# Algebra, Hyperalgebra and Lie-Santilli Theory

**B. Davvaz**[a], **R. M. Santilli**[b], and **T. Vougiouklis**[c]

[a]Department of Mathematics, Yazd University, Yazd, Iran
davvaz@yazduni.ac.ir
bdavvaz@yahoo.com

[b]Institute for Basic Research,
P. O. Box 1577, Palm Harbor, FL 34682, U.S.A.
ibr@gte.net

[c]Democritus University of Thrace, School of Science of Education
68100 Alexandroupolis, Greece
tvougiou@eled.duth.gr

**Abstract.**

**Keywords:**
**PACS:** 02.10.-v, 02.20.-a.

## 1 Introduction

## 2 Hyperrings, hyperfields and hypervector spaces

Let $H$ be a non-empty set and $\circ : H \times H \to \wp^*(H)$ be a *hyperoperation*, where $\wp^*(H)$ is the family of all non-empty subsets of $H$. The couple $(H, \circ)$ is called a *hypergroupoid*. For any two non-empty subsets $A$ and $B$ of $H$ and $x \in H$, we define $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$, $A\circ\{x\} = A\circ x$ and $\{x\}\circ A = x\circ A$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all $a, b, c$ in $H$ we have $(a\circ b)\circ c = a\circ(b\circ c)$. In addition, if for every $a \in H$, $a\circ H = H = H\circ a$, then, $(H, \circ)$ is called a *hypergroup*. A non-empty subset $K$ of a semihypergroup $(H, \circ)$ is called a *sub-semihypergroup* if it is a semihypergroup. In other words, a non-empty subset $K$ of a semihypergroup $(H, \circ)$ is a sub-semihypergroup if $K \circ K \subseteq K$. We say that a hypergroup $(H, \circ)$ is *canonical* if

(1) It is commutative;

(2) It has a scalar identity (also called scalar unit), which means that there exists $e \in H$, for all $x \in H$, $x \circ e = x$;

(3) Every element has a unique inverse, which means that for all $x \in H$, there exists a unique $x^{-1} \in H$ such that $e \in x \circ x^{-1}$;

(4) It is reversible, which means that if $x \in y \circ z$, then $z \in y^{-1} \circ x$ and $y \in x \circ z^{-1}$.

In [3] there are several types of hyperrings and hyperfields. In what follows we shall consider one of the most general types of hyperrings.

The triple $(R, +, \cdot)$ is a *hyperring* if

(1) $(R, +)$ is a canonical hypergroup;

(2) $(R, \cdot)$ is a semihypergroup such that $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$, i.e, 0 is a bilaterally absorbing element;

(3) the hyperoperation " $\cdot$" is distributive over the hyperoperation "+", which means that for all $x, y, z$ of $R$ we have:

$$x \cdot (y + z) = x \cdot y + x \cdot z \text{ and } (x + y) \cdot z = x \cdot z + y \cdot z.$$

EXAMPLE 1. Let $R = \{x,\ y,\ z,\ t\}$ be a set with the following hyperoperations:

| $+$ | $x$ | $y$ | $z$ | $t$ |
|---|---|---|---|---|
| $x$ | $x$ | $y$ | $z$ | $t$ |
| $y$ | $y$ | $x$ | $t$ | $z$ |
| $z$ | $z$ | $t$ | $\{x,\ z\}$ | $\{y,\ t\}$ |
| $t$ | $t$ | $z$ | $\{y,\ t\}$ | $\{x,\ z\}$ |

| $\cdot$ | $x$ | $y$ | $z$ | $t$ |
|---|---|---|---|---|
| $x$ | $x$ | $x$ | $x$ | $x$ |
| $y$ | $x$ | $y$ | $x$ | $y$ |
| $z$ | $x$ | $x$ | $\{x,\ z\}$ | $\{x,\ z\}$ |
| $t$ | $x$ | $y$ | $\{x,\ z\}$ | $\{y,\ t\}$ |

Then, $(R, +, \cdot)$ is a hyperring.

We call $(R, +, \cdot)$ a *hyperfield* if $(R, +, \cdot)$ is a hyperring and $(R - \{0\}, \cdot)$ is a hypergroup.

EXAMPLE 2. Let $F = \{x,\ y\}$ be a set with the following hyperoperations:

| $+$ | $x$ | $y$ |
|---|---|---|
| $x$ | $x$ | $y$ |
| $y$ | $y$ | $\{x,\ y\}$ |

| $\cdot$ | $x$ | $y$ |
|---|---|---|
| $x$ | $x$ | $x$ |
| $y$ | $x$ | $y$ |

Then, $(F, +, \cdot)$ is a hyperfield.

A *Krasner hyperring* is a hyperring such that $(R, +)$ is a canonical hypergroup with identity 0 and $\cdot$ is an operation such that 0 is a bilaterally absorbing element. An exhaustive review updated to 2007 of hyperring theory appears in [3].

**Definition 2.1.** Let $(R, +, \cdot)$ be a hypering. We define the relation $\Gamma$ as follows:

$$x\ \Gamma\ y \Leftrightarrow \exists n \in \mathbb{N}, \exists (k_1, \cdots, k_n) \in \mathbb{N}^n \quad \text{and} \quad [\exists (x_{i1}, \cdots, x_{ik_i}) \in R^{k_i}, (i = 1, \cdots, n)]$$

such that

$$x, y \in \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}).$$

**Theorem 2.2.** *[12, 13] Let $(R, +, \cdot)$ be a hypering $\Gamma^*$ be the transitive closure of $\Gamma$.*

*(1) $\Gamma^*$ is a strongly regular relation both on $(R, +)$ and $(R, \cdot)$.*

*(2) The quotient $R/\Gamma^*$ be a ring.*

*(3) The relation $\Gamma^*$ is the smallest equivalence relation such that the quotient $R/\Gamma^*$ be a ring.*

**Theorem 2.3.** *[5] The relation $\Gamma$ on every hyperfield is an equivalence relation and $\Gamma = \Gamma^*$.*

Remark 1. Let $(F, +, \cdot)$ be a hyperfield. Then, $F/\gamma^*$ is a field. If $\phi : F \to F/\Gamma^*$ is the canonical map, then $\omega_F = \{x \in F \mid \phi(x) = 0\}$, where 0 is the zero of the fundamental field $F/\Gamma^*$.

Let $(R, +, \cdot)$ be a hyperring, $(M, +)$ be a canonical hypergroup and there exists an external map

$$\cdot : R \times M \to \wp^*(M),\ (a, x) \mapsto ax$$

such that for all $a, b \in R$ and for all $x, y \in M$ we have

$$a(x + y) = ax + ay,\ (a + b)x = ax + bx,\ (ab)x = a(bx),$$

then $M$ is called a hypermodule over $R$. If we consider a hyperfield $F$ instead of a hyperring $R$, then $M$ is called a hypervector space.

Remark 2. Note that it is possible in a hypervector space one or more of hyperoperations be ordinary operations.

Example 3. Let $F$ be a field and $V$ be a vector space on $F$. If $S$ is a subspace of $V$, we consider the following external hyperoperation: $a \circ x = ax + S$, for all $a \in F$ and $x \in V$. Then, $V$ is a hypervector space.

## 3 Algebra and Hyperalgebra

**Definition 3.1.** Let $(L, +, \cdot)$ be a hypervector space over the hyperfield $(F, +, \cdot)$. Consider the bracket (commutator) hope:

$$[\ ,\ ] : L \times L \to \wp^*(L) : (x, y) \to [x, y]$$

then $L$ is a Lie hyperalgebra over $F$ if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e. $[\lambda_1 x_1 + \lambda_2 x_2, y] = (\lambda_1[x_1, y] + \lambda_2[x_2, y])$, $[x, \lambda_1 y_1 + \lambda_2 y_2] = (\lambda_1[x, y_1] + \lambda_2[x, y_2])$, for all $x, x_1, x_2, y, y_1, y_2 \in L$, $\lambda_1, \lambda_2 \in F$;

(L2) $0 \in [x, x]$, for all $x \in L$;

(L3) $0 \in \Big([x, [y, z]] + [y, [z, x]] + [z, [x, y]]\Big)$, for all $x, y \in L$.

**Definition 3.2.** Let $A$ be a hypervector space over a hyperfield $F$. Then, $A$ is called a *hyperalgebra* over the hyperfield $F$ if there exists a mapping $\cdot : A \times A \to \wp^*(A)$ (images to be denoted by $x \cdot y$ for $x, y \in A$ such that the following conditions hold:

(1) $(x + y) \cdot z = x \cdot z + y \cdot z$ and $x \cdot (y + z) = x \cdot y + x \cdot z$;

(2) $(cx) \cdot y = c(x \cdot y) = x \cdot (cy)$;

(3) $0 \cdot y = y \cdot 0 = 0$;

for all $x, y, z \in A$ and $c \in F$.

In the above definition, if all hyperoperations are ordinary operations, then we have an *algebra*.

A non-empty subset $A^{'}$ of a hyperalgebra $A$ is called a *sub hyperalgebra* if it is a subhyperspace of $A$ and for all $x, y \in A^{'}$ we have $xy \in A^{'}$.

In connection with the explicit forms of the hyperproduct let us consider an associative hyperalgebra $A$, with hyperproduct $a \cdot b$, over a hyperfield $F$. It is possible to construct a new hyperalgebra, denoted by $A^-$, by means of the anti-commutative hyperproduct

$$[a, b] = a \cdot b - b \cdot a = \bigcup_{\substack{x \in a \cdot b \\ y \in b \cdot a}} x - y. \tag{1}$$

**Lemma 3.3.** *For any non-empty subset $S$ of $A$, we have $0 \in S - S$.*

*Proof.* It is straightforward. □

**Proposition 3.4.** *$A^-$ is a Lie hyperalgebra.*

*Proof.* For all $x, x_1, x_2, y, y_1, y_2 \in L$, $\lambda_1, \lambda_2 \in F$,we have

$$\begin{aligned}
[\lambda_1 x_1 + \lambda_2 x_2, y] &= (\lambda_1 x_1 + \lambda_2 x_2) \cdot y - y \cdot (\lambda_1 x_1 + \lambda_2 x_2) \\
&= (\lambda_1 x_1) \cdot y + (\lambda_2 x_2) \cdot y - y \cdot (\lambda_1 x_1) - y \cdot (\lambda_2 x_2) \\
&= \Big((\lambda_1 x_1) \cdot y - y \cdot (\lambda_1 x_1)\Big) + \Big((\lambda_2 x_2) \cdot y - y \cdot (\lambda_2 x_2)\Big) \\
&= [\lambda_1 x_1, y] + [\lambda_2 x_2, y], \\
&= \lambda_1 [x_1, y] + \lambda_2 [x_2, y],
\end{aligned}$$

and similarly we obtain $[x, \lambda_1 y_1 + \lambda_2 y_2] = (\lambda_1 [x, y_1] + \lambda_2 [x, y_2])$.

Now, we prove (L2). Since $x \cdot x$ is non-empty, there exists $a_0 \in x \cdot x$. hence, $-a_0 \in -x \cdot x$. Thus, $0 \in a_0 - a_0 \subseteq \bigcup_{a \in x \cdot x} a - a = x \cdot x - x \cdot x = [x, x]$.

It remains to show that (L3) is also satisfied. For,

$$\begin{aligned}[x,[y,z]] &= x\cdot[y,z]-[y,z]\cdot x\\ &= x\cdot(y\cdot z-z\cdot y)-(y\cdot z-z\cdot y)\cdot x\\ &= x\cdot y\cdot z-x\cdot z\cdot y-y\cdot z\cdot x-z\cdot y\cdot x.\end{aligned}$$

Hence,

$$\begin{aligned}\Big([x,[y,z]]+[y,[z,x]]+[z,[x,y]]\Big) &= x\cdot y\cdot z-x\cdot z\cdot y-y\cdot z\cdot x-z\cdot y\cdot x\\ &\quad +(y\cdot z\cdot x-y\cdot x\cdot z-z\cdot x\cdot y+x\cdot z\cdot y)\\ &\quad +(z\cdot x\cdot y-z\cdot y\cdot x-x\cdot y\cdot z+y\cdot x\cdot z)\\ &= (x\cdot y\cdot z-x\cdot y\cdot z)+(x\cdot z\cdot y-x\cdot z\cdot y)\\ &\quad +(z\cdot y\cdot x-z\cdot y\cdot x)+(y\cdot z\cdot x-y\cdot z\cdot x)\\ &\quad +(y\cdot x\cdot z-y\cdot x\cdot z)+(z\cdot x\cdot y-z\cdot x\cdot y).\end{aligned}$$

By Lemma 3.3, 0 is belong to the right hand of the abobe equality, so $0\in\Big([x,[y,z]]+[y,[z,x]]+[z,[x,y]]\Big)$. □

**Definition 3.5.** Corresponding to any hyperalgebra $A$ with hyperproduct $a\cdot b$ it is possible to define an anticommutative hyperalgebra $A^-$ which is the same hypervector space as $A$ with the new hyperproduct

$$[a,b]_{A^-}=a\cdot b-b\cdot a. \tag{2}$$

A hyperalgebra $A$ is called *Lie-admissible* if the hyperalgebra $A^-$ is a Lie hyperalgebra.

If $A$ is an associative hyperalgebra, then the hyperproduct (2) coincide with (1) and $A^-$ is a Lie hyperalgebra in its more usual form. Thus, the associative hyperalgebras constitute a basic class of Lie-admissible hyperalgebras.

A Jordan algebra is a (non-associative) algebra over a field whose multiplication satisfies the following axioms:

(1) $xy=yx$ (commutative law);

(2) $(xy)(xx)=x(y(xx))$ (Jordan identity).

**Definition 3.6.** A Jordan hyperalgebra is a (non-associative) hyperalgebra over a hyperfield whose multiplication satisfies the following axioms:

(J1) $x\cdot y=y\cdot x$ (commutative law);

(J2) $(x\cdot y)\cdot(x\cdot x)=x\cdot(y\cdot(x\cdot x))$ (Jordan identity).

Let $A$ be an associative hyperalgebra over a hyperfield $F$. It is possible to construct a new hyperalgebra, denoted by $A^+$, by means of the commutative hyperproduct

$$\{a,b\} = a \cdot b + b \cdot a = \bigcup_{\substack{x \in a\cdot b \\ y \in b \cdot a}} x + y. \tag{3}$$

**Proposition 3.7.** *$A^+$ is a Jordan hyperalgebra.*

*Proof.* It is straightforward. □

**Definition 3.8.** Corresponding to any hyperalgebra $A$ with hyperproduct $a \cdot b$ it is possible to define, as for $A^-$, a commutative hyperalgebra $A^+$ which is the same hypervector space as $A$ but with the new hyperproduct

$$\{a,b\}_{A^+} = a \cdot b + b \cdot a. \tag{4}$$

In this connection the most interesting case occure when $A^+$ is a (commutative) Jordan hyperalgebra.

A hyperalgebra $A$ is said to be *Jordan admissible* if $A^+$ is a (commutative) Jordan hyperalgebra.

If $A$ is an associative hyperalgebra, then the hyperproduct (4) reduces to (3) and $A^+$ is a special Jordan hyperalgebra. Thus, associative hyperalgebras constitute a basis class of Jordan-admissible hyperalgebras.

**Definition 3.9.** The fundamental relation $\epsilon^*$ is defined in a hyperalgebra as the smallest equivalence relation such that the quotient is an algebra.

By using strongly regular relations, we can connect hyperalgebras to algebras. More exactly, starting with a hyperalgebra and using a strongly regular relation, we can construct an algebra structure on the quotient set. An equivalence relation $\rho$ on a hyperalgebra $A$ is called right (resp. left) strongly regular if and only if $x\rho y$ implies that $(x+z)\overline{\overline{\rho}}(y+z)$ and $(x\alpha z)\overline{\overline{\rho}}(y \cdot z)$ for every $z \in A$ (resp. $(z+x)\overline{\overline{\rho}}(z+y)$ and $(z \cdot x)\overline{\overline{\rho}}(z \cdot y)$ ), and $\rho$ is strongly regular if it is both left and right strongly regular.

**Theorem 3.10.** *Let $A$ be a hyperalgebra over the hyperfield $F$. Denote by $\mathcal{U}$ the set of all finite polynomials of elements of $A$ over $F$. We define the relation $\epsilon$ on $A$ as follows:*

$$x \epsilon y \;\text{ if and only if }\; \{x,y\} \subseteq u, \;\text{ where }\; u \in \mathcal{U}.$$

*Then, the $\epsilon^*$ is the transitive closure of $\epsilon$ and is called the fundamental equivalence relation on $A$.*

*Proof.* The proof is similar to the proof of Theorem 3.1 in [8]. □

Remark 3. Note that the relation $\epsilon^*$ is a strongly regular relation.

Remark 4. In $A^-/\epsilon^*$, the binary operations and external operation are defined in the usual manner:

$$\epsilon^*(x) \oplus \epsilon^*(y) = \epsilon^*(z), \text{ for all } z \in \epsilon^*(x) + \epsilon^*(y),$$
$$\epsilon^*(x) \odot \epsilon^*(y) = \epsilon^*(z), \text{ for all } z \in \epsilon^*(x) \cdot \epsilon^*(y),$$
$$\Gamma^*(r) \circ \epsilon^*(x) = \epsilon^*(z), \text{ for all } z \in \Gamma^*(r)\epsilon^*(x).$$

**Theorem 3.11.** *Let $A$ be an associative hyperalgebra over a hyperfield $F$. Then, $A^-/\epsilon^*$ is a Lie-admissible algebra with the following product:*

$$\Big\langle \epsilon^*(x), \epsilon^*(y) \Big\rangle = \epsilon^*(x) \odot \epsilon^*(y) \ominus \epsilon^*(y) \odot \epsilon^*(x). \tag{5}$$

*Proof.* By Definition 3.9 and Theorem 3.10, $A/\epsilon^*$ is an ordinary associative algebra. So, it is enough to show that it is a Lie algebra with the hyperproduce (5). By Proposition 3.4, $A^-$ is a Lie hyperalgebra with the hyperproduct $[a, b] = a \cdot b - b \cdot a$.

(1) By (L1), for all $x_1, x_2, y \in A$, $\lambda_1, \lambda_2 \in F$, we have $[\lambda_1 x_1 + \lambda_2 x_2, y] = \lambda_1[x_1, y] + \lambda_2[x_2, y]$. Hence,

$$(\lambda_1 x_1 + \lambda_2 x_2) \cdot y - y \cdot (\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1(x_1 \cdot y - y \cdot x_1) + \lambda_2(x_2 \cdot y - y \cdot x_2),$$

and so

$$\epsilon^*\Big((\lambda_1 x_1 + \lambda_2 x_2) \cdot y - y \cdot (\lambda_1 x_1 + \lambda_2 x_2)\Big) = \epsilon^*\Big(\lambda_1(x_1 \cdot y - y \cdot x_1) + \lambda_2(x_2 \cdot y - y \cdot x_2)\Big).$$

This implies that

$$\Big(\Gamma^*(\lambda_1) \circ \epsilon^*(x_1) \oplus \Gamma^*(\lambda_2) \circ \epsilon^*(x_2)\Big) \odot \epsilon^*(y) \ominus \epsilon^*(y) \odot \Big(\Gamma^*(\lambda_1) \circ \epsilon^*(x_1) \oplus \Gamma^*(\lambda_2) \circ \epsilon^*(x_2)\Big)$$
$$= \Gamma^*(\lambda_1) \circ (\epsilon^*(x_1) \odot \epsilon^*(y) \ominus \epsilon^*(y) \odot \epsilon^*(x_1)) + \Gamma^*(\lambda_2) \circ (\epsilon^*(x_2) \odot \epsilon^*(y) \ominus \epsilon^*(y) \odot \epsilon^*(x_2)).$$

Therefore,

$$\Big\langle \Gamma^*(\lambda_1) \circ \epsilon^*(x_1) \oplus \Gamma^*(\lambda_2) \circ \epsilon^*(x_2), \epsilon^*(y) \Big\rangle$$
$$= \Gamma^*(\lambda_1) \circ \Big\langle \epsilon^*(x_1), \epsilon^*(y) \Big\rangle \oplus \Gamma^*(\lambda_2) \circ \Big\langle \epsilon^*(x_2), \epsilon^*(y) \Big\rangle.$$

Similarly, for all $x, y_1, y_2 \in A$, $\lambda_1, \lambda_2 \in F$, we obtain

$$\Big\langle \epsilon^*(x), \Gamma^*(\lambda_1) \circ \epsilon^*(y_1) \oplus \Gamma^*(\lambda_2) \circ \epsilon^*(y_2) \Big\rangle$$
$$= \Gamma^*(\lambda_1) \circ \Big\langle \epsilon^*(x), \epsilon^*(y_1) \Big\rangle \oplus \Gamma^*(\lambda_2) \circ \Big\langle \epsilon^*(x), \epsilon^*(y_2) \Big\rangle.$$

(2) By (L2), $0 \in x \cdot x - x \cdot x$, so

$$\begin{aligned}\epsilon^*(0) = \epsilon^*(x \cdot x - x \cdot x) &= \epsilon^*(x) \odot \epsilon^*(x) \ominus \epsilon^*(x) \odot \epsilon^*(x) \\ &= \Big\langle \epsilon^*(x), \epsilon^*(x) \Big\rangle.\end{aligned}$$

(3) By (L3), we have $0 \in \Big([x, [y, z]] + [y, [z, x]] + [z, [x, y]]\Big)$, for all $x, y \in A$. Thus,

$$\epsilon^*(0) = \Big\langle \epsilon^*(x), \Big\langle \epsilon^*(y), \epsilon^*(z) \Big\rangle \Big\rangle + \Big\langle \epsilon^*(y), \epsilon^*(z), \epsilon^*(x) \Big\rangle \Big\rangle + \Big\langle \epsilon^*(z), \Big\langle \epsilon^*(x), \epsilon^*(y) \Big\rangle \Big\rangle.$$

□

**Theorem 3.12.** *Let $A$ be an associative hyperalgebra over a hyperfield $F$. Then, $A^+/\epsilon^*$ is a Jordan-admissible algebra with the following product:*

$$\Big\lceil \epsilon^*(x), \epsilon^*(y) \Big\rceil = \epsilon^*(x) \odot \epsilon^*(y) \oplus \epsilon^*(y) \odot \epsilon^*(x). \tag{6}$$

*Proof.* By Definition 3.9 and Theoremt1, $A/\epsilon^*$ is an ordinary associative algebra. So, it is enough to show that it is a Jordan algebra with the hyperproduct (6). By Proposition 3.7, $A^+$ is a Jordan hyperalgebra with the hyperproduct $\{a, b\} = a \cdot b + b \cdot a$.

(1) By (J1), for all $x, y \in A$, $\{x, y\} = \{y, x\}$. So, $\epsilon^*(x \cdot y + y \cdot x) = \epsilon^*(y \cdot x + x \cdot y)$ which implies that $\epsilon^*(x) \odot \epsilon^*(y) \oplus \epsilon^*(y) \odot \epsilon^*(x) = \epsilon^*(y) \odot \epsilon^*(x) \oplus \epsilon^*(x) \odot \epsilon^*(y)$. Thus, $\Big\lceil \epsilon^*(x), \epsilon^*(y) \Big\rceil = \Big\lceil \epsilon^*(y), \epsilon^*(x) \Big\rceil$

(2) By (J2), for all $x, y \in A$, $\Big\{\{x, y\}, \{x, x\}\Big\} = \Big\{x, \{y, \{x, x\}\}\Big\}$. Hence,

$$\epsilon^*\Big(\Big\{\{x, y\}, \{x, x\}\Big\}\Big) = \epsilon^*\Big(\Big\{x, \{y, \{x, x\}\}\Big\}\Big),$$

which implies that

$$\Big\lceil \lceil \epsilon^*(x), \epsilon^*(y) \rceil, \lceil \epsilon^*(x), \epsilon^*(x) \rceil \Big\rceil = \Big\lceil \epsilon^*(x), \lceil \epsilon^*(y), \lceil \epsilon^*(x), \epsilon^*(x) \rceil \rceil \Big\rceil.$$

This completes the proof.

□

In the same way it is possible to introduce always in terms of the associative product the following bilinear form

$$(a, b) = \lambda a \cdot b + (1 - \lambda) b \cdot a = \lambda [a, b] + b \cdot a, \tag{7}$$

where $\lambda$ is a free element belonging to the hyperfield $F$, which characterizes the $\lambda$-mutations $A(\lambda)$ of $A$. Clearly, $A(1)$ is isomorphic to $A$.

In this connection, a more general bilinear form in terms of the associative hyperproduct is given by

$$(a, b) = \lambda a \cdot b + \mu b \cdot a = \alpha[a, b] + \beta\{a, b\}, \tag{8}$$

where $\lambda = \alpha + \beta$ and $\mu = \beta - \alpha$ are free elements belonging to the hyperfield $F$, which constitutes the basic hyperproduct of the $(\lambda, \mu)$-mutations $A(\lambda, \mu)$ of $A$. Clearly, $A(1, 0)$ is isomorphic to $A$ and $A(1, -1)$ is isomorphic to $A^-$; $A(1, 1)$ is isomorphic to $A^+$ and $A(\lambda, 1 - \lambda)$ is isomorphic to $A(\lambda)$.

Let $A$ be an associative hyperalgebra. We can define another hyperoperation by means an element $T$ which is denoted by $[\cdot, \cdot]_T$ and is defined by

$$\begin{array}{l} [\cdot, \cdot]_T : A \times A \to \mathcal{P}^*(A), \\ [\cdot, \cdot]_T : (x, y) \mapsto [x, y]_T = x \cdot T \cdot y - y \cdot T \cdot x. \end{array} \tag{9}$$

**Proposition 3.13.** *The hyperoperation* $[\cdot, \cdot]_T$ *satisfies the following conditions:*

*(1)* $0 \in [x, x]_T$, *for all* $x \in A$;

*(2)* $0 \in \Big([x, [y, z]_T]_T + [y, [z, x]_T]_T + [z, [x, y]_T]_T\Big)$, *for all* $x, y \in T$.

*Proof.* (1) Since $x \cdot T \cdot x$ is non-empty, there exists $a_0 \in x \cdot T \cdot x$. hence, $-a_0 \in -x \cdot T \cdot x$. Thus, $0 \in a_0 - a_0 \subseteq \bigcup_{a \in x \cdot T \cdot x} a - a = x \cdot T \cdot x - x \cdot T \cdot x = [x, x]_T$.

(2) We have

$$\begin{aligned} [x, [y, z]_T]_T &= x \cdot T \cdot [y, z]_T - [y, z]_T \cdot T \cdot x \\ &= x \cdot T \cdot (y \cdot T \cdot z - z \cdot T \cdot y) - (y \cdot T \cdot z - z \cdot T \cdot y) \cdot T \cdot x \\ &= x \cdot T \cdot y \cdot T \cdot z - x \cdot T \cdot z \cdot T \cdot y - y \cdot T \cdot z \cdot T \cdot x - z \cdot T \cdot y \cdot T \cdot x. \end{aligned}$$

Hence,

$$\begin{aligned} &\Big([x, [y, z]_T]_T + [y, [z, x]_T]_T + [z, [x, y]_T]_T\Big) \\ &= x \cdot T \cdot y \cdot T \cdot z - x \cdot T \cdot z \cdot T \cdot y - y \cdot T \cdot z \cdot T \cdot x - z \cdot T \cdot y \cdot T \cdot x \\ &\quad + (y \cdot T \cdot z \cdot T \cdot x - y \cdot T \cdot x \cdot T \cdot z - z \cdot T \cdot x \cdot T \cdot y + x \cdot T \cdot z \cdot T \cdot y) \\ &\quad + (z \cdot T \cdot x \cdot T \cdot y - z \cdot T \cdot y \cdot T \cdot x - x \cdot T \cdot y \cdot T \cdot z + y \cdot T \cdot x \cdot T \cdot z) \\ &= (x \cdot T \cdot y \cdot T \cdot z - x \cdot T \cdot y \cdot T \cdot z) + (x \cdot T \cdot z \cdot T \cdot y - x \cdot T \cdot z \cdot T \cdot y) \\ &\quad + (z \cdot T \cdot y \cdot T \cdot x - z \cdot T \cdot y \cdot T \cdot x) + (y \cdot T \cdot z \cdot T \cdot x - y \cdot T \cdot z \cdot T \cdot x) \\ &\quad + (y \cdot T \cdot x \cdot T \cdot z - y \cdot T \cdot x \cdot T \cdot z) + (z \cdot T \cdot x \cdot T \cdot y - z \cdot T \cdot x \cdot T \cdot y). \end{aligned}$$

By Lemma 3.3, 0 is belong to the right hand of the abobe equality, so $0 \in \Big([x, [y, z]_T]_T + [y, [z, x]_T]_T + [z, [x, y]_T]_T\Big)$. □

REMARK 5. If $A$ is an algebra and $[\cdot,\cdot]_T : A \times A \to A$, then we have Lie-Santilli braket.

**Definition 3.14.** Corresponding to any hyperalgebra $A$ with hyperproduct $a \cdot b$ it is possible to define a $A^\star$ which is the same hypervector space as $A$ with the new hyperproduct

$$[a,b]_{A^\star} = a \cdot T \cdot b - b \cdot T \cdot a. \tag{10}$$

A hyperalgebra $A$ is called *Lie-Santilli-admissible* if the hyperalgebra $A^\star$ is a Lie hyperalgebra.

**Corollary 3.15.** If $A$ is an associative hyperalgebra, then the hyperproduct (10) coincide with (9) and $A^\star$ is a Lie hyperalgebra.

**Theorem 3.16.** *Let $A$ be an associative hyperalgebra over a hyperfield $F$. Then, $A^\star/\epsilon^*$ is a Lie-Santilli-admissible algebra with the following product:*

$$\Big\langle \epsilon^*(x), \epsilon^*(y) \Big\rangle_T = \epsilon^*(x) \odot \epsilon^*(T) \odot \epsilon^*(y) \ominus \epsilon^*(y) \odot \epsilon^*(T) \odot \epsilon^*(x). \tag{11}$$

*Proof.* By Definition 3.9 and Theorem 3.10, $A/\epsilon^*$ is an ordinary associative algebra. So, it is enough to show that it is a Lie algebra with the hyperproduce (1). By corollary 3.15, $A^\star$ is a Lie hyperalgebra with the hyperproduct $[a,b]_T = a \cdot T \cdot b - b \cdot T \cdot a$.

(1) By (L1), for all $x_1, x_2, y \in$, $\lambda_1, \lambda_2 \in F$, we have $[\lambda_1 x_1 + \lambda_2 x_2, y]_T = \lambda_1 [x_1, y]_T + \lambda_2 [x_2, y]_T$. Hence,

$$(\lambda_1 x_1 + \lambda_2 x_2) \cdot T \cdot y - y \cdot T \cdot (\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 (x_1 \cdot T \cdot y - y \cdot T \cdot x_1) + \lambda_2 (x_2 \cdot T \cdot y - y \cdot T \cdot x_2),$$

and so

$$\epsilon^*\Big((\lambda_1 x_1 + \lambda_2 x_2) \cdot T \cdot y - y \cdot T \cdot (\lambda_1 x_1 + \lambda_2 x_2)\Big) = \epsilon^*\Big(\lambda_1 (x_1 \cdot T \cdot y - y \cdot T \cdot x_1) + \lambda_2 (x_2 \cdot T \cdot y - y \cdot T \cdot x_2)\Big).$$

This implies that

$$\begin{aligned} &\Big(\Gamma^*(\lambda_1) \circ \epsilon^*(x_1) \oplus \Gamma^*(\lambda_2) \circ \epsilon^*(x_2)\Big) \odot \epsilon^*(T) \odot \epsilon^*(y) \\ &\quad \ominus \epsilon^*(y) \odot \epsilon^*(T) \odot \Big(\Gamma^*(\lambda_1) \circ \epsilon^*(x_1) \oplus \Gamma^*(\lambda_2) \circ \epsilon^*(x_2)\Big) \\ &= \Gamma^*(\lambda_1) \circ \Big(\epsilon^*(x_1) \odot \epsilon^*(T) \odot \epsilon^*(y) \ominus \epsilon^*(y) \odot \epsilon^*(T) \odot \epsilon^*(x_1)\Big) \\ &\quad \oplus \Gamma^*(\lambda_2) \circ \Big(\epsilon^*(x_2) \odot \epsilon^*(T) \odot \epsilon^*(y) \ominus \epsilon^*(y) \odot \epsilon^*(T) \odot \epsilon^*(x_2)\Big). \end{aligned}$$

Therefore,

$$\begin{aligned} &\Big\langle \Gamma^*(\lambda_1) \circ \epsilon^*(x_1) \oplus \Gamma^*(\lambda_2) \circ \epsilon^*(x_2), \epsilon^*(y) \Big\rangle_T \\ &= \Gamma^*(\lambda_1) \circ \Big\langle \epsilon^*(x_1), \epsilon^*(y) \Big\rangle_T \oplus \Gamma^*(\lambda_2) \circ \Big\langle \epsilon^*(x_2), \epsilon^*(y) \Big\rangle_T. \end{aligned}$$

Similarly, for all $x, y_1, y_2 \in A$, $\lambda_1, \lambda_2 \in F$, we obtain

$$\begin{aligned}&\Big\langle \epsilon^*(x), \Gamma^*(\lambda_1) \circ \epsilon^*(y_1) \oplus \Gamma^*(\lambda_2) \circ \epsilon^*(y_2) \Big\rangle_T \\ &= \Gamma^*(\lambda_1) \circ \Big\langle \epsilon^*(x), \epsilon^*(y_1) \Big\rangle_T \oplus \Gamma^*(\lambda_2) \circ \Big\langle \epsilon^*(x), \epsilon^*(y_2) \Big\rangle_T.\end{aligned}$$

(2) By (L2), $0 \in x \cdot T \cdot x - x \cdot T \cdot x$, so

$$\begin{aligned}\epsilon^*(0) = \epsilon^*(x \cdot T \cdot x - x \cdot T \cdot x) \quad &= \epsilon^*(x) \odot \epsilon^*(T) \odot \epsilon^*(x) \ominus \epsilon^*(x) \odot \epsilon^*(T) \odot \epsilon^*(x) \\ &= \Big\langle \epsilon^*(x), \epsilon^*(x) \Big\rangle_T.\end{aligned}$$

(3) By (L3), we have $0 \in \Big([x, [y, z]] + [y, [z, x]] + [z, [x, y]]\Big)$, for all $x, y \in A$. Thus,

$$\epsilon^*(0) = \Big\langle \epsilon^*(x), \Big\langle \epsilon^*(y), \epsilon^*(z) \Big\rangle_T \Big\rangle_T + \Big\langle \epsilon^*(y), \epsilon^*(z), \epsilon^*(x) \Big\rangle_T \Big\rangle_T + \Big\langle \epsilon^*(z), \Big\langle \epsilon^*(x), \epsilon^*(y) \Big\rangle_T \Big\rangle_T.$$

□

## References


[1] R. Anderson, A. A. Bhalekar, B. Davvaz, P. S. Muktibodh, V. M. Tangde, A. S. Muktibodh, and T. Vougiouklis, *An introduction to Santilli's isodual theory of antimatter and the open problem of detecting antimatter asteroids*, Numat Bulletin, 6 (2012-13) 1-33.

[2] B. Davvaz, R. M. Santilli and T. Vougiouklis, *Studies of multivalued hyperstructures for the characterization of matter-antimatter systems and their extension*, Algebras Groups and Geometries, 28 (2011) 105-116.

[3] B. Davvaze, V. Leoreanu-Fotea, Hyperring Theory and Applications, *Internationl Academic Press, USA,* 2007.

[4] B. Davvaz, R. M. Santilli and T. Vougiouklis, *Multi-Valued Hypermathematics for Characterization of Matter and Antimatter Systems*, Journal of Computational Methods in Sciences and Engineering (JCMSE), 13 (2013) 37-50.

[5] S. Mirvakili, S.M. Anvariyeh and B. Davvaz, *Transitivity of* $\Gamma$*-relation on hyperfields*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 51(99) (2008) 233-243.

[6] A.H. Rezaei, B. Davvaz and S.O. Dehkordi, *Fundamentals of* $\Gamma$*-Algebra and* $\Gamma$*-dimension*, UPB Scientific Bulletin, Series A, 76(2) (2014) 11-122.

[7] R.M. Santilli, *An introduction to Lie-admissible algebras*, Nuovo Cimento Suppl. (1), 6 (1968) 1225-1249.

[8] R.M. Santilli and T. Vougiouklis, *Lie-admissible hyperalgebra*, Italian Journal of Pure and Applied Mathematics, 31 (20130 239-254.

[9] M. S. Tallini, *Hypervector spaces*, Proc. Fourth Int. Congress on Algebraic Hyperstructures and Applications (AHA 1990), World Scientific, (1991) 167-174.

[10] T. Vougiouklis, *The Lie-hyperalgebras and their fundamental relations*, Southeast Asian Bulletin of Mathematics, 37 (2013) 601-614.

[11] T. Vougiouklis, $H_v$*-vector spaces* Algebraic hyperstructures and applications (Iasi, 1993), 181–190, Hadronic Press, Palm Harbor, FL, 1994.

[12] T. Vougiouklis, *Hyperstructures and Their Representations*, Hadronic Press, Inc, 115, Palm Harber, USA, 1994.

[13] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*, Proc. Fourth Int. Congress on Algebraic Hyperstructures and Ap- plications (AHA 1990), World Scientific, (1991) 203-211.