)$ is a hyperring. Then*

$$\alpha' = \alpha \text{ for all } \alpha \in S^0 \text{ or } \alpha' = -\alpha \text{ for all } \alpha \in S^0,$$

where α' represent the opposite of α, with respect to $\oplus$.

Proof. The result is trivially true if $V = \{0\}$. Assume that $V \neq \{0\}$. Since $G_R(V) \subseteq S$, $1_V \in S$. Then, for every $\alpha \in G_R(V)$ we have

$$1'_V\alpha = (1_V\alpha)' = \alpha' = (\alpha 1_V)' = \alpha 1'_V.$$

By Lemma 3.6.16, $1'_V = a1_V$ for some $a \in Z(R)$. Since $(1'_V)^2 = 1_V$, we obtain $(a1_V)^2 = 1_V$. If $u \in V \setminus \{0\}$ then $a^2u = (a1_V)^2(u) = 1_V(u) = u$. We deduce that $a^2 = 1$, so $a = \pm 1$ since R is a division ring. Consequently, we have $0 \in 1_V \oplus 1_V$ or $0 \in 1_V \oplus (-1_V)$ which implies that

$$0 \in (\alpha \oplus \alpha) \text{ for all } \alpha \in S^0 \quad \text{or} \quad 0 \in (\alpha \oplus -\alpha) \text{ for all } \alpha \in S^0.$$

Hence the theorem is proved. ∎

Corollary 3.6.18. *Let S be a subsemigroup of $M_n(R)$ containing all non-singular matrices in $M_n(R)$. Assume that $\oplus$ is a hyperoperation on S^0 such that $(S^0, \oplus, \cdot)$ is a hyperring. Then*

$$A' = A \textit{ for all } A \in S^0 \quad \textit{or} \quad A' = -A \textit{ for all } A \in S^0$$

where A' is the additive inverse of A in $(S^0, \oplus, \cdot)$.

We know that $G_R(V) \in \mathcal{SHR}$ for any dimension of V. As a consequence, we have that $G_R(V) \in \mathcal{SR}$ only for case $\dim_R V \leq 1$.

Corollary 3.6.19. $G_R(V) \in \mathcal{SR}$ *if and only if* $\dim_R V \leq 1$.

Proof. Since $G_R(V) = \{0\}$ if $\dim_R V$=0 and $G_R(V) \cong R \setminus \{0\}$ if $\dim_R V$=1, we have that $G_R(V) \in \mathcal{SR}$ if $\dim_R V \leq 1$.

Conversely, assume that $G_R(V) \in \mathcal{SR}$ and suppose that $\dim_R V > 1$. Let $\oplus$ be an operation on $G^0_R(V)$ such that $(G^0_R(V), \oplus, \cdot)$ is a ring. By Theorem 3.6.17,

$$\alpha \oplus \alpha = 0 \quad \text{for all} \quad \alpha \in G^0_R(V) \quad \text{or}$$
$$\alpha \oplus (-\alpha) = 0 \quad \text{for all} \quad \alpha \in G^0_R(V).$$

Let B be a basis of V. By assumption, $|B| > 1$. Let u and w be distinct elements of B. Define $\beta \in L_R(V)$ by

$$\beta(v) = \begin{cases} w & \text{if } v = u, \\ u & \text{if } v = w, \\ v & \text{if } v \in B \setminus \{u, w\}. \end{cases}$$

Then $\beta \in G_R(V)$, $\beta \neq 1_V$ and $\beta^2 = 1_V$. Since u and w are linearly independent over R, $w \neq -u$, so $\beta \neq -1_V$. We have that $1_V \oplus \beta \neq 0$ and $1_V \oplus (-\beta) \neq 0$. Then we have

$$\begin{aligned} 0 \neq (1_V \oplus \beta)^2 &= 1_V \oplus \beta \oplus \beta \oplus \beta^2 \\ &= 1_V \oplus \beta \oplus \beta \oplus 1_V \end{aligned}$$

and

$$\begin{aligned} 0 \neq (1_V \oplus \beta)(1_V \oplus (-\beta)) &= 1_V \oplus (-\beta) \oplus \beta \oplus (-\beta^2) \\ &= 1_V \oplus (-\beta) \oplus \beta \oplus (-1_V). \end{aligned}$$

Therefore, we obtain a contradiction. Hence the corollary holds. ■

As an immediate consequences of Corollary 3.6.19, we have

Corollary 3.6.20. *$G_n(R) \in \mathcal{SR}$ if and only if $n = 1$.*

We note here that if $\dim_R V=0$, then $L_R(V)=\{0\}$ and $L_R(V) \setminus G_R(V)=\emptyset$ and if $\dim_R V=1$, then $L_R(V) \cong R$ and $L_R(V) \setminus G_R(V)=\{0\}$.

Theorem 3.6.21. *Assume that $\dim_R V > 1$ and let S be a subsemigroup of $L_R(V)$ containing $L_R(V) \setminus G_R(V)$. If $S \in \mathcal{SHR}$, then $S = L_R(V)$.*

Proof. Let $\oplus$ be a hyperoperation on S such that $(S, \oplus, \cdot)$ is a hyperring where $\cdot$ is the operation on S. In order to show that $S = L_R(V)$, let $\alpha \in L_R(V)$. If $\alpha \notin G_R(V)$, then $\alpha \in L_R(V) \setminus G_R(V) \subseteq S$. Suppose that $\alpha \in G_R(V)$. Let B be a basis of V. Then $|B| \geq 2$. Let $u \in B$ be fixed. Then $\{\alpha(u)\}$ is not a basis of V. Let $\beta, \gamma \in L_R(V)$ be defined by

$$\beta(v) = \begin{cases} \alpha(v) & \text{if } v \in B \setminus \{u\}, \\ 0 & \text{if } v = u. \end{cases}$$

and

$$\gamma(v) = \begin{cases} \alpha(u) & \text{if } v = u, \\ 0 & \text{if } v \in B \setminus \{u\}. \end{cases}$$

Then $\beta, \gamma \in L_R(V) \setminus G_R(V) \subseteq S$, so $\beta \oplus \gamma \subseteq S$. For each $w \in B$, let $\lambda_w \in L_R(V)$ be defined by

$$\lambda_w(v) = \begin{cases} w & \text{if } v = w, \\ 0 & \text{if } v \in B \setminus \{w\}. \end{cases}$$

Then $\lambda_w \in S$ for all $w \in B$, so $(\beta \oplus \gamma)\lambda_w = \beta\lambda_w \oplus \gamma\lambda_w$ for all $w \in B$. We clearly have

$$\begin{aligned} \beta\lambda_u &= 0, \quad (\gamma\lambda_u)(u) = \alpha(u), \\ \beta\lambda_v(v) &= \alpha(v) \text{ for all } v \in B \setminus \{u\} \text{ and} \\ \gamma\lambda_v &= 0 \text{ for all } v \in B \setminus \{u\}. \end{aligned}$$

Consequently, we have

$$(\beta \oplus \gamma)\lambda_u = \{\gamma\lambda_u\}$$

and

$$(\beta \oplus \gamma)\lambda_v = \{\beta\lambda_v\} \text{ for all } v \in B \setminus \{u\}.$$

Let $\eta \in \beta \oplus \gamma$. Then we have

$$\eta\lambda_u = \gamma\lambda_u \text{ and } \eta\lambda_v = \beta\lambda_v \text{ for all } v \in B \setminus \{u\}.$$

These imply that

$$\eta(u) = (\eta\lambda_u)(u) = (\gamma\lambda_u)(u) = \alpha(u)$$

and

$$\eta(v) = (\eta\lambda_v)(v) = (\beta\lambda_v)(v) = \alpha(v) \text{ for all } v \in B \setminus \{u\}.$$

It follows that $\alpha = \eta \in \beta \oplus \gamma \subseteq S$. Hence we prove that $S = L_R(V)$, as required. ∎

As a consequence of Theorem 3.6.21, we have

Corollary 3.6.22. *Assume that $n > 1$ and let S be a subsemigroup of $M_n(R)$ containing all singular matrices in $M_n(R)$. If $S \in \mathcal{SHR}$, then $S = M_n(R)$.*

Now, our purpose is to verify that the direct limit (inverse limit) of an $\mathcal{SHR}$ direct family (respectively, an $\mathcal{SHR}$ inverse family) of $\mathcal{SHR}$ semigroups is an $\mathcal{SHR}$ semigroup. We use the results obtained by V. Leoreanu (see [75]).

Definition 3.6.23. We say that a family $\{< H_i, \circ_i >\}_{i \in I}$ of semigroups is a *direct* (*inverse*) *family* if:

(1) $(I, \leq)$ is a directed partially ordered set;

(2) $\forall (i,j) \in I^2,\ i \neq j \Longrightarrow H_i \cap H_j = \emptyset$;

(3) $\forall (i,j) \in I^2,\ i \leq j$ (respectively, $i \geq j$) there is a homomorphism of semigroups $f_{ij} : H_i \to H_j$, such that if $i \leq j \leq k$ (respectively, $i \geq j \geq k$), then $f_{jk}f_{ij} = f_{ik}$ and $\forall i \in I,\ f_{ii} = 1_{H_i}$.

A direct (inverse) family of canonical hypergroups, a direct (inverse) family of hyperrings, etc. are defined similarly.

Definition 3.6.24. We say that a family of SHR semigroups $\{\langle H_i, \circ_i \rangle\}_{i\in I}$ is an SHR *direct* (*inverse*) *family* if there is an associated *direct* (*inverse*) *family* of hyperrings $\{\langle H_i^0, \oplus_i, \circ_i \rangle\}_{i\in I}$. An SR *direct* (*inverse*) *family* of SR semigroups is defined similarly.

Let $\{\langle H_i, \circ_i \rangle\}_{i\in I}$ be a direct (inverse) family of semigroups. If $(i,j)\in I^2$, $i \leq j$ and $x_i \in H_i$, we set $f_{ij}(x_i) = x_j$. We shall recall the construction of the direct limit of a direct family of semigroups.

Let $H = \bigcup\limits_{i\in I} H_i$ and we consider the following equivalence relation on H:

$$\forall (x_i, y_j) \in H_i \times H_j,\ \text{set } x_i \sim y_j \Longleftrightarrow \exists k \in I,\ k \geq i,\ k \geq j,$$
$$\text{such that } f_{ik}(x_i) = f_{jk}(y_j).$$

Let $\overline{H}$ be the set of all equivalence classes on H and for each $x \in H$, let $\bar{x}$ denote the equivalence class of x.

We define the following operation on $\overline{H}$:

$$\bar{x} \circ \bar{y} = \bar{z} \text{ iff } \exists i \in I,\ \exists x_i \in \bar{x} \cap H_i,\ \exists y_i \in \bar{y} \cap H_i,\ \exists z_i \in \bar{z} \cap H_i,$$
$$\text{such that } x_i \circ_i y_i = z_i.$$

$< \overline{H}, \circ >$ is the direct limit of the direct family $\{< H_i, \circ_i >\}_{i\in I}$. $< \overline{H}, \circ >$ is a semigroup, being the direct limit of a direct family of semigroups.

The direct limit of a direct family of canonical hypergroups, the direct limit of the direct family of hyperrings, etc. are defined similarly.

Theorem 3.6.25. *Let $\{< H_i, \circ_i >\}_{i\in I}$ be a direct family of semigroups with the corresponding family $\{f_{ij} \mid (i,j) \in I^2,\ i \leq j\}$ of homomorphisms, such that $\forall i \in I,\ \exists k \in I,\ k \geq i : < H_k, \circ_k >$ is an $\mathcal{SHR}$ semigroup. Set*

$K=\{k \in I \mid <H_k, \circ_k> \text{ is an } SHR \text{ semigroup}\}$. So, for each $k \in K$, there exists a hyperoperation $\oplus_k$ on H_k^0 such that $\langle H_k^0, \oplus_k, \circ_k\rangle$ is a hyperring. If $\forall (k,\ell) \in K^2$, $k \leq \ell$, $f_{k\ell}$ is a homomorphism of hyperrings, then the direct limit of the direct family of semigroups $\{<H_i, \circ_i>\}_{i\in I}$ is a $\mathcal{SHR}$ semigroup. Hence the direct limit of an $\mathcal{SHR}$ direct family of $\mathcal{SHR}$ semigroups is an $\mathcal{SHR}$ semigroup.

Proof. First of all, let us notice that $\{<H_k, \circ_k>\}_{k\in K}$ is an SHR direct family of $\mathcal{SHR}$ semigroups and the direct families of semigroups $\{<H_i, \circ_i>\}_{i\in I}$ and $\{<H_k, \circ_k>\}_{k\in K}$ have the same direct limit $< \overline{H}, \circ >$.

Since $\{\langle H_k, \circ_k\rangle\}_{k\in K}$ is an $\mathcal{SHR}$ direct family of $\mathcal{SHR}$ semigroups, it follows that the associated family of hyperrings $\{\langle H_k^0, \oplus_k, \circ_k\rangle\}_{k\in K}$ is a direct one, so the family of canonical hypergroups $\{\langle H_k^0, \oplus_k\rangle\}_{k\in K}$ is a direct one, too.

Now, we shall prove that the direct limit $< \overline{H}^0, \oplus >$ of the direct family $\{< H_k^0, \oplus_k >\}_{k\in K}$ of canonical hypergroups is a canonical hypergroup, too. We have $\forall (\bar{x}, \bar{y}) \in (\overline{H}^0)^2$,

$$\bar{x} \oplus \bar{y} = \{\bar{z} \in \overline{H}^0 \mid \exists k \in K,\ \exists x_k \in \bar{x} \cap H_k^0,\ \exists y_k \in \bar{y} \cap H_k^0,\ \exists z_k \in \bar{z} \cap H_k^0 : z_k \in x_k \oplus_k y_k\}.$$

We know that a commutative hypergroup is a canonical one if and only if it is a join space with a scalar identity.

$< \overline{H}^0, \oplus >$ is a join space, being the direct limit of a direct family of join spaces.

Moreover, if e_k is the scalar identity of the canonical hypergroup $\langle H_k^0, \oplus_k\rangle$, then $\overline{e_k}$ is the scalar identity of $< \overline{H}^0, \oplus >$.

Indeed, if $\bar{x} \in \overline{H}^0$ and $\bar{z} \in \bar{x} \oplus \overline{e_k}$, then there are $t \in K$, $z_t \in \bar{z} \cap H_t^0$, $x_t \in \bar{x} \cap H_t^0$ and $e_t \in \overline{e_k} \cap H_t^0$ such that $z_t \in x_t \oplus_t e_t$.

There is $s \in K$, $s \geq t$, $s \geq k$, such that $e_s = f_{ks}(e_k) = f_{ts}(e_t)$. Since $\forall x_k \in H_k^0$, $x_k \oplus e_k = x_k$ it follows that $x_s \oplus_s e_s = x_s$ and since $z_t \in x_t \oplus_t e_t$ it follows that $z_s \in x_s \oplus_s e_s$. So, $z_s = x_s$, whence $\bar{z} = \bar{x}$. Then $\forall \bar{x} \in \overline{H}^0$, $\bar{x} = \bar{x} \oplus \overline{e_k}$, hence $\overline{e_k}$ is the scalar identity of $< \overline{H}^0, \oplus >$. From here, it follows that for any $p \in K$, the scalar identity e_p of the canonical hypergroup $< H_p^0, \oplus_p >$ belongs to the equivalence class $\overline{e_k}$ of e_k.

We denote $\bar{e} = \overline{e_k}$. Therefore, $< \overline{H}^0, \oplus >$ is a join space with a scalar identity $\bar{e}$, hence $< \overline{H}^0, \oplus >$ is a canonical hypergroup.

Let us notice now that $\forall \bar{x} \in \overline{H}^0$, $\bar{e} \circ \bar{x} = \bar{x} \circ \bar{e} = \bar{e}$. Indeed, since $\forall k \in K$, $<H_k^0, \oplus_k, \circ_k>$ is a hyperring, it follows that $\forall x_k \in H_k^0$, $e_k \circ_k x_k = e_k = x_k \circ_k e_k$, where e_k is the scalar identity of $< H_k^0, \oplus_k >$. Hence, $\bar{e} \circ \bar{x} = \bar{e} = \bar{x} \circ \bar{e}$.

Let us notice that $\forall k \in K$, e_k is zero of H_k^0 and $\bar{e}$ is zero of $\overline{H}^0$.

Now, we have to verify only the distributivity of the operation "$\circ$" over the hyperoperation "$\oplus$".

Let $(\bar{x}, \bar{y}, \bar{z}) \in (\overline{H}^0)^3$. We have $\bar{u} \in \bar{x} \circ (\bar{y} \oplus \bar{z}) \Longleftrightarrow \exists \bar{v} \in \bar{y} \oplus \bar{z} : \bar{u} = \bar{x} \circ \bar{v} \Longleftrightarrow \exists (k, i) \in K \times I : v_k \in y_k \oplus_k z_k$ and $u_i = x_i \circ_i v_i \Longleftrightarrow \exists t \in K$, $t \geq k, t \geq i : v_t \in y_t \oplus_t z_t$ and $u_t = x_t \circ_t v_t \Longleftrightarrow \exists t \in K$, $u_t \in x_t \circ_t (y_t \oplus_t z_t) = (x_t \circ_t y_t) \oplus_t (x_t \circ_t z_t)$ (since $< H_t^0, \oplus_t, \circ_t >$ is a hyperring) $\Longleftrightarrow \exists t \in K$, $u_t \in a_t \oplus_t b_t$, where $a_t = x_t \circ_t y_t$ and $b_t = x_t \circ_t z_t \Longleftrightarrow \bar{u} \in \bar{a} \oplus \bar{b}$, where $\bar{a} = \bar{x} \circ \bar{y}$ and $\bar{b} = \bar{x} \circ \bar{z} \Longleftrightarrow \bar{u} \in (\bar{x} \circ \bar{y}) \oplus (\bar{x} \circ \bar{z})$.

Therefore, $< \overline{H}^0, \oplus, \circ >$ is a hyperring, which means that $< \overline{H}, \circ >$ is an $\mathcal{SHR}$ semigroup. ■

Remark 3.6.26. Let $\{< H_i, \circ_i >\}_{i \in I}$ be a direct family of semigroups with the corresponding family $\{f_{ij} \mid (i,j) \in I^2,\ i \leq j\}$ of homomorphisms, such that $\forall i \in I$, $\exists k \in I$, $k \geq i$ for which $< H_k, \circ_k >$ is an $\mathcal{SR}$ semigroup.

Set $K = \{k \in I \mid < H_k, \circ_k > \text{ is an } \mathcal{SR} \text{ semigroup}\}$. So, for every $k \in K$, there is an operation $\oplus_k$ on H_k^0 such that $< H_k^0, \oplus_k, \circ_k >$ is a ring. If $\forall (k, \ell) \in K^2$, $k \leq \ell$, $f_{k\ell}$ is a homomorphism of rings, then the direct limit of the direct family of semigroups $\{< H_i, \circ_i >\}_{i \in I}$ is an $\mathcal{SR}$ semigroup.

Proof. We have that $< \overline{H}^0, \oplus, \circ >$ is a ring, being the direct limit of the direct family of rings $\{< H_k^0, \oplus_k, \circ_k >\}_{k \in K}$, whence it follows that $< \overline{H}, \circ >$ is an SR semigroup. ■

Let $\{< H_i, \circ_i >\}_{i \in I}$ be an $\mathcal{SHR}$ inverse family of $\mathcal{SHR}$ semigroups.

Let us consider the following subset of the direct product $H = \prod_{i \in I} H_i$:

$$\widetilde{H} = \{\widetilde{p} \in H \mid f_{ij}(p_i) = p_j,\ \forall i \geq j\} \text{ where } \widetilde{p} = (p_i)_{i \in I}.$$

If $\widetilde{H} \neq \emptyset$, we define the following operation on $\widetilde{H}$:

$$\forall (\widetilde{x}, \widetilde{y}) \in \widetilde{H}^2,\ \widetilde{x} \Box \widetilde{y} = (x_i \circ_i y_i)_{i \in I}.$$

We have $\widetilde{x}\square\widetilde{y} \in \widetilde{H}$. Indeed, if we denote $z_i = x_i \circ_i y_i$, for every $i \in I$, we have $\forall (i,j) \in I^2$, $i \geq j$, $f_{ij}(z_i) = f_{ij}(x_i \circ_i y_i) = f_{ij}(x_i) \circ_j f_{ij}(y_i) = x_j \circ_j y_j = z_j$, whence $\widetilde{x}\square\widetilde{y} = (z_i)_{i\in I} \in \widetilde{H}$. $< \widetilde{H}, \square >$ is called the *inverse limit* of the inverse family $\{< H_i, \circ_i >\}_{i\in I}$. Note that, if I has a maximum, then $\widetilde{H} \neq \emptyset$.

Theorem 3.6.27. *Let $\{< H_i, \circ_i >\}_{i\in I}$ be an $\mathcal{SHR}$ inverse family of $\mathcal{SHR}$ semigroups, such that $(I, \leq)$ has a maximum. Then the inverse limit $< \widetilde{H}, \square >$, of the above family, is an $\mathcal{SHR}$ semigroup.*

Proof. Since $\{< H_i, \circ_i >\}_{i\in I}$ is an $\mathcal{SHR}$ inverse family of $\mathcal{SHR}$ semigroups, it follows that there is a corresponding inverse family $\{< H_i^0, \boxplus_i >\}_{i\in I}$ of canonical hypergroups. Since I has a maximum, it follows that $\widetilde{H} \neq \emptyset$.

For every $(\widetilde{x}, \widetilde{y}) \in (\widetilde{H}^0)^2$, $\widetilde{x} = (x_i)_{i\in I}$, $\widetilde{y} = (y_i)_{i\in I}$, set $\widetilde{x} \boxtimes \widetilde{y} = (x_i \boxplus_i y_i)_{i\in I}$ and $\widetilde{x} \boxplus \widetilde{y} = \widetilde{x} \boxtimes \widetilde{y} \cap \widetilde{H}^0$. Then $< \widetilde{H}^0, \boxplus >$ is a join space, being the inverse limit of the inverse family of join spaces $\{< H_i^0, \boxplus_i >\}_{i\in I}$, such that $(I, \leq)$ has a maximum.

Moreover, we shall prove that $< \widetilde{H}^0, \boxplus >$ has a scalar identity.

Set $s = \max I$, let e_s be the scalar identity of $< H_s^0, \boxplus_s >$ and set $\widetilde{e} = (f_{sj}(e_s))_{j\in I} \in \widetilde{H}^0$.

Let us notice that if $\widetilde{x} \in \widetilde{H}^0$, $\widetilde{x} = (x_i)_{i\in I}$, then $\forall j \in I$, $x_j \boxplus_j f_{sj}(e_s) = x_j$. Indeed, since $x_s \boxplus_s e_s = x_s$, it follows that $\forall j \in I$, $f_{sj}(x_s) \boxplus_j f_{sj}(e_s) = f_{sj}(x_s)$, that is $x_j \boxplus_j f_{sj}(e_s) = x_j$.

Hence, $\forall \widetilde{x} \in \widetilde{H}^0$, $\widetilde{x} \boxplus \widetilde{e} = \{\widetilde{z} \in \widetilde{H}^0 \mid \forall i \in I,\ z_i \in x_i \boxplus_i f_{si}(e_s) = x_i\}$, which means that $\widetilde{x} \boxplus \widetilde{e} = \widetilde{x}$, whence it follows that $\widetilde{e}$ is the scalar identity of $< \widetilde{H}^0, \boxplus >$. Since $< \widetilde{H}^0, \boxplus >$ is also a join space, we obtain that $< \widetilde{H}^0, \boxplus >$ is a canonical hypergroup. Notice also that $< \widetilde{H}, \square >$ is a semigroup, being the inverse limit of an inverse family of semigroups $\{< H_i, \circ_i >\}_{i\in I}$.

Let us see now that $\forall \widetilde{x} \in \widetilde{H}$, $\widetilde{x}\square\widetilde{e} = \widetilde{e}\square\widetilde{x} = \widetilde{e}$. Indeed, if $\widetilde{z} = \widetilde{x}\square\widetilde{e}$, where $\widetilde{x} = (x_i)_{i\in I}$, $\widetilde{z} = (z_i)_{i\in I}$, then $z_i = x_i \circ_i f_{si}(e_s) = x_i$, as we have seen above, so $\widetilde{z} = \widetilde{x}$. Similarly, it follows that $\widetilde{e}\square\widetilde{x} = \widetilde{x}$.

Let us notice that $\forall i \in I$, e_i is zero of H_i^0 and $\widetilde{e}$ is zero of $\widetilde{H}^0$.

Concerning to the distributivity of the operation "$\square$" over the hyperoperation "$\boxplus$", we have:

for any $\widetilde{x} = (x_i)_{i\in I}$, $\widetilde{y} = (y_i)_{i\in I}$, $\widetilde{z} = (z_i)_{i\in I}$: $\widetilde{u} \in \widetilde{x}\square(\widetilde{y} \boxplus \widetilde{z}) \iff \exists \widetilde{v} \in \widetilde{y} \boxplus \widetilde{z}$, $\widetilde{u} = \widetilde{x}\square\widetilde{v} \iff \forall i \in I$, $f_{si}(v_s) \in y_i \boxplus_i z_i$, $f_{si}(u_s) = x_i \circ_i f_{si}(v_s)$

(where $s = \max I$, $\widetilde{v} = (f_{si}(v_s))_{i\in I}$, $\widetilde{u} = (f_{si}(u_s))_{i\in I} \Longleftrightarrow \forall i \in I$, $f_{si}(u_s) \in x_i \circ_i (y_i \boxplus_i z_i) = (x_i \circ_i y_i) \boxplus_i (x_i \circ_i z_i)$ (since $\forall i \in I$, $< H_i^0, \boxplus_i, \circ_i >$ is a hyperring) $\Longleftrightarrow \widetilde{u} \in (\widetilde{x}\Box\widetilde{y}) \boxplus (\widetilde{x}\Box\widetilde{z})$.

Therefore, $< \widetilde{H}^0, \boxplus, \Box >$ is a hyperring, which means that $< \widetilde{H}, \Box >$ is an $\mathcal{SHR}$ semigroup. ■

Remark 3.6.28. *Let $\{< H_i, \circ_i >\}_{i\in I}$ be an SR inverse family of $\mathcal{SR}$ semigroups, such that $\widetilde{H} \neq \emptyset$. Then the inverse limit $< \widetilde{H}, \Box >$ of the above family, is an $\mathcal{SR}$ semigroup, too.*

Proof. Let $\{< H_i^0, \boxplus_i, \circ_i >\}_{i\in I}$ be an associated inverse family of rings of the inverse family $\{< H_i, \circ_i >\}_{i\in I}$. Since $\widetilde{H} \neq \emptyset$, it follows that $< \widetilde{H}^0, \boxplus, \Box >$ is a ring, being the inverse limit of the inverse family of rings $\{< H_i^0, \boxplus_i, \circ_i >\}_{i\in I}$. Hence, $< \widetilde{H}, \Box >$ is an $\mathcal{SR}$ semiring.■

3.7 Hypernear-rings

In the context of canonical hypergroups some mathematicians studied multi-valued systems whose additive structure are quasicanonical hypergroups. In [29], Dasic introduced the notion of hypernear-ring in a particular case and Gontineac [52] called this type of hypernear-ring zero symmetric. A study of the concept of a hypernear-ring in a general case is done in [52] and [67]. Firstly, we present some fundamental definitions.

Definition 3.7.1. A *hypernear-ring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:
(1) $(R, +)$ is a *quasi canonical hypergroup*, i.e., in $(R, +)$ the following conditions hold:

(a) $x + (y + z) = (x + y) + z$ for all $x, y, z \in R$;

(b) There is $0 \in R$ such that $x + 0 = 0 + x = x$ for all $x \in R$;

(c) For every $x \in R$ there exists one and only one $x' \in R$ such that $0 \in x + x'$, (we shall write $-x$ for x' and we call it the *opposite* of x);

(d) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

(2) $(R, \cdot)$ is a semigroup respect to the multiplication, having an absorbing element 0, i.e., $x \cdot 0 = 0$ for all $x \in R$. But, in general, $0x \neq 0$ for some $x \in R$.
(3) The multiplication is left distributive with respect to the hyperoperation $+$, i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

A hypernear-ring R is called *zero symmetric* if $0x = x0 = 0$ for all $x \in R$.

Note that for all $x, y \in R$, we have $-(-x) = x, 0 = -0, -(x+y) = -y-x$ and $x(-y) = -xy$.

Example 3.7.2. Let $(H, +)$ be a hypergroup and let $M_0(H)$ be the set of all mappings $f : H \longrightarrow H$ such that $f(0) = 0$. For all $f, g \in M_0(H)$ we define $f \oplus g$ as follows:

$$f \oplus g = \{h \in M_0(H) \mid \forall x \in H,\ h(x) \in f(x) + g(x)\}.$$

If $h \in f \oplus g$, then $h(0) \in f(0) + g(0) = 0$, so $h(0) = 0$. Furthermore, $(f \oplus g)(x) = f(x) + g(x)$. As multiplication we consider the composition of mappings. Then $(M_0(H), \oplus, \cdot)$ is a zero symmetric hypernear-ring

Definition 3.7.3. Let R be a hypernear-ring. A nonempty subset S of R is called a *subhypernear-ring* if $(S, +)$ is a subhypergroup of $(R, +)$ and $(S, \cdot)$ is a subsemigroup of $(R, \cdot)$.

Now, we consider the notion of hyper R-subgroup of a hypernear-ring R.

Definition 3.7.4. A *two-sided hyper R-subgroup* of a hypernear-ring R is a subset H of R such that

(1) $(H, +)$ is a subhypergroup of $(R, +)$, i.e.,
  (i) $a, b \in H$ implies $a + b \subseteq H$,
  (ii) $a \in H$ implies $-a \in H$,

(2) $RH \subseteq H$,

(3) $HR \subseteq H$.

If H satisfies (1) and (2), then it is called a *left hyper R-subgroup* of R. If H satisfies (1) and (3), then it is called a *right hyper R-subgroup* of R.

Definition 3.7.5. Let $(R, +, \cdot)$ be a hypernear-ring.

(1) The subset $R_0 = \{x \in R \mid 0x = 0\}$ of R is called a *zero-symmetric part* of R.

(2) The subset $R_c = \{x \in R \mid xy = y,\ \forall y \in R\}$ is called a *constant part* of R.

(3) If $R = R_0$ (respectively, $R = R_c$), we say that R is a *zero-symmetric* (respectively, *constant*) *hypernear-ring,* respectively.

Example 3.7.6. Consider the hypernear-ring $R = \{0, a, b, c\}$ with the addition and multiplication tables given below:

$+$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$	$\{c\}$
b	$\{b\}$	$\{b\}$	$\{0, a, c\}$	$\{b, c\}$
c	$\{c\}$	$\{c\}$	$\{b, c\}$	$\{0, a, b\}$

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	a	b	c
b	0	a	b	c
c	0	a	b	c

Since $R = R_c$, it follows that R is a constant hypernear-ring.

Lemma 3.7.7. *If $0y = y$ for each y in a hypernear-ring R, then R is a constant hypernear-ring.*

Proof. Take $x \in R$. Then $xy = x(0y) = (x0)y = 0y = y$. ■

Lemma 3.7.8. *Let $(R, +, \cdot)$ be a constant hypernear-ring. Then R is the only right hyper R-subgroup of R.*

Proof. Let $(H, +)$ be a subhypergroup of $(R, +)$. If $HR \subseteq H$, then we have $0R \subseteq H$, and so $R \subseteq H$ since $0R = R$. Thus R is the only right hyper R-subgroup of R. ■

Lemma 3.7.9. *Let $(R, +, \cdot)$ be a constant hypernear-ring. If $(H, +)$ is a subhypergroup of $(R, +)$ then H is a left hyper R-subgroup of R.*

Proof. Let $(H, +)$ be a subhypergroup of $(R, +)$. Since $xy = y$ for all $x, y \in R$, we have $RH \subseteq H$, and so H is a left hyper R-subgroup of R. ■

Proposition 3.7.10. *Let R be a hypernear-ring. Then R_0 is a zero symmetric subhypernear-ring of R.*

Proof. Let $x, y \in R_0$ and $z \in x + y$, arbitrary. Then we have $0x = 0$

and $0y = 0$, and so $0z \in 0(x + y) = 0x + 0y = 0 + 0 = 0$. Hence we obtain $z \in R_0$. This implies that $x + y \subseteq R_0$. From $x(-y) = -xy$, we have $0(-x) = -(x0) = -0 = 0$, for all $x \in R_0$. This implies that $-x \in R_0$. Also, we have $0(xy) = (0x)y = 0y = 0$, that is., $R_0R_0 \subset R_0$. This completes the proof.■

Now, let H be a subhypernear-ring of hypernear-ring R. Define the sets by $B_t := \{0x \mid x \in H\}$ and $B := \{0r \mid r \in R\}$.

Theorem 3.7.11. *Let H be a subhypernear-ring of hypernear-ring R. Then we have*

(1) *B_t is a two-sided hyper H-subgroup of H,*

(2) *B_t is a left hyper B-subgroup of B.*

Proof. (1) Let $x, y \in B_t$ and $z \in x + y$. Then there exists $s_1, s_2 \in H$ such that $x = 0s_1, y = 0s_2$. Hence we have $z \in x+y = 0s_1+0s_2 = 0(s_1+s_2) \subseteq B_t$ since $s_1 + s_2 \subseteq H$. This implies that $x + y \subset B_t$. Next, let $x \in B_t$. Then there exists $s \in H$ such that $x = 0s$. Hence we have $-x = -(0s) = 0(-s)$. This implies that $-x \in B_t$ since $-s \in H$. Let $a \in H$ and $b \in B_t$. Then there exists $x \in H$ such that $b = 0x$ and $ab = a(0x) = (a0)x = 0x \in H$. Therefore, $HB_t \subset B_t$. Similarly, let $a_1 \in H$ and $b_1 \in B_t$. Then there exists $x_1 \in H$ such that $b_1 = 0x_1$. Hence $b_1a_1 = (0x_1)a_1 = 0(x_1a_1)$. Since H is a subhypernear-ring, we obtain $x_1a_1 \in H$ for a_1 and x_1 in H. Therefore $b_1a_1 \in B_t$. This implies that $B_tH \subset H$. Hence, B_t is a two-sided hyper H-subgroup of H.

(2) Let $0r, 0s \in B$ and $z \in 0r + 0s$. Then, $z \in 0r + 0s = 0(r + s) \subseteq B$. Hence we have $z \in B$. Also, we have $0r \cdot 0s = 0s$ since $(0r) \cdot (0s) = ((0r)0)s = 0s$. Therefore B is a subhypernear-ring and from (i), $(B_t, +)$ is a subhypergroup of $(B, +)$. It remains to show that $BB_t \subset B_t$. Let $a \in B$ and $b \in B_t$. Then there exists $x \in H$ and $r \in R$ such that $a = 0r, b = 0x$. Thus $ab = (0r) \cdot (0x) = 0(r0)x = 0x \in B_t$. Therefore, we have $BB_t \subset B_t$. This completes the proof. ■

A subhypergroup $A \subseteq R$ is called *normal* if for all $x \in R$, we have $x + A - x \subseteq A$.

Proposition 3.7.12. *Let A is a normal subhypergroup of R. Then*

(1) $A + x = x + A$ *for all* $a \in R$,

(2) $(A + x) + (A + y) = A + x + y$ *for all* $x, y \in R$.

Proof. (1) Suppose that $y \in A + x$. Then there exists $a \in A$ such that $y \in a+x$. Hence $y \in a+x = 0+a+x \subseteq (x-x)+a+x = x+(-x+a+x) \subseteq x+A$, and so $A + x \subseteq x + A$. Similarly $x + A \subseteq A + x$.
(2) We have $(A+x)+(A+y) = A+(x+A)+y = A+(A+x)+y = A+x+y$. ∎

Theorem 3.7.13. *Let R be a hypernear-ring and H be a subhypernear-ring of R. Then $H = (R_0 \cap H) + B_t$ and $(R_0 \cap H) \cap B_t = \{0\}$.*

Proof. We have $(R_0 \cap H) \cap B_t \subset R_0 \cap B$ and $R_0 \cap B = \{0\}$. Thus $(R_0 \cap H) \cap B_t = \{0\}$. Finally, if $a \in H$, we have $0 \in 0a - 0a = 0a - (00)a = 0a - 0(0a) = 0(a - 0a)$. So there exists $y \in a - 0a$ such that $0 = 0y$. Since $a \in H$, we have $a - 0a \subseteq H$, and so $y \in H$. From $y \in a - 0a$, it follows that $a \in y + 0a$. Since $a \in H$, then $0a \in B_t$. Since $0 = 0y$, then $y \in R_0$. Therefore, we have $a \in (R_0 \cap H) + B_t$. ∎

Definition 3.7.14. For an element x of a hypernear-ring R, the (*right*) *annihilator* of x is $Ann(x) = \{r \in R \mid xr = 0\}$. For a nonempty subset B of a hypernear-ring R, the annihilator of B is $Ann(B) = \cap\{Ann(x) \mid x \in B\}$.

Proposition 3.7.15. *For any element x of a zero symmetric hypernear-ring R, $Ann(x)$ is a right R-subgroup of R.*

Proof. Certainly $0 \in Ann(x)$. If $a, b \in Ann(x)$, then $x(a+b) = xa+xb = 0$, so for every $c \in a + b$, we have $xc = 0$ which implies that $a + b \subseteq Ann(x)$. Also, we have $x(-a) = -xa = -0 = 0$ and so $-a \in Ann(x)$. On the other hand, for $r \in R$ and $a \in Ann(x)$, we have $x(ar) = (xa)r = 0r = 0$ so $ar \in Ann(x)$ which yields $Ann(x)R \subseteq Ann(x)$. ∎

Proposition 3.7.16. *If e is any element of a hypernear-ring R, then $eR = \{er \mid r \in R\}$ is a right hyper R-subgroup of R.*

Proof. It is straightforward. ∎

Definition 3.7.17. An element e of a hypernear-ring R is an *idempotent* if $e^2 = e$.

Lemma 3.7.18. *In a hypernear-ring R, if $e \in R_c$ then $e^2 = e$, so e is an idempotent.*

Proof. Since each element of a constant hypernear-ring is a left identity, it follows that it is also an idempotent. ■

Theorem 3.7.19. *Let e be an idempotent element of a zero symmetric hypernear-ring R. Then*

(1) $Ann(e) \cap eR = \{0\}$.
(2) *For all $r \in R$, there exists a unique element $a \in Ann(e)$ and there exists a unique element $b \in eR$ such that $r \in A + b$.*

Proof. (1) Let $x \in Ann(e) \cap eR$. Then $x = er$ for some $r \in R$. So

$$0 = ex = e(er) = (ee)r = er = x,$$

hence $Ann(e) \cap eR = \{0\}$.

(2) For $r \in R$, we have

$$0 \in er - er = er - e^2r = er - e(er) = e(r - er).$$

So there exists $y \in r - er$ such that $0 = ey$. From $y \in r - er$, using condition (d) in Definition 3.7.1, we obtain $r \in y + er$. Since $ey = 0$, then $y \in Ann(e)$. We set $a = y$ and $b = er$. Then $x \in a + b$. If we take another $a' \in Ann(e)$ and $b' \in eR$ with $x \in a' + b'$, then $x \in (a + b) \cap (a' + b')$. From $x \in a' + b'$, we get $b' \in -a' + x$, and so $b' \in -a' + (a + b) = (-a' + a) + b$. Hence there exists $y \in -a' + a$ such that $b' \in y + b$, and so $y \in b' - b$. Therefore $(-a' + a) \cap (b' - b) \neq \emptyset$. Since $-a' + a \subseteq Ann(e)$ and $b' - b \subseteq eR$ and $eR \cap Ann(e) = \emptyset$, we obtain $-a' + a = b' - b = \{0\}$. Therefore $a = a'$ and $b = b'$. ■

As for any algebraic structure, it is natural to introduce the homomorphism notion in the context of hypernear-rings. This is a mapping that preserves some or all properties of a hypernear-ring. One could summarize a lot of research effort by saying that are considered to be relevant to hypernear-rings.

Definition 3.7.20. Let R and R' be two hypernear-rings. Then the map $f : R \rightarrow R'$ is called a *homomorphism* if for all $x, y \in R_1$,

(1) $f(x + y) = f(x) + f(y)$,
(2) $f(x \cdot y) = f(x) \cdot f(y)$,
(3) $f(0) = 0$.

If f is one to one and onto, then f is an isomorphism.

Definition 3.7.21. If f is a homomorphism from R into R', then the *kernel* of f is the set $kerf = \{x \in R \mid f(x) = 0\}$.

It is easy to see that $kerf$ is a left hyper R-subgroup of R, but in general it is not normal in R.

Proposition 3.7.22. *Let $f : R \to R'$ be a homomorphism of hypernear-rings. Then the following statements are true.*

(1) *If f is onto and M is a hyper R-subgroup of R, then $f(M)$ is a hyper R'-subgroup of R'.*

(2) *If N is a hyper R'-subgroup of R', then $f^{-1}(N)$ is a hyper R-subgroup of R.*

(3) $f(R_0) \subseteq R'_0$.

(4) $f(R_c) \subseteq R'_c$.

(5) *If f is an isomorphism, then f^{-1} is an isomorphism, too.*

Proof. The proof is immediate. ■

Definition 3.7.23. A normal subhypergroup A of the hypergroup $(R, +)$ is

(1) a *left hyperidel* of R if $x \cdot a \in A$ for all $x \in R$ and $a \in A$.

(2) a *right ideal* of R if $(x + A) \cdot y - x \cdot y \subseteq A$ for all x and $y \in R$.

(3) a *hyperideal* of R if $(x + A) \cdot y - x \cdot y \cup z \cdot A \subseteq A$ for all x, y and $z \in R$.

Theorem 3.7.24. *Let $(R, +, \cdot)$ be a hypernear-ring.*

(1) *If K is a left hyperideal of R and L is a left hyper R-subgroup of R, then $L + K$ is a left hyper R-subgroup of R.*

(2) *If K is a right hyperideal of R and L is a right hyper R-subgroup of R, then $L + K$ is a right hyper R-subgroup of R.*

Proof. In each case, $L + K = K + L$ is a subhypergroup which is normal if L is normal.

(1) If $RL \subseteq L$ and $RK \subseteq K$, then $r(l + k) = rl + rk \subseteq L + K$ for all $r \in R$, $l \in L$ and $k \in K$. This completes the proof of (1).

(2) Now, assume that K is a right hyperideal and K is a right hyper R-subgroup. Let $r \in R$, $l \in L, k \in K$. Then $(l+k)r - lr \subseteq K$ since K is a right hyperideal of R. So for some $k_1 \in K$, we have

$$(l+k)r = k_1 + lr \subseteq K + LR \subseteq K + L.$$

Hence $L + K = K + L$ is a right hyper R-subgroup. ■

Lemma 3.7.25. *Let R be a hypernear-ring, S a subhypernear-ring of R and H a left (right, two-sided respectively) hyper R-subgroup of R. Then $H \cap S$ is a left (right, two-sided respectively) hyper R-subgroup of R.*

Proof. The proof is immediate. ■

Let H be a normal hyper R-subgroup of hypernear-ring R. If we define a relation

$$x \sim y \ (\text{mod } H) \text{if and only if } (x-y) \cap H \neq \emptyset, \text{ for all } x, y \in H,$$

then this relation is a regular equivalence relation (congruence) on H.

Let $\rho(x)$ be the equivalence class of the element $x \in H$ and denote the quotient set by R/H. Define the hyperoperation $\oplus$ and multiplication $\odot$ on R/H by

$$\rho(a) \oplus \rho(b) = \{\rho(c) \mid c \in \rho(a) + \rho(b)\} \text{ and } \rho(a) \odot \rho(b) = \rho(a \cdot b).$$

Theorem 3.7.26. *$(R/H, \oplus, \odot)$ is a hypernear-ring and it is called a quotient hypernear-ring.*

Lemma 3.7.27. *Let H be a normal hyper R-subgroup of R. Then $\rho(x) = H + x$.*

Proof. Suppose that $y \in H + x$, then there exists $a \in H$ such that $y \in a + x$, which implies that $a \in y - x$ and so $(y-x) \cap H \neq \emptyset$ or $y \in \rho(x)$. Thus $H + x \subseteq \rho(x)$. Similarly, we have $\rho(x) \subseteq H + x$. ■

Theorem 3.7.28. (First isomorphism theorem). *Let f be a homomorphism from R into R' with the kernel K such that K is a normal hyper R-subgroup of R. Then $R/H \cong Imf$.*

Theorem 3.7.29. *Let R be a hypernear-ring and K a normal hyper R-subgroup of R. Then, the following statement are equivalent:*

(1) *K is the kernel of a hypernear-ring homomorphism.*

(2) *$(a+x)y - xy \subseteq K$ for all $x, y \in R$ and all $a \in K$.*

(3) *$-xy + (a+x)y \subseteq K$ for all $x, y \in R$ and all $a \in K$.*

Proof. ($1 \Longrightarrow 2$): Suppose that K is the kernel of a hypernear-ring homomorphism f. Then

$$f((a+x)y - xy) = (f(a) + f(x))f(y) - f(x)f(y) = 0$$

for all $x, y \in R$ and $a \in K$. Hence $(a+x)y - xy \subseteq K$.

($2 \Longrightarrow 1$): For a normal hyper R-subgroup K of R, we consider the quotient hypernear-ring R/K and the natural map $\pi : R \longrightarrow R/K$ where $\pi(x) = x + K$. Clearly, we have $\pi(x+y) = \pi(x) + \pi(y)$ and $\pi(0) = 0$. We show that $\pi(xy) = \pi(x)\pi(y)$, that is, $K + xy = (K+x)(K+y)$ and in order to do this, we need only to show that $(K+x)(K+y) = K + xy$ is a well defined binary operation. We take $K + x' = K + x$ and $K + y' = K + y$. So there are $a, b \in K$ such that $x' \in a + x$ and $y' \in b + y$. Hence $x'y' \in (a+x)(b+y) = (a+x)b + (a+x)y$. Now $x'y' - xy \subseteq (a+x)b + [(a+x)y - xy] \subseteq K + K \subseteq K$. This means that $K + x'y' = K + xy$, which in turn means that $(K+x)(K+y) = K + xy$ is well defined.

($2 \Longrightarrow 3$): For any $a \in K$ and $x, y \in R$, we have

$$-xy + (a+x)y = -[-(a+x)y + xy] = -[(a+y)(-y) - x(-y)] \subseteq K$$

since by (2) $(a+y)(-y) - x(-y) \subseteq K$.

($3 \Longrightarrow 1$): The proof is similar to ($2 \Longrightarrow 3$). ∎

Chapter 4

Multiplicative hyperrings

4.1 The notion of a multiplicative hyperring.

The second type of a hyperring was introduced by R. Rota [108] in 1982. The multiplication is a hyperoperation, while the addition is an operation, that is why she called it a multiplicative hyperring. Let us give the definition.

Definition 4.1.1. A triple $(R, +, \cdot)$ is called a *multiplicative hyperring* if

(1) $(R, +)$ is an abelian group;

(2) $(R, \cdot)$ is a semihypergroup;

(3) for all $a, b, c \in R$, we have $a \cdot (b+c) \subseteq a \cdot b + a \cdot c$ and $(b+c) \cdot a \subseteq b \cdot a + c \cdot a$;

(4) for all $a, b \in R$, we have $a \cdot (-b) = (-a) \cdot b \doteq -(a \cdot b)$.

If in (3) we have equalities instead of inclusions, then we say that the multiplicative hyperring is *strongly distributive.*

An element e in R, such that for all $a \in R$, we have $a \in a \cdot e \cap e \cdot a$, is called a *weak identity* of R.

Let us give some Examples.

Example 4.1.2

(1) Let $(R, +, \cdot)$ be a ring and I be an ideal of it. We define the following hyperoperation on R: For all $a, b \in R$, $a * b = a \cdot b + I$. Then $(R, +, *)$ is a strongly distributive hyperring.

Indeed, first of all, $(R,+)$ is an abelian group. Then, for all $a,b,c \in R$, we have
$a*(b*c) = a*(b\cdot c+I) = \bigcup_{h\in I} a*(b\cdot c+h) = \bigcup_{h\in I} a\cdot(b\cdot c+h)+I = a\cdot b\cdot c+I$
and similarly, we have $(a*b)*c = a\cdot b\cdot c+I$. Moreover, for all $a,b,c \in R$, we have $a*(b+c) = a\cdot(b+c)+I = a\cdot b+a\cdot c+I = a*b+a*c$ and similarly, we have $(b+c)*a = b*a+c*a$. Finally, for all $a,b \in R$, we have $a*(-b) = a\cdot(-b)+I = (-a)\cdot b+I = (-a)*b$ and $-(a*b) = (-a\cdot b)+I = (-a)\cdot b+I = a*(-b)$.

(2) Let K be a field and V be a vectorial space over K. If for all $a,b \in K$ we denote by (a,b) the subspace generated by the subset $\{a,b\}$ of V, then we can consider the following hyperoperation on V: for all $a,b \in V$, $a \circ b = (a,b)$. It follows that $(V,+,\circ)$ is a multiplicative hyperring, which is not strongly distributive.

Indeed, the hyperoperation $\circ$ is associative. For all $a,b,c \in V$, we have $(a\circ b)\circ c = \bigcup_{x\in(a,b)} (x,c)$ and $a\circ(b\circ c) = \bigcup_{x\in(b,c)} (a,x)$.

If $v \in (a\circ b)\circ c$, then there exists $x \in a\circ b$, $v \in x\circ c$, whence $x = \alpha a+\beta b$, $v = \alpha' x+\beta' c$ for some $\alpha,\alpha',\beta,\beta' \in K$. If we set $y = \alpha'\beta b+\beta' c$, then $v = \alpha'\alpha a+y$, which means that $y \in (b,c)$, $v \in (a,y)$ hence $v \in a\circ(b\circ c)$. Thus $(a\circ b)\circ c \subseteq a\circ(b\circ c)$. Similarly, we obtain the converse inclusion. Moreover, $a\circ(b+c) = (a,(b+c))$. If $v \in a\circ(b+c)$, then there exist $\alpha,\beta \in K$ such that $v = \alpha a+\beta(b+c)$, whence $v \in (a,b)+(a,c) = a\circ b+a\circ c$. Finally, for all $a,b \in V$, we have $a\circ(-b) = (a,-b)$; $(-a)\circ b = (-a,b)$; $-(a\circ b) = -(a,b)$ and clearly, $(a,-b) = (-a,b) = -(a,b)$ so we obtain also the condition (4) of Definition 4.1.1. Notice that $(a,0) = (a)$ and so, $(0) = V$.

(3) Let $(R,+,\cdot)$ be a nonzero ring. For all $a,b \in R$ we define the hyperoperation $a*b = \{a\cdot b, 2a\cdot b, 3a\cdot b, ...\}$. Then $(R,+,*)$ is a multiplicative hyperring, which is not strongly distributive. Notice that for all $a \in R$, we have $a*0 = 0*a = \{0\}$.

In what follows, we give some properties of a multiplicative hyperring. These results were obtained by R. Rota [108], D.M. Olson and V.K. Ward [98].

Theorem 4.1.3. *If $(R, +, \cdot)$ is a multiplicative hyperring, then for all $a, b, c \in R$,*

$$a \cdot (b - c) \subseteq a \cdot b - a \cdot c \quad \text{and} \quad (b - c) \cdot a \subseteq b \cdot a - c \cdot a.$$

If $(R, +, \cdot)$ is a strongly distributive, then for all $a, b, c \in R$,

$$a \cdot (b - c) = a \cdot b - a \cdot c \quad \text{and} \quad (b - c) \cdot a = b \cdot a - c \cdot a.$$

Proof. The statement follows from the conditions (3) and (4) of Definition 4.1.1. ■

Theorem 4.1.4. *In a strong distributive hyperring $(R, +, \cdot)$, we have $0 \in a \cdot 0$ and $0 \in 0 \cdot a$, for all $a \in R$.*

Proof. The statement follows from the above theorem, by considering b=c. ■

Theorem 4.1.5. *For a strongly distributive hyperring $(R, +, \cdot)$, the following statements are equivalent:*

(1) *there exists $a \in R$ such that $|0 \cdot a| = 1$,*

(2) *there exists $a \in R$ such that $|a \cdot 0| = 1$,*

(3) $|0 \cdot 0| = 1$,

(4) *$\forall a, b \in R$ such that $|a \cdot b| = 1$,*

(5) *$(R, +, \cdot)$ is a ring.*

Proof. (2)$\Longrightarrow$(3): Suppose $a \neq 0$. For all $a \in R$ we have $0 \cdot 0 = (a - a) \cdot 0 = a \cdot 0 - a \cdot 0$ and so by (2), it follows that $0 \cdot 0 = \{0\}$, whence we obtain (3).

(3)$\Longrightarrow$(4): For all $a \in R$, we have $0 \cdot 0 = a \cdot 0 - a \cdot 0$ and so by (3) it follows that $|a \cdot 0| = 1$, otherwise if we suppose that there exist $x \neq y$ elements of $a \cdot 0$, then $0 \cdot 0$ would contain $x - y \neq 0$ and 0, a contradiction. On the other hand, for all $a, b \in R$ we have $a \cdot 0 = a \cdot (b - b) = a \cdot b - a \cdot b$, whence it follows that $a \cdot b$ contains only an element. The other implications (4)$\Longrightarrow$(5) and (5)$\Longrightarrow$(2) are immediate. Similarly, the condition (1) is equivalent to (3), (4), (5). ■

Corollary 4.1.6. *A strongly distributive hyperring* $(R, +, \cdot)$ *is a ring if and only if there exist* $a_0, b_0 \in R$ *such that* $|a_0 \cdot b_0| = 1$.

Proof. According to the above theorem, it is sufficient to check that $|a_0 \cdot 0|=1$. We have $a_0 \cdot 0 = a_0 \cdot (b_0 - b_0) = a_0 \cdot b_0 - a_0 \cdot b_0$, whence we obtain that $a_0 \cdot 0$ contains only 0. ■

As we can see from Example 4.1.2(3), there exist multiplicative hyperrings, which are not strongly distributive and for which we have $a * 0 = \{0\}$ for all $a \in R$.

Definition 4.1.7. A hyperring $(R, +, \cdot)$ is called *unitary* if it contains an element u, such that $a \cdot u = u \cdot a = \{a\}$ for all $a \in R$.

We obtain

Theorem 4.1.8. *Every unitary strongly distributive hyperring* $(R, +, \cdot)$ *is a ring.*

Proof. If u is the unit element, then we have $u \cdot u = \{u\}$ and according to the above corollary, it follows that R is a ring. ■

Theorem 4.1.9. *In any multiplicative hyperring* $(R, +, \cdot)$, *if there are* $a, b \in R$ *such that* $|a \cdot b| = 1$, *then* $0 \cdot 0 = \{0\}$.

Proof. We have $a \cdot 0 = a \cdot (b - b) \subseteq a \cdot b - a \cdot b = \{0\}$. On the other hand, $0 \cdot 0 = (a - a) \cdot 0 \subseteq a \cdot 0 - a \cdot 0$. But this must also be $\{0\}$, since $a \cdot 0$ is a singleton. ■

Corollary 4.1.10. *In any unitary multiplicative hyperring* $(R, +, \cdot)$, *we have* $0 \cdot 0 = \{0\}$.

Definition 4.1.11. Let $(R, +, \cdot)$ be a multiplicative hyperring and H be a nonempty subset of R. We say that H is a *subhyperring* of $(R, +, \cdot)$ if $(H, +, \cdot)$ is a multiplicative hyperring.

In other words, H is a subhyperring of $(R, +, \cdot)$ if $H - H \subseteq H$ and for all $x, y \in H$, $x \cdot y \subseteq H$.

Definition 4.1.12. We say that H is a *hyperideal* of $(R, +, \cdot)$ if $H - H \subseteq H$ and for all $x, y \in H$, $r \in R$, $x \cdot r \cup r \cdot x \subseteq H$.

The intersection of two subhyperrings of a multiplicative hyperring $(R,+,\cdot)$ is a subhyperring of R. The intersection of two hyperideals of a multiplicative hyperring $(R,+,\cdot)$ is a hyperideal of R. Moreover, any intersection of subhyperrings of a multiplicative hyperring is a subhyperring, while any intersection of hyperideals of a multiplicative hyperring is a hyperideal. In this manner, we can consider the *hyperideal generated by any subset* S of $(R,+,\cdot)$, which is the intersection of all hyperideals of R, which contain S.

For each multiplicative hyperring $(R,+,\cdot)$, the *zero hyperideal* is the hyperideal generated by the additive identity 0. Contrary to what happens in ring theory, the zero hyperideal can contain other elements than 0. If we denote the zero hyperideal of R by $< 0 >$, then we have

$$\langle 0\rangle = \Big\{\sum_i x_i + \sum_j y_j + \sum_k z_k \mid \text{each sum is finite and for each } i,j,k \text{ there exist } r_i, s_j, t_k, u_k \in R \text{ such that } x_i \in r_i\cdot 0,\ y_j \in 0\cdot s_j,\ z_k \in t_k\cdot 0\cdot u_k\Big\}.$$

Denote by $H \oplus K$ the hyperideal generated by $H \cup K$, where H and K are hyperideals of $(R,+,\cdot)$.

Theorem 4.1.13. *If H and K are hyperideals of R, then*

$$H \oplus K = \{h + k \mid h \in H, k \in K\}.$$

Proof. Denote the set $\{h+k \mid h \in H, k \in K\}$ by I. Then I is a hyperideal of R, which contains H and K.

Moreover, if J is a hyperideal of R, containing H and K, then $I \subseteq J$. Hence we have $I = H \oplus K$. ∎

Notice that the above theorem can be extended to an whichever family of hyperideals.

If we denote by $\mathcal{I}$ the set of all hyperideals of a multiplicative hyperring $(R,+,\cdot)$, then $(\mathcal{I},\subseteq)$ is a complete lattice. The infimum of any family of hyperideals is their intersection, while the supremum is the hyperideal generated by their union.

4.2 Homomorphisms between multiplicative hyperrings

Now, it is natural to speak about homomorphisms.

Definition 4.2.1. A *homomorphism (good homomorphism)* between two multiplicative hyperrings $(R, +, \circ)$ and $(R', +', \circ')$ is a map $f : R \longrightarrow R'$ such that for all x, y of R, we have $f(x + y) = f(x) +' f(y)$ and $f(x \circ y) \subseteq f(x) \circ' f(y)$ $(f(x \circ y) = f(x) \circ' f(y)$ respectively).

Denote by $Hom((R, +, \circ), (R', +', \circ'))$ the set of all multiplicative hyperring homomorphisms from $(R, +, \circ)$ to $(R', +', \circ')$.

Moreover, denote $Hom((R, +, \circ), (R, +, \circ))$ by $Hom(R, +, \circ)$.

Multiplicative hyperrings that we shall analyze in this paragraph, are defined as follows (see Example 4.1.2(1)). Let $(R, +, \cdot)$ be a ring and I be an ideal of R. For all x, y of R, we consider $x \circ y = xy + I$. Then $(R, +, \circ)$ is a strongly distributive multiplicative hyperring. For convenience, we shall denote this multiplicative hyperring by $(R, +, I)$. For $I = R$, we obtain $x \circ y = R$ for all x, y of R and hence for any $f : R \longrightarrow R$ and x, y of R, we have $f(x \circ y) = f(R) \subseteq R = f(x)f(y) + R = f(x) \circ f(y)$. From here, we obtain the following result, that we shall use later.

Theorem 4.2.2. *If $(R, +, \cdot)$ is a ring, then $Hom(R, +, R) = Hom(R, +)$.*

In what follows, we characterize the homomorphisms of the multiplicative hyperrings $(\mathbb{Z}, +, p\mathbb{Z})$ and $(\mathbb{Z}_n, +, p\mathbb{Z}_n)$, where p is a prime number and n is a positive integer. These results were obtained by C. Namnak, N. Triphop and Y. Kemprasit [94].

First, for $a \in \mathbb{Z}$, consider the maps $g_a : \mathbb{Z} \longrightarrow \mathbb{Z}$ and $h_{\overline{a}} : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$, defined by $g_a(x) = ax$ and $h_{\overline{a}}(\overline{x}) = \overline{ax}$ for all $x \in \mathbb{Z}$.

Theorem 4.2.3. *The map $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ is a homomorphism of the multiplicative hyperring $(\mathbb{Z}, +, p\mathbb{Z})$ if and only if there exists an element $a \in \mathbb{Z}$, such that $f = g_a$ and either $p|a$ or $p|(a - 1)$.*

Proof. Suppose that f is a homomorphism of $(\mathbb{Z}, +, p\mathbb{Z})$. Since $Hom(\mathbb{Z}, +) = \{g_a \mid a \in \mathbb{Z}\}$ and $Hom(\mathbb{Z}, +, p\mathbb{Z}) \subseteq Hom(\mathbb{Z}, +)$, it follows that $f = g_a$, for some $a \in \mathbb{Z}$. From $f(1 \circ 1) \subseteq f(1) \circ f(1)$, we obtain

$$a + ap\mathbb{Z} = a(1 \cdot 1 + p\mathbb{Z}) = g_a(1 \circ 1) \subseteq g_a(1) \circ g_a(1) = a \circ a = a^2 + p\mathbb{Z}.$$

This implies that $a - a^2 + ap\mathbb{Z} \subseteq p\mathbb{Z}$, whence $a - a^2 \in p\mathbb{Z}$. This means that $p|(a - a^2)$, so either $p|a$ or $p|(a-1)$.

Conversely, suppose that $f = g_a$, for some $a \in \mathbb{Z}$ and $p|a$ or $p|(a-1)$. Then $p|(a - a^2)$. Let $t \in \mathbb{Z}$ be such that $a^2 - a = pt$. On the other hand, for all x, y in $\mathbb{Z}$, we have

$$\begin{aligned} f(x \circ y) &= g_a(x \circ y) = axy + ap\mathbb{Z} = (a^2 - pt)xy + ap\mathbb{Z} \\ &\subseteq a^2xy + p\mathbb{Z} = g_a(x) \circ g_a(y) = f(x) \circ f(y). \end{aligned}$$

which means that f is a homomorphism of $(\mathbb{Z}, +, p\mathbb{Z})$. ■

From the proof of the above theorem, we see that the converse holds if p is any nonzero integer, which is not necessarily prime.

Corollary 4.2.4. *The following statements are true:*

(1) $Hom(\mathbb{Z}, +, p\mathbb{Z}) = \{g_a \mid a \in p\mathbb{Z} \cup (p\mathbb{Z} + 1)\}$.
In particular, $Hom(\mathbb{Z}, +, 2\mathbb{Z}) = Hom(\mathbb{Z}, +)$.

(2) $|Hom(\mathbb{Z}, +, p\mathbb{Z})| = \chi_0$.

Proof. We obtain (1) from the above theorem and the fact that for $a \in \mathbb{Z}$, we have $p|a$ or $p|(a-1)$ if and only if $a \in p\mathbb{Z} \cup (p\mathbb{Z}+1)$. Since $\mathbb{Z} = 2\mathbb{Z} \cup (2\mathbb{Z}+1)$, it follows that $Hom(\mathbb{Z}, +, 2\mathbb{Z}) = \{g_a \mid a \in \mathbb{Z}\} = Hom(\mathbb{Z}, +)$.

(2) For all distinct elements a, b of $\mathbb{Z}$, we have $g_a \neq g_b$, whence $|p\mathbb{Z} \cup (p\mathbb{Z} + 1)| = \chi_0$, so we have $|Hom(\mathbb{Z}, +, p\mathbb{Z})| = \chi_0$. ■

Theorem 4.2.5. *Let* $f : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ *. Then the following statements hold:*

(1) *If* $p|n$, *then* f *is a homomorphism of the multiplicative hyperring* $(\mathbb{Z}_n, +, p\mathbb{Z}_n)$ *if and only if there exists an element* $a \in \mathbb{Z}$, *such that* $f = h_{\overline{a}}$ *and either* $p|a$ *or* $p|(a-1)$.

(2) *If* p *does not divide* n, *then* f *is a homomorphism of* $(\mathbb{Z}_n, +, p\mathbb{Z}_n)$ *if and only if* $f = h_{\overline{a}}$ *for all* $a \in \mathbb{Z}$.

Proof. (1) Suppose that f is a homomorphism of $(\mathbb{Z}_n, +, p\mathbb{Z}_n)$. Since

$$Hom(\mathbb{Z}_n, +, p\mathbb{Z}_n) \subseteq Hom(\mathbb{Z}_n, +),$$

we have $f = h_{\overline{a}}$ for some $a \in \mathbb{Z}$. From $f(\overline{1} \circ \overline{1}) \subseteq f(\overline{1}) \circ f(\overline{1})$, it follows that

$$\overline{a} + ap\mathbb{Z} = \overline{a}(\overline{1} \cdot \overline{1} + p\mathbb{Z}_n) = h_{\overline{a}}(\overline{1} \circ \overline{1}) \subseteq h_{\overline{a}}(\overline{1}) \circ h_{\overline{a}}(\overline{1}) = \overline{a}^2 + p\mathbb{Z}_n.$$

Hence $\overline{a} - \overline{a}^2 + ap\mathbb{Z}_n \subseteq p\mathbb{Z}_n$ and thus $\overline{a} - \overline{a}^2 = p\overline{s}$ for some $s \in \mathbb{Z}$. From here, we obtain $a^2 - a + ps = nt$ for some $t \in \mathbb{Z}$. Since $p|n$, it follows $p|(a^2 - a)$. Moreover, p is prime, so we obtain $p|a$ or $p|(a-1)$. Conversely, suppose that $f = h_{\overline{a}}$ for some $a \in \mathbb{Z}$ and either $p|a$ or $p|(a-1)$. Then $p|a(a-1)$. Let $t \in \mathbb{Z}$ be such that $a^2 - a = pt$. It follows that $\overline{a}^2 - \overline{a} = p\overline{t}$. If $x, y \in \mathbb{Z}$, then $h_{\overline{a}}(\overline{x} \circ \overline{y}) = h_{\overline{a}}(\overline{xy} + p\mathbb{Z}_n) = \overline{a}(\overline{xy} + p\mathbb{Z}_n) = \overline{axy} + ap\mathbb{Z}_n$ and $h_{\overline{a}}(\overline{x}) \circ h_{\overline{a}}(\overline{y}) = \overline{ax} \circ \overline{ay} = \overline{a^2xy} + p\mathbb{Z}_n$. We have

$$\begin{aligned} f(\overline{x} \circ \overline{y}) &= h_{\overline{a}}(\overline{x} \circ \overline{y}) \\ &= \overline{axy} + ap\mathbb{Z}_n \\ &\subseteq \overline{a^2xy} + p\mathbb{Z}_n + ap\mathbb{Z}_n \\ &= \overline{a^2xy} + p\mathbb{Z}_n \\ &= h_{\overline{a}}(\overline{x}) \circ h_{\overline{a}}(\overline{y}) \\ &= f(\overline{x}) \circ f(\overline{y}). \end{aligned}$$

Hence f is a homomorphism of $(\mathbb{Z}_n, +, p\mathbb{Z}_n)$.

(2) If p does not divide n, then $p\mathbb{Z}_n = \mathbb{Z}_n$, so

$$Hom(\mathbb{Z}_n, +, p\mathbb{Z}_n) = Hom(\mathbb{Z}_n, +, \mathbb{Z}_n) = Hom(\mathbb{Z}_n, +) = \{h_{\overline{a}} \mid a \in \mathbb{Z}\}. \blacksquare$$

The converse of (1) of the above theorem holds if p is any nonzero integer.

Corollary 4.2.6. *The following statements hold:*

(1) $Hom(\mathbb{Z}_n, +, p\mathbb{Z}_n) = \{h_{\overline{a}} \mid a \in p\mathbb{Z} \cup (p\mathbb{Z}+1)\}$ *if* $p|n$ *and* $Hom(\mathbb{Z}_n, +, p\mathbb{Z}_n) = \{h_{\overline{a}} \mid a \in \mathbb{Z}\} = Hom(\mathbb{Z}_n, +)$ *if* p *does not divide* n;

(2) $|Hom(\mathbb{Z}_n, +, p\mathbb{Z}_n)| = 2n/p$ *if* $p|n$ *and* $|Hom(\mathbb{Z}_n, +, p\mathbb{Z}_n)| = n$ *if* p *does not divide* n.

Proof. This follows from the above theorem and the fact that for $a \in \mathbb{Z}$, we have $p|a$ or $p|(a-1)$ if and only if $a \in p\mathbb{Z} \cup (p\mathbb{Z}+1)$.

(2) Suppose that $p|n$. Then

$$\begin{aligned} Hom(\mathbb{Z}_n, +, p\mathbb{Z}_n) &= \{h_{\overline{a}} \mid a \in p\mathbb{Z} \cup (p\mathbb{Z}+1)\} \\ &= \{h_{\overline{px}} \mid x \in \mathbb{Z}\} \cup \{h_{\overline{1+px}} \mid x \in \mathbb{Z}\} \end{aligned}$$

Since $h_{\overline{a}} \neq h_{\overline{b}}$ for $\overline{a} \neq \overline{b}$ in $\mathbb{Z}_n$, we have $\{\overline{px} \mid x \in \mathbb{Z}\} = p\mathbb{Z}_n$ and $\{\overline{1+px} \mid x \in \mathbb{Z}\} = p\mathbb{Z}_n = \overline{1}+p\mathbb{Z}_n$, hence $|Hom(\mathbb{Z}_n, +, p\mathbb{Z}_n)| = |p\mathbb{Z}_n|+|\overline{1}+p\mathbb{Z}_n| = 2n/p$. If p does not divide n, then the assertion follows from (1). ∎

4.3 Regular equivalences on a multiplicative hyperring. Quotient multiplicative hyperrings.

On this topic, R. Rota, D.M. Olson and V.K. Ward have been worked. We present here some of their results.

Regular equivalences on a multiplicative hyperring.

Definition 4.3.1. Let $(R, +, \cdot)$ be a multiplicative hyperring and ρ be an equivalence relation on R. We say that ρ is *regular* (or *congruence*) if the following implication holds:

$$a\rho b,\ c\rho d \implies (a+c)\rho(b+d) \text{ and } a\cdot c\overline{\rho} b\cdot d,$$

and ρ is called *strongly regular* if the following implication holds:

$$a\rho b,\ c\rho d \implies (a+c)\rho(b+d) \text{ and } a\cdot c\overline{\overline{\rho}} b\cdot d \quad \text{(see 2.5)}.$$

According to Theorems 2.5.2 and 2.5.5, we have that

Theorem 4.3.2.

(1) *The equivalence relation ρ is regular if and only if R/ρ is a multiplicative hyperring.*

(2) *The equivalence relation ρ is strongly regular if and only if R/ρ is a ring.*

Now, let I be a hyperideal of a multiplicative hyperring $(R, +, \cdot)$. We define the following relation on R:

$$a\rho_I b \iff b - a \in I.$$

Theorem 4.3.3. *The equivalence relation* ρ_I *is regular on* $(R, +, \cdot)$.

Proof. First, since $(R, +)$ is an abelian group, it follows that ρ_I is an equivalence relation which preserves the addition in R. Now, consider $a\rho_I b$ and $c\rho_I d$. Then $b - a,\ d - c \in I$. Hence there exist h and k in I, such that $b = a + h$ and $d = c + k$. We have

$$b \cdot d = (a + h) \cdot (c + k) \subseteq a \cdot c + a \cdot k + h \cdot c + h \cdot k \subseteq a \cdot c + I.$$

For any $x \in b \cdot d$, there exists $y \in a \cdot c$ and $t \in I$, such that $x = y + t$, which means that $x\rho_I y$. Similarly, by writing $a = b - h$ and $c = d - k$, it follows that for any $y \in a \cdot c$ there exists $x \in b \cdot d$, such that $y\rho_I x$. Moreover, the equivalence class of 0 is I. ■

Corollary 4.3.4. *There exists an one-to-one correspondence between the set of regular equivalences, for which the equivalence class of* 0 *is an hyperideal and the set of hyperideals of a multiplicative hyperring.*

Clearly, to any hyperideal I we associate the regular equivalence ρ_I. Conversely, to any regular equivalence, for which the equivalence class of 0 is a hyperideal, we associate the equivalence class of 0. The above two correspondences are inverses each other.

Quotient multiplicative hyperrings

Let $(R, +, \cdot)$ be a multiplicative hyperring and I be a hyperideal of it. We consider the usual addition of cosets and the multiplication defined as:

$$(a + I) * (b + I) = \{c + I \mid c \in a \cdot b\},$$

on the set $R/I = \{a + I \mid a \in R\}$ of all cosets of I. Then $(R/I, +, *)$ is a multiplicative hyperring and it is strongly distributive if R is so.

Theorem 4.3.5. *If* I *is an hyperideal of a multiplicative hyperring* $(R, +, \cdot)$, *then for any element* $a + I \in R/I$, *we have* $|(a + I) * (0 + I)| = 1$.

Proof. If $a + I$ is an element of R/I, then we have $(a+I)*(0+I) = \{x + I \mid x \in a \cdot 0\}$. Since $0 \in I$, it follows that $a \cdot 0 \subseteq I$. Hence $(a + I) * (0 + I)$ contains only the zero element of R/I. ■

Corollary 4.3.6. *If* $(R, +, \cdot)$ *is a strongly distributive multiplicative hyperring and* I *is a hyperideal of* R*, then* R/I *is a ring.*

Proof. $(R/I, +, *)$ is a strongly distributive multiplicative hyperring, for which there exists a hyperproduct of two elements containing only an element, hence it is a ring. ∎

Example 4.3.7. If $(R, +, \cdot)$ is a ring, H is an ideal of it and for all a, b of R, we define $a * b = a \cdot b + H$, then $(R, +, *)$ is a strongly distributive multiplicative hyperring. If I is an ideal of the ring R, such that $H \subseteq I$, then the quotient R/I of $(R, +, *)$ is a ring.

The fundamental isomorphism theorem

Let $f : R_1 \longrightarrow R_2$ be a good homomorphism of multiplicative hyperrings. The kernel of f is the inverse image of $< 0 >$, the hyperideal generated by the zero in R_2. It is denoted by $Kerf$. Since the inverse images of hyperideals are hyperideals, it follows that the kernel is a hyperideal. Similarly as in ring theory, we have $f(< 0 >) \subseteq < 0 >$, which means that $< 0 > \subseteq Kerf$.

Theorem 4.3.8.(Fundamental isomorphism theorem). *Let* R *and* S *be multiplicative hyperrings. If* $f : R \longrightarrow S$ *is a good epimorphism, then there exists an isomorphism* $R/Kerf \cong S/ < 0 >$.

Proof. Let $f : R \longrightarrow S$ be a good epimorphism of multiplicative hyperrings. Denote $K = Kerf$. We define $\varphi : R/K \longrightarrow S/ < 0 >$ by $\varphi(r + K) = f(r)+ < 0 >$. First, we show that φ is well-defined. Let $r + K = s + K$. Then $r - s \in K$, which implies that $f(r) - f(s) = f(r - s) \in < 0 >$. Hence $f(r)+ < 0 >= f(s)+ < 0 >$. Let us show now that φ is one to one. Suppose that $\varphi(r + K) = \varphi(s + K)$. Then $f(r)+ < 0 >= f(s)+ < 0 >$, whence $f(r-s) \in < 0 >$ and so $r-s \in K$. On the other hand, φ is onto. Indeed, if $s+ < 0 > \in S/ < 0 >$ then there exists $r \in R$, such that $f(r) = s$. We have $\varphi(r + K) = f(r)+ < 0 >= s+ < 0 >$. Finally, we check that φ is an isomorphism. Since f is a good homomorphism, it follows that $\varphi(r + K + s + K) = \varphi(r + K) + \varphi(s + K)$. Now, if $y \in \varphi((r + K)(s + K)) = \{\varphi(c + K) \mid c \in rs\}$, then $y = \varphi(c+K) = f(c)+ < 0 >$ for some $c \in rs$. We have $f(c) \in f(r)f(s)$ and so, $y = f(c)+ < 0 > \in (f(r)+ < 0 >)(f(s)+ < 0 >) = \varphi(r + K)\varphi(s + K)$.

Conversely, let $y \in \varphi(r+K)\varphi(s+K) = (f(r)+ <0>)(f(s)+ <0>)$. Then $y = c+ <0>$ for some $c \in f(r)f(s) = f(rs)$. Hence there exists $x \in rs$ such that $y = f(x)+ <0> = \varphi(x+K)$. We obtain $y = \varphi(x+K) \in \varphi((r+K)(s+K))$. Hence φ is a good homomorphism. ■

Notice that $a - b \in Kerf = K$ does not imply that $f(a) = f(b)$, as in ring theory, since the zero hyperideal may contain more than just zero. Hence the fundamental isomorphism theorem has not the same form as in ring theory. Surprisingly, the second and third isomorphism theorems have a similar form, as in ring theory.

Definition 4.3.9. Let R and S be multiplicative hyperrings. We say that R is *semi-isomorphic* to S if there exists a good epimorphism $f : R \longrightarrow S$ such that $Kerf = <0>$.

Theorem 4.3.10. *If R is a multiplicative hyperring, then R is semi-isomorphic to $R/<0>$.*

Proof. Define $f : R \longrightarrow R/<0>$ by $f(r) = r+ <0>$. Clearly, f is a good epimorphism. Since $<<0>> = \{<0>\}$, it follows that $Kerf = f^{-1}(\{<0>\})$. Let $r \in <0>$. Then $f(r) = r+ <0> = <0>$ implies that $r \in Kerf$. Conversely, if $r \in Kerf$, then $f(r) = <0>$. But $f(r) = r+ <0>$. Hence $r \in <0>$. Consequently, $Kerf = <0>$. ■

Corollary 4.3.11. *If R is a multiplicative hyperring, such that $<0> = \{0\}$ in R, then R is isomorphic to $R/<0>$.*

Proof. We show that $f : R \longrightarrow R/<0>$ defined by $f(r) = r+ <0>$, is one to one. Suppose $f(r) = f(s)$. Then $r+ <0> = s+ <0>$ and so $r - s \in <0> = \{0\}$. Hence $r=s$. Therefore, R is isomorphic to $R/<0>$. ■

Theorem 4.3.12. (Second isomorphism theorem). *If I and J are hyperideals in a multiplicative hyperring R, then $I/(I \cap J) \cong (I+J)/J$.*

Proof. Define $f : I \longrightarrow (I+J)/J$ by $f(r) = r+J$. For all r, s of I, we have $f(r+s) = r+s+J = r+J+s+J = f(r)+f(s)$ and $f(rs) = \{f(c) \mid c \in rs\} = \{c+J \mid c \in rs\} = (r+J)(s+J) = f(r)f(s)$. Hence f is a good homomorphism. In order to show that f is onto, choose $r+J \in (I+J)/J$. Then $r = i+j$ for some $i \in I, j \in J$. So $f(i) = i+J = r-j+J = r+J$. Since $<J> = \{J\}$ in $(I+J)/J$

$$Kerf = f^{-1}(J) = \{r \in I \mid f(r) = J\} = \{r \in I \mid r + J = J\} = I \cap J.$$

According to the fundamental isomorphism theorem, we have $I/(I \cap J) \cong R/J$, where $R = (I + J)/J$. Since $< J >= \{J\}$ in $(I + J)/J$ and according to the above corollary, it follows that $R \cong R/J$. Therefore $I/(I \cap J) \cong (I + J)/J$. ■

Theorem 4.3.13. *If I, K are hyperideals in a multiplicative hyperring R, with $K \subseteq I$, then I/K is a hyperideal in R/K.*

Proof. Let $a + K, b + K$ be elements of I/K, where a, b are in I. Then $(a + K) - (b + K) = a - b + K$, since $a - b \in I$. Let $r + K \in R/K$. We have $(r + K)(a + K) = \{c + K \mid c \in ra \subseteq I\} \subseteq I/K$. Similarly, we obtain $(a + K)(r + K) \subseteq I/K$. ■

Theorem 4.3.14. (Third isomorphism theorem). *If I and K are hyperideals in a multiplicative hyperring R with $K \subseteq I$, then $(R/K)/(I/K) \cong R/I$.*

Proof. We define the map $f : R/K \longrightarrow R/I$ by $f(r+K) = r+I$. First, we check that f is well-defined. Indeed, if $r + K = s + K$, then $r - s \in K \subseteq I$ and so $r + I = s + I$. Now, we check that f is a good homomorphism. If r, s are elements of R, then

$$f(r+K+s+K) = f(r+s+K) = r+s+I = r+I+s+I = f(r+K)+f(s+K)$$

$$f((r+K)(s+K)) = \{f(c+K) \mid c \in rs\} = (r+I)(s+I) = f(r+K)f(s+K).$$

On the other hand, f is clearly onto. The kernel of f is $\{r + K \in R/K \mid r \in I\} = I/K$. According to the fundamental isomorphism theorem and the last corollary, we obtain $(R/K)/(I/K) \cong (R/I)/\{I\} \cong R/I$. ■

4.4 Polynomials over multiplicative hyperrings

In this paragraph, we present the multiplicative hyperring of polinyomials over a multiplicative hyperring. We use the results obtained by R. Procesi Ciampi and R. Rota (see [106]).

Let $(R, +, \cdot)$ be a commutative multiplicative hyperring, such that for all a in R, we have $a \cdot 0 = \{0\}$. Moreover, we consider an element x which does not belong to R. By a polynomial in x over R we mean any expression of the form:

$$f(x) = a_0x^0 + a_1x^1 + a_2x^2 + ... = \sum_{k\in\mathbb{N}}^{*} a_kx^k,$$

where for all k in $\mathbb{N}$, we have $a_K \in R$, " + " is a connective and the symbol $*$ means that only a finite number of a_k are nonzero. The degree of $f(x)$ is $max\{k \in \mathbb{N} \mid a_k \neq 0\}$. Denote the set of all polynomials in x over R by $R[x]$. We define the addition in $R[x]$, as follows:

$$\sum_{k\in\mathbb{N}}^{*} a_kx^k + \sum_{k\in\mathbb{N}}^{*} b_kx^k = \sum_{k\in\mathbb{N}}^{*}(a_k + b_k)x^k.$$

Notice that the symbol + was used for simplicity in three different meanings: the sum in R, the connective in $R[x]$ and the sum in $R[x]$. We obtain that $(R[x], +)$ is an abelian group, for which the zero element is

$$0x^0 + 0x^1 + 0x^2 + ...$$

Now, we define a hyperproduct in $R[x]$, that generalizes the usual polynomials product in the following way:

for all $f(x) = \sum_{k=0}^{n} a_kx^k$ and $g(x) = \sum_{k=0}^{m} b_kx^k$ elements of $R[x]$, we consider

$$f(x) * g(x) = \left\{ \sum_{k=0}^{n+m} c_kx^k \mid c_k \in \sum_{i+j=k} a_ib_j \right\}.$$

We obtain the following

Theorem 4.4.1. *The hyperstructure* $(R[x], +, *)$ *is a commutative multiplicative hyperring.*

Proof. First, notice that the hyperoperation $*$ is commutative. Then, we check that $*$ is also associative.

Let $f(x) = \sum_{k=0}^{n} a_k x^k$, $g(x) = \sum_{k=0}^{m} b_k x^k$ and $h(x) = \sum_{k=0}^{r} d_k x^k$ be elements of $R[x]$. We have

$$\begin{aligned}(f(x) * g(x)) * h(x) &= \left\{ \sum_{k=0}^{n+m} c_k x^k \mid c_k \in \sum_{i+j=k} a_i b_j \right\} * h(x) \\ &= \left\{ \sum_{k=0}^{n+m+r} t_k x^k \mid t_k \in \sum_{u+v=k} c_u d_v,\ c_u \in \sum_{i+j=u} a_i b_j \right\}\end{aligned}$$

and

$$\begin{aligned}f(x) * (g(x) * h(x)) &= f(x) * \left\{ \sum_{k=0}^{m+r} c_k x^k \mid c_k \in \sum_{i+j=k} b_i d_j \right\} \\ &= \left\{ \sum_{k=0}^{n+m+r} s_k x^k \mid s_k \in \sum_{u+v=k} a_u e_v,\ e_v \in \sum_{i+j=v} b_i d_j \right\}.\end{aligned}$$

Notice that

$$t_k \in \sum_{u+v=k} c_u d_v \subseteq \sum_{u+v=k} \sum_{i+j=u} a_i b_j d_v$$

$$s_k \in \sum_{u+v=k} a_u e_v \subseteq \sum_{u+v=k} \sum_{i+j=v} a_u b_i d_j.$$

From $\sum_{u+v=k} \sum_{i+j=u} a_i b_j d_v = \sum_{u+v=k} \sum_{i+j=v} a_u b_i d_j$, it follows the associative property.

On the other hand, we have

$$\begin{aligned}(f(x) + g(x)) * h(x) &= \sum_{k=0}^{max(n,m)} (a_k + b_k) x^k * h(x) \\ &= \left\{ \sum_{k=0}^{max(n,m)+r} t_k x^k \mid t_k \in \sum_{u+v=k} (a_u + b_u) d_v \right\}\end{aligned}$$

and

$$\begin{aligned} f(x)*h(x) &+ g(x)*h(x) \\ &= \left\{\sum_{k=0}^{n+r} t_k x^k \mid t_k \in \sum_{u+v=k} a_u d_v\right\} + \left\{\sum_{k=0}^{m+r} s_k x^k \mid s_k \in \sum_{u+v=k} b_u d_v\right\} \\ &= \left\{\sum_{k=0}^{max(n,m)+r} (t_k + s_k)x^k \mid t_k + s_k \in \sum_{u+v=k} a_u d_v + \sum_{u+v=k} b_u d_v\right\}. \end{aligned}$$

From $\sum\limits_{u+v=k} (a_u + b_u)d_v \subseteq \sum\limits_{u+v=k} a_u d_v + \sum\limits_{u+v=k} b_u d_v$, it follows the distributive property. Finally, we have

$$\begin{aligned} f(x)*(-g(x)) &= \left\{\sum_{k=0}^{n+m} t_k x^k \mid t_k \in \sum_{u+v=k} a_u(-b_v)\right\} \\ &= \left\{\sum_{k=0}^{n+m} t_k x^k \mid t_k \in -\sum_{u+v=k} a_u b_v\right\} \\ &= -(f(x)*g(x)). \end{aligned}$$

Similarly we have $(-f(x))*g(x) = -(f(x)*g(x))$, which completes the proof. ■

If u is a weak identity for $(R,+,\cdot)$, then u is a weak identity for $(R[x],+,*)$, too.

We can define another hyperproduct in $R[x]$, in order to obtain a multiplicative hyperring. For all $f(x) = \sum\limits_{k=0}^{n} a_k x^k$ and $g(x) = \sum\limits_{k=0}^{m} b_k x^k$ elements of $R[x]$, we define $f(x) \circ g(x) = \left\{\sum\limits_{k=0}^{min(n,m)} c_k x^k \mid c_k \in a_k b_k\right\}$. In this definition, the product is made component by component, hence the properties of $\circ$ are obtained as a consequence of the analogue in $(R,+,\cdot)$. Thus, the following theorem holds:

Theorem 4.4.2. *The hyperstructure $(R[X], +, \circ)$ is a commutative multiplicative hyperring.*

If u is a weak identity in $(R, +, \circ)$, then we notice that u is not a weak identiy in $(R[x], +, \circ)$ anymore. However, we can construct the polynomial $u_n(x) = \sum_{k=0}^{n} ux^k$, where $n \in \mathbb{N}$, such that we have $f(x) \in u_n(x) \circ f(x)$, for all $f(x) = \sum_{k=0}^{m} a_k x^k$, where $m \leq n$.

The hypothesis $a \cdot 0 = \{0\}$ for all a, is basic in the construction of a polynomial multiplicative hyperring. First, we give two examples of multiplicative hyperrings, which satisfy this condition.

Example 4.4.3.

(1) Let $(R, +, \cdot)$ be a unitary commutative ring and we define the following hyperproduct on R:

$a \circ b = \{ab, kab\}$ where $k \in \mathbb{Z}$, $char R = 0$ or $char R$ does not divide $(k-1)$.

Then $(R, +, \circ)$ is a commutative multiplicative hyperring.

(2) Let $(R, +, \cdot)$ be a unitary commutative ring, S be one of its unitary subrings and $aS^*b = abS$ for all a, b of R. Then $(R, +, S^*)$ is a commutative multiplicative hyperring, such that for all a of R, we have $aS^*0 = 0S^*a = \{0\}$. Moreover, for all a, b of R, $0 \in aS^*b$ and we have $aS^*b = \{0\}$ if and only if $ab = 0$.

In what follows, starting from a particular multiplicative hyperring, we construct a new multiplicative hyperring, satisfying the property: for all a, we have $a \cdot 0 = \{0\}$.

Theorem 4.4.4. *If $(R, +, \circ)$ is a strongly distributive commutative multiplicative hyperring and for all $a, b \in R \setminus \{0\}$, we define $a \otimes b = a \circ b$ and for all $a \in R$, we define $a \otimes 0 = \{0\}$, then $(R, +, \otimes)$ is a multiplicative hyperring.*

Proof. First, we check the associativity of the hyperoperation $\otimes$.

If $a, b, c \in R \setminus \{0\}$, then we consider the following two situations:

$$\text{(i) } 0 \notin a \circ b; \quad \text{(ii) } 0 \in a \circ b.$$

(i)

$$\begin{aligned}(a \otimes b) \otimes c &= (a \circ b) \otimes c = \bigcup_{x \in a \circ b} x \otimes c = \bigcup_{x \in a \circ b} x \circ c \\ &= (a \circ b) \circ c = a \circ (b \circ c) = \bigcup_{y \in b \circ c} a \circ y\end{aligned}$$

and

$$a \otimes (b \otimes c) = a \otimes (b \circ c) = \bigcup_{y \in b \circ c} a \otimes y.$$

At this point, we have two possibilities: $0 \notin b \circ c$ and $0 \in b \circ c$.

If $0 \notin b \circ c$, then for every $y \in b \circ c$, $y \neq 0$ we have $a \otimes y = a \circ y$ and so

$$\bigcup_{y \in b \circ c} a \circ y = \bigcup_{y \in b \circ c} a \otimes y.$$

If $0 \in b \circ c$, then $b \circ c = 0 \circ 0$, and so

$$\begin{aligned}\bigcup_{y \in b \circ c} a \otimes y &= \bigcup_{y \in 0 \circ 0} a \otimes y = \bigcup_{y \in 0 \circ 0, y \neq 0} a \otimes y \cup a \otimes 0 = \bigcup_{y \in 0 \circ 0, y \neq 0} a \circ y \cup \{0\} \\ &= 0 \circ 0 = b \circ c = a \circ (b \circ c).\end{aligned}$$

Hence $(a \otimes b) \otimes c = a \otimes (b \otimes c)$.

Similarly, we can prove the associativity in the case (ii).

We check now the distributive property. We have the following seven cases:

(1) $a, b, c \neq 0,\ a \neq -b$;

(2) $a, b, c \neq 0,\ a = -b$;

(3) $a = 0,\ b, c \neq 0$;

(4) $a + b \neq 0,\ c = 0$;

(5) $a + b = 0,\ c = 0$;

(6) $a = 0$ or $b = 0, c = 0$;

(7) $a = b = c = 0$.

We have:

(1) $(a + b) \otimes c = (a + b) \circ c = a \circ c + b \circ c = a \otimes c + b \otimes c$.

(2) $(a+b) \otimes c = 0 \otimes c = \{0\} \subseteq -b \circ c + b \circ c = a \circ c + b \circ c = a \otimes c + b \otimes c$.

(3) $(0 + b) \otimes c = b \otimes c = b \circ c = b \circ c + \{0\} = 0 \otimes c + b \otimes c$.

(4) $(a + b) \otimes 0 = \{0\} = a \otimes 0 + b \otimes 0$.

The cases (5), (6) and (7) are immediate. Finally, if $a, b \neq 0$ then

$$-(a \otimes b) = -(a \circ b) = (-a) \circ b = a \circ (-b) = (-a) \otimes b = a \otimes (-b).$$

If $a = 0$, then $-(0 \otimes b) = \{0\} = 0 \otimes (-b)$. Similarly, for $b = 0$, we obtain $-(a \otimes 0) = \{0\} = (-a) \otimes 0$, and this completes the proof. ∎

Example 4.4.5. We can start from a well known example of strongly distributive multiplicative hyperring and apply the above theorem. Let $(R, +, \cdot)$ be a ring, I be one of its ideals and $(R, +, \circ)$ be the strongly distributive multiplicative hyperring, for which $a \circ b = ab + I$. Thus $(R, +, \otimes)$ is a multiplicative hyperring, for which we have $a \otimes b = a \circ b$ if $a, b \neq 0$ and $a \otimes b = \{0\}$ if $a = 0$ or $b = 0$.

The following theorem provides us a new situation when we obtain a multiplicative hyperring $(R, +, \otimes)$, such that $a \otimes 0 = \{0\}$ for all a of R.

Theorem 4.4.6. *Let $(R, +, \circ)$ be a multiplicative commutative hyperring such that for all $a, b \in R \setminus \{0\}$, $0 \notin a \circ b$. Then $(R, +, \otimes)$ is a multiplicative commutative hyperring, where for all $a, b \in R \setminus \{0\}$, $a \otimes b = a \circ b$ and for all $a \in R$, $a \otimes 0 = \{0\}$.*

Proof. In order to prove that $\otimes$ is associative, we need to consider two different cases: $a, b, c \in R \setminus \{0\}$ and ($a = 0$ or $b = 0$ or $c = 0$). In the first case, we have:

$$\begin{aligned}(a \otimes b) \otimes c &= (a \circ b) \otimes c = \bigcup_{x \in a \circ b} x \otimes c = \bigcup_{x \in a \circ b} x \circ c \\ &(a \circ b) \circ c = a \circ (b \circ c) = \bigcup_{y \in b \circ c} a \circ y \\ &\bigcup_{y \in b \otimes c} a \otimes y = a \otimes (b \otimes c).\end{aligned}$$

We have used that $0 \notin a \circ b$ and $0 \notin b \circ c$. In the second case, we can suppose that $a = 0$. We have

$$0 \otimes (b \otimes c) = \bigcup_{y \in b \otimes c} 0 \otimes y = \{0\}$$

and

$$(0 \otimes b) \otimes c = \{0\} \otimes c = \{0\}.$$

Hence, the associativity holds.

For the distributivity property, we consider all the above seven different cases. We check only (1) and (2).

(1) $(a + b) \otimes c = (a + b) \circ c \subseteq a \circ c + b \circ c = a \otimes c + b \otimes c$.

(2) $(a + b) \otimes c = 0 \otimes c = \{0\} \subseteq a \otimes c + b \otimes c$, since, $a \otimes c = a \circ c = (-b) \circ c = -(b \circ c)$ and $b \otimes c = b \circ c$.

Finally, for all $a, b \in R \setminus \{0\}$ we have

$$a \otimes (-b) = a \circ (-b) = -(a \circ b) = -(a \otimes b), \; 0 \otimes (-b) = \{0\} = -(0 \otimes b) \quad \blacksquare$$

4.5 Hyperoperations that induce strongly distributive multiplicative hyperring structures on an abelian group

The main result of this paragraph is a complete description of hyperoperations that induce a strongly distributive multiplicative hyperring structure on an abelian group. The following results were obtained by S. Feigelstock [48, 49]. In these papers, he analyzed also the hyperoperations that induce strongly left distributive multiplicative hypernear-ring structures on a group.

Denote by $SDMH$ the class of all strongly distributive multiplicative hyperrings.

Lemma 4.5.1. *If* $(R, +, \cdot) \in SDMH$, *then for all* $a \in R$, $a \cdot 0$ *is a subgroup of* $(R, +)$.

Proof. Since $a \cdot 0 = a \cdot (0 - 0) = a \cdot 0 - a \cdot 0$ it follows that $a \cdot 0$ is such that for all x, y of $a \cdot 0$, we have $x - y \in a \cdot 0$, which means that $a \cdot 0$ is a subgroup of $(R, +)$. ■

Lemma 4.5.2. *If $(R, +, \cdot) \in SDMH$, then for all $a, b \in R$, $a \cdot b$ is a coset of $a \cdot 0$.*

Proof. Let $c \in a \cdot b$. For any $x \in a \cdot b$, we have $x - c \in a \cdot b - a \cdot b = a \cdot 0$, which means that $x + a \cdot 0 = c + a \cdot 0$. Therefore $a \cdot b = a \cdot (b + 0) = a \cdot b + a \cdot 0 = \bigcup\limits_{x \in a \cdot b} x + a \cdot 0 = c + a \cdot 0$. Similarly, it can be shown that for all $a, b \in R$, $a \cdot b$ is a coset of $0 \cdot b$. ■

From this lemma we obtain the following corollary:

Corollary 4.5.3. *If $(R, +, \cdot) \in SDMH$, then for all $a, b, c, d \in R$, the sets $a \cdot b$ and $c \cdot d$ have the same cardinality.*

Corollary 4.5.4. *If $(R, +, \cdot) \in SDMH$, then for all $a, b \in R$, the set $a \cdot b$ is a coset of $0 \cdot 0$.*

Proof. $a \cdot b$ is a coset of $a \cdot 0$, and $a \cdot 0$ is a coset of $0 \cdot 0$. ■

Now, we can establish the main theorem of this paragraph.

Theorem 4.5.5. *Let $(R, +)$ be an abelian group, let S be a subgroup of R, and let $f : R \longrightarrow Hom(R, R/S)$ be a homomorphism. If we define $a \cdot b = f(a)(b)$ for all $a, b \in R$, then the structure $(R, +, \cdot) \in SDMH$. Conversely, every $(R, +, \cdot) \in SDMH$ is obtained in this manner.*

Proof. For all $a, b, c \in R$, we have

$$\begin{aligned} &a(b + c) = f(a)(b + c) = f(a)(b) + f(a)(c) = ab + ac, \\ &(b + c)a = f(b + c)(a) = (f(b) + f(c))(a) = ba + ca \text{ and} \\ &a(-b) = f(a)(-b) = -f(a)(b) = -(ab). \end{aligned}$$

Therefore, $(R, +, \cdot) \in SDMH$. Conversely, let $(R, +, \cdot) \in SDMH$. For all $a \in R$, consider $S = 0 \cdot 0$ and for all $a, b \in R$, define $f(a)(b) = a \cdot b$. Then $f(a)(b) \in R/S$, according to Corollary 4.5.4. Moreover, all $a \in R$, the map $f(a)$ is a homomorphism. ■

For an abelian group $(R,+)$, denote by $Mult_{SDMH}(R)$ the set of all hyperoperations "$\cdot$" for which $(R,+,\cdot) \in SDMH$. Then the above theorem can be restated as follows:

Theorem 4.5.6. *If $(R,+)$ is an abelian group, then there is an one-to-one correspondence between $Mult_{SDMH}(R)$ and $\bigcup\limits_{S \leq R} Hom(R, Hom(R, R/S))$.*

If we denote by $Mult_r(R)$ the set of all hyperoperations "$\cdot$" for which $(R,+,\cdot)$ is a ring, then we obtain the following result:

Corollary 4.5.7. *If $(R,+)$ is an abelian group, then there is an one-to-one correspondence between $Mult_r(R)$ and $Hom(R, End(R))$.*

Chapter 5

General hyperrings

5.1 Feeble hyperrings

In this section we introduce a new type of hyperrings. Both addition and multiplication are hyperoperations, that satisfy a set of conditions. This definition was introduced by Corsini [20] and he used it in order to define and study feeble hypermodules.

First, recall that a regular hypergroup $(H, \circ)$ is a hypergroup which has at least an identity and any element of H has at least an inverse. In other words, there exists $e \in H$, such that for all $x \in H$, we have $x \in x \circ e \cap e \circ x$ and there exists $x' \in H$ such that $e \in x \circ x' \cap x' \circ x$.

In what follows, ω_H or simply ω denotes the heart of the hypergroup $(H, \circ)$.

Definition 5.1.1. A hyperstructure $(R, +, \cdot)$ is called a *feeble hyperring* (*F-hyperring*) if $(R, +)$ is a regular hypergroup and $\cdot$ is a hyperoperation on R, such that the following conditions hold for all $a, b, c \in R$:

(1) $a(b + c) \subseteq ab + ac + \omega_R$,

(2) $(a + b)c \subseteq ac + bc + \omega_R$,

(3) $(ab)c \subseteq a(bc) + \omega_R$,

(4) $\exists (ab)' \subseteq R,\ (ab)' \neq \emptyset$, such that $ab + (ab)' \subseteq \omega_R$.

We denote the heart of the hypergroup $(R,+)$ by ω_R.

Remark 5.1.2. There are hyperstructures which satisfy the above conditions (1), (2), (3), but do not satisfy (4).

Indeed, we can consider the associativity hyperring of the Cayley-Dickson algebra, defined by the following hyperoperations: $x \oplus y = A(x+y)$, $x \circ y = A(xy)$, where A is the associative closure.

Lemma 5.1.3. *If $a, b \in R$ and $x, y \in ab$, then $x \in y + \omega_R$.*

Proof. First, if y' is an inverse of y, then for any $(ab)'$ we have $y' \in (ab)' + \omega_R$. Indeed, from $ab + (ab)' \subseteq \omega_R$, it follows that there exists $y^* \in (ab)'$ such that $y + y^* \subseteq \omega_R$. Then for all inverse y' of y, we have

$$y' + y + y^* \subseteq y' + \omega_R \Rightarrow y^* + \omega_R = y' + \omega_R \Rightarrow y' \in y^* + \omega_R \subseteq (ab)' + \omega_R.$$

Hence

$$x + y' \subseteq ab + (ab)' + \omega_R \Rightarrow x + y' + y \subseteq \omega_R + y \Rightarrow x \in y + \omega_R.$$

On the other hand, if $x, y \in ab$, then $x + \omega_R = y + \omega_R$. ■

Lemma 5.1.4. *If $a, b \in R$ and b' is an inverse of b, then for all $(ab)'$, we have $ab' + \omega_R = (ab)' + \omega_R$.*

Proof. If 0 is an identity of $(R,+)$, then we have $a0 \subseteq \omega_R$. Indeed, if $q \in R$, then $aq \subseteq a0 + aq + \omega_R$ whence $\omega_R \supseteq aq + (aq)' \subseteq a0 + aq + (aq)' + \omega_R \subseteq a0 + \omega_R$. Then there exists $x \in a0$, $x \in \omega_R$. According to Lemma 5.1.3, we obtain $a0 \subseteq \omega_R$. On the other hand, we have $a0 \subseteq ab' + ab + \omega_R$, whence $\omega_R + ab' + ab + (ab)' \supseteq \omega_R + (ab)'$, so $ab' + \omega_R \supseteq (ab)'$. If $x \in (ab)'$, then there exist $y \in ab', u \in \omega_R$, such that $x \in y + u$. We obtain $y \in x + \omega_R \subseteq (ab)' + \omega_R$. According to Lemma 5.1.3, we have $ab' \subseteq (ab)' + \omega_R$, so we obtain $ab' + \omega_R = (ab)' + \omega_R$. ■

Example 5.1.5. Let $(H,+)$ be a commutative hypergroup. We can endow the set $P^*(H)$ of nonempty subsets of H with a hypergroup structure, as follows: $A + B = \{C : C \subseteq A + B\}$.

We denote the set of hypergroup homomorphisms $f : H \longrightarrow P^*(H)$ by $F(H)$. Now we endow $F(H)$ with a feeble hyperring structure, by defining:

$$f + g = \{h \in F(H) : h(x) \subseteq f(x) + g(x), \ \forall x\},$$

$$f \circ g = \{h \in F(H) : \ h(x) \subseteq f(g(x)), \ \forall x\},$$

where $f(g(x))$ denotes the set $\bigcup_{y \in g(x)} f(y)$. Notice that we have $\omega_{F(H)}{=}F(H)$.

Example 5.1.6. Let R be a ring and let I be an ideal of R. We define the following hyperoperations on R:

$$a \oplus b = a + b + I, \ a \circ b = ab + I.$$

Then $(R, \oplus, \circ)$ is a feeble hyperring and for all $a, b \in R$, we can take $(a \circ b)' = -ab + I$.

If $(R, +, \cdot)$ is a feeble hyperring, then according to Corollary 2.5.6, Theorem 2.5.9 and Definition 2.5.18, $(R/\omega_R, \oplus)$ is a group, where for all $a, b \in R$ and $c \in a + b$, we have $(a + \omega_R) \oplus (b + \omega_R) = c + \omega_R$. Moreover, according to Lemma 5.1.3., for all $a, b \in R$ and $u \in a \cdot b$ the following operation is well defined $(a + \omega_R) \odot (b + \omega_R) = u + \omega_R$.

In what follows, we suppose that the feeble hyperring $(R, +, \cdot)$ satisfies the next conditions:

(1) for all $a, b \in R$ we have $a + b \subseteq b + a + \omega_R$;

(2) $\exists 1 \in R : \ a \in a \cdot 1 + \omega_R, \ a \in 1 \cdot a + \omega_R, \forall a \in R$.

Theorem 5.1.7. *If $(R, +, \cdot)$ is a feeble hyperring, then $(R/\omega_R, \oplus, \odot)$ is a ring.*

Proof. Let a, b, c be elements of R. According to Lemmas 5.1.3. and 5.1.4, we have

$$(a + \omega_R) \odot (b + \omega_R) \oplus (a + \omega_R) \odot (c + \omega_R) = ab + ac + \omega_R = z + \omega_R,$$

for all $z \in u + v, \ u \in ab, \ v \in ac$. On the other hand,

$$(a + \omega_R) \odot ((b + \omega_R) \oplus (c + \omega_R)) = (a + \omega_R) \odot (t + \omega_R) = w + \omega_R,$$

for all $t \in b + c, w \in at$. Since $a(b + c) \subseteq ab + ac + \omega_R$, we obtain

$$(a{+}\omega_R){\odot}(b{+}\omega_R){\oplus}(a{+}\omega_R){\odot}(c{+}\omega_R) = (a{+}\omega_R){\odot}((b{+}\omega_R){\oplus}(c{+}\omega_R)) = w{+}\omega_R,$$

for all $w \in a(b + c)$.

Similarly, we obtain that

$$(b+\omega_R)\odot(a+\omega_R)\oplus(c+\omega_R)\odot(a+\omega_R) = ((b+\omega_R)\oplus(c+\omega_R))\odot(a+\omega_R) = z+\omega_R,$$

for all $z \in (b+c)a$.

Finally, we have

$$((a+\omega_R)\odot(b+\omega_R))\odot(c+\omega_R) = (ab+\omega_R)\cdot(c+\omega_R)$$
$$= (ab)c+\omega_R \dot{\subseteq} a(bc)+\omega_R = (a+\omega_R)\cdot(bc+\omega_R) = (a+\omega_R)\cdot(y+\omega_R),$$

for all $y \in bc$ and

$$(a+\omega_R)\cdot(y+\omega_R) = ay+\omega_R = z+\omega_R,$$

for all $z \in ay$. Hence,

$$a(bc)+\omega_R = z+\omega_R,$$

for all $z \in a(bc)$, whence

$$((a+\omega_R)\odot(b+\omega_R))\odot(c+\omega_R) = (a+\omega_R)\odot((b+\omega_R)\odot(c+\omega_R)).$$

Therefore $(R/\omega_R, \oplus, \odot)$ is a ring. ■

5.2 A particular type of general hyperings and hyperskewfields

General hyperrings (also called superrings by J. Mittas [90]) are the widest generalization of the concept of a ring. In this paragraph, a special interest is dedicated to the complete hyperrings and hyperskewfields. Recall that complete semihypergroups are semihypergroups for which any hyperproduct is a complete part. We present here several ways of constructing complete hyperrings and we prove some properties of hyperideals. These results were obtained by M. De Salvo [43].

Let us see first what a hyperringoid is.

Definition 5.2.1. A hyperstructure $(H, \oplus, \odot)$ is called a *hyperringoid* if both $\oplus, \odot$ are binary hyperoperations.

Definition 5.2.2. A hyperringoid $(H, \oplus, \odot)$ is called a *hyperring* if the following conditions are satisfied:

(1) $(H, \oplus)$ is a commutative hypergroup;

(2) $(H, \odot)$ is a semihypergroup;

(3) for all $x, y, z \in H, (x \oplus y) \odot z = (x \odot z) \oplus (y \odot z), z \odot (x \oplus y) = (z \odot x) \oplus (z \odot y)$;

(4) for all $x \in H$ and all $u \in \omega_{(H,\oplus)}$, $x \odot u \subseteq \omega_{(H,\oplus)} \supseteq u \odot x$.

The hyperring $(H, \oplus, \odot)$ is called *commutative* if for all $x, y \in H$,

$$x \odot y = y \odot x.$$

In what follows, we denote the heart of $(H, \oplus)$ by ω and $H \setminus \omega$ by H^*.

Definition 5.2.3. A (commutative) hyperring $(H, \oplus, \odot)$ is called *hyperfield (hyperskewfield)* if $H^* \neq \emptyset$ and $(H^*, \odot)$ is a hypergroup.

If $(H, \oplus, \odot)$ is a hyperfield, then we denote the heart of $(H^*, \odot)$ by ω^*.

Definition 5.2.4. Let $(H, \oplus, \odot)$ be a hyperring. If $(H, \oplus)$ is complete, then we say that H is $\oplus$ *complete*. If $(H, \odot)$ is complete, then we say that H is $\odot$ *complete* and if both $(H, \oplus)$,$(H, \odot)$ are complete, then we say that H is *complete*.

Examples 5.2.5.

(1) All rings (skewfields) are hyperrings (hyperskewfields). The hyperrings (hyperskewfields) which are not rings (skewfields) are called *proper*.

(2) Let $H = (\{a, b\}, \oplus, \odot)$, where

$\oplus$	a	b
a	a	a, b
b	a, b	a, b

$\odot$	a	b
a	a	a, b
b	a	a, b

(3) The hyperringoid $H = (\{a,b,c,d,e\}, \oplus, \odot)$ defined as follows:

$\oplus$	a	b	c	d	e
a	a	b,c	b,c	d	e
b	b,c	d	d	e	a
c	b,c	d	d	e	a
d	d	e	e	a	b,c
e	e	a	a	b,c	d

$\odot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b,c	b,c	d	e
c	a	b,c	b,c	d	e
d	a	d	d	a	d
e	a	e	e	d	b,c

is a commutative complete hyperring, but it is not a hyperskewfield. Notice that the subset $\{a,d\}$ is a hyperideal of H.

(4) The hyperringoid $H = (\{a,b,c,d\}, \oplus, \odot)$ defined as follows:

$\oplus$	a	b	c	d
a	a	a,b	c,d	c,d
b	a,b	a,b	c,d	c,d
c	c,d	c,d	a,b	a,b
d	c,d	c,d	a,b	a,b

$\odot$	a	b	c	d
a	a,b	a,b	a,b	a,b
b	a,b	a,b	a,b	a,b
c	a,b	a,b	c,d	c,d
d	a,b	a,b	c,d	c,d

Definition 5.2.6. Let $(H, \oplus, \odot)$ be a hyperring (hyperskewfield) and let K be a nonempty subset of it. We say that K is a *subhyperring* (*subhyperskewfield*) of H if it satisfies the following conditions:

(1) $(K, \oplus)$ is a subhypergroup of $(H, \oplus)$;

(2) $(K, \odot)$ is a subsemihypergroup of $(H, \odot)$ $((K^*, \odot)$ is a subsemihypergroup of $(H^*, \odot))$.

Definition 5.2.7. Let $(H, \oplus, \odot)$ be a hyperring and let I be a nonempty subset of it. We say that I is a *left* (*right*) *hyperideal* of H if it satisfies the following conditions:

(1) $(I, \oplus)$ is a subhypergroup of $(H, \oplus)$;

(2) for all $x \in I, a \in H, a \odot x \subseteq I (x \odot a \subseteq I)$.

I is a *hyperideal* if it is a left and right hyperideal.

Notice that any hyperring has always two hyperideals: ω and H.

Remark 5.2.8. There are not proper hyperskewfields with less than three elements.

Let $H = (\{a, b\}, \oplus, \odot)$ be a hyperskewfield. From definition 5.1.3, it follows that $\mid \omega \mid= 1$. Set $\omega = \{a\}$. We have $a \oplus a = a$, $a \oplus b = b \oplus a = b$ $b \oplus b = a$. Moreover, we have $H^* = \{b\}$, whence $b \odot b = b$, $a \odot a = a \odot b = b \odot a = a$. Therefore, H is the field $\mathbb{Z}_2$.

Theorem 5.2.9. *Let $(H, \oplus)$ be the total hypergroup. A hyperringoid $(H, \oplus, \odot)$ is a hyperring if and only if $(H, \oplus)$ is a commutative hypergroup and $\omega = H$.*

Proof. Let H be a hypering. Then $\omega \odot H \subseteq \omega$. Since $(H, \odot)$ is the total hypergroup, we obtain $\omega \odot H = H$ whence $\omega = H$. Similarly, we prove the converse implication. ∎

Definition 5.2.10. A hyperring $(H, \oplus, \odot)$ is called an *integral hyperdomain* if for all $(x, y) \in H^2$, such that $x \odot y \subseteq \omega$, it follows that $x \in \omega$ or $y \in \omega$.

Theorem 5.2.11. *The $\odot$-complete hyperskewfields are integral hyperdomains.*

Proof. Let $(H, \oplus, \odot)$ be a $\odot$-complete hyperskewfield. Let $x \odot y \subseteq \omega$, $y \notin \omega$ $(*)$. Since $(H, \odot)$ is a complete semihypergroup, we have two situations:

(1) $C_{(H,\odot)}(x) = \{x\}$,

(2) there exist $u, v \in H$, such that $C_{(H,\odot)}(x) = u \odot v$.

(1) Since $(H^*, \odot)$ is a hypergroup, we obtain $x \notin H^*$ and so $x \in \omega$.
(2) We have $\{u, v\} \cap \omega \neq \emptyset$ since if $\{u, v\} \subseteq H^*$ then it would follow $x \odot y \subseteq (u \odot v) \odot y \subseteq H^* \odot y = H^*$, which contradicts $(*)$. From $\{u, v\} \cap \omega \neq \emptyset$ it follows $u \odot v \subseteq \omega$ and so $x \in \omega$. ∎

Theorem 5.2.12. *If $(H, \oplus, \odot)$ is a $\odot$-complete hyperskewfield, then $(H, \odot)$ is an extension of $(H^*, \odot)$ and for all $x \in H^*$, $C_{(H^*,\odot)}(x) = C_{(H,\odot)}(x)$.*

Proof. For all $x \in H^*$, there are $u, v \in H^*$ such that $C_{(H,\circ)}(x) = u \circ v$, whence $u \circ v = C_{(H^*,\circ)}(u \circ v) = C_{(H^*,\circ)}(x)$. Hence $C_{(H^*,\circ)}(x) = C_{(H,\circ)}(x)$. ∎

Corollary 5.2.13. *If $(H, \oplus, \odot)$ is a $\odot$-complete hyperskewfield, then $(H^*, \odot)$ is a complete hypergroup.*

Proof. According to the above theorem, we obtain: for all $a, b \in H^*$ and for all $u \in a \circ b$,

$$a \circ b = \mathcal{C}_{(H,\circ)}(a \circ b) = \mathcal{C}_{(H,\circ)}(u) = \mathcal{C}_{(H^*,\circ)}(u) = \mathcal{C}_{(H^*,\circ)}(a \circ b). \blacksquare$$

Theorem 5.2.14. *Let $(H, \oplus, \odot)$ be a $\odot$-complete hyperskewfield. Then for all $a, b, c \in H$, the following implications hold:*

$$a \circ b = a \circ c, \quad a \notin \omega \Longrightarrow \{b, c\} \subseteq \omega \text{ or } b\beta^*_{(H,\circ)}c$$
$$b \circ a = c \circ a, \quad a \notin \omega \Longrightarrow \{b, c\} \subseteq \omega \text{ or } b\beta^*_{(H,\circ)}c.$$

Proof. There are two possibilities to analyze:

(1) $\{b, c\} \cap \omega \neq \emptyset$;

(2) $\{b, c\} \cap \omega = \emptyset$.

(1) Let $b \in \omega$. Then $a \circ b \subseteq \omega$ and so $a \circ c \subseteq \omega$, whence $c \in \omega$, according to Theorem 5.2.11.

(2) By Corollary 5.2.13, $(H^*, \circ)$ is a complete hypergroup and so it is regular. Let a' be an inverse of a in $(H^*, \circ)$. We have $a' \circ a \circ b = a' \circ a \circ c$, whence $\omega^* \circ b = \omega^* \circ c$. Then $\mathcal{C}_{(H^*,\circ)}(b) = \mathcal{C}_{(H^*,\circ)}(c)$ which means that $\mathcal{C}_{(H,\circ)}(b) = \mathcal{C}_{(H,\circ)}(c)$.

Similarly we show the second implication. ■

Definition 5.2.15. A hyperring (hyperskewfield) $(H, \oplus, \circ)$ is called a Δ-*hyperring* (Δ-*hyperskewfield*) if the following condition holds:

$$(\Delta) \qquad \forall u \in \omega,\ \forall v \in H, \quad u \circ v = v \circ u = \omega.$$

Remark 5.2.16. A $\circ$-complete hyperring $(H, \oplus, \circ)$ is a Δ-hyperring if and only if there exists $(u, v) \in (\omega \times H) \cup (H \times \omega)$ such that $u \circ v = \omega$.
ω is an equivalence class modulo $\beta^*_{(H,\circ)}$ in $\circ$-complete Δ-hyperrings.

Corollary 5.2.17. *Let $H = (H, \oplus, \circ)$ be a $\circ$-complete Δ-hyperskewfield. Then, for all $a, b, c \in H$, the following implications hold:*

$$a \circ b = a \circ c, \quad a \notin \omega \Longrightarrow b\beta^*_{(H,\circ)}c;$$
$$b \circ a = c \circ a, \quad a \notin \omega \Longrightarrow b\beta^*_{(H,\circ)}c.$$

Corollary 5.2.18. *Let $(H, \oplus, \circ)$ be a $\circ$-complete Δ-hyperring such that for all $a, b, c \in H$, the following implications hold:*

$$a \circ b = a \circ c, \quad a \notin \omega \Longrightarrow b\beta^*_{(H,\circ)}c;$$
$$b \circ a = c \circ a, \quad a \notin \omega \Longrightarrow b\beta^*_{(H,\circ)}c.$$

Then H is an integral hyperdomain.

Proof. Let $a \circ b = \omega$ and $a \notin \omega$. For all $u \in \omega$, $a \circ b = a \circ u = \omega$, whence $b\beta^*_{(H,\circ)}u$, which means that $b \in \mathcal{C}_{(H,\circ)}(u) = \omega$. ■

Remark 5.2.19.

(1) If $(H, \oplus, \circ)$ is a hyperring, such that $|\omega| = 1$, then H is a Δ-hyperring.

(2) Let $(H, \oplus, \circ)$ be a hyperring such that $\omega = H$. Then H is a Δ-hyperring if and only if $(H, \circ)$ is a total hypergroup.

Construction of complete hyperrings

Starting with a ring,we construct a hyperring in the following manner:
Let $(R, +, \cdot)$ be a ring and let $\{A(g)\}_{g \in R}$ be a family of nonempty sets, such that:

(1) $\forall g, g' \in R, \; g \neq g' \Longrightarrow A(g) \cap A(g') = \emptyset$.

(2) $g \notin R \cdot R \Longrightarrow |A(g)| = 1$.

Set $H_R = \bigcup_{g \in R} A(g)$ and define the following hyperoperations $\oplus, \circ$ on H_R:

$$\forall a, b \in H_R, \; \exists\, g, g' \in R \text{ such that } a \in A(g), \; b \in A(g').$$

Set $a \oplus b = A(g + g')$, $a \circ b = A(gg')$.

Lemma 5.2.20. *For all $g, g' \in R$, $\forall u \in A(g)$, $\forall v \in A(g')$, we have:*

(1) $u \oplus v = A(g + g') = A(g) \oplus A(g')$;

(2) $u \circ v = A(gg') = A(g) \circ A(g')$.

Theorem 5.2.21. *H_R is a complete Δ-hyperring.*

Proof. It is sufficient to check the distributivity and the condition (4) of Definition 5.2.2.

Let $x, y, z \in H_R$ and let $x \in A(g)$, $y \in A(g')$, $z \in A(g'')$. Then

$$(x \oplus y) \circ z = A(g+g') \circ z = \bigcup_{u \in A(g+g')} u \circ z = A[(g+g')g''].$$

Moreover,

$$(x \circ z) \oplus (y \circ z) = A(gg'') \oplus A(g'g'') = A(gg'' + g'g'') = A[(g+g')g''].$$

Hence $(x \oplus y) \circ z = (x \circ z) \oplus (y \circ z)$ and similarly it follows the other distributive law. On the other hand, $\omega = A(0_R)$ and so $\forall u \in \omega$, $\forall x \in H_R$ we have $u \circ x = x \circ u = A(0_R) = \omega$. Therefore, H_R is a complete Δ-hyperring. ■

5.3 Fundamental relations in hyperrings

We analyze here the fundamental relation in the context of general hyperrings. Using this, we shall define general hyperfields. The most of the results presented in this paragraph were obtained by T. Vougiouklis [131], [132].

In what follows, we shall use the following definition:

Definition 5.3.1. A multivalued system $(R, +, \cdot)$ is a *(general) hyperring* if

(1) $(R, +)$ is a hypergroup;

(2) $(R, \cdot)$ is a semihypergroup;

(3) $(\cdot)$ is (strong) distributive with respect to $(+)$, i.e., for all x, y, z in R we have $x \cdot (y+z) = x \cdot y + x \cdot z$ and $(x+y) \cdot z = x \cdot z + y \cdot z$.

In this paragraph, we shall use the term of a hyperring, instead of the term of a general hyperring, intending the above definition. A hyperring may be commutative with respect to $(+)$ or $(\cdot)$. If R is commutative with respect to both $(+)$ and $(\cdot)$, then we call it a commutative hyperring.

The above definition contains the class of multiplicative hyperrings and additive hyperrings as well.

Example 5.3.2. The set $H = \{0, 1\}$ endowed with the following hyperoperations is a commutative hyperring.

+	0	1
0	0	0,1
1	0,1	1

$\cdot$	0	1
0	0	0
1	0	0,1

In the above hyperstructures, we introduce an equivalence relation γ^*, which is similar to the relation β^*, defined in every hypergroup. Using this relation we obtain a general definition of a hyperfield.

Definition 5.3.3. Let $(R, +, \cdot)$ be a hyperring. We define the relation γ as follows:

$a\gamma b$ if and only if $\{a, b\} \subseteq u$ where u is a finite sum of finite products of elements of R.

We denote the transitive closure of γ by γ^*. The equivalence relation γ^* is called the *fundamental equivalence relation* in R. We denote the equivalence class of the element a (also called the *fundamental class of* a) by $\gamma^*(a)$.

According to the distributive law, every set which is the value of a polynomial in elements of R is a subset of a sum of products in R.

Let $\mathcal{U}$ be the set of all finite sums of products of elements of R. We can rewrite the definition of γ^* on R as follows:

$a\gamma^*b$ if and only if $\exists z_1, ..., z_{n+1} \in R$ with $z_1{=}a, z_{n+1}{=}b$ and $u_1, ..., u_n \in \mathcal{U}$ such that $\{z_i, z_{i+1}\} \subseteq u_i$ for $i \in \{1, ..., n\}$.

Theorem 5.3.4. *Let $(R, +, \cdot)$ be a hyperring. Then the relation γ^* is the smallest equivalence relation in R such that the quotient R/γ^* is a ring.*

R/γ^* is called the *fundamental ring*.

Proof. First, we prove that R/γ^* is a ring. The product $\odot$ and the sum $\oplus$ in R/γ^* are defined as follows:

$$\gamma^*(a) \oplus \gamma^*(b) = \{\gamma^*(c) : c \in \gamma^*(a) + \gamma^*(b)\},$$

$$\gamma^*(a) \odot \gamma^*(b) = \{\gamma^*(d) : d \in \gamma^*(a) \cdot \gamma^*(b)\}.$$

Let $a' \in \gamma^*(a), b' \in \gamma^*(b)$. Hence

$a'\gamma^*a$ implies that $\exists x_1, ..., x_{m+1}$ with $x_1{=}a', x_{m+1}{=}a$ and $u_1, ..., u_m \in \mathcal{U}$ such that $\{x_i, x_{i+1}\} \subseteq u_i$ for $i \in \{1, ..., m\}$;

$b'\gamma^*b$ implies that $\exists y_1, ..., y_{n+1}$ with $y_1 = b', y_{n+1} = b$ and $v_1, ..., v_n \in \mathcal{U}$ such that $\{y_j, y_{j+1}\} \subseteq v_j$ for $j \in \{1, ..., n\}$.

We obtain

$$\{x_i, x_{i+1}\} + y_1 \subseteq u_i + v_1, \; i \in \{1, ..., m-1\}$$
$$x_{m+1} + \{y_j, y_{j+1}\} \subseteq u_m + v_j, \; j \in \{1, ..., n\}.$$

The sets $u_i + v_1 = t_i, \; i \in \{1, ..., m-1\}$ and $u_m + v_j = t_{m+j-1}$, $j \in \{1, ..., n\}$ are elements of $\mathcal{U}$. We choose the elements $z_1, ..., z_{m+n}$ such that $z_i \in x_i + y_1, i \in \{1, ..., m\}$ and $z_{m+j} \subseteq x_{m+1} + y_{j+1}, j \in \{1, ..., n\}$. We obtain $\{z_k, z_{k+1}\} \subseteq t_k, \; k \in \{1, ..., m+n-1\}$. Hence, any element $z_1 \in x_1 + y_1 = a' + b'$ is γ^* equivalent to any element $z_{m+n} \in x_{m+1} + y_{n+1} = a{+}b$. Thus, $\gamma^*(a) \oplus \gamma^*(b)$ is singleton and we have $\gamma^*(a) \oplus \gamma^*(b) = \gamma^*(c)$ for all $c \in \gamma^*(a) + \gamma^*(b)$. According to the distributive law, we have $u \cdot v \in \mathcal{U}$ for all $u, v \in \mathcal{U}$. Similarly, we obtain $\gamma^*(a) \odot \gamma^*(b) = \gamma^*(d)$ for all $d \in \gamma^*(a) \cdot \gamma^*(b)$. Therefore, it is immediate that R/γ^* is a ring.

Now, let ρ be an equivalence relation in R, such that R/ρ is a ring and let $\rho(a)$ be the equivalence class of the element a. Then $\rho(a) \oplus \rho(b)$ and $\rho(a) \odot \rho(b)$ are singletons for all $a, b \in R$, which means that for all $a, b \in R$, for all $c \in \rho(a) + \rho(b)$, for all $d \in \rho(a) \cdot \rho(b)$ we have

$$\rho(a) \oplus \rho(b) = \rho(c), \;\; \rho(a) \odot \rho(b) = \rho(d).$$

The above equalities are called the *fundamental properties* in $(R/\rho, \oplus, \odot)$. Hence, for all $a, b \in R$ and $A \subseteq \rho(a), \; B \subseteq \rho(b)$ we have

$$\rho(a) \oplus \rho(b) = \rho(a+b) = \rho(A+B) \text{ and } \rho(a) \odot \rho(b) = \rho(a \cdot b) = \rho(A \cdot B).$$

By induction, we extend the above equalities to finite sums and products. Now, set $u \in \mathcal{U}$, which means that there exist the finite sets of indices J and I_j and the elements $x_i \in R$ such that:

$$u = \sum_{j \in J} \left(\prod_{i \in I_j} x_i \right).$$

For all I_j, the set $\prod_{i\in I_j} x_i$ is a subset of one class, say $\rho(a_j)$. Thus, for all $a \in \sum_{j\in J} a_j$ we have

$$u \subseteq \sum_{j\in J} \rho(a_j) = \rho\left(\sum_{j\in J} a_j\right) = \rho(a).$$

Therefore, for all $x, y \in R$, $x\gamma y$ implies $x\rho y$, whence $x\gamma^* y$ implies $x\rho y$. Hence, for all $a \in R$, $\gamma^*(a) \subseteq \rho(a)$, which means that γ^* as the smallest equivalence relation in R such that the quotient R/γ^* is a ring. ■

Remark 5.3.5. If $u = \sum_{j\in J}\left(\prod_{i\in I_j} x_i\right) \in \mathcal{U}$ then for all $z \in u$,

$$\gamma^*(u) = \oplus\sum_{j\in J}\left(\odot\prod_{i\in I_j}\gamma^*(x_i)\right) = \gamma^*(z),$$

where $\oplus\sum$ and $\odot\prod$ denote the sum and the product of classes.

In order to speak about canonical maps, we need the following notion:

Definition 5.3.6. Let R_1 and R_2 be two hyperrings. The map $f : R_1 \to R_2$ is called an *inclusion homomorphism* if for all $x, y \in R$, the following conditions hold:

$$f(x + y) \subseteq f(x) + f(y) \quad \text{and} \quad f(x \cdot y) \subseteq f(x) \cdot f(y).$$

f is called a *strong homomorphism* if for all $x, y \in R$, we have

$$f(x + y) = f(x) + f(y) \quad \text{and} \quad f(x \cdot y) = f(x) \cdot f(y).$$

Let R be a hyperring. We denote by $\beta_.$ and β_+ the following binary relations:

$x\beta_. y$ if and only if $\exists z_1, ..., z_n \in R$ such that $\{x, y\} \subseteq z_1 \cdot ... \cdot z_n$ and

$x\beta_+ y$ if and only if $\exists z_1, ..., z_n \in R$ such that $\{x, y\} \subseteq z_1 + ... + z_n$.

We denote the transitive closures of the relations $\beta_.$ and β_+ by $\beta_.^*$ and β_+^*, and we call $\beta^*_{\cdot}$ and β^*_+ the *fundamental relations* with respect to multiplication and addition, respectively. For all $a \in R$ we denote the corresponding equivalence classes of a by $\beta^*_{\cdot}(a)$ and $\beta^*_+(a)$ and we have

$$\beta^*_{\cdot}(a) \subseteq \gamma^*(a), \ \beta^*_+(a) \subseteq \gamma^*(a).$$

Let us consider the following canonical maps

$$\begin{array}{rl} \varphi_{\cdot} : R \longrightarrow R/\beta^*_{\cdot}, & \varphi_{\cdot}(x) = \beta^*_{\cdot}(x), \\ \varphi_+ : R \longrightarrow R/\beta^*_+, & \varphi_+(x) = \beta^*_+(x), \\ \varphi^* : R \longrightarrow R/\gamma^*, & \varphi^*(x) = \gamma^*(x). \end{array}$$

We notice that the maps $\varphi_+ : (R,+) \to (R/\beta^*_+, \oplus)$, $\varphi_{\cdot} : (R,\cdot) \to (R/\beta^*_{\cdot}, \odot)$, $\varphi^* : (R,+,\cdot) \to (R/\gamma^*, \oplus, \odot)$ are strong homomorphisms.

We denote by ω_+, ω^* the kernels of φ_+, φ^*, respectively. If $\bar{0}$ is the zero element of R/β^*_+ or R/γ^*, then

$$\begin{array}{l} \omega_+ = ker\varphi_+ = \{x \in R : \varphi_+(x) = \bar{0}\}, \\ \omega^* = ker\varphi^* = \{x \in R : \varphi^*(x) = \bar{0}\}. \end{array}$$

We have $\omega_+ \subseteq \omega^*$.

Let us see what a hyperideal is in a hyperring.

Definition 5.3.7 Let R be a hyperring. A nonempty subset S of R is called a *hyperideal* if following axioms hold:

(1) $(S,+)$ is a subhypergroup of $(R,+)$;

(2) $(S \cdot R) \cup (R \cdot S) \subseteq S$.

Theorem 5.3.8. *Let $(R,+,\cdot)$ be a hyperring. Then,*

(1) *$R\omega^* \subseteq \omega^*$, $\omega^* R \subseteq \omega^*$;*

(2) *If $(R,+)$ is a regular hypergroup, then ω^* is a hyperideal of R.*

Proof. (1) For all $r \in R$, $x \in \omega^*$, $a \in rx$ we have

$$\varphi^*(a) = \varphi^*(rx) = \varphi^*(r) \odot \varphi^*(x) = \varphi^*(r) \odot \bar{0} = \bar{0}.$$

(2) If $a, b \in \omega^*$ i.e. $\varphi^*(a) = \varphi^*(b) = \bar{0}$ then

$$\varphi^*(a+b) = \varphi^*(a) \oplus \varphi^*(b) = \bar{0}^* \oplus \bar{0} = \bar{0},$$

so $a + b \in \omega^*$. Suppose that $(R, +)$ is regular and let e be an identity of it. Then, $e \in \omega_+ \subseteq \omega^*$. Set $x \in \omega^*$. Then, for all $x' \in R$, such that $e \in x + x'$, we have

$$\bar{0} = \varphi^*(e) = \varphi^*(x + x') = \varphi^*(x) \oplus \varphi^*(x') = \bar{0} \oplus \varphi^*(x').$$

So $\varphi^*(x') = \bar{0}$ whence $x' \in \omega^*$. Therefore, for all $y \in \omega^*$ we have

$$y \in e + y \subseteq (x + x') + y = x + (x' + y)$$

and from $x' + y \in \omega^*$ we obtain $y \in x + \omega^*$. Hence $\omega^* \subseteq x + \omega^*$, whence ω^* is a subhypergroup of $(R, +)$. ■

Theorem 5.3.9. *For all additive hyperrings we have* $\gamma^* = \beta^*_+$.

Proof. In an additive hyperring R, any product of elements of R is singleton. Hence, for all

$$u = \sum_{j \in J} \left(\prod_{i \in I_j} x_i \right) \in \mathcal{U}$$

we consider the elements $y_j = \prod_{i \in I_j} x_i$ for all $j \in J$ and we have $u = \sum_{j \in J} y_j$. This means that $a\gamma^* b$ if and only if $a\beta^*_+ b$. ■

Theorem 5.3.10. *Let* $(R, +, \cdot)$ *be a hyperring. Then* R/γ^* *and* $(R/\beta^*_\cdot)/\beta^*_\oplus$ *are isomorphic, where* $\beta^*_\oplus$ *is the fundamental relation defined in* $(R/\beta^*_\cdot, \oplus)$ *by setting* $\beta^*_\cdot(a) \oplus \beta^*_\cdot(b) = \{\beta^*_\cdot(c) : \ c \in \beta^*_\cdot(a) + \beta^*_\cdot(b)\}$.

Proof. The quotient $(R/\beta^*_\cdot)/\beta^*_\oplus$ is a ring. We denote the equivalence relation associated with the projection $p : R \to (R/\beta^*_\cdot)/\beta^*_\oplus$ by σ. Since p is ring homomorphism, we obtain $\gamma^*(a) \subseteq \sigma(a)$ for all $a \in R$. On the other hand, for all $x \in R$ we have $\beta^*_\cdot(x) \subseteq \gamma^*(x)$, hence

$$\bigcup_{\beta^*_\cdot(z) \in \beta^*_\cdot(x) \oplus \beta^*_\cdot(y)} \beta^*_\cdot(z) = \bigcup_{z \in \beta^*_\cdot(x) + \beta^*_\cdot(y)} \beta^*_\cdot(z) \subseteq \bigcup_{z \in \gamma^*(x) + \gamma^*(y)} \gamma^*(z).$$

From the fundamental property in R/γ^*, for all $w \in x+y$ we have $\gamma^*(x)+\gamma^*(y)=\gamma^*(w)$, whence

$$\bigcup_{z \in \beta^*_\cdot(x) \oplus \beta^*_\cdot(y)} \beta^*_\cdot(z) \subseteq \gamma^*(w).$$

Therefore, for every finite set $\{x_i : i \in I\}$ of elements of $R/\beta^*_\cdot$ and for all $w \in \sum\limits_{i\in I} x_i$ we have

$$\bigcup_{z \in \oplus \sum_{i\in I} \beta^*_\cdot(x_i)} \beta^*_\cdot(z) \subseteq \gamma^*(w).$$

Since γ^* is transitive, for all $a \in R$ we have

$$\sigma(a) = \bigcup_{z:\ \beta^*_\cdot(z) \beta^*_\oplus \beta^*_\cdot(a)} \beta^*_\cdot(z) \subseteq \gamma^*(a).$$

Therefore, $\sigma = \gamma^*$. ■

Definition 5.3.11. A hypering $(R,+,\cdot)$ is a *hyperfield* if either R/γ^* is a field or $\omega^* = R$, which means that R/γ^* consists only of the zero element.

Theorem 5.3.12. *Every hyperring in the sense of Krasner is a hyperfield.*

Proof. Let us recall that $(R,+,\cdot)$ is a hyperfield in the sense of Krasner, if $(R,+)$ is a canonical hypegroup, $(R\setminus\{0\},\cdot)$ is a group, where 0 is the scalar identity of $(R,+)$, which is a bilateraly absorbing element of R and the (strong) distributive law is satisfied. We denote by 1 the unit element of $(R\setminus\{0\},\cdot)$. For all $r \in R$ we have $\varphi^*(r) = \varphi^*(1\cdot r) = \varphi^*(1)\odot\varphi^*(r)$, hence $\varphi^*(1)$ is the unit element of $(R/\gamma^* \setminus \{\omega^*\},\odot)$. If $1 \in \omega^*$ then we obtain $R = \omega^*$ since ω^* is a hyperideal of R. If $1 \notin \omega^*$, then $\varphi^*(1)$ is the unit element of R/γ^*. Hence, for all $x \in R - \omega^*$ we take the element x^{-1}, such that

$$\varphi^*(1) = \varphi^*(x\cdot x^{-1}) = \varphi^*(x) \odot \varphi^*(x^{-1}),$$

which means that $\varphi^*(x^{-1})$ is the inverse of $\varphi^*(x)$ in $(R/\gamma^* - \{\omega^*\},\odot)$. Therefore, R/γ^* is a field. ■

5.4 The (H,R)-hyperrings

Now, we present a way to obtain new hyperrings, starting with other hyperrings [43, 44, 119].

Let $(H,\circ,\Box)$ be a hyperring and let $\{A_i\}_{i\in R}$ be a family of nonempty sets such that:

(1) $(R,+,\cdot)$ is a ring;

(2) $A_{0_R}=H$;

(3) $\forall i,j\in R,\ A_i\cap A_j=\emptyset$.

Let $K=\bigcup\limits_{i\in R}A_i$ and define the following hyperoperations on K:

$$\begin{array}{ll}\forall x,y\in H, & x\oplus y=x\circ y\\ & x\odot y=H.\\ \forall x\in A_i, & \forall y\in A_j,\ \text{such that } A_i\times A_j\neq H\times H,\\ & x\oplus y=A_k\ \text{if } i+j=k\\ & x\odot y=A_m\ \text{if } i\cdot j=m.\end{array}$$

We have $\omega=H$ and $H\odot K=H=K\odot H$. The structure $(K,\oplus,\odot)$ is a hyperring. We shall say that K is an *(H,R)–hyperring*. Denote $R^*=R\setminus\{0_R\}$.

Remark 4.4.1. If K is an (H,R)–hyperring, then K is commutative if and only if R is commutative.

Lemma 5.4.2. *Let K be an (H,R)–hyperring. Then K is an integral hyperdomain if and only if R is an integral domain.*

Proof. Let $K=\bigcup\limits_{i\in R}A_i$ and let R be an integral domain. Let $x\in A_i$, $y\in A_j$ and $x\odot y=\omega=H$. From the definition of $\odot$ it follows $i=0_R$ or $j=0_R$ and so $x\in H$ or $y\in H$, which means that K is integral. Similarly, we obtain the converse implication. ■

Theorem 5.4.3. *Let $(K,\oplus,\odot)$ be an (H,R)–hyperring. If R is a skewfield (field), then K is a hyperskewfield (hyperfield).*

Proof. We have to check that $(K^*, \odot)$ is a hypergroup. Let $\{x, y\} \subseteq K^*$. Then there exists $\{i, j\} \subseteq R^*$ such that $x \in A_i$, $y \in A_j$. From Lemma 5.4.2, we have $x \odot y \subseteq K^*$, since R is integral. Moreover, there exists $p \in R^*$, such that $i = jp$. If $z \in A_p$, then $y \odot z = A_i \ni x$. Similarly, there exists $w \in K^*$, such that $x \in w \odot y$. Hence K is a hyperskewfield. Finally, if R is a field, then K is a hyperfield. ■

By induction on n, we can show that:

Lemma 5.4.4. *Let* $(K, \oplus, \odot)$ *be an* (H, R)*–hyperring, such that* $K = \bigcup_{i \in R} A_i$. *Then* $\forall n \in \mathbb{N}^* \setminus \{1\}$, $\forall x_1, ..., x_n \in K$ *the following cases are possible:*

$$\begin{array}{l} \exists\, j \in R : \oplus\Pi x_i = A_j; \\ \exists\, B \in \mathcal{P}(H) \setminus \{\emptyset\} : \oplus\Pi x_i = B; \\ \exists\, p \in R : \odot\Pi x_i = A_p. \end{array}$$

From Lemma 5.4.4, we obtain the next lemma:

Lemma 5.4.5 *Let* $(K, \oplus, \odot)$ *be an* (H, R)*–hyperring, such that* $K = \bigcup_{i \in R} A_i$. *Then* $\forall i \in R$, A_i *is a complete part of* $(K, \oplus)$ *and of* $(K, \odot)$.

From Lemma 5.4.2 and Lemma 5.4.5, we obtain

Lemma 5.4.6. *Let* $(K, \oplus, \odot)$ *be an* (H, R)*–hyperring, such that* $K = \bigcup_{i \in R} A_i$. *Then*

$$\begin{array}{l} \forall i \in R,\ \forall a \in A_i,\ C_{(K,\oplus)}(a) = A_i; \\ \forall p \in R \text{ such that } p \in R \cdot R,\ \forall b \in A_p,\ C_{(K,\odot)}(b) = A_p; \\ \forall s \in R \text{ such that } s \notin R \cdot R,\ \forall c \in A_s,\ C_{(K,\odot)}(c) = \{c\}. \end{array}$$

Remark 5.4.7. By Lemma 5.4.2 and Lemma 5.4.5, it follows that any (H, R)–hyperring is a $\odot$–complete Δ–hyperring.

Lemma 5.4.8. *Let* $(K, \oplus, \odot)$ *be an* (H, R)*–hyperring. If* K *is a hyperskewfield, then:*

(1) $(K^*, \odot)$ *is a complete hypergroup;*

(2) $\omega^* = A_u$, *where* $u = 1_R$.

Proof. (1) It follows from Remark 5.4.7 and Corollary 5.2.13.

(2) Let $\alpha \in \omega^*$. There exist $v, w \in K^*$ such that $\alpha \in v \odot w$. From Lemma 5.4.4, there exists $u \in R$ such that $v \odot w = A_u$. From Lemma 5.4.6, we have $\mathcal{C}_{(K,\odot)}(\alpha) = A_u$ and from Theorem 5.2.12 we obtain $\omega^* = A_u$. Since ω^* is the set of bilateral identities of $(K^*, \odot)$, we have $\forall i \in R$, $\forall a \in A_i$, $\forall e \in A_u$, $a \in a \odot e$ and so $a \odot e = A_i$. Similarly, $e \odot a = A_i$. We obtain $u = 1_R$. ∎

Theorem 5.4.9. *Let $(K, \oplus, \odot)$ be an (H,R)–hyperring. Then K is a hyperskewfield (hyperfield) if and only if R is a skewfield (field).*

Proof. According to Theorem 5.4.3, it suffices to check only an implication. Let K be a hyperskewfield. According to Lemma 5.4.8, R is a unitary ring, so we have to check that any element of R^* has a unique multiplicative inverse. Again, by Lemma 5.4.8, $(K^*, \odot)$ is a complete hypergroup, so it is regular.

For all $i \in R^*$ and for all $a \in A_i$, there exists $j \in R^*$ and $a' \in A_j$, such that $a \odot a' = a' \odot a = \omega^*$. According to Lemma 5.4.8, $\omega^* = A_u$ where $u = 1_R$. By the definition of $\odot$, we have $i \cdot j = j \cdot i = 1_R$. By the way of contraposition, suppose that there exists $m \in R^*$, $m \neq j$, such that $m \cdot i = i \cdot m = 1_R$. By Theorem 5.2.11 and Lemma 5.4.2, it follows that R is an integral domain and so, from $i \cdot m = 1_R = i \cdot j$ we obtain $j = m$. ∎

Lemma 5.4.10 *Let $(H, \circ, \Box)$ be a $\Box$–complete Δ–hyperskewfield and let I be a hyperideal of H. Then I is proper if and only if $I \cap \omega^* = \emptyset$.*

Proof. If $I \cap \omega^* = \emptyset$, then $I \neq H$.

Conversely, let I be proper and by the way of contraposition suppose there exists $x \in \omega^* \cap I$. Then, for all $u \in H$, we have two possibilities:

(i) $u \in \omega$;

(ii) $u \in H^*$.

In the case (i), from (Δ) it follows $u \Box x = \omega$ and since I is a hyperideal, $u \Box x \subseteq H \Box I \subseteq I$, whence $u \in \omega \subseteq I$.

In the case (ii), we have:

$$u \Box x = u \Box \mathcal{C}_{(H^*,\Box)}(x) = u \Box \omega^* = \mathcal{C}_{(H^*,\Box)}(u) \ni u.$$

But $u \Box x \subseteq I$ and so $u \in I$. Hence we would obtain $H = I$, a contradiction. ∎

Theorem 5.4.11 *If* $(H, \circ, \Box)$ *is a* $\Box$*-complete* Δ*-hyperskewfield, then* ω *is the unique proper hyperideal of it.*

Proof. By the way of contraposition, suppose that H has an proper hyperideal $I \neq \omega$ and let $x \in I - \omega$. Since H is $\Box$-complete, it follows that there exists an inverse x' of x in H^*, such that $x \Box x' = \omega^*$. Since I is a hyperideal, we obtain $x \Box x' \subseteq I \Box H \subseteq I$. So $\omega^* \cap I \neq \emptyset$, which contradicts Lemma 5.4.10. ∎

Lemma 5.4.12. *Let R be ring with the unity u and let $(K, \oplus, \odot)$ be an (H, R)-hyperring. If we denote* $E((K, \odot)) = \{e \in K \mid \forall x \in K,\ x \in e \odot x \cap x \odot e\}$, *then* $E((K, \odot)) = A_u$.

Proof. Let $y \in A_u$. For all $x \in K$, if $x \in A_i$, then $x \odot y = y \odot x = A_i \ni x$. Hence $y \in E((K, \odot))$. Conversely, let $e \in E((K, \odot))$. Then there exists $j \in R^*$, such that $e \in A_j$. Let $z \in A_u$. We have $z \in z \odot e = e \odot z = A_j$ and so $A_u \cap A_j \neq \emptyset$, whence $A_u = A_j$. ∎

Lemma 5.4.13. *Let $(K, \circ, \Box)$ be a hyperring and let $x \in K$. If we set $I = K \Box x$, then I is a left hyperideal of K if and only if for all $y \in I$, $I \circ y = I = y \circ I$.*

Proof. If $u, v \in I$, then there exist $a, b \in K$ such that $u \in a \Box x$, $v \in b \Box x$. We have $u \circ v \subseteq (a \Box x) \circ (b \Box x) = (a \circ b) \Box x \subseteq K \Box x$. Hence I is a subsemihypergroup of $(K, \circ)$. Moreover, $K \Box I = K \Box (K \Box x) = (K \Box K) \Box x \subseteq K \Box x = I$, which means that I is a left hyperideal of K. Conversely, it is immediate. ∎

Theorem 5.4.14. *Let $(K, \oplus, \odot)$ be an (H, R)-hyperring and let $x \in K$. Then $K \odot x$ is a left hyperideal of K.*

Proof. By Lemma 5.4.13, it is sufficient to show that $K \odot x$ is a subquasihypergroup of $(K, \oplus)$. If $x \in H$, then $K \odot x = H = \omega$ and ω is a hyperideal of K. Let $x \in K - H$, $x \in A_i$. If $u, v \in K \odot x$, then there exist $h_1, h_2 \in K$ such that $u \in h_1 \odot x$, $v \in h_2 \odot x$.

We have two possibilities:

(i) $h_1, h_2 \in H$; (ii) $h_1 \in A_j$, $h_2 \in A_m$, and $A_j \times A_m \neq H \times H$.

(i) We have $u, v \in H$, so there exists $t \in H$, such that $u \in v \oplus t$. But $K \odot x \supset H \odot x = H$ and so $t \in K \odot x$.

(ii) Let $j \cdot i = k$ and $m \cdot i = p$. Then $u \in A_k$ and $v \in A_p$. Set $s = k - p$ and let $t \in A_s$. We have $u \in v \oplus t = A_k$. Moreover, $z \in A_{j-m}$ and we have $z \odot x = A_s$, whence $t \in K \odot x$. Similarly, we show that there exists $t' \in K \odot x$ such that $u \in t' \oplus v$. ∎

Theorem 5.4.15. *Let R be an integral domain and let $(K, \oplus, \odot)$ be an (H, R)-hyperring. Then K is a hyperskewfield if and only if the only proper hyperideal of K is ω.*

Proof. By Theorem 5.4.11 and Remark 5.4.7, it follows the implication "$\Longrightarrow$".

Conversely, we show that $(K^*, \odot)$ is a hypergroup. By Lemma 5.4.2, K is integral and so $\forall a, b \in K^*$, $a \odot b \subseteq K^*$. Let $x \in K^*$. By Theorem 5.4.14, $K \odot x$ is a hyperideal of K. By Lemma 5.4.12, $x \in K \odot x$. Since $x \notin \omega$ and the fact the unique proper hyperideal of K is ω, it follows that $K \odot x = K$ for all $x \in K^*$.

Let $a \in K^*$. Then there exists $b \in K$, such that $a \in b \odot x$ and, by the definition of $\odot$, we have $b \in K^*$. ∎

From Theorems 5.4.9 and 5.4.15, we obtain:

Corollary 5.4.16. *Let R be an integral domain and let $(K, \oplus, \odot)$ be an (H, R)-hyperring. Then the following conditions are equivalent:*

(1) *K is a hyperfield;*

(2) *R is a field;*

(3) *the only proper hyperideal of K is ω.*

Theorem 5.4.17. *Let $(H, \circ, \square)$ be a complete hyperring, such that $\beta^*_{(H,\circ)} = \beta^*_{(H,\square)}$. Then, for all $x \in H$, $H \square x$ is a left hyperideal of H.*

Proof. By Lemma 5.4.13, it is sufficient to show that $H \square x$ is a subquasihypergroup of $(H, \circ)$. Let $u, v \in H \square x$. Then there exists $h_1, h_2 \in H$, such that $u \in h_1 \square x$, $v \in h_2 \square x$. Let h_2' be an inverse of h_2 in $(H, \circ)$ and let $t \in (h_2' \circ h_1) \square x$. Then

$$\begin{aligned} v \circ t &\subseteq (h_2 \Box x) \circ [(h_2' \circ h_1) \Box x] \\ &= [h_2 \circ (h_2' \circ h_1)] \Box x \\ &= (\omega \circ h_1) \Box x \\ &= \mathcal{C}_{(H,\circ)}(h_1) \Box x \\ &= \mathcal{C}_{(H,\Box)}(h_1) \Box x. \end{aligned}$$

Hence

$$(v \circ t) \cap (\mathcal{C}_{(H,\Box)}(h_1) \Box x) \neq \emptyset. \qquad (*)$$

We show that $v \circ t$ is a complete part of $(H, \Box)$. Let $\Box \Pi z_i \cap (v \circ t) \neq \emptyset$. Then there exists $\alpha \in \Box \Pi z_i$, $\alpha \in v \circ t$. Let $\gamma \in \Box \Pi z_i$. We have $\alpha \beta^*_{(H,\Box)} \gamma$, whence $\alpha \beta^*_{(H,\circ)} \gamma$. Since $\mathcal{C}_{(H,\circ)}(\alpha) = v \circ t$, we have $\gamma \in v \circ t$ and so $v \circ t$ is a complete part of $(H, \Box)$. Moreover, there exist $r, s \in H$, such that $\mathcal{C}_{(H,\Box)}(h_1) = r \Box s$ since $(H, \Box)$ is complete.

From $(*)$, we obtain $\mathcal{C}_{(H,\Box)}(h_1) \Box x = (r \Box s) \Box x \subseteq v \circ t$ and since $u \in \mathcal{C}_{(H,\Box)}(h_1) \Box x$, it follows that $H \Box x$ is a right subquasihypergroup of $(H, \circ)$.

Similarly, we show that $H \Box x$ is a left subquasihypergroup of $(H, \circ)$. ■

5.5 $(H, HypR)$-Hyperrings

The hyperrings introduced in this paragraph generalize (H, R)-hyperrings, studied in Section 5.4. We consider R a hypering, instead of a ring. Then, we study quotients of such structures, with respect to two-sided hyperideal. These results were obtained by Mahmoud [78].

Let $(H, *, \circ)$ and $(R, +, \cdot)$ be hyperings, such that $(R, +)$ has a unique identity, denoted by 0. Moreover, suppose that 0 is a two sided absorbing element. Let $\{A_i\}_{i \in R}$ be a family of nonempty sets, such that for all $i, j \in R$, $i \neq j$, $A_i \cap A_j = \emptyset$ and $A_0 = H$.

Set $K = \bigcup\limits_{i \in R} A_i$ and define the hyperoperations $\oplus$ and $\odot$ on K as follows:

$$\forall (x, y) \in H^2, \ x \oplus y = x * y, \ x \odot y = x \circ y,$$

$$\forall (x, y) \in A_i \times A_j \neq H^2, \ x \oplus y = A_{i+j} = \bigcup_{t \in i+j} A_t, \ x \odot y = A_{i \cdot j} = \bigcup_{t \in i \cdot j} A_t.$$

Theorem 5.5.1. $(K, \oplus, \odot)$ *is a hyperring.*

Proof. First, notice that $(K, \oplus)$ is a hypergroup. Indeed, for all $x \in A_i$, $y \in A_j$, $z \in A_r$ we have

$$(x\oplus y)\oplus z = A_{i+j}\oplus z = \bigcup_{t\in i+j} A_t\oplus z = \bigcup_{w\in(i+j)+r} A_w = \bigcup_{w\in i+(j+r)} A_w = x\oplus(y\oplus z)$$

and

$$\begin{aligned} x \oplus K &= \bigcup_{k\in K} x \oplus k = \bigcup_{j\in R} A_{i+j} = \bigcup_{t\in i+j, j\in R} A_t = \bigcup_{t\in i+R} A_t \\ &= \bigcup_{t\in R+i} A_t = K \oplus x = \bigcup_{t\in R} A_t = K. \end{aligned}$$

Similarly, we check that the hyperoperation $\odot$ is associative. Finally, it is easy to check that $\odot$ is distributive with respect to $\oplus$. Therefore, $(K, \oplus; \odot)$ is a hyperring. ■

Definition 5.5.2. The hyperring $(K, \oplus, \odot)$ is called $(H, Hyp(R))$-*hyperring* with support $K = \bigcup_{i\in R} A_i$.

We say that $(K, \oplus, \odot)$ is commutative if both $(H, *, \circ)$ and $(R, +, \cdot)$ are commutative.

Remark 5.5.3. If $i, u, j \in R$ such that $x \in A_i$ and $x \in A_{u+j}$ then $i \in u+j$. Indeed, $x \in A_{u+j}$ implies $x \in \bigcup_{t\in u+j} A_t$, which means that there exists $r \in u + j$, such that $x \in A_r$. So $i = r$, whence $i \in u + j$.

Theorem 5.5.4. *If $(K, \oplus, \odot)$ is a $(H, Hyp(R))$-hyperring with support $K = \bigcup_{i\in R} A_i$, then K/γ^*_K and R/γ^*_R are isomorphic.*

Proof. Let $a \in K$. If $a \in A_r$, then denote a by a_r. Let $x_t \in \gamma^*_K(a_r)$, where $x_t \in A_t$, $t \in R$. Then there exist $z_{r_1} \in A_{r_1}, ..., z_{r_{n+1}} \in A_{r_{n+1}}$ such that $z_{r_1} = x_t$, $z_{r_{n+1}} = a_r$ and for all $i \in I_n = \{1, ..., n\}$, there exists $y_{t_{j_i}} \in A_{t_{j_i}}$, where $t_{j_i} \in R$, $j_i \in I_{m_i}$ and m_i is a nonzero natural number, such that

$$\{z_{r_i}, z_{r_{i+1}}\} \subseteq \oplus \sum_{m_i\in M} \left(\odot \prod_{j_i\in I_{m_i}} y_{t_{j_i}} \right) = A_{\sum\limits_{m_i\in M} \left(\prod\limits_{j_i\in I_{m_i}} t_{j_i} \right)}, \quad i \in I_n.$$

According to Remark 5.5.3., we have

$$\{r_i, r_{i+1}\} \subseteq \sum_{m_i \in M} \left(\prod_{j_i \in I_{m_i}} t_{j_i} \right), \ i \in I_n.$$

In other words, $r_1 \in \gamma^*(r_{n+1})$, which means that $t \in \gamma_R^*(r)$. Hence, there is an isomorphism $\sigma : K/\gamma_K^* \longrightarrow R/\gamma_R^*,\ \gamma_K^*(a_r) \mapsto \gamma_R^*(r)$. ∎

Corollary 5.5.5. *If $(R, +, \cdot)$ is a ring, then K/γ_K^* is isomorphic to R.*

Definition 5.5.6. Let $(K, \oplus, \odot)$ be a (H, Hyp(R))-hyperring. An element $x' \in K$ is called an *opposite* of $x \in K$ if and only if $H \subseteq x \oplus x' \cap x' \oplus x$.

Definition 5.5.7. A hypering $(R, +, \cdot)$ is called *regular* if $(R, +)$ is regular.

Theorem 5.5.8. *Let $(K, \oplus, \odot)$ be a $(H, Hyp(R))$-hyperring and F be a nonempty subset of K. If $(R, +, \cdot)$ is a regular hyperring, then F is a hyperideal of K if and only if $F = \bigcup_{i \in E} A_i$ where E is a hyperideal of R.*

Proof. If F is a hyperideal of K, then $H = F \odot H \subsetneq F$, since 0 is a two sided absorbing element. Let $x \in F \setminus H$. Then there exists $i \in R - \{0\}$ such that $x \in A_i$. For all $y \in H$, we have $x \oplus y = A_i \subseteq F$. Since R is regular, it follows that there exists an inverse i' of i. If $z \in A_{i'}$, then $x \oplus z = A_{i+i'} = \bigcup_{t \in i+i'} A_t$, hence $H \subseteq x \oplus z$. This means that z is an opposite of x in K and since $(F, \oplus)$ is a subhypergroup of $(K, \oplus)$ it follows that $z \in F$. Therefore, $A_{i'} = z \oplus y \subseteq F$, whence there exists a nonempty subset E of R, such that $F = \bigcup_{i \in E} A_i$.

Let $i, j \in E$. Then, there exist $x \in A_i$, $y \in A_j$ such that $x \oplus y = A_{i+j} \subseteq F$. Hence $i + j \subseteq E$, which means that $E + E \subseteq E$. Hence, $(E, +)$ is a regular subhypergroup of $(R, +)$. In a similar way, we can show that $i + E = E = E + i$.

We still have to check that if $i \in E, j \in R \setminus \{0\}$, then $i \cdot j \subseteq E$. For all $x \in A_i \subseteq F, y \in A_j$ we have $x \odot y = A_{i \cdot j} \subseteq F$, which means that $i \cdot j \subset E$. Therefore, E is a regular hyperideal of R.

Conversely, let $F = \bigcup_{i \in E} A_i$, where E is a regular hyperideal of R. If

$x \in F, k \in K$ then there exist $i \in E, r \in R$ such that $x \in A_i, k \in A_r$. Hence, $x \odot k = A_{i \cdot r}$ and since $i \cdot r \subset E$ we obtain $x \odot k = \bigcup_{t \in i \cdot r} A_t \subseteq \bigcup_{t \in E} A_t = F$.

Finally, let i' be an opposite of i in E. Then $H \subseteq A_{i+i'} \subseteq F$. Moreover, any $x' \in A_{i'}$ is an opposite of x in F, since $H \subseteq A_{i+i'} = x \oplus x'$. By an immediate check, we obtain $F \oplus F \subseteq F$ and $x \oplus F = F = F \oplus x$ for all $x \in F$. Therefore, F is a regular hyperideal of K. ■

Theorem 5.5.9. *If $F = \bigcup_{i \in E} A_i$ is a hyperideal of K and E is an invertible subhypergroup of $(R, +)$, then F is an invertible subhypergroup of $(K, \oplus)$.*

Proof. Let $a \in F \oplus b$, where $b \notin F$. Hence $b \in A_j$ for some $j \in R \setminus E$. Then $a \in \bigcup_{t \in E+j} A_t$ implies that $a \in A_i$ for some $i \in E + j$. Since E is invertible in R, it follows that $j \in E + i$, whence $b \in \bigcup_{r \in E+i} A_r = F \oplus a$, which means that F is an invertible subhypergroup of $(K, \oplus)$. ■

In what follows, we shall use the next theorem of immediate check.

Theorem 5.5.10. *Let K be a hyperideal of $(R, +, \cdot)$, such that K is an invertible subhypergroup of $(R, +)$. Then $(R/K, \bullet, \circ)$ is a hyperring, where for all $x, y \in R$ we have*

$$(x + K) \bullet (y + K) = \{z + K : \ z \in x + K + y + K\} \quad \text{and}$$
$$(x + K) \circ (y + K) = \{w + K \mid w \in (x + K) \cdot (y + K)\} = \{w + K \mid w \in x \cdot y\}.$$

Recall that a map $f : A \to B$ is a *homomorphism* between the hyperrings $(A, +, \cdot)$ and $(B, \oplus, \odot)$ if for all $x, y \in A$, we have $f(x+y) \subseteq f(x) \oplus f(y)$ and $f(x \cdot y) \subseteq f(x) \odot f(y)$.

Theorem 5.5.11. *Let $(R, +, \cdot)$ be a hyperring, $(K, \oplus, \odot)$ be a $(H, Hyp(R))$-hyperring. Let F be a hyperideal of K and E be the corresponding hyperideal of R, i.e. $F = \bigcup_{i \in E} A_i$. Moreover, suppose that E is an invertible subhypergroup of $(R, +)$. Then there is a hyperring homomorphism $\psi : (K/F, \bullet, \circ) \to (R/E, \bullet, \circ)$.*

Proof. According to the above theorem, $(K/F, \bullet, \circ)$ and $(R/E, \bullet, \circ)$ are hyperrings. For all $x, y \in K$, there exist $i, j \in R$ such that $x \in A_i, y \in A_j$ and

$$(x \oplus F) \bullet (y \oplus F) = \bigcup_{t \in i+E+j+E} A_t,$$

$$(x \oplus F) \circ (y \oplus F) = \bigcup_{w \in i \cdot j+E} A_w = \bigcup_{w \in (i+E)\cdot(j+E)} A_w.$$

Define $\psi : (K/F, \bullet, \circ) \to (R/E, \bullet, \circ)$ by $\psi(x \oplus F) = \{r+E : r \in i+E\}$. We have $\psi[(x \oplus F) \bullet (y \oplus F)] = \{t+E : t \in i+E+j+E\} = (i+E) \bullet (j+E) \subseteq \psi(x \oplus F) \bullet \psi(y \oplus F)$. Similarly, we have $\psi[(x \oplus F) \circ (y \oplus F)] \subseteq \psi(x \oplus F) \circ \psi(y \oplus F)$. Therefore, ψ is a hyperring homomorphism. ■

5.6 Hyperring of series and hyperring of polynomials

In this paragraph we construct a hyperring of series and a hyperring of polynomials over a general hyperring. Then we mention some properties of series hyperrings and polynomial hyperrings. The results presented in this paragraph were obtained by B. Davvaz and A. Koushky [39].

First, we do some notations. Let R be a general hyperring.

(1) A *series with coefficients in* R is an infinite sequence $(a_0, a_1, ..., a_n, ...)$ in which all a_i belong to R. The set of all series with coefficients in R will be denoted as usual by $R[[x]]$. Two series $(a_0, a_1, ..., a_n, ...)$ and $(b_0, b_1, ..., b_n, ...)$ are equal if and only if $a_i = b_i$ for all non-negative i.

(2) The *addition* is defined by

$$(a_0, a_1, ..., a_n, ...) \oplus (b_0, b_1, ..., b_n, ...) = \{(c_0, c_1, ..., c_n, ...) \mid c_k \in a_k + b_k\}.$$

(3) The *multiplication* is defined by

$$(a_0, a_1, ..., a_n, ...) \odot (b_0, b_1, ..., b_n, ...) = \left\{ (c_0, c_1, ..., c_n, ...) \mid c_k \in \sum_{i+j=k} a_i \cdot b_j \right\}.$$

The element $f = (a_0, a_1, ..., a_n, ...)$ is frequently written $f = \sum_{n=0}^{\infty} a_n x^n$. This notation is very convenient for dealing with the calculations, although it is purely formal.

We consider R, R_1 and R_2 commutative hyperrings and $(R, +)$, $(R_1, +)$ and $(R_2, +)$ regular hypergroups.

Lemma 5.6.1. *$\oplus, \odot$ defined above are well defined hyperoperations.*

Theorem 5.6.2. *$(R[[x]], \oplus, \odot)$ is a general hyperring.*

Proof. It is easy to see that $\oplus, \odot$ are associative. Let us check that $(R[[x]], \oplus)$ is a quasihypergroup. For all $f \in R[[x]]$, we show that

$$f \oplus R[[x]] = R[[x]].$$

Suppose that $f = \sum_{i=0}^{\infty} a_i x^i$ and $g = \sum_{i=0}^{\infty} b_i x^i$. Since $(R, +)$ is regular, we can assume that $\bar{a}_i$ is an inverse of a_i. We define

$$\bar{f} = \bar{a_0} + \bar{a_1} x + \bar{a_2} x^2 + ... + \bar{a_n} x^n + ...,$$

and we have $f \oplus \bar{f} \subseteq f \oplus R[[x]]$. Since $\bar{a_i}$ is an inverse of a_i, there exists an identity $e_i \in a_i + \bar{a}_i$. If we set

$$e = e_0 + e_1 x + e_2 x^2 + ... + e_n x^n + ...,$$

then we obtain $e \in f \oplus \bar{f}$ (in general, e is not unique). On the other hand $\bar{f} \oplus g \subseteq R[[x]]$, whence $f \oplus (\bar{f} \oplus g) \subseteq f \oplus R[[x]]$. Therefore

$$g \in e \oplus g \subseteq (f \oplus \bar{f}) \oplus g = f \oplus (\bar{f} \oplus g) \subseteq f \oplus R[[x]],$$

and so $R[[x]] \subseteq f \oplus R[[x]]$. Hence $R[[x]] = f \oplus R[[x]]$. Now, we check the distributivity of $\odot$ with respect to $\oplus$. For simplicity of notations, we shall write $\sum_{k=0}^{\infty} A_k x^k$ instead of $\left\{ \sum_{k=0}^{\infty} c_k x^k \mid c_k \in A_k \right\}$. Suppose that $f = \sum_{i=0}^{\infty} a_i x^i$, $g = \sum_{i=0}^{\infty} b_i x^i$ and $h = \sum_{i=0}^{\infty} c_i x^i$. Then, $g \oplus h = \sum_{i=0}^{\infty} (b_i + c_i) x^i$

and so $f \odot (g \oplus h) = a_0 \cdot (b_0 + c_0) + [a_0 \cdot (b_1 + c_1) + a_1 \cdot (b_0 + c_0)]x + \ldots + [a_0 \cdot (b_m + c_m) + a_1 \cdot (b_{m-1} + c_{m-1}) + \ldots + a_m \cdot (b_0 + c_0)]x^m + \ldots$. Also, we have $(f \odot g) \oplus (f \odot h) = (a_0 \cdot b_0 + a_0 \cdot c_0) + (a_0 \cdot b_1 + a_1 \cdot b_0 + a_0 \cdot c_1 + a_1 \cdot c_0)x + \ldots + (a_0 \cdot b_m + a_1 \cdot b_{m-1} + \ldots + a_m \cdot b_0 + a_0 \cdot c_m + a_1 \cdot c_{m-1} + \ldots a_m \cdot c_0)x^m + \ldots$. Since R is commutative, we obtain $f \odot (g \oplus h) = (f \odot g) \oplus (f \odot h)$, and in the similar way, we get $(f \oplus g) \odot h = (f \odot h) \oplus (g \odot h)$. Thus, $(R[[x]], \oplus, \odot)$ is a general hyperring. ■

Theorem 5.6.3. *If $g : R_1 \longrightarrow R_2$ is a strong homomorphism, then g induces a strong homomorphism $\widehat{g} : R_1[[x]] \longrightarrow R_2[[x]]$.*

Proof. We define $\widehat{g}\left(\sum\limits_{i=0}^{\infty} a_i x^i\right) = \sum\limits_{i=0}^{\infty} g(a_i)x^i$. Obviously, $\widehat{g}$ is well defined.

Suppose that $f = \sum\limits_{i=0}^{\infty} a_i x^i$, $h = \sum\limits_{i=0}^{\infty} b_i x^i$ are two arbitrary elements of $R_1[[x]]$.

Then, $f \oplus h = \left\{\sum\limits_{i=0}^{\infty} c_i x^i \mid c_i \in a_i + b_i\right\}$, and so

$$\widehat{g}(f \oplus h) = \left\{\sum_{i=0}^{\infty} g(c_i)x^i \mid c_i \in a_i + b_i\right\}.$$

On the other hand $\widehat{g}(f) = \sum\limits_{i=0}^{\infty} g(a_i)x^i$ and $\widehat{g}(h) = \sum\limits_{i=0}^{\infty} g(b_i)x^i$ which imply that

$$\begin{aligned}\widehat{g}(f) \oplus \widehat{g}(h) &= \left\{\sum_{i=0}^{\infty} d_i x^i \mid d_i \in g(a_i) + g(b_i)\right\} = \left\{\sum_{i=0}^{\infty} d_i x^i \mid d_i \in g(a_i + b_i)\right\} \\ &= \left\{\sum_{i=0}^{\infty} g(c_i)x^i \mid c_i \in a_i + b_i\right\}.\end{aligned}$$

Therefore, we obtain $\widehat{g}(f \oplus h) = \widehat{g}(f) \oplus \widehat{g}(h)$.

Also, we have $f \odot h = \left\{\sum\limits_{i=0}^{\infty} c_i x^i \mid c_i \in \sum\limits_{k+l=i} a_k \cdot b_l\right\}$ and so

$$\widehat{g}(f \odot h) = \left\{\sum_{i=0}^{\infty} g(c_i)x^i \mid c_i \in \sum_{k+l=i} a_k \cdot b_l\right\}.$$

On the other hand,

$$\begin{aligned}
&\widehat{g}(f) \odot \widehat{g}(h) \\
&= \left\{ \sum_{i=0}^{\infty} d_i x^i \mid d_i \in \sum_{k+l=i} g(a_k) \cdot g(b_l) \right\} = \left\{ \sum_{i=0}^{\infty} d_i x^i \mid d_i \in \sum_{k+l=i} g(a_k \cdot b_l) \right\} \\
&= \left\{ \sum_{i=0}^{\infty} d_i x^i \mid d_i \in g(\sum_{k+l=i} a_k \cdot b_l) \right\} = \left\{ \sum_{i=0}^{\infty} g(c_i) x^i \mid c_i \in \sum_{k+l=i} a_k \cdot b_l \right\}.
\end{aligned}$$

Therefore, we get $\widehat{g}(f) \odot \widehat{g}(h) = \widehat{g}(f \odot h)$ and the theorem is proved. ■

Let $R[x]$ denote the set of all polynomials $(a_0, a_1, ..., a_n, ...)$ of R such that $a_i = 0$ except a finite number of indices i. Then, we have

Theorem 5.6.4. *$R[x]$ is a subhyperring of $R[[x]]$.*

The hyperring $R[x]$ is called the *hyperring of polynomials* over R. The natural mapping $\psi : R \longrightarrow R[x]$ where $\psi(a) = a$ is a strong homomorphism.

Theorem 5.6.5. *If $\phi : R \longrightarrow R/\gamma^*$ is the canonical map, then the map $\theta : R[x] \longrightarrow (R/\gamma^*)[x]$ defined by $\theta\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} \gamma^*(a_i) x^i$ is a strong homomorphism.*

Proof. Suppose that $f = \sum_{i=0}^{n} a_i x^i$ and $g = \sum_{i=0}^{n} b_i x^i$ are two arbitrary elements of $R[x]$, then

$$\begin{aligned}
\theta(f \oplus g) &= \theta\left(\left\{ \sum_{i=0} c_i x^i \mid c_i \in a_i + b_i \right\}\right) = \left\{ \sum_{i=0}^{n} \gamma^*(c_i) x^i \mid c_i \in a_i + b_i \right\} \\
&= \sum_{i=0}^{n} \gamma^*(c_i) x^i \quad \text{for all} \quad c_i \in a_i + b_i,
\end{aligned}$$

and

$$\begin{aligned}
\theta(f) \oplus' \theta(g) &= \sum_{i=0}^{n} \gamma^*(a_i) x^i \oplus' \sum_{i=0}^{n} \gamma^*(b_i) x^i = \sum_{i=0}^{n} (\gamma^*(a_i) \oplus' \gamma^*(b_i)) x^i \\
&= \sum_{i=0}^{n} \gamma^*(a_i + b_i) x^i = \sum_{i=0}^{n} \gamma^*(c_i) x^i \quad \text{for all} \quad c_i \in a_i + b_i.
\end{aligned}$$

Also, we have

$$\theta(f \odot g) \;= \theta\left(\left\{\sum_i c_i x^i \mid c_i \in \sum_{k+l=i} a_k \cdot b_l\right\}\right) = \sum_i \gamma^*(c_i)x^i$$

$$\text{for all } \; c_i \in \sum_{k+l=i} a_k \cdot b_l$$

and

$$\begin{aligned}
\theta(f) \odot \theta(g) \; &= \sum_i \gamma^*(a_i)x^i \odot' \sum_i \gamma^*(b_i)x^i \\
&= \sum_i A_i x^i, \; A_i = \sum_{k+l=i} (\gamma^*(a_k) \odot' \gamma^*(b_l)) \\
&= \sum_i A_i x^i, \; A_i = \gamma^*\left(\sum_{k+l=i} a_k \cdot b_l\right) \\
&= \sum_i \gamma^*(c_i)x^i \; \text{ for all } \; c_i \in \sum_{k+l=i} a_k \cdot b_l.
\end{aligned}$$

Therefore, θ is a strong homomorphism. ■

Corollary 5.6.6. *The following diagram is commutative, i.e.,* $\theta\psi = \psi'\rho$.

$$\begin{array}{ccc}
R\psi & \longrightarrow & R[x] \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \theta} \\
R/\gamma^* & \xrightarrow{\psi'} & R[x]/\gamma^*
\end{array}$$

Chapter 6

H_v-rings

6.1 H_v-groups

H_v-structures were introduced by Vougiouklis at the Fourth AHA congress (1990)[132]. The concept of a H_v-structure constitutes a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule and so on). Actually some axioms concerning the above hyperstructures such as the associative law, the distributive law and so on are replaced by their corresponding weak axioms. Since the quotients of the H_v-structures with respect to the fundamental equivalence relations (β^*, γ^*, ϵ^*, ets.) are always ordinary structures.

Since then the study of H_v-structure theory has been pursued in many directions by T. Vougiouklis, B. Davvaz, S. Spartalis, A. Dramalidis, Š. Hošková, and others. We invite the reader to consult the references for an in depth exposition of the theory and its applications.

Definition 6.1.1. The hyperstructure $(H, \cdot)$ is called an *H_v-group* if

(1) $x \cdot (y \cdot z) \cap (x \cdot y) \cdot z \neq \emptyset$ for all $x, y, z \in H$,

(2) $a \cdot H = H \cdot a = H$ for all $a \in H$.

A motivation to study the above structures is given by the following examples:

Example 6.1.2.

(1) Let $(G, \cdot)$ be a group and R an equivalence relation on G. In G/R consider the hyperoperation $\odot$ defined by $\overline{x} \odot \overline{y} = \{\overline{z} \mid z \in \overline{x} \cdot \overline{y}\}$, where $\overline{x}$ denotes the equivalence class of the element x. Then $(G, \odot)$ is an H_v-group which is not always a hypergroup.

(2) On the set $\mathbb{Z}_{mn}$ consider the hyperoperation $\oplus$ defined by setting $0 \oplus m = \{0, m\}$ and $x \oplus y = x + y$ for all $(x, y) \in \mathbb{Z}^2_{mn} - \{(0, m)\}$. Then $(\mathbb{Z}_{mn}, \oplus)$ is an H_v-group. $\oplus$ is weak associative but not associative.

(3) Consider the group $(\mathbb{Z}^n, +)$ and take $m_1, ..., m_n \in \mathbb{N}$. We define a hyperoperation $\oplus$ in $\mathbb{Z}^n$ as follows:

$$
\begin{aligned}
(m_1, 0, ..., 0) \oplus (0, 0, ..., 0) &= \{(m_1, 0, ..., 0), (0, 0, ..., 0)\},\\
(0, m_1, ..., 0) \oplus (0, 0, ..., 0) &= \{(0, m_1, ..., 0), (0, 0, ..., 0)\},\\
(0, 0, ..., m_n) \oplus (0, 0, ..., 0) &= \{(0, 0, ..., m_n), (0, 0, ..., 0)\},
\end{aligned}
$$

and $\oplus = +$ in the remaining cases. Then $(\mathbb{Z}^n, \oplus)$ is an H_v-group.

Definition 6.1.3. Let $(H_1, \cdot)$, $(H_2, *)$ be two H_v-groups. A map $f : H_1 \to H_2$ is called an *H_v-homomorphism* or a *weak homomorphism* if

$$f(x \cdot y) \cap f(x) * f(y) \neq \emptyset \quad \text{for all } x, y \in H_1.$$

f is called an *inclusion homomorphism* if

$$f(x \cdot y) \subseteq f(x) * f(y) \quad \text{for all } x, y \in H_1.$$

Finally, f is called a *strong homomorphism* if

$$f(x \cdot y) = f(x) * f(y) \quad \text{for all } x, y \in H_1.$$

If f is onto, one to one and strong homomorphism, then it is called an *isomorphism*. Moreover, if the domain and the range of f are the same H_v-group, then the isomorphism is called an *automorphism*. We can easily verify that the set of all automorphisms of H, defined by $AutH$, is a group.

Several H_v-structures can be defined on a set H. A partial order on these hyperstructures can be introduced, as follows:

Definition 6.1.4. Let $(H,\cdot)$, $(H,*)$ be two H_v-groups defined on the same set H. We say that $(\cdot)$ less than or equal to $(*)$, and we write $\cdot \leq *$, if there is $f \in Aut(H,*)$ such that $x \cdot y \subseteq f(x * y)$ for all $x, y \in H$.

If a hyperoperation is weak associative then every greater hyperoperation, defined on the same set is also weak associative. In [136], the set of all H_v-groups with a scalar unit defined on a set with three elements is determined using this property.

Let $(H,\cdot)$ be an H_v-group. The relation β^* is the smallest equivalence relation on H such that the quotient H/β^* is a group. β^* is called the *fundamental equivalence relation* on H.

If $\mathcal{U}$ denotes the set of all finite products of elements of H, then a relation β can be defined on H whose transitive closure is the fundamental relation β^*. The relation β is defined as follows: for x and y in H we write $x\beta y$ if and only if $\{x,y\} \subseteq u$ for some $u \in \mathcal{U}$. We can rewrite the definition of β^* on H as follows:

> $a\beta^*b$ if and only if there exist $z_1, \ldots, z_{n+1} \in H$ with $z_1 = a$, $z_{n+1} = b$ and $u_1, \ldots, u_n \in \mathcal{U}$ such that $\{z_i, z_{i+1}\} \subseteq u_i \ \ (i = 1, \ldots, n)$.

Suppose that $\beta^*(a)$ is the equivalence class containing $a \in H$. Then the product $\odot$ on H/β^* is defined as follows:

$$\beta^*(a) \odot \beta^*(b) = \{\beta^*(c) |\ c \in \beta^*(a) \cdot \beta^*(b)\} \ \text{ for all } \ a, b \in H.$$

It is not difficult to see that $\beta^*(a) \odot \beta^*(b)$ is the singleton $\{\beta^*(c)\}$ for all $c \in \beta^*(a) \cdot \beta^*(b)$. In this way H/β^* becomes a group.

Now, we define a very large subclass of H_v-structures as follows:

Definition 6.1.5. An H_v-structures H is called an *H_b-structure* if there exists at least a subset A of H, such that the hyperoperations of H are operations on A and A becomes an ordinary structure, endowed with these operations. A will be called a *b-structure* and its operations are called *b-operations*. Obviously, any H_b-structure may contain more than one b-structure.

Examples (2,3) in Example 6.1.2, are H_b-groups.

6.2 H_v-rings and some examples

Definition 6.2.1. A multi-valued system $(R, +, \cdot)$ is an *H_v-ring* if

(1) $(R, +)$ is an H_v-group,

(2) $(R, \cdot)$ is an H_v-semigroup,

(3) $(\cdot)$ is weak distributive with respect to $(+)$, i.e., for all x, y, z in R we have

$$(x \cdot (y + z)) \cap (x \cdot y + x \cdot z) \neq \emptyset \quad \text{and} \quad ((x + y) \cdot z) \cap (x \cdot z + y \cdot z) \neq \emptyset.$$

An H_v-ring may be commutative with respect either to $(+)$ or $(\cdot)$. If H is commutative with respect to both $(+)$ and $(\cdot)$, then we call it a *commutative H_v-ring.* If there exists $u \in R$ such that $x \cdot u = u \cdot x = \{x\}$ for all $x \in R$, then u is called the *scalar unit* of R and it is denoted by 1.

Example 6.2.2. Let $(R, +, \cdot)$ be a ring and $\mu : R \longrightarrow [0, 1]$ be a function. We define the hyperoperations $\uplus, \otimes, *$ on R as follows:

$$\begin{array}{l} x \uplus y = \{t \mid \mu(t) = \mu(x + y)\}, \\ x \otimes y = \{t \mid \mu(t) = \mu(x \cdot y)\}, \\ x * y = y * x = \{t \mid \mu(x) \leq \mu(t) \leq \mu(y)\}, \quad (\text{if } \mu(x) \leq \mu(y)). \end{array}$$

Then $(R, *, *)$, $(R, *, \otimes)$, $(R, *, \uplus)$, $(R, \uplus, *)$, and $(R, \uplus, \otimes)$ are H_v-rings.

Definition 6.2.3. An H_v-ring $(R, +, \cdot)$ is called a *dual H_v-ring* if $(R, \cdot, +)$ is an H_v-ring. If both $(+)$, $(\cdot)$ are weak commutative then R is called a *weak commutative dual H_v-ring.*

Proposition 6.2.4. *If $(H, *)$ is an H_v-group, then for every hyperoperation $(\circ)$ such that $\{x, y\} \subseteq x \circ y$ for all $x, y \in H$, the hyperstructure $(H, *, \circ)$ is a dual H_v-ring.*

Proof. First we prove that $(H, *, \circ)$ is an H_v-ring. For every x, y, z in H, we have

$$\begin{array}{l} \{x\} \cup (y * z) \subseteq x \circ (y * z) \\ (x * x) \cup (x * z) \cup (y * x) \cup (y * z) = \{x, y\} * \{x, z\} \subseteq (x \circ y) * (x \circ z) \end{array}$$

therefore $y * z \subseteq (x \circ (y * z)) \cap ((x \circ y) * (x \circ z)) \neq \emptyset$. Thus the left and similarly the right weak distributivity are valid and the rest axioms can be easily verified.

Now, we prove that $(H, \circ, *)$ is an H_v-ring. For every x, y, z in H, we have

$$(x * y) \cup (x * z) = x * \{y, z\} \subseteq x * (y \circ z)$$
$$(x * y) \cup (x * z) \subseteq (x * y) \circ (x * z)$$

therefore $(x * y) \cup (x * z) \subseteq (x * (y \circ z)) \cap ((x * y) \circ (x * z)) \neq \emptyset$. So, $*$ is a left weak distributive with respect to $\circ$ and the rest axioms are easily verified. ■

Proposition 6.2.5. *Let $(H, +)$ be an H_v-group with a scalar zero element 0. Then for every hyperoperation $\odot$ such that*

$$\{x, y\} \subseteq x \odot y \text{ for all } x, y \text{ in } H \setminus \{0\},\ x \odot 0 = 0 \odot x = 0 \text{ for all } x \text{ in } H,$$

the hyperstructure $(H, +, \odot)$ is an H_v-ring.

Proof. For every nonzero elements x, y, z in H, we have

$$y + z \subseteq (x \odot (y + z)) \cap ((x \odot y) + (x \odot z)) \neq \emptyset.$$

Moreover, if one of the elements x, y, z is zero, then the strong distributivity is valid. The rest of the weak axioms are also valid. ■

Proposition 6.2.6.[46]. *We define the following three hyperoperations on the set $\mathbb{R}^n$, where $\mathbb{R}$ is the set of real numbers:*

$$x \oplus y = \{r(x + y) |\ r \in [0, 1]\},$$
$$x \odot y = \{x + r(y - x) |\ r \in [0, 1]\},$$
$$x \bullet y = \{x + ry |\ r \in [0, 1]\}.$$

Then the hyperstructure $(\mathbb{R}^n, *, \circ)$ is a weak commutative dual H_v-ring where $*, \circ \in \{\oplus, \odot, \bullet\}$.

Proof. The associativity:

1) We have

$$x \oplus (y \oplus z) = \{rx + rmy + rmz \mid r, m \in [0, 1]\}$$
$$(x \oplus y) \oplus z = \{tnx + tny + tz \mid t, n \in [0, 1]\}.$$

If $r = t = 0$ then $\{0\} \subseteq (x \oplus (y \oplus z)) \cap ((x \oplus y) \oplus z)$. If $m = n = 1$ then

$$\begin{aligned}\{r(x+y+z) \mid r \in [0,1]\} &= \{t(x+y+z) \mid t \in [0,1]\} \\ &\subseteq (x \oplus (y \oplus z)) \cap ((x \oplus y) \oplus z).\end{aligned}$$

We claim that for all $m, n \in [0, 1)$ and for all $r, t \in (0, 1]$ the following assertion is valid:

$$x \oplus (y \oplus z) \neq (x \oplus y) \oplus z.$$

Indeed, if there exist $m, n \in [0, 1)$ and $r, t \in (0, 1]$ such that we have the equality in the above condition to be hold, then $r = tn$, $rm = tn$, $rm = t$ which imply $rm = r$, $tn = t$. So $m = n = 1$, which is a contradiction.

(2) We have

$$\begin{aligned}x \odot (y \odot z) &= \{(1-r)x + r(1-m)y + rmz \mid r, m \in [0,1]\} \\ (x \odot y) \odot z &= \{(1-t)(1-n)x + t(1-n)y + nz \mid t, n \in [0,1]\}\end{aligned}$$

We claim that the above two sets are equal. Let $r, m \in [0, 1]$. Then we have

$$(1-n)(1-t) = 1-r, \; t(1-n) = r(1-m), \; n = rm$$

if and only if

$$(1-rm)(1-t) = 1-r, \; t(1-rm) = r(1-m), \; n = rm.$$

It is obvious that $n = rm \in [0, 1]$. Now, we shall prove that $t \in [0, 1]$. If $rm = 1$ then $r = m = 1$, so we obtain $n = 1$ and $0t = 0$ which is valid for all $t \in [0, 1]$. If $rm \neq 1$ then we have $t = \dfrac{r(1-m)}{1-rm}$ and $r \neq 1$ or $m \neq 1$. Let $m \neq 1$, then $0 \leq rm < 1$, so we have $1 - rm > 0$. Now, from $r \leq 1$ we obtain

$$r - rm \leq 1 - rm \iff \frac{r(1-m)}{1-rm} \leq 1 \iff t \leq 1.$$

Obviously, $r(1-m) \geq 0$, so $t \geq 0$.

Let $n, t \in [0, 1]$. Using the same technique, we can easily show that $r, m \in [0, 1]$.

(3) We have

$$x \bullet (y \bullet z) = \{x + ry + rmz \mid r, m \in [0,1]\}$$
$$(x \bullet y) \bullet z = \{x + ny + tz \mid t, n \in [0,1]\}$$

By setting $r = n$ and $rm = t$ we obtain $x \bullet (y \bullet z) \subseteq (x \bullet y) \bullet z$.

The commutativity: For the first two hyperoperations we have:

$$x \oplus y = [0, x+y] = [0, y+x] = y \oplus x \quad \text{and} \quad x \odot y = [x,y] = [y,x] = y \odot x.$$

For the third one, we have

$$x \bullet y = \{x + ry \mid r \in [0,1]\} \quad \text{and} \quad y \bullet x = \{mx + y \mid m \in [0,1]\}.$$

The above two sets have common elements for all $x, y \in \mathbb{R}^n$, only in the case $r = m = 1$.

The reproduction axiom: We can see easily that, for all $x \in \mathbb{R}^n$

$$x \oplus \mathbb{R}^n = \mathbb{R}^n \oplus x = \mathbb{R}^n, \quad x \odot \mathbb{R}^n = \mathbb{R}^n \odot x = \mathbb{R}^n, \quad x \bullet \mathbb{R}^n = \mathbb{R}^n \bullet x = \mathbb{R}^n.$$

The distributivity: The following assertions hold for all $x, y, z \in \mathbb{R}^n$.

$$(x \oplus (y \oplus z)) \cap ((x \oplus y) \oplus (x \oplus z)) \neq \emptyset,$$
$$((x \oplus y) \oplus z) \cap ((x \oplus z) \oplus (y \oplus z)) \neq \emptyset,$$
$$x \oplus (y \odot z) = (x \oplus y) \odot (x \oplus z),$$
$$(x \odot y) \oplus z = (x \odot z) \oplus (y \odot z),$$
$$(x \oplus (y \bullet z)) \cap ((x \oplus y) \bullet (x \oplus z)) \neq \emptyset,$$
$$((x \bullet y) \oplus z) \cap ((x \oplus z) \bullet (y \oplus z)) \neq \emptyset,$$
$$(x \odot (y \odot z) = (x \odot y) \odot (x \odot z),$$
$$(x \odot y) \odot z) = (x \odot z) \odot (y \odot z),$$
$$((x \odot (y \oplus z)) \cap ((x \odot y) \oplus (x \odot z)) \neq \emptyset,$$
$$((x \oplus y) \odot z) \cap ((x \odot z) \oplus (y \odot z)) \neq \emptyset,$$
$$((x \odot (y \bullet z)) \cap ((x \odot y) \bullet (x \odot z)) \neq \emptyset,$$
$$((x \bullet y) \odot z) \cap ((x \odot z) \bullet (y \odot z)) \neq \emptyset,$$
$$((x \bullet (y \bullet z)) \cap ((x \bullet y) \bullet (x \bullet z)) \neq \emptyset,$$
$$((x \bullet y) \bullet z) \cap ((x \bullet z) \bullet (y \bullet z)) \neq \emptyset,$$
$$x \bullet (y \odot z) = (x \bullet y) \odot (x \bullet z),$$
$$(x \odot y) \bullet z = (x \bullet z) \odot (y \bullet z),$$
$$(x \bullet (y \oplus z)) \cap ((x \bullet y) \oplus (x \bullet z)) \neq \emptyset,$$
$$((x \oplus y) \bullet z) \cap ((x \bullet z) \oplus (y \bullet z)) \neq \emptyset.$$

Among the 9 cases, we shall prove here the last one. The rest of them can be proved in a similar way

$$x \bullet (y \oplus z) = \{x + mty + mtz \mid m, t \in [0,1]\},$$
$$(x \bullet z) \oplus (y \bullet z) = \{2rx + rny + rkz \mid r, n, k \in [0,1]\}.$$

If $m = n = k = 0$ and $r = \frac{1}{2}$ then $\{x\} \subseteq (x \bullet (y \oplus z)) \cap ((x \bullet y) \oplus (x \bullet z))$. Also, $((x \oplus y) \bullet z) \cap ((x \bullet z) \oplus (y \bullet z)) \neq \emptyset$, for all $x, y, z \in \mathbb{R}^n$. ■

Now, we present some general constructions which can be useful in the theory of representations of several classes of H_v-groups.

Let $(H, \circ)$ be a hypergroupoid; by $\triangle_H$ we mean the diagonal of the Cartesian product $H \times H$, i.e., $\triangle_H = \{[x,x] \mid x \in H\}$. Let us define a mapping $D : H \longrightarrow H \times H$ by $D(x) = [x,x]$ for all $x \in H$, i.e., $\triangle_H = D(H)$.

Lemma 6.2.7. *Let $(H, \circ)$ be a hypergroupoid. Define a hyperoperation $*$ on the diagonal $\triangle_H$ as follows:*

$$[x,x] * [y,y] = D(x \circ y \cup y \circ x) = \{[u,u] \mid u \in x \circ y \cup y \circ x\}$$

for any pair $[x,x], [y,y] \in \triangle_H$. Then the following assertions hold:

(1) *For any hypergroupoid $(H, \circ)$ we have that $(\triangle_H, *)$ is a commutative hypergroupoid.*

(2) *If $(H, \circ)$ is a weakly associative hypergroupoid, then the hypergroupoid $(\triangle_H, *)$ is weakly associative as well.*

(3) *If $(H, \circ)$ is a quasihypergroup, the the hypergroupoid $(\triangle_H, *)$ satisfies also the reproduction axiom, i.e., it is a quasihypergroup.*

(4) *If $(H, \circ)$ is associative, then the hypergroupoid $(\triangle_H, *)$ is weakly associative (but not associative in general).*

Proof. The assertion (1) follows immediately from the above definition of the hyperoperation $*$.

(2) Suppose that $[x,x],[y,y],[z,z] \in \triangle_H$. Then

$$
\begin{aligned}
&([x,x]*[y,y])*[z,z]\\
&= D(x\circ y \cup y\circ x)*[z,z] = (D(x\circ y)\cup D(y\circ x))*[z,z]\\
&= (D(x\circ y)*[z,z])\cup(D(y\circ x)*[z,z])\\
&= \left(\bigcup_{u\in x\circ y}[u,u]*[z,z]\right)\cup\left(\bigcup_{v\in y\circ x}[v,v]*[z,z]\right)\\
&= \left(\bigcup_{u\in x\circ y}D(u\circ z\cup z\circ u)\right)\cup\left(\bigcup_{v\in y\circ x}D(v\circ z\cup z\circ v)\right)\\
&= \left(\bigcup_{u\in x\circ y}D(u\circ z)\right)\cup\left(\bigcup_{u\in x\circ y}D(z\circ u)\right)\cup\left(\bigcup_{v\in y\circ x}D(v\circ z)\right)\cup\left(\bigcup_{v\in y\circ x}D(z\circ u)\right)\\
&= D\left(\bigcup_{u\in x\circ y}u\circ z\right)\cup D\left(\bigcup_{u\in x\circ y}z\circ u\right)\cup D\left(\bigcup_{v\in y\circ x}v\circ z\right)\cup D\left(\bigcup_{v\in y\circ x}z\circ u\right)\\
&= D(x\circ y\circ z)\cup D(z\circ x\circ y)\cup D(y\circ x\circ z)\cup D(z\circ y\circ x)\\
&= D(x\circ y\circ z\cup z\circ y\circ x)\cup D(z\circ x\circ y\cup y\circ x\circ z).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
[x,x]*([y,y]*[z,z]) &= ([z,z]*[y,y])*[x,x]\\
&= D(z\circ y\circ x\cup x\circ z\circ y\cup y\circ z\circ x\cup x\circ y\circ z)\\
&= D(x\circ y\circ z\cup z\circ y\circ x)\cup D(x\circ z\circ y\cup y\circ z\circ x).
\end{aligned}
$$

Thus $([x,x]*[y,y])*[z,z]\cap[x,x]*([y,y]*[z,z]) \supseteq D(x\circ y\circ z)\cup D(z\circ y\circ x) \neq \emptyset$.

(3) Let $x \in H$ be an arbitrary element. Then $x\circ H = H = H\circ x$ and we have

$$
\begin{aligned}
&[x,x]*\triangle_H = \bigcup_{y\in H}([x,x]*[y,y]) = \bigcup_{y\in H}D(x\circ y\cup y\circ x)\\
&= \left(\bigcup_{y\in H}D(x\circ y)\right)\cup\left(\bigcup_{y\in H}D(y\circ x)\right) = D\left(\bigcup_{y\in H}x\circ y\right)\cup D\left(\bigcup_{y\in H}y\circ x\right)\\
&= D(x\circ H)\cup D(H\circ x) = D(H) = \triangle_H.
\end{aligned}
$$

Since a semihypergroup is also weakly associative, the assertion (4) follows from (2). ■

Let $(R, +, \cdot)$ be an H_v-ring. We define the hyperoperations $\oplus$ and $\odot$ on the diagonal $D(R) = \triangle_R$ by

$$[x,x] \oplus [y,y] = \{[u,u] \mid u \in (x+y) \cup (y+x)\},$$
$$[x,x] \odot [y,y] = \{[v,v] \mid v \in (x \cdot y) \cup (y \cdot x)\}$$

for all $x, y \in R$. Then we have:

Proposition 6.2.8. *Let $(R, +, \cdot)$ be an H_v-ring. Then $(D(R), \oplus, \odot)$ is a commutative H_v-ring.*

Proof. According to Lemma 6.2.7, we obtain that $(D(R), \oplus)$ is a commutative weakly associative hypergroupoid satisfying the reproduction axiom, thus it is a commutative H_v-group. Similarly, $(D(R), \odot)$ is a commutative H_v-semigroup. Thus it remains to prove that

$$[x,x] \odot ([y,y] \oplus [z,z]) \cap ([x,x] \odot [y,y]) \oplus ([x,x] \odot [z,z]) \neq \emptyset$$

for arbitrary elements $x, y, z \in R$.

Indeed, we have $[y,y] \oplus [z,z] = \{[u,u] \mid u \in (y+z) \cup (z+y)\}$ and

$$[x,x] \odot ([y,y] \oplus [z,z]) = \bigcup_{u \in (y+z) \cup (z+y)} [x,x] \odot [u,u]$$
$$= \left(\bigcup_{u \in y+z} [x,x] \odot [u,u] \right) \cup \left(\bigcup_{u \in z+y} [x,x] \odot [u,u] \right)$$
$$= \left(\bigcup_{u \in y+z} \{[v,v] \mid v \in x \cdot u \cup u \cdot x\} \right) \cup \left(\bigcup_{u \in z+y} \{[v,v] \mid v \in x \cdot u \cup u \cdot x\} \right)$$
$$= \{[v,v] \mid v \in x \cdot (y+x)\} \cup M(x,y,z),$$

where $M(x,y,z) = \bigcup_{u \in y+z} \{[v,v] \mid v \in u \cdot x\} \cup \bigcup_{u \in z+y} \{[v,v] \mid v \in x \cdot u \cup u \cdot x\}$.

On the other hand,

$$[x,x] \odot [y,y] = \{[v,v] \mid v \in x \cdot y \cup y \cdot x\}$$
$$= \{[v,v] \mid v \in x \cdot y\} \cup \{[v,v] \mid v \in y \cdot x\},$$
$$[x,x] \odot [z,z] = \{[v,v] \mid v \in x \cdot z\} \cup \{[v,v] \mid v \in z \cdot x\}$$

and then

$$\begin{aligned}
&([x,x]\odot[y,y])\oplus([x,x]\odot[z,z])\\
&=(\{[v,v]|v\in x\cdot y\}\cup\{[v,v]|v\in x\cdot z\})\oplus(\{[v,v]|v\in z\cdot x\}\cup\{[v,v]|v\in y\cdot x\})\\
&=(\{[v,v]|v\in x\cdot y\}\oplus\{[v,v]|v\in x\cdot z\})\cup(\{[v,v]|v\in x\cdot y\}\oplus\{[v,v]|v\in z\cdot x\})\\
&\cup(\{[v,v]|v\in y\cdot x\}\oplus\{[v,v]|v\in x\cdot z\})\oplus(\{[v,v]|v\in y\cdot x\}\cup\{[v,v]|v\in z\cdot x\})\\
&=\left(\bigcup_{\substack{v\in x\cdot y\\ u\in x\cdot z}}[v,v]\oplus[u,u]\right)\cup\left(\bigcup_{\substack{v\in x\cdot y\\ u\in z\cdot x}}[v,v]\oplus[u,u]\right)\\
&\cup\left(\bigcup_{\substack{v\in y\cdot x\\ u\in x\cdot z}}[v,v]\oplus[u,u]\right)\cup\left(\bigcup_{\substack{v\in y\cdot x\\ u\in z\cdot x}}[v,v]\oplus[u,u]\right)\\
&=\bigcup_{\substack{v\in y\cdot x\\ u\in x\cdot z}}\{[t,t]\mid t\in(v+u)\cup(u+v)\}\cup K(x,y,z),
\end{aligned}$$

where

$$K(x,y,z)=$$

$$\left(\bigcup_{\substack{v\in x\cdot y\\ u\in z\cdot x}}[v,v]\oplus[u,u]\right)\cup\left(\bigcup_{\substack{v\in y\cdot x\\ u\in x\cdot z}}[v,v]\oplus[u,u]\right)\cup\left(\bigcup_{\substack{v\in y\cdot x\\ u\in z\cdot x}}[v,v]\oplus[u,u]\right).$$

Now, we have

$$\begin{aligned}
&([x,x]\odot[y,y])\oplus([x,x]\odot[z,z])\\
&=\left(\bigcup_{\substack{v\in x\cdot y\\ u\in x\cdot z}}[t,t]\mid t\in u+v\}\right)\cup\left(\bigcup_{\substack{v\in x\cdot y\\ u\in x\cdot z}}[t,t]\mid t\in u+v\}\right)\cup K(x,y,z).
\end{aligned}$$

From $(x\cdot y+x\cdot z)\cap x\cdot(y+x)\neq\emptyset$, it follows that $[t_0,t_0]\in\{[v,v]\mid v\in x\cdot(y+z)\}$ for some $t_0\in x\cdot y+x\cdot z$, thus

$$\{[v,v]\mid v\in x\cdot(y+z)\}\cap\{[t,t]\mid t\in x\cdot y+x\cdot z\}\neq\emptyset,$$

consequently the sets $[x,x]\odot([y,y]\oplus[z,z])$ and $([x,x]\odot[y,y])\oplus([x,x]\odot[z,z])$ have a nonempty intersection. ■

From the above proof, it follows that only one (right or left) of the weak distributivity laws for $(R,+,\cdot)$ ensures the weak distributivity of $(D(R),\oplus,\odot)$.

Definition 6.2.9. Let R_1, R_2 be two H_v-rings. The map $f : R_1 \longrightarrow R_2$ is called an *H_v-homomorphism* or a *weak homomorphism* if, for all $x, y \in R_1$ the following conditions hold:

$$f(x+y)\cap(f(x)+f(y))\neq\emptyset \text{ and } f(x\cdot y)\cap f(x)\cdot f(y)\neq\emptyset.$$

f is called an *inclusion homomorphism* if, for all $x, y \in R$, the following relations hold:

$$f(x+y)\subseteq f(x)+f(y) \text{ and } f(x\cdot y)\subseteq f(x)\cdot f(y).$$

Finally, f is called a *strong homomorphism* if for all x, y in R_1 we have

$$f(x+y)=f(x)+f(y) \text{ and } f(x\cdot y)=f(x)\cdot f(y).$$

If R_1 and R_2 are H_v-rings and there exists a strong one to one and onto homomorphism from R_1 to R_2, then R_1 and R_2 are called *isomorphic.*

Corollary 6.2.10. *Let $(R,+,\cdot)$ be an H_v-ring and $r_R(x)=[x,x]\in D(R)$ for any $x\in R$. Then the mapping $r_R : (R,+,\cdot)\longrightarrow(D(R),\oplus,\odot)$ is an incusion homomorphism of H_v-rings.*

Theorem 6.2.11. *For any pair of H_v-rings $(R,+,\cdot)$, $(S,+,\cdot)$ and for any inclusion H_v-ring homomorphism $f:(R,+,\cdot)\longrightarrow(S,+,\cdot)$ there exists exactly one inclusion homomorphism $\psi:(D(R),\oplus,\odot)\longrightarrow(D(S),\oplus,\odot)$ such that the diagram*

$$\begin{array}{ccc}
(R,+,\cdot) & \xrightarrow{\;f\;} & (S,+,\cdot) \\
{\scriptstyle r_R}\downarrow & & \downarrow{\scriptstyle r_S} \\
(D(R),\oplus,\odot) & \xrightarrow{\;\psi\;} & (D(S),\oplus,\odot)
\end{array}$$

is commutative.

Proof. Consider an arbitrary inclusion homomorphism $f : R \longrightarrow S$ and define $\psi : D(R) \longrightarrow D(S)$ as the restriction of the mapping $f \times f : R \times R \longrightarrow S \times S$ onto $D(R) \subseteq R \times R$, i.e., $\psi = (f \times f)|_{D(R)}$, hence $\psi([x,x]) = [f(x), f(x)]$ for any $x \in R$. Now, we have

$$\begin{aligned}\psi([x,x] \oplus [y,y]) &= \psi(\{[u,u] \mid u \in (x+y) \cup (y+x)\} \\ &= \{[f(u), f(u)] \mid u \in (x+y) \cup (y+x)\} \\ &= \{[v,v] \mid v \in f(x+y) \cup f(y+x)\} \\ &\subseteq \{[v,v] \mid v \in (f(x)+f(y)) \cup (f(y)+f(x))\} \\ &= [f(x)+f(y)] \oplus [f(y)+f(x)] \\ &= \psi([x,x]) \oplus \psi([y,y])\end{aligned}$$

for any elements $x, y \in R$ and similarly $\psi([x,x] \odot [y,y]) \subseteq \psi([x,x]) \cdot \psi([y,y])$, which is obtained as above. . Now, we show that the above diagram commutes.

Let us suppose that $f : R \longrightarrow S$ is an inclusion homomorphism. Then evidently $\psi : D(R) \longrightarrow D(S)$ is an inclusion homomorphism as well. For an arbitrary x in R, we have

$$\begin{aligned}(r_S \circ f)(x) &= r_S(f(x)) = [f(x), f(x)] = (f \times f)(x,x) \\ &= \psi([x,x]) = \psi(r_R(x)) = (\psi \circ r_R)(x),\end{aligned}$$

and so $r_S \circ f = \psi \circ r_R$. Now, let $g : D(R) \longrightarrow D(S)$ be an inclusion homomorphism such that $r_S \circ f = g \circ r_R$. Since $r_R : R \longrightarrow D(R)$ and $r_S : S \longrightarrow D(S)$ are bijections, there exist the maps $r_R^{-1} : D(R) \longrightarrow R$ and $r_S^{-1} : D(S) \longrightarrow S$. Then, we obtain

$$\psi = \psi \circ id_{D(R)} = \psi \circ r_R \circ r_R^{-1} = r_S \circ f \circ r_R^{-1} = g \circ r_R \circ r_R^{-1} = g \circ id_{D(R)} = g. \blacksquare$$

From the above results we obtain the following theorem.

Theorem 6.2.12 *Let $\mathcal{H}_v\mathcal{R}$ be the category of all H_v-rings and their inclusion homomorphisms and $\mathcal{AH}_v\mathcal{R}$ be its full subcategory of all commutative H_v-rings. Then there exists the functor $\phi : \mathcal{H}_v\mathcal{R} \longrightarrow \mathcal{AH}_v\mathcal{R}$ defined by*

$$\phi(R,+,\cdot) = (D(R), \oplus, \odot), \quad \phi(f) = \psi \text{ for any } (R,+,\cdot) \in Ob(\mathcal{H}_v\mathcal{R}),$$

and any morphism $f \in Mor(\mathcal{H}_v\mathcal{R})$, $f : (R,+,\cdot) \longrightarrow (S,+,\cdot)$ is a reflector; more precisely the pair $(r_R, (\triangle_R, \oplus, \odot))$ is an $\mathcal{AH}_v\mathcal{R}$-reflection for any

$(R,+,\cdot) \in Ob(\mathcal{H}_v\mathcal{R})$. *Thus* $\mathcal{AH}_v\mathcal{R}$ *is a reflective full subcategory of the category* $\mathcal{H}_v\mathcal{R}$.

6.3 Fundamental relations in H_v-rings

In what follows, we focus our attention on the β^* and γ^* relations defined on H_v-rings. Notice that two kinds of β^* relations can be defined on H_v-rings. We denote them by β^*_+ and $\beta^*_\cdot$. They are β^* relations with respect to addition and multiplication, respectively. If $(R,+,\cdot)$ is an H_v-ring, then the relations β_+ and $\beta_\cdot$ are defined as follows:

$x\beta_+y$ if and only if there exist $z_1,...,z_n \in R$ such that $\{x,y\} \subseteq z_1+...+z_n$,
$x\beta_\cdot y$ if and only if there exist $z_1,...,z_n \in R$ such that $\{x,y\} \subseteq z_1 \cdot ... \cdot z_n$.

β^*_+ and $\beta^*_\cdot$ are the transitive closures of the relations β_+ and $\beta_\cdot$ Note that the quotient hyperstructures with respect to β^*_+ and $\beta^*_\cdot$ are H_v-rings. In this section, the fundamental relations defined on an H_v-ring are studied. Especially, some connections among different types of fundamental relations are obtained.

Definition 6.3.1. Let $(R,+,\cdot)$ be an H_v-ring. We define γ^* as the smallest equivalence relation such that the quotient R/γ^* is a ring. γ^* is called the *fundamental equivalence relation* and R/γ^* is called the *fundamental ring*. An H_v-ring is called an *H_v-field* if its fundamental ring is a field.

Let us denote the set of all finite polynomials of elements of R over $\mathbb{N}$ by $\mathcal{U}$. We define the relation γ as follows:

$$x\gamma y \text{ if and only if } \{x,y\} \subseteq u, \text{ where } u \in \mathcal{U}.$$

Theorem 6.3.2. *The fundamental equivalence relation* γ^* *is the transitive closure of the relation* γ.

Proof. Let $\widehat{\gamma}$ be the transitive closure of the relation γ. We denote the equivalence class of a by $\widehat{\gamma}(a)$. First, we prove that the quotient set $R/\widehat{\gamma}$ is a ring. The sum $\oplus$ and the product $\odot$ are defined in $R/\widehat{\gamma}$ in the usual manner:

$$\widehat{\gamma}(a) \oplus \widehat{\gamma}(b) = \{\widehat{\gamma}(c) \mid c \in \widehat{\gamma}(a) + \widehat{\gamma}(b)\},$$
$$\widehat{\gamma}(a) \odot \widehat{\gamma}(b) = \{\widehat{\gamma}(d) \mid d \in \widehat{\gamma}(a) \cdot \widehat{\gamma}(b)\}.$$

Take $a' \in \widehat{\gamma}(a)$ and $b' \in \widehat{\gamma}(b)$. Then, we have $a'\widehat{\gamma}a$ if and only if there exist $x_1, ..., x_{m+1}$ with $x_1{=}a'$, $x_{m+1}{=}a$ and $u_1, ..., u_m{\in}\mathcal{U}$ such that $\{x_i, x_{i+1}\} \subseteq u_i$ $(i = 1, ..., m)$, and $b'\widehat{\gamma}b$ if and only if there exist $y_1, ..., y_{n+1}$ with $y_1 = b'$, $y_{n+1} = b$ and $v_1, ..., v_n \in \mathcal{U}$ such that $\{y_j, y_{j+1}\} \subseteq v_j$ $(j = 1, ..., n)$. Now, we obtain

$$\{x_i, x_{i+1}\} + y_1 \subseteq u_i + v_1 \quad (i = 1, ..., m-1),$$
$$x_{m+1} + \{y_j, y_{j+1}\} \subseteq u_m + v_j \quad (j = 1, ..., n).$$

The sums $u_i{+}v_1{=}t_i$ $(i{=}1, ..., m{-}1)$ and $u_m{+}v_j{=}t_{m+j-1}$ $(j{=}1, ..., n)$ are polynomials and so $t_k \in \mathcal{U}$ for all $k \in \{1, ..., m{+}n{-}1\}$. Now, pick up the elements $z_1, ..., z_{m+n}$ such that $z_i \in x_i + y_1$ $(i = 1, ..., m)$ and $z_{m+j} \in x_{m+1} + y_{j+1}$ $(j = 1, ..., n)$. Hence, we obtain $\{z_k, z_{k+1}\} \subseteq t_k$ $(k = 1, ..., m + n - 1)$. Therefore, every element $z_1 \in x_1 + y_1 = a' + b'$ is $\widehat{\gamma}$ equivalent to every element $z_{m+n} \in x_{m+1} + y_{n+1} = a + b$. Thus, $\widehat{\gamma}(a) \oplus \widehat{\gamma}(b) = \widehat{\gamma}(c)$ for all $c \in \widehat{\gamma}(a) + \widehat{\gamma}(b)$. In a similar way, it is proved that $\widehat{\gamma}(a) \odot \widehat{\gamma}(b) = \widehat{\gamma}(d)$ for all $d \in \widehat{\gamma}(a) \cdot \widehat{\gamma}(b)$.

The weak associativity and the weak distributivity on R guarantee that the associativity and distributivity are valid in the quotient $R/\widehat{\gamma}$. Therefore, $R/\widehat{\gamma}$ is a ring.

Now, let σ be an equivalence relation on R such that R/σ is a ring. Denote the equivalence class of a by $\sigma(a)$. Then, $\sigma(a) \oplus \sigma(b)$ and $\sigma(a) \odot \sigma(b)$ are singletons for all $a, b \in R$, i.e., $\sigma(a) \oplus \sigma(b) = \sigma(c)$ for all $c \in \sigma(a) + \sigma(b)$ and $\sigma(a) \odot \sigma(b) = \sigma(d)$ for all $d \in \sigma(a) \cdot \sigma(b)$. Thus, for every $a, b \in R$ and $A \subseteq \sigma(a)$, $B \subseteq \sigma(b)$ we can write

$$\sigma(a) \oplus \sigma(b) = \sigma(a + b) = \sigma(A + B) \text{ and } \sigma(a) \odot \sigma(b) = \sigma(a \cdot b) = \sigma(A \cdot B).$$

By induction, we extend these equalities on finite sums and products. So, for every $u \in \mathcal{U}$ and for all $x \in u$ we have $\sigma(x) = \sigma(u)$. Therefore, for every $a \in R$,

$$x \in \widehat{\gamma}(a) \text{ implies } x \in \sigma(a).$$

Since σ is transitive, we obtain that

$$x \in \gamma(a) \text{ implies } x \in \sigma(a).$$

This means that the relation $\widehat{\gamma}$ is the smallest equivalence relation on R such that $R/\widehat{\gamma}$ is a ring, i.e., $\widehat{\gamma} = \gamma^*$. ■

Definition 6.3.3. We define the γ_1^*, γ_2^* relations as the transitive closures of the relations γ_1, γ_2 respectively, which are defined as follows:
$x\gamma_1 y$ if and only if there exist $a_i \in R$ and I_k, K finite sets of indices such that

$$\{x,y\} \subseteq \sum_{k\in K}\left(\prod_{i\in I_k} a_i\right)$$

and $x\gamma_2 y$ if and only if there exist $b_j \in R$ and J_s, S finite sets of indices such that

$$\{x,y\} \subseteq \prod_{s\in S}\left(\sum_{j\in J_s} b_j\right).$$

In a multiplicative H_v-ring, the addition is an operation, while in an additive H_v-ring, the multiplication is an operation.

Proposition 6.3.4.

(1) *R/γ_1^* is a multiplicative H_v-ring.*

(2) *R/γ_2^* is an additive H_v-ring.*

Proof. We prove only (1) and similarly (2) can be proved. The sum of the classes is

$$\gamma_1^*(x) \oplus \gamma_1^*(y) = \{\gamma_1^*(z) \mid z \in \gamma_1^*(x) + \gamma_1^*(y)\}.$$

In the definition of γ_1^*, expressions of the type $v = \sum(\prod)$ are used. In the definition of $\oplus$, the element z belongs to the sums v of the above type, which means that z belongs to a sum of products. In other words, all the elements z are in the same γ_1^* class. So, the sum of γ_1^*-classes is a singleton. Therefore, R/γ_1^* is a multiplicative H_v-ring. ■

Note that the γ_1^* classes are greater than the β^* classes. Actually, the γ_1^* is not the smallest equivalence relation such that $(R/\gamma_1^*, \oplus)$ is a group. In order to see this, consider a multiplicative H_v-ring R. Then $R/\beta_+^* \cong R$, but R/γ_1^* is not isomorphic to R.

Proposition 6.3.5. *For all additive H_v-rings, we have $\gamma_1^* = \beta_+^*$. For all multiplicative H_v-rings, we have $\gamma_2^* = \beta^*$.*

Proof. We present the proof for a multiplicative H_v-ring R. In this case, every sum of elements of R is singleton. Therefore,

$$\prod_{s\in S}\left(\sum_{j\in J_s} b_j\right) = \prod_{s\in S} d_s, \quad \text{where } d_s = \sum_{j\in J_s} b_j.$$

This means that $x\gamma_2^* y$ if and only if $x\beta_\cdot^* y$. ■

Using the above propositions, it follows that $(R/\gamma_1^*)/\gamma_2^* = (R/\gamma_1^*)/\beta_\cdot^*$ is a multiplication H_v-ring and $(R/\gamma_2^*)/\gamma_1^* = (R/\gamma_2^*)/\beta_+^*$ is an additive H_v-ring.

Theorem 6.3.6. *Let $(R,+,\cdot)$ be an H_v-ring. Then $R/\gamma^* \cong (R/\beta_\cdot^*)/\beta_\uplus^*$, where $\beta_\uplus^*$ is the fundamental relation defined in $(R/\beta_\cdot^*,\uplus)$ by setting $\beta_\cdot^*(a) \uplus \beta_\cdot^*(b) = \{\beta_\cdot^*(c) \mid c \in \beta_\cdot^*(a) + \beta_\cdot^*(b)\}$.*

Proof. The quotient of the additive H_v-ring $(R/\beta_\cdot^*, \uplus, \otimes)$ with respect to $\beta_\uplus^*$ is a ring. Let us denote the equivalence relation associated to the projection $\psi : R \longrightarrow (R/\beta_\cdot^*)/\beta_\uplus^*$ by σ. Since ψ is a ring homomorphism, then we obtain $\gamma^*(a) \subseteq \sigma(a)$ for all $a \in R$. On the other hand, since $\beta_\cdot^*(x) \subseteq \gamma^*(x)$ for all $x \in R$, we have

$$\bigcup_{\beta_\cdot^*(z)\in\beta_\cdot^*(x)\uplus\beta_\cdot^*(y)} \beta_\cdot^*(z) = \bigcup_{z\in\beta_\cdot^*(x)+\beta_\cdot^*(y)} \beta_\cdot^*(z) \subseteq \bigcup_{z\in\gamma^*(x)+\gamma^*(y)} \gamma^*(z).$$

From the fundamental property in $(R/\gamma^*, \oplus, \odot)$, we know that $\gamma^*(x)\oplus\gamma^*(y)$ is a singleton, so $\gamma^*(x) \oplus \gamma^*(y) = \gamma^*(w)$, where $w \in x + y$. Therefore,

$$\bigcup_{\beta_\cdot^*(z)\in\beta_\cdot^*(x)\uplus\beta_\cdot^*(y)} \beta_\cdot^*(z) \subseteq \gamma^*(w), \text{ where } w \in x + y.$$

Consequently, for every finite sum of elements in $R/\beta_\cdot^*$, we have

$$\bigcup_{z\in\uplus\sum_{i\in I}\beta_\cdot^*(x_i)} \beta_\cdot^*(z) \subseteq \gamma^*(w), \text{ where } w \in \sum_{i\in I} x_i.$$

Moreover, since γ^* is transitive, we have

$$\sigma(a) = \bigcup_{\{z|(\beta_\cdot^*(z))\ \beta_\uplus^*\ (\beta_\cdot^*(a))\}} \beta_\cdot^*(z) \subseteq \gamma^*(a) \text{ for all } a \in R.$$

Therefore, $\sigma = \gamma^*$. ■

Theorem 6.3.7. *If $(H, \Diamond)$ is an H_v-group, then for every hyperoperation $\triangledown$ such that $\{x, y\} \subseteq x \triangledown y$ for all $x, y \in H$, the hyperstructures $(H, \Diamond, \triangledown)$ and $(H, \triangledown, \Diamond)$ are H_v-rings.*

Proof. Every hyperoperation $\triangledown$ that satisfies the condition of hypothesis is weak associative, weak commutative and H/γ^* is a singleton. Moreover, every element $x \in H$ is a unit element, i.e., $y \in x \triangledown y \cap y \triangledown x$ for all $y \in H$, and every element $x \in H$ is symmetric with respect to the unit x, i.e., $x \in x \triangledown y \cap y \triangledown x$.

In order to prove that $(H, \Diamond, \triangledown)$ is an H_v-ring we need only to prove the weak distributivity on the left. For every x, y, z in H we have

$$x \triangledown (y \Diamond z) \supseteq \{x\} \cup (y \Diamond z)$$

and

$$(x \triangledown y) \Diamond (x \triangledown z) \supseteq \{x, y\} \Diamond \{x, z\} = (x \Diamond x) \cup (x \Diamond z) \cup (y \Diamond x) \cup (y \Diamond z),$$

therefore, $y \Diamond z \subseteq [x \triangledown (y \Diamond z)] \cap [(x \triangledown y) \Diamond (x \triangledown z)] \neq \emptyset$. Thus, the left and similarly the right weak distributivity are valid.

Similarly, we need to prove the weak distributivity on the left for $(H, \triangledown, \Diamond)$. For every x, y, x in H we have

$$x \Diamond (y \triangledown z) \supseteq x \Diamond \{y, z\} = (x \Diamond y) \cup (x \Diamond z)$$

and

$$(x \Diamond y) \triangledown (x \Diamond z) \supseteq (x \Diamond y) \cup (x \Diamond z).$$

So the left distributivity is valid, because

$$(x \Diamond y) \cup (x \Diamond z) \subseteq [x \Diamond (y \triangledown z)] \cap [(x \Diamond y) \triangledown (x \Diamond z)] \neq \emptyset.$$

H_v-rings $(H, \triangledown, \Diamond)$ and $(H, \Diamond, \triangledown)$ are called *associated H_v-rings.* ■

In the theory of representations of the hypergroups in the sense of Marty, there are three types of associated hyperrings $(H, \oplus, \cdot)$ with the hypergroup $(H, \cdot)$. The hyperoperation $\oplus$ is defined, respectively, for all x, y in H, as follows:

type a: $x \oplus y = \{x, y\}$,
type b: $x \oplus y = \beta^*(x) \cup \beta^*(y)$,
type c: $x \oplus y = H$.

In all the above types, the strong associativity and strong or inclusion distributivity are valid. However, in H_v-structures there exists only one class of associated H_v-rings instead of three types.

Theorem 6.3.8. *Let $(H, +)$ be an H_v-group with a scalar zero element 0. Then, for every hyperoperation $\otimes$ such that*

$$\{x, y\} \subseteq x \otimes y \text{ for all } x, y \text{ in } H \setminus \{0\},$$

$$x \otimes 0 = 0 \otimes x = 0 \text{ for all } x \text{ in } H,$$

the hyperstructure $(H, +, \otimes)$ is an H_v-ring.

Proof. For every nonzero elements x, y, z in H, we have

$$y + z \subseteq [x \otimes (y + z)] \cap [(x \otimes y) + (x \otimes z)] \neq \emptyset.$$

Moreover, if one of the elements x, y, z is zero, then the strong distributivity is valid. The rest of the weak axioms are also valid. ■

Theorem 6.3.9. *Let $(H, \cdot)$ be an H_v-group. Take an element $0 \notin H$ and denote $H' = H \cup \{0\}$. We define the hyperoperation $+$ as follows:*

$$0 + 0 = 0, \quad 0 + x = H = x + 0, \quad x + y = 0 \text{ for all } x, y \in H,$$

and we extend the hyperoperation $\cdot$ in H' by putting

$$0 \cdot 0 = 0 \cdot x = x \cdot 0 = 0 \text{ for all } x \in H.$$

Then, the hyperstructure $(H', +, \cdot)$ is an H_v-field with $H'/\gamma^* \cong \mathbb{Z}_2$, where 0 is an absorbing and $\gamma^*(0)$ is a singleton.

Proof. From the definition it is clear that 0 is an absorbing element. The hyperoperation $+$ is (strongly) associative because if in any triple (x, y, z) of elements of H' there are one or three nonzero elements, then their hypersum is 0; in the other cases, the result is H.

The $\cdot$ is weak associative because 0 is an absorbing and $(H, \cdot)$ is an H_v-group. The strong distributivity of $+$ with respect to $\cdot$ is valid, because the only one nonzero case is for $x, y \in H$ in which we have

$$x \cdot (0 + y) = (0 + y) \cdot x = x \cdot 0 + x \cdot y = 0 \cdot x + y \cdot x = H.$$

Finally, one can check that $\gamma^*(0)$ is a singleton and that there are only two fundamental classes in H'. Thus, $(H', +, \cdot)$ is an H_v-field and $H'/\gamma^* \cong \mathbb{Z}_2$. ■

Notice that if the H_v-group $(H, \cdot)$ is strongly associative, then $(H', +, \cdot)$ is a hyperfield instead of an H_v-field; moreover, the strong distributivity is valid.

Since $\gamma^*(0)$ is a singleton, the H_v-fields of this type are very useful. This happens always in an H_v-group H, that we need to represent, for which the cardinality of the hyperproducts of the elements is equal to a power of $cardH$. On the other hand, the representations are normally of lower dimension and $cardH$ is a small number. The H_v-groups of constant length, such as the P-hypergroups, can be also represented on these H_v-fields.

Now, one can prove the following theorem. There is no need to check if the weak axioms are valid since they are obvious. Notice that non-degenerate fundamental rings or fields, which are desired actually, are obtained using this construction.

Theorem 6.3.10. *Let $(R, +, \cdot)$ be a ring and J be an ideal. Then, we can define two H_b-operations $\boxplus$ and $\boxdot$ greater than $+$ and $\cdot$, respectively, for all x, y in R as follows:*

$$x \boxplus y \subseteq x + y + J \quad and \quad x \boxdot y \subseteq xy + J.$$

Then, the hyperstructure $(R, \boxplus, \boxdot)$ is an H_v-ring for which the fundamental ring $R = /\gamma^*$ is a subring of R/J.

Notice that the maximum of the above hyperadditions, i.e. $x \boxplus y = x + y + J$, is a P-hyperoperation so that the H_v-ring $(R, \boxplus, \boxdot)$ can be a P-H_v-ring (see Section 6.4). Remark that for any maximal ideal J, one obtains $R/\gamma^* = R/J$. This construction leads to an enormous number of H_v-rings. Let us point out that if the products of the ring R are enlarged, then all hyperproducts with any cardinality can be represented and the main theorem of this theory is not trivial.

6.4 H_v-rings endowed with P-hyperoperations

In this paragraph, we study a wide class of H_v-rings obtained from an arbitrary ring by using P-hyperoperations. We use the results obtained by

S. Spartalis [120, 122].

Let $(R, +, \cdot)$ be a ring and P_1, P_2 be nonempty subsets of R. We shall use of the following right P-hyperoperations:

$$xP_1^*y = x + y + P_1, \quad xP_2^*y = xyP_2 \text{ for all } x, y \in R.$$

We denote the center of the semigroup $(R, \cdot)$ by $Z(R)$.

Theorem 6.4.1. *If $0 \in P_1$ and $P_2 \cap Z(R) \neq \emptyset$, then (R, P_1^*, P_2^*) is an H_v-ring called a P-H_v-ring or an H_v-ring with P-hyperoperations.*

Proof. The proof is straightforward. ■

Let J be an H_v-ideal of (R, P_1^*, P_2^*). Since $0 \in RP_2^*J \cap JP_2^*R \subseteq J$ and $P_1 = 0P_1^*0 \subseteq J$, we have $JP_1^*x = J + x = xP_1^*J$ for all $x \in R$. Moreover, the addition $\oplus$ and the multiplication $\odot$ between classes are defined in a usual manner:

$$(JP_1^*x) \oplus (JP_1^*y) = \{JP_1^*z \mid z \in (JP_1^*x)P_1^*(JP_1^*y)\} = \{J + x + y\},$$

$$(JP_1^*x) \odot (JP_1^*y) = \{J + w \mid w \in (JP_1^*x)P_2^*(JP_1^*y)\} = \{J + w \mid w \in xyP_2\}.$$

Theorem 6.4.2. *If (R, P_1^*, P_2^*) is a P-H_v-ring and J is an H_v-ideal, then $(R/J, \oplus, \odot)$ is a multiplicative H_v-ring.*

Proof. Obviously, $(R/J, \oplus)$ is an abelian group. Moreover, $(R/J, \odot)$ is an H_v-semigroup, because $\odot$ is well defined and for all $x, y, z \in R$, we have

$$\begin{array}{l} (JP_1^*x) \odot [(JP_1^*y) \odot (JP_1^*z)] = \{J + v \mid v \in xyzP_2P_2\}, \\ [(JP_1^*x) \odot (JP_1^*y)] \odot (JP_1^*z) = \{J + u \mid u \in xyP_2zP_2\}. \end{array}$$

But, since $zP_2P_2 \cap P_2zP_2 \neq \emptyset$, it follows that the multiplication is weak associative. Finally,

$$\begin{array}{rl} (JP_1^*x) \odot [(JP_1^*y) \oplus (JP_1^*z)] & = \{J + v \mid v \in x(y + z)P_2\} \\ & \subseteq \{J + u \mid u \in xyP_2 + xzP_2\} \\ & = [(JP_1^*x) \odot (JP_1^*y)] \oplus [(JP_1^*x) \odot (JP_1^*z)]. \end{array}$$

In the same way, the right distributivity is proved and so $(R/J, \oplus, \odot)$ is a multiplicative H_v-ring. ■

Theorem 6.4.3. *Let (R, P_1^*, P_2^*) be a P-H_v-ring over the ring $(R, +, \cdot)$ and J be an ideal of the ring R, containing P. Then $(R/J, \oplus, \odot)$ is a multiplicative H_v-ring, which is a P-H_v-ring.*

Proof. It is easy to see that J is an H_v-ideal of (R, P_1^*, P_2^*). So, according to the previous theorem, (R, P_1^*, P_2^*) is a multiplicative H_v-ring. On the other hand, consider the quotient ring $(R/J, +, \cdot)$ and take $L_1 = \{J\}$, $L_2 = \{J + a \mid a \in P_2\}$. Then, for all $x \in R$, we have

$$(J + x)L_2L_2 = \{J + v \mid v \in xP_2P_2\},$$
$$L_2(J + x)L_2 = \{J + w \mid w \in P_2xP_2\}.$$

Since for all $x \in R$, $xP_2P_2 \cap P_2xP_2 \neq \emptyset$, it follows that

$$(J + x)L_2L_2 \cap L_2(J + x)L_2 \neq \emptyset.$$

Consequently, since J is the zero element of R/J, from Theorem 6.4.1 it follows that $(R/J, L_1^*, L_2^*)$ is a P-H_v-ring over the ring $(R/J, +, \cdot)$. Therefore, the hyperoperation $\odot$ is the P-hyperoperation L_2^* and $\oplus$ is the degenerate P-hyperoperation L_1^*. ■

Theorem 6.4.4. *Let (R, P_1^*, P_2^*) be a P-H_v-ring and J be an H_v-ideal of R. If H is an H_v-subring of R containing P_1, then $HP_1^*J/J \cong H/H \cap J$.*

Proof. It is easy to see that H, J are subgroups of $(R, +)$ and so $HP_1^*J = H + J$ and $H \cap J$ are two groups containing P_1. Since

$$(HP_1^*J)P_2^*(HP_1^*) = (H + J)(H + J)P_2 \subseteq HHP_2 + J$$

we have that (HP_1^*J, P_1^*, P_2^*) is an H_v-subring of (R, P_1^*, P_2^*). Moreover, $J, J \cap H$ are H_v-ideals of the H_v-rings HP_1^*J and H respectively. On the other hand, the quotients

$$HP_1^*J/J = \{J + x \mid x \in H\} \text{ and } H/H \cap J = \{(H \cap J) + y \mid y \in H\}$$

are multiplicative H_v-rings. We consider the bijection map $f : HP_1^*J/J \longrightarrow H/H \cap J$ such that $J + x \longmapsto (H \cap J) + x$. The map f is a homomorphism, since for all $x, y \in R$, we have

$$f(J{+}x \oplus J{+}y) = f(J+x+y) = (H\cap J)+x+y = f(J+x)\oplus f(J+y),$$
$$f(J{+}x \odot J{+}y) = f(J+s \mid s\in xyP_2\}) = \{(H\cap J)+s \mid s\in xyP_2\}$$
$$= f(J+x)\odot f(J+y).$$

Consequently, f is an isomorphism. ■

Theorem 6.4.5. *Let J, K be two H_v-ideals of the $P-H_v$-ring (R, P_1^*, P_2^*). If $J\subseteq K$, then $(R/J)/(K/J)\cong R/K$.*

Proof. The quotients R/J, K/J and R/K are multiplicative H_v-rings and K/J is an H_v-ideal of $(R/J,\oplus,\odot)$. Indeed, $K/J\subseteq R/J$ and for all $x\in K$, $y\in R$, we have

$$(J+x)\odot(J+y) = \{J+z \mid z\in xP_2^*y\}\subseteq K/J.$$

Therefore, $K/J\odot R/J\subseteq K/J$. Similarly, $R/J\odot K/J\subseteq K/J$. On the other hand, $((R/J)/(K/J),\diamond,*)$ is a multiplicative H_v-ring, where $\diamond$ and $*$ are the usual addition and multiplication of classes. Now, we consider the map $f:(R/J)/(K/J)\longrightarrow R/K$ such that $(K/J)\oplus(J+x)\longmapsto K+x$. Since, for all $x,y\in R$,

$$\begin{aligned}
&(K/J)\oplus(J+x) = (K/J)\oplus(J+y)\\
&\Longleftrightarrow \{J+z \mid z\in K+x\} = \{J+w \mid w\in K+y\}\\
&\Longleftrightarrow y-x\in K\\
&\Longleftrightarrow K+x = K+y
\end{aligned}$$

it follows that f is well defined and one to one. Obviously, it is onto, so it remains to prove that f is a homomorphism. Indeed, for all $x,y\in R$ we have

$$\begin{aligned}
&f[(K/J)\oplus(J+x)\diamond(K/J)\oplus(J+y)]\\
&= f(\{(K/J)\oplus(J+z) \mid J+z\in(K/J)\oplus(J+x)\oplus(K/J)\oplus(J+y)\})\\
&= f(K/J\oplus(J+x+y)) = K+x+y = (K+x)\oplus(K+y)\\
&= f((K/J)\oplus(J+x))\oplus f((K/J)\oplus(J+y))
\end{aligned}$$

and

$$\begin{aligned}
&f[(K/J)\oplus(J+x)*(K/J)\oplus(J+y)]\\
&= f[\{(K/J)\oplus(J+w) \mid J+w\in(J+x)\odot(J+y)\}]\\
&= f(\{(K/J)\oplus(J+w) \mid w\in xyP_2\}] = \{K+w \mid w\in xP_2^*y\}\\
&= (K+x)\odot(K+y) = f((K/J)\oplus(J+x))\odot f((K/J)\oplus(J+y)).
\end{aligned}$$

Hence f is an isomorphism. ■

Theorem 6.4.6. *Let R, A be rings, $f \in Hom(R, A)$ and (R, P_1^*, P_2^*) be an H_v-ring with P-hyperoperations. Then the following assertions hold:*

(1) *If (A, L_1^*, L_2^*) is an H_v-ring such that $f(P_2) \cap L_2 \neq \emptyset$, then $f : (R, P_1^*, P_2^*) \longrightarrow (A, L_1^*, L_2^*)$ is an H_v-homomorphism.*

(2) *If $f(P_2) \cap Z(A) \neq \emptyset$, then $f : (R, P_1^*, P_2^*) \longrightarrow (A, f(P_1)^*, f(P_2)^*)$ is a strong homomorphism.*
A particular case: $f : (R, P_1^*, P_2^*) \longrightarrow (Imf, f(P_1)^*, f(P_2)^*)$.

Proof. (1) For all $x, y \in R$ we have

$$f(xP_1^*y) = f(x) + f(y) + f(P_1) \text{ and } f(x)L_1^*f(y) = f(x) + f(y) + L_1.$$

From the hypothesis it follows that $0 \in P_1$ and so

$$0 = f(0) \in f(P_1) \cap L_1 \text{ and } f(xP_1^*y) \cap f(x)L_1^*f(y) \neq \emptyset.$$

Moreover, the condition $f(xP_2^*y) \cap f(x)L_2^*f(y) \neq \emptyset$ holds obviously. Hence f is an H_v-homomorphism.

(2) The structure $(A, f(P_1)^*, f(P_2)^*)$ is an H_v-ring, because $0 \in f(P_1)$ and $f(P_2) \cap Z(A) \neq \emptyset$. The H_v-homomorphism f is strong, since for all $x, y \in R$, we have

$$f(xP_1^*y) = f(x) + f(y) + f(P_1) = f(x)f(P_1)^*f(y),$$
$$f(xP_2^*y) = f(x)f(y)f(P_2) = f(x)f(P_2)^*f(y).$$

In the particular case when f is the H_v-homomorphism from (R, P_1^*, P_2^*) to $(Imf, f(P_1)^*, f(P_2)^*)$, from $P_2 \cap Z(R) \neq \emptyset$, we can deduce easily that $f(P_2) \cap Z(Imf) \neq \emptyset$. Hence, (2) is valid. ■

Proposition 6.4.7. *Let (R, P_1^*, P_2^*) be an H_v-ring with P-hyperoperations. If $\alpha \in Z(R)$, then the translation of the semigroup $(R, \cdot)$ by α:*

$$f_\alpha : x \longrightarrow \alpha x$$

is a multiplicatively strong homomorphism from $(R, L_1^*, (\alpha P_2)^*)$ to (R, P_1^*, P_2^*), where $0 \in L_1 \subsetneq R$.

Proof. First, we can observe that the structure $(R, L_1^*, (\alpha P_2)^*)$ is an H_v-ring because $0 \in L_1$ and $\alpha P_2 \cap Z(R) \neq \emptyset$. Moreover, for all $x, y, z \in R$, we have

$$f_\alpha(xL_1^*y) = \alpha(x+y+L_1) = \alpha x+\alpha y+\alpha L_1 \text{ and } f_\alpha(x)P_1^*f_\alpha(y) = \alpha x+\alpha y+P_1$$

and since $0 \in \alpha L_1 \cap P_1$, it follows that

$$f_\alpha(xL_1^*y) \cap f_\alpha(x)P_1^*f_\alpha(y) \neq \emptyset.$$

Moreover,

$$f_\alpha(x(\alpha P_2)^*y) = \alpha(xy\alpha P_2) = (\alpha x)(\alpha y)P_2 = f_\alpha(x)P_2^*f_\alpha(y).$$

Hence, f_α is a multiplicatively strong homomorphism. ■

Proposition 6.4.8. *Let (R, P_1^*, P_2^*) be an H_v-ring with P-hyperoperations and $\alpha \in Z(R)$. If the element α is simplificable and reproductive in $(R, \cdot)$ then for each subset L_1 of R such that $\alpha L_1 = P_1$, we have*

$$(R, L_1^*, (\alpha P_2)^*) \cong (R, P_1^*, P_2^*).$$

Proof. Let us consider the translation of the semigroup $(R, \cdot)$ by α

$$f_\alpha : (R, L_1^*, (\alpha P_2)^*) \longrightarrow (R, P_1^*, P_2^*), \quad f_\alpha(x) = \alpha x,$$

which is a multiplicatively strong homomorphism of H_v-rings. Moreover, f_α is additively strong because, for all $x, y \in R$,

$$f_\alpha(xL_1^*y) = \alpha x + \alpha y + \alpha L_1 = \alpha x + \alpha y + P_1 = f_\alpha(x)P_1^*f_\alpha(y).$$

Finally, according to the hypothesis, the map f_α is one to one and onto, hence f_α is an isomorphism of H_v-rings. ■

Suppose that the conditions of Proposition 6.4.8 hold and $\alpha^2 P_1 = P_1$. Therefore, every translation of the sets P_1, P_2 by $\alpha \in Z(R)$ gives an isomorphism of H_v-rings, i.e., $(R, (\alpha P_1)^*, (\alpha P_2)^*) \cong (R, P_1^*, P_2^*)$. In case the H_v-ring (R, P_1^*, P_2^*) is derived from a ring with unit 1, we obtain the following isomorphism: $(R, (-P_1)^*, (-P_2)^*) \cong (R, P_1^*, P_2^*)$, since -1 satisfies the hypothesis of Proposition 6.4.8 and $(-1)^2 P_1 = P_1$.

Now, we calculate the number of H_v-rings with P-hyperoperations, which can be constructed starting from a finite ring $(R, +, \cdot)$. By Proposition 6.4.7, it follows that this number can be substantially reduced, because some of these H_v-rings are isomorphic.

Proposition 6.4.9. *Let $(R, +, \cdot)$ be a ring with $cardR = n$, $n > 1$ and $cardZ(R)=m$. The number of H_v-rings (R, P_1^*, P_2^*) is at most $2^{n-1}(2^n - 2^{n-m})$.*

Proof. The number of the subsets P_1 of R, which satisfy the condition $0 \in P_1$, is 2^{n-1}. The number of the subsets P_2 of R, which satisfy the condition $P_2 \cap Z(R) \neq \emptyset$, is $2^{n-1} + 2^{n-2} + ... + 2^{n-m} = 2^n - 2^{n-m}$. Hence, the number of H_v-rings is at most $2^{n-1}(2^n - 2^{n-m})$. ■

If $\cdot$ is commutative , the above number is $2^{2n-1} - 2^{n-1}$.

Proposition 6.4.10. *If $(R, +, \cdot)$ is a commutative ring with nonzero divisors and $cardR = n$, $n > 1$, then the number of hyperrings (R, P_1^*, P_2^*) which are not rings is at most $5 \cdot 2^{n-1} - n - 4$.*

Proof. First of all it is easy to check that the structure (R, P_1^*) is a hypergroup. Moreover, because of the commutativity of the multiplication, we have

$$xP_2^*(yP_2^*z) = xyzP_2P_2 = (xP_2^*y)P_2^*z,$$

for all $x, y, z \in R$. The necessary and sufficient condition for the validity of the inclusion distributivity is $RP_2P_1 \subseteq P_1$. We suppose that there exist $P_1, P_2 \subseteq R$, $P_1 \neq R$, satisfying the previous condition. Then, for any $p_1 \in P_1$, $p_2 \in P_2$, $p_1 \neq 0 \neq p_2$, the condition $Rp_2p_1 \subseteq P_1$ is valid and so, there exist $a, b \in R$, $a \neq b$ such that $ap_2p_1 = bp_2p_1$. Therefore, $(a - b)p_2p_1 = 0$, i.e., p_1 or p_2 is a zero divisor, which is a contradiction. Hence, the only cases, in which the condition of distributivity is satisfied, are $P_1 = R$, $P_1 = \{0\}$ and $P_2 = \{0\}$. Therefore, the number of hyperrings (R, P_1^*, P_2^*) is at most $(2^n - 1 + 2^n - 1 + 2^{n-1}) - 2 = 5 \cdot 2^{n-1} - 4$, because the hyperrings $(R, \{0\}^*, \{0\}^*)$ and $(R, R^*, \{0\}^*)$ are calculated twice. Finally, the number of hyperrings which are not rings is at most $5 \cdot 2^{n-1} - n - 4$. ■

We remark that the following facts are also valid.

- There are $2^n - 1$ hyperrings of the form (R, R^*, P_2^*), where $P_2 \subseteq R$, because $xR^*y = R$ holds for all $x, y \in R$.

- There are $2^n - n - 1$ multiplicative hyperrings $(R, \{0\}^*, P_2^*)$ in the sense of Rota, where $P \subseteq R$. Indeed, the hyperoperation $\{0\}^*$ of each hyperring of the above form is the addition $+$ of the ring $(R, +, \cdot)$. Moreover, we have $(-x)P_2^*y = -(xP_2^*y) = xP_2^*(-y)$.

- If $\cdot \equiv \circ$, where $x \circ y = 0$, for all $x, y \in R$, then the number of H_v-rings is at most 2^{n-1}, because $(R, P_1^*, \circ) \cong (R, P_1^*, \{0\}^*)$.

Example 6.4.11. In the case of the ring $(\mathbb{Z}_p, +, \cdot)$, where p is a prime number, there are at most $2^{2p-1} - 2^{p-1}$ H_v-rings, from which $5 \cdot 2^{n-1} - n - 4$ are hyperrings, that are not rings. In the particular case $p = 3$ we have 28 H_v-rings. Observe that 13 H_v-rings are hyperrings which are not rings. We have the following isomorphisms:

$$
\begin{aligned}
&(\mathbb{Z}_3, \{0\}^*, \{1\}^*) \cong (\mathbb{Z}_3, \{0\}^*, \{2\}^*),\\
&(\mathbb{Z}_3, \{0\}^*, \{0,1\}^*) \cong (\mathbb{Z}_3, \{0\}^*, \{0,2\}^*),\\
&(\mathbb{Z}_3, \{0,1\}^*, \{0\}^*) \cong (\mathbb{Z}_3, \{0,2\}^*, \{0\}^*),\\
&(\mathbb{Z}_3, \{0,1\}^*, \{1\}^*) \cong (\mathbb{Z}_3, \{0,2\}^*, \{2\}^*),\\
&(\mathbb{Z}_3, \{0,1\}^*, \{2\}^*) \cong (\mathbb{Z}_3, \{0,2\}^*, \{1\}^*),\\
&(\mathbb{Z}_3, \{0,1\}^*, \{0,1\}^*) \cong (\mathbb{Z}_3, \{0,2\}^*, \{0,2\}^*),\\
&(\mathbb{Z}_3, \{0,1\}^*, \{0,2\}^*) \cong (\mathbb{Z}_3, \{0,2\}^*, \{0,1\}^*),\\
&(\mathbb{Z}_3, \{0,1\}^*, \{1,2\}^*) \cong (\mathbb{Z}_3, \{0,2\}^*, \{1,2\}^*),\\
&(\mathbb{Z}_3, \{0,1\}^*, \mathbb{Z}_3^*) \cong (\mathbb{Z}_3, \{0,2\}^*, \mathbb{Z}_3^*),\\
&(\mathbb{Z}_3, \mathbb{Z}_3^*, \{1\}^*) \cong (\mathbb{Z}_3, \mathbb{Z}_3^*, \{2\}^*),\\
&(\mathbb{Z}_3, \mathbb{Z}_3^*, \{0,1\}^*) \cong (\mathbb{Z}_3, \mathbb{Z}_3^*, \{0,2\}^*).
\end{aligned}
$$

So, the number of H_v-rings is reduced to 17 and observe that 9 of them are hyperrings which are not rings.

Example 6.4.12. We consider the finite field $(F, +, \cdot)$ and suppose that $cardF = p^n$, where p is a prime number, $p > 2$, $n \geq 1$. Consequently, if we consider the translation of F

$$f_{a^{-1}b} : x \longmapsto a^{-1}bx, \text{ for all } a, b \in R \setminus \{0\},$$

then we have $f_{a^{-1}b}(a) = b$. Using Proposition 2.4.8, we obtain the following isomorphisms between H_v-rings with P-hyperoperations:

$$\begin{array}{l}(F,\{0\}^*,\{a\}^*)\cong(F,\{0\}^*,\{b\}^*),\\(F,F^*,\{a\}^*)\cong(F,F^*,\{b\}^*),\\(F,\{0\}^*,\{0,a\}^*)\cong(F,\{0\}^*,\{0,b\}^*),\\(F,F^*,\{0,a\}^*)\cong(F,F^*,\{0,b\}^*),\end{array}$$

So, the number of H_v-rings (R,P_1^*,P_2^*) is at most $2^{2p^n-1}-2^{p^n-1}-4(p^n-2)$.

Theorem 6.4.13. *Let (R,P_1^*,P_2^*) be an H_v-ring with the P-hyperoperations P_1^*,P_2^* over the ring $(R,+,\cdot)$. Consider the subgroup $<P_1>$ of $(R,+)$ generated by P_1. Then for all a in R, we have $\beta_+^*(a)=<P_1>+a$ and R/β_+^* is a multiplicative H_v-ring with the inclusion distributivity.*

Proof. We denote the fundamental class of $a\in R$ by $\beta_+^*(a)$ and any hypersum with respect to the P_1^* by $\overset{*}{\sum}$. Let $a\in R$ and $x\in\beta_+^*(a)$. Then, there exist $z_1,\dots,z_{n+1}$ and there are $y_j\in R$ and the finite sets of indices I_i, $i=1,\dots,n$ such that

$$\{z_i,z_{i+1}\}\subseteq\overset{*}{\sum_{j\in I_i}}y_j \text{ for } i=1....,n.$$

Set $u_i=\overset{*}{\sum\limits_{j\in I_i}}y_j$ and $s_i=cardI_i$. Then, $\{z_i,z_{i+1}\}\subseteq u_i+(s_i-1)P_1$ for $i=1,\dots,n$. Therefore, for $i=1,\dots,n-1$ we have

$$z_{i+1}\in(u_i+(s_i-1)P_1)\cap(u_{i+1}+(s_{i+1}-1)P_1)$$

and so $u_i\in z_{i+1}-(s_i-1)P_1$.

Consequently, $u_i\in u_{i+1}+(s_{i+1}-1)P_1-(s_i-1)P_1$. We obtain

$$u_1\in u_n+(s_2+s_3+\dots+s_n-n+1)P_1-(s_1+s_2+\dots+s_{n-1}-n+1)P_1.$$

But $z_{n+1}\in u_n+(s_n-1)P_1$, so $u_n\in z_{n+1}-(s_n-1)P_1$.

Moreover, $z_1\in u_1+(s_1-1)P_1$. Thus, we have

$$z_1\in z_{n+1}+(s_1+s_2+\dots+s_n-n)P_1-(s_1+s_2+\dots+s_n-n)P_1.$$

Finally, $x \in (s_1+s_2+...+s_n-n)(P_1-P_1)$. This means that $x \in <P_1> +a$, so $\beta_+^* \subseteq <P_1> +a$.

Now, let $a \in R$ and take $x \in <P_1> +a$. Then, there exists $s \in \mathbb{N}$ such that $x \in s(P_1-P_1)+a$. So $\{x,a\} \subseteq s(P_1-P_1)+a = aP_1^*(-P_1)...P_1^*(-P_1)$ which means that $x\beta_+^*$. Therefore, we proved that $\beta_+^*(a) = <P_1> +a$. The sum $\uplus$ and the product $\otimes$ of the elements of R/β_+^* are defined in the usual manner, and $(R/\beta_+^*, \uplus)$ is a group. Moreover, the weak associativity of $\otimes$ is valid. Finally, for all $x,y,z \in R$, we have

$$\begin{aligned}\beta_+^*(x) \otimes (\beta_+^*(y) \uplus \beta_+^*(z)) &= \beta_+^*(x) \otimes \beta_+^*(y+z) \\ &= \{\beta_+^*(u) \mid u \in (<P_1> +x)(<P_1> +y+z)P_2\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&(\beta_+^*(x) \otimes \beta_+^*(y)) \uplus (\beta_+^*(x) \otimes \beta_+^*(z)) \\ &= \{\beta_+^*(v) \mid v \in (<P_1> +x)(<P_1> +y)P_2\} \\ &\qquad \uplus\{\beta_+^*(w) \mid w \in (<P_1> +x)(<P_1> +z)P_2\} \\ &= \{\beta_+^*(v+w) \mid v+w \in (<P_1> +x)(<P_1> +y)P_2 \\ &\qquad +(<P_1> +x)(<P_1> +z)P_2\}.\end{aligned}$$

Consequently, the inclusion distributivity is valid. ■

Let (R, P_1^*, P_2^*) be an H_v-ring with the P-hyperoperations P_1^*, P_2^* over the ring $(R,+,\cdot)$ such that $RP_2 \subseteq P_2$. Denote the set of all finite polynomials of elements of R by A. Then, for every $a_i \in A$, $i \in \mathbb{N}$, there exist $r_i \in R$, I_i finite set of indices, $P_{2j} \in \mathcal{P}(P_2)$, $j \in I_i$ and $s_i \in \mathbb{N}$ such that

$$a_i = r_i + \sum_{j \in I_i} P_{2j} + s_i P_i,$$

where $s_i P_1 = \underbrace{P_1 + ... + P_1}_{s_i}$.

Theorem 6.4.14. *Let (R, P_1^*, P_2^*) be an H_v-ring with the P-hyperoperations P_1^*, P_2^* over the ring $(R,+,\cdot)$. If $RP_2 \subseteq P_2$ and $<P_1,P_2>$ is the subgroup of $(R,+)$ generated by $P_1 \cup P_2$, then for all $x \in R$, $\gamma^*(x) \subseteq <P_1,P_2> +x$.*

Proof. Let $x \in R$ and $y \in \gamma^*(x)$. Then there exist $z_1,...,z_{m+1} \in R$ with $z_1 = y$, $z_{m+1} = x$ and $a_1,...,a_m \in A$ such that $\{z_i, z_{i+1}\} \subseteq a_i$, $(i = 1,...,m)$.

Then, for $i = m$, we have $x = z_{m+1} \in r_m + \sum_{j \in I_m} P_{2j} + s_m P_1$. So $r_m \in x + t_m(-P_2) + s_m(-P_1)$, where $t_m = cardI_m$. Moreover, for all $i = 1, ..., m$, we have

$$z_{i+1} \in (r_i + \sum_{j \in I_i} P_{2j} + s_i P_1) \cap (r_{i+1} + \sum_{j \in I_{i+1}} P_{2j} + s_{i+1} P_1)$$

and hence $r_i \in r_{i+1} + t_{i+1} P_2 + t_i(-P_2) + s_{i+1} P_1 + s_i(-P_1)$, where $t_{i+1} = cardI_{i+1}$, $t_i = cardI_i$. We obtain $y = z_1 \in x + t(P_2 - P_2) = s(P_1 - P_1)$, where $t = t_1 + ... + t_m$, $t_i = cardI_i$, $i \in \{1, ..., m\}$, $s = s_1, ..., s_m$. This means that $y \in x+ < P_2 > + < P_1 >$. Hence, $\gamma^*(x) \subseteq < P_1, P_2 > +x$. ■

Theorem 6.4.15. *Let (R, P_1^*, P_2^*) be an H_v-ring with the P-hyperoperations P_1^*, P_2^* over the unitary ring $(R, +, \cdot)$. If P_2 is a right ideal and $< P_1 >$ is the subgroup of $(R, +)$ generated by P_1, then $R/\gamma^* = R/(< P_1 > +P_2)$.*

Proof. Suppose that $x \in R$. From the previous theorem, we have that $\gamma^*(x) \subseteq < P_1, P_2 > +x$. Since P_2 is a subgroup of $(R, +)$, it follows that $\gamma^*(x) \subseteq < P_1 > +P_2 + x$.

Conversely, for all $z \in < P_1 > +P_2 + x$ there exist $p_2 \in P_2$ and $n \in \mathbb{N}$ such that $z \in x + p_2 + n(P_1 - P_1)$. Moreover, $p_2 \in P_2 = 1P_2^*1$ and so

$$\{z, x\} \subseteq x P_1^*(1P_2^*1)P_1^*(-P_1)P_1^* ... P_1^*(-P_1)$$

where $P_1^*(-P_1)$ appears n times. Hence $\gamma^*(x) = < P_1 > +P_2 + x$. ■

Corollary 6.4.16. *If $(R, +, \cdot)$ is a ring and P_2 is a right ideal, then for all multiplicative $P - H_v$-rings over R, we have $R/\gamma^* = R/P_2$.*

Proof. Suppose that (R, P_1^*, P_2^*) is a multiplicative $P - H_v$-ring over the ring $(R, +, \cdot)$. Then, $cardP_1 = 1$ and since the necessary and sufficient condition for the weak distributivity is $0 \in P_1$, we have $P_1 = \{0\}$. From the previous theorem, it follows that $R/\gamma^* = R/P_2$. ■

6.5 (H, R)-H_v-rings

In this section, we introduce the following new structures [121]:

Let $(H, *, \circ)$ be an H_v-ring, $(R, +, \cdot)$ be a ring with the zero element denoted by 0 and $\{A_i\}_{i\in R}$ be a family of nonempty sets indexed in R such that $A_0 = H$ and for all $i, j \in R$, $i \neq j$, $A_i \cap A_j = \emptyset$. Moreover, for all A_i, $i \in R^*$ there exists a set of indices I_i and a unique family $\{B_k\}_{k\in I_i}$ such that $B_k \subseteq A_i$ and $\bigcup_{k\in I_i} B_k \neq \emptyset$. Set $K = \bigcup_{i\in R} A_i$ and consider the hyperoperations $\oplus$, $\odot$ defined in K as follows:

$$\forall (x,y) \in H^2,\ x \oplus y = x * y,\ x \odot y = x \circ y,$$

$$\forall (x,y) \in A_i \times A_j \neq H^2,\ x \oplus y = A_{i+j},\ x \odot y = \begin{cases} H & \text{if } ij = 0 \\ \bigcup\limits_{k\in I_{ij}} B_k & \text{if } ij \neq 0. \end{cases}$$

It is clear that $(K, \oplus)$ is an H_v-group. Moreover, $(K, \odot)$ is an H_v-semigroup since the hyperoperation $\circ$ is weak associative and for all $(x, y, z) \in A_i \times A_j \times A_r \neq H^3$, we have

(1) if $i = 0 = j$ (similarly, if $i = 0, j \neq 0 \neq r$), then $x \odot (y \odot z) \subseteq (x \odot y) \odot z = H$;

(2) if $j = 0 = r$ (similarly, if $r = 0, i \neq 0 \neq j$), then $H = x \odot (y \odot z) \supseteq (x \odot y) \odot z$;

(3) if $i = 0 = r$, then $(x \circ H = x \odot (y \odot z)) \cap ((x \odot y) \odot z = H \circ z) \neq \emptyset$;

(4) if $j = 0, i \neq 0 \neq r$, then $x \odot (y \odot z) = H = (x \odot y) \odot z$;

(5) if $i \neq 0 \neq j \neq 0 \neq r$, then $x \odot (y \odot z) \cap (x \odot y) \odot z \neq \emptyset$.

Finally, the weak distributive law is verified and so $(K, \oplus, \odot)$ is an H_v-ring.

Definition 6.5.1. The previouse H_v-ring $(K, \oplus, \odot)$ is called $(H, R) - H_v$-*ring with the support* $K = \bigcup_{i\in R} A_i$.

Theorem 6.5.2. *If γ^* is the fundamental equivalence relation in K, then $K/\gamma^* \cong R$.*

Proof. Let $a \in K$. Then there exists $r \in R$ such that $a \in A_r$. In order to determine $\gamma^*(a)$, we consider $x \in \gamma^*(a)$. Then, there exist $z_1, \ldots, z_{n+1} \in K$ such that $z_1 = x$, $z_{n+1}=a$ and $u_i \in \mathcal{U}$, $i \in \{1, \ldots, n\}$ such that $\{z_i, z_{i+1}\} \subseteq u_i$,

$(i = 1, ..., n)$. Moreover, it is clear that for all $u_i \in \mathcal{U}$, $i = 1, ..., n$ there exists an appropriate $r_i \in R$, such that $u_i \subseteq A_{r_i}$. Consequently, $A_{r_i} = A_{r_{i+1}}$, $i = 1, ..., n-1$, because $z_{i+1} \in A_{r_i} \cap A_{r_{i+1}}$. Hence, $\{x, a\} \subseteq A_{r_n} = A_r$ and so $\gamma^*(a) \subseteq A_r$.

Conversely, let $y \in A_r$. If $r \in R^*$, then we consider $u \in H$, $w \in A_r$, and we have $\{y, a\} \subseteq u \oplus w = A_r$. Hence $y\gamma^* a$, i.e., $y \in \gamma^*(a)$. If $r = 0$, then we have $\{y, a\} \subseteq u \oplus w = H$, i.e., $y \in \gamma^*(a)$. Therefore, $A_r \subseteq \gamma^*(a)$ and consequently, $\gamma^*(a) = A_r$.

Finally, the map $f : K/\gamma^* \longrightarrow R$ such that $f(A_i) = i$ is an isomorphism and so $K/\gamma^* \cong R$. ■

We denote the kernel of the canonical map $\phi_K : K \longrightarrow K/\gamma^*$ such that $\phi_K(x)=\gamma^*(x)$ by ω_k. According to the previous theorem, for all $i\in R$, $x\in A_i$, we have $\gamma^*(x) = A_i$ and hence, we can write $K/\gamma^* = \{\phi_K(A_i) \mid i \in R\}$. Consequently, $\omega_K = H$.

Now, we consider the (H_1, R_1)-H_v-ring $(K_1, \oplus, \odot)$ with the support $K_1 = \bigcup\limits_{i\in R_1} A_i$ and the (H_2, R_2)-H_v-ring $(K_2, \boxplus, \boxdot)$ with support $K_2 = \bigcup\limits_{j\in R_2} G_j$. We prove the following theorems:

Theorem 6.5.3. *If $f : K_1 \longrightarrow K_2$ is an inclusion homomorphism, then*

(1) *$f(\gamma^*(x)) \subseteq \gamma^*(f(x))$ for all $x \in K_1$.*

(2) *We define the induced homomorphism $f^* : K_1/\gamma^* \longrightarrow K_2/\gamma^*$ of f by $f^*(\phi_{K_1}(x)) = \phi_{K_2}(f(x))$.*

(3) *$f(H_1) \subseteq H_2$.*

Proof. (1) Let $x \in \gamma^*(x) = A_i$, $i \in R$. Then, for $y \in A_j$, $z \in A_{i-j}$, we have

$$f(x) \in f(\gamma^*(x)) = f(y \oplus z) \subseteq f(y) \boxplus f(z) = \gamma^*(f(x)).$$

Therefore, $f(\gamma^*(x)) \subseteq \gamma^*(f(x))$.

(2) The map f^* is well defined. In fact, if $\phi_{K_1}(x) = \phi_{K_2}(y)$, then $x\gamma^* y$ and so $f(x)\gamma^* f(y)$, i.e., $\phi_{K_1}(f(x)) = \phi_{K_2}(f(y))$ and hence $f^*(\phi_{K_1}(x)) = f^*(\phi_{K_2}(y))$. Moreover, f^* is a homomorphism, because for all $x, y \in K_1$, $z \in x \oplus y$, $w \in x \odot y$, we obtain

$$\begin{aligned} f^*(\phi_{K_1}(x) + \phi_{K_2}(y)) &= f^*(\phi_{K_1}(z)) = \phi_{K_2}(f(z)) = \phi_{K_2}(f(x) \boxplus f(y)) \\ &= \phi_{K_2}(f(x)) + \phi_{K_2}(f(y)) = f^*(\phi_{K_1}(x)) + f^*(\phi_{K_2}(y)) \end{aligned}$$

and

$$\begin{aligned} f^*(\phi_{K_1}(x)\cdot\phi_{K_2}(y)) &= f^*(\phi_{K_1}(w)) = \phi_{K_2}(f(w)) = \phi_{K_2}(f(x)\boxdot f(y)) \\ &= \phi_{K_2}(f(x))\cdot\phi_{K_2}(f(y)) = f^*(\phi_{K_1}(x))\cdot f^*(\phi_{K_2}(y)). \end{aligned}$$

(3) From (1) and (2) it follows that $f(x) = \phi_{K_2}^{-1}(f^*(\phi_{K_1}(x)))$. Therefore, if $x \in H$ and $\bar{0}_{K_1/\gamma^*}$, $\bar{0}_{K_2/\gamma^*}$ are the zero elements of the rings K_1/γ^*, K_2/γ^*, respectively, then

$$f(x) \in \phi_{K_2}^{-1}(f^*(\bar{0}_{K_1/\gamma^*})) = \phi_{K_2}^{-1}(\bar{0}_{K_2/\gamma^*}) = H_2,$$

which implies that $f(H_1) \subseteq H_2$. ■

Theorem 6.5.4. *If $K_1 \cong K_2$, then $H_1 \cong H_2$ and $R_1 \cong R_2$.*

Proof. From the previous theorem it follows that for all $x \in K_1$, $f(\gamma^*(x)) = \gamma^*(f(x))$ and f^* is a homomorphism. Therefore, $K_1/\gamma^* \cong K_2/\gamma^*$. Consequently, from Theorem 6.5.2, we obtain $R_1 \cong R_2$. Next we consider the map $g : H_1 \longrightarrow H_2$ defined by $g(x) = f(x)$. The map g is well defined, one to one and onto. We show that it is a strong homomorphism. In fact, for all $(x, y) \in H^2$, we obtain

$$\begin{aligned} g(x *_1 y) &= f(x \oplus y) = f(x) \boxplus f(y) = g(x) *_2 g(y), \\ g(x \circ_1 y) &= f(x \odot y) = f(x) \boxdot f(y) = g(x) \circ_2 g(y). \end{aligned}$$ ■

Theorem 6.5.5. *If $g : H_1 \longrightarrow H_2$, $f : R_1 \longrightarrow R_2$ are homomorphisms and for all $i \in R_1^*$, $cardA_i \leq cardG_{f(i)}$, then there is an additively strong and one to one homomorphism from K_1 to K_2.*

Proof. Let $i \in R_1^*$ and let $F_i = F_i(A_i, G_{f(i)})$ be the set of all the one to one maps from A_i to $G_{f(i)}$. If $\{B_k\}_{k\in I_i}$, $\{B_s\}_{s\in I_{f(i)}}$ are the families of subsets of A_i and $G_{f(i)}$, respectively, then we denote

$$F_i^* = \left\{ h_{f(i)} \in F_i \mid h_{f(i)}\left(\bigcup_{k\in I_i} B_k\right) \cap \left(\bigcup_{s\in I_{f(i)}} B_s\right) \neq \emptyset \right\}$$

and we consider the map

$$t : K_1 \longrightarrow K_2 : \quad x \mapsto t(x) = \begin{cases} g(x) & \text{if } x \in H_1 \\ h_{f(i)}(x) \text{ where } h_{f(i)} \in F_i^* & \text{if } x \in A_i \neq H_1. \end{cases}$$

The map t is well defined and one to one. We will show that it is an additively strong homomorphism.

If $(x, y) \in H^2$, then

$$t(x \oplus y) = g(x *_1 y) = g(x) *_2 g(y) = t(x) \boxplus t(y)$$
$$t(x \odot y) = g(x \circ_1 y) = g(x) \circ_2 g(y) = t(x) \boxdot t(y).$$

If $(x, y) \in A_i \times A_j \neq H^2$, then

$$t(x \oplus y) = t(A_{i+j}) = G_{f(i+j)} \text{ and } t(x) \boxplus t(y) = G_{f(i)+f(j)} = G_{f(i+j)}.$$

In order to check the multiplications, we notice that:

If $ij = 0$, then $t(x \odot y) = g(H_1) = H_2$ and $t(x) \boxdot t(y) = G_{f(ij)} = H_2$.

If $ij \neq 0$, then $t(x \odot y) = h_{f(ij)}(\bigcup_{r \in I_{ij}} B_r)$, where $B_r \subseteq A_{ij}$,

$$t(x) \boxdot t(y) = \bigcup_{m \in I_{ij}} B_m, \qquad \text{where } B_m \subseteq G_{f(ij)}.$$

Since $h_{f(ij)}\left(\bigcup_{r \in I_{ij}} B_r\right) \cap \left(\bigcup_{m \in I_{f(ij)}} B_m\right) \neq \emptyset$, it follows that

$$t(x \odot y) \cap t(x) \boxdot t(y) \neq \emptyset. \blacksquare$$

6.6 The H_v-ring of fractions

It is well-known that if S is a multiplicatively closed subset of a commutative ring R, then there is a natural way to define the ring of fractions of R with respect to S. This ring is denoted by $S^{-1}R$ and the detail of its construction is given in Example 1.2.2 (9). A natural question that arises, is the following one: how the H_v-ring of fractions can be defined? In this section, our aim is to answer the above question and obtain some properties of the H_v-ring of fractions. We use the results obtained by M.R. Darafsheh and B. Davvaz [28].

Throughout this section, R is a commutative (general) hyperring with a unit denoted by 1. Recall that a hyperstructure $(R, +, \cdot)$ is a hyperring if $(R,+)$ is a hypergroup, $(\cdot)$ is an associative hyperoperation and the distributivity is valid, that is $x\cdot(y+z)=x\cdot y+x\cdot z$, $(x+y)\cdot z=x\cdot z+y\cdot z$ for all $x,y,z\in R$. Under the above conditions, we define the H_v-ring of fractions of R.

Definition 6.6.1. A nonempty subset S of R is called a *strong multiplicatively closed subset* (s.m.c.s.) if the following axioms hold:

(1) $1\in S$,

(2) $a\cdot S=S\cdot a=S$ for all $a\in S$.

Now, as we have indicated earlier, suppose that R is a commutative hyperring with scalar unit. Furthermore we assume that S is a s.m.c.s. of R .Let $\mathcal{M}$ be the set of all the ordered pairs (r,s) where $r\in R$, $s\in S$. For $A\subseteq R$ and $B\subseteq S$, we denote the set $\{(a,b)|a\in A,\ b\in B\}$ by (A,B). We define the following relation $\sim$ on $\mathcal{P}(\mathcal{M})$:
$(A,B)\sim(C,D)$ if and only if there exists a subset X of S such that $X\cdot(A\cdot D)=X\cdot(B\cdot C)$.

Lemma 6.6.2. *$\sim$ is an equivalence relation on $\mathcal{P}(\mathcal{M})$.*

Proof. Obviously $\sim$ is reflexive and symmetric. To verify that $\sim$ is transitive, we assume $(A_1,B_1)\sim(A_2,B_2)$ and $(A_2,B_2)\sim(A_3,B_3)$, where $(A_i,B_i)\in\mathcal{P}(\mathcal{M})$, $1\leq i\leq 3$. By definition of $\sim$ there exist the subsets X_1 and X_2 of S such that

$$X_1\cdot(A_1\cdot B_2)=X_1\cdot(A_2\cdot B_1),\qquad (I)$$

$$X_2\cdot(A_2\cdot B_3)=X_2\cdot(A_3\cdot B_2).\qquad (II)$$

Multiplying both sides of (II) by $X_1.B_1$ we get $X_1\cdot X_2\cdot A_2\cdot B_3\cdot B_1=X_1\cdot X_2\cdot A_3\cdot B_2\cdot B_1$ which implies that $X_2\cdot(X_1\cdot A_2\cdot B_1)\cdot B_3=X_1\cdot X_2\cdot A_3\cdot B_2\cdot B_1$. Using (I), we obtain $X_2\cdot(X_1\cdot A_1\cdot B_2)\cdot B_3=X_1\cdot X_2\cdot A_3\cdot B_2\cdot B_1$ which implies that $(X_1\cdot X_2\cdot B_2)\cdot(A_1\cdot B_3)=(X_1\cdot X_2\cdot B_2)\cdot(A_3\cdot B_1)$. If we take $X=X_1\cdot X_2\cdot B_2$, then $X\cdot(A_1\cdot B_3)=X\cdot(A_3\cdot B_1)$ which implies that $(A_1,B_1)\sim(A_3,B_3)$. ∎

We consider the restriction of the relation $\sim$ on $\mathcal{M}$. We obtain the following two corollaries.

Corollary 6.6.3. *For $(r, s), (r_1, s_1) \in \mathcal{M}$, we have $(r, s) \sim (r_1, s_1)$ if and only if there exists $A \subseteq S$ such that $A \cdot (r \cdot s_1) = A \cdot (r_1 \cdot s)$.*

Corollary 6.6.4. *$\sim$ is an equivalence relation on $\mathcal{M}$.*

In $\mathcal{M}$, the equivalence class containing (r, s) is denoted by $[r, s]$ and we denote the set of all the equivalence classes by $S^{-1}R$.

In $\mathcal{P}(\mathcal{M})$, the equivalence class containing (A, B) is denoted by $\| A, B \|$. We define:

$$\ll A, B \gg = \bigcup_{(A_1,B_1)\in\|A,B\|} \{[a_1, b_1] | a_1 \in A_1, b_1 \in B_1\}.$$

Now, we define the following hyperoperations on $S^{-1}R$,

$$\begin{aligned}
[r_1, s_1] \uplus [r_2, s_2] &= \bigcup_{(A,B)\in\|r_1\cdot s_2+r_2\cdot s_1, s_1\cdot s_2\|} \{[r, s] | r \in A, s \in B\} \\
&= \ll r_1 \cdot s_2 + r_2 \cdot s_1, s_1 \cdot s_2 \gg, \\
[r_1, s_1] \otimes [r_2, s_2] &= \bigcup_{(A,B)\in\|r_1\cdot r_2, s_1\cdot s_2\|} \{[r, s] | r \in A, s \in B\} \\
&= \ll r_1 \cdot r_2, s_1 \cdot s_2 \gg .
\end{aligned}$$

Lemma 6.6.5. *$\uplus$ and $\otimes$ defined above are well-defined hyperoperations.*

Proof. Suppose that $[r_1, s_1] = [a_1, t_1]$ and $[r_2, s_2] = [a_2, t_2]$. Then there exist subsets A and B of S such that

$$A \cdot r_1 \cdot t_1 = A \cdot a_1 \cdot s_1 \quad \text{(I)}$$

$$B \cdot r_2 \cdot t_2 = B \cdot a_2 \cdot s_2. \quad \text{(II)}$$

Multiplying (I) by $B \cdot s_2 \cdot t_2$ and (II) by $A \cdot t_1 \cdot s_1$ we obtain $A \cdot B \cdot s_1 \cdot s_2 \cdot t_2 \cdot a_1 = A \cdot B \cdot s_2 \cdot t_2 \cdot t_1 \cdot r_1$ and $A \cdot B \cdot s_1 \cdot s_2 \cdot t_1 \cdot a_2 = A \cdot B \cdot s_1 \cdot t_1 \cdot t_2 \cdot r_2$. Adding the above equalities, we obtain

$$A \cdot B \cdot (s_1 \cdot s_2 \cdot (t_2 \cdot a_1 + t_1 \cdot a_2)) = A \cdot B \cdot (t_1 \cdot t_2 \cdot (s_2 \cdot r_1 + s_1 \cdot r_2)).$$

Therefore $\| r_1 \cdot s_2 + r_2 \cdot s_1, s_1 \cdot s_2 \| = \| a_1 \cdot t_2 + a_2 \cdot t_1, t_1 \cdot t_2 \|$ which implies that

$$\ll r_1 \cdot s_2 + r_2 \cdot s_1, s_1 \cdot s_2 \gg = \ll a_1 \cdot t_2 + a_2 \cdot t_1, t_1 \cdot t_2 \gg,$$

hence $\uplus$ is well defined.

Now, multiplying (I) with (II) we obtain $A \cdot B \cdot (r_1 \cdot r_2) \cdot (t_1 \cdot t_2) = A \cdot B \cdot (a_1 \cdot a_2) \cdot (s_1 \cdot s_2)$ and so $\| r_1 \cdot r_2, s_1 \cdot s_2 \| = \| a_1 \cdot a_2, t_1 \cdot t_2 \|$ which implies that

$$\ll r_1 \cdot r_2, s_1 \cdot s_2 \gg = \ll a_1 \cdot a_2, t_1 \cdot t_2 \gg,$$

therefore $\otimes$ is well defined. ■

Corollary 6.6.6. *For all $r \in R$, $s \in S$, we have $\ll r, s \gg = \ll r \cdot s, s \cdot s \gg$.*

Theorem 6.6.7. *$(S^{-1}R, \uplus, \otimes)$ is an H_v-ring, that we shall call the H_v-ring of fractions.*

Proof. If $[r_1, s_1], [r_2, s_2], [r_3, s_3] \in S^{-1}R$, then we have:

$$\{[r, s] \mid r \in r_1 \cdot (s_2 \cdot s_3) + (r_2 \cdot s_3 + r_3 \cdot s_2) \cdot s_1, s \in s_1 \cdot (s_2 \cdot s_3)\}$$
$$\subseteq [r_1, s_1] \uplus ([r_2, s_2] \uplus [r_3, s_3]),$$
$$\{[r, s] \mid r \in (r_1 \cdot s_2 + r_2 \cdot s_1)s_3 + r_3(s_1 \cdot s_2), s \in (s_1 \cdot s_2) \cdot s_3\}$$
$$\subseteq ([r_1, s_1] \uplus [r_2, s_2]) \uplus [r_3, s_3].$$

Since R is associative and distributive, we obtain that $(S^{-1}R, \uplus)$ is weak associative. The weak distributivity of $(S^{-1}R, \otimes)$ can be proved in a similar way.

Now, we prove the reproduction axioms for $(S^{-1}R, \uplus)$.

For every $[r, s], [r_1, s_1] \in S^{-1}R$, we have $s \in S, s_1 \in S$ and then by the definition of S there exists $s_2 \in S$ such that $s \in s_1 \cdot s_2$. On the other hand, since reproduction axioms hold for the additive law in R, we obtain $r_1 \cdot s_2 + (s_1 + 1)R = R$. Therefore, there exists $r_2 \in R$ such that $r \in r_1 \cdot s_2 + s_1 \cdot r_2 + r_2$ which implies that $r \in r_1 \cdot s_2 + (r_2 + r_2 \cdot s_3) \cdot s_1$ where $1 \in s_3 \cdot s_1$. Therefore, there exists $a \in r_2 + r_2 \cdot s_3$ such that $r \in r_1 \cdot s_2 + a \cdot s_1$. Hence

$$[r, s] \in [r_1, s_1] \uplus [a, s_2] = \ll r_1 \cdot s_2 + a \cdot s_1, s_1 \cdot s_2 \gg$$

which implies that $S^{-1}R \subseteq [r_1, s_1] \uplus S^{-1}R$, therefore $S^{-1}R = [r_1, s_1] \uplus S^{-1}R$.

Finally, we prove the weak distributivity of $\otimes$ with respect to $\uplus$. We have $\{[r, s] | r \in s_1 \cdot r_1 \cdot (r_2 \cdot s_3 + s_2 \cdot r_3), s \in s_1 \cdot s_1 \cdot (s_2 \cdot s_3)\} \subseteq [r_1, s_1] \otimes ([r_2, s_2] \uplus [r_3, s_3]),$

by Corollary 6.6.6, and hence

$$\{[r,s] \mid r \in (r_1 \cdot r_2) \cdot (s_1 \cdot s_3) + (r_1 \cdot r_3) \cdot (s_1 \cdot s_2), s \in (s_1 \cdot s_2) \cdot (s_1 \cdot s_3)\} \\ \subseteq ([r_1, s_1] \otimes [r_2, s_2]) \uplus ([r_1, s_1] \otimes [r_3, s_3]).$$

Therefore,

$$[r_1, s_1] \otimes ([r_2, s_2] \uplus [r_3, s_3]) \cap ([r_1, s_1] \otimes [r_2, s_2]) \uplus ([r_1, s_1] \otimes [r_3, s_3]) \neq \emptyset$$

and in the similar way we obtain

$$(([r_1, s_1] \uplus [r_2, s_2]) \otimes [r_3, s_3]) \cap (([r_1, s_1] \otimes [r_3, s_3]) \uplus ([r_2, s_2] \otimes [r_3, s_3])) \neq \emptyset$$

thus, $(S^{-1}R, \uplus, \otimes)$ is an H_v-ring. ■

Theorem 6.6.8. *Let R_1 and R_2 be two commutative hyperrings with scalar unit and S be a s.m.c.s. of R_1 and let $g : R_1 \longrightarrow R_2$ be a strong homomorphism of H_v-rings such that $g(1) = 1$. Then g induces an H_v-homomorphism $\overline{g} : S^{-1}R_1 \longrightarrow g(S)^{-1}R_2$ by setting*

$$\overline{g}([r,s]) = [g(r), g(s)].$$

Proof. It is clear that $g(S)$ is a s.m.c.s. of R_2. First, we prove that $\overline{g}$ is well defined. If $[r,s] = [r_1, s_1]$ then there exists $A \subseteq S$ such that $A \cdot r \cdot s_1 = A \cdot r_1 \cdot s$ which implies that $g(A \cdot r \cdot s_1) = g(A \cdot r_1 \cdot s)$ or $g(A) \cdot g(r) \cdot g(s_1) = g(A) \cdot g(r_1) \cdot g(s)$. Since $g(A) \subseteq g(S)$, we obtain $[g(r), g(s)] = [g(r_1), g(s_1)]$ or $\overline{g}([r,s]) = \overline{g}([r_1, s_1])$. Thus, $\overline{g}$ is well defined.

Moreover, $\overline{g}$ is an H_v-homomorphism because we have

$$\{[a,b] \mid a \in g(r_1 \cdot s_2 + r_2 \cdot s_1), b \in g(s_1 \cdot s_2)\} \subseteq \overline{g}([r_1, s_1] \uplus [r_2, s_2])$$

and

$$\{[a,b] | a \in g(r_1) \cdot g(s_2) + g(r_2) \cdot g(s_1), b \in g(s_1) \cdot g(s_2)\} \subseteq \overline{g}([r_1, s_1]) \uplus \overline{g}([r_2, s_2]).$$

Therefore,

$$\overline{g}([r_1, s_1] \uplus [r_2, s_2]) \cap (\overline{g}([r_1, s_1]) \uplus \overline{g}([r_2, s_2])) \neq \emptyset$$

and similarly we obtain

$$\overline{g}([r_1, s_1] \otimes [r_2, s_2]) \cap (\overline{g}([r_1, s_1]) \otimes \overline{g}([r_2, s_2])) \neq \emptyset,$$

which proves that $\overline{g}$ is an H_v-homomorphism. ■

Definition 6.6.9. Let R be an H_v-ring. A nonempty subset I of R is called an *H_v-ideal* if the following conditions hold:

(1) $(I, +)$ is an H_v-subgroup of $(R, +)$,

(2) $I \cdot R \subseteq R$ and $R \cdot I \subseteq I$.

An H_v-ideal I is called an *H_v-isolated ideal* if it satisfies the following axiom,

- For all $X \subseteq I, Y \subseteq S$ if $(M, N) \in \| X, Y \|$, then $M \subseteq I$.

Lemma 6.6.10. *If I is an H_v-isolated ideal of R, then the set $S^{-1}I = \{[a, s] \mid a \in I, s \in S\}$ is an H_v-ideal of $S^{-1}R$.*

Proof. First, we prove that $(S^{-1}I, \uplus)$ is an H_v-subgroup of $(S^{-1}R, \uplus)$. For every $[a_1, s_1], [a_2, s_2] \in S^{-1}I$, we have

$$[a_1, s_1] \uplus [a_2, s_2] = \bigcup_{(A,B) \in \| a_1 \cdot s_2 + a_2 \cdot s_1, s_1 \cdot s_2 \|} \{[a, s] | a \in A, s \in B\}.$$

From $a_1, a_2 \in I$ we obtain $a_1 \cdot s_2 + a_2 \cdot s_1 \subseteq I$ and since I is an H_v-isolated ideal of R, it follows that $A \subseteq I$. Therefore, $[a_1, s_1] \uplus [a_2, s_2] \subseteq S^{-1}I$.

Now, we prove the equality $S^{-1}I = [a_1, s_1] \uplus S^{-1}I$, for all $[a_1, s_1] \in S^{-1}I$. Suppose that $[a, s] \in S^{-1}I,\ a \in I$. Since $s, s_1 \in S$, there exists $s_2 \in S$ such that $s \in s_1 \cdot s_2$. Moreover, since I is an H_v-ideal, we have $a_1 \cdot s_2 + (s_1 + 1)I = I$. Hence there exists $a_2 \in I$, such that $a \in a_1 \cdot s_2 + s_1 \cdot a_2 + a_2$ and so $a \in a_1 \cdot s_2 + (a_2 + a_2 \cdot s_3) \cdot s_1$, whence $1 \in s_3 \cdot s_1$. So there exists $b \in a_2 + a_2 \cdot s_3$ such that $a \in a_1 \cdot s_2 + b \cdot s_1$, therefore $[a, s] \in [a_1, s_1] \uplus [b, s_2]$ implying $S^{-1}I \subseteq [a_1, s_1] \uplus S^{-1}I$.

It remains to prove the second condition of the definition of an H_v-ideal. In order to do this, suppose that $[a, t] \in S^{-1}I$ and $[r, s] \in S^{-1}R$. Then

$$[a, t] \otimes [r, s] = \bigcup_{(A,B) \in \| a \cdot r, t \cdot s \|} \{[x, y] | x \in A, y \in B\}.$$

Since $a \in I$ and I is an H_v-isolated ideal of R, we have $a \cdot r \subseteq I$ and so $A \subseteq I$. Consequently $[a,t] \otimes [r,s] \subseteq S^{-1}I$. Therefore, $S^{-1}I$ is an H_v-ideal of $S^{-1}R$. ■

Lemma 6.6.11. *If I, J are two H_v-isolated ideals of R, then*

(1) $S^{-1}(I \cap J) = S^{-1}I \cap S^{-1}J$,

(2) $S^{-1}(I \cdot J) = S^{-1}I \otimes S^{-1}J$,

(3) $S^{-1}(I + J) \subseteq S^{-1}I \uplus S^{-1}J$.

Proof. The proof is straightforward and is omitted. ■

The *natural mapping* $\psi : R \longrightarrow S^{-1}R$, where $\psi(r) = [r, 1]$, is an inclusion homomorphism.

Theorem 6.6.12. Let I be an H_v-isolated ideal of R. Then $S \cap I \neq \emptyset$ if and only if $S^{-1}I = S^{-1}R$.

Proof. If $t \in S \cap I$, then $[t,t] = [1,1] \in S^{-1}I$. Therefore, for every $[r,s] \in S^{-1}R$, we have $[1,1] \otimes [r,s] \subseteq S^{-1}I$. From $[r,s] \in [1,1] \otimes [r,s]$ we obtain $[r,s] \in S^{-1}I$ and this prove that $S^{-1}R \subseteq S^{-1}I$.

Conversely, assume that $S^{-1}I = S^{-1}R$. If we consider the natural inclusion homomorphism $\psi : R \longrightarrow S^{-1}R$, then $\psi(1) = [1,1]$. On the other hand, $\psi(1) \in S^{-1}R$, consequently $\psi(1) \in S^{-1}I$ and so $\psi(1) = [a,s]$ for some $a \in I, s \in S$. Now, we have $[1,1] = [a,s]$, therefore, there exists $A \subseteq S$ such that $A \cdot s = A \cdot a$. Since $A \cdot s \subseteq S$ and $A \cdot a \subseteq I$, we get $I \cap S \neq \emptyset$. ■

Theorem 6.6.13. *Let I be an H_v-isolated ideal of R. Then the following assertions hold:*

(1) $I \subseteq \psi^{-1}(S^{-1}I)$,

(2) *If $I = \psi^{-1}(J)$ for some H_v-ideal J of $S^{-1}R$, then $S^{-1}I = J$.*

Proof. The proof of (1) is obvious. In order to prove (2), let $I = \psi^{-1}(J)$ where J is an H_v-ideal of $S^{-1}R$. Then $[r,s] \in S^{-1}I$ implies $r \in I$ and so $\psi(r) = [r,1] \in J$. Therefore, $[1,s] \otimes [r,1] \subseteq J$. Since $[r,s] \in [1,s] \otimes [r,1]$, we obtain $[r,s] \in J$ which implies that $S^{-1}I \subseteq J$. Now, let $[r,s] \in J$. Then $\psi(r) = [r,1] \in [r,1] \otimes [s,s] = [r,s] \otimes [s,1] \subseteq J$. Therefore $r \in \psi^{-1}(J) = I$, hence $[r,s] \in S^{-1}I$, and this proves that $J \subseteq S^{-1}I$. ■

Definition 6.6.14. Let A be a commutative H_v-ring. An H_v-ideal P is called an H_v-*prime ideal* of A, if $a \cdot b \subseteq P$ implies $a \in P$ or $b \in P$.

Theorem 6.6.15. *If P is an H_v-isolated prime ideal of R such that $S \cap P = \emptyset$, then $S^{-1}P$ is an H_v-prime ideal of $S^{-1}R$ and $\psi^{-1}(S^{-1}P) = P$.*

Proof. By Lemma 6.6.10, $S^{-1}P$ is an H_v-ideal of $S^{-1}R$. Now, we check that $S^{-1}P$ is prime. If $[r, s] \otimes [r_1, s_1] \subseteq S^{-1}P$, then $\{[b, s_2] \mid b \in r \cdot r_1, s_2 \in s \cdot s_1\} \subseteq \ll r \cdot r_1, s \cdot s_1 \gg \subseteq S^{-1}P$. It follows that for every $b \in r \cdot r_1$, $s_2 \in s \cdot s_1$ there exists $a \in P$ and $t \in S$ such that $[b, s_2] = [a, t]$. Therefore, there exists a subset A of S such that $A \cdot b \cdot t = A \cdot a \cdot s_2$. Since $A \cdot a \cdot s_2 \subseteq P$, we have $A \cdot b \cdot t \subseteq P$. Now, for every $x \in A \cdot t$ we obtain $x \cdot b \subseteq P$. Since $A \cdot t \subseteq S$ and $S \cap P = \emptyset$, it follows that $x \notin P$ and so $b \in P$. Consequently $r \cdot r_1 \subseteq P$ which implies that $r \in P$ or $r_1 \in P$. Therefore, $[r, s] \in S^{-1}P$ or $[r_1, s_1] \in S^{-1}P$.

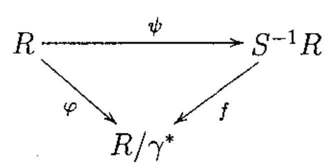
On the other hand, by Theorem 6.6.13, we have $P \subseteq \psi^{-1}(S^{-1}P)$.

Conversely, assume that $r \in \psi^{-1}(S^{-1}P)$. Then, $\psi(r) \in S^{-1}P$ and since $\psi(r) = [r, 1]$, there exists $a \in P, t \in S$ such that $[r, 1] = [a, t]$. Therefore, there is a subset A of S such that $A \cdot r \cdot t = A \cdot a$. Since $A \cdot a \subseteq P$, we have $A \cdot r \cdot t \subseteq P$. Now, for every $x \in A \cdot t$, we obtain $x \cdot r \subseteq P$. Since $A \cdot t \subseteq S$, it follows that $x \notin P$ and so, $r \in P$. Therefore, $\psi^{-1}(S^{-1}P) = P$. ∎

Lemma 6.6.16. *For every $a \in S$, $\gamma^*(a)$ is invertible in R/γ^*.*

Proof. Since $1 \in R$, then $\gamma^*(1) \in R/\gamma^*$. Now, for every $\gamma^*(x) \in R/\gamma^*$, we have $\gamma^*(x) \odot \gamma^*(1) = \gamma^*(1) \odot \gamma^*(x) = \gamma^*(x)$, i.e. $\gamma^*(1)$ is the identity of the ring R/γ^*. On the other hand, by the definition of S, for every $a \in S$ there exists $b \in S$ such that $1 \in a \cdot b = b \cdot a$. Therefore, $\gamma^*(1) = \gamma^*(a \cdot b) = \gamma^*(b \cdot a)$ and so $\gamma^*(1) = \gamma^*(a) \odot \gamma^*(b) = \gamma^*(b) \odot \gamma^*(a)$ which implies that $\gamma^*(b)$ is the inverse of $\gamma^*(a)$. ∎

Theorem 6.6.17. *If all subsets A of S are finite polynomials of elements of R over $\mathbb{N}$, then there exists an H_v-homomorphism $f : S^{-1}R \longrightarrow R/\gamma^*$ such that $f\psi = \varphi$, i.e., the following diagram is commutative.*



Proof. We define $f : S^{-1}R \longrightarrow R/\gamma^*$ by setting $f([r,s]) = \gamma^*(r) \odot \gamma^*(s)^{-1}$. First, we prove that f is well defined. If $[r,s] = [r_1, s_1]$, then there exists $A \subseteq S$ such that $A \cdot r \cdot s_1 = A \cdot r_1 \cdot s$ and so $\varphi(A \cdot r \cdot s_1) = \varphi(A \cdot r_1 \cdot s)$ which implies that $\gamma^*(A) \odot \gamma^*(r) \odot \gamma^*(s_1) = \gamma^*(A) \odot \gamma^*(r_1) \odot \gamma^*(s)$. By hypothesis $\gamma^*(A) = \gamma^*(a)$ for every $a \in A$, so we obtain $\gamma^*(a) \odot \gamma^*(r) \odot \gamma^*(s_1) = \gamma^*(a) \odot \gamma^*(r_1) \odot \gamma^*(s)$.

Multiplying the above relation by $\gamma^*(a)^{-1} \odot \gamma^*(s)^{-1} \odot \gamma^*(s_1)^{-1}$, we have $\gamma^*(r) \odot \gamma^*(s)^{-1} = \gamma^*(r_1) \odot \gamma^*(s_1)^{-1}$. Therefore, $f([r,s]) = f([r_1, s_1])$. Thus, f is well defined. Moreover, f is an H_v-homomorphism, because we have

$$\begin{array}{l}
\gamma^*(r_1 \cdot s_2 + r_2 \cdot s_1) \odot \gamma^*(s_1 \cdot s_2)^{-1} \in f([r_1, s_1)] \uplus [r_2, s_2]), \\
(\gamma^*(r_1) \odot \gamma^*(s_1)^{-1}) \oplus (\gamma^*(r_2) \odot \gamma^*(s_2)^{-1}) = f([r_1, s_1]) \oplus f([r_2, s_2]), \\
\gamma^*(r_1 \cdot r_2) \odot \gamma^*(s_1 \cdot s_2)^{-1} \in f([r_1, s_1]) \otimes [r_2, s_2]), \\
(\gamma^*(r_1) \odot \gamma^*(s_1)^{-1}) \odot (\gamma^*(r_2) \odot \gamma^*(s_2)^{-1}) = f([r_1, s_1]) \odot f([r_2, s_2]).
\end{array}$$

Finally, it is clear that $f\psi = \varphi$. ∎

Let γ_s^* be the fundamental equivalence relation on $S^{-1}R$ and let $\mathcal{U}_s$ denotes the set of finite polynomials of elements of $S^{-1}R$ over $\mathbb{N}$.

Theorem 6.6.18. *There exists a homomorphism* $h : R/\gamma^* \longrightarrow S^{-1}R/\gamma_s^*$.

Proof. We define $h(\gamma^*(r)) = \gamma_s^*([r,1])$. First, we prove that h is well defined. Suppose that $\gamma^*(r_1) = \gamma^*(r_2)$, so $r_1 \gamma^* r_2$. Hence there exist $x_1, ..., x_{m+1} \in R$; $u_1, ..., u_m \in \mathcal{U}$ with $x_1 = r_1$, $x_{m+1} = r_2$ such that $\{x_i, x_{i+1}\} \subseteq u_i$, $i = 1, ..., m$ which implies that

$$\{[x_i, 1], [x_{i+1}, 1]\} \subseteq \ll u_i, 1 \gg \in \mathcal{U}_s.$$

Therefore, $[r_1, 1] \gamma_s^* [r_2, 1]$ and so $\gamma_s^*([r_1, 1]) = \gamma_s^*([r_2, 1])$. Thus, h is well defined. h is a homomorphism, because

$$h(\gamma^*(a) \oplus \gamma^*(b)) = h(\gamma^*(c)) = \gamma_s^*([c,1]) \text{ for all } c \in \gamma^*(a) + \gamma^*(b)$$

and

$$\begin{aligned}
h(\gamma^*(a)) \oplus h(\gamma^*(b)) &= \gamma_s^*([a,1]) \oplus \gamma_s^*([b,1]) \\
&= \gamma_s^*([d,s]) \text{ for all } [d,s] \in \gamma_s^*([a,1]) \uplus \gamma_s^*([b,1]).
\end{aligned}$$

Thus, setting $d = c \in a + b,\ s = 1$, we obtain

$$h(\gamma^*(a) \oplus \gamma^*(b)) = h(\gamma^*(a)) \oplus h(\gamma^*(b)).$$

Similarly, we obtain

$$h(\gamma^*(a) \odot \gamma^*(b)) = h(\gamma^*(a)) \odot h(\gamma^*(b)).$$

Therefore, h is a homomorphism of rings. ■

Corollary 6.6.19. *The following diagram is commutative, i.e., $\varphi_s\psi = h\varphi$ where φ and φ_s are the canonical projections.*

$$\begin{array}{ccc} R & \xrightarrow{\psi} & S^{-1}R \\ \varphi\downarrow & & \downarrow\varphi_s \\ R/\gamma^* & \xrightarrow{h} & S^{-1}R/\gamma_s^* \end{array}$$

Corollary 6.6.20. *If $\varphi : R \longrightarrow R/\gamma^*$ is the canonical projection, then the map $\theta : S^{-1}R \longrightarrow \varphi(S)^{-1}(R/\gamma^*)$ defined by $\theta([r,s]) = [\gamma^*(r), \gamma^*(s)]$ is an H_v-homomorphism.*

Corollary 6.6.21. *The following diagram is commutative, i.e., $\theta\psi = \psi_1\varphi$.*

$$\begin{array}{ccc} R & \xrightarrow{\psi} & S^{-1}R \\ \varphi\downarrow & & \downarrow\theta \\ R/\gamma^* & \xrightarrow{\psi_1} & \varphi(S)^{-1}(R/\gamma^*) \end{array}$$

6.7 Rough sets in a fundamental ring

In this paragraph, we present a connection between rough sets and H_v-rings. We use the results obtained by B. Davvaz [30].

Let $(R, +, \cdot)$ be an H_v-ring. For a subset $A \subseteq R$ we define two approximations of A relative to the fundamental equivalence relation γ^*:

$$\underline{\gamma^*}(A) = \{x \in R \mid \gamma^*(x) \subseteq A\} \quad \text{and} \quad \overline{\gamma^*}(A) = \{x \in R \mid \gamma^*(x) \cap A \neq \emptyset\}.$$

The set $\underline{\gamma^*}(A)$ is called the γ^*-*lower approximation* of A, and the set $\overline{\gamma^*}(A)$ is called the γ^*-*upper approximation* of A. In the following proposition, we collect the basic properties of the approximations of A, which follow directly from their definitions.

Proposition 6.7.1.

(1) $\underline{\gamma^*}(A) \subseteq A \subseteq \overline{\gamma^*}(A)$.

(2) $\underline{\gamma^*}(\underline{\gamma^*}(A)) = \underline{\gamma^*}(A)$ and $\overline{\gamma^*}(\overline{\gamma^*}(A)) = \overline{\gamma^*}(A)$.

The difference $\widehat{\gamma^*(A)} = \overline{\gamma^*}(A) - \underline{\gamma^*}(A)$ is called the γ^*-*boundary region* of A. In the case when $\widehat{\gamma^*(A)} = \emptyset$ the set A is said to be γ^*-*exact*; otherwise A is γ^*-*rough.*

Proposition 6.7.2. *If A and B are nonempty subsets of R, then*

(1) $\overline{\gamma^*}(A) + \overline{\gamma^*}(B) \subseteq \overline{\gamma^*}(A + B)$.

(2) $\overline{\gamma^*}(A) \cdot \overline{\gamma^*}(B)) \subseteq \overline{\gamma^*}(A \cdot B)$.

Proof. We prove only (1), because the proof of (2) is similar to (1). Suppose that c is an element of $\overline{\gamma^*}(A) + \overline{\gamma^*}(B)$. Then, $c \in a + b$ where $a \in \overline{\gamma^*}(A)$ and $b \in \overline{\gamma^*}(B)$. Thus, there exist the elements $x, y \in R$ such that $x \in \gamma^*(a) \cap A$ and $y \in \gamma^*(b) \cap B$. Therefore,

$$x + y \subseteq \gamma^*(a) + \gamma^*(b) \subseteq \gamma^*(a + b).$$

Since $x + y \subseteq A + B$, we have $x + y \subseteq \gamma^*(a + b) \cap (A + B)$ and so $\gamma^*(a + b) \cap (A + B) \neq \emptyset$. Therefore, for every $c \in a + b$ we have $\gamma^*(c) \cap (A + B) \neq \emptyset$ which implies that $c \in \overline{\gamma^*}(A + B)$. Therefore, $a + b \subseteq \overline{\gamma^*}(A + B)$. Thus, we have

$$\overline{\gamma^*}(A) + \overline{\gamma^*}(B) \subseteq \overline{\gamma^*}(A + B). \blacksquare$$

There exists another way to characterize a rough set by a membership function.

For any $A \subseteq R$, we define a *rough membership function* as follows:

$$\mu_A(x) = \frac{|A \cap \gamma^*(x)|}{|\gamma^*(x)|},$$

where $|\ |$ denotes the cardinality of a set. By the definition, the elements in the same equivalence class have the same degree of membership. One can notice the similarity between a rough membership function and a conditional probability. The rough membership value $\mu_A(x)$ may be interpreted as the probability of x belonging to A given that x belongs to an equivalence class.

The following proposition collects the basic properties of the rough membership functions.

Proposition 6.7.3. *The rough membership functions of the form μ_A have the following properties:*

(1) $\mu_A(x) = 1$ *if and only if* $x \in \underline{\gamma^*}(A)$,

(2) $\mu_A(x) = 0$ *if and only if* $x \in \underline{\gamma^*}(A^c)$,

(3) $0 < \mu_A(x) < 1$ *if and only if* $x \in \widehat{\gamma^*(A)}$,

(4) $\mu_A(x) = 1 - \mu_{A^c}(x)$,

(5) $\mu_{A\cup B}(x) \geq max\{\mu_A(x), \mu_B(x)\}$,

(6) $\mu_{A\cap B}(x) \geq min\{\mu_A(x), \mu_B(x)\}$.

The lower and upper approximations can be presented in an equivalent form as shown below. Let A be a nonempty subsets of R. Then,

$$\underline{\gamma^*(A)} = \{\gamma^*(x) \in R/\gamma^* \mid \gamma^*(x) \subseteq A\}$$

and

$$\overline{\gamma^*(A)} = \{\gamma^*(x) \in R/\gamma^* \mid \gamma^*(x) \cap A \neq \emptyset\}.$$

We can interpret these sets as subsets of the fundamental ring R/γ^* of an H_v-ring R.

Proposition 6.7.4. *Let A and B are nonempty subsets of R. Then the following assertions:*

(1) $\overline{\gamma^*(A \cup B)} = \overline{\gamma^*(A)} \cup \overline{\gamma^*(B)}$,

(2) $\underline{\gamma^*(A \cap B)} = \underline{\gamma^*(A)} \cap \underline{\gamma^*(B)}$,

(3) $A \subseteq B$ *implies* $\overline{\gamma^*(A)} \subseteq \overline{\gamma^*(B)}$,

(4) $A \subseteq B$ *implies* $\underline{\gamma^*(A)} \subseteq \underline{\gamma^*(B)}$,

(5) $\underline{\gamma^*(A)} \cup \underline{\gamma^*(B)} \subseteq \underline{\gamma^*(A \cup B)}$,

(6) $\overline{\gamma^*(A \cap B)} \subseteq \overline{\gamma^*(A)} \cap \overline{\gamma^*(B)}$.

Proof.

$$
\begin{aligned}
(1) \quad \gamma^*(x) \in \overline{\gamma^*(A \cup B)} &\Longleftrightarrow \gamma^*(x) \cap (A \cup B) \neq \emptyset \\
&\Longleftrightarrow (\gamma^*(x) \cap A) \cup (\gamma^*(x) \cap B) \neq \emptyset \\
&\Longleftrightarrow \gamma^*(x) \cap A \neq \emptyset \ \text{ or } \ \gamma^*(x) \cap B \neq \emptyset \\
&\Longleftrightarrow \gamma^*(x) \in \overline{\gamma^*(A)} \ \text{ or } \ \gamma^*(x) \in \overline{\gamma^*(B)} \\
&\Longleftrightarrow \gamma^*(x) \in \overline{\gamma^*(A)} \cup \overline{\gamma^*(B)}.
\end{aligned}
$$

Thus, $\overline{\gamma^*(A \cup B)} = \overline{\gamma^*(A)} \cup \overline{\gamma^*(B)}$.

$$
\begin{aligned}
(2) \quad \gamma^*(x) \in \underline{\gamma^*(A \cap B)} &\Longleftrightarrow \gamma^*(x) \subseteq A \cap B \\
&\Longleftrightarrow \gamma^*(x) \subseteq A \ \text{ and } \ \gamma^*(x) \subseteq B \\
&\Longleftrightarrow \gamma^*(x) \in \underline{\gamma^*(A)} \ \text{ and } \ \gamma^*(x) \in \underline{\gamma^*(B)} \\
&\Longleftrightarrow \gamma^*(x) \in \underline{\gamma^*(A)} \cap \underline{\gamma^*(B)}.
\end{aligned}
$$

Thus, $\underline{\gamma^*(A \cap B)} = \underline{\gamma^*(A)} \cap \underline{\gamma^*(B)}$.

(3) Since $A \subseteq B$ if and only if $A \cup B = B$, by (1) we obtain $\overline{\gamma^*(B)} = \overline{\gamma^*(A \cup B)} = \overline{\gamma^*(A)} \cup \overline{\gamma^*(B)}$. This implies that $\overline{\gamma^*(A)} \subseteq \overline{\gamma^*(B)}$.

(4) Since $A \subseteq B$ if and only if $A \cap B = A$, by (2) we obtain $\underline{\gamma^*(A)} = \underline{\gamma^*(A \cap B)} = \underline{\gamma^*(A)} \cap \underline{\gamma^*(B)}$. This implies that $\underline{\gamma^*(A)} \subseteq \underline{\gamma^*(B)}$.

(5) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (4) we obtain $\underline{\gamma^*(A)} \subseteq \underline{\gamma^*(A \cup B)}$ and $\underline{\gamma^*(B)} \subseteq \underline{\gamma^*(A \cup B)}$, which yields $\underline{\gamma^*(A)} \cup \underline{\gamma^*(B)} \subseteq \underline{\gamma^*(A \cup B)}$.

(6) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by (3) we obtain $\overline{\gamma^*(A \cap B)} \subseteq \overline{\gamma^*(A)}$ and $\overline{\gamma^*(A \cap B)} \subseteq \overline{\gamma^*(B)}$, which yields $\overline{\gamma^*(A \cap B)} \subseteq \overline{\gamma^*(A)} \cap \overline{\gamma^*(B)}$. ■

In what follows, we denote the identity of the group $(R/\gamma^*, \oplus)$ by ω_R, too.

Theorem 6.7.5. *If A is an H_v-subgroup of $(R, +)$, then $\overline{\gamma^*(A)}$ is a subgroup of $(R/\gamma^*, \oplus)$.*

Proof. First, we show that $\omega_R \in \overline{\gamma^*(A)}$. Since A is an H_v-subgroup of $(R, +)$, then for every $a \in A$ we have $a + A = A$. Therefore, $a \in a + A$ and so there exists $b \in A$ such that $a \in a + b$ which implies that $\gamma^*(a) = \gamma^*(a + b) = \gamma^*(a) \oplus \gamma^*(b)$. Therefore, $\gamma^*(b) = \omega_R$ and so $b \in \omega_R \cap A$ which implies that $\omega_R \cap A \neq \emptyset$. Therefore $\omega_R \in \overline{\gamma^*(A)}$.

Now, suppose that $\gamma^*(x), \gamma^*(y) \in \overline{\gamma^*(A)}$. We show that $\gamma^*(x) \oplus \gamma^*(y) \in \overline{\gamma^*(A)}$. We have $\gamma^*(x) \cap A \neq \emptyset$ and $\gamma^*(y) \cap A \neq \emptyset$ whence there exist $a \in \gamma^*(x) \cap A$ and $b \in \gamma^*(y) \cap A$. Thus, $a \in \gamma^*(x)$, $a \in A$, $b \in \gamma^*(y)$, $b \in A$ and so

$$a + b \subseteq \gamma^*(x) + \gamma^*(y) \subseteq \gamma^*(x + y) = \gamma^*(x) \oplus \gamma^*(y).$$

For every $c \in x + y$ we have $\gamma^*(c) = \gamma^*(x) \oplus \gamma^*(y)$. Therefore, we obtain

$$a + b \subseteq \gamma^*(c) \text{ and } a + b \subseteq A.$$

Therefore, $\gamma^*(c) \cap A \neq \emptyset$ which yields $\gamma^*(c) \in \overline{\gamma^*(A)}$ or $\gamma^*(x) \oplus \gamma^*(y) \in \overline{\gamma^*(A)}$.

Finally, if $\gamma^*(x) \in \overline{\gamma^*(A)}$ then we show that $-\gamma^*(x) \in \overline{\gamma^*(A)}$. Since $\omega_R \cap A \neq \emptyset$, then there exists $r \in \omega_R \cap A$ and since $\gamma^*(x) \cap A \neq \emptyset$, then there exists $y \in \gamma^*(x) \cap A$. By the reproduction axiom we obtain $r \in y + A$. Then there exists $a \in A$ such that $r \in y + a$ which implies that $\gamma^*(r) = \gamma^*(y) \oplus \gamma^*(a)$. Since $r \in \omega_R$ then $\gamma^*(r) = \omega_R$. Therefore, $\omega_R = \gamma^*(y) \oplus \gamma^*(a) = \gamma^*(x) \oplus \gamma^*(a)$ which yields $\gamma^*(a) = -\gamma^*(x)$. Since $a \in A$ and $a \in \gamma^*(a)$ then $\gamma^*(a) \cap A \neq \emptyset$ and so $\gamma^*(a) \in \overline{\gamma^*(A)}$. Therefore, $\overline{\gamma^*(A)}$ is a subgroup of $(R/\gamma^*, \oplus)$. ∎

Lemma 6.7.6. *If A and B are nonempty subsets of R, then*

$$\overline{\gamma^*(A)} \oplus \overline{\gamma^*(B)} \subseteq \overline{\gamma^*(A + B)}.$$

Proof. We have

$$\begin{aligned} \overline{\gamma^*(A)} \oplus \overline{\gamma^*(B)} &= \{\gamma^*(a) \oplus \gamma^*(b) \mid \gamma^*(a) \in \overline{\gamma^*(A)}\ ,\ \gamma^*(b) \in \overline{\gamma^*(B)}\} \\ &= \{\gamma^*(a) \oplus \gamma^*(b) \mid \gamma^*(a) \cap A \neq \emptyset\ ,\ \gamma^*(b) \cap B \neq \emptyset\}. \end{aligned}$$

Therefore, $(\gamma^*(a) + \gamma^*(b)) \cap (A + B) \neq \emptyset$. Since $\gamma^*(a) + \gamma^*(b) \subseteq \gamma^*(a + b)$ we obtain

$$\gamma^*(a + b) \cap (A + B) \neq \emptyset.$$

Thus, $\gamma^*(a+b)=\gamma^*(a)\oplus\gamma^*(b)\in\overline{\gamma^*(A+B)}$ and so $\overline{\gamma^*(A)}\oplus\overline{\gamma^*(B)}\subseteq\overline{\gamma^*(A+B)}$. ∎

Lemma 6.7.7. *If A is a nonempty subset of R and B is an H_v-ideal of R, then*

$$\overline{\gamma^*(A)} \odot \overline{\gamma^*(B)} \subseteq \overline{\gamma^*(B)}.$$

Proof. We have

$$\begin{aligned}\overline{\gamma^*(A)} \odot \overline{\gamma^*(B)} &= \{\oplus \sum_{finite} (\gamma^*(a) \odot \gamma^*(b)) |\ \gamma^*(a) \in \overline{\gamma^*(A)}\ ,\ \gamma^*(b) \in \overline{\gamma^*(B)}\} \\ &= \{\oplus \sum_{finite} (\gamma^*(a) \odot \gamma^*(b)) |\ \gamma^*(a) \cap A \neq \emptyset\ ,\ \gamma^*(b) \cap B \neq \emptyset\}.\end{aligned}$$

Therefore, $\gamma^*(a) \cdot \gamma^*(b) \cap A \cdot B \neq \emptyset$. Since $\gamma^*(a) \cdot \gamma^*(b) \subseteq \gamma^*(a \cdot b)$ we obtain

$$\gamma^*(a \cdot b) \cap A \cdot B \neq \emptyset.$$

Since B is an H_v-ideal of R, we have $A \cdot B \subseteq B$ and so

$$\gamma^*(a \cdot b) \cap B \neq \emptyset.$$

Thus, $\gamma^*(a \cdot b) = \gamma^*(a) \odot \gamma^*(b) \in \gamma^*(B)$. Since B is an H_v-subgroup of $(R, +)$, then $\overline{\gamma^*(B)}$ is a subgroup of $(R/\gamma^*, \oplus)$, therefore $\oplus \sum_{finite} \gamma^*(a) \odot \gamma^*(b) \in \overline{\gamma^*(B)}$ and so $\overline{\gamma^*(A)} \odot \overline{\gamma^*(B)} \subseteq \overline{\gamma^*(B)}$. ∎

Corollary 6.7.8. *If A and B are H_v-ideals of R, then*

$$\overline{\gamma^*(A)} \odot \overline{\gamma^*(B)} \subseteq \overline{\gamma^*(A)} \cap \overline{\gamma^*(B)}.$$

Theorem 6.7.9. *If A is an H_v-ideal of R, then $\overline{\gamma^*(A)}$ is an ideal of R/γ^*.*

Proof. Suppose that A is an H_v-ideal of R. Then $(\overline{\gamma^*(A)}, \oplus)$ is a subgroup of $(R/\gamma^*, \oplus)$. Notice that $\overline{\gamma^*(R)} = R/\gamma^*$. Then we have

$$R/\gamma^* \odot \overline{\gamma^*(A)} = \overline{\gamma^*(R)} \odot \overline{\gamma^*(A)} \subseteq \overline{\gamma^*(A)},$$

$$\overline{\gamma^*(A)} \odot R/\gamma^* = \overline{\gamma^*(A)} \odot \overline{\gamma^*(R)} \subseteq \overline{\gamma^*(A)} \blacksquare$$

Definition 6.7.10. Let A, B and C be H_v-ideals of R. The sequence of strong homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be *exact* if for every $x \in A$,

$$g \circ f(x) \in \omega_R.$$

Theorem 6.7.11. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of H_v-ideals of R. Then, the sequence*

$$\overline{\gamma^*(A)} \xrightarrow{F} \overline{\gamma^*(B)} \xrightarrow{G} \overline{\gamma^*(C)}$$

is an exact sequence of ideals of R/γ^* where

$$F(\gamma^*(a)) = \gamma^*(f(a)) \text{ for all } a \in A,$$

$$G(\gamma^*(b)) = \gamma^*(g(b)) \text{ for all } b \in B.$$

Proof. First, we prove that F is well defined. Suppose that $\gamma^*(a)=\gamma^*(b)$ then there exist $x_1, ..., x_{m+1}$ and $u_1, ..., u_m \in \mathcal{U}$ with $x_1 = a$, $x_{m+1} = b$ such that

$$\{x_i, x_{i+1}\} \subseteq u_i, \quad i = 1, ..., m$$

which implies that

$$\{f(x_i), f(x_{i+1})\} \subseteq f(u_i), \quad i = 1, ..., m.$$

Since f is a homomorphism and $u_i \in \mathcal{U}$ we obtain $f(u_i) \in \mathcal{U}$. Therefore, $f(a)\gamma^* f(b)$ which means that $F(\gamma^*(a)) = F(\gamma^*(b))$. On the other hand if $\gamma^*(a) \in \overline{\gamma^*(A)}$, then $\gamma^*(a) \cap A \neq \emptyset$ and so there exists $b \in \gamma^*(a) \cap A$. Thus, $b\gamma^* a$ and $b \in A$ which yield $f(b)\gamma^* f(a)$ and $f(b) \in B$. So $f(b) \in \gamma^*(f(a))$ and $f(b) \in B$ whence $\gamma^*(f(a)) \cap B \neq \emptyset$ and so $\gamma^*(f(a)) \in \overline{\gamma^*(B)}$ which means that $F(\gamma^*(a)) \in \overline{\gamma^*(B)}$. Thus, F is well defined. Similarly G is well defined.

Now, we show that F is a homomorphism. We have

$$\begin{aligned} F(\gamma^*(a) \oplus \gamma^*(b)) &= F(\gamma^*(a+b)) \\ &= \gamma^*(f(a+b)) \\ &= \gamma^*(f(a) + f(b)) \\ &= \gamma^*(f(a)) \oplus \gamma^*(f(b)) \\ &= F(\gamma^*(a)) \oplus F(\gamma^*(b)) \end{aligned}$$

and

$$\begin{aligned} F(\gamma^*(a) \odot \gamma^*(b)) &= F(\gamma^*(a \cdot b)) \\ &= \gamma^*(f(a \cdot b)) \\ &= \gamma^*(f(a) \cdot f(b)) \\ &= \gamma^*(f(a)) \odot \gamma^*(f(b)) \\ &= F(\gamma^*(a)) \odot F(\gamma^*(b)) \end{aligned}$$

Therefore, F is a homomorphism. Similarly, G is a homomorphism.

Finally, it is enough to show that $ImF = KerG$. We have

$$\begin{aligned} \gamma^*(b) \in ImF &\Longrightarrow F(\gamma^*(a)) = \gamma^*(b) \text{ for some } a \in A \\ &\Longrightarrow \gamma^*(f(a)) = \gamma^*(b) \\ &\Longrightarrow G(\gamma^*(f(a)) = G(\gamma^*(b)) \\ &\Longrightarrow \gamma^*(g(f(a))) = G(\gamma^*(b)) \\ &\Longrightarrow G(\gamma^*(b)) \subseteq \gamma^*(\omega_R) \\ &\Longrightarrow G(\gamma^*(b)) = \gamma^*(g(b)) \subseteq \omega_R \\ &\Longrightarrow G(\gamma^*(b)) = \omega_R \\ &\Longrightarrow \gamma^*(b) \in KerG \end{aligned}$$

and so $ImF \subseteq KerG$.

Conversely, we can show that $KerG \subseteq ImF$. Therefore, $ImF{=}KerG$. ∎

6.8 H_v-group rings

In an H_v-group, several convolutions can be defined. In this paragraph, we present a convolution and obtain an H_v-group ring. We use the results obtained by S. Spartalis, A. Dramalidis and T. Vougiouklis [123]. Examples and applications in known classes of hyperstructures are also investigated.

Definition 6.8.1. Let $(H, \cdot)$ be a hypergroupoid. The following set is called a *set of fundamental maps on H with respect to* $\cdot$:

$$\Theta = \{\theta : H \times H \longrightarrow^{onto} H \mid \theta(x, y) \in x \cdot y\}.$$

Any subset $\Theta_\mu \subseteq \Theta$ define a hyperoperation $\circ_\mu$ on H as follows:

$$x \circ_\mu y = \{z \mid z = \theta(x, y) \text{ for some } \theta \in \Theta_\mu\}.$$

Obviously, $\circ_\mu \leq \cdot$ and $\Theta_\mu \subseteq \Theta_{\circ_\mu}$, where $\Theta_{\circ_\mu}$ denotes of the fundamental maps on H with respect to $\circ_\mu$. A set $\Theta_\alpha \subseteq \Theta$ is called *associative* (respectively weak associative) if and only if for every subset $\Theta_\mu \subseteq \Theta_\alpha$ the hyperoperation $\circ_\mu$ is associative (respectively weak associative). A hypergroupoid $(H,\cdot)$ will be called Θ-WASS if there exists an element $\theta_\circ \in \Theta$ which defines an associative operation $\circ$ in H.

- For every $\theta \in \Theta$ we have $1 \leq |\theta^{-1}(g)| \leq n^2 - n + 1$ for all $g \in H$ where $\theta^{-1}(g)$ is the inverse image of g. However, for every $\theta \in \Theta$ we have $\sum\limits_{g\in H} |\theta^{-1}(g)| = n^2$.
- If $(H,\cdot)$ is Θ-WASS, then every greater hypergroupoid is Θ-WASS.
- All H_b-groups are Θ-WASS.
- Any H_b-semigroup, which has a b-structure with the property $H^2 = H$ is Θ-WASS.
- If $(H,\cdot)$ is an H_v-group containing a scalar, then all the maps $f : H \times H \longrightarrow H$ with $f(x,y) \in xy$ for all $x, y \in H$, are onto, i.e., $f \in \Theta$. Indeed, if s is a scalar, then $f(s,H) = H$.

Example 6.8.2. An example of a type of b-semigroups is defined as follows: Fix an element $s \in H$. We define the product $\circ$ by setting:

$$x \circ y = \begin{cases} s & \text{if } x \neq y \\ x & \text{if } x = y. \end{cases}$$

Then $H \circ H = H$ and $\circ$ is associative, since

$$(x \circ y) \circ z = x \circ (y \circ z) = \begin{cases} x & \text{if } x = y = z \\ s & \text{otherwise.} \end{cases}$$

Definition 6.8.3. Let $(H,\cdot)$ be Θ-WASS with $|G| = n$ and $\theta_\circ \in \Theta$ be associative. Let F be a field and $F[H]$ be the set of formal linear combinations of elements of H with coefficients from F. In $F[H]$ the ordinary addition $+$ can be defined by setting

$$(f_1 + f_2)(g) = f_1(g) + f_2(g) \text{ for all } g \in H \text{ and } f_1, f_2 \in F[H].$$

Furthermore, consider the hyperproduct $*$, called *convolution*, defined for every f_1, f_2 of $F[H]$ as follows:

$$f_1 * f_2 = \left\{ f_\theta \mid f_\theta(g) = \sum_{\theta(x,y)=g} f_1(x) f_2(y);\ \theta \in \Theta \right\}.$$

Theorem 6.8.4. *The structure $(F[H], +, *)$ is a multiplicative H_v-ring where the inclusion distributivity is valid.*

Proof. Obviously, $(F[H], +)$ is a group. Let $f_1, f_2, f_3 \in F[H]$. Then

$$\begin{aligned}(f_1 * f_2) * f_3 &= \left\{ f_\theta \mid f_\theta(u) = \sum_{\theta(x,y)=u} f_1(x) f_2(y);\ \theta \in \Theta \right\} * f_3 \\ &= \left\{ f_\varphi \mid f_\varphi(g) = \sum_{\varphi(u,z)=g} \sum_{\theta(x,y)=u} f_1(x) f_2(y) f_3(z);\ \theta, \varphi \in \Theta \right\} \ni f_{\theta_\circ}\end{aligned}$$

where $f_{\theta_\circ}(g) = \sum\limits_{\theta_\circ(u,z)=g} \sum\limits_{\theta_\circ(x,y)=u} f_1(x) f_2(y) f_3(z)$. On the other hand,

$$f_1 * (f_2 * f_3) = \left\{ f_{\underline{\varphi}} \mid f_{\underline{\varphi}}(g) = \sum_{\underline{\varphi}(x,v)=g} \sum_{\underline{\theta}(y,z)=v} f_1(x) f_2(y) f_3(z);\ \underline{\theta}, \varphi \in \Theta \right\} \ni f_{\theta_\circ}$$

where $f_{\theta_\circ}(g) = \sum\limits_{\theta_\circ(x,v)=g} \sum\limits_{\theta_\circ(y,z)=v} f_1(x) f_2(y) f_3(z)$. Since $\theta_\circ$ is associative. Thus, $(f_1 * f_2) * f_3 \cap f_1 * (f_2 * f_3) \neq \emptyset$. Now, for the left distributivity, we have

$$\begin{aligned}f_1 * (f_2 + f_3) &= \left\{ f_\theta \mid f_\theta(g) = \sum_{\theta(x,y)=g} f_1(x)[(f_2 + f_3)(y)];\ \theta \in \Theta \right\} \\ &= \left\{ f_\theta \mid f_\theta(g) = \sum_{\theta(x,y)=g} [f_1(x) f_2(y) + f_1(x) f_3(y)];\ \theta \in \Theta \right\}\end{aligned}$$

and

$$\begin{aligned} f_1 * f_2 + f_1 * f_3 &= \left\{ f_\rho \mid f_\rho(g) = \sum_{\rho(x,y)=g} f_1(x) f_2(y);\ \rho \in \Theta \right\} \\ &\quad + \left\{ f_\varphi \mid f_\varphi(g) = \sum_{\varphi(x,y)=g} f_1(x) f_3(y);\ \varphi \in \Theta \right\} \\ &= \left\{ f_{\rho\varphi} = f_\rho + f_\varphi \mid f_{\rho\varphi}(g) = \sum_{\rho(x,y)=g} f_1(x) f_2(y) \right. \\ &\qquad \left. + \sum_{\varphi(r,s)=g} f_1(r) f_3(s);\ \rho, \varphi \in \Theta \right\}. \end{aligned}$$

Thus, $f_1 * (f_2 + f_3) \subseteq f_1 * f_2 + f_1 * f_3$, which is the left inclusion distributively. Similarly, the right inclusion distributivity is valid. ■

Definition 6.8.5. The above H_v-ring is called a *hypergroupoid H_v-algebra* or an *H_v-group ring.*

Given a hypergroupoid $(H, \cdot)$ one can define an H_v-group ring by "enlarging" the hyperoperation $\cdot$ as follows: Take any $*$ on H such that $(H, *)$ is Θ-WASS hypergroupoid then take the union $\lozenge = \cdot \cup *$, i.e., $x \lozenge y = (xy) \cup (x * y)$ for all $x, y \in H$. An H_v-group ring is defined on $(H, \lozenge)$.

The most important condition in order to define the H_v-group ring is the Θ-WASS condition. Therefore, in what follows we focus on our attention on classes which satisfy the Θ-WASS condition.

Now, we prove the following theorem.

Theorem 6.8.6. *In every H_v-group ring $(F[H], +, *)$ we have*

$$(-f_1) * f_2 = -(f_1 * f_2) = f_1 * (-f_2), \quad \textit{for all } f_1, f_2 \in F[H].$$

and there exists an absorbing element.

Proof. For every $f_1, f_2 \in F[H]$, we have

$$\begin{aligned} -f_1 * f_2 &= \left\{ f_\theta \mid f_\theta(h) = \sum_{\theta(x,y)=h} (-f_1)(x) f_2(y);\ \theta \in \Theta \right\} \\ &= \left\{ -f_\theta \mid f_\theta(h) = \sum_{\theta(x,y)=h} f_1(x) f_2(y);\ \theta \in \Theta \right\} \\ &= -(f_1 * f_2). \end{aligned}$$

Take the element $f_0 \in F[H]$ such that $f_0(h) = 0$, for all $h \in H$. Then, for every $f \in F[H]$ we have

$$\begin{aligned} f_0 * f &= \left\{ f_\theta \mid f_\theta(h) = \sum_{\theta(x,y)=h} f_0(x) f(y);\ \theta \in \Theta \right\} \\ &= \{ f_\theta \mid f_\theta(h) = 0;\ h \in H,\ \theta \in \Theta \} \\ &= f_0. \end{aligned}$$

Therefore, f_0 is the absorbing element. ■

Example 6.8.7. Consider the H_b-group $(\mathbb{Z}_3, \oplus)$ which has the b-group $(\mathbb{Z}_3, +)$ and the non-singleton products are: $\overline{1} \oplus \overline{1} = \{\overline{0}, \overline{1}\}$, $\overline{2} \oplus \overline{2} = \{\overline{0}, \overline{1}, \overline{2}\}$. The $\overline{0}$ is scalar. Therefore, every map $\theta : \mathbb{Z}_3^2 \longrightarrow \mathbb{Z}_3$ with $\theta(x,y) \in x \oplus y$, is an element of Θ. Thus, $|\Theta| = 2 \cdot 3 = 6$ and we see $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6\}$. Then, for all $\theta \in \Theta$, we have

$$\theta(\overline{0},\overline{0}) = \overline{0},\ \theta(\overline{0},\overline{1}) = \theta(\overline{1},\overline{0}) = \overline{1},\ \theta(\overline{0},\overline{2}) = \theta(\overline{2},\overline{0}) = \overline{2},\ \theta(\overline{1},\overline{2}) = \theta(\overline{2},\overline{1}) = \overline{0},$$

and

$$\begin{aligned} &\theta_1(\overline{1},\overline{1}) = \overline{0},\ \theta_1(\overline{2},\overline{2}) = \overline{0}; \theta_2(\overline{1},\overline{1}) = \overline{0},\ \theta_2(\overline{2},\overline{2}) = \overline{1}; \theta_3(\overline{1},\overline{1}) = \overline{0},\ \theta_3(\overline{2},\overline{2}) = \overline{2}; \\ &\theta_4(\overline{1},\overline{1}) = \overline{1},\ \theta_4(\overline{2},\overline{2}) = \overline{0}; \theta_5(\overline{1},\overline{1}) = \overline{1},\ \theta_5(\overline{2},\overline{2}) = \overline{1}; \theta_6(\overline{1},\overline{1}) = \overline{1},\ \theta_6(\overline{2},\overline{2}) = \overline{2}. \end{aligned}$$

Every hyperproduct of elements of $F[\mathbb{Z}_3]$ has at most 6 elements. Let $r, s \in F[\mathbb{Z}_3]$, then

$$r * s = \left\{ t_\theta \mid t_\theta(g) = \sum_{\theta(x,y)=g} r(x) s(y);\ \theta \in \Theta \right\}.$$

For every $\theta \in \Theta$, we have to calculate 9 products of the form $r(x)s(y)$ in order to obtain t_θ.

$$\begin{aligned}
t_{\theta_1}(\overline{0}) &= r(\overline{0})s(\overline{0}) + r(\overline{1})s(\overline{2}) + r(\overline{2})s(\overline{1}) + r(\overline{1})s(\overline{1}) + r(\overline{2})s(\overline{2}),\\
t_{\theta_1}(\overline{1}) &= r(\overline{0})s(\overline{1}) + r(\overline{1})s(\overline{0}),\\
t_{\theta_1}(\overline{2}) &= r(\overline{0})s(\overline{2}) + r(\overline{2})s(\overline{0}),\\
t_{\theta_2}(\overline{0}) &= r(\overline{0})s(\overline{0}) + r(\overline{1})s(\overline{2}) + r(\overline{2})s(\overline{1}) + r(\overline{1})s(\overline{1}),\\
t_{\theta_2}(\overline{1}) &= r(\overline{0})s(\overline{1}) + r(\overline{1})s(\overline{0}) + r(\overline{2})s(\overline{2}),\\
t_{\theta_2}(\overline{2}) &= r(\overline{0})s(\overline{2}) + r(\overline{2})s(\overline{0}),\\
t_{\theta_3}(\overline{0}) &= r(\overline{0})s(\overline{0}) + r(\overline{1})s(\overline{2}) + r(\overline{2})s(\overline{1}) + r(\overline{1})s(\overline{1}),\\
t_{\theta_3}(\overline{1}) &= r(\overline{0})s(\overline{1}) + r(\overline{1})s(\overline{0}),\\
t_{\theta_3}(\overline{2}) &= r(\overline{0})s(\overline{2}) + r(\overline{2})s(\overline{0}) + r(\overline{2})s(\overline{2}).
\end{aligned}$$

and similar for the $t_{\theta_4}, t_{\theta_5} t_{\theta_6}$.

Example 6.8.8. Consider the H_b-group $(\mathbb{Z}_{mn}, \oplus)$ defined in Example 6.1.2. (2). Then, Θ has only two elements $\Theta = \{\theta_1, \theta_2\}$. For all $\theta \in \Theta$, we have $\theta(x,y) = x + y$ if $(x,y) \neq (\overline{0}, \overline{m})$ and $\theta_1(\overline{0}, \overline{m}) = \overline{0}$, $\theta_2(\overline{0}, \overline{m}) = \overline{m}$. The map θ_2 leads to the known convolution on $(\mathbb{Z}_{mn}, +)$. For every element $g \neq \overline{0}$ and m, the sum $t_{\theta_1}(g) = \sum\limits_{\theta_1(x,y)=g} r(x)s(y)$ has mn elements. Moreover,

$$t_{\theta_1}(\overline{0}) = \sum_{\theta_1(x,y)=\overline{0}} r(x)s(y)$$

is a sum of $mn + 1$ terms of the form $r(x)s(y)$ and

$$t_{\theta_1}(\overline{m}) = \sum_{\theta_1(x,y)=\overline{m}} r(x)s(y)$$

is a sum of $mn - 1$ terms of $r(x)s(y)$.

Let $(H, \circ)$ be an H_v-group, $(G, +)$ be a group with the zero element 0, $\{A_i\}_{i \in G}$ be a family of non empty sets with $A_0 = H$ and $A_i \cap A_j = \emptyset$, for all $i, j \in G, i \neq j$. Set $K = \bigcup\limits_{i \in G} A_i$ and consider the hyperoperation $\odot$ defined in K as follows:

$$x \odot y = \begin{cases} x \circ y & \text{if } (x,y) \in H^2 \\ A_{i+j} & \text{if } (x,y) \in A_i \times A_j \neq H^2. \end{cases}$$

Then, $(K, \odot)$ becomes an H_v-group. which is called an (H, G)-*H_v-group*. It is easy to see that $K/\beta^* \cong G$.

Theorem 6.8.9. *If $cardA_i = n$ for all $i \in G$ and $(H, \circ)$ is Θ-WASS, then $(K, \odot)$ is Θ-WASS. Moreover,*

$$card\Theta_K \geq (n!)^{m-1} card\Theta_H,$$

where $m = cardG$.

Proof. We consider a family of one to one maps $\{p_i\}_{i\in G}$ such that $p_i : H \longrightarrow A_i$, $i \neq 0$ and p_0 is the identity map. Notice that all these maps are also onto. Take $\theta \in \Theta_H$ which defines an associative operation $\cdot$ and consider the mapping $\theta' : K \times K \longrightarrow K$ which defines the operation $\Diamond$ in K, as follows:

$$x \Diamond y = p_{i+j}(p_i^{-1}(x) \cdot p_j^{-1}(y)) \text{ for all } x \in A_i, y \in A_j.$$

This mapping is, obviousely, onto, so it remain to prove that $\Diamond$ is associative. Suppose that $(x, y, z) \in A_i \times A_j \times A_r$, we have

$$\begin{aligned} x\Diamond(y\Diamond z) &= x \Diamond p_{j+r}(p_j^{-1}(y) \cdot p_r^{-1}(z)) \\ &= p_{i+(j+r)}[p_i^{-1}(x) \cdot (p_j^{-1}(y) \cdot p_r^{-1}(z))] = (x\Diamond y)\Diamond z. \end{aligned}$$

Therefore, $(K, \oplus)$ is Θ-WASS. Now, we remark that the number of the families p_i, $i \in G$, $i \neq 0$ of bijective maps is $(n!)^{m-1}$. Therefore,

$$card\Theta_K \geq (n!)^{m-1} card\Theta_H. \blacksquare$$

Theorem 6.8.10 *Let $(K, \cdot)$ be Θ-WASS such that $cardA_i = 1$, for all $i \neq 0$. Then, for all $\theta \in \Theta_K$ which define an associative operation $\Diamond$ in K, there exists an element $x \in H$ such that*

$$y \Diamond x = x = x \Diamond y, \text{ for all } y \in H,$$
$$z \Diamond w = x, \text{ for all } (z, w) \in A_r \times A_s \neq H^2 \text{ for which } r + s = 0.$$

Proof. Take $(z, w) \in A_r \times A_s \neq H^2$ for which $r+s = 0$ and set $z \Diamond w = x \in H$. Then, for all $y \in H$,

$$y \Diamond x = y \Diamond (z \Diamond w) = (y \Diamond z) \Diamond w = z \Diamond w = x.$$

Similarly, $x\Diamond y = x$. This element x is unique, because if there exists another element x' such that $(u,v) \in A_p \times A_q \neq H^2$ with $p+q=0$ and $u\Diamond v = x'$, then $x = x\Diamond x' = x'$. ∎

Note that in the above theorem, the operation induced by the restriction of θ to $H \times H$ is weak associative. This means that $(H,\circ)$ is Θ-WASS.

From the above theorem we obtain the following construction.

Theorem 6.8.11. *Let $(H,\circ)$ be such that there exists an associative operation $\Diamond$ on H and a special element $x \in H$ such that $x\Diamond y = x = y\Diamond x$ for all $y \in H$. Then there exists a Θ-WASS (H,G)-H_v-group $(K,\odot)$ with $cardA_i = 1$ for all $i \in G$, $i \neq 0$.*

Proof. Consider the extension of $\Diamond$ to K for which $z\Diamond w = x$ for all $(z,w) \in A_i \times A_j \neq H$ with $i+j=0$. It is easy to check that this operation is associative on K. ∎

In the case of the (H,G)-H_v-groups with $cardA_i = 1$, for all $i \in G\setminus\{0\}$, the cardinality of the set of onto maps

$$K \times K \longrightarrow K: \ (x,y) \mapsto z \in x \odot y$$

is less or equal to n^{m+n^2-1}, where $n = cardH$ and $m = cardG$.

Definition 6.8.12. Let $\{S_i\}_{i\in I}$ be a pairwise disjoint family of H_v-semigroups where $|I| > 1$. We define a hyperoperation $\otimes$, called an *S-hyperoperation* on the set $S = \bigcup\limits_{i\in I} S_i$ as follows:

$$\begin{array}{l} x_i \otimes y_i = x_i y_i \ \text{ for all } (x_i,y_i) \in S_i^2, \\ x_i \otimes x_j = S_i \cup S_j \text{ for all } (x_i,x_j) \in S_i \times S_j, \ i \neq j. \end{array}$$

Then, the hyperstructure $(S,\otimes)$, called an *S-construction*, is an H_v-group.

Let $\{S_i\}_{i\in I}$ be a family of pairwise disjoint sets, where $cardI > 1$. On S_i we consider the *total hyperoperation* $ab = S_i$, for all $a,b \in S_i$ or the *least incidence hyperoperation* $ab = \{a,b\}$, for all $a,b \in S_i$. In each case, we obtain the S-construction $(S,\otimes)$.

In what follows, we consider the finite case. Let $cardI{=}n$ and $cardS_i{=}s_i$,

$i \in I$ and suppose that for each $i \in I$, S_i is a group or a groupoid with the associated set of fundamental maps $\Theta_i \neq \emptyset$. Let Θ_i be the set of fundamental maps on S with respect to $\otimes$. We ontain

$$card\Theta = 2 \cdot \prod_{i,j\in I}^{i<j} (s_i + s_j).$$

In the particular case when $S_i = \{x_i\}$, $i \in I$, the S-construction coincides with the incidence operation and we obtain $cardI = n$, we have

$$card\Theta = 2 \cdot 2^{n(n-1)}.$$

Theorem 6.8.13. *If each one of the following conditions is valid:*

(1) *S_i is a group for all $i \in I$;*

(2) *S_i is a semigroup such that $S_i^2 = S_i$ for all $i \in I$;*

(3) *Every S_i has a scalar element;*

then every map $S \times S \longrightarrow S: \ (x, y) \mapsto z \in xy$ is onto.

Proof. It is clear. ∎

Theorem 6.8.14. *For each one of the cases of Theorem* 6.8.13, *we have*

$$card\Theta = \left(\prod_{i,j\in I}^{i<j} (n_i + n_j)^{n_i n_j} \right)^2.$$

Proof. It is clear. ∎

In the particular case when $S_i = \{x_i\}$, the S-construction concides with the incidence operation and we obtain $card\Theta = 2^{n(n-1)}$.

Theorem 6.8.15. *Every S-construction, where $(S, \cdot)$ is an H_b-semigroup with a b-semigroup S_i, is Θ-WASS.*

Proof. Suppose that the b-operation of $(S_i, \cdot)$ is $\circ$. Consider the operation $\otimes$ on S, where $\otimes$ coincides with $\circ$ on S_i and

$$s_i \otimes s_j = s_j \circ s_i = s_j \ \text{ for } \ s_i \in S_i, \ s_j \in S_j \text{ and } i \neq j.$$

The operation $\otimes$ guarantees that $\Theta \neq \emptyset$ and it is easy to check that it is associative. ∎

Chapter 7

Commutative rings obtained from hyperrings

The commutativity of the addition in rings is connected to the existence of the multiplication unit. If e is the unit in a ring then for all elements a, b we have

$$(a+b)(e+e) = (a+b)e + (a+b)e = a+b+a+b,$$
$$(a+b)(e+e) = a(e+e) + b(e+e) = a+a+b+b.$$

So $a+b+a+b = a+a+b+b$ whence $b+a = a+b$. Therefore, when we consider a hyperring $(R, +, \cdot)$, the hyperoperation $(+)$ is not commutative and there is not a multiplication unit. So the commutativity, as well as the existence of the unit, it is not assumed in the fundamental ring. Of course, we know there exist many rings without unit. Sometimes, we need that the fundamental ring is commutative with respect to both sum and product, that is, the fundamental ring is an ordinary commutative ring. We use the results obtained by B. Davvaz, T. Vougiouklis, S.M. Anvariyeh and S. Mirvakili [5, 41, 87, 88].

7.1 α^*-relations

We begin this paragraph with the following definition:

Definition 7.1.1. Let R be a hyperring. We define the relation α as follows:

$x \;\alpha\; y \iff \exists n \in \mathbb{N},\ \exists(k_1, ..., k_n) \in \mathbb{N}^n,\ \exists \sigma \in \mathbb{S}_n$ and $[\exists(x_{i1}, ..., x_{ik_i}) \in R^{k_i}$, $\exists \sigma_i \in \mathbb{S}_{k_i},\ (i = 1, ..., n)]$ such that

$$x \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right) \quad \text{and} \quad y \in \sum_{i=1}^{n} A_{\sigma(i)},$$

where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$.

The relation α is reflexive and symmetric. Let α^* be the transitive closure of α. Then

Lemma 7.1.2. *α^* is a strongly regular relation both on $(R, +)$ and on $(R, \cdot)$.*

Proof. Clearly α^* is an equivalence relation. In order to prove that it is strongly regular, it is enough to show that

$$x\alpha y \implies \begin{cases} x + a \;\overline{\overline{\alpha}}\; y + a, & a + x \;\overline{\overline{\alpha}}\; a + y, \\ x \cdot a \;\overline{\overline{\alpha}}\; y \cdot a, & a \cdot x \;\overline{\overline{\alpha}}\; a \cdot y, \end{cases}$$

for every $a \in R$. If $x\alpha y$, then $\exists n \in \mathbb{N},\ \exists(k_1, ..., k_n) \in \mathbb{N}^n,\ \exists \sigma \in \mathbb{S}_n$ and $[\exists(x_{i1}, ..., x_{ik_i}) \in R^{k_i},\ \exists \sigma_i \in \mathbb{S}_{k_i},\ (i = 1, ..., n)]$ such that

$$x \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right) \quad \text{and} \quad y \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)} \right),$$

and so

$$x + a \subseteq \left(\sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right) \right) + a \quad \text{and} \quad y + a \subseteq \left(\sum_{i=1}^{n} \left(\prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)} \right) \right) + a.$$

Now, let $k_{n+1} = 1$, $x_{n+1\ 1} = a$, $\sigma_{n+1} = id$ and τ be the permutation of $\mathbb{S}_{n+1}$ such that

$$\tau(i) = \sigma(i) \ \ \forall i = 1, ..., n \ \text{ and } \ \tau(n+1) = n + 1.$$

Thus

$$x + a \subseteq \sum_{i=1}^{n+1} \left(\prod_{j=1}^{k_i} x_{ij} \right) \quad \text{and} \quad y + a \subseteq \sum_{i=1}^{n+1} \left(\prod_{j=1}^{k_{\tau(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)} \right).$$

Therefore for all $u \in x + a$ and $v \in y + a$, we have $u\alpha v$. Thus $x + a \,\overline{\overline{\alpha}}\, y + a$. In the same way we can show that $a + x \,\overline{\overline{\alpha}}\, a + y$. It is easy to see that

$$x + a \,\overline{\overline{\alpha^*}}\, y + a \quad \text{and} \quad a + x \,\overline{\overline{\alpha^*}}\, a + y.$$

Now, notice that

$$x \cdot a \subseteq \left(\sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right) \right) \cdot a \quad \text{and} \quad y \cdot a \subseteq \left(\sum_{i=1}^{n} \left(\prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)} \right) \right) \cdot a.$$

which yields that

$$x \cdot a \subseteq \sum_{i=1}^{n} \left(\left(\prod_{j=1}^{k_i} x_{ij} \right) \cdot a \right) \quad \text{and} \quad y \cdot a \subseteq \sum_{i=1}^{n} \left(\left(\prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)} \right) \cdot a \right).$$

We set $k'_i = k_i + 1$, $x_{ik'_i} = a$ and we define

$$\tau_i(r) = \sigma_i(r) \quad (\forall r = 1, ..., k_i) \quad \text{and} \quad \tau_i(k_i + 1) = k_i + 1.$$

Hence $\tau_i \in \mathbb{S}_{k'_i}$ $(i = 1, ..., n)$. Thus

$$x \cdot a \subseteq \sum_{i=1}^{n} \left(\prod_{j=1}^{k'_i} x_{ij} \right) \quad \text{and} \quad y \cdot a \subseteq \sum_{i=1}^{n} \left(\prod_{j=1}^{k'_{\sigma(i)}} x_{\sigma(i)\ \tau_{\sigma(i)}(j)} \right).$$

Therefore for all $u \in x \cdot a$ and $v \in y \cdot a$, we have $u\alpha v$. Thus $x \cdot a \,\overline{\overline{\alpha^*}}\, y \cdot a$ and $a \cdot x \,\overline{\overline{\alpha^*}}\, a \cdot y$. ■

Theorem 7.1.3. *The quotient R/α^* is a commutative ring.*

Proof. We define $\oplus$ and $\otimes$ on R/α^* in the usual manner:

$$\alpha^*(a) \oplus \alpha^*(b) = \{\alpha^*(c) \mid c \in \alpha^*(a) + \alpha^*(b)\},$$
$$\alpha^*(a) \otimes \alpha^*(b) = \{\alpha^*(d) \mid d \in \alpha^*(a) \cdot \alpha^*(b)\}.$$

Let $a'\alpha^*a$ and $b'\alpha^*b$. Then we have

$$a'\alpha^*a \iff \exists a_1, ..., a_{p+1},\ a_1{=}a',\ a_{p+1}{=}a \text{ such that } a_r\alpha a_{r+1}\ (r{=}1, ..., p),$$
$$b'\alpha^*b \iff \exists b_1, ..., b_{q+1},\ b_1 = b',\ b_{q+1} = b \text{ such that } b_s\alpha b_{s+1}\ (s = 1, ..., q),$$

and so

$a_r\, \alpha\, a_{r+1} \iff \exists n_r{\in}\mathbb{N}$, $\exists(k_{r1}, ..., k_{rn_r}){\in}\mathbb{N}^{n_r}$, $\exists\sigma \in \mathbb{S}_{n_r}$ and $[\exists(x_{ri1}, ..., x_{rik_{ri}}){\in}R^{k_{ri}}$, $\exists\sigma_{ri} \in \mathbb{S}_{k_{ri}}$ $(i = 1, ..., n_r)]$ such that

$$a_r \in \sum_{i=1}^{n_r} \left(\prod_{j=1}^{k_{ri}} x_{rij} \right), \quad \text{and} \quad a_{r+1} \in \sum_{i=1}^{n_r} \left(\prod_{j=1}^{k_{r\sigma(i)}} x_{r\sigma(i)\sigma_{r\sigma(i)}(j)} \right);$$

$b_s\, \alpha\, b_{s+1} \iff \exists m_s \in \mathbb{N}$, $\exists(t_{s1}, ..., t_{sm_s}){\in}\mathbb{N}^{m_s}$, $\exists\tau{\in}\mathbb{S}_{m_s}$ and $[\exists(y_{si1}, ..., y_{sit_{si}}){\in}R^{t_{si}}$, $\exists\tau_{si} \in \mathbb{S}_{t_{si}}$ $(i = 1, ..., m_s)]$ such that

$$b_s \in \sum_{i=1}^{m_s} \left(\prod_{j=1}^{t_{si}} y_{sij} \right) \quad \text{and} \quad b_{s+1} \in \sum_{i=1}^{m_s} \left(\prod_{j=1}^{t_{s\sigma(i)}} y_{s\tau(i)\tau_{s\tau(i)}(j)} \right).$$

Therefore, we obtain

$$a_r + b_1 \subseteq \sum_{i=1}^{n_r} \left(\prod_{j=1}^{k_{ri}} x_{rij} \right) + \sum_{i=1}^{m_1} \left(\prod_{j=1}^{t_{1i}} y_{1ij} \right),$$

$$a_{r+1} + b_1 \subseteq \sum_{i=1}^{n_r} \left(\prod_{j=1}^{k_{ri}} x_{r\sigma(i)\sigma_{ri}(j)} \right) + \sum_{i=1}^{m_1} \left(\prod_{j=1}^{t_{1i}} y_{1ij} \right),$$

and

$$a_{p+1} + b_s \subseteq \sum_{i=1}^{n_r} \left(\prod_{j=1}^{k_{r\sigma(i)}} x_{r\sigma(i)\sigma_{r\sigma(i)}(j)} \right) + \sum_{i=1}^{m_s} \left(\prod_{j=1}^{t_{si}} y_{sij} \right),$$

$$a_{p+1} + b_{s+1} \subseteq \sum_{i=1}^{n_r} \left(\prod_{j=1}^{k_{r\sigma(i)}} x_{r\sigma(i)\sigma_{r\sigma(i)}(j)} \right) + \sum_{i=1}^{m_s} \left(\prod_{j=1}^{t_{s\tau(i)}} y_{s\tau(i)\tau_{s\tau(i)}(j)} \right).$$

Now, we choose the elements $c_1, ..., c_{p+q}$ such that

$$c_r \in a_r + b_1 \quad (r = 1, ..., p),$$
$$c_{p+s} \in a_{p+1} + b_{s+1} \quad (s = 1, ..., q),$$

and using the above inclusions we get $c_r \alpha c_{r+1}$. Therefore, every element $c_1 \in a_1 + b_1 = a' + b'$ is α^*-equivalent to every element $c_{p+q} \in a_{p+1} + b_{q+1} = a + b$. Therefore,

$$\alpha^*(a) \oplus \alpha^*(b) = \alpha^*(c), \quad \forall c \in \alpha^*(a) + \alpha^*(b).$$

In a similar way, it is proved that

$$\alpha^*(a) \otimes \alpha^*(b) = \alpha^*(d), \quad \forall d \in \alpha^*(a) \cdot \alpha^*(b).$$

The associativity and distributivity on R guarantee that the associativity and distributivity are valid for R/α^*. Suppose that σ is the permutation of $\mathbb{S}_2$ such that $\sigma(1) = 2$. For every $x \in x_1 + x_2$, $a \in x_1 \cdot x_2$ and $y \in x_{\sigma(1)} + x_{\sigma(2)}$, $b \in x_{\sigma(1)} \cdot x_{\sigma(2)}$, we have $x\alpha y$ and $a\alpha b$. Thus $x\alpha^* y$ and $a\alpha^* b$, and so

$$\alpha^*(x_1) \oplus \alpha^*(x_2) = \alpha^*(x) = \alpha^*(x_2) \oplus \alpha^*(x_1),$$
$$\alpha^*(x_1) \otimes \alpha^*(x_2) = \alpha^*(a) = \alpha^*(x_2) \otimes \alpha^*(x_1).$$

Therefore R/α^* is a commutative ring. ■

Notice that we use the Greek letter α for the relation studied in this paragraph, because α corresponds to the letter "a" from abelian.

Theorem 7.1.4. *The relation α^* is the smallest equivalence relation such that the quotient R/α^* is a commutative ring.*

Proof. Let θ be an equivalence relation such that R/θ is a commutative ring and let $\phi : R \longrightarrow R/\theta$ be the canonical projection. If $x\alpha y$ then there exist $n \in \mathbb{N}$, $(k_1, ..., k_n) \in \mathbb{N}^n$, $\sigma \in \mathbb{S}_n$ and there exist $(x_{i1}, ..., x_{ik_i}) \in R^{k_i}$ and $\sigma_i \in \mathbb{S}_{k_i}$ $(i = 1, ..., n)$ such that

$$x \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right) \quad \text{and} \quad y \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)} \right),$$

whence

$$\phi(x) = \oplus\sum_{i=1}^{n}\left(\otimes\prod_{j=1}^{k_i} x_{ij}\right) \quad \text{and} \quad \phi(y) = \oplus\sum_{i=1}^{n}\left(\otimes\prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)}\right).$$

By the commutativity of R/θ, it follows that $\phi(x) = \phi(y)$. Thus $x\alpha y$ implies that $x\theta y$. Finally, let $x\alpha^* y$. Since θ is transitive, we obtain

$$x \in \alpha^*(y) \implies x \in \theta(y).$$

Therefore $\alpha^* \subseteq \theta$. ∎

Let $(R, +, \cdot)$ be a hyperring. Then we define the relations Γ and Γ_+ on R as follows:

$$x\Gamma y \iff \exists n \in \mathbb{N},\ \exists(z_1, ..., z_n) \in R^n, \exists\sigma \in \mathbb{S}_n :\ x \in \prod_{i=1}^{n} z_i,\quad y \in \prod_{i=1}^{n} z_{\sigma(i)},$$

$$x\Gamma_+ y \iff \exists m \in \mathbb{N},\ \exists(y_1, ..., y_m) \in R^m, \exists\tau \in \mathbb{S}_m :\ x \in \sum_{i=1}^{m} y_i,\ \ y \in \sum_{i=1}^{m} y_{\tau(i)}.$$

We denote the transitive closures of Γ and Γ_+ by Γ^* and Γ_+^*, respectively. The equivalence relation Γ^* was introduced on hypergroups and semi-hypergroups by Freni [51]. We have $\Gamma^* \cup \Gamma_+^* \subseteq \alpha^*$.

Theorem 7.1.5. *For all additive hyperrings we have* $\alpha^* = \Gamma_+^*$.

Proof. In an additive hyperring R, every product of elements of R is a singleton. Thus, for every

$$A = \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij}\right)$$

we can consider the elements $y_i = \prod_{j=1}^{k_i} x_{ij}$ $(i = 1, ..., n)$ of R, for which we have $A = \sum_{i=1}^{n} y_i$. This means that $a\alpha^* b$ if and only if $a\Gamma_+^* b$. ∎

Since the Krasner hyperring is an additive hyperring, it follows that the above theorem can be applied to those hyperrings.

Let $(R, +, \cdot)$ be a hyperring. We can consider the additive hyperring $(R/\Gamma^*, \uplus, \odot)$, where

$$\begin{aligned}\Gamma^*(a) \uplus \Gamma^*(b) &= \{\Gamma^*(c) \mid c \in \Gamma^*(a) + \Gamma^*(b)\},\\ \Gamma^*(a) \odot \Gamma^*(b) &= \Gamma^*(d) \text{ for all } d \in \Gamma^*(a) \cdot \Gamma^*(b).\end{aligned}$$

Theorem 7.1.6. *Let $(R, +, \cdot)$ be a hyperring. Then*

$$R/\alpha^* \cong (R/\Gamma^*)/\Gamma^*_{\uplus}.$$

Proof. Let $\varphi : R \longrightarrow (R/\Gamma^*)/\Gamma^*_{\uplus}$ be the canonical projection. We denote by θ the equivalence relation associated to φ. For every $a \in R$ we have $\alpha^*(a) \subseteq \theta(a)$. On the other hand, since $\Gamma^*(x) \subseteq \alpha^*(x)$ for all $x \in R$, we have

$$\bigcup_{\Gamma^*(z) \in \Gamma^*(x) \uplus \Gamma^*(y)} \Gamma^*(z) = \bigcup_{z \in \Gamma^*(x) + \Gamma^*(y)} \Gamma^*(z) \subseteq \bigcup_{z \in \alpha^*(x) + \alpha^*(y)} \alpha^*(z) = \alpha^*(w)$$

for all $w \in x + y$. Consequently, we get

$$\bigcup_{z \in \sum\limits_{i=1}^{n} \Gamma^*(x_i)} \Gamma^*(z) \subseteq \alpha^*(w), \text{ where } w \in \sum_{i=1}^{n} x_i.$$

Moreover, since α^* is transitive, we have

$$\theta(a) = \bigcup_{\{z \mid \Gamma^*(z)\ \Gamma^*_{\uplus}\ \Gamma^*(a)\}} \Gamma^*(z) \subseteq \alpha^*(a) \text{ for all } a \in R.$$

Therefore $\theta = \alpha^*$. ■

Lemma 7.1.7. *Let R_1 and R_2 be two hyperrings, $a, c \in R_1$, $b, d \in R_2$ and $\sigma \in \mathbb{S}_n$. Then*

$$(a, b) \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} (x_{ij}, y_{ij}) \right) \text{ and } (c, d) \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_{\tau(i)}} (x_{\tau(i)\sigma_{\tau(i)}(j)}, y_{\tau(i)\sigma_{\tau(i)}(j)}) \right)$$

if and only if

$$a \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij}\right), \quad c \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_{\tau(i)}} x_{\tau(i)\sigma_{\tau(i)}(j)}\right)$$

and

$$b \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} y_{ij}\right), \quad d \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_{\tau(i)}} y_{\tau(i)\sigma_{\tau(i)}(j)}\right),$$

for some $x_{ij} \in R_1,\ y_{ij} \in R_2$.

Corollary 7.1.8. *Let* R_1 *and* R_2 *be two hyperrings,* $\alpha^*_{R_1}$, $\alpha^*_{R_2}$ *and* $\alpha^*_{R_1\times R_2}$ *be* α^**-relations on* R_1, R_2 *and* $R_1 \times R_2$ *respectively. Then*

$$(a,b)\ \alpha^*_{R_1\times R_2}\ (c,d) \ \text{ if and only if } \ a\ \alpha^*_{R_1}\ c \ \text{ and } \ b\ \alpha^*_{R_2}\ d.$$

Theorem 7.1.9. *Let* R_1 *and* R_2 *be two hyperrings,* $\alpha^*_{R_1}$, $\alpha^*_{R_2}$ *and* $\alpha^*_{R_1\times R_2}$ *be* α^**-relations on* R_1, R_2 *and* $R_1 \times R_2$ *respectively. Then*

$$(R_1 \times R_2)/\alpha^*_{R_1\times R_2} \cong R_1/\alpha^*_{R_1} \ \times \ R_2/\alpha^*_{R_2}.$$

Proof. We define the map $\xi : (R_1 \times R_2)/\alpha^*_{R_1\times R_2} \longrightarrow R_1/\alpha^*_{R_1} \ \times \ R_2/\alpha^*_{R_2}$ by

$$\xi(\alpha^*_{R_1\times R_2})(a,b) = (\alpha^*_{R_1}(a), \alpha^*_{R_2}(b)).$$

By Lemma 7.1.7 and Corollary 7.1.8, it is not difficult to see that ξ is an isomorphism. ■

Let $\phi : R \longrightarrow R/\alpha^*$ be the canonical projection and let $D(R)$ be the *kernel* of ϕ. If we denote the zero element of R/α^* by $\overline{0}$, then $D(R) = \phi^{-1}(\overline{0})$.

Lemma 7.1.10. *Let* R *be a hyperring. Then*

$$R \cdot D(R) \subseteq D(R) \ \text{ and } \ D(R) \cdot R \subseteq D(R).$$

Proof. For all $a \in R \cdot D(R)$ there exist $r \in R$ and $x \in D(R)$ such that $a \in rx$. So $\alpha^*(a) = \alpha^*(rx) = \alpha^*(r) \otimes \alpha^*(x) = \alpha^*(r) \otimes \overline{0} = \overline{0}$. ■

Lemma 7.1.11. *If R is a Krasner hyperring, then*

$$\alpha^*(0) = \overline{0} \text{ and } \alpha^*(-x) = -\alpha^*(x) \text{ for all } x \in R.$$

Proof. It is straightforward. ■

Theorem 7.1.12. *If R is a Krasner hyperring, then $D(R)$ is a hyperideal of R.*

Proof. We have $0 \in D(R)$. Let $x, y \in D(R)$. Then for every $z \in x + y$, we have $\alpha^*(z) = \alpha^*(x) \oplus \alpha^*(y) = D(R) \oplus D(R) = D(R)$ which yields that $z \in D(R)$, and so $x + y \subseteq D(H)$. On the other hand, since $x \in D(R)$, then there exists $-x \in R$ such that $0 \in x - x$. So

$$D(R) = \alpha^*(0) = \alpha^*(x - x) = \alpha^*(x) \oplus \alpha^*(-x) = D(R) \oplus \alpha^*(-x) = \alpha^*(-x),$$

and hence $-x \in D(R)$. ■

Let $(H, \circ)$ be a hypergroupoid. The hyperoperation $(\circ)$ is called *weak commutative* if

$$x \circ y \cap y \circ x \neq \emptyset \text{ for all } x, y \in H.$$

We denote the weak commutativity by COW.

A COW hyperring $(R, +, \cdot)$ is a hyperring for which both $(+)$ and $(\cdot)$ are weak commutative.

Theorem 7.1.13. *If R is a COW hyperring, then $\alpha^* = \gamma^*$.*

Proof. By definition, γ^* is the smallest equivalence relation such that R/γ^* is a (fundamental) ring. Since R is a *COW hyperring*, we have

$$(x + y) \cap (y + x) \neq \emptyset \text{ and } x \cdot y \cap y \cdot x \neq \emptyset \text{ for all } x, y \in R.$$

Therefore, there exist $a \in (x + y) \cap (y + x)$ and $b \in x \cdot y \cap y \cdot x$ which yield that

$$\gamma^*(a) = \gamma^*(x) \oplus \gamma^*(y) = \gamma^*(y) \oplus \gamma^*(x),$$
$$\gamma^*(b) = \gamma^*(x) \odot \gamma^*(y) = \gamma^*(y) \odot \gamma^*(x),$$

that is R/γ^* is a commutative ring. Since $\gamma \subseteq \alpha$, we obtain $\gamma^* \subseteq \alpha^*$. Moreover, α^* is the smallest equivalence relation on R such that R/α^* is a commutative ring, hence $\gamma^* = \alpha^*$. ■

Corollary 7.1.14. *Let R be a Krasner hyperring. If R is commutative or if it has a unit, then $\gamma^* = \alpha^*$.*

Theorem 7.1.15. *Let R be a commutative ring and $(A, \uplus, \odot)$ be a (H, R)-hyperring with the support $A = \bigcup_{r \in R} S_r$. Then*

$$A/\alpha^* \cong R.$$

Proof. Let $a \in A$. Then there exists $c \in R$ such that $a \in S_c$. Suppose that $x \in \alpha^*(a)$. Then

$$\exists a_1, ..., a_{p+1},\ a_1 = a',\ a_{p+1} = a \text{ such that } a_r \alpha a_{r+1}\ (r = 1, ..., p),$$

and so $\exists n_r \in \mathbb{N}$, $\exists (k_{r1}, ..., k_{rn_r}) \in \mathbb{N}^{n_r}$, $\exists \sigma \in \mathbb{S}_{n_r}$ and $[\exists (x_{ri1}, ..., x_{rik_{ri}}) \in R^{k_{ri}}$, $\exists \sigma_{ri} \in \mathbb{S}_{k_{ri}}$ $(i = 1, ..., n_r)]$ such that

$$a_r \in \sum_{i=1}^{n_r} \left(\prod_{j=1}^{k_{ri}} x_{rij} \right), \quad \text{and} \quad a_{r+1} \in \sum_{i=1}^{n_r} \left(\prod_{j=1}^{k_{r\sigma(i)}} x_{r\sigma(i)\sigma_{r\sigma(i)}(j)} \right);$$

From the definition of the hyperoperations $\oplus$ and $\odot$ it follows that for all $r = 1, ..., p$, there exist an appropriate $c_r \in R$ and $\pi \in S_r$ such that

$$\sum_{i=1}^{n_r} \left(\prod_{j=1}^{k_{ri}} x_{rij} \right) \subseteq S_{c_r} \quad \text{and} \quad \sum_{i=1}^{n_r} \left(\prod_{j=1}^{k_{r\sigma(i)}} x_{r\sigma(i)\sigma_{r\sigma(i)}(j)} \right) \subseteq S_{\pi(c_r)}.$$

Since R is commutative we get $S_{c_r} = S_{\pi(c_r)}$, and so $a_{r+1} \in S_{c_r} \cap S_{c_{r+1}}$. Consequently, $S_{c_r} = S_{c_{r+1}}$ $(i = 1, ..., p-1)$. Therefore $\alpha^*(a) \subseteq S_c$.

Now, we show that $\alpha^*(a) = S_c$. Let $z \in S_c$.

> If $c \in R \setminus \{0\}$, then we consider $y \in H$, $u \in S_c$, for which we have $\{z, a\} \subseteq y \oplus u = S_c$, and so $z \in \alpha^*(a)$.
>
> If $c = 0$, then we consider $i \in R \setminus \{0\}$ and $y \in S_i$, $u \in S_{-i}$, and we obtain $\{z, a\} \subseteq y \oplus u = H$, that is, $z \in \alpha^*(a)$.

The map $\psi : A/\alpha^* \longrightarrow R$, where $S_c \longrightarrow c$ is an isomorphism. ■

Let us denote the kernel of the canonical map $\phi_A : A \longrightarrow A/\alpha^*$ by $D(A)$. According to the previous theorem, for all $c \in R$ and $x \in S_c$, the equality $\alpha^*(x) = S_c$ holds and hence $A/\alpha^* = \{\phi_A(S_c) \mid c \in R\}$. Consequently $D(A) = H$.

Theorem 7.1.16. *Let R_1, R_2 be two commutative ring. We consider the (H_1, R_1)-hyperring $(A_1, \uplus_1, \odot_1)$ and the (H_2, R_2)-hyperring $(A_2, \uplus_2, \odot_2)$ with the supports $A_1 = \bigcup_{c\in R_1} S_c$ and $A_2 = \bigcup_{d\in R_2} T_d$. If $\zeta : A_1 \longrightarrow A_2$ is an inclusion homomorphism, then*

(1) $\zeta(\alpha_1^*(x)) \subseteq \alpha_2^*(\zeta(x))$ *for all $x \in A_1$.*

(2) *We can define the ζ-induced homomorphism $\zeta^* : A_1/\alpha_1^* \longrightarrow A_2/\alpha_2^*$ of ζ by*
$$\zeta^*(\phi_{A_1})(x)) = \phi_{A_2}(\zeta(x)).$$

(3) $\zeta(H_1) \subseteq H_2$.

Proof. It is straightforward. ■

Corollary 7.1.17. *If $A_1 \cong A_2$ then $H_1 \cong H_2$ and $R_1 \cong R_2$.*

Let $(R, +, \cdot)$ be an H_v-ring. For $n \in \mathbb{N}$ and $(k_1, ..., k_n) \in \mathbb{N}^n$ and $(x_{i1}, ..., x_{ik_i}) \in R^{ki}$ $(i = 1, ..., n)$, we define the set $\mathcal{U}^n_{i,(k_1,...,k_i)}[x_{i1}, ..., x_{ik_i}]$ as follows:

For $1 \leq i \leq n$, we can define a k_i-ary hyperproduct, induced by $(\cdot)$ by inserting $k_i - 2$ parentheses in the sequence of elements $x_{i1}, ..., x_{ik_i}$. Let us denote such a pattern of k_{i-2} parentheses by $p_i(x_{i1}, ..., x_{ik_i})$ and then we set

$$P_{i,k_i} = \{p_i(x_{i\sigma(1)}, ..., x_{i\sigma(k_i)}) \mid \sigma \in \mathbb{S}_{k_i}\}.$$

For example, if $k_i = 3$, then we obtain

$$\begin{aligned} P_{i,3} = \ & \{x_{i1} \cdot (x_{i2} \cdot x_{i3}),\ x_{i1} \cdot (x_{i3} \cdot x_{i2}),\ x_{i2} \cdot (x_{i1} \cdot x_{i3}),\ x_{i2} \cdot (x_{i3} \cdot x_{i1}), \\ & x_{i3} \cdot (x_{i1} \cdot x_{i2}),\ x_{i3} \cdot (x_{i2} \cdot x_{i1}),\ (x_{i1} \cdot x_{i2}) \cdot x_{i3},\ (x_{i2} \cdot x_{i1}) \cdot x_{i3}, \\ & (x_{i1} \cdot x_{i3}) \cdot x_{i2},\ (x_{i3} \cdot x_{i1}) \cdot x_{i2},\ (x_{i2} \cdot x_{i3}) \cdot x_{i1},\ (x_{i3} \cdot x_{i2}) \cdot x_{i1}\}. \end{aligned}$$

Let $X_i \in P_{i,k_i}$ $(i = 1, ..., n)$. Similarly, we can define an n-ary hypersum can be defined induced by $(+)$ by inserting $n-2$ parentheses in the sequence of $X_1, ..., X_n$ in a standard position. Let us denote such a pattern of $n-2$ parentheses by $q(X_1, ...X_n)$. Then

$$\mathcal{U}^n_{i,(k_1,...,k_i)}[x_{i1}, ..., x_{ik_i}] = \{q(x_{\tau(1)}, ..., X_{\tau(n)}) \mid \; X_i \in P_{i,k_i}, \tau \in S_n, \\ q(X_1, ..., X_n) \text{ is a pattern}\}.$$

Definition 7.1.18. We define the relation $\alpha_{n,(k_1,...,k_n)}$ as follows:

$$x\alpha_{n,(k_1,...,k_n)}y \iff \exists (x_{i1}, ..., x_{ik_i}) \in R^{k_i} \; (i = 1, ..., n) \text{ and} \\ \exists A, B \in \mathcal{U}^n_{i,(k_1,...,k_i)}[x_{i1}, ..., x_{ik_i}] \text{ such that } x \in A, \; y \in B.$$

Now, we define

$$\alpha = \bigcup_{n,k_1,...,k_n} \alpha_{n,(k_1,...,k_n)},$$

and let α^* be the transitive closure of α.

Lemma 7.1.19. *Let $(R, +, \cdot)$ be an H_v-ring. Then α^* is a strongly regular relation both on $(R, +)$ and on $(R, \cdot)$.*

Theorem 7.1.20. *Let $(R, +, \cdot)$ be an H_v-ring. Then the quotient R/α^* is a commutative ring.*

Proof. The proof is similar to the proof of Theorem 7.1.3. ■

Theorem 7.1.21. *Let $(R, +, \cdot)$ be an H_v-ring. Then the relation α^* is the smallest equivalence relation such that the quotient R/α^* is a commutative ring.*

Proof. The proof is similar to the proof of Theorem 7.1.4. ■

7.2 Transitivity conditions of α

In this section, we state the conditions that are equivalent to the transitivity of the relation α and we characterized the complete hyperring.

If R is a hyperring, then we set:

$$\alpha_0 = \{(x, x) | x \in R\}$$

and, for every integer $n \geq 1$, α_n is the relation defined as follows:
$x \ \alpha_n \ y \iff \exists(k_1, k_2, ..., k_n) \in \mathbb{N}^n$, $\exists\sigma \in \mathbb{S}_n$ and $[\exists(x_{i1}, ..., x_{ik_i}) \in R^{k_i}$, $\exists\sigma_i \in \mathbb{S}_{k_i}$, $(i = 1, ..., n)]$ such that

$$x \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij}\right) \quad \text{and} \quad y \in \sum_{i=1}^{n} A_{\sigma(i)},$$

where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$.

Obviously, for every $n \geq 1$, the relation α_n are symmetric, and the relation $\alpha = \bigcup_{n\geq 1} \alpha_n$ is reflexive and symmetric. Let α^* be the transitive closure of α. Then

- α^* is a strongly regular relation both on $(R, +)$ and on $(R, \cdot)$.
- The quotient R/α^* is a commutative ring.
- The relation α^* is the smallest equivalence relation such that the quotient R/α^* is a commutative ring.

Lemma 7.2.1. *If R is a hyperring and $n \geq 1$ then $\alpha_n \subseteq \alpha_{n+1}$.*

Proof. If $x \, \alpha_n \, y$ then $\exists(k_1, k_2, ..., k_n) \in \mathbb{N}^n$, $\exists\sigma \in \mathbb{S}_n$ and $[\exists(x_{i1}, ..., x_{ik_i}) \in R^{k_i}$, $\exists\sigma_i \in \mathbb{S}_{k_i}$, $(i = 1, ..., n)]$ such that

$$x \in \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}) \quad \text{and} \quad y \in \sum_{i=1}^{n} A_{\sigma(i)},$$

where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$.

$x_{nk_n} \in R$ and since $(R, +)$ is a hypergroup, it follows that there exist $t, s \in R$ such that $x_{nk_n} = t + s$. Set $x'_{ij} = x_{ij}$, $i = 1, ..., n-1$, $j = 1, ..., k_i$, $k'_{n+1} = k_n$, $k'_i = k_i$, $i = 1, ..., n$, $x'_{nj} = x_{nj}$, $j = 1, ..., k_n - 1$, $x'_{n+1,j} = x_{nj}$,

$i = 1, ..., k_{n+1} - 1$, $x'_{nk_n} = t$, $x'_{n+1,k_{n+1}} = s$, $\sigma' = (n, n+1)\sigma$, $\sigma'_i = \sigma_i$, $i = 1, ..., n$, $\sigma'_{n+1} = \sigma_n$. So

$$x \in \sum_{i=1}^{n+1} \left(\prod_{j=1}^{k'_i} x'_{ij} \right) \quad \text{and} \quad y \in \sum_{i=1}^{n+1} A'_{\sigma'(i)},$$

where $A'_i = \prod_{j=1}^{k'_i} x'_{i\sigma'_i(j)}$. Therefore $x \ \alpha_{n+1} \ y$. ■

Lemma 7.2.2. *If $x \ \alpha_n \ y$ then for every $a \in R$, $x + a \ \overline{\overline{\alpha_n}} \ y + a$ and $xa \ \overline{\overline{\alpha_n}} \ ya$*

Proof. It is straightforward. ■

Now, we determine some necessary and sufficient conditions so that the relation α is transitive. First, we define the notion of an α-part.

Definition 7.2.3. Let M be a nonempty subset of a hyperring R. We say that M is an *α-part* if for every $n \in \mathbb{N}$, $i = 1, 2, ..., n$, $\forall k_i \in \mathbb{N}$, $\forall (z_{i1}, z_{i2}, ..., z_{ik_i}) \in R^{k_i}$, $\forall \sigma \in \mathbb{S}_n, \forall \sigma_i \in \mathbb{S}_{k_i}$, we have

$$\sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} z_{ij} \right) \bigcap M \neq \emptyset \Longrightarrow \sum_{i=1}^{n} A_{\sigma(i)} \subseteq M$$

where $A_i = \prod_{j=1}^{k_i} z_{i\sigma_i(j)}$.

Proposition 7.2.4. *Let M be a nonempty subset of a hyperring R. The following conditions are equivalent:*

(1) *M is a α-part of R;*

(2) $x \in M, x \ \alpha \ y \Rightarrow y \in M$;

(3) $x \in M, x \ \alpha^* \ y \Rightarrow y \in M$.

Proof. (1$\Longrightarrow$2): If $(x, y) \in R^2$ is a pair such that $x \in M$ and $x \ \alpha \ y$, then $\exists n \in \mathbb{N}$, $\exists (k_1, ..., k_n) \in \mathbb{N}^n$, $\exists \sigma \in \mathbb{S}_n$ and $[\exists (z_{i1}, ..., z_{ik_i}) \in R^{k_i}$, $\exists \sigma_i \in \mathbb{S}_{k_i}$,

$(i = 1, ..., n)]$ such that $x \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} z_{ij}\right)\bigcap M$ and $y \in \sum_{i=1}^{n} A_{\sigma(i)}$ where $A_i = \prod_{j=1}^{k_i} z_{i\sigma_i(j)}$. Since M is a α-part of R, we have $\sum_{i=1}^{n} A_{\sigma(i)} \subseteq M$ and $y \in M$.

$(2 \Longrightarrow 3)$: Let $(x, y) \in R^2$ be such that $x \in M$ and $x\ \alpha^*\ y$. Obviously, there exist $m \in \mathbb{N}$ and $(x = w_0, w_1, ..., w_{m-1}, w_m = y) \in R^{m+1}$ such that $x = w_0\ \alpha\ w_1\ \alpha ... \alpha\ w_{m-1}\ \alpha\ w_m = y$. Since $x \in M$, we obtain $y \in M$, by applying (2) m times.

$(3 \Longrightarrow 1)$: Let $\sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} z_{ij}\right)\bigcap M \neq \emptyset$, and $x \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} z_{ij}\right)\bigcap M$. For every $\sigma \in \mathbb{S}_n$ and every $\sigma_i \in \mathbb{S}_{k_i}, i = 1, 2, ..., n$ and for every $y \in \sum_{i=1}^{n} A_{\sigma(i)}$ where $A_i = \prod_{j=1}^{k_i} z_{i\sigma_i(j)}$, we have $x\alpha y$. Thus $x \in M$ and $x\alpha^* y$. Finally, by (3), we obtain $y \in M$, where $\sum_{i=1}^{n} A_{\sigma(i)} \subseteq M$ and $A_i = \prod_{j=1}^{k_i} z_{i\sigma_i(j)}$. ■

Before proving the next theorem, we introduce the following notations. For every element x of a hyperrring R, set:

$$[x]^n_{k_1,k_2,...,k_n} = \{(x_{i1}, x_{i2}, ..., x_{ik_i}) \in R^{k_i} | i = 1, 2..., n\}$$

$$T_n(x) = \left\{ [x]^n_{k_1,k_2,...,k_n} \mid x \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij}\right)\right\}$$

$$P_n(x) = \bigcup \left\{ \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{\sigma(i)\sigma_{\sigma(i)}(j)}\right) \mid \sigma \in \mathbb{S}_n, \sigma_i \in \mathbb{S}_{k_i}, [x]^n_{k_1,k_2,...,k_n} \in T_n(x)\right\}$$

$$P(x) = \bigcup_{n \geq 1} P_n(x)$$

From the above notations and definitions, we obtain:

Lemma 7.2.5. *For every $x \in R$, $P(x) = \{y \in R \mid x\ \alpha\ y\}$.*

Proof. For every pair (x, y) of elements of R we have:

$$\begin{aligned} x\,\alpha\,y \iff & \exists n \in \mathbb{N},\ \exists (k_1, \ldots, k_n) \in \mathbb{N}^n,\ \exists \sigma \in \mathbb{S}_n[\exists (z_{i1}, \ldots, z_{ik_i}) \in R^{k_i}, \exists \sigma_i \in \mathbb{S}_{k_i}, \\ & (i{=}1, \ldots, n)] : x {\in} \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right), y {\in} \sum_{i=1}^{n} A_{\sigma(i)} \text{ where } A_i {=} \prod_{j=1}^{k_i} z_{i\sigma_i(j)} \\ \iff & \exists n \in \mathbb{N} : y \in P_n(x) \\ \iff & y \in P(x) \ \blacksquare \end{aligned}$$

Theorem 7.2.6. *Let R be a hyperring. Then the following conditions are equivalent:*

(1) *α is transitive;*

(2) *for every $x \in R, \alpha^*(x) = P(x)$;*

(3) *for every $x \in R$, $P(x)$ is an α-part of R.*

Proof. (1$\Longrightarrow$2): By Lemma 7.2.5, for every pair (x, y) of elements of R we have:

$$y \in \alpha^*(x) \iff x\alpha^* y \iff x\alpha y \iff y \in P(x).$$

(2$\Longrightarrow$3): If M is a nonempty subset of R, then M is an α-part of R if and only if it is a union of equivalence classes modulo α^*. Particularly, every equivalence class modulo α^* is an α-part.

(3$\Longrightarrow$1): If $x\ \alpha\ y$ and $y\ \alpha\ z$, then $\exists (n, m) \in \mathbb{N} \times \mathbb{N}$, $\exists\ [x]^n_{k_1,k_2,\ldots,k_n} \in T_n(x)$, $\exists\ [y]^m_{k'_1,k'_2,\ldots,k'_m} \in T_m(y)$, $\exists\ \sigma \in \mathbb{S}_n$ and $\exists\ \tau \in \mathbb{S}_m$, $\exists \sigma_i \in \mathbb{S}_{k_i}$, $(i = 1, \ldots, n)]$ and $\exists\ \tau_i \in \mathbb{S}_{k'_i}$, $(i = 1, \ldots, m)]$ such that

$$x \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right), \quad y \in \sum_{i=1}^{n} A_{\sigma(i)} \cap \sum_{i=1}^{m} \left(\prod_{j=1}^{k_i} y_{ij} \right) \quad \text{and} \quad z \in \sum_{i=1}^{m} B_{\tau(i)}$$

where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$ and $B_i = \prod_{j=1}^{k'_i} y_{i\tau_i(j)}$. Since $P(x)$ is an α-part of R, we have:

$$x \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right) \bigcap P(x) \Longrightarrow \sum_{i=1}^{n} A_{\sigma(i)} \subseteq P(x) \Longrightarrow y \in \sum_{i=1}^{m} \left(\prod_{j=1}^{k'_i} y_{ij} \right) \bigcap P(x)$$

$$\Longrightarrow \sum_{i=1}^{m} B_{\tau(i)} \subseteq P(x) \Longrightarrow z \in P(x) \Longrightarrow \exists k \in \mathbb{N} \ : z \in P_k(x) \Longrightarrow z \ \alpha \ x.$$

Therefore α is transitive. ■

Definition 7.2.7. A hyperring R is said *n-complete* if $\forall (k_1, ..., k_n) \in \mathbb{N}^n$, $\forall (x_{ij}, ..., x_{ik_i}) \in R^{k_i}$ we have

$$\gamma \left(\sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right) \right) = \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}).$$

Definition 7.2.8. A hyperring R is said α_n*-complete* if $\forall (k_1, ..., k_n) \in \mathbb{N}^n$, $\forall (x_{ij}, ..., x_{ik_i}) \in R^{k_i}$, $\forall \sigma \in \mathbb{S}_n$, $\forall \sigma_i \in \mathbb{S}_{k_i}$, $i = 1, ..., n$ we have

$$\alpha \left(\sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right) \right) = \sum_{i=1}^{n} A_{\sigma(i)},$$

where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$.

Corollary 7.2.9. *If R is a commutative hyperring then R is an α_n-complete hyperring if and only if R is an n-complete hyperring .*

Proof. Since R is a commutative hyperring then we have $\Gamma = \alpha$. ■

Proposition 7.2.10. *A hyperring R is α_n-complete if and only if* $\forall (k_1, ..., k_n) \in \mathbb{N}^n$, $\forall (x_{ij}, ..., x_{ik_i}) \in R^{k_i}, \forall \sigma \in \mathbb{S}_n, \forall \sigma_i \in \mathbb{S}_{k_i}, i = 1, ..., n$ *and for* *every* $x \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij}\right)$, *we have* $\alpha(x) = \sum_{i=1}^{n} A_{\sigma(i)}$, *where* $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$.

Proof . Suppose that R is α_n-complete. From $x \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij}\right)$ it follows that

$$\alpha(x) \subseteq \bigcup_{t\in\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})} \alpha(t) = \alpha\left(\sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij}\right)\right) = \sum_{i=1}^{n} A_{\sigma(i)},$$

and so $\alpha(x) \subseteq \sum_{i=1}^{n} A_{\sigma(i)}$.

Now, if $y \in \sum_{i=1}^{n} A_{\sigma(i)}$, then

$$x\alpha_n y \Longrightarrow x\alpha y \Longrightarrow y \in \alpha(x),$$

whence $\sum_{i=1}^{n} A_{\sigma(i)} \subseteq \alpha(x)$. Hence $\alpha(x) = \sum_{i=1}^{n} A_{\sigma(i)}$.

Conversely, for every $[(k_1, ..., k_n)\in\mathbb{N}^n,\ (x_{ij}, ..., x_{ik_i})\in R^{k_i},\ \sigma\in\mathbb{S}_n,\ \sigma_i\in\mathbb{S}_{k_i}]$, and for all $x \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij}\right)$ we have $\alpha(x) = \sum_{i=1}^{n} A_{\sigma(i)}$. Hence

$$\alpha\left(\sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij}\right)\right) = \bigcup_{x\in\sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij}\right)} \alpha(x) = \sum_{i=1}^{n} A_{\sigma(i)}.$$

Therefore R is α_n-*complete.* ∎

Proposition 7.2.11. *If* R *is an* α_n*-complete hyperring then* $\alpha = \alpha_n$.

Proof . It is suffices to prove that $\alpha \subseteq \alpha_n$. Suppose that $x\alpha y$. Thus $\exists m \in \mathbb{N}$ such that $x\alpha_m y$. If $m \leq n$ then by Lemma 7.2.1, we have $\alpha_m \subseteq \alpha_n$. If $m > n$ then $[\exists(k_1, ..., k_n) \in \mathbb{N}^n, (x_{ij}, ..., x_{ik_i}) \in R^{k_i}, \sigma \in \mathbb{S}_n, \sigma_i \in \mathbb{S}_{k_i}]$ such that $x \in \sum_{i=1}^{m} \left(\prod_{j=1}^{k_i} x_{ij} \right)$ and $y \in \sum_{i=1}^{m} A_{\sigma(i)}$, where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)})$. It follows that there exists $s \in R$ such that

$$s \in \sum_{i=n}^{m} \left(\prod_{j=1}^{k_i} x_{ij} \right) \text{ and } x \in \sum_{i=1}^{n-1} \left(\prod_{j=1}^{k_i} x_{ij} \right) + s.$$

We set $z_{ij} = x_{ij}, i = 1, ..., n-1, k_n = 1, z_{nk_n} = s$. Hence $x \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} z_{ij} \right)$. By Proposition 7.2.10, for every $\sigma \in \mathbb{S}_n, \sigma_i \in \mathbb{S}_{k_i}, (i = 1, ..., n)$, we have $y \in \alpha(x) = \sum_{i=1}^{n} A_{\sigma(i)}$ where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$. Hence $x \alpha_n y$. ■

Proposition 7.2.12. *If R is an α_n-complete hyperring then for all $[(k_1, ..., k_n) \in \mathbb{N}^n, (x_{ij}, ..., x_{ik_i}) \in R^{k_i}, \sigma \in \mathbb{S}_n, \sigma_i \in \mathbb{S}_{k_i}]$, $\sum_{i=1}^{n} A_{\sigma(i)}$ is an α-part of R, where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)})$.*

Proof. We set $M = \sum_{i=1}^{n} A_{\sigma(i)}$. Consider $m \in \mathbb{N}, (k'_1, ..., k'_m) \in \mathbb{N}^m$ and $(y_{ij}, ..., y_{ik'_i}) \in R^{k_i}$ such that $\sum_{i=1}^{m} \left(\prod_{j=1}^{k'_i} y_{ij} \right) \bigcap M \neq \emptyset$. Then there exists $a \in \sum_{i=1}^{m} \left(\prod_{j=1}^{k'_i} y_{ij} \right) \bigcap M$. If $A'_i = \prod_{j=1}^{k'_i} y_{i\sigma_i(j)}$, then for every $\sigma' \in \mathbb{S}_m, \sigma'_i \in \mathbb{S}_{k'_i}$ and for every $y \in \sum_{i=1}^{m} A'_{\sigma'(i)}$, we have $a\ \alpha_m\ y$, so $y \in \alpha_m(a)$. By Proposition 7.2.11, we have $y \in \alpha(a) = \sum_{i=1}^{n} A_{\sigma(i)} = M$, thus $\sum_{i=1}^{m} A'_{\sigma'(i)} \subseteq \sum_{i=1}^{n} A_{\sigma(i)} = M$. ■

7.3 Applications of the α^*-relation to Krasner hyperrings

Let $(R, +, \cdot)$ be a Krasner hyperring. We can define the relation Γ as follows: $a\Gamma b$ if and only if $\{a, b\} \subseteq u$, where u is a finite sum of finite products of elements of R. The relation γ^* is the transitive closure of γ. The operations $\oplus$ and $\odot$ are defined on R/γ^* as follows:

$$\gamma^*(a) \oplus \gamma^*(b) = \gamma^*(c) \text{ for all } c \in \gamma^*(a) + \gamma^*(b),$$
$$\gamma^*(a) \odot \gamma^*(b) = \gamma^*(ab).$$

Also, we can define the relation β_+^* as the smallest equivalence relation such that the quotient R/β_+^* is a group. Let us consider the following relation on R.

$a\beta_+ b$ if and only if there exists $(c_1, ..., c_n) \in R^n$ such that $\{a, b\} \subseteq c_1 + ... + c_n$.

It is clear that $\beta_+^* = \beta_+$. We define the relation α^* as the smallest equivalence relation on R such that the quotient R/α^*, the set of all equivalence classes, is a commutative ring. R/α^* is called a commutative fundamental ring. Suppose that $\alpha^*(a)$ is the equivalence class of $a \in R$. Then both the sum $\uplus$ and the product $\otimes$ are defined in R/α^* as follows:

$$\alpha^*(a) \uplus \alpha^*(b) = \alpha^*(c) \text{ for all } c \in \alpha^*(a) + \alpha^*(b),$$
$$\alpha^*(a) \otimes \alpha^*(b) = \alpha^*(ab).$$

Recall that the relation α is the following one.
$x\alpha y \iff \exists n \in \mathbb{N}, \exists (k_1, ..., k_n) \in \mathbb{N}^n, \exists \sigma \in \mathbb{S}_n, \exists (x_{i1}, ..., x_{ik_i}) \in R^{k_i}$ and $\exists \sigma_i \in \mathbb{S}_{k_i}, (i = 1, ..., n)$ such that

$$x \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right) \quad \text{and} \quad y \in \sum_{i=1}^{n} A_{\sigma(i)},$$

where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$. Also, we can define the relation Γ_+^* as the smallest equivalence relation such that the quotient R/Γ_+^* is an abelian group. Since all Krasner hyperrings are additive, we have $\Gamma^* = \beta^*_+$. The relational notation $A \approx B$ is used to *assert* that the sets A and B have an element in common, that is, $A \cap B \neq \emptyset$.

Theorem 7.3.1. *If $(R, +, \cdot)$ is a Krasner hyperring and $(R, \cdot)$ is commutative then we have $\alpha^* = \Gamma^*_+$.*

Proof. Every product of elements of R is a singleton in a Krasner hyperring. Thus, for every $A = \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij}\right)$ we can consider the elements $y_i = \prod_{j=1}^{k_i} x_{ij}$ $(i = 1, ..., n)$ of R, for which we have $A = \sum_{i=1}^{n} y_i$. Since the semigroup $(R, \cdot)$ is commutative, this means that $a\alpha^* b$ if and only if $a\Gamma^*_+ b$. ■

Recall that $\varphi : R \longrightarrow R/\Gamma^*$ and $\phi : R \longrightarrow R/\alpha^*$ be the canonical projections.

Lemma 7.3.2. *The following assertions hold:*

(1) $\omega_R = \varphi^{-1}(0_{R/\Gamma^*})$ *and* $D(R) = \phi^{-1}(0_{R/\alpha^*})$,

(2) $\Gamma^*(-x) = -\Gamma^*(x)$ *and* $\alpha^*(-x) = -\alpha^*(x)$ *for all* $x \in R$,

(3) $\Gamma^*(0) = 0_{R/\Gamma^*}$ *and* $\alpha^*(-x) = 0_{R/\alpha^*}$.

Proof. It is straightforward. ■

The next lemma and theorem follow from Lemma 7.1.10 and Theorem 7.1.12.

Lemma 7.3.3. *Let R be a Krasner hyperring. Then*

$$R \cdot D(R) \subseteq D(R) \quad \text{and} \quad D(R) \cdot R \subseteq D(R).$$

Theorem 7.3.4. *If R is a Krasner hyperring, then $D(R)$ is a hyperideal of R.*

Theorem 7.3.5. *If R is a Krasner hyperring, then $D(R)$ is a normal hyperideal of R.*

Proof. If $y \in x+D(R)-x$ then there exists $a \in D(R)$ such that $y \in x+a-x$, thus $\phi(y) = \phi(x+a-x) = \phi(x)+\phi(a)+\phi(-x) = \phi(x)+0_{R/\alpha^*}-\phi(x) = 0_{R/\alpha^*}$. Therefore, we have $y \in \phi^{-1}(0_{R/\alpha^*}) = D(R)$. ∎

Recall that if A is a normal hyperideal of a hyperring, then the relation A^* is defined as follows: $x \equiv y(modA)$ if and only if $(x-y) \cap A \neq \emptyset$ (see Definition 3.2.7).

Theorem 7.3.6. *If R is a Krasner hyperring and $A = D(R)$ then we have $A^* = \alpha^*$.*

Proof. Let xA^*y. Thus there exists $z \in A = D(R)$ such that $z \in x-y$ and so $\alpha^*(x-y) = \alpha^*(z) = 0_{R/\alpha^*}$. Hence $\alpha^*(x) = \alpha^*(y)$ and $x\ \alpha^*\ y$, thus $A^* \subseteq \alpha^*$. For converse if $\alpha^*(x) = \alpha^*(y)$ then $\alpha^*(x-y) = 0_{R/\alpha^*}$. So $x-y \subseteq D(R) = A$, and therefore $x\ A^*\ y$. Thus $A^* = \alpha^*$. ∎

Theorem 7.3.7. *If R is a Krasner hyperring and $B = \omega_R$ then we have*

(1) *B is a normal hyperideal of R,*

(2) *the equivalence relation B^* is equal to the fundamental relation γ^*.*

Recall that if A is a normal hyperideal of a hyperring R, then $[R : A^*] = \{A^*(x) \mid x \in R\}$ is a hyperring.

Remark 7.3.8. If $A = D(R)$ then $[R : A^*] = R/\alpha^*$ and so $[R : A^*]$ is a commutative ring. If $B = \omega_R$ then $[R : B^*] = R/\gamma^*$ and $[R : B^*]$ is a ring.

Theorem 7.3.9. *Let R be a Krasner hyperring. Then $I = D(R)/\gamma^*$ is an ideal of the ring $S = R/\gamma^*$ and we have $S/I \cong R/\alpha^*$.*

Proof. Set $A{=}D(R)$ and $B{=}\omega_R$. By Remark 7.3.8, we have $[R : B^*]{=}R/\gamma^*$, $[R : A^*]{=}R/\alpha^*$ and $[D(R) : B^*]{=}D(R)/\gamma^*$. Now, by the third isomorphism theorem, $[A : B^*]$ is a normal hyperideal of $[R : B^*]$ and $[[R : B^*] : [A : B^*]] \cong [R : A^*]$. But $[R : B^*] = R/\gamma^*$ is a ring and so $[A : B^*] = D(R)/\gamma^*$ is an ideal of R/γ^*. Therefore $[R/\gamma^* : D(R)/\gamma^*] \cong R/\alpha^*$. ∎

The following results are obtained from Lemma 7.1.7, Corollary 7.1.8 and Theorem 7.1.9.

Lemma 7.3.10. *Let R_1 and R_2 be two Krasner hyperrings, $a, c \in R_1$, $b, d \in R_2$ and $\sigma \in \mathbb{S}_n$. Then*

$$(a,b) \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} (x_{ij}, y_{ij}) \right) \text{ and } (c,d) \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_{\tau(i)}} (x_{\tau(i)\sigma_{\tau(i)}(j)}, y_{\tau(i)\sigma_{\tau(i)}(j)}) \right)$$

if and only if

$$a \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right), \quad c \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_{\tau(i)}} x_{\tau(i)\sigma_{\tau(i)}(j)} \right)$$

and

$$b \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} y_{ij} \right), \quad d \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_{\tau(i)}} y_{\tau(i)\sigma_{\tau(i)}(j)} \right),$$

for some $x_{ij} \in R_1, \ y_{ij} \in R_2$.

Corollary 7.3.11. *Let* R_1 *and* R_2 *be two Krasner hyperrings,* $\alpha^*_{R_1}$, $\alpha^*_{R_2}$ *and* $\alpha^*_{R_1 \times R_2}$ *be* α^**-relations on* R_1, R_2 *and* $R_1 \times R_2$ *respectively. Then*

$$(a,b) \ \alpha^*_{R_1 \times R_2} \ (c,d) \ \text{ if and only if } \ a \ \alpha^*_{R_1} \ c \ \text{ and } \ b \ \alpha^*_{R_2} \ d.$$

Theorem 7.3.12. *Let* R_1 *and* R_2 *be two Krasner hyperrings,* $\alpha^*_{R_1}$, $\alpha^*_{R_2}$ *and* $\alpha^*_{R_1 \times R_2}$ *be* α^**-relations on* R_1, R_2 *and* $R_1 \times R_2$ *respectively. Then*

$$(R_1 \times R_2)/\alpha^*_{R_1 \times R_2} \cong R_1/\alpha^*_{R_1} \ \times \ R_2/\alpha^*_{R_2},$$

Lemma 7.3.13. *If* A, B *are normal hyperideals of* R_1, R_2 *respectively, then*

$$[(R_1 \times R_2) : (A \times B)^*] \cong [R_1 : A^*] \times [R_2 : B^*].$$

Proof. The proof is straightforward and we omit it. ■

Corollary 7.3.14. *If* A, B *are normal hyperideals of* R_1, R_2 *respectively, and* α_1^*, α_2^* *and* α^* *are the fundamental equivalence relations on* $[R_1 : A^*]$, $[R_2 : B^*]$ *and* $[(R_1 \times R_2) : (A \times B)^*]$ *respectively, then*

$$[(R_1 \times R_2) : (A \times B)^*]/\alpha^* \cong [R_1 : A^*]/\alpha_1^* \times [R_2 : B^*]/\alpha_2^*.$$

Proof. The proof is obtained directly from Theorem 7.3.12 and Lemma 7.3.13. ■

Definition 7.3.15. Let f be a strong homomorphism from R_1 to R_2 and let α_1^*, α_2^* be the fundamental relations on R_1, R_2 respectively. We define

$$\overline{kerf} = \{\alpha_1^*(x) \mid x \in R_1,\ \alpha_2^*(f(x)) = D(R_2)\}.$$

Lemma 7.3.16. *$\overline{kerf}$ is an ideal of the commutative fundamental ring R_1/α_1^*.*

Proof. Assume that $\alpha_1^*(x), \alpha_1^*(y) \in \overline{kerf}$. Then for every $z \in x - y$ we have $\alpha_1^*(z) = \alpha_1^*(x) \uplus \alpha_1^*(-y)$. On the other hand, we have

$$\begin{aligned}\alpha_2^*(f(z)) &= \alpha_2^*(f(x) + f(-y)) = \alpha_2^*(f(x)) \uplus \alpha_2^*(f(-y)) \\ &= \alpha_2^*(f(x)) \uplus (-\alpha_2^*(f(y))) = D(R_2) \uplus D(R_2) = D(R_2).\end{aligned}$$

Therefore $\alpha_1^*(z) \in \overline{kerf}$. Now, for $\alpha_1^*(r) \in R_1/\alpha_1^*$ and $\alpha_1^*(x) \in \overline{kerf}$ we have

$$\alpha_2^*(f(r\cdot x)) = \alpha_2^*(f(r)\cdot f(x)) = \alpha_2^*(f(r)) \otimes \alpha_2^*(f(x)) = \alpha_2^*(f(r)) \otimes D(R_2) = D(R_2),$$

and so $\alpha_1^*(r) \otimes \alpha_1^*(x) \in \overline{kerf}$. Therefore, $\overline{kerf}$ is an ideal of R_1/α^*. ■

Theorem 7.3.17. *Let $(R, +, \cdot)$ be a Krasner hyperring such that $(R, \cdot)$ be commutative. Let A, B be two normal hyperideals of R with $A \subseteq B$ and let $\phi : [R : A^*] \longrightarrow [R : B^*]$ be the canonical map. Suppose that α_A^*, α_B^* are the fundamental equivalence relations on $[R : A^*]$, $[R : B^*]$ respectively. Then*

$$([R : A^*]/\alpha_A^*)/\overline{ker\phi} \cong [R : B^*]/\alpha_B^*.$$

Proof. We define the map

$$\rho : [R : A^*]/\alpha_A^* \longrightarrow [R : B^*]/\alpha_B^*$$

by

$$\rho : \alpha_A^*(A + x) \longmapsto \alpha_B^*(B + x) \ \ (\text{for all } x \in R).$$

We must check that ρ is well-defined, i.e., if $x, y \in R$ and $\alpha_A^*(A + x) = \alpha_A^*(A + y)$ then $\alpha_B^*(B + x) = \alpha_B^*(B + y)$. Using Theorem 7.3.1, we have

$\alpha_A^* = (\Gamma_+^*)_A$ and $\alpha_B^* = (\Gamma_+^*)_B$. Now, $(\Gamma_+^*)_A(A+x) = (\Gamma_+^*)_A(A+y)$ if and only if there exist $(A+x_1, A+x_2, ..., A+x_n) \in [R : A^*]^n$ and $\sigma \in S_n$ such that $A+x \in \oplus\sum_{i=1}^n A+x_i$ and $A+y \in \oplus\sum_{i=1}^n A+x_{\sigma(i)}$. We have

$$\oplus\sum_{i=1}^n A+x_i = \left\{A+z \mid z \in \sum_{i=1}^n x_i\right\}.$$

Therefore, we have $A+x = A+z_1$ and $A+y = A+z_2$ for some $z_1 \in \sum_{i=1}^n x_i$, $z_2 \in \sum_{i=1}^n x_{\sigma(i)}$. So there exist $a \in (x-z_1) \cap A$ and $b \in (y-z_2) \cap A$, whence $x \in a+z_1$ and $y \in b+z_2$. Hence $B+x \in (B+a) \oplus (B+z_1)$ and $B+y \in (B+b) \oplus (B+z_2)$. Since $a, b \in A \subseteq B$, it follows that $B+a = B$, $B+b = B$. Since $B \oplus (B+z_1) = B+z_1$ and $B \oplus (B+z_2) = B+z_2$, we have $B+x = B+z_1$ and $B+y = B+z_2$. Finally, from

$$B+z_1 \in \left\{B+z \mid z \in \sum_{i=1}^n x_i\right\} \text{ and } B+z_2 \in \left\{B+z \mid z \in \sum_{i=1}^n x_{\sigma(i)}\right\},$$

we obtain

$$B+x \in \left\{B+z \mid z \in \sum_{i=1}^n x_i\right\} = \oplus\sum_{i=1}^n (B+x_i),$$

$$B+y \in \left\{B+z \mid z \in \sum_{i=1}^n x_{\sigma(i)}\right\} = \oplus\sum_{i=1}^n (B+x_{\sigma(i)}).$$

Therefore $(\Gamma_+^*)_B(B+x) = (\Gamma_+^*)(B+y)$. This means that ρ is well-defined. Moreover ρ is a strong homomorphism. Indeed, if $x, y \in R_1$ then

$$\begin{aligned}
\rho(\alpha_A^*(A+x) \uplus \alpha_A^*(A+y)) &= \rho(\alpha_A^*(A+x+y)) = \alpha_B^*(B+x+y) \\
&= \alpha_B^*(B+x) \uplus \alpha_B^*(B+y) \\
&= \rho(\alpha_A^*(A+x)) \uplus \rho(\alpha_A^*(A+y)), \\
\rho(\alpha_A^*(A+x) \otimes \alpha_A^*(A+y)) &= \rho(\alpha_A^*(A+xy)) = \alpha_B^*(B+xy) \\
&= \alpha_B^*(B+x) \otimes \alpha_B^*(B+y) \\
&= \rho(\alpha_A^*(A+x)) \otimes \rho(\alpha_A^*(A+y)),
\end{aligned}$$

and $\rho(D([R:A^*])) = \rho(\alpha^*_A(A)) = \alpha^*_B(B) = D([R:B^*])$. Clearly, ρ is onto. Now, we show that $ker\rho = \overline{ker\phi}$. We have

$$\begin{aligned} ker\rho &= \{\alpha^*_A(A+x) \mid \rho(\alpha^*_A(A+x)) = D([R:B^*])\} \\ &= \{\alpha^*_A(A+x) \mid \alpha^*_B(B+x) = D([R:B^*])\} \\ &= \{\alpha^*_A(A+x) \mid \Gamma^*_B(\phi(A+x)) = D([R:B^*])\} \\ &= \overline{ker\phi}. \end{aligned}$$

Finally, we apply the first isomorphism theorem. ■

Theorem 7.3.18. *Let R be a Krasner hyperring and $a_1,...,a_m,b_1,...,b_m \in R$ such that $a_j \,\alpha\, b_j$ for all j=1, ...m. Then for all $x \in \sum_{i=1}^{m} \delta_i a_i$ and for all $y \in \sum_{i=1}^{m} \delta_i b_i$ where $\delta_i \in \{1,-1\}$ (i=$\{1,...,m\}$), we have $x \,\alpha\, y$.*

Proof. Suppose that $a_j \;\alpha\; b_j$ for all $j = 1,...,m$. Then there exist $n_j \in \mathbb{N}$, $(k_{j1},...,k_{jn_j}) \in \mathbb{N}^{n_j}$ and $(z_{ji1},...,z_{jik_{ji}}) \in R^{k_{ji}}$ and there exist $\sigma_j \in \mathbb{S}_{n_j}$ and $\sigma_{ji} \in \mathbb{S}_{k_{ji}}$ when $(i = 1,...,n_j)$ such that

$$a_j \in \sum_{i=1}^{n_j} \left(\prod_{l=1}^{k_{ji}} x_{jil} \right) \quad \text{and} \quad b_j \in \sum_{i=1}^{n_j} \left(\prod_{l=1}^{k_{j\sigma(i)}} x_{j\sigma(i)\sigma_{j\sigma(i)}(l)} \right);$$

Therefore

$$\sum_{j=1}^{m} a_j \subseteq \sum_{j=1}^{m} \left(\sum_{i=1}^{n_j} \left(\prod_{l=1}^{k_{ji}} x_{jil} \right) \right), \quad \sum_{j=1}^{m} b_j \subseteq \sum_{j=1}^{m} \left(\sum_{i=1}^{n_j} \left(\prod_{l=1}^{k_{j\sigma(i)}} x_{j\sigma(i)\sigma_{j\sigma(i)}(l)} \right) \right).$$

If we rename x_{ijl}'s, then we conclude that there exists $n, k_i \in \mathbb{N}$ and $\tau \in \mathbb{S}_n$ and $\tau_i \in \mathbb{S}_{k_i}$ such that

$$\sum_{j=1}^{m} a_j \subseteq \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right), \qquad \sum_{j=1}^{m} b_j \subseteq \sum_{i=1}^{n} A_{\sigma(i)},$$

where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$, and so for all $x \in \sum_{j=1}^{m} a_j$ and for all $y \in \sum_{j=1}^{m} b_j$ we get $x \;\alpha\; y$. Finally, note that if $a_j \;\alpha\; b_j$ then $(-a_j) \;\alpha\; (-b_j)$. ■

Theorem 7.3.19. *Let R be a Krasner hyperring. Then $x, y \in \alpha^*(0)$ if and only if there exist $A, A', B, B' \subseteq \alpha^*(z)$ for some $z \in R$ such that $(x+A) \approx B$ and $(y + A') \approx B'$.*

Proof. Suppose that there exist $A, A' \subseteq \alpha^*(z)$ and $B, B' \subseteq \alpha^*(-z)$ for some $z \in R$ such that $(x + A) \approx B$ and $(y + A') \approx B'$. Then we have

$$(\alpha^*(x) \oplus \{\alpha^*(a) \mid a \in A\}) \approx \{\alpha^*(b) \mid b \in B\},$$
$$(\alpha^*(y) \oplus \{\alpha^*(a') \mid a' \in A'\}) \approx \{\alpha^*(b') \mid b' \in B'\}.$$

Therefore, we obtain $\alpha^*(x)\oplus\alpha^*(z) = \alpha^*(z)$ and $\alpha^*(y)\oplus\alpha^*(z) = \alpha^*(z)$, which imply that $\alpha^*(x) = \alpha^*(y) = \alpha^*(z) \oplus \alpha^*(-z) = \alpha^*(0)$. Hence $x, y \in \alpha^*(0)$.

For the converse, take $A = A' = B = B' = \alpha^*(0)$, then $\alpha^*(0) \subseteq \alpha^*(0)$ and $\{x\} \subseteq \alpha^*(0)$ which imply that $(x + \alpha^*(0)) \subseteq \alpha^*(0)$ or $(x + A) \approx A$. Similarly, we obtain $(y + A) \approx A$. This complete the proof. ∎

If $r \in \mathbb{N}^*$ and $a \in R$ then $ra = \sum_{i=1}^{r} a$.

Theorem 7.3.20. *Let R be a finite Krasner hyperring. For every $a \in R$, there exist $r, s \in \mathbb{N}$ such that $0 < s < r$, $ra \approx sa$ and $(r - s)a \subseteq \alpha^*(0)$.*

Proof. Since R is finite it follows that there exist $r, s \in \mathbb{N}$ such that $0 < s < r$ and $ra \approx sa$. From $ra \approx sa$ we obtain $\phi(ra)=\phi(sa)$, and so $r\alpha^*(a)=s\alpha^*(a)$. Since $r\alpha^*(a)$ and $s\alpha^*(a)$ are the elements of the commutative ring R/α^*, we obtain

$$(r - s)\alpha^*(a) = 0_{R/\alpha^*} = \alpha^*(0)$$

which implies that $\phi((r - s)a) = \alpha^*(0)$, and so $(r - s)a \subseteq \alpha^*(0)$. ∎

Let M be a nonempty subset of a Krasner hyperring R. We say that M is an *α-part* of R, if for every $n\in\mathbb{N}$, $i=1, 2, \ldots, n$, $\forall k_i\in\mathbb{N}$, $\forall(z_{i1}, z_{i2}, \ldots, z_{ik_i})\in R^{n_i}$, $\forall\sigma\in\mathbb{S}_n$, $\forall\sigma_i \in \mathbb{S}_{k_i}$, we have

$$\sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} z_{ij}\right) \approx M \Rightarrow \sum_{i=1}^{n} A_{\sigma(i)} \subseteq M \text{ where } A_i = \prod_{j=1}^{k_i} z_{i\sigma_i(j)}.$$

Let A be a nonempty subset of R. The intersection of all α-parts of R which contain A is called the *α-closure* of A in R. It will be denoted by $C_\alpha(A)$. We have

(1) If A is an α-part of R then $A+B$, $B+A$, AB and BA are α-parts of R for every $B \in \mathcal{P}^*(R)$.

(2) Let $A \in \mathcal{P}^*(R)$. Then A is an α-part of R if and only if $A+D(R)=A$.

(3) If $A \in \mathcal{P}^*(R)$, then $D(R)+A = A+D(R) = C_\alpha(A)$.

(4) $D(R)$ is an α-part of R.

Let R be a Krasner hyperring, $X = <\mathcal{P}^*(R), \uplus>$ be the set of all nonempty subsets of R endowed with the hyperoperation $\uplus$ defined as follows:

$$A \uplus B = \{C \in \mathcal{P}^*(R) \mid C \subseteq A+B\} \text{ for all } (A,B) \in \mathcal{P}^*(R)^2.$$

Let $\sum(R)$ be the set of all hypersums of elements of R.

Theorem 7.3.21. *If $(R,+,\cdot)$ is a Krasner hyperring, then $<\sum(R), \uplus, \cdot>$ is a Krasner hyperring, too.*

Proof. It is clear that $\uplus$ is associative. If $E=\{0\}$, then for all $A \in \sum(R)$ we have $A \uplus E = E \uplus A = A$. We define the function $-I$ as follows:

$$-I : \sum(R) \longrightarrow \sum(R) \qquad -I\Big(\sum_{i=1}^{n} x_i\Big) = \sum_{i=1}^{n}(-x_i).$$

Now, let $X = \sum_{i=1}^{m} x_i$, $Y = \sum_{i=1}^{n} y_i$ and $Z = \sum_{i=1}^{p} z_i$ be elements of $\sum(R)$ such that $X \in Y \uplus Z$. Let $x \in X$ and $y \in Y$ be arbitrary. Then there exists $z \in Z$ such that $x \in y+z$. Indeed, if we suppose that for every $z \in Z$ we have $x \notin y+z$ then $\{x\} \notin Y \uplus Z$ which is a contradiction. From $x \in y+z$, we obtain $y \in x-z$, and so $y \in \sum_{i=1}^{n} x_i + \sum_{i=1}^{p}(-z_i)$ for every $y \in Y$. Therefore $Y \in X \uplus -I(Z)$. Similarly, we obtain $Z \in -I(Y) \uplus X$. Now, we prove that $\left(\sum(R), \cdot\right)$ is a semigroup. Suppose that $X = \sum_{i=1}^{m} x_i$, $Y = \sum_{i=1}^{n} y_i$, we have

$$X \cdot Y = \left(\sum_{i=1}^{m} x_i\right) \cdot \left(\sum_{i=1}^{n} y_i\right) = \sum_{j=1}^{m}\sum_{i=1}^{n} x_i y_j \in \sum(R)$$

and $0 \cdot X = X \cdot 0 = 0$. The operation $\cdot$ is distributive over the hyperoperation $\uplus$. Indeed, if $X = \sum_{i=1}^{m} x_i$, $Y = \sum_{i=1}^{n} y_i$ and $Z = \sum_{i=1}^{p} z_i$ are elements of $\sum(R)$, then

$$X \cdot Y + X \cdot Z = \sum_{i=1}^{m} x_i \sum_{i=1}^{n} y_i + \sum_{i=1}^{m} x_i \sum_{i=1}^{p} z_i = \sum_{i=1}^{m} x_i \left(\sum_{i=1}^{n} y_i + \sum_{i=1}^{p} z_i \right) = X \cdot (Y + Z),$$

whence $X \cdot Y \uplus X \cdot Z = X \cdot (Y \uplus Z)$. Indeed, if $A \in X \cdot Y \uplus X \cdot Z$, then $A \subseteq X \cdot Y + X \cdot Z = X \cdot (Y + Z)$. Hence there exists $C \subseteq Y + Z$ such that $A = X \cdot C$. We obtain $C \in Y \uplus Z$ whence $A \in X \cdot (Y \uplus Z)$. Conversely, if $A \in X \cdot (Y \uplus Z)$ then there exists $B \in Y \uplus Z$, such that $A = X \cdot B$. Hence $B \subseteq Y + Z$ and so $A = X \cdot (Y + Z) = X \cdot Y + X \cdot Z$, whence $A \in X \cdot Y \uplus X \cdot Z$. Thus $< \sum(R), \uplus, \cdot >$ is a Krasner hyperring. ■

Corollary 7.3.22. *If A is a subhyperring of R and A belongs to $\sum(R)$, then A is contained in $D(R)$.*

The following example show that not all subhyperrings of a Krasner hyperring R are in $\sum(R)$.

Example 7.3.23. Let $(R, +, \cdot)$ be a Krasner hyperring such that $+$ is given by the following table:

$+$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	c	$\{a, b, d\}$	$\{c, d\}$
d	d	d	$\{c, d\}$	$\{a, b, c\}$

and the semigroup $(R, \cdot)$ has the following operation: $x \cdot y = a$ for all $x, y \in R$. It is clear that $A = \{a, b\}$ is a subhyperring of R, but $A \notin \sum(R)$. Moreover $D(R) = c + d + d = R \in \sum(R)$.

If R is a Krasner hyperring, we denote the set of all hypersums A of elements of R such that $C_\alpha(A) = A$ by $\sum_{C_\alpha}(P)$.

Theorem 7.3.24. *Let R be a Krasner hyperring and $(x_1, ..., x_n) \in R^n$ be such that $\sum\limits_{i=1}^{n} x_i \in \sum\limits_{C_\alpha}(R)$. Then there exists $(y_1, ..., y_n) \in R^n$ such that $\sum\limits_{i=1}^{n} x_i + \sum\limits_{i=1}^{n} y_i = D(R)$.*

Proof. For $1 \leq j \leq n$, let a_j be an element of $D(R)$. Then there exists $y_j \in R$ such that $a_j \in x_j + y_j$. Since $D(R)$ is an α-part, we have $x_j + y_j \subseteq D(R)$. Therefore

$$\begin{aligned}\sum_{i=1}^{n} x_i + y_n &= D(R) + \sum_{i=1}^{n} x_i + y_n = \sum_{i=1}^{n-1} x_i + D(R) + x_n + y_n \\ &= \sum_{i=1}^{n-1} x_i + D(R) = D(R) + \sum_{i=1}^{n-1} x_i,\end{aligned}$$

and so

$$\sum_{i=1}^{n} x_i + y_n + y_{n-1} = D(R) + \sum_{i=1}^{n-2} x_i + x_{n-1} + y_{n-1} = D(R) + \sum_{i=1}^{n-2} x_i.$$

Going on in the same way, one arrives to $\sum\limits_{i=1}^{n} x_i + \sum\limits_{i=2}^{n} y_i = D(R) + x_1$ whence finally

$$\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i = D(R) + x_1 + y_1 = D(R). \blacksquare$$

Theorem 7.3.25. *Let R be a Krasner hyperring. If $R \backslash D(R)$ is a hyper-sum, then $D(R)$ is also a hyper-sum.*

Proof. Since $D(R)$ is an α-part, so $R \backslash D(R)$ also is an α-part. Now, by using Theorem 7.3.24, the proof is completed. ■

Chapter 8

Outline of applications of algebraic hyperstructures

In [27], several of the numerous applications of hyperstructures are presented, especially those that were found and studied in the last fifteen years. There are applications to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, median algebra, relation algebras, combinatorics, codes, artificial intelligence and probabilities.

In this section, we intend to present some other recent applications of hyperstructures in chemistry and physics.

8.1 Chemical hyperstructures

In this paragraph, we give some examples of hypergroups associated with chemistry. These chemical examples are connected to chain reactions and were considered and analyzed by B. Davvaz and A. Dehghan-Nezhad (see [37]).

Chain reactions

An atom of a group of atoms, possessing an odd (unpaired) electron, is called a *free radical*. Several examples are the following ones:

$$Cl,\ CH_3,\ C_2H_5.$$

The chlorination of methane is an example of a chain reaction, a reaction that involves a series of steps and each of them generates a reactive substance that leads to the next step. Even chain reactions may be widely different in their details, they all have certain fundamental characteristics in common.

(1) $Cl_2 \longrightarrow 2Cl^o$
(1) is called a *Chain-initiating step.*

(2) $Cl^o + CH_4 \longrightarrow HCl + CH_3^o$

(3) $CH_3^o + Cl_2 \longrightarrow CH_3Cl + Cl^o$
then (2), (3), (2), (3), etc, until the end.
(2) and (3) are called *Chain-propagating steps.*

(4) $Cl^o + Cl^o \longrightarrow Cl_2$ or

(5) $CH_3^o + CH_3^o \longrightarrow CH_3CH_3$ or

(6) $CH_3^o + Cl^o \longrightarrow CH_3Cl.$
(4), (5) and (6) are called *Chain-terminating steps.*

During the chain-initiating step, the energy is absorbed and a reactive particle is generated. In the chlorination of methane, the chain-initiating step is the cleavage of chlorine into atoms (step 1).

There are one or more chain-propagating steps and each of them consumes a reactive particle and generates another one. In the present reaction, the chain-propagating steps are the reaction of chlorine atoms with methane (step 2), and of methyl radicals with chlorine (step 3).

Finally, there are chain-terminating steps, in which reactive particles are consumed, but they are not generated new particles. In the present reaction, the chain-terminating steps are the union of two of the reactive particles, or the capture of one of them by the walls of the reaction vessel.

The Halogens F, CL, Br, and I

The halogens are typical non-metals. Although their physical forms differ-fluorine and chlorine are gases, bromine is a liquid and iodine is a solid at room temperature, each of them consists of diatomic molecules:

F_2, Cl_2, Br_2 and I_2. The halogens react with hydrogen and form gaseous components, with the formulas HF, HCL, HBr, and HI. All of them are very soluble in water. The halogens react with metals and give halides.

$$:\ddot{\underset{..}{F}} - \ddot{\underset{..}{F}}:, \qquad :\ddot{\underset{..}{CL}} - \ddot{\underset{..}{CL}}:, \qquad :\ddot{\underset{..}{Br}} - \ddot{\underset{..}{Br}}:, \qquad :\ddot{\underset{..}{I}} - \ddot{\underset{..}{I}}:$$

During the chain reaction

$$A_2 + B_2 \xleftrightarrow{\text{Heat or Light}} 2AB$$

the molecules A_2, B_2, AB occur, whose fragment parts are A^o, B^o. The elements of this collection can combine each other. All probable combinations for the set $\mathcal{H} = \{A^o, B^o, A_2, B_2, AB\}$, that can be obtained without energy, can be displayed as follows:

$+$	A^o	B^o	A_2	B_2	AB
A^o	A^o, A_2	A^o, B^o, AB	A^o, A_2	A^o, B_2, B^o, AB	A^o, AB, A_2, B^o
B^o	A^o, B^o, AB	B^o, B_2	A^o, B^o, AB, A_2	B^o, B_2	A^o, B^o, AB, B_2
A_2	A^o, A_2	A^o, B^o, AB, A_2	A^o, A_2	A^o, B^o, A_2, B_2, AB	A^o, B^o, A_2, AB
B_2	A^o, B^o, B_2, AB	B^o, B_2	A^o, B^o, A_2, B_2, AB	B^o, B_2	A^o, B^o, B_2, AB
AB	A^o, AB, A_2, B^o	A^o, B^o, AB, B_2	A^o, B^o, A_2, AB	A^o, B^o, B_2, AB	A^o, B^o, A_2, B_2, AB

Theorem 8.1.1. *$(\mathcal{H}, +)$ is an H_v-group.*

Proof. Clearly, the reproduction axiom and weak associativity are valid. As a sample of how to calculate the weak associativity, we illustrate several cases:

$$\begin{cases} (AB + A_2) + B_2 = \{AB, A_2, A^o, B^o\} + B_2 = \{B_2, AB, A_2, A^o, B^o\}, \\ AB + (A_2 + B_2) = AB + \{A_2, B_2, A^o, B^o, AB\} = \{A_2, B_2, AB, A^o, B^o\}, \end{cases}$$

$$\begin{cases} (AB + A^o) + A^o = \{AB, A^o, A_2, B^o\} + A^o = \{A_2, A^o, AB, B^o\}, \\ AB + (A^o + A^o) = AB + \{A_2, A^o\} = \{A_2, AB, A^o, B^o\}, \end{cases}$$

$$\begin{cases} (A_2 + B^o) + B_2 = \{AB, A^o, A_2, B^o\} + B_2 = \{B_2, AB, B^o, A^o, A_2\}, \\ A_2 + (B^o + B_2) = A_2 + \{B_2, B^o\} = \{A_2, A^o, AB, B^o, B_2\}. \quad \blacksquare \end{cases}$$

Corollary 8.1.2. *$\mathcal{H}_1 = \{A^0, A_2\}$ and $\mathcal{H}_2 = \{B^0, B_2\}$ are only H_v-subgroups of $(\mathcal{H}, +)$.*

If we consider $A = H$ and $B \in \{F, CL, Br, I\}$ (for example $B = I$), the complete reaction table becomes:

$+$	H^o	I^o	H_2	I_2	HI
H^o	H^o, H_2	H^o, I^o, HI	H^o, H_2	H^o, I_2, I^o, HI	H^o, HI, H_2, I^o
I^o	H^o, I^o, HI	I^o, I_2	H^o, I^o, HI, H_2	I^o, I_2	H^o, I^o, HI, I_2
H_2	H^o, H_2	H^o, I^o, HI, I_2	H^o, H_2	H^o, I^o, H_2, I_2, HI	H^o, I^o, H_2, HI
I_2	H^o, I^o, I_2, HI	H^o, I_2	H^o, I^o, H_2, I_2, HI	H^o, I_2	H^o, I^o, I_2, HI
HI	H^o, HI, H_2, I^o	H^o, I^o, HI, I_2	H^o, I^o, H_2, HI	H^o, I^o, H_2, HI	H^o, I^o, H_2, I_2, HI

8.2 e-hyperstructures and their applications

e- hyperstructures are a special kind of hyperstructures and, in what follows, we shall see that they can be interpreted as a generalization of two important concepts for physics: Isotopies and Genotopies. On the other hand, biological systems such as cells or organisms at large are open and irreversible because they grow. The representation of more complex systems, such as neural networks, requires more advances methods, such as hyperstructures. In this manner, e-hyperstructures can play a significant rôle for the representation of complex systems in physics and biology, such as nuclear fusion, the reproduction of cells or neural systems.

These applications were investigated by R.M. Santilli and T. Vougiouklis and we mention here some of their results and examples (see [116], [114]). Firstly, we shall define and analyze several types of e-hyperstructures.

Definition 8.2.1. A hypergroupoid $(H, \cdot)$ is called an *e-hypergroupoid* if H contains a scalar identity (also called unit) e, which means that for all $x \in H$, $x \cdot e = e \cdot x = x$.

In an e-hypergroupoid, an element x' is called *inverse* of a given element $x \in H$ if $e \in x \cdot x' = x' \cdot x$.

Clearly, if a hypergroupoid contains a scalar unit, then it is unique, while the inverses are not necessarily unique. In what follows, we use some examples which are obtained as follows: Take a set where an operation "$\cdot$"

is defined, then we "enlarge" the operation putting more elements in the products of some pairs. Thus a hyperoperation "$\circ$" can be obtained, for which we have $x \cdot y \in x \circ y$, $\forall x, y \in H$. Recall that the hyperstructures obtained in this way are H_b-structures (see Definition 6.1.5.).

Examples 8.2.2.

(1) Consider the usual multiplication on the subset $\{1, -1, i, -i\}$ of complex numbers. Then we can consider the hyperoperation $\circ$ defined in the following table:

$\circ$	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i,-i
i	i	-i	-1	1
-i	-i	i	1,i	-1,i

Notice that we enlarged the products $(-1)\cdot(-i)$, $(-i)\cdot i$ and $(-i)\cdot(-i)$ by setting $(-1)\circ(-i)=\{i,-i\}$, $(-i)\circ i=\{1,i\}$ and $(-i)\circ(-i)=\{-1,i\}$. We obtain an e-hypergroupoid, with the scalar unit 1. The inverses of the elements $-1, i, -i$ are $-1, -i, i$ respectively. Moreover, the above structure is an H_v-abelian group, which means that the hyperoperation $\circ$ is weak associative, weak commutative and the reproductive axiom holds.

(2) Consider the set $H=\{f_i \mid i\in\{1,2,3,4,5,6\}\}$ of real functions, defined from the real open interval $(0,1)$ to $(0,1)$, where $f_1(x) = x$, $f_2(x)=(1-x)^{-1}$, $f_3(x)=1-x^{-1}$, $f_4(x)=x^{-1}$, $f_5(x)=1-x$, $f_6(x)=x(1-x)^{-1}$. Let the multiplication on H be the usual composition of functions. We can obtain a hyperoperation $\circ$, given by the following table:

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	6
f_2	f_2	f_3	f_1	f_6, f_5	f_4, f_6	f_5, f_4
f_3	f_3	f_1	f_2	f_5, f_6	f_6, f_4	f_4, f_5
f_4	f_4	f_5	f_6	f_1	f_2	f_3
f_5	f_5	f_6	f_4	f_3, f_2	f_1	f_2
f_6	f_6	f_4	f_5	f_2, f_3	f_3, f_2	f_1

We obtain an e-hypergroupoid, with the scalar unit f_1. The inverses of the elements f_2, f_3, f_4, f_5, f_6 are f_3, f_2, f_4, f_5, f_6 respectively. Moreover, the above structure is an H_v-abelian group.

(3) Consider now the finite noncommutative quaternion group

Q={1,-1,i,-i,j,-j,k,-k},

for which the multiplication is given by the following table:

o	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	-k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

Denote $\bar{i} = \{i, -i\}, \bar{j} = \{j, -j\}$ and $\bar{k} = \{k, -k\}$. We can obtain a hyperoperation $\circ$, given by the following table:

o	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	-k
i	i	-i	-1	1	k	$\bar{k}$	$\bar{j}$	j
-i	-i	i	1	-1	$\bar{k}$	k	j	$\bar{j}$
j	j	-j	$\bar{k}$	k	-1	1	i	$\bar{i}$
-j	-j	j	k	$\bar{k}$	1	-1	$\bar{i}$	i
k	k	-k	j	$\bar{j}$	$\bar{i}$	i	-1	1
-k	-k	k	$\bar{j}$	j	i	$\bar{i}$	1	-1

We obtain an e-hypergroupoid, with the scalar unit 1. The inverses of the elements $-1, i, -i, j, -j, k, -k$ are $-1, -i, i, -j, j, -k, k$ respectively. Moreover, the above structure is an H_v-abelian group, too.

It is immediate the following basic result, that holds for all the above examples:

Theorem 8.2.3. *The weak associativity is valid for all H_b-structures with associative basic operations.*

We are interested now in another kind of an e-hyperstructure, which is the *e-hyperfield.*

Definition 8.2.4. A set F, endowed with an operation "+", which we call addition and a hyperoperation, called multiplication "$\cdot$", is said to be an *e-hyperfield* if the following axioms are valid:

(1) $(F, +)$ is an abelian group where 0 is the additive unit;

(2) the multiplication $\cdot$ is weak associative;

(3) the multiplication $\cdot$ is weak distributive with respect to $+$, i.e. $\forall x, y, z \in F,\ x(y+z) \cap (xy+xz) \neq \emptyset,\ (x+y)z \cap (xz+yz) \neq \emptyset$;

(4) 0 is an absorbing element, i.e. $\forall x \in F,\ 0 \cdot x = x \cdot 0 = 0$;

(5) there exists a multiplicative scalar unit 1, i.e. $\forall x \in F,\ 1 \cdot x = x \cdot 1 = x$;

(6) for every element $x \in F$ there exists an inverse x^{-1}, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

The elements of an e-hyperfield $(F, +, \cdot)$ are called *e-hypernumbers*.

Examples 8.2.5.

(1) Starting with the ring $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$, we can obtain a hyperring by enlarging the product $\bar{2} \circ \bar{2} = \{\bar{1}\}$ to $\bar{2} \circ \bar{2} = \{\bar{1}, \bar{2}\}$. In other words, we obtain the following table:

$\circ$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{1}, \bar{2}$

The above structure is an e-hyperfield.

(2) In the above example, only a hyperproduct is not a singleton. These hyperstructures, for which only a hyperproduct is not a singleton, are called *very thin* and they are useful to the theory of representations of H_v-groups by hypermatrices. Hence, a way to obtain a very thin hyperstructure is the following one: we take a classical structure and we choose two elements a, b, then we can enlarge the product $a \cdot b$. Therefore, in order to obtain a very thin e-hyperfield we can take a field and enlarge only one product of two, nonzero and non-unit elements. This simple change of the operation leads to enourmous changes to the algebraic hyperstructure, so it looks like a chain reaction in physics.

(3) Another large class of e-hyperfields can be obtained by using H_b-structures. For instance, we can take the field of real numbers $\mathbb{R}$, or the field of complex numbers $\mathbb{C}$ or the field of quaternions Q and then we can enlarge all products of nonzero and nonunit elements by adding nonzero elements and so we obtain e-hyperfields.

(4) We can use the above method starting from an e-hyperfield or a ring, not necessarily a field. For instance, we can take the ring $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ of integers modulo 6. We consider a hyperoperation $\circ$ given by the following table:

$\circ$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{4}$	$\bar{0}, \bar{1}$	$\bar{2}, \bar{3}$	$\bar{4}, \bar{5}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{0}, \bar{1}$	$\bar{3}, \bar{2}$	$\bar{0}, \bar{5}$	$\bar{3}, \bar{4}$
$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{2}, \bar{3}$	$\bar{0}, \bar{5}$	$\bar{4}, \bar{1}$	$\bar{2}$
$\bar{5}$	$\bar{0}$	$\bar{5}$	$\bar{4}, \bar{5}$	$\bar{3}, \bar{4}$	$\bar{2}$	$\bar{1}$

Then $(\mathbb{Z}_6, +, \circ)$ is an e-hyperfield, for which the multiplication is not closed in $\mathbb{Z}_6 - \{\bar{0}\}$.

For the following example, we recall a *P-hyperstructure* notion, given more details (see 6.4.).

Definition 8.2.6. Let $(G, \cdot)$ be a semigroup and $P \subseteq G$, $P \neq \emptyset$. The following hyperoperations are called *P-hyperoperations*:

$$\forall x, y \in G,\ xP^*y = xPy,\ xP_r^*y = xyP,\ xP_l^*y = Pxy.$$

If in a set they are defined P-hyperoperations, then we obtain P-hyperstructures.

The P-hyperoperation P^* is associative, so (G, P^*) is a semihypergroup. If $P \subseteq Z(G)$, where $Z(G)$ is the center of G, then the above three hyperoperations coincide. P-hyperoperations can be defined in groupoids or hypergroupoids as well, and so we obtain a large class of hyperstructures. We can also define P-hyperoperations in sets with partial operations. Moreover, in structures with more than one operations, we can define more P-hyperoperations. In a P-hypergroup the set of left or right units is P. The set of left inverses of x with respect to the unit p_0^{-1} is $p_0^{-1}x^{-1}P^{-1}$ and similarly the set of right inverses of x with respect to the unit p_0^{-1} is $P^{-1}x^{-1}p_0^{-1}$.

Example 8.2.7. Let F be a field and P be a set such that $1 \in P \subseteq F - \{0\}$. We define the following hypermultiplication:

$xP^*y = xPy,\ \forall x, y \in F - \{0, 1\}$ and $xP^*y = xy$ otherwise.

For instance, if we take the field $\mathbb{Z}_7$ of integers modulo 7 and the set $P = \{\bar{1}, \bar{3}\}$, then the hypermultiplication P^* is given by the following table:

P^*	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{4}, \bar{5}$	$\bar{6}, \bar{4}$	$\bar{1}, \bar{3}$	$\bar{3}, \bar{2}$	$\bar{5}, \bar{1}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{6}, \bar{4}$	$\bar{2}, \bar{6}$	$\bar{5}, \bar{1}$	$\bar{1}, \bar{3}$	$\bar{4}, \bar{5}$
$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{1}, \bar{3}$	$\bar{5}, \bar{1}$	$\bar{2}, \bar{6}$	$\bar{6}, \bar{4}$	$\bar{3}, \bar{2}$
$\bar{5}$	$\bar{0}$	$\bar{5}$	$\bar{3}, \bar{2}$	$\bar{1}, \bar{3}$	$\bar{6}, \bar{4}$	$\bar{4}, \bar{5}$	$\bar{2}, \bar{6}$
$\bar{6}$	$\bar{0}$	$\bar{6}$	$\bar{5}, \bar{1}$	$\bar{4}, \bar{5}$	$\bar{3}, \bar{2}$	$\bar{2}, \bar{6}$	$\bar{1}, \bar{3}$

Then $(\mathbb{Z}_7, +, P^*)$ is an e-hyperfield.

Definition 8.2.8. An *e-hypermatrix* is a matrix with entries elements of an e-hyperfield.

We can define the product of two e-matrices in an usual manner: the elements of product of two e-matrices $(a_{ij}), (b_{ij})$ are $c_{ij} = \sum a_{ik} \circ b_{kj}$, where the sum of products is the usual sum of sets.

If we consider the e-hyperfield given in Example 8.2.5(1), then we have:

$$\begin{bmatrix} \bar{2} & \bar{1} \\ \bar{2} & \bar{0} \end{bmatrix} \circ \begin{bmatrix} \bar{2} & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix} = \begin{bmatrix} \bar{2}\circ\bar{2}+\bar{1}\circ\bar{1} & \bar{2}\circ\bar{1}+\bar{1}\circ\bar{1} \\ \bar{2}\circ\bar{2}+\bar{0}\circ\bar{1} & \bar{2}\circ\bar{1}+\bar{0}\circ\bar{1} \end{bmatrix}$$

$$= \begin{bmatrix} \{\bar{1},\bar{2}\}+\bar{1} & \bar{2}+\bar{1} \\ \{\bar{1},\bar{2}\}+\bar{0} & \bar{2}+\bar{0} \end{bmatrix}$$

$$= \begin{bmatrix} \{\bar{2},\bar{0}\} & \bar{0} \\ \{\bar{1},\bar{2}\} & \bar{2} \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} \bar{2} & \bar{0} \\ \bar{1} & \bar{2} \end{bmatrix}, \begin{bmatrix} \bar{2} & \bar{0} \\ \bar{2} & \bar{2} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{2} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{2} & \bar{2} \end{bmatrix} \right\}$$

Moreover, notice that the product of an e-hypernumber with an e-hypermatrix is also a hyperoperation. For instance, again on the above hyperfield, we have

$$\bar{2}\circ \begin{bmatrix} \bar{2} & \bar{1} \\ \bar{2} & \bar{2} \end{bmatrix} = \begin{bmatrix} \bar{2}\circ\bar{2} & \bar{2}\circ\bar{1} \\ \bar{2}\circ\bar{2} & \bar{2}\circ\bar{2} \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} \bar{1} & \bar{2} \\ \bar{1} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{2} \\ \bar{1} & \bar{2} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{2} \\ \bar{2} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{2} \\ \bar{2} & \bar{2} \end{bmatrix}, \begin{bmatrix} \bar{2} & \bar{2} \\ \bar{1} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{2} & \bar{2} \\ \bar{1} & \bar{2} \end{bmatrix}, \begin{bmatrix} \bar{2} & \bar{2} \\ \bar{2} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{2} & \bar{2} \\ \bar{2} & \bar{2} \end{bmatrix} \right\}.$$

This remark is useful for the definition of an *e-hypervector space.*

Definition 8.2.9. Let $(F,+,\cdot)$ be an e-hyperfield. An ordered set $a = (a_1, a_2, ..., a_n)$ of n e-hypernumbers of F is called an *e-hypervector* and the e-hypernumbers a_i, $i \in \{1, 2, ..., n\}$ are called *components* of the e-hypervector a.

Two e-hypervectors are equals if they have equal corresponding components. The hypersums of two e-hypervectors a, b is defined as follows:

$$a + b = \{(c_1, c_2, ..., c_n) \mid c_i \in a_i + b_i,\ i \in \{1, 2, ..., n\}\}.$$

The scalar hypermultiplication of an e-hypervector a by an e-hypernumber λ is defined in a usual manner:

$$\lambda \circ a = \{(c_1, c_2, ..., c_n) \mid c_i \in \lambda \cdot a_i,\ i \in \{1, 2, ..., n\}\}.$$

The set F^n of all e-hypervectors with elements of F, endowed with the hypersum and the scalar hypermultiplication is called *n-dimensional e-hypervector space*. The set of $m \times n$ hypermatrices is an mn-dimensional e-hypervector space.

The next proposition can be easily verified.

Proposition 8.2.10. *Let F be an e-hyperfield and F^n be its n-dimensional e-hypervector space. Then the following assertions hold:*

(1) *the additive unit is the zero e-hypervector $0 = (0, 0, ..., 0)$;*
(2) $\lambda \circ (a + b) \cap (\lambda \circ a + \lambda \circ b) \neq \emptyset,\ \forall \lambda \in F, \forall a, b \in F^n$;
(3) $(\lambda + \alpha) \circ a \cap (\lambda \circ a + \alpha \circ a) \neq \emptyset,\ \forall \lambda, \alpha \in F, \forall a \in F^n$;
(4) $\lambda \circ (\alpha \circ a) \cap (\lambda \cdot \alpha) \circ a. \neq \emptyset,\ \forall \lambda, \alpha \in F, \forall a \in F^n$;
(5) $1 \circ a = a,\ \lambda \circ 0 = 0,\ \forall \lambda \in F, \forall a \in F^n$.

Notice that by $(\lambda + \alpha) \cdot a_i$ we intend $\bigcup\limits_{t \in \lambda + \alpha} t \cdot a_i$, while $\lambda \cdot a_i + \alpha \cdot a_i$ means $\bigcup\limits_{x \in \lambda \cdot a_i, y \in \alpha \cdot a_i} (x + y)$.

Definition 8.2.11. An *e-hyperalgebra* over an e-hyperfield $(F, +, \cdot)$ is an n-dimensional e-hypervector space F^n, endowed with a multiplication of e-hypervectors $\odot$, such that $(F^n, +, \odot)$ is an e-hyperring and for all $\lambda \in F$ and all $x, y \in F^n$, we have

$$\lambda \circ (x \odot y) = (\lambda \circ x) \odot y = x \odot (\lambda \circ y).$$

The most important example of an e-hyperalgebra is the algebra of $n \times n$ square e-hypermatrices.

As it is well know, Lie's theory is at the foundation of all physical theories, including classical and quantum mechanics, particle physics, nuclear physics, superconductivity, chemistry, astrophysics, etc. Despite the mathematical and physical consistency, by no means Lie's theory can represent the totality of systems existing in the universe. We conclude the presentation of e-hyperstructures with the definition of an e-hyper-Lie-algebra.

Definition 8.2.12. Let $(L,+)$ be an e-hypervector space over an e-hyperfield $(F,+,\cdot)$. Consider any bracket or commutator hyperoperation:

$$[\ ,\ ] : L\times L \to P(L):\ (x,y)\mapsto [x,y].$$

Then L is an *e-hyper-Lie-algebra* over F if the following axioms hold:

(1) the bracket hyperoperation is bilinear, i.e.

$$\begin{array}{l}\forall x, x_1, x_2, y, y_1, y_2 \in L, \forall \alpha_1, \alpha_2, \beta_1, \beta_2 \in F,\\ [\alpha_1 x_1 + \alpha_2 x_2, y] \cap (\alpha_1[x_1,y] + \alpha_2[x_2,y]) \neq \emptyset,\\ [x, \beta_1 y_1 + \beta_2 y_2] \cap (\beta_1[x,y_1] + \beta_2[x,y_2]) \neq \emptyset;\end{array}$$

(2) $\forall x \in L, 0 \in [x,x]$;

3) $\forall x,y,z \in L, 0 \in ([x,[y,z]] + [y,[z,x]] + [z,[x,y]])$.

The most important thing in studying e-hyper-Lie-algebras is to check if a subset is closed under the Lie bracket. This is so, because the product of hypermatrices normally has an enourmous number of elements. However, for some interesting subclasses it is easy to check if they are closed or not.

Examples 8.2.13.

(1) Consider the Lie bracket of the two traceless e-hypermatrices, over the e-hyperfield given in Example 8.2.5(1):

$$A = \begin{bmatrix} \bar{2} & \bar{2} \\ \bar{1} & \bar{1} \end{bmatrix},\quad B = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{2} & \bar{2} \end{bmatrix}.$$

We obtain

$$\begin{aligned}
[A,B] &= \begin{bmatrix} \bar{2} & \bar{2} \\ \bar{1} & \bar{1} \end{bmatrix}\cdot\begin{bmatrix} \bar{1} & \bar{0} \\ \bar{2} & \bar{2} \end{bmatrix} - \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{2} & \bar{2} \end{bmatrix}\cdot\begin{bmatrix} \bar{2} & \bar{2} \\ \bar{1} & \bar{1} \end{bmatrix}\\
&= \begin{bmatrix} \bar{2}+\{\bar{1},\bar{2}\} & \bar{0}+\{\bar{1},\bar{2}\} \\ \bar{2}+\bar{2} & \bar{0}+\bar{2} \end{bmatrix} - \begin{bmatrix} \bar{2}+\bar{0} & \bar{2}+\bar{0} \\ \{\bar{1},\bar{2}\}+\bar{2} & \{\bar{1},\bar{2}\}+\bar{2} \end{bmatrix}\\
&= \begin{bmatrix} \{\bar{0},\bar{1}\} & \{\bar{1},\bar{2}\} \\ \bar{1} & \bar{2} \end{bmatrix} - \begin{bmatrix} \bar{2} & \bar{2} \\ \{\bar{0},\bar{1}\} & \{\bar{0},\bar{1}\} \end{bmatrix}\\
&= \begin{bmatrix} \{\bar{1},\bar{2}\} & \{\bar{2},\bar{0}\} \\ \{\bar{1},\bar{0}\} & \{\bar{2},\bar{1}\} \end{bmatrix}.
\end{aligned}$$

We notice that the Lie bracket of A and B consists of 16 e-hypermatrices and some of them are not traceless. For example,

$$\begin{bmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{bmatrix}, \quad \begin{bmatrix} \bar{2} & \bar{2} \\ \bar{0} & \bar{2} \end{bmatrix}.$$

Hence the set of traceless e-hypermatrices is not closed.

(2) Now, consider the e-hyperfield based on $\mathbb{Z}_7$, where the multiplication is replaced by the P-hyperoperation given in Example 8.2.7. Take the following e-hypermatrices:

$$A = \begin{bmatrix} \bar{1} & \bar{2} \\ \bar{0} & \bar{3} \end{bmatrix}, \quad B = \begin{bmatrix} \bar{4} & \bar{5} \\ \bar{0} & \bar{6} \end{bmatrix}.$$

Then the Lie bracket is:

$$\begin{aligned} [A, B] &= \left[\begin{bmatrix} \bar{1} & \bar{2} \\ \bar{0} & \bar{3} \end{bmatrix}, \begin{bmatrix} \bar{4} & \bar{5} \\ \bar{0} & \bar{6} \end{bmatrix} \right] \\ &= \begin{bmatrix} \bar{1} \cdot \bar{4} + \bar{2} \cdot \bar{0} & \bar{1} \cdot 5 + \bar{2} \cdot \bar{6} \\ \bar{0} \cdot \bar{4} + \bar{3} \cdot \bar{0} & \bar{0} \cdot \bar{5} + \bar{3} \cdot \bar{6} \end{bmatrix} - \begin{bmatrix} \bar{4} \cdot \bar{1} + \bar{5} \cdot \bar{0} & \bar{4} \cdot \bar{2} + \bar{5} \cdot \bar{3} \\ \bar{0} \cdot \bar{1} + \bar{6} \cdot \bar{0}\} & \bar{0} \cdot \bar{2} + \bar{6} \cdot \bar{3} \end{bmatrix} \\ &= \begin{bmatrix} \{\bar{4} & \{\bar{3}, \bar{6}\} \\ \bar{0} & \{\bar{4}, \bar{5}\} \end{bmatrix} - \begin{bmatrix} \bar{4} & \{\bar{2}, \bar{4}, \bar{6}\} \\ \bar{0} & \{\bar{4}, \bar{5}\} \end{bmatrix} \\ &= \begin{bmatrix} \bar{0} & \{\bar{1}, \bar{6}, \bar{4}, \bar{2}, \bar{0}\} \\ \bar{0} & \{\bar{0}, \bar{6}, \bar{1}\} \end{bmatrix}. \end{aligned}$$

The Lie bracket of A and B has 15 elements among them 5 are strictly upper triangular e-hypermatrices. Therefore the set of upper triangular e-hypermatrices is closed under the Lie bracket hyperoperation.

Now, we connect the above e-hyperstructures to isotopies and genotopies. We give now an idea about these topics, which were constructed for physical needs.

Isotopies can be traced back to the early stages of set theory, where two Latin squares were said to be *isotopically related* when they can be made to coincide via permutations. Since Latin squares can be interpreted as the multiplication table of quasigroups, the isotopies propagated to quasigroups and then to Jordan algebras. Santilli used the term *isotopy* from its Greak meaning of *preserving the topology* and interpreted them as axiom-preserving. In fact, the new and old structures are indistinguishable at the abstract. Nowadays, the term "isotopies" denotes nonlinear, nonlocal and nonhamiltonian liftings of any given linear, local and Hamiltonian structure, which preserves linearity, locality and canonicity in generalized spaces over generalized fields.

The main novelty of the isotopies studied by Santilli with respect to the preceding ones is the lifting of the trivial n-dimensional unit $I = diag(1, 1, ..., 1)$ of a conventional theory into a nowhere singular, symmetric, real-valued, positive-defined and n-dimensional matrix:

$$\hat{I} = (\hat{I}_{i,j}) = (\hat{I}_{j,i}) = \hat{I}^{-1} = (\hat{I}_{i,j})^{-1} = (\hat{I}_{j,i})^{-1},\ i, j \in \{1, 2, ..., n\},$$

whose elements have a smooth but otherwise arbitrary functional dependence on the local coordinates x, their derivates $\dot{x}$, $\ddot{x}$, ... with respect to an independent variable t and any needed additional local quantity,

$$\hat{I} \to \hat{I}(x,\ \dot{x},\ \ddot{x},\ ...).$$

The original theory is reconstructed in such a way to admit $\hat{I}$ as the new left and right unit. Thus, if $(F, +, \cdot)$ is a field of characteristic zero, then we can construct an *isofield* $\hat{F} = (\hat{F}, +, \circ)$, whose elements have the form $\hat{a} = a \cdot \hat{1}$, where $a \in F$ and $\hat{1}$ is a positive-defined element generally outside F. The new multiplication $\circ$, called *isomultiplication* is defined as follows:

$$\forall \hat{a}, \hat{b},\ \hat{a} \circ \hat{b} = \hat{a} \cdot \hat{1} \cdot \hat{b}.$$

The element $\hat{1} = \hat{1}^{-1}$ is the left and right unit of $\hat{F}$. The structure $(\hat{F}, +, \circ)$ is a new field and it is called an *isotope* of F, while the lifting $F \to \hat{F}$ is called an *isotopy*. For instance, we obtain the isofields $(\mathbb{R}, +, \circ)$ of *isoreal numbers*, $(\mathbb{C}, +, \circ)$ of *isocomplex numbers*, $(Q, +, \circ)$ of *isoquaternions*. Notice that $\hat{1}^{\hat{n}} = \underbrace{\hat{1} \circ ... \circ \hat{1}}_{n} = \hat{1},\ \hat{1}/\hat{1} = \hat{1}$.

Genotopies were introduced by Santilli from the Greak meaning of *inducing topology* and interpreted them as liftings of a given theory verifying certain axioms into a form which verifies broader axioms admitting the original ones as particular cases. The main difference between isotopies and the genotopies is that the isomultiplication of two isonumbers $\hat{a}, \hat{b}$ has no ordering, while for the genotopies one must assume an ordering.

The multiplication of two quantities is ordered on the right and it is denoted by the symbol $>$, when the first quantity multiplies the second one on the right, while it is ordered on the left and denoted by the symbol $<$, when the second quantity multiplies the first one on the left.

The genotopies are based on the property that the restriction of the multiplication on the right in an ordered field permits the preservation of all axioms of a field. We obtain two fields $(F^>, +, >)$ and $(^<F, +, <)$, based on the multiplication on the right and on the left respectively. The genotopies emerge when the multiplication on the right is assumed to be different from that on the left. Hence we have two different generalized units, one for the multiplication on the right $\hat{I}^>$ and one for the multiplication on the left $^<\hat{I}$. For isotopies, we have the same isounit $\hat{I}$ for both isomultiplications. Isotopies are a particular case of genofields.

If $(F, +, \cdot)$ is a field of characteristic zero, then we can construct an *genofield on the right* $\hat{F}^> = (\hat{F}^>, +, \hat{>})$, whose elements have the form $\hat{a}^> = \hat{a} \cdot \hat{1}^>$ and are called *genonumbers on the right*, where $a \in F$ and $\hat{1}^>$ is a quantity generally outside F and $\hat{F}$. The new multiplication $\hat{>}$, called *genomultiplication on the right* is defined as follows:

$$\forall \hat{a}, \hat{b},\ \hat{a} \hat{>} \hat{b} = \hat{a} \cdot \hat{Q} \cdot \hat{b}.$$

The element $\hat{Q}^{-1}$ is the left and right unit $\hat{1}^>$ of $\hat{F}^>$. In other words, for all $\hat{a}^> \in \hat{F}^>$, $\hat{1}^> \hat{>} \hat{a}^> = \hat{a}^> = \hat{a}^> \hat{>} \hat{1}^>$.

A *genofield on the left* $^<\hat{F} = (^<\hat{F}, +, \hat{<})$ is the image of $\hat{F}^> = (\hat{F}^>, +, \hat{>})$, under the replacement of the genomultiplication on the right $\hat{>}$ with the genomultiplication on the left:

$$^<\hat{a} \hat{<} {}^<\hat{b} = {}^< \hat{a} \cdot \hat{P} \cdot {}^< \hat{b},$$

with the corresponding *genounit on the left* $^<\hat{I} = \hat{P}^{-1}$, i.e. for all $^<\hat{a} \in {}^< \hat{F}$, we have $^<\hat{1} \hat{<} {}^<\hat{a} = {}^< \hat{a} = {}^< \hat{a} \hat{<} {}^<\hat{1}$.

For $P = Q$ we obtain isotopies.

The unit $\hat{I}$ of the isotopies and the units ${}^{<}\hat{I}$, $\hat{I}^{>}$ of the genotopies have a realization, e.g. via a given function or matrix. A first class of e-hyperstructures can be introduced as hyperstructures with hypermultiplication, for which e verifies the weak condition to be a left and a right unit and this class leads us to isotopies, while a second class of e-hyperstructures requires the further differentiation of the hypermultiplication on the right from that on the left and this leads us to genotopies. Hence isotopies and genotopies represent particular cases of e-hyperstructures. As a remark, we considered multiplicative hyperfields in order to define an e-hyperfield, since the sum is not lifted in isotopies.

8.3 Transposition hypergroups of Fredholm integral operators

Another motivation for the study of hyperstructures comes from physical phenomenon as the nuclear fission. This motivation and the results of this paragraph were presented by S. Hošková, J. Chvalina and P. Račková (see [56], [57]).

Nuclear fission occurs when a heavy nucleus, such as U^{235}, splits or fissions into two smaller nuclei. By this fission process we can get several dozens of different combinations of two medium-mass elements and several neutrons (as barium Ba^{141} and krypton Kr^{92} and 3 neutrons; strontium Sr^{94}, xenon Xe^{140} and 2 neutrons; lanthanum La^{147}, bromum Br^{87} and 2 neutrons; Sn^{132}, Mo^{101} and 3 neutrons and so on). The input of this reaction is always the same, but the result is in general different. The heavy uranium is bombarded with neutrons and there are about 90 different daughter nuclei that can be formed. Moreover, two or three neutrons are released per event. In any fission equation, there are many combinations of fission fragments, but they satisfy always the requirements of conservation of energy and charge. Similar situations appear during several nuclear fissions. The result depends on conditions. Although the input two particles are the same, the output can be variant. It can differ both in the number of arising particles and in their kind.

An example when the interaction result between two particles is the whole set of particles is the interaction between a photon with a certain energy and an electron. The result of this interaction is not deterministic. A photo-electric effect or Coulomb repulsion effect or changeover of photon onto a pair electron and positron can arise.

Moreover, some ideas leading naturally to hyperstructures come also from quantum mechanics, quantum optics, quantum cryptography or development of quantum computers. In fact, quantum object can be simulated in more different states simultaneously. States of quantum object possess the property that they are not spacely localized. Quantum particles are situated in many places at the same time and they are coming through several trajectories simultaneously. Quantum computers give a technology which is of the great interest worldwide. These computers should be able to provide extremely quick computation thanks to their possibility to be localized in more states together. In spite of the fact this technology is developed on the basis of single-photon sources, it seems to be natural that the theory of suitable modified hyperstructures can serve as mathematical background in the field of quantum communication systems.

According to the well known physicist John Archibald Wheeler, one of the coauthors of the *hypothesis of more worlds*, the reality is decomposed in more parallel branches, by the collapse of wave function.

Now, we present shortly some basic ideas concerning a scalable quantum computer chip. A scalable quantum computer chip for atomic qubits was built for the first time at the University of Michigan, offering hopes for making a practical quantum computer using conventional semiconductor manufacturing technology. Exploiting the strange rules of the atomic world, quantum computers could potentially break top-secret codes and perform certain kinds of searches more quickly than the conventional computers can. The building blocks of quantum computers are called *qubits* or *quantum bits* made of atoms or photons. Multiple qubits are connected via an electrostatic or other suitable interaction in a quantum computer, similar to how a traditional computer is made by wiring together individual transistors. Unlike a conventional computer's bits, which can have values of either 0 or 1, a qubit can possess value of 0 and 1 simultaneously. The Michigan group of researches chose an individual cadmium ion for their qubit, held in free space by a number of electrodes inside a postage-stamp-sized gallium arsenide semiconductor chip. Additional electric fields are able to manipulate the position of the ion and laser beams could control the qubit value in the ion. Ions pose an advantage over other potential qubits, such as photons and electron dots. They are easier to isolate and shield from external disturbances that can disrupt their operation. An integrated semiconductor chip is a markedly different environment for ion qubits, which were previously held in hand-mad ion traps, that could not be easily scaled up or mass produced. Making a quantum computer would require scaling up a single chip, so that it contains enough electrodes to trap many ions simultaneously.

The description and solution of many problems from technical sciences and from various fields of applied mathematics use some results and methods from the theory of linear integral equations. We consider integral operators that correspond to Fredholm integral equations of the second kind and of the first kind, as well. Fredholm integral equations can be considered as a modification of linear equation systems, thus this topic has algebraic roots.

In what follows, we describe a certain construction of transposition hypergroups of integral operators on spaces of continuous functions. Transposition hypergroups were introduced and investigated firstly by J. Jantosciak [62], [61]. Many well-known hypergroups forming wide classes as join spaces, weak cogroups, double coset spaces, polygroups and canonical hypergroups, including ordinary groups and also some geometrically motivated noncommutative hyperstructures are all transposition hypergroups.

Using operators which correspond to Fredholm equations we will construct ordered groups determining transposition hypergroups. Moreover, using certain subhypergroups of these transposition groups we obtain hyperstructures of operator blocks. In particular, by the decomposition of the group of all Fredholm integral operators of the second kind by its subgroup of operators of the first kind we obtain a quasihypergroup of operator blocks.

Definition 8.3.1. An integral equation of the form

$$\varphi(x) - \lambda \int_a^b K(x,s)\varphi(s)ds = f(x)$$

is called a *Fredholm integral equation of the second kind*, whereas the integral equation of the form

$$\int_a^b K(x,s)\varphi(s)ds = f(x)$$

is called a *Fredholm integral equation of the first kind*, where $x \in (a,b)$; K is a real or complex valued function, called *kernel*; f is a function called a *free* or an *absolute member*; λ is a numerical parameter and φ is an unknown function.

It is known that under some conditions, the solution of a Fredholm integral equation can be expressed in the form of a sum of Neumann's series. Usually, Fredholm integral equations with a nondegenerate Lebesgue square integrable kernel $K(x,s)$ are considered, i.e. Lebesgue integral $\int\int_M |K(x,s)|^2 dxds$ is convergent, where $M = (a,b) \times (a,b) \subseteq R \times R$.

We will construct transposition hypergroups on the set of operators of the type $F(\lambda, K, f)$, with continuous positive functions f, K (absolute member and kernel) and a nonzero parameter λ.

So, we consider operators $F(\lambda, K, f) : C(a,b) \to C(a,b)$

$$F(\lambda, K, f)(\varphi(x)) = \lambda \int_a^b K(x,s)\varphi(s)ds + f(x), \tag{I}$$

where $C(a,b)$ means the set of continuous functions on $(a,b) \subseteq R$.

The mentioned operator occurs in the construction of a series of functions which approximate the solution of the Fredholm equation of the second kind.

Definition 8.3.2. A hypergroup $(H, \cdot)$ is called a *transposition hypergroup* or a *noncommutative join space* if it satisfies the transposition axiom: for all $a, b, c, d \in H$ the condition $b \setminus a \cap c/d \neq \emptyset$ implies $a \cdot d \cap b \cdot c \neq \emptyset$, where sets $b \setminus a = \{x \in H \mid a \in b \cdot x\}$, $c/d = \{x \in H \mid c \in x \cdot d\}$ are called *left* and *right extensions* respectively.

Definition 8.3.3. An *quasi-ordered semigroup* is a triple $(G, \cdot, \leq)$, where $(G, \cdot)$ is a semigroup and the binary relation $\leq$ is a quasiordering (reflexive and transitive) on the set G, such that for all $x, y, z \in G$, the condition $x \le y$ implies $x \cdot z \le y \cdot z,\ z \cdot x \le z \cdot y$.

An *ordered (semi)group* is a triple $(G, \cdot, \le)$, where $(G, \cdot)$ is a (semi) group and $\le$ is a reflexive, antisymmetrical and transitive binary relation on the set G, such that for any $x, y, z \in G$, the condition $x \le y$ implies $x \cdot z \le y \cdot z,\ z \cdot x \le z \cdot y$. Further, $[a)_\le = \{x \in G \mid a \le x\}$ is called a *principal end* generated by $a \in G$.

Now, we obtain the following important lemma:

Lemma 8.3.4. *Let $(G, \cdot, \le)$ be a quasi-ordered semigroup. We define the following hyperoperation on G:*

$* : G \times G \longrightarrow P^*(G)$ by $a * b = [a \cdot b)_\le = \{x \in G \mid a \cdot b \le x\}$ for all $a, b \in G$.

(1) *Then $(G, *)$ is a semihypergroup which is commutative if the semigroup $(G, \cdot)$ is commutative.*

(2) *If $(G, *)$ is the above defined semihypergroup, then $(G, *)$ is a hypergroup if and only if for any elements $a, b \in G$ there exist $c, c' \in G$ such that $a \cdot c \le b,\ c' \cdot a \le b$.*

Proof.

(1) We have to verify the associativity law. If $a, b, c, x, y \in G$ and $x \le y$, then $a \cdot x \le a \cdot y$ and $x \cdot a \le y \cdot a$ whence $[a \cdot y)_\le \subseteq [a \cdot x)_\le$ and $[y \cdot a)_\le \subseteq [x \cdot a)_\le$. We have

$$\begin{aligned} a * (b * c) &= \bigcup_{x \in b * c} a * x = \bigcup_{x \in [b \cdot c)_\le} [a \cdot x)_\le \\ &= [a \cdot b \cdot c)_\le \cup \bigcup_{x > b \cdot c} [a \cdot x)_\le \\ &= [a \cdot b \cdot c)_\le = [a \cdot b \cdot c)_\le \cup \bigcup_{y > a \cdot b} [y \cdot c)_\le \\ &= \bigcup_{y \in [a \cdot b)_\le} [y \cdot c)_\le = \bigcup_{y \in a * b} y * c \\ &= (a * b) * c, \end{aligned}$$

since $[a\cdot x)_\leq \subseteq [a\cdot b\cdot c)_\leq$ for all x such that $x \geq b\cdot c$ and $[y\cdot c)_\leq \subseteq [a\cdot b\cdot c)_\leq$ for all y such that $y \geq a\cdot b$. Hence, $(G, *)$ is a semihypergroup. Clearly, if the semigroup $(G, \cdot)$ is commutative, then the semihypergroup $(G, *)$ is also commutative.

(2) If $t \in G$, then the set inclusions $t * G \subseteq G$, $G * t \subseteq G$ hold. We must prove the opposite inclusions. For any $s, t \in G$ there exist $c, c' \in G$ such that $t \cdot c \leq s$, $c' \cdot t \leq s$. From this, we obtain

$$s \in [t\cdot c)_\leq \cap [c' \cdot t)_\leq = (t * c) \cap (c' * t) \subseteq \left(\bigcup_{x\in G} t * x\right) \cap \left(\bigcup_{x\in G} x * t\right)$$
$$= (t * G) \cap (G * t),$$

whence $G \subseteq t * G$, $G \subseteq G * t$. So, the reproduction axiom is fulfilled. Now, if $(G, *)$ is a hypergroup and $a, b \in G$ are arbitrary elements, then $b * G = G = G * b$, whence

$$a \in b * G = \bigcup_{t\in G} b * t = \bigcup_{t\in G} [b \cdot t)_\leq,$$

which means that $a \in [b \cdot c)_\leq$ for an appropriate element $c \in G$, i.e. $b \cdot c \leq a$. Similarly, $a \in G * b$ which implies that $c' \cdot b \leq a$ for an appropriate element $c' \in G$. ■

Remark 8.3.5. If the binary relation $\leq$ is an ordering, then the commutativity of $(G, *)$ implies the commutativity of $(G, \cdot)$.

Corollary 8.3.6. *Let $(G, \cdot, \leq)$ be an ordered group. Define a hyperoperation $*$ on G, as follows: $* : G \times G \to P^*(G)$ by $a * b = [a \cdot b)_\leq = \{x \in G \mid a \cdot b \leq x\}$ for all elements $a, b \in G$. Then $(G, *)$ is a hypergroup which is commutative if and only if the group $(G, \cdot)$ is commutative.*

In what follows, we shall construct join spaces of operators, based on ordered groups. Let us denote the sets of continuous functions on J, $J \times J$, by $C(J)$, $C(J \times J)$ respectively, where $J \subseteq R$ is an interval and $f(x) \neq 0$ for all $x \in J$.

Proposition 8.3.7. *Let $J = (a, b)$. We consider the set $\mathcal{F} = \{F(\lambda, K, f) \mid K \in C(J \times J),\ f \in C_+(J),\ \lambda \neq 0\}$, where $F(\lambda, K, f)$ is given by (I). For any operators $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2)$ of $\mathcal{F}$, we define*

$$F(\lambda_1, K_1, f_1) \cdot F(\lambda_2, K_2, f_2) = F(\lambda_1\lambda_2, K_2 f_1 + K_1, f_1 f_2)$$

and $F(\lambda_1, K_1, f_1) \leq F(\lambda_2, K_2, f_2)$ *if and only if* $\lambda_1 = \lambda_2,\ f_1(x) = f_2(x)$ *and* $K_1(x,s) \leq K_2(x,s)$, *for any* $(x,s) \in J \times J$. *Then* $(\mathcal{F}, \cdot, \leq)$ *is a noncommutative ordered group.*

Proof. Let $F(\lambda_i, K_i, f_i) \in \mathcal{F}$ for all $i = 1,2,3$. Then

$$\begin{aligned}
&F(\lambda_1, K_1, f_1) \cdot (F(\lambda_2, K_2, f_2) \cdot F(\lambda_3, K_3, f_3))\\
&= F(\lambda_1, K_1, f_1) \cdot F(\lambda_2\lambda_3, K_3 f_2 + K_2, f_2 f_3)\\
&= F(\lambda_1\lambda_2\lambda_3, K_3 f_2 f_1 + K_2 f_1 + K_1, f_1 f_2 f_3)\\
&= F(\lambda_1\lambda_2, K_2 f_1 + K_1, f_1 f_2) \cdot F(\lambda_3, K_3, f_3)\\
&= (F(\lambda_1, K_1, f_1) \cdot F(\lambda_2, K_2, f_2)) \cdot F(\lambda_3, K_3, f_3),
\end{aligned}$$

which means that the binary operation $\cdot$ is associative. Moreover, for any operator $F(\lambda, K, f) \in \mathcal{F}$ we have

$$F(\lambda, K, f) \cdot F(1,0,1) = F(\lambda, K, f) = F(1,0,1) \cdot F(\lambda, K, f),$$

thus the operator $F(1,0,1)$ is the unit of the semigroup $(\mathcal{F}, \cdot)$.

On the other hand, for any operator $F(\lambda, K, f) \in \mathcal{F}$ we have $\lambda \neq 0$ and $f(x) > 0$ for all $x \in J \subseteq R$. Then the operator $F(1/\lambda, -K/f, 1/f)$ is well defined and belongs to $\mathcal{F}$. Moreover,

$$F(1/\lambda, -K/f, 1/f) \cdot F(\lambda, K, f) = F(1,0,1) = F(\lambda, K, f) \cdot F(1/\lambda, -K/f, 1/f),$$

which means that $F(1/\lambda, -K/f, 1/f)$ is the inverse of $F(\lambda, K, f)$. Hence $(\mathcal{F}, \cdot)$ is a group. Clearly, the binary operation $\cdot$ is noncommutative on $\mathcal{F}$.

From the definition of the relation $\leq$ it follows that this relation is reflexive, antisymmetrical and transitive on $\mathcal{F}$, hence the pair $(\mathcal{F}, \leq)$ is an ordered set.

It remains to verify the compatibility of the ordering $\leq$ with the binary operation $\cdot$ on $\mathcal{F}$. Let $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2) \in \mathcal{F}$ be integral operators, such that $F(\lambda_1, K_1, f_1) \leq F(\lambda_2, K_2, f_2)$ and $F(\lambda, K, f) \in \mathcal{F}$ be an arbitrary operator. Then $0 < f_1(x) = f_2(x),\ \lambda_1 = \lambda_2 \neq 0,\ K_1(x,s) \leq K_2(x,s)$ for any $(x,s) \in (a,b) \times (a,b)$, which implies that

$$\begin{aligned}&\lambda\lambda_1 = \lambda\lambda_2,\\ &f(x)f_1(x) = f(x)f_2(x),\\ &K_1(x,s)f(x) + K(x,s) \le K_2(x,s)f(x) + K(x,s),\\ &K(x,s)f_1(x) + K_1(x,s) \le K(x,s)f_2(x) + K_2(x,s)\end{aligned}$$

for each $(x, s) \in J \times J$. Hence

$$\begin{aligned}F(\lambda, K, f) \cdot F(\lambda_1, K_1, f_1) &= F(\lambda\lambda_1, K_1 f + K, f f_1)\\ &\le F(\lambda\lambda_2, K_2 f + K, f f_2)\\ &= F(\lambda, K, f) \cdot F(\lambda_2, K_2, f_2)\end{aligned}$$

and

$$\begin{aligned}F(\lambda_1, K_1, f_1) \cdot (F(\lambda, K, f) &= F(\lambda_1\lambda, K f_1 + K_1, f_1 f)\\ &\le F(\lambda_2\lambda, K f_2 + K_2, f_2 f)\\ &= F(\lambda_2, K_2, f_2) \cdot F(\lambda, K, f).\end{aligned}$$

Consequently, $(\mathcal{F}, \cdot, \le)$ is a noncommutative ordered group. ■

We apply the construction of a hypergroup from Lemma 8.3.4. onto this considered concrete case of integral operators, as follows. We define the following hyperoperation on $\mathcal{F}$: for all $F(\lambda_1, K_1, f_1),\ F(\lambda_2, K_2, f_2) \in \mathcal{F}$,

$$\begin{aligned}&F(\lambda_1, K_1, f_1) * F(\lambda_2, K_2, f_2)\\ &= \{F(\lambda, K, f) \in \mathcal{F} \mid F(\lambda_1, K_1, f_1) \cdot F(\lambda_2, K_2, f_2) \le F(\lambda, K, f)\}\\ &= \{F(\lambda, K, f) \in \mathcal{F} \mid F(\lambda_1\lambda_2, K_2 f_1 + K_1, f_1 f_2) \le F(\lambda, K, f)\}\\ &= \{F(\lambda_1\lambda_2, K, f_1 f_2) \mid K_2(x,s)f_1(x) + K_1(x,s) \le K(x,s),\ (x,s) \in J \times J\}.\end{aligned}$$

By Proposition 8.3.7 and Lemma 8.3.4 we obtain:

Proposition 8.3.8. *Let $J = (a, b) \subseteq R$ and $* : \mathcal{F} \times \mathcal{F} \to P^*(\mathcal{F})$ be the above defined binary hyperoperation. Then the hypergroupoid $(\mathcal{F}, *)$ is a noncommutative hypergroup satisfying the transposition axiom, so it is a transposition hypergroup.*

Now, we use the following construction for groups of integral operators.

Lemma 8.3.9. *Let $\mathcal{R}$ be an equivalence relation on an arbitrary set S. For all $x, s \in S$ we define a hyperoperation $* : S \times S \to P^*(S)$ as follows: $x * y = \bar{x} \cup \bar{y}$, where $\bar{x}, \bar{y}$ are the equivalence classes of x, y respectively. Then $(S, *)$ is a join space.*

Proof. Notice that $(S,*)$ is a commutative hypergroup. Indeed, if $x,y,z\in S$, then

$$\begin{aligned} x*(y*z) &= x*(\bar{y}\cup\bar{z}) = (x*\bar{y})\cup(x*\bar{z}) \\ &= \left(\bigcup_{t\in\bar{y}} x*t\right)\cup\left(\bigcup_{u\in\bar{z}} x*u\right) = (\bar{x}\cup\bar{y})\cup(\bar{x}\cup\bar{z}) \\ &= \bar{x}\cup\bar{y}\cup\bar{z} = \bar{z}\cup\bar{x}\cup\bar{y} = z*(x*y) = (x*y)*z. \end{aligned}$$

Notice that the reproduction axiom holds. Indeed,

$$x*S = S*x = \bigcup_{t\in S} t*x = \bigcup_{t\in S} \bar{x}*\bar{t} = S.$$

For $u,v\in S$ we have $u/v = \{x\in S \mid u\in x*v\} = \bar{u}$, if u and v are not in the relation R, otherwise $u/v = S$.

It remains to check the transposition axiom, i.e. if $u/v\cap z/t\neq\emptyset$, then $u*t\cap v*z\neq\emptyset$, which means that $(\bar{u}\cup\bar{t})\cap(\bar{v}\cup\bar{z})\neq\emptyset$. Suppose that $u/v=\bar{u}$ and $z/t=\bar{z}$. We have $\bar{u}\cap\bar{z}\neq\emptyset$, thus $\bar{u}=\bar{z}$. Hence $u*t\cap v*z\neq\emptyset$. Now, suppose that $u/v=\bar{u}$ and $z/t=S$. We have $\bar{z}=\bar{t}$. Hence $u*t\cap v*z\neq\emptyset$. The remaining cases can be verified in a similar way. ■

In what follows, we obtain two groups of Fredholm integral operators. The first group is the following one: Let $\mathcal{F}_1$, $\mathcal{F}_2$ be subsets of $\mathcal{F}$ formed by integral operators of the form $F(1,K,f)$, $F(\lambda,K,1)$ respectively.

Lemma 8.3.10. $(\mathcal{F}_1,\cdot)$, $(\mathcal{F}_2,\cdot)$ *are normal subgroups of the group* $(\mathcal{F},\cdot)$.

Proof. Let $F(1,K,f)$, $F(1,K_1,f_1)\in\mathcal{F}_1$ be arbitrary operators. Then

$$\begin{aligned} F(1,K,f)\cdot F^{-1}(1,K_1,f_1) &= F(1,K,f)\cdot F(1,-K_1/f_1,1/f_1) \\ &= F(1,K-K_1f/f_1,f/f_1)\in\mathcal{F}_1, \end{aligned}$$

whence $(\mathcal{F}_1,\cdot)$ is a subgroup of the group $(\mathcal{F},\cdot)$.

Similarly, for $F(\lambda,K,1)$, $F(\bar{\lambda},\bar{K},1)\in\mathcal{F}_2$, we have

$$\begin{aligned} F(\lambda,K,1)\cdot F^{-1}(\bar{\lambda},\bar{K},1) &= F(\lambda,K,1)\cdot F(1/\bar{\lambda},-\bar{K},1) \\ &= F(\lambda/\bar{\lambda},K-\bar{K},1)\in\mathcal{F}_2. \end{aligned}$$

So, $(\mathcal{F}_2,\cdot)$ is a subgroup of the group $(\mathcal{F},\cdot)$, too.

Now, let $F(\lambda, K, f) \in \mathcal{F}$, $F(1, K_1, f_1) \in \mathcal{F}_1$. Then

$$\begin{aligned}&F(\lambda, K, f) \cdot F(1, K_1, f_1) \cdot F^{-1}(\lambda, K, f)\\&= F(\lambda, K_1 f + K, f f_1) \cdot F(1/\lambda, -K/f, 1/f)\\&= F(1, K_1 f - K(f_1 - 1), f_1) \in \mathcal{F}_1.\end{aligned}$$

Therefore, the subgroup $(\mathcal{F}_1, \cdot)$ is a normal subgroup of the group $(\mathcal{F}, \cdot)$.

Analogously, if $F(\lambda, K, f) \in \mathcal{F}$, $F(\bar{\lambda}, \bar{K}, 1) \in \mathcal{F}_2$, then

$$\begin{aligned}&F(\lambda, K, f) \cdot F(\bar{\lambda}, \bar{K}, 1) \cdot F^{-1}(\lambda, K, f)\\&= F(\lambda\bar{\lambda}, \bar{K} f + K, f) \cdot F(1/\lambda, -K/f, 1/f)\\&= F(\bar{\lambda}, \bar{K} f, 1) \in \mathcal{F}_2.\end{aligned}$$

Hence $(\mathcal{F}_2, \cdot)$ is a normal subgroup of the group $(\mathcal{F}, \cdot)$, too. ■

The second group of Fredholm integral operators is obtained as follows: We consider the set $\mathcal{G} = \{F(\lambda, K, f) \mid \lambda \neq 0,\ K \neq 0\}$ for arbitrary $(x, s) \in J \times J$, $K \in C(J \times J)$, $f \in C(J)$. We define

$$F(\lambda_1, K_1, f_1) \odot F(\lambda_2, K_2, f_2) = F(\lambda_1\lambda_2, K_1K_2, \lambda_1 \hat{K}_1 f_2 + f_1),$$

where $\hat{K}(x, s) = K(x, x)$.

Lemma 8.3.11. *$(\mathcal{G}, \odot)$ is a noncommutative group.*

Proof. Firstly, we check the associativity law. Let $F(\lambda_i, K_i, f_i) \in \mathcal{G}$, for $i = 1, 2, 3$. Then

$$\begin{aligned}&(F(\lambda_1, K_1, f_1) \odot F(\lambda_2, K_2, f_2)) \odot F(\lambda_3, K_3, f_3)\\&= F(\lambda_1\lambda_2, K_1K_2, \lambda_1\hat{K}_1 f_2 + f_1) \odot F(\lambda_3, K_3, f_3)\\&= F(\lambda_1\lambda_2\lambda_3, K_1K_2K_3, \lambda_1\lambda_2\hat{K}_1\hat{K}_2 f_3 + \lambda_1\hat{K}_1 f_2 + f_1)\\&= F(\lambda_1\lambda_2\lambda_3, K_1K_2K_3, \lambda_1\hat{K}_1(\lambda_2\hat{K}_2 f_3 + f_2) + f_1)\\&= F(\lambda_1, K_1, f_1) \odot (F(\lambda_2\lambda_3, K_2K_3, \lambda_2\hat{K}_2 f_3 + f_2))\\&= F(\lambda_1, K_1, f_1) \odot (F(\lambda_2, K_2, f_2) \odot F(\lambda_3, K_3, f_3)).\end{aligned}$$

Moreover, for any operator $F(\lambda, K, f) \in \mathcal{G}$ we have

$$F(\lambda, K, f) \odot F(1, 1, 0) = F(\lambda, K, f) = F(1, 1, 0) \odot F(\lambda, K, f).$$

Finally, if $F(\lambda, K, f) \in \mathcal{G}$ is an arbitrary operator, then its inverse is the operator $F^{-1}(\lambda, K, f) = F(\lambda^{-1}, 1/K, -\lambda^{-1} f/\hat{K})$. Therefore, $(\mathcal{G}, \odot)$ is a group, which is clearly noncommutative. ■

Now, we describe the left and right decompositions of the group $(\mathcal{G}, \odot)$, determined by its subgroup $(\mathcal{G}_0, \odot)$, which are created by the left and right translation of this subgroup:

$$\mathcal{L}(\mathcal{G}) = \{F(\lambda, K, f) \odot \mathcal{G}_0 \mid F(\lambda, K, f) \in \mathcal{G}\},$$
$$\mathcal{R}(\mathcal{G}) = \{\mathcal{G}_0 \odot F(\lambda, K, f) \mid F(\lambda, K, f) \in \mathcal{G}\}.$$

The corresponding equivalence relations are:

$$E_L=\{(F(\lambda_1, K_1,f_1), F(\lambda_2, K_2,f_2))\in\mathcal{G}\times\mathcal{G}|F^{-1}(\lambda_1, K_1,f_1)\odot F(\lambda_2, K_2,f_2)\in\mathcal{G}_0\},$$
$$E_R=\{(F(\lambda_1, K_1,f_1), F(\lambda_2, K_2,f_2))\in\mathcal{G}\times\mathcal{G}|F(\lambda_2, K_2,f_2)\odot F^{-1}(\lambda_1, K_1,f_1)\in\mathcal{G}_0\}.$$

Clearly, we have $F(\lambda_1, K_1, f_1)E_L F(\lambda_2, K_2, f_2)$ if and only if there exists $F(1, K, 0) \in \mathcal{G}_0$, such that $F(\lambda_1, K_1, f_1) \odot F(1, K, 0) = F(\lambda_2, K_2, f_2)$, so

$$F(\lambda_1, KK_1, f_1) = F(\lambda_2, K_2, f_2)$$

whence $\lambda_1 = \lambda_2,\ KK_1 = K_2,\ f_1 = f_2$ hold. Whereas

$$F(\lambda_1, K_1,f_1)E_R F(\lambda_2, K_2,f_2) \text{ if and only if } F(\lambda_2, K_2,f_2)=F(\lambda_1, KK_1, \hat{K}f_1)$$

for a suitable $F(1, K, 0) \in \mathcal{G}_0$, i.e. there exists $F(1, K, 0) \in \mathcal{G}_0$ such that $K_2 = KK_1,\ \lambda_1 = \lambda_2,\ f_2 = \hat{K}f_1$.

Now, we describe the decomposition of the group $(\mathcal{F}, \cdot)$ determined by its subgroups $(\mathcal{F}_1, \cdot)$ and $(\mathcal{F}_2, \cdot)$.

As it was mentioned above the subgroup $(\mathcal{F}_1, \cdot)$ is normal in $(\mathcal{F}, \cdot)$, so the left and right decompositions are equal:

$$\mathcal{F}/_R\mathcal{F}_1 = \mathcal{F}/_L\mathcal{F}_1 = \{\mathcal{F}_1 \cdot F(\lambda, K, f) = F(\lambda, K, f) \cdot \mathcal{F}_1 \mid F(\lambda, K, f) \in \mathcal{F}\}.$$

The corresponding equivalence relation is:

$$\begin{aligned} E_1 = \{(F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2))\in\mathcal{F} \times \mathcal{F} \mid F^{-1}(\lambda_1, K_1, f_1)\cdot F(\lambda_2, K_2, f_2) \\ \in \mathcal{F}_1 \text{ or } F(\lambda_2, K_2, f_2) \cdot F^{-1}(\lambda_1, K_1, f_1) \in \mathcal{F}_1\}. \end{aligned}$$

Clearly, we have $F(\lambda_1, K_1, f_1)\ E_1\ F(\lambda_2, K_2, f_2)$ if and only if there exists $F(1, K, f) \in \mathcal{F}_1$ such that $F(\lambda_2, K_2, f_2) = F(1, K, f) \cdot F(\lambda_1, K_1, f_1)$, which means that $K_2 = K_1 f + K,\ \lambda_1 = \lambda_2,\ f_2 = ff_1$.

On the other hand, the subgroup $(\mathcal{F}_2, \cdot)$ is also normal in $(\mathcal{F}, \cdot)$, so the left and right decompositions are equal.

$$\mathcal{F}/_R\mathcal{F}_2 = \mathcal{F}/_L\mathcal{F}_2 = \{\mathcal{F}_2 \cdot F(\lambda, K, f) = F(\lambda, K, f) \cdot \mathcal{F}_2 \mid F(\lambda, K, f) \in \mathcal{F}\}.$$

Here, the corresponding equivalence relation is:

$$E_2 = \{(F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2)) \in \mathcal{F} \times \mathcal{F} \mid F^{-1}(\lambda_1, K_1, f_1) \cdot F(\lambda_2, K_2, f_2) \in \mathcal{F}_2 \text{ or } F(\lambda_2, K_2, f_2) \cdot F^{-1}(\lambda_1, K_1, f_1) \in \mathcal{F}_2\}.$$

Clearly, we have $F(\lambda_1, K_1, f_1)\ E_2\ F(\lambda_2, K_2, f_2)$ if and only if there exists $F(\lambda, K, 1) \in \mathcal{F}_2$ such that $K_2 = K_1 + K$, $\lambda\lambda_1 = \lambda_2, f_2 = f_1$. The above obtained decompositions $\mathcal{G}/_R\mathcal{G}_0$, $\mathcal{G}\ /_L\mathcal{G}_0$, $\mathcal{F}\ /_R\mathcal{F}_1 = \mathcal{F}\ /_L\mathcal{F}_1$, $\mathcal{F}\ /_R\mathcal{F}_2 = \mathcal{F}\ /_L\mathcal{F}_2$ make possible to construct certain hypergroups.

Theorem 8.3.12. *The hyperstructures $(\mathcal{F}, *_i)$ for $i = 1, 2$ and $(\mathcal{G}, *)$ are join spaces, where*

$$x *_i y = \bar{x} \cup \bar{y} \quad \text{if } \bar{x}, \bar{y} \in \mathcal{F}/\mathcal{F}_i,$$
$$x * y = \bar{x} \cup \bar{y} \quad \text{if } \bar{x}, \bar{y} \in \mathcal{G}/_L\mathcal{G}_0.$$

Proof. It follows by Lemma 8.3.9. ■

In what follows, we consider

$$\mathcal{G}_+ = \{F(\lambda, K, f) \mid \lambda \in R, \lambda > 0, K \in C_+(J \times J)\}.$$

We define a binary hyperoperation on the factor set $L(\mathcal{G}_+)$. Recall that for all $\bar{x}, \bar{y} \in L(\mathcal{G}_+)$, we put

$$\bar{x} \odot \bar{y} = \{F(\lambda, K, f) \mid F(\lambda, K, f) = F(\lambda_1, K_1, f_1) \odot F(\lambda_2, K_2, f_2), F(\lambda_1, K_1, f_1) \in \bar{x},\ F(\lambda_2, K_2, f_2) \in \bar{y}\}.$$

Then for all $\bar{x}, \bar{y} \in L(\mathcal{G}_+)$ we define

$$\bar{x} * \bar{y} = \{\bar{z} \in L(\mathcal{G}_+) \mid \bar{z} \cap (\bar{x} \odot \bar{y}) \neq \emptyset\}.$$

Note that this is an usual construction in algebraic hyperstructure theory.

Proposition 8.3.13. *The hypergroupoid $L(\mathcal{G}_+), *)$ is a quasi-hypergroup with the following properties:*

(1) *For all* $\bar{x} \in L(\mathcal{G}_+)$ *we have* $\bar{x} * \mathcal{G}_0 = \{\bar{x}\}$, *i.e.* $\mathcal{G}_0$ *is the right unit of* $(L(\mathcal{G}_+), *)$ *and* $\bar{x} \in \mathcal{G}_0 * \bar{x}$.

(2) *If* $\bar{x}, \bar{y} \in L(\mathcal{G}_+)$ *are blocks (equivalence classes) with the representing operators* $F(\lambda, K_0, f) \in \bar{x}$, $F(\mu, P_0, g) \in \bar{y}$ *such that* $f, g \in C_+(J)$ *we have*

$$\bar{x} * \bar{y} = \{[F(\lambda\mu, KP, h)] \mid K, P \in C_+(J \times J),\ h \in C(J),\ h > f\},$$

where by [ ] *we mean the block with representing operator* $\cdot$.

Proof. Let $\bar{x} \in L(\mathcal{G}_+)$ be an arbitrary block and let $F(\lambda, K, f) \in \bar{x}$ be its arbitrary representing operator. For all $F(1, P, 0) \in \mathcal{G}_0$ we have $F(\lambda, K, f) \odot F(1, P, 0) = F(\lambda, KP, f) \in \bar{x}$, thus $\bar{x} \odot \mathcal{G}_0 \subseteq \bar{x}$. Then

$$\bar{x} * \mathcal{G}_0 = \{\bar{z} \mid \bar{z} \in L(\mathcal{G}_+),\ \bar{z} \cap (\bar{x} \odot \mathcal{G}_0) \neq \emptyset\} = \{\bar{x}\}.$$

Moreover $F(1, P, 0) \odot F(\lambda, K, f) = F(\lambda, KP, \hat{P}f)$. Since any function $U \in C_+(J \times J)$ can be represented in the form $U = KP$ with the fixed function K given above, it follows that the set $\mathcal{G}_0 \odot \bar{x}$ is a union of some blocks of $\mathcal{G}/\mathcal{G}_0$. If $P = 1$ we obtain $F(\lambda, K, f) \in \mathcal{G}_0 \odot \bar{x}$, consequently $\bar{x} \in \mathcal{G}_0 * \bar{x}$.

Let us denote $\phi = \{[F(\lambda\mu, KP, h)] \mid K, P \in C_+(J \times J),\ h \in C(J),\ h > f\}$. Let $\bar{z} \in \bar{x} * \bar{y}$, where $\bar{z} \in L(\mathcal{G}_+)$. Then $\bar{z} = [F(\xi, U, \varphi)]$, where $\xi = \lambda\mu$, $U = KP$ for suitable $K, P \in C_+(J \times J)$ and $\varphi = \lambda\hat{K}g + f > f$ since $\lambda\hat{K}g > 0$ on J, which implies that $\bar{x} * \bar{y} \subseteq \phi$. Suppose that $[F(\xi, U, \varphi)] \in \phi$. Then $\xi = \lambda\mu$ and there exist $K_1, P_1 \in C_+(J \times J)$ such that $U = K_1P_1$, moreover φ is a continuous functions on J, $f(x) < \varphi(x)$ for any $x \in J$. Consider a function $K \mid J \times J \to R$ defined in the following way: Denote $\psi(x) = (\varphi(x) - f(x))/g(x)$ for $x \in J$. Since g is a positive function, it follows that ψ is well defined and it is also positive and continuous on the segment J. Hence $\psi \in C_+(J)$. Define $K(x, s) = \psi(x)/\lambda$, $x, s \in J$. We have $K(x_1, s) = K(x_2, s)$ for all $x_1, x_2 \in J$, if $x_1 \neq s \neq x_2$, i.e. the function K is continuously extended from the diagonal $J \times J$ in such a way that K is constant on the segments $s = s_0$, $x = t$, $t \in J$. Thus $K \in C_+(J \times J)$ and $F(\lambda, K, f)\, E_L\, F(\lambda, K_0, f)$, i.e. $F(\lambda, K, f) \in \bar{x}$. Define a function $P_2 : J \times J \to R$ by $P_2 = (K_1P_1)/K = U/K$. Then $F(\mu, P_2, g) \in \bar{y}$

and we have $[F(\xi, U, \varphi)] = [F(\lambda\mu, KP_2, \psi g+f)] = [F(\lambda\mu, KP_2, \lambda\hat{K}g+f)] = [F(\lambda, K, f) \odot F(\mu, P_2, g)] \in \bar{x} * \bar{y}$, thus $\phi \subseteq \bar{x} * \bar{y}$. Therefore, the equality $\bar{x} * \bar{y} = \phi$ holds. ■

We recall the following definition, that we shall use in what follows.

Definition 8.3.14. A subhypergroup $(S, \cdot)$ of a hypergroup $(H, \cdot)$ is called

(1) *closed* if $a/b \subseteq S$ and $b \setminus a \subseteq S$ for all $a, b \in S$,

(2) *invertible* if $a/b \cap S \neq \emptyset$ implies $b/a \cap S \neq \emptyset$, and $b \setminus a \cap S \neq \emptyset$ implies $a \setminus b \cap S \neq \emptyset$ for all $a, b \in H$,

(3) *reflexive* if $a \setminus S = S/a$ for all $a \in H$,

(4) *normal* if $a \cdot S = S \cdot a$ for all $\in H$.

Note that the notion of a subhypergroup, and each of the properties for a subhypergroup in the above definition is self-dual. In a transposition hypergroup, a closed and normal subhypergroup is reflexive. Thus in a transposition hypergroup, an invertible and normal subhypergroup is closed and reflexive. Now, we are going to verify that the subhypergroupoid $(\mathcal{F}_1, *)$ of the transposition hypergroup $(\mathcal{F}, *)$ has the properties mentioned above. In the proof we will need next lemma.

Lemma 8.3.15. *Let $F(\lambda_1, K_1, f_1)$, $F(\lambda_2, K_2, f_2) \in \mathcal{F}$ be arbitrary operators, i.e., elements of the hypergroup $(\mathcal{F}, *)$. Then*

(1) $F(\lambda_1, K_1, f_1)/F(\lambda_2, K_2, f_2) = \{F(\lambda_1/\lambda_2, K, f_1/f_2) \mid K(x, s) \leq K_1(x, s) - K_2(x, s)f_1(x)/f_2(x), (x, s) \in J\}$,

(2) $F(\lambda_2, K_2, f_2) \setminus F(\lambda_1, K_1, f_1) = \{F(\lambda_1/\lambda_2, K, f_1/f_2) \mid K(x, s) \leq (K_1(x, s) - K_2(x, s))/f_2(x), (x, s) \in J\}$.

Proof. (1) Using that the function f_2 is positive and $\lambda_2 \neq 0$, for all operators $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2) \in \mathcal{F}$ we obtain that

$$\begin{aligned}
&F(\lambda_1, K_1, f_1)/F(\lambda_2, K_2, f_2)\\
&= \{F(\lambda, K, f) \mid F(\lambda_1, K_1, f_1) \in F(\lambda, K, f) * F(\lambda_2, K_2, f_2)\}\\
&= \{F(\lambda, K, f) \mid F(\lambda_1, K_1, f_1) \geq F(\lambda, K, f) \cdot F(\lambda_2, K_2, f_2)\}\\
&= \{F(\lambda, K, f) \mid F(\lambda_1, K_1, f_1) \geq F(\lambda\lambda_2, K_2 f + K, f f_2)\}\\
&= \{F(\lambda_1/\lambda_2, K, f_1/f_2) \mid K \leq K_1 - K_2 f_1/f_2\}\\
&= \{F(\lambda_1/\lambda_2, K, f_1/f_2) \mid K(x, s) \leq K_1(x, s) - K_2(x, s)f_1(x)/f_2(x)\}.
\end{aligned}$$

(2) We have

$$\begin{aligned}
&F(\lambda_2, K_2, f_2) \setminus F(\lambda_1, K_1, f_1) \\
&= \{F(\lambda, K, f) \mid F(\lambda_1, K_1, f_1) \in F(\lambda_2, K_2, f_2) * F(\lambda, K, f)\} \\
&= \{F(\lambda, K, f) \mid F(\lambda_1, K_1, f_1) \geq F(\lambda_2, K_2, f_2) \cdot F(\lambda, K, f)\} \\
&= \{F(\lambda, K, f) \mid F(\lambda_1, K_1, f_1) \geq F(\lambda_2\lambda, K f_2 + K_2, f_2 f)\} \\
&= \{F(\lambda_1/\lambda_2, K, f_1/f_2) \mid K \leq (K_1 - K_2)/f_2\} \\
&= \{F(\lambda_1/\lambda_2, K, f_1/f_2) \mid K(x,s) \leq (K_1(x,s) - K_2(x,s))/f_2(x)\}. \blacksquare
\end{aligned}$$

Theorem 8.3.16. *The subhypergroupoid $(\mathcal{F}_1, *)$ is a closed, invertible, reflexive and normal subhypergroup of $(\mathcal{F}, *)$.*

Proof. Firstly, notice that the subhypergroupoid $(\mathcal{F}_1, *)$ is a subhypergroup, since it satisfies the axiom of reproduction, i.e.,

$$F * \mathcal{F}_1 = \mathcal{F}_1 * F = \mathcal{F}_1 \text{ for all } F = F(1, K, f) \in \mathcal{F}_1.$$

Clearly, $F(1, K, f) * \mathcal{F}_1 \subseteq \mathcal{F}_1$. In order to prove that $\mathcal{F}_1 \subseteq F(1, K, f) * \mathcal{F}_1$, let us recall that

$$\begin{aligned}
F(1, K, f) * \mathcal{F}_1 &= F(1, K, f) * \{F_2(1, K_2, f_2)\} \\
&= \bigcup_{K_2} \bigcup_{f_2} \{F(1, K_L, f_2 f \mid K_2 f + K \leq K_L\}.
\end{aligned}$$

For any $F_3(1, K_3, f_3) \in \mathcal{F}_1$ let us choose $f_2 = f_3/f$ and $K_2 \leq (K_3 - K)/f$. Then $F_3 \in F * F_2 \subseteq F * \mathcal{F}_1$. The second equality $\mathcal{F}_1 * F = \mathcal{F}_1$ can be proved analogously. Now it is enough to show that the subhypergroup $(\mathcal{F}_1, *)$ is invertible and normal, because invertibility and normality implies closeness and reflexivity (see 2.3.)

Invertibility of $(\mathcal{F}_1, *)$: Suppose that $F_1(\lambda_1, K_1, f_1)$, $F_2(\lambda_2, K_2, f_2) \in \mathcal{F}$ are operators satisfying $F_1(\lambda_1, K_1, f_1)/F_2(\lambda_2, K_2, f_2) \cap \mathcal{F}_1 \neq \emptyset$. By Lemma 8.3.15, we have

$$F_1(\lambda_1, K_1, f_1)/F_2(\lambda_2, K_2, f_2) = \{F(1, K, f_1/f_2) \mid K \leq K_1 - K_2 f_1/f_2\} \cap \mathcal{F}_1 \neq \emptyset$$

i.e., $\lambda_1 = \lambda_2$. Then $F_2(\lambda_2, K_2, f_2)/F_1(\lambda_1, K_1, f_1) = \{F(1, K, f_2/f_1) \mid K \leq K_2 - K_1 f_2/f_1\} \subseteq \mathcal{F}_1$. Mainly, $F_2(\lambda_2, K_2, f_2)/F_1(\lambda_1, K_1, f_1) \cap \mathcal{F}_1 \neq \emptyset$.

Similarly,

$$F_2(\lambda_2, K_2, f_2) \setminus F_1(\lambda_1, K_1, f_1) \cap \mathcal{F}_1 \neq \emptyset,$$

$F_2(\lambda_2, K_2, f_2) \setminus F_1(\lambda_1, K_1, f_1) = \{F(1, K, f_1/f_2) \mid K \leq (K_1 - K_2)/f_2\} \cap \mathcal{F}_1 \neq \emptyset$, i.e. $\lambda_1 = \lambda_2$. Further, $F_1(\lambda_1, K_1, f_1) \setminus F_2(\lambda_2, K_2, f_2) = \{F(1, K, f_2/f_1) \mid K \leq (K_2 - K_1)/f_1\} \subseteq \mathcal{F}_1$. Thus, $(\mathcal{F}_1, *)$ is an invertible subhypergroup of $(\mathcal{F}, *)$.

Normality of $(\mathcal{F}_1, *)$: If $F(\lambda, K, f) \in \mathcal{F}$, then

$$\begin{aligned} F(\lambda, K, f) * \mathcal{F}_1 &= \bigcup \{F(\lambda, K, f) * F_2(1, K_2, f_2) \mid F_2(1, K_2, f_2) \in \mathcal{F}_1\} \\ &= \bigcup_{f_2} \bigcup_{K_2} \{F(\lambda, K_L, f f_2) \mid K_L \geq K_2 f + K\} \end{aligned}$$

and similarly

$$\mathcal{F}_1 * F(\lambda, K, f) = \bigcup_{f_3} \bigcup_{K_3} \{F(\lambda, K_{LL}, f_3 f) \mid K_{LL} \geq K f_3 + K_3\}.$$

For $f_2 = f_3$ we have

$$\bigcup_{K_2} \{F(\lambda, K_L, f f_2) \mid K_L \geq K_2 f + K\} = \bigcup_{K_3} \{F(\lambda, K_{LL}, f_3 f) \mid K_{LL} \geq K f_3 + K_3\}$$

and $F(\lambda, K, f) * \mathcal{F}_1 = \mathcal{F}_1 * F(\lambda, K, f)$ holds for any operator $F(\lambda, K, f) \in \mathcal{F}$, hence $(\mathcal{F}_1, *)$ is normal. ∎

Now, we intend to construct a discrete transformation hypergroup, in fact an action of the hypergroup of integral operators on the space of continuous functions. Let us recall that according to Proposition 8.3.7, $(\mathcal{F}, \cdot)$ is the group of Fredholm integral operators of the second kind.

Denote the following determinant by $D^{f_1, f_2}_{K_1, K_2}(x, s)$:

$$\begin{vmatrix} f_1(x) & f_2(x) \\ K_1(x, s) & K_2(x, s) \end{vmatrix}$$

Lemma 8.3.17. *The Fredholm integral operators $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2)$ are commuting in the group $(\mathcal{F}, \cdot)$ if and only if*

$$D^{f_1, f_2}_{K_1, K_2}(x, s) = K_2(x, s) - K_1(x, s) \text{ for all } x, s \in (a, b).$$

Proof. Let $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2) \in \mathcal{F}$ be commuting integral operators, i.e.,

$$F(\lambda_1, K_1, f_1) \cdot F(\lambda_2, K_2, f_2) = F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1),$$
$$F(\lambda_1\lambda_2, K_2 f_1 + K_1, f_1 f_2) = F(\lambda_2\lambda_1, K_1 f_2 + K_2, f_2 f_1).$$

Then $K_2 f_1 + K_1 = K_1 f_2 + K_2$, i.e. $f_1 K_2 - f_2 K_1 = K_2 - K_1$, hence $D^{f_1,f_2}_{K_1,K_2}(x,s) = K_2(x,s) - K_1(x,s)$ for all elements $x, s \in (a,b)$. Clearly, the procedure can be reversed. ■

Definition 8.3.18. Let X be a set, $(G, \cdot)$ be a semihypergroup and $\pi \mid X \times G \to X$ be a mapping such that $\pi(\pi(x,t),s) \in \pi(x, t\cdot s)$, where $\pi(x, t \cdot s) = \{\pi(x,u) \mid u \in t \cdot s)\}$ for each $x \in X,\ s,t \in G$. Then (X, G, π) is called a *discrete transformation semihypergroup* or an *action* of the semihypergroup G on the set X. The mapping π is usually called an *action*.

The above condition is called *Generalized Mixed Associativity Condition* (GMAC).

More generally, it is possible to consider the situation when the set X is endowed with some additional structure.

By a *centralizer* of an element a of the group G, we mean the subgroup $C_G(a) = \{x \in G \mid ax = xa\}$. A centralizer of an element $F(\lambda, K, f) \in \mathcal{F}$ is the subgroup

$$\begin{aligned} C_F(F(\lambda,K,f)) &= \{F(\mu,L,g) \mid F(\lambda,K,f)\cdot F(\mu,L,g) \\ &= F(\mu,L,g)\cdot F(\lambda,K,f)\} \\ &= \{F(\mu,L,g) \mid D^{f,g}_{K,L}(x,s) = L(x,s) - K(x,s)\} \end{aligned}$$

for all $x, s \in (a,b)$.

Definition 8.3.19. Let $F(\lambda_0, K_0, f_0) \in \mathcal{F}$ be an arbitrary, but fixed operator. Denote the centralizer of the operator $F(\lambda_0, K_0, f_0)$ in the group $(\mathcal{F}, \cdot)$ by $C_F(F(\lambda_0, K_0, f_0)) = C_F$. Let us define a hyperoperation $* : C_F \times C_F \to P^*(C_F)$, as follows:

$$\begin{aligned} &F(\lambda_1, K_1, f_1) * F(\lambda_2, K_2, f_2) \\ &= \{F^n(\lambda_0, K_0, f_0) \cdot F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1) \mid n \in \mathbb{N}\} \end{aligned}$$

for any pair of operators $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2) \in C_F(F(\lambda_0, K_0, f_0))$.

Denote $M(F(\lambda_0, K_0, f_0)) = (C(a,b), (C_F, *), \delta)$, where the mapping $\delta : C(a,b) \times C_F \to C(a,b)$ is defined by

$$\begin{aligned}\delta(\varphi, F(\lambda, K, f)) &= (F(\lambda_0, K_0, f_0) \cdot F(\lambda, K, f))(\varphi(x)) \\ &= F(\lambda_0\lambda, Kf_0 + K_0, f_0 f)(\varphi(x)).\end{aligned}$$

Proposition 8.3.20. *The system* $M(F(\lambda_0, K_0, f_0)) = (C(a,b), (C_F, *), \delta)$ *is a discrete transformation of the semihypergroup* $(C_F, *)$ *on the set* $C(a,b)$.

Proof. We show first that $(C_F, *)$ is a semihypergroup. We consider the binary relation $r \subseteq C_F \times C_F$, defined by $F(\lambda_1, K_1, f_1)\ r\ F(\lambda_2, K_2, f_2)$ if and only if $F(\lambda_2, K_2, f_2) = F^n(\lambda_0, K_0, f_0) \cdot F(\lambda_1, K_1, f_1)$ for some $n \in N$. We obtain that (C_F, r) is a quasiordered monoid. Clearly, the relation r is a quasi-ordering on C_F. Further, for all $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2), F(\lambda_3, K_3, f_3) \in C_F$, such that $F(\lambda_1, K_1, f_1)\ r\ F(\lambda_2, K_2, f_2)$, i.e.,

$$F(\lambda_2, K_2, f_2) = F^n(\lambda_0, K_0, f_0) \cdot F(\lambda_1, K_1, f_1),$$

for a suitable $n \in \mathbb{N}$, we have

$$F^n(\lambda_0, K_0, f_0) \cdot F(\lambda_1, K_1, f_1) \cdot F(\lambda_3, K_3, f_3) = F(\lambda_2, K_2, f_2) \cdot F(\lambda_3, K_3, f_3),$$

which means that

$$(F(\lambda_1, K_1, f_1) \cdot F(\lambda_3, K_3, f_3))\ r\ (F(\lambda_2, K_2, f_2) \cdot F(\lambda_3, K_3, f_3))$$

and similarly

$$F(\lambda_3, K_3, f_3) \cdot F(\lambda_2, K_2, f_2) = F(\lambda_3, K_3, f_3) \cdot F^n(\lambda_0, f_0) \cdot F(\lambda_1, K_1, f_1),$$

i.e., $(F(\lambda_3, K_3, f_3) \cdot F(\lambda_1, K_1, f_1))\ r\ (F(\lambda_3, K_3, f_3) \cdot F(\lambda_2, K_2, f_2))$, therefore $(C_F, \cdot, r)$ is a quasi-ordered monoid.

Now, we define a binary hyperoperation $*$ by

$$\begin{aligned}&F(\lambda_1, K_{1,1}) * F(\lambda_2, K_2, f_2) \\ &= \{F^n(\lambda_0, K_0, f_0) \cdot F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1) \mid n \in \mathbb{N}\}\end{aligned}$$

and we obtain

$$\begin{aligned}F(\lambda_1, K_{1,1}) * F(\lambda_2, K_2, f_2) &= r(F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1)) \\ &= [F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1)]_r,\end{aligned}$$

whence it follows that that $(C_F, *)$ is a semihypergroup (non commutative, in general).

It remains to show that GMAC is satisfied. Let $\varphi \in C(a,b)$ be an arbitrary function, $F(\lambda_1, K_1, f_1)$, $F(\lambda_2, K_2, f_2) \in C_F$ be arbitary operators. We have

$$\begin{aligned}
&\delta(\delta(\varphi, F(\lambda_1, K_1, f_1)), F(\lambda_2, K_2, f_2)) \\
&= \delta(F(\lambda_0, K_0, f_0) \cdot (F(\lambda_1, K_1, f_1)(\varphi(x)), F(\lambda_2, K_2, f_2)) \\
&= \delta(F(\lambda_0\lambda_1, K_1 f_0 + K_0, f_0 f_1)(\varphi(x)), F(\lambda_2, K_2, f_2)) \\
&= (F(\lambda_0, K_0, f_0) \cdot F(\lambda_2, K_2, f_2) \cdot F(\lambda_0\lambda_1, K_1 f_0 + K_0, f_0 f_1))(\varphi(x)) \\
&= F(\lambda_0^2\lambda_1\lambda_2, K_1 f_0^2 f_2 + K_2 f_0^2 + K_0 f_0 + K_0, f_0^2 f_1 f_2)(\varphi(x)).
\end{aligned}$$

On the other hand, for

$$\begin{aligned}
F &= F(\lambda_0, K_0, f_0) \cdot F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1) \\
&= F(\lambda_0\lambda_1\lambda_2, K_1 f_0 f_2 + K_2 f_0 + K_0, f_0 f_1 f_2)
\end{aligned}$$

and for an arbitrary function $\varphi \in C(a, b)$ we have

$$\begin{aligned}
&F(\lambda_0^2\lambda_1\lambda_2, K_2 f_0^2 f_1 + K_1 f_0^2 + K_0 f_0 + K_0, f_0^2 f_1 f_2)(\varphi(x)) \\
&= (F(\lambda_0, K_0, f_0) \cdot F(\lambda_0\lambda_1\lambda_2, K_1 f_0 f_2 + K_2 f_0 + K_0, f_0 f_1 f_2))(\varphi(x)) \\
&\qquad \in \{(F(\lambda_0, K_0, f_0) \cdot F_n)(\varphi(x)) \mid \\
&\qquad\qquad F_n \in F^n(\lambda_0, K_0, f_0) \cdot F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1),\, n \in \mathbb{N}\} \\
&= \{\delta(\varphi, F_n) \mid F_n \in \{F^n(\lambda_0, K_0, f_0) \cdot F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1) \mid n \in N\}\} \\
&= \{\delta(\varphi, F_n) \mid F_n \in F(\lambda_1, K_1, f_1) * F(\lambda_2, K_2, f_2)\} \\
&= \delta(\varphi, F(\lambda_1, K_1, f_1) * F(\lambda_2, K_2, f_2)). \blacksquare
\end{aligned}$$

We end this paragraph by presenting separable kernels and we obtain the construction of multiautomata of a general form.

As it was mentioned above, linear integral equations are the continuous analog of systems of algebraic equations. Fredholm theory is the theory of integral equations that have kernels, which can be approximated arbitrarily accurately by separable kernels. These kernels can be written in the form $K(x,s) = \sum_{i=1}^{n} f_i(x) \cdot g_i(s)$. It is useful to suppose that the sum has been reduced to a form in which the functions f_i are linearly independent and the functions g_i are also linearly independent.

We give the following example (see [102]). Let us consider the equation:

$$u(x) = f(x) + c\int_{-1}^{1} cos\pi(x-y)u(y)dy, \tag{II}$$

where f is a given integrable function. By expanding the cosine, we can rewrite this, as follows:

$$u(x) = f(x) + cu_1\ sin(\pi x) + cu_2\ cos(\pi x),$$

where $u_1 = \int_{-1}^{1} sin(\pi y)u(y)dy$, $u_2 = \int_{-1}^{1} cos(\pi y)u(y)dy$. Multiplying the equation by $sin(\pi x)$ and integrating over the interval, and then doing the same with $cos(\pi x)$, we obtain:

$$u_1 = f_1 + cu_1,\ u_2 = f_2 + cu_2. \tag{III}$$

For $c = 1$, we have two possibilities. If either f_1 or f_2 is not zero, the above mentioned system (III) has no solution and so, the integral equation (II) has no solution, as well. The alternative is that $f_1 = f_2 = 0$. These are the *Fredholm conditions* on $f(x)$. If they are satisfied, then (III) is satisfied whatever values u_1 and u_2 may be. Hence, for $c = 1$, either the integral equation (II) has no solution or, if Fredholm conditions hold, there is a solution, but it is not unique.

The kernel in this problem is *separable*. For any separable kernel, the problem is reduced to a matter of linear algebraic equations.

Now, we consider the subgroup $\mathcal{G}_0$ of all operators of the form $F(1, K, 0)$, where K is a positive continuous function on the interval $J \times J$. For any continuous function $\varphi \in C(J)$ we have

$$F(1, K, 0)(\varphi) = \int_a^b K(x, s)\varphi(s)ds.$$

For an arbitrary fixed positive integer $n \in N$, we denote the set of all functions of the form $K(x, s) = \sum_{i=1}^{n} k_{1i}(x)k_{2i}(s)$ by $Sp_n(J \times J)$, where $k_{1i}, k_{2i} \in C(J)$, $i = 1, , 2, ..., n$. In other words, $Sp_n(J \times J)$ is the set of all separable kernels of the same length n.

We shall construct a hypergroupoid $(Sp_n(J \times J), \circ)$, acting on the ring of continuous functions $C(J)$.

For all functions $K(x,s)=\sum_{i=1}^{n} k_{1i}(x)k_{2i}(s)$, $G(x,s)=\sum_{i=1}^{n} g_{1i}(x)g_{2i}(s)$ of $Sp_n(J\times J)$, we define the hyperproduct $K\circ G$ as being the set of all functions $H(x,s)=\sum_{i=1}^{n} h_{1j}(x)h_{2j}(s)$, where $h_{1j}, h_{2j} \in C(J)$, $i=1,2,...,n$, such that

$$F(1,H,0)(\varphi) = \int_a^b H(x,s)\varphi(s)ds = \sum_{j=1}^{n} c_j h_{1j}(x), \ c_j = \int_a^b h_{2j}(s)\varphi(s)ds,$$

where $g_{1j}(x) \le h_{1j}(x)$ for all $j=1,2,...,n$, $x \in J$ and $\sum_{i=1}^{n} a_i b_{ij} \le c_j$,

$$a_i = \int_a^b k_{2i}\varphi(x)dx, \ b_{ij} = \int_a^b k_{1i}(x)g_{2j}(x)dx, \quad i,j = 1,2,...,n.$$

Thus, a_i, b_{ij} are scalar products $a_i = (k_{2i}, \varphi)$, $b_{1i} = (k_{1i}, g_{2j})$. Clearly, $(Sp_n(J\times J), \circ)$ is a noncommutative hypergroupoid of separable kernels of integral operators $F(1,K,0)$, $K \in Sp_n(J\times J)$.

We define the transition function $\delta_j : C(J) \times Sp_n(J\times J) \to C(J)$ in the following way: for all $K(x,s) = \sum_{i=1}^{n} k_{1i}(x)k_{2i}(s) \in Sp_n(J\times J)$ and all $\varphi \in C(J)$, we put

$$\begin{aligned} \delta_j(\varphi, K) &= F(1,K,0)(\varphi) = \int_a^b K(x,s)\varphi(s)ds \\ &= \sum_{i=1}^{n} k_{1i}(x) \int_a^b k_{2i}(s)\varphi(s)ds = \sum_{i=1}^{n} a_i k_{1i}(x), \end{aligned}$$

where $a_i = \int_a^b k_{2i}(s)\varphi(s)ds = (k_{2i}, \varphi)$, $i = 1,2,...,n$. We show that the hypergroupoid $(Sp_n(J\times J), \circ)$ acts on the ring $C(J)$, i.e., the Generalized Mixed Associativity Condition (GMAC) holds. In order to check this, we show that for all $[K, G, \varphi] \in Sp_n(J\times J) \times Sp_n(J\times J) \times C(J)$ we have $\delta_j(\delta_j(\varphi, K), G) \in \delta_j(\varphi, K\circ G) = \{\delta_j(\varphi, H) \mid H \in K\circ G\}$.

Indeed, for all $K(x,s) = \sum_{i=1}^{n} k_{1i}(x)k_{2i}(s)$, $G(x,s) = \sum_{i=1}^{n} g_{1i}(x)g_{2i}(s)$ and all $\varphi \in C(J)$, we have

$$\delta_j(\delta_j(\varphi, K), G) = \delta_j\left(\sum_{i=1}^{n} a_i k_{2i}(x), G\right)$$
$$= \int_a^b \left(\sum_{i=1}^{n} g_{1i}(x) \int_a^b g_{2i}(s)\right)\left(\sum_{i=1}^{n} a_i k_{2i}(x)\right) ds$$
$$= \int_a^b (g_{11}(x)g_{21}(s)+g_{12}(x)g_{22}(s) + \ldots + g_{1n}(x)g_{2n}(s)) \cdot (a_1k_{11}(s)+a_2k_{12}(s) + \ldots + a_nk_{1n}(s))ds$$
$$= \int_a^b (a_1g_{11}(x)g_{21}(s)k_{11}(s)+a_1g_{12}(x)g_{22}(s)k_{11}(s)+\ldots+a_1g_{1n}(x)g_{2n}(s)k_{11}(s) +a_2g_{11}(x)g_{21}(s)k_{12}(s) + \ldots + a_2g_{1n}(x)g_{2n}(s)k_{12}(s) + \ldots +a_ng_{11}(x)g_{21}(s)k_{1n}(s) + \ldots + a_ng_{1n}(x)g_{2n}(s)k_{1n}(s))ds$$
$$= a_1g_{11}(x)\int_a^b g_{21}(s)k_{11}(s)ds + a_1g_{12}(x)\int_a^b g_{22}(s)k_{11}(s)ds + \ldots +a_1g_{1n}(x)\int_a^b g_{2n}(s)k_{11}(s)ds + a_2g_{11}(x)\int_a^b g_{21}(s)k_{12}(s)ds + \ldots +a_2g_{12}(x)\int_a^b g_{22}(s)k_{12}(s)ds + \ldots + a_2g_{1n}(x)\int_a^b g_{2n}(s)k_{12}(s)ds + \ldots +a_ng_{11}(x)\int_a^b g_{21}(s)k_{1n}(s)ds + \ldots + a_ng_{1n}(x)\int_a^b g_{2n}(s)k_{1n}(s)ds$$
$$= a_1b_{11}g_{11}(x) + a_1b_{12}g_{12}(x) + \ldots + a_1b_{1n}g_{1n}(x) + a_2b_{21}g_{11}(x) + a_2b_{22}g_{12}(x) +\ldots + a_2b_{2n}g_{1n}(x) + \ldots + a_nb_{n1}g_{11}(x) + \ldots + a_nb_{nn}g_{1n}(x)$$
$$= (a_1b_{11} + a_2b_{21} + \ldots + a_nb_{n1})g_{11}(x) + (a_1b_{12} + a_2b_{22} + \ldots + a_nb_{n2})g_{12}(x) +\ldots + (a_1b_{1n} + a_2b_{2n} + \ldots + a_nb_{nn})g_{1n}(x)$$
$$= \left(\sum_{i=1}^{n} a_ib_{i1}\right) g_{11}(x) + \left(\sum_{i=1}^{n} a_ib_{i2}\right) g_{12}(x) + \ldots + \left(\sum_{i=1}^{n} a_ib_{in}\right) g_{1n}(x)$$
$$= \sum_{i=1}^{n}\left(\sum_{i=1}^{n} a_ib_{ij}\right) g_{1j}(x)$$
$$\in \left\{\sum_{i=1}^{n} c_jh_{1j}(x); \sum_{i=1}^{n} a_ib_{ij} \le c_j,\ g_{1j}(x) \le h_{1j}(x),\ h_{1j} \in C(J),\ j=1,2,\ldots,n,\ x \in J\right\}$$
$$= \{H(x,s)(\varphi);\ H \in K \circ G\} = \delta_j(\varphi, K \circ G).$$

Therefore the Generalized Mixed Associativity Condition is satisfied. Consequently, $(C(J), Sp_n(J \times J), \delta_j)$ is a multiautomaton with the state set $C(J)$ and the input hypergroupoid $(Sp_n(J \times J), \circ)$ of separable kernels $K(x,s) = \sum_{i=1}^{n} k_{1i}(x)k_{2i}(s) \in C(J \times J)$. In other words, δ_j is an action of the hypergroupoid $(Sp_n(J \times J), \circ)$ on the set $C(J)$.

Multiautomata constructed from integral operators are of the so-called type [ ]. The input hyperstructures are constructed using centralizers of given operators.

8.4 Hyperstructures determined by differential rings

The systematic study of algebraic aspects of transformations of differential and difference operators applied to investigation of differential and difference equations is of a persistent interest. General algebraic approach to the transformation theory is described in [96], more in detail see also [97] and other related papers of Neuman. In this paragraph, we give construction of hyperstructures determined by quasiorders defined by means of derivation operators on differential rings. The results of this paragraph were obtained by J. Vhvalina and L. Chvalinová [12].

Definition 8.4.1. Let $(R, +, \cdot, \triangle_R)$ be a commutative *differential ring*, i.e., $(R, +, \cdot)$ is a commutative ring, $\triangle_R$ is a set of derivations on the set R, which means that $\triangle_R$ is a subset of the endomorphism monoid $End(R, +)$ satisfying the differentiation rule. Thus for $d \in \triangle_R$ and any pair of elements x, y of R we have $d(x+y) = d(x) + d(y)$ and $d(x \cdot y) = d(x) \cdot y + x \cdot d(y)$. Moreover, we suppose that any $d : R \longrightarrow R$ is surjective. A differential structure $\triangle_R$ of a ring can be endowed with the Lie multiplication $d_1 \Diamond_L d_2 = d_1 d_2 - d_2 d_1$. Then $(R, +, \Diamond_L)$ is a Lie ring of derivations. If $\triangle_R = \{d\}$ is a singleton, then we say that this differential structure is *monogenous*.

Examples 8.4.2.

(1) Let $J = (a, b) \subsetneq \mathbb{R}$ (possibly $J = \mathbb{R}$) and $C^\infty(J)$ be the ring of real functions $f : J \longrightarrow \mathbb{R}$ with continuous derivatives of all orders.

If $\Delta_R = \left\{ \dfrac{d}{dx} \right\}$, where $\dfrac{df}{dx} = f'$ is the usual derivative of a function $f \in C^\infty(J)$, then $(C^\infty(J), +, \cdot, \Delta)$ is a differential ring with a monogenous differential structure.

(2) Let $\mathbb{R}[x_1, ..., x_n]$ (for a fixed integer n) be the ring of all polynomials with coefficients in the field $\mathbb{R}$. Denoting

$$\Delta = \left\{ \sum_{k=1}^{n} \lambda_k \cdot \frac{\partial}{\partial x_k} \mid \lambda_1, ..., \lambda_n \in \mathbb{R} \right\},$$

we obtain $(R[x_1, ..., x_n], +, \cdot, \Delta)$ as an example of a differential ring.

Let $(R, +, \cdot, \Delta_R)$ be a differential (non necessary commutative) ring, $M(\Delta_R)$ be the free monoid over Δ_R (i.e., $M(\Delta_R)$ is the set of all finite words $d_1...d_n, d_k \in \Delta_R$ including the empty word Λ, identified with the identity operator id_R, endowed with the binary operation of concatenation). We define $d_1...d_n(x) = d_n(...(d_1(x))...)$ which means the application of the composition of the operators $d_1, ..., d_n$ - in this order - to the element $x \in R$.

Theorem 8.4.3. *Let $(R, +, \cdot, \Delta_R)$ be a differential ring. Let*

$$\begin{aligned} x * y &= \{d_1...d_n(z) \mid z \in \{x, y\}, d_k \in \Delta_R, n \in \mathbb{N}\} \\ &= \{\delta(z) \mid z \in \{x, y\},\ \delta \in M(\Delta_R)\}. \end{aligned}$$

Then we have

(1) *$(R, *)$ is a commutative hypergroup such that any differential endomorphism of the ring $(R, +, \cdot, \Delta_R)$ (i.e., $f \in End(R, +, \cdot)$ with $f(d_k(x)) = d_k(f(x))$, $x \in R$) is a good endomorphism of $(R, *)$.*

(2) *The hypergroup $(R, *)$ satisfies the transposition law, hence it is a join space if and only if for any pair of elements x, y of R such that there exists a pair of words $(\delta, \sigma) \in M(\Delta_R) \times M(\Delta_R)$ and a suitable element $z \in R$ with $\delta(z) = x, \sigma(z) = y$, we have $\tau(x) = \omega(y)$ for some pair of words $\tau \in M(\Delta_R)$, $\omega \in M(\Delta_R)$.*

Proof. Define a binary relation $r \subseteq R \times R$ by xry whenever there exists an m-tuple of derivation operators $d_1, ..., d_m \in \triangle_R$, i.e., a word $\delta = d_1 ... d_m \in M(\triangle_R)$ such that $y = \delta(x)$. The relation r is reflexive (if $d_1 = ... = d_m = id_R$) and transitive: For $x, y, z \in R$ such that xry, yrz, i.e., $y = \delta(x)$, $z = \sigma(y)$ for suitable words $\delta, \sigma \in M(\triangle_R)$ we obtain $z = \sigma\delta(x)$ with $\sigma\delta \in M(\triangle_R)$, thus xrz. If for arbitrary pair $x, y \in R$ we define

$$\begin{aligned} x * y &= \{\delta(z) \mid z \in \{x, y\}, \delta \in M(\triangle_R)\} \\ &= \{\delta(x) \mid \delta \in M(\triangle_R)\} \cup \{\delta(y) \mid \delta \in M(\triangle_R)\} \\ &= r(x) \cup r(y), \end{aligned}$$

then $(R, *)$ is a commutative hypergroup. Further, if $f : R \longrightarrow R$ is a differential endomorphism of the ring $(R, +, \cdot, \triangle_R)$ which means that $f \in End(R, +, \cdot)$ and $f(d(x)) = d(f(x))$ for any $d \in \triangle_R$ and any $x \in R$, then by induction $f(\delta(x)) = \delta(f(x))$ for any word $\delta \in M(\triangle_R)$ and each element $x \in R$. Thus for any pair $x, y \in R$ we have

$$\begin{aligned} f(x * y) &= \{f(\delta(z)) \mid z \in \{x, y\}, \delta \in M(\triangle_R)\} \\ &= \{\delta(f(z)) \mid z \in \{x, y\}, \delta \in M(\triangle_R)\} \\ &= f(x) * f(y). \end{aligned}$$

Hence the assertion (1) is true.

Finally, the monoid $M(\triangle_R)$ acts on the set R. One can show that the hypergroup $(R, *)$ is a join space if and only if for every pair of elements $x, y \in R$ such that there exists a pair of words $\delta, \sigma \in M(\triangle_R)$ and an element $z \in R$ with $\delta(z) = x$, $\sigma(z) = y$, we have $\tau(x) = \omega(y)$ for suitable words $\tau, \omega \in M(\triangle_R)$. Thus we obtain the assertion (2). ∎

Remark 8.4.4. Using principal ideals a differential ring $(R, +, \cdot, \{d\})$ with a monogenous differential structure, we can construct a countable set (in general) of commutative extensive hypergroups $(R, \circ_m)$ with the same carrier R. (Extensivity of a hyperoperation $\circ_m$ means that $x, y \in x \circ_m y$ for all $x, y \in R$.) More in detail, for a given positive integer $m \in \mathbb{N}$ we define

$$x \circ_m y = \{z \in R \mid x \cdot d^m(R) \subseteq z \cdot d^m(R) \text{ or } y \cdot d^m(R) \subseteq z \cdot d^m(R)\},$$

where $d^m(R) = \{d^m(x) \mid x \in R\}$. Then we obtain that (R, om) is a commutative extensive hypergroup.

Theorem 8.4.5. *Let $(R, +, \cdot, \Delta_R)$ be a commutative differential ring with a monogenous differential structure $\Delta_R = \{d\}$. Let $(R, *_d)$ be a commutative hypergroupoid defined by the indefinite integral $x *_d y = d^{-1}(x + y)$ for all $x, y \in R$. Then $(R, *_d)$ is a commutative quasihypergroup such that $(x + y)/(u + v) = x/u + y/v$ for any quadruple $(x, y, u, v) \in R^4$ and for arbitrary triad $(x, y, z) \in R^3$ we have*

(1) $x/y = d(x) - y$,

(2) $d(x) = (x + y)/z - y/z$,

(3) $d(x/y) = d(x)/d(y)$,

(4) $d(x *_d x + y *_d y) = d(x *_d y) + d(x *_d y)$.

Proof. We show first that the hypergroupoid $(R, *_d)$ satisfies the reproduction axiom.

Let $a \in R$ be an arbitrary element. Since $a *_d R \subseteq R$ and $(R, *_d)$ is commutative if suffices to prove the inclusion $R \subseteq a *_d R$. For any $x \in R$ the set

$$d^{-1}(x) = I(x) = \{y \in R \mid d(y) = x\}$$

is called the *indefinite integral* of x. Now, for arbitrary $b \in R$ we denote $x_b = d(b) - a$. Then $d(b) = a + x_b$, i.e.,

$$b \in d^{-1}(a + x_b) = I(a + x_b) = a *_d x_b \subseteq \bigcup_{x \in R} a *_d R$$

hence $a *_d R = R = R *_d a$ for any $a \in R$. It is easy to see that $(R, *_d)$ is not associative in general, thus $(R, *_d)$ is a commutative quasihypergroup. Further, for any $x, y, u, v \in R$ we have

(1) $x/y = \{z \in R \mid x \in z *_d y\} = \{z \in R \mid x \in I(z + y)\}$, thus $x \in z *_d y$ if and only if $d(x) = z + y$, thus $z = d(x) - y$, hence we obtain that $z \in x/y$ if and only if $z = d(x) - y$ consequently $x/y = d(x) - y$ which is a singleton. Now

$$x/u + y/v = d(x) - u + d(y) - v = d(x + y) - (u + v) = (x + y)/(u + v).$$

(2) For any $x, y, z \in R$ we have $(x+y)/z = d(x+y)-z = d(x)+d(y)-z = d(x)+y/z$, therefore $d(x) = (x+y)/z - y/z$.

(3) Similarly, we have $d(x/y) = d(d(x)-y) = d(d(x))-d(y) = d(x)/d(y)$.

(4) We have

$$\begin{aligned} d(x *_d y) + d(x *_d y) &= d(d^{-1}(x+y)) + d(d^{-1}(x+y)) \\ &= x+y+x+y \\ &= x+x+y+y \\ &= d(d^{-1}(x+x+y+y)) \\ &= d(d^{-1}(x+x)+d^{-1}(y+y)) \\ &= d^{-1}(x *_d x + y *_d y). \quad \blacksquare \end{aligned}$$

Now, we consider the classical differential rings of real functions $f \in C^\infty(J)$, $J = (a,b) \subseteq R$ (not excluding the case $J = R$) with the usual differentiation. For any $f \in C^\infty(J)$ we denote the set of all primitive functions to f by $\int f(x)dx$, i.e., $\int f(x)dx = \{F : J \longrightarrow R \mid F'(x) = f(x), x \in J\}$. For any pair of functions $\varphi, \psi \in C^\infty(J)$ we define a hyperoperation $*$ on the ring $C^\infty(J)$ by

$$f *_{(\varphi,\psi)} g = \int (\varphi'(x)f(x) + \psi'(x)g(x))dx, \quad f, g \in C^\infty(J).$$

Evidently, $(C^\infty(J), *_{(\varphi,\psi)})$ is a hypergroupoid (noncommutative in general).

Theorem 8.4.6. *Let $J \subseteq \mathbb{R}$ be an open interval, $\varphi, \psi \in C^\infty(J)$ be a pair of strictly monotone functions (i.e., $\varphi'(x)\psi'(x) \neq 0$ for all $x \in J$). Then the hypergroupoid $(C^\infty(J), *_{(\varphi,\psi)})$ is a quasihypergroup (i.e., it satisfies the reproduction axiom) which is commutative if and only if the difference $\varphi - \psi$ on the interval J is a constant function on the interval J.*

Proof. Clearly, for any pair $f, g \in C^\infty(J)$ and any function $h \in f *_{(\varphi_1,\varphi_2)} g$ we have $h \in C^\infty(J)$. Suppose that $f \in C^\infty(J)$ is an arbitrary function. Then

$$f *_{(\varphi_1,\varphi_2)} C^\infty(J) = \cup\{f *_{(\varphi_1,\varphi_2)} g \mid g \in C^\infty(J)\} \subseteq C^\infty(J)$$

and

$$C^\infty(J) *_{(\varphi_1,\varphi_2)} f \subseteq C^\infty(J).$$

We prove the opposite inclusions. Suppose that $g \in C^\infty(J)$ is an arbitrary function. Define

$$h_1(x) = \frac{1}{\varphi_2'(x)}(g'(x) - \varphi_1'(x)f(x)), \quad x \in J.$$

Since $\varphi_1'(x)\cdot\varphi_2'(x)\neq 0$ for each $x \in J$, it follows that $\varphi_2'(x)^{-1}\neq 0$ for any $x\in J$, thus the function $\frac{1}{\varphi_2'(x)}$ is defined on the interval J and $\frac{1}{\varphi_2'(x)}\in C^\infty(J)$, $g'(x) - \varphi_1'(x)f(x) \in C^\infty(J)$, hence $h_1 \in C^\infty(J)$. Then

$$f *_{(\varphi_1,\varphi_2)} h_1 = \int(\varphi_1'(x)f(x)+\varphi_2'(x)h_1(x))dx = \int g'(x)dx = \{g(x)+c \mid c \in \mathbb{R}\},$$

thus

$$g \in f *_{(\varphi_1,\varphi_2)} h_1 \subseteq \cup\{f *_{(\varphi_1,\varphi_2)} h \mid h \in C^\infty(J)\}.$$

Similarly, if we define

$$h_2(x) = \frac{1}{\varphi_1'(x)}(g'(x) - \varphi_2'(x)f(x)), \quad x \in J,$$

then the assumption $\varphi_1'(x) \neq 0$ for any $x \in J$ and $f, g, \varphi_1, \varphi_2 \in C^\infty(J)$ imply that $h_2 \in C^\infty(J)$. Further,

$$h_2 *_{(\varphi_1,\varphi_2)} f = \int(\varphi_1'(x)h_2(x)+\varphi_2'(x)f(x))dx = \int g'(x)dx = \{g(x)+c \mid c \in \mathbb{R}\},$$

thus we have

$$g \in h_2 *_{(\varphi_1,\varphi_2)} f \subseteq \cup\{h *_{(\varphi_1,\varphi_2)} f \mid h \in C^\infty(J)\} = C^\infty(J) *_{(\varphi_1,\varphi_2)} f.$$

Hence

$$C^\infty(J) \subseteq (f *_{(\varphi_1,\varphi_2)} C^\infty(J)) \cap (C^\infty(J) *_{(\varphi_1,\varphi_2)} f),$$

consequently the hypergroupoid $(C^\infty(J), *_{(\varphi_1,\varphi_2)})$ satisfies the reproduction axiom. Therefore it is a quasihypergroup.

Now, suppose that $\varphi_1(x) - \varphi_2(x) = c$ for some real number $c \in \mathbb{R}$. Then

$\varphi_1' = \varphi_2'$ and $f *_{(\varphi_1,\varphi_2)} g = g *_{(\varphi_1,\varphi_2)} f$ for any pair of functions $f, g \in C^\infty(J)$. On the contrary, if the hyperoperation $*_{(\varphi_1,\varphi_2)}$ is commutative, then

$$\int (\varphi_1'(x)f(x) + \varphi_2'(x)g(x))dx = \int (\varphi_1'(x)g(x) + \varphi_2'(x)f(x))dx$$

which is equivalent to

$$\int (\varphi_1'(x) - \varphi_2'(x))(f(x) - g(x))dx = 0. \tag{I}$$

For $f(x) = g(x)+1$, $x \in J$, equality (I) gives $\int (\varphi_1'(x)-\varphi_2'(x))dx = 0$ which implies that $\varphi_1'(x) - \varphi_2'(x)=0$ thus $\varphi_1'(x)=\varphi_2'(x)$ is a constant function. ■

Remark 8.4.7. It is easy to see that the hyperoperation

$$*_{(\varphi_1,\varphi_2)} : C^\infty(J) \times C^\infty(J) \longrightarrow \mathcal{P}^*(C^\infty(J))$$

is not associative. In the special case $\varphi_1(x) = \varphi_2(x) = x$, $x \in J$, where $f * g = \int (f(x) + g(x))dx$, for any pair $f, g \in C^\infty(J)$, we obtain

$$f(x)/g(x) = \frac{df(x)}{dx} - g(x),$$

$$\frac{d}{dx}(f(x)/g(x)) = \frac{df(x)}{dx} \Big/ \frac{dg(x)}{dx},$$

$$\frac{df(x)}{dx} = (f(x) + g(x))/h(x) - g(x)/h(x),$$

for arbitrary $f, g, h \in C^\infty(J)$.

Moreover, for any quadruple $(f, g, u, v) \in C^\infty(J)^4$ we obtain

$$(f(x) + g(x))/(u(x) + v(x)) = f(x)/u(x) + g(x)/v(x).$$

Using derivatives of functions from $C^\infty(J)$ we can express certain sufficient conditions for validity of transposition law for the quasihypergroup $(C^\infty(J), *_{(\varphi_1,\varphi_2)})$. Moreover, transposition hypergroups can be constructed from quasiordered groups and monoids of some transformation operators of rings of continuously differentiable functions.

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This monograph is devoted especially to the study of Hyperring Theory. It begins with some basic results concerning ring theory and algebraic hyperstructures, which represent the most general algebraic context, in which the reality can be modeled. Several kinds of hyperrings are introduced and analyzed in the following chapters: Krasner hyperrings, multiplicative hyperrings, general hyperrings. Another class of hyperstructures, which has a lot of applications, is Hv-ring class, introduced by T. Vougiouklis. An interesting connection between rings and hyperrings is presented in the seventh chapter. The volume ends with an outline of applications in Chemistry and Physics, analizying several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. The theory of suitable modified hyperstructures can serve as a mathematical background in the field of quantum communication systems.

The volume is highly recommended to mathematicians and theoreticians in pure and applied mathematics.

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