Santilli's Lie-Admissible Mechanics. The Only Option Commensurate with Irreversibility and Nonequilibrium Thermodynamics

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Concluding Remarks

Introduction

The real and virtual motions are described by mathematical equations. Traditionally there are two different ways to analyze whether they are reversible or irreversible descriptions. The dynamical irreversibility is determined and understood using a yard stick of time reversal symmetry. The second yard stick of irreversibility originates from the second law of thermodynamics[1] and also via Boltzmann integro-differential equation[2].

[1] See for example S. M. Blinder, *Advanced Physical Chemistry*. New York: Collier-Macmillan, September 1969

[2] S. Chapman and T. G. Cowling, *Mathematical Theory of Non-uniform Gases: An Account of the Kinetic Theory of Viscosity, Thermal Conduction and Diffusion in Gases*, 3rd ed. Cambridge: Cambridge University Press, 1970

- **Indeed, the dynamical reversibility and irreversibility can be worked out in principle both at microscopic and macroscopic levels of considerations.**
- **Whereas the thermodynamic reversibility and irreversibility are determined at the macroscopic level only.**
- **It is believed that the reversibility and irreversibility aspects of mathematical descriptions are well understood using and they are sound too.**
- **Santilli has already illustrated and stressed the existing lacuna in them. However, we are elaborating in this presentation a few more existing ambiguities in case we still continue with the 20^{th} century mathematics and physics.**

For instance, one striking example is that of the following unresolved paradox, the Loschmidt paradox (J. Loschmidt, *Sitzungsber. Kais. Akad. Wiss. Wien, Math. Naturwiss.* Classe 73, 128 -142 (1876)).

Consider the case of a non-uniform dilute monatomic gas. Its molecules follow the reversible dynamics, that is **these molecules follow the invariant to time reversal Newtonian equations**.

But when the time variation of Boltzmann-H function is computed, which involves the kinetic theory based averaging for all molecules over all velocities, the **result is its negative definite time rate**. This is the well known **Boltzmann H-theorem**.

That is the **bi-directional dynamics** of individual molecules produces **uni-directional evolution** of the system.

This computed behaviour with time of H-function is then juxtaposed with the law of monotonic increase of entropy of thermodynamics and then it is proposed that the Boltzmann constant times the negative of this H-function is equal to Clausius' entropy.

Notice that **this approach only side tracks the above paradox it is not at all its solution**.

The purpose of this presentation is to elaborate the ambiguities that result on applying the time reversal symmetry test to a good number of well known dynamical equations from fluid dynamics, discuss thermodynamical and dynamical reversibility (irreversibility) and then conclude that all the existing shortcomings get resolved if one adopts the genomathematics based dynamics[3].

[3] A. A. Bhalekar, "Geno-nonequilibrium thermodynamics. I. Background and preparative aspects," *Hadronic J.*, 2013, (In press)

Thermodynamic Reversibility

The *thermodynamic reversibility* gets mathematically described as follows,

$$
\begin{aligned}
dS = \frac{dQ}{T} \quad &\Longrightarrow \quad (-dS) = \frac{(-dQ)}{T} \Longrightarrow dS = \frac{dQ}{T} \\
dU = dQ - p\,dV \quad &\Longrightarrow \quad (-dU) = (-dQ) - p\,(-dV) \\
&\Longrightarrow \quad dU = dQ - p\,dV \\
dU = TdS - p\,dV \quad &\Longrightarrow \quad (-dU) = T(-dS) - p\,(-dV) \\
&\Longrightarrow \quad dU = TdS - p\,dV \qquad (1)
\end{aligned}
$$

Notice that dQ is the differential amount of heat exchanged by the system in the forward direction and $(-dQ)$ is that in the reverse direction, but dQ is not the state function.

Thus an equilibrium state of a closed system gets described by any two state functions out of the five offered by $dU = TdS - p\,dV$, namely U, S, V, T, p but in this list obviously dQ doesn't appear.

This is an **amazing outcome** that demonstrates the **tight hold of thermodynamic principles on the state of a system in equilibrium**[4].

This result is well tested in laboratory.

[4] See for example S. M. Blinder, *Advanced Physical Chemistry*. New York: Collier-Macmillan, September 1969

Thus a **thermodynamic reversible path** is the one on which a system can be equally carried along forward and reverse directions that guarantees the system to assume the identically same equilibrium states but in reverse order on reversing the transition.

Hence on reversing the transition the magnitudes of heat and work exchanged by the system simply change their signs without altering their magnitudes. **Notice that this thermodynamic reversibility is not a result of the act of time reversal as there is no time parameter involved**.

Dynamic Reversibility and Irreversibility

On the other hand, **all the time dependent processes are irreversible**. This is so because the past times are unattainable.

However, for such processes the traditionally derived mathematical descriptions are based on Newtonian and Hamiltonian mechanics. Obviously they involve time parameter.

This then offers an opportunity to analyze them for their behaviour against the time reversal operation. Amazingly, this exercise produces both **reversible** and **irreversible** results.

For example, Newtonian dynamics of a particle or a rigid body in motion reads as[5]:

$$m\frac{d\mathbf{u}}{dt} = \mathbf{F}, \quad \mathbf{v} = \mathbf{u} + (\mathbf{F}/m)t \tag{2}$$

where $\mathbf{u}$ and $\mathbf{v}$ are the velocities, $\mathbf{F}$ is the body force acting on the particle of mass m and t is time.

Notice that eq.(2) when describes the forward motion say from state A to state B the same equation does describe the motion from B to A following the same trajectory in the reverse order.

But this then means that one would attain the past times.

[5]G. K. Batchelor, *An Introduction to Fluid Dynamics*. Cambridge CB2 2RU: Cambridge University Press, 1967

However, the forward motion is irreversible in time and hence the corresponding equation needs to be irreversible in time. This property is not inherited in eq.(2).

Let us further illustrate it by an example.

In practice if one throws a stone vertically straight up, it travels through a succession of positions upward and while coming down it follows the same succession of heights and positions but in reverse order.

It is considered to demonstrates the dynamical reversibility of motion.

Notice that in doing so the time factor (that is it is still increasing) and the directional nature of velocity associated with the particle have been ignored.

The reversibility of Newtonian equation say, $\mathbf{v} = \mathbf{u} + \mathbf{f}\,t$ (c.f. eq.(2)), where now $\mathbf{f}$ is the per unit mass force $(= \mathbf{F}/m)$, is traditionally described as follows.

The above so-called (or deceptive) dynamic reversibility is usually considered being described well by time reversal symmetry of the preceding equation just by substituting $-t$ for t that reverses the sign of velocities,

$$\mathbf{u} = \frac{d\mathbf{r}}{dt} \xrightarrow{t \to -t} \frac{d\mathbf{r}}{d(-t)} = -\mathbf{u}. \tag{3}$$

Hence one is led to,

$$\mathbf{v} = \mathbf{u} + \mathbf{f}\, t \overset{T}{\Longrightarrow} -\mathbf{v} = -\mathbf{u} + \mathbf{f}\,(-t) \longrightarrow \mathbf{v} = \mathbf{u} + \mathbf{f}\, t \tag{4}$$

where the transformation $(\mathbf{r},\, \mathbf{u},\, t) \to (\mathbf{r},\, -\mathbf{u},\, -t)$ has been symbolically represented by $\overset{T}{\Longrightarrow}$.

As stated above the **catch in the above argument** is as follows.

- Indeed, $\mathbf{v} = \mathbf{u} + \mathbf{f}\,t$ holds good both for a motion forward and its backward.
- However, in the said upward motion the velocities have positive sign whereas during downward motion all velocities have negative sign.
- Hence, at each position the dynamical state is not identically same during upward and downward motions. Moreover, during downward motion time t never becomes $-t$ instead it still remains positive and increases continuously.
- Hence, in reality there is no time reversal at all.

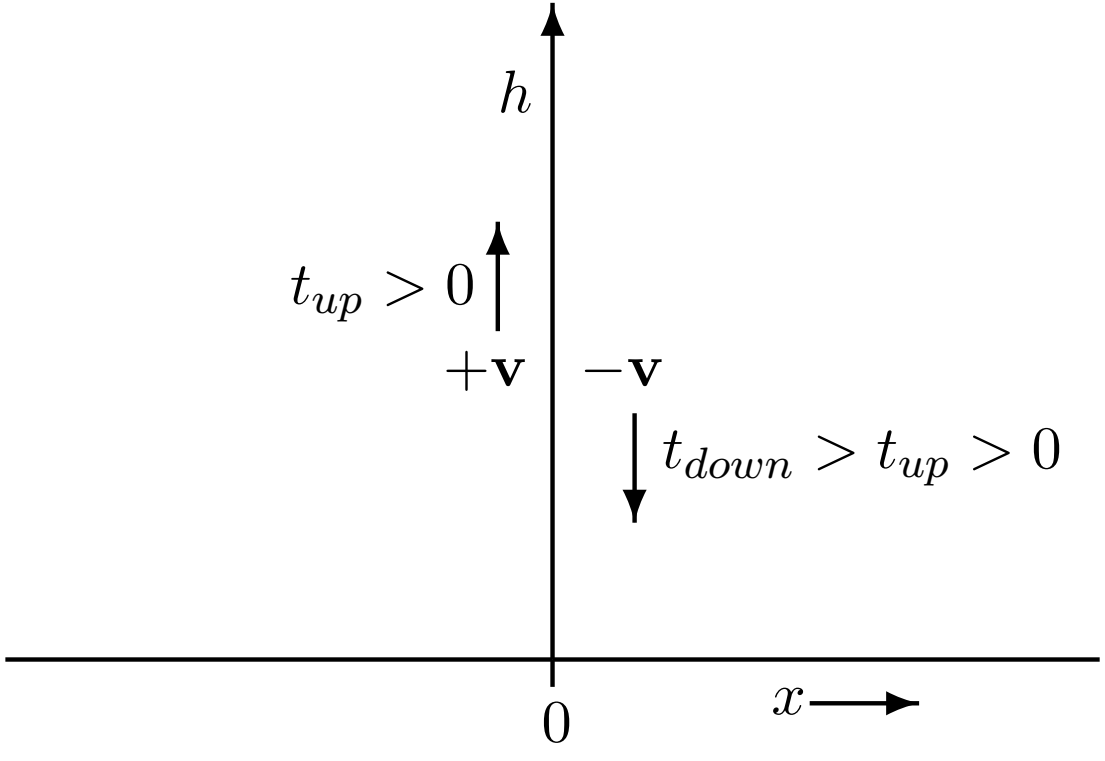

Figure 1: A schematic view of the upward and downward motion.

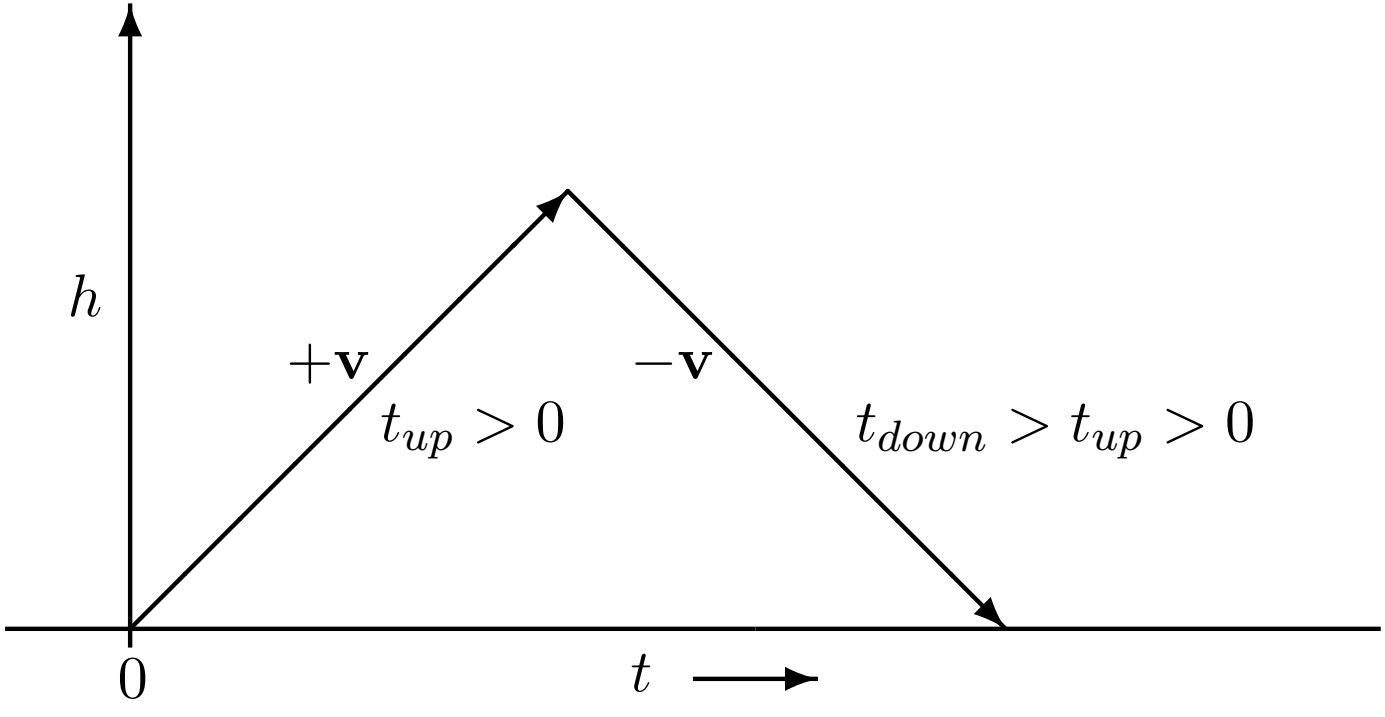

Figure 2: A schematic view of the upward and downward motion in time frame.

In Figures 1 and 2 we have represented the considered motion in two frames. One is in the spatial frame and the other in the time frame.

Notice that Figure 2 clearly illustrates that even during downward motion of the particle time increases.

Thus though in the spatial representation the downward motion appears as a backward motion but in the time frame it is indeed a forward motion as there is no time reversal.

In other words, **there is seemingly spatial reversibility of motion because time and velocity reversal have been ignored**. Of course, the motion is described throughout by one and the same equation $\mathbf{v} = \mathbf{u} + \mathbf{f}\, t$.

From the above discussion it is clear that every time dependent motion is, in actuality, **irreversible** in time frame.

However, traditionally we describe all real motions using equations on which time reversal operation can be applied.

In other words, we are using time reversible equations to describe irreversible motions.

Thus incompatible mathematical equations are being used.

On the other hand, the time reversal operation $\stackrel{T}{\Longrightarrow}$ can be executed at any stage of the motion and its result would be the reversal of motion both in space and in time. However, this is a paper and pencil operation because for the observer time is still increasing.

Or in other words, **the time reversed motion is an imaginary motion**.

That is, **the time reversal operation is not a physical reality**.

However, the time reversal operation gives an opportunity to **visualize past situations** without actually going into the past as going into the past is impossible.

Moreover, we will see below that there are **a large number of dynamic situations whose traditional mathematical descriptions when subjected to the time reversal operation produces ambiguous results**.

This then questions the **very basis of time reversal operation from its physical utility point of view**.

Precisely the above fundamental facts led Professor R. M. Santilli to discover a proper mathematics.

Following decades of systematic and sustained research Santilli[6] has achieved a **genotopic** (that is, axiom inducing) lifting of 20^{th} century mathematics with a Lie algebra structure as used for reversible systems into a **covering Lie-admissible form which is irreversible** beginning with basic multiplications and related units, today is known as **Santilli genomathematics**.

[6] R. M. Santilli, *Elements of Hadronic Mechanics, Vol. I: Mathematical Foundations*, 2nd ed. Kiev 4, 252601, Ukraine: Ukraine Academy of Sciences, Institute for Theoretical Physics, Naukova Dumka Publishers, 3 Tereshchenkivska Street, 1995, (1^{st} edition 1993)

Besides this he achieved an isotopic (that is, axiom-preserving) lifting of 20^{th} century Lie mathematics with a Lie-isotopic structure, today known as **Santilli's isomathematics.**

Finally, Santilli has discovered an anti-Hermitean map, called **isoduality**, for the description of antimatter, resulting in the novel **isodual genomathematics** and **isodual isomathematics**, that have permitted the first known initiation of thermodynamical studies for antimatter.

One of the fundamental and striking discoveries of Santilli is the **No Reduction Theorem**[7] that reads as,

[7] I. Gandzha and J. Kadisvily, *New Sciences for a New Era*. Sankata Printing Press, Kathmandu, Nepal, 2011

A classical system that is irreversible over time cannot be consistently reduced to a finite number of elementary particles all reversible over time and, vice-versa, a finite number of elementary particles all in reversible conditions cannot yield a macroscopic irreversible event under the correspondence or other principles.

1. The above theorem has far reaching implications on **bridging macroscopic and microscopic domains of physical descriptions**.
2. Santilli's new mathematics removed almost all inadequacies of all branches of physics, which were so far presented using 20^{th} century mathematics and mechanics. Hence, it is no exaggeration to state that now we have **New Sciences for A New Era**.

Time Reversal Symmetry of Dynamical Equations

Notice that we have used in introduction only an elementary equation of Newtonian mechanics to illustrate the incompatibility of it with the observed irreversibility in time frame.

However, there we encounter other inconsistencies and even ambiguities when we use fluid dynamical equations, Boltzmann integro-differential equation and Boltzmann H-theorem with respect to time reversal operation.

Let us illustrate it.

Determination of Time Reversal Symmetry. Simple Equations

Let us first apply the above test to certain basic equations from fluid dynamics[8] which are obtained as symmetric to time reversal operation.

[8](a) S. R. De Groot and P. Mazur, *Non-Equilibrium Thermodynamics*. Amsterdam: North Holland, 1962; (b) G. K. Batchelor, *An Introduction to Fluid Dynamics*. Cambridge CB2 2RU: Cambridge University Press, 1967

Relation between **convective and local time derivatives**:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \overset{T}{\Longrightarrow} \frac{d}{d(-t)} = \frac{\partial}{\partial(-t)} + (-\mathbf{u}) \cdot \nabla$$

$$\Rightarrow \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

Relation between **convective and local time derivatives of mass density**, ρ:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho \overset{T}{\Longrightarrow} \frac{d\rho}{d(-t)} = \frac{\partial \rho}{\partial(-t)} + (-\mathbf{u}) \cdot \nabla \rho$$

$$\Rightarrow \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho$$

Relation between **convective and local time derivatives of velocity vector**, $\mathbf{u}$:

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \overset{T}{\Longrightarrow} \frac{d(-\mathbf{u})}{d(-t)} = \frac{\partial(-\mathbf{u})}{\partial(-t)} + (-\mathbf{u}) \cdot \nabla(-\mathbf{u})$$

$$\Rightarrow \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}$$

Eulerian equation of continuity:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u}) \overset{T}{\Longrightarrow} \frac{\partial \rho}{\partial(-t)} = -\nabla \cdot (\rho(-\mathbf{u}))$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$

Lagrangian equation of continuity:

$$\frac{d\rho}{dt} = -\rho\,\nabla\cdot\mathbf{u} \overset{T}{\Longrightarrow} \frac{d\rho}{d(-t)} = -\rho\,\nabla\cdot(-\mathbf{u})$$

$$\Rightarrow \frac{d\rho}{dt} = -\rho\,\nabla\cdot\mathbf{u}$$

Lagrangian equation of continuity in terms of specific volume, $v = \rho^{-1}$:

$$\rho\,\frac{dv}{dt} = \nabla\cdot\mathbf{u} \overset{T}{\Longrightarrow} \rho\,\frac{dv}{d(-t)} = \nabla\cdot(-\mathbf{u})$$

$$\Rightarrow \rho\,\frac{dv}{dt} = \nabla\cdot\mathbf{u}$$

Relation between Lagrangian and Eulerian dynamics of a property, say z, a per unit mass quantity:

$$\rho\frac{dz}{dt} = \frac{\partial \rho z}{\partial t} + \nabla\cdot(\rho z\mathbf{u}) \overset{T}{\Longrightarrow} \rho\frac{dz}{d(-t)} = \frac{\partial \rho z}{\partial(-t)} + \nabla\cdot(\rho z(-\mathbf{u}))$$

$$\Rightarrow \rho\frac{dz}{dt} = \frac{\partial \rho z}{\partial t} + \nabla\cdot(\rho z\mathbf{u})$$

Now assume that only the conservative body forces act upon a **rigid body in motion**, that means:

$$\mathbf{F} = -\nabla\psi, \quad \frac{\partial\psi}{\partial t} = 0, \quad \frac{\partial \mathbf{F}}{\partial t} = 0$$

Equation of motion in Eulerian form:

$$\frac{\partial \rho \mathbf{u}}{\partial t} = -\nabla \cdot (\rho \mathbf{u}\mathbf{u}) + \rho\, \mathbf{F}$$

$$\stackrel{T}{\Longrightarrow} \frac{\partial \rho(-\mathbf{u})}{\partial(-t)} = -\nabla \cdot (\rho(-\mathbf{u})(-\mathbf{u})) + \rho\, \mathbf{F}$$

$$\Rightarrow \frac{\partial \rho \mathbf{u}}{\partial t} = -\nabla \cdot (\rho \mathbf{u}\mathbf{u}) + \rho\, \mathbf{F}$$

Equation of motion in Lagrangian form:

$$\rho\frac{d\mathbf{u}}{dt} = \rho\mathbf{F} \overset{T}{\Longrightarrow} \rho\frac{d(-\mathbf{u})}{d(-t)} = \rho\mathbf{F}$$

$$\Rightarrow \rho\frac{d\mathbf{u}}{dt} = \rho\mathbf{F}$$

Kinetic energy balance equation in Eulerian form:

$$\frac{\partial\frac{1}{2}\rho\mathbf{u}^2}{\partial t} = -\nabla\cdot\left(\frac{1}{2}\rho\mathbf{u}^2\mathbf{u}\right) + \rho\,\mathbf{F}\cdot\mathbf{u}$$

$$\overset{T}{\Longrightarrow} \frac{\partial\frac{1}{2}\rho\mathbf{u}^2}{\partial(-t)} = -\nabla\cdot\left(\frac{1}{2}\rho\mathbf{u}^2(-\mathbf{u})\right) + \rho\,\mathbf{F}\cdot(-\mathbf{u})$$

$$\Rightarrow \frac{\partial\frac{1}{2}\rho\mathbf{u}^2}{\partial t} = -\nabla\cdot\left(\frac{1}{2}\rho\mathbf{u}^2\mathbf{u}\right) + \rho\,\mathbf{F}\cdot\mathbf{u}$$

Kinetic energy in Lagrangian form:

$$\rho\frac{d\frac{1}{2}\mathbf{u}^2}{dt} = \rho\,\mathbf{F}\cdot\mathbf{u} \quad \overset{T}{\Longrightarrow} \quad \rho\frac{d\frac{1}{2}\mathbf{u}^2}{d(-t)} = \rho\,\mathbf{F}\cdot(-\mathbf{u})$$

$$\Rightarrow \rho\frac{d\frac{1}{2}\mathbf{u}^2}{dt} = \rho\,\mathbf{F}\cdot\mathbf{u}$$

Potential energy in Eulerian form:

$$\frac{\partial\rho\psi}{\partial t} = -\nabla\cdot(\rho\psi\mathbf{u}) - \rho\,\mathbf{F}\cdot\mathbf{u}$$

$$\overset{T}{\Longrightarrow} \quad \frac{\partial\rho\psi}{\partial(-t)} = -\nabla\cdot(\rho\psi(-\mathbf{u})) - \rho\,\mathbf{F}\cdot(-\mathbf{u})$$

$$\Rightarrow \frac{\partial\rho\psi}{\partial t} = -\nabla\cdot(\rho\psi\mathbf{u}) - \rho\,\mathbf{F}\cdot\mathbf{u}$$

Potential energy in Lagrangian form:

$$\rho\frac{d\psi}{dt} = -\rho\,\mathbf{F}\cdot\mathbf{u} \quad \overset{T}{\Longrightarrow} \quad \rho\frac{d\psi}{d(-t)} = -\rho\,\mathbf{F}\cdot(-\mathbf{u})$$

$$\Rightarrow \rho\frac{d\psi}{dt} = -\rho\,\mathbf{F}\cdot\mathbf{u}$$

Total mechanical energy in Eulerian form:

$$\frac{\partial\rho\left(\frac{1}{2}\mathbf{u}^2+\psi\right)}{\partial t} = -\nabla\cdot\left[\rho\left(\frac{1}{2}\mathbf{u}^2+\psi\right)\mathbf{u}\right]$$

$$\overset{T}{\Longrightarrow} \quad \frac{\partial\rho\left(\frac{1}{2}\mathbf{u}^2+\psi\right)}{\partial(-t)} = -\nabla\cdot\left[\rho\left(\frac{1}{2}\mathbf{u}^2+\psi\right)(-\mathbf{u})\right]$$

$$\Rightarrow \frac{\partial\rho\left(\frac{1}{2}\mathbf{u}^2+\psi\right)}{\partial t} = -\nabla\cdot\left[\rho\left(\frac{1}{2}\mathbf{u}^2+\psi\right)\mathbf{u}\right]$$

Total mechanical energy in Lagrangian form:

$$\rho \frac{d\left(\frac{1}{2}\mathbf{u}^2 + \psi\right)}{dt} = 0$$

All the **above equations are seen as time reversal symmetric** but as said above the question still remains that of its physical utility.

Now in the next subsection we will examine more involved fluid dynamical equation for time reversal symmetry and we would see that it produces **ambiguous results**.

Determination of Time Reversal Symmetry. Equations Producing Ambiguous Results

Total energy, e (per unit mass), in Eulerian form:

$$\frac{\partial \rho e}{\partial t} = -\nabla \cdot (\rho e \mathbf{u}) - \nabla \cdot \mathbf{J}_e$$

$$\stackrel{T}{\Longrightarrow} \frac{\partial \rho e}{\partial (-t)} = -\nabla \cdot (\rho e(-\mathbf{u})) - \nabla \cdot (-\mathbf{J}_e)$$

$$\Rightarrow \frac{\partial \rho e}{\partial t} = -\nabla \cdot (\rho e \mathbf{u}) - \nabla \cdot \mathbf{J}_e$$

Notice that the the conductive flux, $\mathbf{J}_e$, has been assumed to reverse its sign on time reversal as it is a flux.

However, we will see below while discussing corresponding internal energy balance equation that the time reversal symmetry or antisymmetry for the heat flux density is not an unambiguous choice and hence the same remains true of $\mathbf{J}_e$ as it is none else than the heat flux density.

Total energy in Lagrangian form:

$$\rho\frac{de}{dt} = -\nabla\cdot\mathbf{J}_e \overset{T}{\Longrightarrow} \rho\frac{de}{d(-t)} = -\nabla\cdot(-\mathbf{J}_e)$$

$$\Rightarrow \rho\frac{de}{dt} = -\nabla\cdot\mathbf{J}_e$$

The internal energy, u (per unit mass), is given by:

$$u = e - \left(\frac{1}{2}\mathbf{u}^2 + \psi\right)$$

Hence, **internal energy in Eulerian form** reads as:

$$\frac{\partial \rho u}{\partial t} = \frac{\partial \rho \left(e - \left(\frac{1}{2}\mathbf{u}^2 + \psi\right)\right)}{\partial t} = -\nabla \cdot (\rho u \mathbf{u} + \mathbf{J}_u)$$

where the convective flux of internal energy gets quantified as: $\rho u \mathbf{u} = \rho \left(e - \left(\frac{1}{2}\mathbf{u}^2 + \psi\right)\right) \mathbf{u}$ whereas in the present case there is no conductive flux of mechanical energy hence $\mathbf{J}_e = \mathbf{J}_u$.

Therefore, the time reversal operation produces,

$$\frac{\partial \rho u}{\partial t} = -\nabla \cdot (\rho u \mathbf{u} + \mathbf{J}_u) \overset{T}{\Longrightarrow} \frac{\partial \rho u}{\partial (-t)} = -\nabla \cdot (\rho u(-\mathbf{u}) + (-\mathbf{J}_u))$$

$$\Rightarrow \frac{\partial \rho u}{\partial t} = -\nabla \cdot (\rho u \mathbf{u} + \mathbf{J}_u)$$

Notice that $\mathbf{J}_u$ also has been assumed to change its sign on time reversal as it is none else but the $\mathbf{J}_e$ in the present case.

The **Lagrangian form of internal energy balance equation** in the present case reads as:

$$\rho \frac{du}{dt} = -\nabla \cdot \mathbf{J}_u = -\nabla \cdot \mathbf{q}$$

Thus $\mathbf{J}_u$ is nothing else but the heat flux density, $\mathbf{q}$.

Now we have,

$$\rho \frac{du}{dt} \overset{T}{\Longrightarrow} \rho \frac{du}{d(-t)} = -\rho \frac{du}{dt}$$

it implies that: $-\nabla \cdot \mathbf{q} \overset{T}{\Longrightarrow} -(-\nabla \cdot \mathbf{q})$

this can happen only when: $\mathbf{q} \overset{T}{\Longrightarrow} -\mathbf{q}.$

Moreover, in kinetic theory[9] the expression of heat flux is $\mathbf{q} = \overline{\rho \frac{1}{2} \mathbf{C}^2 \mathbf{C}}$ for a non-uniform monatomic gas, where $\mathbf{C}(= \mathbf{c} - \mathbf{u})$ is the chaotic velocity of a molecule and $\mathbf{c}$ is the molecular velocity. In view of this definition $\mathbf{C}$ would change its sign on time reversal as the sign of both $\mathbf{c}$ and $\mathbf{u}$ changes on time reversal. Therefore, $\mathbf{q} = \overline{\rho \frac{1}{2} \mathbf{C}^2 \mathbf{C}}$ would change it sign on time reversal.

[9](a) S. Chapman and T. G. Cowling, *Mathematical Theory of Non-uniform Gases: An Account of the Kinetic Theory of Viscosity, Thermal Conduction and Diffusion in Gases*, 3rd ed. Cambridge: Cambridge University Press, 1970; (b) H. Grad, "Priciples of the kinetic theory of gases," in *Handbuch der Physik*, S. Flügge, Ed., vol. XII. Berlin: Springer-Verlag, 1958, pp. 205 – 294; (c) L. S. García-Colín, *Teoría Cinética de Los Gases*. Colección CBI, Universidad Autónoma Metropolitana - Iztapalapa, 1990, vol. 2.

This is mathematically illustrated below,

$$\mathbf{q} = \overline{\rho\frac{1}{2}\mathbf{C}^2\mathbf{C}} \overset{T}{\Longrightarrow} \mathbf{q}' = \overline{\rho\frac{1}{2}(-\mathbf{C})^2(-\mathbf{C})} = -\overline{\rho\frac{1}{2}\mathbf{C}^2\mathbf{C}} = -\mathbf{q}$$

Thus $\mathbf{q}$ is obtained as time reversal antisymmetric.

That is **on time reversed path heat flux has opposite sign**.

Now let us consider Maxwell-Cattaneo-Vernotte (MCV) equation[10], namely:

$$\tau \frac{d\mathbf{q}}{dt} + \mathbf{q} = -\lambda \nabla T$$

where τ is the relaxation time and λ is the thermal conductivity.

However, as heat flux reverses its direction on time reversal the MCV equation is obtained as not time invariant on time reversal, as shown below:

[10](a) D. D. Joseph and L. Preziosi, "Heat waves," *Rev. Mod. Phys.*, vol. 61, no. 1, pp. 41 – 74, 1989; (b) ——, "Addendum to the paper "heat waves" [Rev. Mod. Phys. 61, 41 (1989)]," *Rev. Mod. Phys.*, vol. 62, no. 2, pp. 375 – 391, 1990

$$\tau\frac{d\mathbf{q}}{dt}+\mathbf{q}=-\lambda\nabla T \overset{T}{\Longrightarrow} \tau\frac{d(-\mathbf{q})}{d(-t)}+(-\mathbf{q})=-\lambda\nabla T$$

$$\Rightarrow \tau\frac{d\mathbf{q}}{dt}-\mathbf{q}=-\lambda\nabla T$$

However, the MCV equation may be considered a description that describes approach to corresponding nonequilibrium stationary state (NSS) and hence at NSS $\left(\frac{d\mathbf{q}}{dt}=0\right)$ MCV equation correctly produces the Fourier law, $\mathbf{q}=-\lambda\nabla T$.

While on the time reversed path we see from above that the Fourier law is not recovered if $\mathbf{q}$ has changed its sign. Therefore, **does it indicate that $\mathbf{q}$ is time reversal invariant**? Even if this is accepted the MCV equation is not obtained as time reversal invariant,

$$\tau\frac{d\mathbf{q}}{dt}+\mathbf{q}=-\lambda\nabla T \overset{T}{\Longrightarrow} \tau\frac{d(\mathbf{q})}{d(-t)}+(\mathbf{q})=-\lambda\nabla T$$

$$\Rightarrow\ -\tau\frac{d\mathbf{q}}{dt}+\mathbf{q}=-\lambda\nabla T$$

Furthermore, in classical irreversible thermodynamics (CIT)[11] the entropy source strength, σ_s, due to heat transfer is quantified as $\sigma_s = \mathbf{q} \cdot \nabla T^{-1} > 0$. Thus if on time reversal $\mathbf{q}$ changes its sign then the second law of thermodynamics appears to stand violated on the time reversed path. Whereas, in extended irreversible thermodynamics (EIT)[12] we have $\sigma_s = \beta_{\mathbf{q}} \mathbf{q}^2 > 0$ hence even if $\mathbf{q}$ changes sign on time reversed path the second law is not violated.

[11] S. R. De Groot and P. Mazur, *Non-Equilibrium Thermodynamics*. Amsterdam: North Holland, 1962

[12] D. Jou, J. Casas-Vázquez, and G. Lebon, *Extended Irreversible Thermodynamics*, 2nd ed. Berlin: Springer-Verlag, 1996

From the above elaboration **one may perhaps assume that, in general $\mathbf{q}$ is not time reversal symmetric and hence the MCV equation is not time reversal symmetric**.

But this is not the unambiguous option because as stated above **if we consider the corresponding expression of entropy source strength of CIT, on time reversal the second law of thermodynamics gets violated as heat flux changes its sign on time reversed path. Whereas in the case of EIT given expression of entropy source strength the time reversal does not violate the second law of thermodynamics**. Thus there does exist an ambiguity.

Now let us proceed to describe a few more fluid dynamical equations where no unambiguous conclusion about the time reversal symmetry can be made.

A Viscous Fluid in Motion:
In this case the **Eulerian form of momentum balance equation** is:

$$\frac{\partial \rho \mathbf{u}}{\partial t} = -\nabla \cdot (\rho \mathbf{u}\mathbf{u} - \mathcal{T}) + \rho \mathbf{F}$$

where $\mathcal{T}$ is the stress tensor, which measures the non-convective momentum flux density and alternately measures the normal and tangential forces per unit area. To be specific, the component, T_{ij}, of the stress tensor quantify the following:

$$
\begin{aligned}
T_{ij} &= (i - \text{component of force per unit area} \\
&\quad \text{across a surface perpendicular to } \mathbf{e}_j) \\
&= (i - \text{component of momentum that crosses} \\
&\quad \text{a unit area which is perpendicular to } \mathbf{e}_j, \text{per unit} \\
&\quad \text{time, with the crossing being from } -x^j \text{ to } +x^j)
\end{aligned}
$$

The corresponding equation in Lagrangian form reads as:

$$\rho \frac{d\mathbf{u}}{dt} = \nabla \cdot \mathcal{T} + \rho \mathbf{F}$$

With the above physical explanations of $\mathcal{T}$ we are led to the following conclusions about its time reversal symmetry.

- Now as the components of $\mathcal{T}$ represent forces then on time reversal it should not change sign. Whereas its interpretation as a momentum flux suggests that it should change its sign on time reversal.
- In kinetic theory[13] we have $\mathcal{T} = \overline{\rho \mathbf{C}\mathbf{C}}$ hence as $\mathbf{C}$ changes sign on time reversal the sign of $\mathcal{T}$ would not change.

Thus we see that **there is no unambiguous way to decide the symmetry property of $\mathcal{T}$ on time reversal operation**.

[13] S. Chapman and T. G. Cowling, *Mathematical Theory of Non-uniform Gases: An Account of the Kinetic Theory of Viscosity, Thermal Conduction and Diffusion in Gases*, 3rd ed. Cambridge: Cambridge University Press, 1970

In the premise of thermodynamics one uses fluid dynamical version of the first law of thermodynamics hence let us consider it,

$$\rho\frac{du}{dt} = -\nabla\cdot\mathbf{q} - p\rho\frac{dv}{dt} + \mathbf{\Pi} : \nabla\mathbf{u} + \sum_k \mathbf{J}_k \cdot \mathbf{F}_k$$

where $\mathbf{J}_k$ is the diffusion flux density of the component k and $\mathbf{F}_k$ is the body force per unit mass exerted on the chemical component k. Notice that on the right hand side of the preceding equation there is a term that contains $\mathbf{\Pi}$ in it, in view of the above discussion, the time reversal symmetry of the preceding equation remains ambiguous.

On the other hand, the first law of thermodynamics for spatially uniform systems in nonequilibrium reads as,

$$\frac{dU}{dt} = \frac{dQ}{dt} - p\frac{dV}{dt}$$

and it is seen that this equation is symmetric to time reversal (as $dU,\, dQ,\, dV \overset{T}{\Longrightarrow} -dU,\, -dQ,\, -dV$ and $t \overset{T}{\Longrightarrow} -t$).

In view of the above illustration, **one would then use, in nonequilibrium thermodynamics, two different types of equations representing the first law of thermodynamics — one symmetric to time reversal and the other ambiguous on this count**. It would be then a bad commentary on thermodynamics.

Thermodynamic Irreversibility

The origin of the description of thermodynamic irreversibility is in the second law of thermodynamics. At the **global level of consideration** it is stated as follows. **The entropy of a system can change on two and only two counts. The one is by the exchange of entropy, which can be positive or negative and the second is by the production of entropy, which is positive definite**. Mathematically it is stated as[14]:

$$\frac{dS}{dt} = \frac{d_eS}{dt} + \frac{d_iS}{dt}, \quad \frac{d_eS}{dt} \gtrless 0, \quad \frac{d_iS}{dt} \geq 0$$

[14] I. Prigogine and R. Defay, *Chemical Thermodynamics*. London: Longmans Green, 1954, translated by D. H. Everett

where S is the entropy of the system, $\frac{d_e S}{dt}$ is the rate of exchange of entropy which can be positive or negative and $\frac{d_i S}{dt}$ is the rate of entropy production which is always positive definite. Whereas at the **local level of consideration** it is stated as **Clausius-Duhem inequality**[15]:

$$\rho \frac{ds}{dt} + \nabla \cdot \mathbf{J}_s = \sigma_s \geq 0$$

[15](a) W. Muschik, "Formulations of the second law - Recent developments," *J. Phys. Chem. Solids*, vol. 49, no. 6, pp. 709 – 720, 1988, (b) ——, *Six Lectures on Fundamentals and Method: Aspects of Non-Equilibrium Thermodynamics*, 1st ed., ser. Series in Theoretical and Applied Mechanics, R. K. T. Hsieh, Ed. Singapore: World Scientific, 1990, no. 9

where s is per unit mass entropy, $\mathbf{J}_s$ is the entropy flux density and σ_s is the entropy source strength.

- The existence of $\mathbf{J}_s$ and σ_s at the local level implies the **existence of thermodynamic irreversibility**.
- However, for the **existence of thermodynamic irreversibility** the non-zero and positive definite σ_s at the local level or $\frac{d_i S}{dt}$ at the global level is the necessary and sufficient requirement.
- Moreover, at every stage of evolution of the system **the second law of thermodynamics guarantees the non-zero and positive definite values of σ_s at the local level and positive definite non-zero value of $\frac{d_i S}{dt}$ at the global level**.

- It, therefore, means that **even at a past stage of evolution** of a thermodynamic system σ_s and $\frac{d_i S}{dt}$ **were positive definite**.
- Hence, **if the time reversal implies leading us to past times** then σ_s and $\frac{d_i S}{dt}$ **cannot change their sign**.
- That is, σ_s and $\frac{d_i S}{dt}$ **remain time reversal invariant**,

$$\frac{dS}{dt} = \frac{d_e S}{dt} + \frac{d_i S}{dt} \overset{T}{\Longrightarrow} -\frac{dS}{dt} = -\frac{d_e S}{dt} + \frac{d_i S}{dt}$$

$$\rho \frac{ds}{dt} + \nabla \cdot \mathbf{J}_s = \sigma_s \overset{T}{\Longrightarrow} -\rho \frac{ds}{dt} - \nabla \cdot \mathbf{J}_s = \sigma_s$$

- Because the entropy flux $\mathbf{J}_s$ changes its sign on time reversal it implies that the time reversed path so obtained is not identically the same one that was traversed in forward direction.
- Thus attaining of the past states through time reversal is not achieved.
- We see that the **dynamic irreversibility demonstrated by time reversal operation of entropy balance equations** is the result of **thermodynamic irreversibility** (invariance of entropy production term on time reversal).

On the other hand, at a **nonequilibrium stationary state** (NSS) from the corresponding above two equations we have,

$$-\frac{d_e S}{dt} = \frac{d_i S}{dt} = constant > 0$$

and

$$\nabla \cdot \mathbf{J}_s = \sigma_s = constant > 0$$

That is there we have non-zero positive definite values of σ_s at the local level and $\frac{d_i S}{dt}$ at the global level.

Thus as **a NSS is the time independent state one should not apply the time reversal operation on equation describing it** although the entropy is being produced and exchanged at a constant rate but compensating each other.

Let us examine the standard nonequilibrium thermodynamic expressions of entropy production and entropy exchange. In CIT[16] we have the following equation of entropy balance,

$$\rho\frac{ds}{dt} = -\nabla\cdot\left(\frac{\mathbf{q}-\sum_k \mu_k\mathbf{J}_k}{T}\right)+\mathbf{q}\cdot\nabla\left(\frac{1}{T}\right)+T^{-1}\mathbf{\Pi}:\nabla\mathbf{u}$$
$$-\frac{1}{T}\sum_k\mathbf{J}_k\cdot\left(T\,\nabla\left(\frac{\mu_k}{T}\right)-\mathbf{F}_k\right)+\sum_\gamma\frac{\mathcal{A}^\gamma}{T}\frac{d\xi^\gamma}{dt}$$

[16] S. R. De Groot and P. Mazur, *Non-Equilibrium Thermodynamics*. Amsterdam: North Holland, 1962

where the entropy flux, $\mathbf{J}_s$, and entropy source strength, σ_s in this case are given by,

$$\mathbf{J}_s = \left(\frac{\mathbf{q} - \sum_k \mu_k \mathbf{J}_k}{T} \right) \gtrless 0$$

$$\begin{aligned} \sigma_s &= \mathbf{q} \cdot \nabla \left(\frac{1}{T} \right) + T^{-1} \mathbf{\Pi} : \nabla \mathbf{u} \\ &\quad - \frac{1}{T} \sum_k \mathbf{J}_k \cdot \left(T \nabla \left(\frac{\mu_k}{T} \right) - \mathbf{F}_k \right) + \sum_\gamma \frac{\mathcal{A}^\gamma}{T} \frac{d\xi^\gamma}{dt} \geq 0 \end{aligned}$$

On time reversal, for example, we will have,

$$-\rho\frac{ds}{dt} = +\nabla\cdot\left(\frac{\mathbf{q}-\sum_k \mu_k\mathbf{J}_k}{T}\right) - \mathbf{q}\cdot\nabla\left(\frac{1}{T}\right) + T^{-1}\mathbf{\Pi}:\nabla\mathbf{u}$$

$$+\frac{1}{T}\sum_k \mathbf{J}_k\cdot\left(T\,\nabla\left(\frac{\mu_k}{T}\right)-\mathbf{F}_k\right) + \sum_\gamma \frac{\mathcal{A}^\gamma}{T}\frac{d\xi^\gamma}{dt}$$

$$\mathbf{J}'_s = -\mathbf{J}_s = -\left(\frac{\mathbf{q}-\sum_k \mu_k\mathbf{J}_k}{T}\right) \geqslant 0$$

$$\begin{aligned}\sigma_s' &= -\mathbf{q}\cdot\nabla\left(\frac{1}{T}\right)+T^{-1}\mathbf{\Pi}:\nabla\mathbf{u}\\ &\quad+\frac{1}{T}\sum_k\mathbf{J}_k\cdot\left(T\,\nabla\left(\frac{\mu_k}{T}\right)-\mathbf{F}_k\right)+\sum_\gamma\frac{\mathcal{A}^\gamma}{T}\frac{d\xi^\gamma}{dt}\neq\sigma_s\end{aligned}$$

Thus we see that on time reversal except viscous dissipation and chemical reaction terms all terms are assumed to reverse their sign. Of course, we could have assumed invariance of $\mathbf{\Pi}$ on time reversal on considering it consisting of components of force. Thus it leads to reversing of the sign of corresponding contribution to entropy source strength, that contradicts the second law of thermodynamics. Let us also consider why we have time reversal invariance of the chemical reaction term. This we illustrate by considering that there proceeds a single reaction only.

Let us consider a chemical reaction $A \underset{k_b}{\overset{k_f}{\rightleftharpoons}} B$ with k_f and k_b are the forward and backward rate constants and A and B are the reactant and the product respectively. For this chemical reaction the chemical affinity $\mathcal{A}$ reads as,

$$\mathcal{A} = \mu_A - \mu_B$$

where μ_A and μ_B are the respective chemical potentials, and rate of reaction $\dfrac{d\xi}{dt}$ based on the law of mass action is given by,

$$\frac{d\xi}{dt} = k_f C_A - k_b C_B$$

where C_A and C_B are the instantaneous concentrations of A and B.

Now on time reversal none of the terms μ_A, μ_B, k_f, k_b, C_A and C_B change their sign nor can instantly change their numerical values.

Hence on time reversal neither $\mathcal{A}$ nor $\frac{d\xi}{dt}$ change their sign.

Hence chemical reaction doesn't get reversed on time reversal.

This explains how second law of thermodynamics remains valid in this case even after time reversal.

Perhaps, the same behaviour should have been shown by the other terms of entropy source strength that originate from spatial non-uniformity.

This difference in behaviour suggests that **CIT has erred in admitting spatial non-uniformity in the domain of the applicability of Gibbs relation**, which has been postulated as a prescription of local equilibrium.

On the other hand, in EIT[17] corresponding expression of σ_s to the first approximation (in absence of body forces and assuming MCV type of constitutive equations for diffusion fluxes) is,

$$\sigma_s = \beta_{\mathbf{q}}\mathbf{q}\cdot\mathbf{q} + \beta_{\mathbf{\Pi}}\mathbf{\Pi}:\mathbf{\Pi} + \sum_k \beta_{\mathbf{J}_k}\mathbf{J}_k\cdot\mathbf{J}_k + \sum_\gamma \frac{\mathcal{A}^\gamma}{T}\frac{d\xi^\gamma}{dt} \geq 0$$

Thus we see that on time reversal, irrespective of fluxes change their sign or not, the first three terms on the right hand side of entropy source strength do not change their sign. Also as demonstrated above the Chemical reaction term is time reversal invariant.

[17] D. Jou, J. Casas-Vázquez, and G. Lebon, *Extended Irreversible Thermodynamics*, 2nd ed. Berlin: Springer-Verlag, 1996

Hence, if we use EIT expression of entropy source strength the second law of thermodynamics is not flouted on the time reversed path.

Time Reversal Symmetry of Boltzmann Equation and Boltzmann's H-theorem

The Boltzmann integro-differential equation for monatomic gas on neglecting body forces reads as[18],

$$\frac{\partial f}{\partial t} + \mathbf{c} \cdot \frac{\partial f}{\partial \mathbf{r}} = J(f|f)$$

where f is the distribution function corresponding to the molecular velocity $\mathbf{c}$, $\mathbf{r}$ is the position vector and $J(f|f)$ is the Boltzmann collisional integral given by:

$$J(f|f) = \int \int (f_1' f' - f_1 f) g \sigma(g, \Omega) d\Omega d\mathbf{c}_1$$

[18] R. L. Liboff, *Kinetic Theory: Classical, Quantum, and Relativistic Descriptions*, ser. Graduate Texts in Contemporary Physics. New York: Springer, 2003

where $\sigma(g, \Omega)$ is the collision cross section that depends on the relative velocity $g = |\mathbf{c}_1 - \mathbf{c}|$ and the scattering solid angle Ω. The primes and indices in the distribution function have the usual meaning, namely, $f_1' = f(\mathbf{c}_1', \mathbf{r}, t)$ is the distribution function of particle 1 after collision, etc.

The left hand side of the Boltzmann equation changes sign on time reversal operation, namely:

$$\frac{\partial f}{\partial t} + \mathbf{c} \cdot \frac{\partial f}{\partial \mathbf{r}} \overset{T}{\Longrightarrow} \frac{\partial f}{\partial (-t)} + (-\mathbf{c}) \cdot \frac{\partial f}{\partial \mathbf{r}} \Rightarrow -\left(\frac{\partial f}{\partial t} + \mathbf{c} \cdot \frac{\partial f}{\partial \mathbf{r}} \right)$$

but the right hand side of Boltzmann equation is obtained as time reversal invariant, namely:

$$J(f|f) = \int\int (f_1' f' - f_1 f) g\sigma(g,\Omega) d\Omega d\mathbf{c}_1 \stackrel{T}{\Longrightarrow}$$

$$J(f|f) = \int\int (f_1' f' - f_1 f) g\sigma(g,\Omega) d\Omega d\mathbf{c}_1$$

because it does not contain t explicitly. **Thus it appears that the Boltzmann integro-differential equation is obtained as antisymmetric to time reversal**.

So far so good but let us analyze it further. On instantaneous time reversal all molecules would reverse their motion then the distribution functions before collision should become those of after collision and vice versa.

Then we would have $(f_1' f' - f_1 f) \stackrel{T}{\Longrightarrow} (f_1 f - f_1' f') \Rightarrow -(f_1' f' - f_1 f)$. However, the effecting of this time reversal operation does imply $f(\mathbf{c}, \mathbf{r}, t) = f(-\mathbf{c}, \mathbf{r}, -t)$. With this insight the **Boltzmann equation is obtained as time reversal invariant**.

But in essence the time reversal operation means the substitution of $-t$ for t without any alteration in the equation. Thus we see that both the options of time reversal operation have their own shortcomings.

With these two options for time reversal symmetry of Boltzmann integro-differential equation, now let us examine the famous Boltzmann's H-theoem.

Boltzmann defined kinetic theory based H function[19] whose negative is the kinetic theory definition of entropy function[20]. It reads as,

$$\rho\, s = -k_B \int f(\ln f - 1) d\mathbf{c}$$

where k_B is the Boltzmann constant.

[19](a) S. Chapman and T. G. Cowling, *Mathematical Theory of Non-uniform Gases: An Account of the Kinetic Theory of Viscosity, Thermal Conduction and Diffusion in Gases*, 3rd ed. Cambridge: Cambridge University Press, 1970

[20]Actually the H-function has been defined by Boltzmann as,

$$H = \int f\, \ln f d\mathbf{c}$$

which was later found to relate with the entropy function given in the main text.

On differentiation of above equation with respect to time and further effecting the standard mathematical manipulations produces,

$$\frac{\partial \rho s}{\partial t} = -div\left(\rho s \mathbf{u} + \mathbf{J}_s\right) + \sigma_s$$

where the entropy flux density, $\mathbf{J}_s$, is given by,

$$\mathbf{J}_s = -k_B \int \mathbf{C} f \left(\ln f - 1\right) d\mathbf{c}$$

where $\mathbf{C}$ is the molecular peculiar or chaotic velocity, and the entropy source strength, σ_s, is given by,

$$\sigma_s = -k_B \int J(f|f) \ln f d\mathbf{c}$$

The entropy source strength, σ_s, is further mathematically manipulated to produce,

$$\sigma_s = \frac{1}{4} k_B \int\int\int \left(\ln \frac{f' f'_1}{f f_1} \right) \left(f'_1 f' - f_1 f \right) g^3 \sigma(g, \Omega) d\Omega d\Omega' dg d\mathbf{c}_{(1)} \geq 0$$

where $\mathbf{c}_{(1)}$ is the center of mass velocity of the colliding molecular pair. The positive definiteness shown in the preceding equation stems from the fact that if $\ln \frac{f' f'_1}{f f_1} > 0$ then we have $(f'_1 f' - f_1 f) > 0$ and if the former is < 0 then the latter is also < 0 the rest of the terms of the integrands are positive numbers by definition. This is the **Boltzmann's H-theorem**[21].

[21] S. R. De Groot and P. Mazur, *Non-Equilibrium Thermodynamics*. Amsterdam: North Holland, 1962

Now we see that the positive definiteness of the preceding equation is time reversal invariant in both the above stated options of change or no change of the roles of the distribution functions on time reversal operation.

This is a **paradoxical** situation because **irrespective of the Boltzmann equation is taken time reversal invariant or variant the H-theorem that results from it is obtained as time reversal invariant**.

Therefore, in literature it is taken as the kinetic theory version of second law of thermodynamics[22]

[22] S. R. De Groot and P. Mazur, *Non-Equilibrium Thermodynamics*. Amsterdam: North Holland, 1962

Conclusion

The above described elaborations unambiguously surface out the fact that on applying time reversal symmetry test to various fluid dynamical equations including the Boltzmann integro-differential equation we are led to the ambiguous results.

Indeed, there is a distinction between **thermodynamic and dynamic reversibility** that is in the former case the identically same states are encountered on reversing the direction because the state of the system doesn't depend on the sign of the exchange differentials.

Whereas in the latter case **the direction of a vector associated with the system changes its sign hence seemingly traverses the same path in reverse direction. But the dynamic state of a system is also determined by its associated vector.**

Hence, the time reversed paths consist of different dynamic states.

This then demands a mathematical apparatus inherited with unidirectionality of the description. **This demand is met by Santilli's genomathematics and the corresponding mechanics**.

In Santilli's genomathematics for forward and backward motions one uses different multiplying genounits namely, for forward motion it is $\hat{I}^{>}$ and for the backward motion it is ${}^{<}\hat{I}$ and $\hat{I}^{>} \neq {}^{<}\hat{I}$.

It means we have separate mathematical descriptions for forward and backward motions as they in reality are not reverse of each other.

That is no time reversal symmetry test is required.

Hence, we conclude that the Santilli Lie-admissible mechanics or Santilli-Newtonian mechanics[23] turns out as the **only option for mathematical description of irreversibility and hence for developing nonequilibrium thermodynamics too**.

[23](a) R. M. Santilli, *Elements of Hadronic Mechanics, Vol. I: Mathematical Foundations*, 2nd ed. Kiev 4, 252601, Ukraine: Ukraine Academy of Sciences, Institute for Theoretical Physics, Naukova Dumka Publishers, 3 Tereshchenkivska Street, 1995, (1^{st} edition 1993); (b) R. M. Santilli, *Foundations of Hadronic Chemistry. With Applications to New Clean Energies and Fuels*. P. O. Box 17, 3300 AA Dordrecht, The Netherlands: Kluwer Academic Publishers, Dordrecht/Boston/London, 2001; (c) I. Gandzha and J. Kadisvily, *New Sciences for a New Era*. Sankata Printing Press, Kathmandu, Nepal, 2011

New Mathematics

In geno-space there is no need to test time reversal symmetry of equations as in this setup all equations are generated as unidirectional in conformity with time's arrow associated with irreversible processes. **My preliminary attempt as well as a short paper on these lines** have recently appeared[24].

[24](a) A. A. Bhalekar, "A preliminary attempt on the reframing of thermodynamics of irreversible processes using Santilli's hadronic mechanics," *Hadronic J.*, vol. 35, no. 2, pp. 221–236, April 2012, (b) ——, "On the geno-GPITT framework," in *Numerical Analysis and Applied Mathematics ICNAAM 2012: Post Ph. D. Seminar Course on Hadronic Mechanics, September 19-25*, ser. AIP Conference Proceedings, T. Simos, G. Psihoyios, C. Tsitouras, and Z. Anastassi, Eds., vol. 1479, no. 1. KOS, Greece: American Institute of Physics Press, February 28 2012, pp. 1002–1005

Geno-mathematics

The main idea of genomathematics is the selection of **two different generalized units** called **genounits**,

- the first $\hat{I}^{>}$ for the **ordered multiplication to the right** $A > B$, called **forward genoproduct**, and
- the second ${}^{<}\hat{I}$ for the **ordered multiplication to the left** $A < B$, called **backward genoproduct**,

according to the general rules[25]:

[25](a) R. M. Santilli, "On a possible Lie-admissible covering of the Galilei relativity in Newtonian mechanics for non-conservative and Galilei form-noninvariant systems," *Hadronic J.*, vol. 1, pp. 223 – 423, 1978; (b) ——, "Invariant Lie-admissible formulation of quantum deformations," *Foundation of Physics*, vol. 27, no. 8, pp. 1159 – 1177, 1997

$$\hat{I}^{>} = 1/\hat{S},\ A > B = A \times \hat{S} \times B,\ \hat{I}^{>} > A = A > \hat{I}^{>} = A, \quad (5)$$

$$^{<}\hat{I} = 1/\hat{R},\ A < B = A \times \hat{R} \times B,\ ^{<}\hat{I} < A = A <^{<} \hat{I} = A, \quad (6)$$

where $\hat{S}$ and $\hat{R}$ are respective **genotopic elements**, that is for the forward and backward multiplications respectively.

$$A = A^{\dagger},\ B = B^{\dagger},\ \hat{R} = \hat{S}^{\dagger}, \text{ and } \hat{R} \neq \hat{S} \quad (7)$$

hence we have by definition:

$$\hat{I}^{>} \neq {}^{<}\hat{I}$$

The **genounits** are conveniently generated from the isounit, $\hat{I}$, as follows,

$$\hat{I}^{>} = V \times \hat{I} \times V^{\dagger} > 0, \quad {}^{<}\hat{I} = W \times \hat{I} \times W^{\dagger} > 0 \quad (8)$$

Hence, we have,

$$\begin{aligned} \hat{S} &= (V \times \hat{I} \times V^{\dagger})^{-1} = V^{-1} \times \hat{I}^{-1} \times (V^{\dagger})^{-1} \\ &= V^{-1} \times \hat{T} \times (V^{\dagger})^{-1} = V^{-1} \hat{\times} (V^{\dagger})^{-1} > 0 \end{aligned} \tag{9}$$

Similarly,

$$\begin{aligned} \hat{R} &= (W \times \hat{I} \times W^{\dagger})^{-1} = W^{-1} \times \hat{I}^{-1} \times (W^{\dagger})^{-1} \\ &= W^{-1} \times \hat{T} \times (W^{\dagger})^{-1} = W^{-1} \hat{\times} (W^{\dagger})^{-1} > 0 \end{aligned} \tag{10}$$

That is we have opted,

$$V \times V^{\dagger} \neq 1, \quad W \times W^{\dagger} \neq 1$$

The **genotopic lifting** of *iso-multiplication of iso-numbers* get expressed as,

$$\begin{aligned}\hat{A}\hat{\times}\hat{B} \quad \longrightarrow \quad & V \times \left(\hat{A}\hat{\times}\hat{B}\right) \times V^{\dagger} \\ & = V \times \hat{A} \times \times V^{\dagger} \times (V^{\dagger})^{-1}\hat{\times}(V)^{-1} \times V \times \hat{B} \times \times V^{\dagger} \\ & = \hat{A}^{>} \times V^{-1} \times \hat{T} \times (V^{\dagger})^{-1} \times \hat{B}^{>} \\ & = \hat{A}^{>} \times \hat{S} \times \hat{B}^{>} = \hat{A}^{>}\hat{\times}^{>}\hat{B}^{>} \qquad (11)\end{aligned}$$

and

$$\begin{aligned}\hat{A}\hat{\times}\hat{B} \quad \longrightarrow \quad & W \times \left(\hat{A}\hat{\times}\hat{B}\right) \times W^{\dagger} \\ & = W \times \hat{A} \times \times W^{\dagger} \times (W^{\dagger})^{-1}\hat{\times}(W)^{-1} \times W \times \hat{B} \times \times W^{\dagger} \\ & = {}^{<}\hat{A} \times W^{-1} \times \hat{T} \times (W^{\dagger})^{-1} \times {}^{<}\hat{B} \\ & = {}^{<}\hat{A} \times \hat{R} \times {}^{<}\hat{B} = {}^{<}\hat{A}\,{}^{<}\hat{\times}\,{}^{<}\hat{B} \end{aligned} \tag{12}$$

Thus we see that,

$$\hat{\times}^{>} = \times\hat{S}\times = \times V^{-1} \times \hat{T} \times (V^{\dagger})^{-1}\times \tag{13}$$

$${}^{<}\hat{\times} = \times\hat{R}\times = \times W^{-1} \times \hat{T} \times (W^{\dagger})^{-1}\times \tag{14}$$

The **genotopic lifting** of an *iso-quotient of iso-numbers* reads as,

$$\begin{aligned}\hat{A}\hat{\div}\hat{B} \quad \longrightarrow \quad & V\times(\hat{A}\hat{\div}\hat{B})\times V^{\dagger}=V\times\left(\frac{\hat{A}}{\hat{B}}\times\hat{I}\right)\times V^{\dagger}\\ &=\frac{V\times\hat{A}\times V^{\dagger}}{V\times\hat{B}\times V^{\dagger}}\times V\times\hat{I}\times V^{\dagger}\\ &=\frac{\hat{A}^{>}}{\hat{B}^{>}}\times\hat{I}^{>}=\hat{A}^{>}\,\hat{\div}^{>}\,\hat{B}^{>} \qquad (15)\end{aligned}$$

or alternatively,

$$\begin{aligned}\hat{A}\hat{\div}\hat{B} \quad \longrightarrow \quad & V\times(\hat{A}\hat{\div}\hat{B})\times V^{\dagger}=V\times(\hat{A}(\div\times\hat{I})\hat{B})\times V^{\dagger}\\ &=\Big((V\times\hat{A}\times V^{\dagger})\div(V\times\hat{B}\times V^{\dagger})\Big)\times V\times\hat{I}\times V^{\dagger}\\ &=\hat{A}^{>}\,\hat{\div}^{>}\,\hat{B}^{>} \qquad (16)\end{aligned}$$

That means we have for **genotopic division**,

$$\hat{\div}^{>} = \div \times \hat{I}^{>}, \quad \hat{/}^{>} = (/) \times \hat{I}^{>} \tag{17}$$

and

$$^{<}\hat{\div} = \div \times {}^{<}\hat{I}, \quad {}^{<}\hat{/} = (/) \times {}^{<}\hat{I} \tag{18}$$

- The broader genomathematics is then given by:

  The lifting of isofields $\hat{F}(\hat{a}, \hat{+}, \hat{x})$ into the **forward and backward geno-fields** $\hat{F}^{>}(\hat{a}^{>}, \hat{+}^{>}, >)$ and ${}^{<}\hat{F}({}^{<}\hat{c}, {}^{<}\hat{+}, <)$ with **forward and backward genonumbers** $\hat{a}^{>} = a \times \hat{I}^{>}$ and ${}^{<}\hat{a} = {}^{<}\hat{I} \times a$, and related operations[26];

[26] ——, "Isonumbers and genonumbers of dimension 1, 2, 4, 8, their isoduals, and pseudoduals, and "hidden numbers" of dimension 3, 5, 6, 7," *Algebras, Groups and Geometries*, vol. 10, pp. 273 – 322, 1993

- The lifting of isofunctions $\hat{f}(\hat{r})$ on $\hat{F}$ into the **forward and backward genofunctions** $\hat{f}^{>}(\hat{r}^{>})$ and ${}^{<}\hat{f}({}^{<}\hat{r})$ on $\hat{F}^{>}$ and ${}^{<}\hat{F}$, respectively, such as $(\hat{e}^{>})^{\hat{X}^{>}} = \left(e^{\hat{X}^{>}\times\hat{S}}\right) \times \hat{I}^{>}$ and ${}^{<}\hat{e}^{{}^{<}\hat{X}} = {}^{<}\hat{I} \times e^{\hat{R}\times{}^{<}\hat{X}}$, with consequential genotopies of transforms and functional analysis at large[27];

[27] ——, *Hadronic Mathematics, Mechanics and Chemistry. Vol. III: Iso-, Geno-, Hyper-Formulations for Matter and Their Isoduals for Antimatter*. Palm Harbor, FL 34682, U.S.A.: International Academic Press, 2008

- The lifting of the **isodifferential calculus** into the **forward and backward genodifferential calculus** with main forward rules $\hat{d}^{>}\hat{r}^{>k} = \hat{I}_i^{>k} \times d\hat{r}^{>i}$, $\hat{d}^{>}\hat{p}_k^{>} = \hat{S}_k^{>i} \times d\hat{p}_i^{>}$, $\hat{\partial}^{>}/\hat{\partial}^{>}\hat{r}^{>i} = \hat{S}_i^{>j} \times \partial/\partial\hat{r}^{>j}$, $\hat{\partial}^{>}/\hat{\partial}^{>}\hat{p}_k^{>} = \hat{S}_k^{>i} \times \partial/\partial\hat{p}_i^{>}$, $\hat{\partial}^{>}\hat{r}^{>i}/\hat{\partial}^{>}\hat{r}^{>j} = \hat{\delta}_j^{>i} = \delta_j^i \times \hat{I}^{>}$, etc., and corresponding backward rules easily obtainable via conjugation [refer the footnote 14 for references]
- The lifting of isotopologies, isogeometries, etc. into the dual forward and backward genotopic forms;

Some of the Examples of Geno-Equations

Some of the representative geno-equations are:

- **Forward Newton-Santilli genoequation for matter**[28]:

$$\hat{m}_a^> > \frac{\hat{d}^> \hat{v}_{ak}^>}{\hat{d}^> \hat{t}^>} = -\frac{\hat{\partial}^> \hat{V}^> (\hat{r})}{\hat{\partial}^> \hat{r}_{ak}^>}, \tag{19}$$

[28] ———, *Hadronic Mathematics, Mechanics and Chemistry. Vol. III: Iso-, Geno-, Hyper-Formulations for Matter and Their Isoduals for Antimatter*. Palm Harbor, FL 34682, U.S.A.: International Academic Press, 2008

- **Geno-internal energy balance equation**[29]:

$$\rho^{>} > \frac{d^{>}u^{>}}{d^{>}t^{>}} = -\nabla^{>} \odot^{>} \mathbf{q}^{>} - p^{>} > \rho^{>} > \frac{d^{>}v^{>}}{d^{>}t^{>}} + \mathbf{\Pi}^{>} \odot^{>} \nabla^{>} \mathbf{u}^{>} \tag{20}$$

[29] A. A. Bhalekar, "Geno-nonequilibrium thermodynamics. II. Spatially uniform, spatially non-uniform, complex and antimatter systems, and thermodynamic time's arrows," *Hadronic J.*, 2013, (In press)

- **Geno-Clausius Inequality for closed systems**[30]:

$$\oint^{>} \frac{d^{>}Q^{>}}{T_R} < 0 \qquad \text{(closed systems)} \qquad (21)$$

[30] A. A. Bhalekar, "Geno-nonequilibrium thermodynamics. II. Spatially uniform, spatially non-uniform, complex and antimatter systems, and thermodynamic time's arrows," *Hadronic J.*, 2013, (In press)

- **Geno-Clausius Inequality for open systems**[31]:

$$\oint^{>} \left[\frac{1^{>}}{T_{R}} > \frac{d^{>}Q^{>}}{d^{>}t^{>}} \right] > d^{>}t^{>}$$

$$\equiv \oint^{>} \left[\int_{A^{>}}^{>} -\frac{\mathbf{Q}^{>}(\mathbf{A}^{>})}{T^{>}(\mathbf{A}^{>})} \odot^{>} \mathbf{d}^{>}\mathbf{A}^{>} \right] > d^{>}t^{>} < 0 \quad \text{(closed systems)} \tag{22}$$

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[31] A. A. Bhalekar, "Geno-nonequilibrium thermodynamics. II. Spatially uniform, spatially non-uniform, complex and antimatter systems, and thermodynamic time's arrows," *Hadronic J.*, 2013, (In press)

Concluding Remarks

We have seen that:

- Thermodynamic reversibility is uniquely determined. Though the thermodynamic forward and reversed paths are not real ones but it guarantees assuming of the same equilibrium states in reverse order on the said reversed path. Of course the equilibrium states are real ones.
- Same is not possible with dynamical equations because herein there exists an additional parameter of time.
- However, the explicit existence of time parameter offers a possibility to test the dynamic equations for **time reversal symmetry**. Thus the time reversal symmetry and the thermodynamic reversibility are determined entirely on two different footings.

- We have seen that even if a dynamical equation is symmetric to time reversal the reversed path are not obtained as composed of the same dynamic states. This is so because the reversal of time reverses velocity vector and hence the dynamic states are not identically same as those encountered on the forward path.
- Thus in the case of symmetric to time reversal dynamical equations what one obtains is the spatial reversibility. Hence it is a partial description of reversibility and there is no physical utility of this test.
- Therefore, even if an equation of motion is symmetric to time reversal the forward and reverse paths it describe are two different irreversible motions and in time frame both are forward motions.

- Further, from the physical utility point of view those equations not found symmetric to time reversal produce additional ambiguities. For example, whether the sign of say momentum flux changes on time reversal or not remains an unanswered or ambiguously answered question. Hence, what happens to the corresponding dissipative term on time reversal cannot be predicted uniquely. This also has repercussions even on the validity of the second law of thermodynamics on the time reversed paths.

- Recall that thermodynamic irreversibility is determined by the existence of entropy source strength with positive definite sign and hence it is termed as entropy production. Because of the above we get caught into a trap that suggest that even the thermodynamic irreversibility doesn't get properly described on a time reversed path.

- That is thermodynamic irreversibility is at the verge of loosing sanctity on time reversed paths.
- We are astonished to see that whether the Boltzmann integro-differential equation is symmetric to time reversal or not the Boltzmann H-theorem remains time reversal invariant.
- Most importantly we need to keep in mind is that the time reversal test we perform on the equations of motion and not on the actual motion. All actual motions are inherently irreversible whether the equation used to describe them are symmetric to time reversal or not. Mathematical description of motions that produce ambiguous results on time reversal operation are also inherently irreversible.

- Thus the time reversal test is of no practical utility particularly in better understanding of either dynamic irreversibility or thermodynamic irreversibility.
- Therefore, we conclude that the mathematics of 20^{th} century and the mechanics based on it are not compatible with the real physical motions. Hence, one needs to use a mathematics which is inherently irreversible to describe dynamics of a real motion and also to develop nonequilibrium thermodynamics.

- The above requirement is perfectly met by genomathematics and the mechanics based on it, which has been developed by Professor Ruggero Maria Santilli.
- This is so because herein one uses two different multiplying units $\hat{I}^{>}$ and ${}^{<}\hat{I}$ for forward and backward motions with $\hat{I}^{>} \neq {}^{<}\hat{I}$. Hence, there is no requirement of testing time reversal symmetry of equations.

THANK YOU VERY MUCH FOR YOUR INTEREST

The forthcoming event on Hadronic Mechanics is:

2013 INTERNATIONAL WORKSHOP ON HADRONIC CHEMISTRY, MATHEMATICS AND PHYSICS
at Department of Chemistry, R. T. M. Nagpur University,
during October 21 - 26, 2013.