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Generalization of the *PCT* theorem to all discrete space-time symmetries in quantum field theories

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The methodology of the *PCT* theorem is applied to all discrete space-time symmetries in the framework of the Wightman formalism. New conditions which hold at Jost points only are introduced and their significance is pointed out. A general theorem is proven according to which in a field theory satisfying the Wightman axioms, with the possible exception of local commutativity, if any discrete space-time symmetry holds at all points then there is a corresponding condition which holds at Jost points and vice versa. Possible links among various symmetries are investigated and two corollaries of the above theorem are formulated. Finally, some applications to both symmetry-preserving and symmetry-violating theories are discussed.

I. INTRODUCTION

The discrete space-time transformations P (space inversion), C (charge conjugation), and T (time inversion), as well as their combinations PC, CT, PT, and PCT, have played a relevant role in elementary-particle physics.

At first it was believed that all the above transformations constituted universal symmetries of nature. Soon, however, evidence began to appear for the violation of one or more of those symmetries.

It was first discovered in 1956 that P and C symmetries were violated by weak interactions involving leptons.[1] Then, for some time it was generally assumed that the P and C symmetry violations compensated each other in such a way that the combined PC transformation was an exact symmetry. In 1964, however, a PC symmetry violation in the very weak interactions was discovered in the K^0 decay.[2] Therefore, for some time a very weak T symmetry violation was assumed in such a way that PCT was an exact symmetry.

At the present time, the validity of all discrete space-time symmetries is under investigation at all levels of interaction.

In view of the above situation, an analytic approach to all discrete space-time symmetries emerges as an essential prerequisite for a systematic attempt to investigate possible relationships among different symmetries.

The celebrated PCT theorem[3,4] undoubtedly constitutes the best example for such an approach. This theorem ultimately relates the PCT condition to the weak local commutativity (WLC) condition by using the connectivity properties of the $L_+(C)$ invariance group in the framework of the holomorphic extension of the vacuum expectation values (VEV's), where the identity can be continuously connected to the total inversion of the separations. In this way, a direct link between the PCT condition and a new relation, the WLC condition, is established.

The main objective of the present paper is to investigate whether the methodology of the PCT theorem can be applied to all discrete space-time symmetries.

In Sec. II we review the discrete space-time symmetries in the framework of the Wightman formalism. In Sec. III we introduce new conditions which are valid at Jost points only, as it occurs for the WLC condition. In Sec. IV we introduce a theorem of general validity for all discrete space-time symmetries with two corollaries. Finally, in Sec. V we discuss some application to both symmetry-preserving and symmetry-violating field theories.

II. DISCRETE SPACE-TIME CONDITIONS OF THE FIRST KIND

Let us consider the celebrated P, C, and T operators and their combinations PC, PT, CT, and PCT (to which we add, for completeness, the trivial identity I) in the framework of the Wightman formalism.

Let

$$\phi_{(\alpha)(\dot\beta)}(x_1), \ldots, \psi_{(\gamma)(\dot\delta)}(x_{n+1}), \quad (\alpha)=\alpha_1\cdots\alpha_J, \ (\dot\beta)=\dot\beta_1\cdots\dot\beta_K$$

be a set of fields[5] transforming according to a general irreducible representation of $SL(2,C)$ and satisfying the Wightman axioms with the possible exception of local commutativity (LC).

As is known,[6] the requirement that the fields have definite transformation laws under the P, C, and T operators uniquely fixes those operators if the phase factors are chosen in such a way that

$$P\Psi_0=\Psi_0\,,\quad C\Psi_0=\Psi_0\,,\quad T\Psi_0=\Psi_0\,, \tag{2.1}$$

where Ψ_0 is the vacuum.[7]

If the C operator defines a symmetry, then the condition between VEV's[8]

$$\langle \phi_{(\alpha)(\dot{\beta})}(x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(x_{n+1})\rangle_0 = \langle \phi^{\dagger}_{(\dot{\alpha})(\beta)}(x_1)\cdots\psi^{\dagger}_{(\dot{\gamma})(\delta)}(x_{n+1})\rangle_0 \tag{2.2}$$

holds for all products in all orders. By using the Hermiticity condition, we can also write for the C symmetry the condition

$$\langle \phi_{(\alpha)(\dot{\beta})}(x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(x_{n+1})\rangle_0 = \langle \psi_{(\dot{\gamma})(\delta)}(x_{n+1})\cdots\phi_{(\dot{\alpha})(\beta)}(x_1)\rangle_0^{*}\,. \tag{2.3}$$

If the P operator defines a symmetry, we have the relation

$$\langle \phi_{(\alpha)(\dot{\beta})}(x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(x_{n+1})\rangle_0 = i^{F}(-1)^{J}\zeta^{J+K}\langle \phi_{(\dot{\alpha})(\beta)}(I_s x_1)\cdots\psi_{(\dot{\gamma})(\delta)}(I_s x_{n+1})\rangle_0\,, \tag{2.4}$$

where F is the number of half-odd-integer-spin fields, J is the total number of undotted indices in $(\alpha),\ldots,(\gamma)$, K is the total number of dotted indices in $(\dot{\beta}),\ldots,(\dot{\delta})$, I_s represents the space inversion, $\zeta=i\tau_2$, and ζ^{J+K} represents the direct product of $J+K$ ζ's acting on the spinor indices.

For the T symmetry we have the condition

$$\langle \phi_{(\alpha)(\dot{\beta})}(x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(x_{n+1})\rangle_0 = \zeta^{J+K}\langle \phi_{(\alpha)(\dot{\beta})}(I_t x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(I_t x_{n+1})\rangle_0^{*}\,, \tag{2.5}$$

where I_t represents time inversion.

In a similar manner, if the operators PC, CT, PT, and PCT constitute symmetries, corresponding conditions among VEV's hold.

In Table I we list, for the reader's convenience, all the above discrete space-time conditions on VEV's for the case of arbitrary fields. In Table II we list the same conditions for the case of scalar fields.

We shall term all these conditions on VEV's the *"discrete space-time conditions of the first kind."*

It is relevant for our purpose to recall that, as it occurs for the PCT condition, all first-kind conditions can be continued analytically into the extended tube τ_n', and they hold at any separation.

III. DISCRETE SPACE-TIME CONDITIONS OF THE SECOND KIND

Our objective is to apply the methodology of the PCT theorem to all discrete space-time conditions of the first kind. Clearly, this demands the search for new conditions valid, as the WLC condition does, at Jost points only.

Let us consider the combination of the C condition with the WLC condition, which we shall term $\bar{I}$, i.e.,

$$\bar{I}=C(\mathrm{WLC})\,. \tag{3.1}$$

Since the WLC condition is given by

$$\langle \phi_{(\alpha)(\dot{\beta})}(x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(x_{n+1})\rangle_0 = i^{F}\langle \psi_{(\gamma)(\dot{\delta})}(x_{n+1})\cdots\phi_{(\alpha)(\dot{\beta})}(x_1)\rangle_0\,, \tag{3.2}$$

by combining (3.2) with (2.3) we get for $\bar{I}$ the condition

$$\langle \phi_{(\alpha)(\dot{\beta})}(x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(x_{n+1})\rangle_0 = i^{F}\langle \phi_{(\dot{\alpha})(\beta)}(x_1)\cdots\psi_{(\dot{\gamma})(\delta)}(x_{n+1})\rangle_0^{*}\,, \tag{3.3}$$

which obviously holds at Jost points only in order to be compatible with WLC.

We now introduce the following set of conditions

TABLE I. The discrete space-time conditions of the first kind for the case of arbitrary fields, where F is the number of half-odd-integer-spin fields; J is the total number of undotted indices in $(\alpha),\ldots,(\gamma)$; K is the total number of dotted indices in $(\dot{\beta}),\ldots,(\dot{\delta})$; $\zeta=i\tau_2$; I_s, I_t, and I_{st} represent space inversion, time inversion, and total inversion of the separation, respectively, and ζ^{J+K} represents the direct product of $J+K$ ζ's acting on the spinor indices.

C	$\langle \phi_{(\alpha)(\dot{\beta})}(x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(x_{n+1})\rangle_0 = \langle \psi_{(\dot{\gamma})(\delta)}(x_{n+1})\cdots\phi_{(\dot{\alpha})(\beta)}(x_1)\rangle_0^{*}$
P	$\langle \phi_{(\alpha)(\dot{\beta})}(x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(x_{n+1})\rangle_0 = i^{F}(-1)^{J}\zeta^{J+K}\langle \phi_{(\dot{\alpha})(\beta)}(I_s x_1)\cdots\psi_{(\dot{\gamma})(\delta)}(I_s x_{n+1})\rangle_0$
T	$\langle \phi_{(\alpha)(\dot{\beta})}(x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(x_{n+1})\rangle_0 = \zeta^{J+K}\langle \phi_{(\alpha)(\dot{\beta})}(I_t x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(I_t x_{n+1})\rangle_0^{*}$
I	$\langle \phi_{(\alpha)(\dot{\beta})}(x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(x_{n+1})\rangle_0 = \langle \phi_{(\alpha)(\dot{\beta})}(x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(x_{n+1})\rangle_0$
PC	$\langle \phi_{(\alpha)(\dot{\beta})}(x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(x_{n+1})\rangle_0 = i^{F}(-1)^{J}\zeta^{J+K}\langle \psi_{(\gamma)(\dot{\delta})}(I_s x_{n+1})\cdots\phi_{(\alpha)(\dot{\beta})}(I_s x_1)\rangle_0^{*}$
CT	$\langle \phi_{(\alpha)(\dot{\beta})}(x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(x_{n+1})\rangle_0 = \zeta^{J+K}\langle \psi_{(\dot{\gamma})(\delta)}(I_t x_{n+1})\cdots\phi_{(\dot{\alpha})(\beta)}(I_t x_1)\rangle_0$
PT	$\langle \phi_{(\alpha)(\dot{\beta})}(x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(x_{n+1})\rangle_0 = i^{F}(-1)^{J}\langle \phi_{(\dot{\alpha})(\beta)}(I_{st} x_1)\cdots\psi_{(\dot{\gamma})(\delta)}(I_{st} x_{n+1})\rangle_0^{*}$
PCT	$\langle \phi_{(\alpha)(\dot{\beta})}(x_1)\cdots\psi_{(\gamma)(\dot{\delta})}(x_{n+1})\rangle_0 = i^{F}(-1)^{J}\langle \psi_{(\gamma)(\dot{\delta})}(I_{st} x_{n+1})\cdots\phi_{(\alpha)(\dot{\beta})}(I_{st} x_1)\rangle_0$

TABLE II. The discrete space-time conditions of the first kind for the case of scalar fields $\varphi(x_1), \ldots, \varphi(x_{n+1})$, with $W(\xi_1, \ldots, \xi_n) = \langle \varphi(x_1), \ldots, \varphi(x_{n+1}) \rangle_0$; $\xi_K = x_K - x_{K+1}$, $K = 1, 2, \ldots, n$; and the phase selection $\eta_A = +1$, $A = P, C, T$.

C	$W(\xi_1, \ldots, \xi_n) = W(I_{st}\xi_n, \ldots, I_{st}\xi_1)^*$
P	$W(\xi_1, \ldots, \xi_n) = W(I_s\xi_1, \ldots, I_s\xi_n)$
T	$W(\xi_1, \ldots, \xi_n) = W(I_t\xi_1, \ldots, I_t\xi_n)^*$
I	$W(\xi_1, \ldots, \xi_n) = W(\xi_1, \ldots, \xi_n)$
PC	$W(\xi_1, \ldots, \xi_n) = W(I_t\xi_n, \ldots, I_t\xi_1)^*$
CT	$W(\xi_1, \ldots, \xi_n) = W(I_s\xi_n, \ldots, I_s\xi_1)$
PT	$W(\xi_1, \ldots, \xi_n) = W(I_{st}\xi_1, \ldots, I_{st}\xi_n)^*$
PCT	$W(\xi_1, \ldots, \xi_n) = W(\xi_n, \ldots, \xi_1)$

defined by combining the $\overline{I}$ conditions with the first-kind conditions[9]:

$$\begin{aligned}
\overline{C} &= (C)\overline{I} = (\mathrm{WLC}),\\
\overline{P} &= (P)\overline{I} = (PC)(\mathrm{WLC}),\\
\overline{T} &= (T)\overline{I} = (TC)(\mathrm{WLC}),\\
\overline{I} &= (I)\overline{I} = C(\mathrm{WLC}),\\
\overline{PC} &= (PC)\overline{I} = P(\mathrm{WLC}),\\
\overline{CT} &= (CT)\overline{I} = T(\mathrm{WLC}),\\
\overline{PT} &= (PT)\overline{I} = (PCT)(\mathrm{WLC}),\\
\overline{PCT} &= (PCT)\overline{I} = PT(\mathrm{WLC}).
\end{aligned} \tag{3.4}$$

In Tables III and IV we list, for the reader's convenience, all conditions (3.4) for the cases of arbitrary fields and scalar fields, respectively.

We shall term all conditions (3.4) on VEV's the *discrete space-time conditions of the second kind*.

A few remarks are now in order. All second-kind conditions are valid at Jost points only. This property arises from the fact that the second-kind conditions can be introduced as a combination of a suitably chosen first-kind condition (valid at any separation) with WLC which holds at Jost points only. For instance, $\overline{P} = (PC)(\mathrm{WLC})$, etc.

Furthermore, the second-kind conditions are new conditions which do not necessarily demand the validity of the first-kind conditions and the $\overline{I}$ condition to hold. For instance, the relation $\overline{P} = (P)\overline{I} = (PC)\overline{C} = (PC)(\mathrm{WLC})$ essentially shows the validity of $\overline{P}$ when the simultaneous validity of P and $\overline{I}$ (or PC and $\overline{C} = \mathrm{WLC}$) occurs, but those latter conditions can be violated and $\overline{P}$ can hold.[10]

Finally, the question which ultimately arises is whether the second-kind conditions on VEV's can be introduced in terms of corresponding operators by acquiring in this way a direct physical meaning as transformations of individual fields, as it occurs for the first-kind conditions.

To answer this question, let us suppose that there exists an antiunitary "$\overline{C}$ operator" such that

$$\begin{aligned}
\overline{C}\varphi(x)\overline{C}^{-1} &= \zeta_{\overline{C}}\varphi^\dagger(x),\\
\overline{C}\Psi_0 &= \Psi_0,
\end{aligned} \tag{3.5}$$

where $\zeta_{\overline{C}} = \pm 1$, Ψ_0 is the physical vacuum, and $\varphi(x)$ is, for instance, a scalar field.

Such a "$\overline{C}$ operator," together with the Hermiticity condition, would indeed produce the $\overline{C}(=\mathrm{WLC})$ condition on VEV's whenever it holds. However, in view of the invariance of the vacuum in (3.5), the $\overline{C}$ condition derived in the above way would hold not only at Jost points, but also at all physical points, which, in turn, would imply the vanishing of the field. Therefore, no physically significant "$\overline{C}$ operator" can be introduced. A similar situation occurs for all other second-kind conditions which retain their significance only as "conditions" on VEV's at Jost points.

The reader can verify through tedious but straightforward calculations that the second-kind conditions satisfy all the Wightman axioms.

TABLE III. The discrete space-time conditions of the second kind for the case of arbitrary fields, where we have used the same notations as those of Table I.

$\overline{C}$	$\langle \phi_{(\alpha)(\dot\beta)}(x_1) \cdots \psi_{(\gamma)(\dot\delta)}(x_{n+1})\rangle_0 = i^F \langle \psi_{(\gamma)(\dot\delta)}(x_{n+1}) \cdots \phi_{(\alpha)(\dot\beta)}(x_1)\rangle_0$
$\overline{P}$	$\langle \phi_{(\alpha)(\dot\beta)}(x_1) \cdots \psi_{(\gamma)(\dot\delta)}(x_{n+1})\rangle_0 = (-1)^J \zeta^{J+K} \langle \phi_{(\alpha)(\dot\beta)}(I_s x_1) \cdots \psi_{(\gamma)(\dot\delta)}(I_s x_{n+1})\rangle_0^*$
$\overline{T}$	$\langle \phi_{(\alpha)(\dot\beta)}(x_1) \cdots \psi_{(\gamma)(\dot\delta)}(x_{n+1})\rangle_0 = i^F \zeta^{J+K} \langle \phi_{(\dot\alpha)(\beta)}(I_t x_1) \cdots \psi_{(\gamma)(\dot\delta)}(I_t x_{n+1})\rangle_0$
$\overline{I}$	$\langle \phi_{(\alpha)(\dot\beta)}(x_1) \cdots \psi_{(\gamma)(\dot\delta)}(x_{n+1})\rangle_0 = i^F \langle \phi_{(\dot\alpha)(\beta)}(x_1) \cdots \psi_{(\dot\gamma)(\delta)}(x_{n+1})\rangle_0^*$
$\overline{PC}$	$\langle \phi_{(\alpha)(\dot\beta)}(x_1) \cdots \psi_{(\gamma)(\dot\delta)}(x_{n+1})\rangle_0 = (-1)^J \zeta^{J+K} \langle \psi_{(\dot\gamma)(\delta)}(I_s x_{n+1}) \cdots \phi_{(\dot\alpha)(\beta)}(I_s x_1)\rangle_0$
$\overline{CT}$	$\langle \phi_{(\alpha)(\dot\beta)}(x_1) \cdots \psi_{(\gamma)(\dot\delta)}(x_{n+1})\rangle_0 = i^F \zeta^{J+K} \langle \psi_{(\gamma)(\dot\delta)}(I_t x_{n+1}) \cdots \phi_{(\alpha)(\dot\beta)}(I_t x_1)\rangle_0^*$
$\overline{PT}$	$\langle \phi_{(\alpha)(\dot\beta)}(x_1) \cdots \psi_{(\gamma)(\dot\delta)}(x_{n+1})\rangle_0 = (-1)^J \langle \phi_{(\alpha)(\dot\beta)}(I_{st} x_1) \cdots \psi_{(\gamma)(\dot\delta)}(I_{st} x_{n+1})\rangle_0$
$\overline{PCT}$	$\langle \phi_{(\alpha)(\dot\beta)}(x_1) \cdots \psi_{(\gamma)(\dot\delta)}(x_{n+1})\rangle_0 = (-1)^J \langle \psi_{(\dot\gamma)(\delta)}(I_{st} x_{n+1}) \cdots \phi_{(\dot\alpha)(\beta)}(I_{st} x_1)\rangle_0^*$

TABLE IV. The discrete space-time conditions of the second kind for the case of scalar fields, where we have used the same notations as those of Table II.

$\overline{C}$	$W(\xi_1,\ldots,\xi_n)=W(I_{st}\xi_n,\ldots,I_{st}\xi_1)$
$\overline{P}$	$W(\xi_1,\ldots,\xi_n)=W(I_s\xi_1,\ldots,I_s\xi_n)^*$
$\overline{T}$	$W(\xi_1,\ldots,\xi_n)=W(I_t\xi_1,\ldots,I_t\xi_n)$
$\overline{I}$	$W(\xi_1,\ldots,\xi_n)=W(\xi_1,\ldots,\xi_n)^*$
$\overline{PC}$	$W(\xi_1,\ldots,\xi_n)=W(I_t\xi_n,\ldots,I_t\xi_1)$
$\overline{CT}$	$W(\xi_1,\ldots,\xi_n)=W(I_s\xi_n,\ldots,I_s\xi_1)^*$
$\overline{PT}$	$W(\xi_1,\ldots,\xi_n)=W(I_{st}\xi_1,\ldots,I_{st}\xi_n)$
$\overline{PCT}$	$W(\xi_1,\ldots,\xi_n)=W(\xi_n,\ldots,\xi_1)^*$

IV. ANALYTIC APPROACH TO DISCRETE SPACE-TIME SYMMETRIES

In our formulation, the PCT theorem ultimately relates two conditions, one of the first kind (PCT) and one of the second kind ($\overline{C}$=WLC), such that their combination $(PCT)(\overline{C})$ is the $\overline{PT}$ condition. This illustrates the key role played by the $\overline{PT}$ condition which is ultimately represented by the total inversion of the separations in the framework of the customary analytic extension into the extended tube.

The above pattern can be followed for all conditions. Indeed we can regroup all pairs of first- and second-kind conditions in such a way that, as for the $PCT \leftrightarrow \overline{C}$ case, their combination is the $\overline{PT}$ condition, according to the following relations[11]:

$$\begin{aligned}
&(C)\overline{PCT}=\overline{PT}\,,\\
&(P)\overline{T}=\overline{PT}\,,\\
&(T)\overline{P}=\overline{PT}\,,\\
&(I)\overline{PT}=\overline{PT}\,,\\
&(PC)\overline{CT}=\overline{PT}\,,\\
&(CT)\overline{PC}=PT\,,\\
&(PT)\overline{I}=\overline{PT}\,,\\
&(PCT)\overline{C}=\overline{PT}\,.
\end{aligned}\tag{4.1}$$

The above reclassification opens the way for a generalization of the PCT theorem to all discrete space-time symmetry conditions. Indeed, as for the $PCT \leftrightarrow \overline{C}$ case, relations (4.1) point out which is the second-kind condition which can be linked to a first-kind condition by means of the total inversion of the separation and vice versa. Furthermore, this pairing of first- and second-kind conditions is unique in the sense that given an arbitrary first-kind condition (e.g., P) there is only one second-kind condition ($\overline{T}$) such that their combination is the $\overline{PT}$ condition, and vice versa.

We can thus introduce the following definition:

Definition. Two conditions, one of the first kind and one of the second kind, are called *dual conditions* if and only if their combination is the $\overline{PT}$ condition.

For the convenience of the reader we list in Table V all pairs of dual conditions.

We are now equipped to formulate our main result.

Theorem. In a field theory satisfying the Wightman axioms, with the possible exception of the local commutativity (LC) condition, if one of the first-kind conditions holds at all points, then its dual condition holds at Jost points. Conversely, if one of the second-kind conditions holds in a (real) neighborhood of a Jost point, then its dual condition holds everywhere.

Proof. Let us consider first the case of a scalar field $\varphi(x)$ with VEV:

$$W(\xi_1,\ldots,\xi_n)=\langle\varphi(x_1)\cdots\varphi(x_{n+1})\rangle_0\,,\qquad \xi_K=x_K-x_{K+1}\,,\quad K=1,2,\ldots,n\,.\tag{4.2}$$

The cases $PCT \leftrightarrow \overline{C}$ and $CT \leftrightarrow \overline{PC}$ can be proved on equivalent grounds since we deal with relations among holomorphic functions simply defined at different points of the extended tube τ_n' (Ref. 12).

Let us represent the PCT and CT conditions with the unified notation

$$W(\xi_1,\ldots,\xi_n)=W(\theta\xi_n,\ldots,\theta\xi_1)\,,\tag{4.3}$$

where for $\theta=I$ (identity) we have the PCT condition and for $\theta=I_s$ (space inversion) we have the CT condition.

TABLE V. The pairing of all "dual conditions" such that their combination is equivalent to the $\overline{PT}$ conditions. According to the theorem of Sec. IV, each condition can be linked to its dual (and vice versa) through the customary analytic continuation and the use of the connectivity properties of the $L_+(C)$ invariance group in a way equivalent to the $PCT \leftrightarrow \overline{C}$(=WLC) link of the celebrated PCT theorem. No link can in general be established among nondual conditions (e.g., P and $\overline{PC}$ or P and PC).

$C \leftrightarrow \overline{PCT}$
$P \leftrightarrow \overline{T}$
$T \leftrightarrow \overline{P}$
$I \leftrightarrow \overline{PT}$
$PC \leftrightarrow \overline{CT}$
$CT \leftrightarrow \overline{PC}$
$PT \leftrightarrow \overline{I}$
$PCT \leftrightarrow \overline{C}$

Let $\underline{W}(z_1, \ldots, z_n)$ be a holomorphic function in the complex vectors $z_1, \ldots, z_n$, with $z_K = \xi_K - i\eta_K$, $\xi_K = x_K - x_{K+1}$, $K = 1, 2, \ldots, n$, holomorphic in the extended tube τ_n' and such that

$$\lim_{\substack{\eta_1, \ldots, \eta_n \to 0 \\ \in V_+}} \underline{W}(z_1, \ldots, z_n) = W(\xi_1, \ldots, \xi_n). \tag{4.4}$$

Then $\underline{W}(z_1, \ldots, z_n) - \underline{W}(\theta z_n, \ldots, \theta z_1)$ is holomorphic through τ_n', and since from (4.3) it vanishes at real separations, it vanishes everywhere. Therefore (4.3) is equivalent to the condition

$$\underline{W}(z_1, \ldots, z_n) = \underline{W}(\theta z_n, \ldots, \theta z_1) \tag{4.5}$$

valid throughout τ_n'.

But W is invariant under $L_+(C)$ transformations, i.e.,

$$\underline{W}(z_1, \ldots, z_n) = \underline{W}(\Lambda z_1, \ldots, \Lambda z_n), \quad \Lambda \in L_+(C), \; z_1, \ldots, z_n \in \tau_n'. \tag{4.6}$$

Consequently, from the connectivity properties of $L_+(C)$ and for $\Lambda = -1 \in L_+(C)$, we can write from (4.5) and (4.6)

$$\underline{W}(z_1, \ldots, z_n) = \underline{W}(-\theta z_n, \ldots, -\theta z_1). \tag{4.7}$$

At Jost points (4.7) is a relation between vacuum expectation values, i.e.,

$$W(\xi_1, \ldots, \xi_n) = W(-\theta \xi_n, \ldots, -\theta \xi_1), \tag{4.8}$$

which is the $\overline{C}$ condition (WLC) for $\theta = I$ and the $\overline{PC}$ condition for $\theta = I_s$.

Note that at real points other than Jost points (4.7) would not give rise to a relation between vacuum expectation values in view of the discontinuity across the cut. Indeed, $z_1, \ldots, z_n$ would approach real vectors from V_+, while $-\theta z_n, \ldots, -\theta z_1$ would approach real vectors from V_-. This also shows the validity of the $\overline{PC}$ condition at Jost points only, as it occurs for $\overline{C}$.

Conversely, if (4.8) holds in a real neighborhood of a Jost point, then it also holds in a complex neighborhood, and, by analytical continuation, (4.7) holds throughout τ_n'. By using the $L_+(C)$ invariance property (4.6) for $\Lambda = -1$, then (4.5) follows. By passing to the boundary in τ_n' we recover (4.3) at any separation, namely, the PCT condition for $\theta = I$ and the CT condition for $\theta = I_s$.

This proves the $PCT \leftrightarrow \overline{C}$ and $CT \leftrightarrow \overline{PC}$ cases.

Consider now the $P \leftrightarrow \overline{T}$ and $I \leftrightarrow \overline{PT}$ cases. Suppose that the condition

$$W(\xi_1, \ldots, \xi_n) = W(\theta \xi_1, \ldots, \theta \xi_n) \tag{4.9}$$

holds for any separation. Then (4.9) is the P condition for $\theta = I_s$ and the I condition for $\theta = I$.

But condition (4.9) is equivalent to the condition

$$\underline{W}(z_1, \ldots, z_n) = \underline{W}(\theta z_1, \ldots, \theta z_n), \tag{4.10}$$

valid at all points of τ_n', and by using $L_+(C)$ invariance we can write

$$\underline{W}(z_1, \ldots, z_n) = \underline{W}(-\theta z_1, \ldots, -\theta z_n). \tag{4.11}$$

At Jost points (4.11) is a relation between vacuum expectation values, i.e.,

$$W(\xi_1, \ldots, \xi_n) = W(-\theta \xi_1, \ldots, -\theta \xi_n), \tag{4.12}$$

which gives the $\overline{T}$ condition for $\theta = I_s$ and the $\overline{PT}$ condition for $\theta = I$. Again at real points other than Jost points, (4.11) does not give rise at the boundary to a relation between VEV's in view of the discontinuity across the cut. This also proves that $\overline{T}$ and $\overline{PT}$, as for $\overline{C}$, are valid at Jost points only.

Vice versa, if (4.12) holds in a real neighborhood of a Jost point, then (4.11) holds at all points of τ_n'. By using $L_+(C)$ invariance, (4.10) follows, and by passing to the boundary in τ_n' one recovers (4.9) at any separation which is the P condition for $\theta = I_s$ and the I condition for $\theta = I$.

This proves the $P \leftrightarrow \overline{T}$ and $I \leftrightarrow \overline{PT}$ cases.

The remaining cases $C \leftrightarrow \overline{PCT}$, $PC \leftrightarrow \overline{CT}$, $PT \leftrightarrow \overline{I}$, and $T \leftrightarrow \overline{P}$ need a different treatment in view of the operation of complex conjugation appearing in the right-hand side.

Let us consider first the cases $C \leftrightarrow \overline{PCT}$ and $PC \leftrightarrow \overline{CT}$ which can be treated simultaneously. Conditions C and PC can be represented with the unified relation

$$W(\xi_1, \ldots, \xi_n) = W(\Xi \xi_n, \ldots, \Xi \xi_1)^*, \tag{4.13}$$

where we have the C condition for $\Xi = I_{st}$ (total inversion) and the PC condition for $\Xi = I_t$ (time inversion).

Together with the function $\underline{W}(z_1, \ldots, z_n)$ analytic in τ_n' and satisfying condition (4.4), we consider a new function $\underline{W}'(z_1^*, \ldots, z_n^*)$ (see Ref. 13) analytic in $\overline{\tau}_n'$, the domain complex conjugate to τ_n', and such that

$$\lim_{\substack{\eta_1, \ldots, \eta_n \to 0 \\ \in V_+}} \underline{W}'(z_1^*, \ldots, z_n^*) = W(\xi_1, \ldots, \xi_n)^*. \tag{4.14}$$

Clearly $\underline{W}'$ is also invariant under $L_+(C)$ transformations, i.e.,

$$\underline{W}'(z_1^*, \ldots, z_n^*) = \underline{W}'((\Lambda z_1)^*, \ldots, (\Lambda z_n)^*), \quad \Lambda \in L_+(C). \tag{4.15}$$

Then, for $\Lambda = -1 \in L_+(C)$ we have

$$\underline{W}'(z_1^*, \ldots, z_n^*) = \underline{W}'(-z_1^*, \ldots, -z_n^*). \tag{4.16}$$

We can thus say that each side of (4.13) is a boundary value of a holomorphic function. But in view of the theorem of the Appendix, the function

$\underline{W}(z_1,\ldots,z_n)-\underline{W}(\Xi z_n^*,\ldots,\Xi z_1^*)$ is holomorphic, and since it vanishes in a real environment by virtue of (4.13), it vanishes everywhere. Consequently, (4.13) implies the relation

$$\underline{W}'(z_1^*,\ldots,z_n^*)=\underline{W}'(\Xi z_n^*,\ldots,\Xi z_1^*). \qquad (4.17)$$

Conversely, if (4.17) holds, then by passing to the boundary (4.13) follows at any separation. Therefore, (4.17) is equivalent to (4.13). By using (4.16) we can write

$$\underline{W}'(z_1^*,\ldots,z_n^*)=\underline{W}'(-\Xi z_n^*,\ldots,-\Xi z_1^*). \qquad (4.18)$$

At Jost points, (4.18) is a relation between VEV's, i.e.,

$$W(\xi_1,\ldots\xi_n)=W(-\Xi\xi_n,\ldots,-\Xi\xi_1)^*, \qquad (4.19)$$

which is the $\overline{PCT}$ condition for $\Xi=I_{st}$ and the $\overline{CT}$ condition for $\Xi=I_t$.

At real points other than Jost points, (4.18) does not give rise to a relation between VEV's. Indeed, we can write

$$\underline{W}(z_1,\ldots,z_n)-\underline{W}(-z_1,\ldots,-z_n)$$
$$=\underline{W}'(-\Xi z_n^*,\ldots,-\Xi z_1^*)-\underline{W}'(\Xi z_n^*,\ldots,\Xi z_1^*), \qquad (4.20)$$

which at the boundary for timelike separations would imply that

$$\mathrm{Disc}\,W(\xi_1,\ldots,\xi_n)=[\mathrm{Disc}\,W(-\Xi\xi_n,\ldots,-\Xi\xi_1)]^*, \qquad (4.21)$$

which is in contradiction with the purely imaginary nature of the discontinuity itself.

Finally, if (4.19) holds in a (real) neighborhood of a Jost point, then (4.18) and (4.17) hold and (4.13) follows at any separation. This proves the $C\leftrightarrow\overline{PCT}$ and $PC\leftrightarrow\overline{CT}$ cases.

An alternative proof for the above cases can be given by considering a function $\underline{\tilde{W}}(z_1,\ldots,z_n)$ analytic in τ_n' and such that[14]

$$\lim_{\substack{\eta_1,\ldots,\eta_n\to 0\\ \in V_+}}\underline{\tilde{W}}(z_1,\ldots,z_n)=\tilde{W}(\xi_1,\ldots,\xi_n)$$
$$=W(\xi_1,\ldots,\xi_n)^*. \qquad (4.22)$$

This essentially implies that in the transition from W to $\tilde{W}$ the coefficients of the (absolutely) convergent power series expansion are substituted by their complex conjugates.

The proof of the $C\leftrightarrow\overline{PCT}$ and $PC\leftrightarrow\overline{CT}$ cases then carries through as for the previous cases which do not involve complex conjugation.

Finally, let us consider the condition

$$W(\xi_1,\ldots,\xi_n)=W(\Xi\xi_1,\ldots,\Xi\xi_n)^*, \qquad (4.23)$$

which represents the PT condition for $\Xi=I_{st}$ and the T condition for $\Xi=I_t$.

But (4.23) can be analytically continued to the condition between holomorphic functions

$$\underline{W}(z_1,\ldots,z_n)=\underline{W}'(\Xi z_1^*,\ldots,\Xi z_n^*). \qquad (4.24)$$

By using $L_+(C)$ invariance we can write

$$\underline{W}(z_1,\ldots,z_n)=\underline{W}'(-\Xi z_1^*,\ldots,-\Xi z_n^*), \qquad (4.25)$$

which at Jost points is a condition between VEV's, i.e.,

$$W(\xi_1,\ldots,\xi_n)=W(-\Xi\xi_1,\ldots,-\Xi\xi_n)^*. \qquad (4.26)$$

This is the $\overline{I}$ condition for $\Xi=I_{st}$ and the $\overline{P}$ condition for $\Xi=I_t$. Again at real points other than Jost points (4.25) does not give rise to a condition between VEV's.

Finally, if (4.26) holds, then (4.25) and (4.24) follow, and, at the boundary, one recovers (4.23) at any separation. This latter relation is the PT condition for $\Xi=I_{st}$ and the T condition for $\Xi=I_t$.

This concludes the proof of the theorem for the case of a scalar field.

The extension of the proof to the case of arbitrary fields $\phi_{(\alpha)(\dot\beta)}(x_1),\ldots,\psi_{(\gamma)(\dot\delta)}(x_{n+1})$ transforming according to a general irreducible representation of the Lorentz group is straightforward. In this case, however, each set of dual conditions must be treated individually in accordance with Tables I and III.

The $PCT\leftrightarrow\overline{C}$ case is well known in the literature.[3] The case $I\leftrightarrow\overline{PT}$ is also known.[9]

For the sake of brevity, we shall consider only one representative case, such as the $P\leftrightarrow\overline{T}$ case.

Let $\underline{W}_{\mu\ldots\nu}(z_1,\ldots,z_n)$ be a function holomorphic in τ_n' with the transformation law under $SL(2,C)\otimes SL(2,C)$ (Ref. 3):

$$\sum_{\mu'\cdots\nu'}S_{\mu\mu'}(A,B)\cdots S_{\nu\nu'}(A,B)\underline{W}_{\mu'\ldots\nu'}(z_1,\ldots,z_n)$$
$$=\underline{W}_{\mu\ldots\nu}(\Lambda z_1\ldots\Lambda z_n), \qquad (4.27)$$
$$\Lambda=\Lambda(A,B)\in SL(2,C)\otimes SL(2,C),$$

and such that

$$\lim_{\substack{\eta_1,\ldots,\eta_n\to 0\\ \in V_+}}\underline{W}_{\mu\ldots\nu}(z_1,\ldots,z_n)$$
$$=\langle\phi_{(\alpha)(\dot\beta)}(x_1)\cdots\psi_{(\gamma)(\dot\delta)}(x_{n+1})\rangle_0. \qquad (4.28)$$

Then, in view of the property

$$S_{\mu\mu'}(-1,1)\cdots S_{\nu\nu'}(-1,1)=(-1)^J\delta_{\mu\mu'}\cdots\delta_{\nu\nu'}, \qquad (4.29)$$

we have the $\overline{PT}$ condition for holomorphic functions

$$\underline{W}_{\mu\ldots\nu}(z_1,\ldots,z_n)=(-1)^J\underline{W}_{\mu\ldots\nu}(-z_1,\ldots,-z_n), \qquad (4.30)$$

which holds for all points of analyticity.

Let $\underline{\hat{W}}_{\rho\cdots\sigma}(z_1,\ldots,z_n)$ be another function analytic in τ_n' and such that

$$\lim_{\substack{\eta_1,\ldots,\eta_n\to 0\\ \in V_+}} \underline{\hat{W}}_{\rho\cdots\sigma}(z_1,\ldots,z_n) = \langle \phi_{(\dot\alpha)(\beta)}(x_1)\cdots\psi_{(\dot\gamma)(\delta)}(x_{n+1})\rangle_0 . \quad (4.31)$$

Then the P condition of Table I is equivalent to the condition between holomorphic functions

$$\underline{W}_{\mu\cdots\nu}(z_1,\ldots,z_n) = i^F(-1)^J\zeta^{J+K}\underline{\hat{W}}_{\rho\cdots\sigma}(I_s z_1,\ldots,I_s z_n). \quad (4.32)$$

By using (4.30) we can write

$$\underline{W}_{\mu\cdots\nu}(z_1,\ldots,z_n) = i^F\zeta^{J+K}\underline{\hat{W}}_{\rho\cdots\sigma}(-I_s z_1,\ldots,-I_s z_n). \quad (4.33)$$

At Jost points (4.33) is a relation between VEV's, namely the $\overline{T}$ condition of Table III. At real points other than Jost points (4.33) does not produce a relation between VEV's since it would approach real vectors from the plus tube in the left-hand side and real vectors from the minus tube in the right-hand side.

Vice versa, if the $\overline{T}$ condition of Table III holds in a real neighborhood of a Jost point, then (4.33) holds and (4.32) follows. But (4.32) is equivalent to the P condition at any separation. This proves the $P \leftrightarrow \overline{T}$ case.

All other cases follow accordingly, the conditions implying complex conjugation being treated as for the case of scalar fields. Q.E.D.

We now investigate possible links among different conditions.

In any field theory satisfying the Wightman axioms (with the possible exception of LC) the conditions I and $\overline{PT}$ always hold. We shall term those conditions the "*trivial conditions*."

We shall term "*equivalent conditions*" two or more conditions which differ only by phase factors.

When two nonequivalent conditions hold, we shall term "*complementary condition*" the third condition which follows from the simultaneous validity of the previous two. For instance, the condition complementary to P and PCT is CT.

Consider a field theory where a nontrivial condition (e.g., P) holds. Then four conditions always hold, namely, the original condition (P), its dual ($\overline{T}$), and the two trivial conditions ($I, \overline{PT}$). Notice that in this case no new complementary condition arises [$PI=P$, $P(\overline{PT})=T$, $(\overline{T})I=\overline{T}$, $\overline{T}(\overline{PT})=P$].

As a direct consequence of our theorem we have the following first corollary:

Corollary 1. Consider a field theory satisfying the Wightman axioms with the possible exception of LC. If two nontrivial, nondual, and nonequivalent conditions hold, then eight conditions, four of the first kind and four of the second kind, hold.

For instance, if the P and T conditions hold, then their dual conditions $\overline{T}$ and $\overline{P}$, their complementary conditions, PT and $\overline{PT}$, and the duals of the complementary conditions, $\overline{I}$ and I, hold. The same set of eight conditions also holds if instead of P and T the original conditions are any one pair of the set: $(P,\overline{P})$, $(P,\overline{I})$, (P,PT), $(\overline{T},T)$, $(\overline{T},\overline{P})$, $(\overline{T},PT)$, $(\overline{T},\overline{I})$.

Therefore, all first- and second-kind conditions can be regrouped into seven sets of conditions, each including eight conditions which are mutually correlated when two nontrivial, nonequivalent, and nondual conditions of the set hold, according to the scheme of Table VI.

The proof of the above corollary can also be done in the framework of the Wightman formalism for what concerns the transition from a given condition to its dual in accordance with the previous theorem. The inclusion of the other conditions then trivially follows from the simultaneous validity of two nonequivalent conditions.

It is interesting to remark, however, that within the framework of the Wightman formalism an analytic continuation from any condition into all the other conditions of the same set of Table VI is forbidden in view of the lack of sufficiently broad connectivity property of the $L_+(C)$ invariance group. More explicitly, consider a PCT-invariant theory where CT is a symmetry. Then the P, $\overline{PC}$, $\overline{C}$, $\overline{T}$, I, and $\overline{PT}$ conditions hold. Nevertheless, the customary analytic continuation allows the transition separately from P to $\overline{T}$ and CT to $\overline{PC}$

TABLE VI. The grouping of all first- and second-kind conditions according to Corollary 1 of Sec. IV into seven sets of eight conditions which hold when two nontrivial, nondual, and nonequivalent conditions hold. For instance, all conditions of set 7 hold when any one of the pairs of conditions (PC,CT), (PC,PT), (CT,PT), $(PC,\overline{PC})$, $(PC,\overline{I})$, $(CT,\overline{CT})$, $(CT,\overline{I})$, $(PT,\overline{CT})$, $(PT,\overline{PC})$, $(\overline{CT},\overline{PC})$, $(\overline{CT},\overline{I})$, $(\overline{PC},\overline{I})$ holds. In this case, pairs such as (PC,I), $(CT,\overline{PT})$ are excluded since I and $\overline{PT}$ are trivial conditions. Similarly, pairs of dual conditions, such as $(PC,\overline{CT})$, are equally excluded.

1	P	C	PC	I	$\overline{T}$	$\overline{PCT}$	$\overline{CT}$	$\overline{PT}$
2	P	T	PT	I	$\overline{T}$	$\overline{P}$	$\overline{I}$	$\overline{PT}$
3	C	T	CT	I	$\overline{PCT}$	$\overline{P}$	$\overline{PC}$	$\overline{PT}$
4	P	CT	PCT	I	$\overline{T}$	$\overline{PC}$	$\overline{C}$	$\overline{PT}$
5	C	PT	PCT	I	$\overline{PCT}$	$\overline{I}$	$\overline{C}$	$\overline{PT}$
6	T	PC	PCT	I	$\overline{P}$	$\overline{CT}$	$\overline{C}$	$\overline{PT}$
7	PC	CT	PT	I	$\overline{CT}$	$\overline{PC}$	$\overline{I}$	$\overline{PT}$

and not from *CT* to *P*, this latter condition being allowed by the simultaneous validity of *PCT* and *CT*.[15]

A simple generalization of our results gives rise to the following corollary:

Corollary 2. Consider a field theory satisfying the Wightman axioms with the possible exception of LC. If three nontrivial, nondual, nonequivalent and noncomplementary conditions hold, then all first- and second-kind conditions hold.

This is the case if, for instance, *P* and *C* are symmetries of the theory. Then, the requirement that *T* is also a symmetry trivially implies that all first-kind conditions are symmetries, and, by using our theorem, all second-kind conditions hold. Corollary 2, however, tells us that this is also the case if, instead of *P* and *C*, the original conditions are *P* and $\overline{PCT}$ or $\overline{T}$ and *C* or $\overline{T}$ and $\overline{PCT}$ and if, instead of *T*, the additional condition is $\overline{P}$. Similarly, if *PC* and *PCT* are symmetries, then the validity of any one among the $\overline{PC}$, $\overline{I}$, $\overline{T}$, $\overline{PCT}$ conditions implies the validity of all first- and second-kind conditions. In this latter example, $\overline{CT}$ and $\overline{C}$ are excluded since they are the duals of *PC* and *PCT*, respectively; similarly, $\overline{P}$ is excluded since it is the dual of the condition complementary to *PC* and *PCT*, namely *T*.

In Table VII we list a representative set of three conditions in accordance with the requirements of corollary 2.

TABLE VII. Some representative sets of three conditions for which, according to Corollary 2 of Sec. IV, when they hold simultaneously and independently, all first- and second-kind conditions hold. Each set is composed of three nontrivial, nonequivalent, nondual, and noncomplementary conditions. According to Table I, there are seven nontrivial (i.e., excluding the identity *I*) first-kind conditions. They can be combined into 43 sets of three conditions. As indicated in this table, by including the dual conditions, each of the above sets produces seven nonequivalent new sets. Therefore there are 344 sets of three conditions which satisfy the requirement of Corollary 2.

P,C,T	P,C,CT	P,T,PCT
$P,\overline{PCT},T$	$P,\overline{PCT},CT$	$P,T,\overline{C}$
$P,C,\overline{P}$	$P,C,\overline{PC}$	$P,\overline{P},PCT$
$\overline{T},C,T$	$\overline{T},C,CT$	$\overline{T},T,PCT$
$P,\overline{PCT},\overline{P}$	$P,\overline{PCT},\overline{PC}$	$P,\overline{P},\overline{C}$
$\overline{T},C,\overline{P}$	$\overline{T},C,\overline{PC}$	$\overline{T},\overline{P},PCT$
$\overline{T},\overline{PCT},T$	$\overline{T},\overline{PCT},CT$	$\overline{T},T,\overline{C}$
$\overline{T},\overline{PCT},\overline{P}$	$\overline{T},\overline{PCT},\overline{PC}$	$\overline{T},\overline{P},\overline{C}$

V. CONCLUDING REMARKS

According to the theorem of Sec. IV, schematically represented in Table V, each of the eight discrete space-time first-kind conditions on VEV's can be linked to its dual (and vice versa) through the customary analytic continuation into the extended tube τ_n' and use of the connectivity properties of the $L_+(C)$ invariance group.

The case $PCT \leftrightarrow \overline{C}$ is the familiar *PCT* theorem.[16] The case $I \leftrightarrow \overline{PT}$ is a trivial case in the sense that for all field theories in the framework of the Wightman formalism the $\overline{PT}$ condition[9] and, trivially, the *I* condition always hold. The remaining cases of Table V are treated in a way equivalent to the $PCT \leftrightarrow \overline{C}$ case, and they have equivalent implications.

We shall consider first symmetry-preserving field theories.

Let us remark in this respect that our second-kind conditions are not assumed *ad hoc* to create the necessary framework for our results. On the contrary, as implied by our theorem, they are a direct consequence of the validity of the customary space-time symmetries in the framework of the Wightman axioms.

Therefore, *the significance of our theorem for symmetry-preserving field theories lies in the fact that if the customary space-time symmetry conditions hold and the Wightman axioms (with the possible exception of LC) are preserved, then new conditions hold at Jost points.*

For instance, if charge conjugation is a symmetry of the system and the Wightman axioms are preserved, then a new condition ($\overline{PCT}$) holds at Jost points. Similarly, if *T* is a symmetry, then a new condition ($\overline{P}$) holds at Jost points.

The case for which *PT* is a symmetry has a particular significance for nonspinorial field theories. Indeed in this case the dual condition $\overline{I}$ (see Ref. 17) is simply the reality condition.

Therefore, *for nonspinorial field theories satisfying the Wightman axioms with the possible exception of LC where PT is a symmetry, the VEV's at Jost points are real valued. Vice versa; if the VEV's at Jost points are real valued, then the PT condition holds.*[18]

It is essential to emphasize that if the restriction of the validity of the Wightman axioms is removed in such a way that the customary analytic continuation into the extended tube and/or the $L_+(C)$ invariance fails, then the above remarks are no longer true. Indeed in this case the validity of a first-kind condition does not necessarily imply the validity of its dual and vice versa.

We shall consider now symmetry-violating field theories.

In this respect let us first emphasize that our theorem *does not* imply that all discrete space-time transformations are symmetries. The theorem simply states that "if" a condition holds, its dual also holds and vice versa.

The well established P, C, and PC violations have been recalled in the Introduction. Furthermore, Lagrangian model field theories violating any of the first- or second-kind conditions can be easily constructed. This, however, does not imply either a contradiction of our theorem or a violation of the Wightman axioms.

Indeed, it is well known that if a Lagrangian model field theory violates the PCT condition, this does not imply a contradiction of the PCT theorem or a necessary violation of the Wightman axioms. In relation to our theorem a similar situation occurs for P, C, or PC violating theories or for theories violating any condition.

If, however, a condition is violated for field theories satisfying the requirements of our theorem then its dual condition must be violated too. Indeed, if this is not the case, then starting from the dual condition one can recover the original condition through the customary analytic continuation, which is in contradiction with its assumed violation.

Therefore, *the significance of our theorem for symmetry-violating field theories lies in the fact that if any of the customary discrete space-time symmetry conditions on VEV's is violated and the Wightman axioms (with the possible exception of LC) are preserved, then its dual condition must be violated too.*

The implications of the above property in the framework of the K^0 system are as follows: If one constructs a field theory for the K^0 system preserving the Wightman axioms (with the possible exception of LC), then the violation of the PC condition on VEV's implies *the violation of a new condition,* namely $\overline{CT}$. Not surprisingly, however, our theorem offers no direct evidence for deriving a T violation from the well established PC violation without the assumption of PCT conservation.[19]

Similarly, in the framework of weak interactions involving leptons, if the Wightman axioms (with the possible exception of LC) are preserved, the violation of the P and C conditions on VEV's implies *the violation of new conditions,* namely $\overline{T}$ and $\overline{PCT}$.[20]

Nevertheless, if the restriction on the preservation of the Wightman axioms is removed, then the above results do not necessarily hold.

Let us note from Table V that the combination of any of the first-kind condition with its dual is always equal to $\overline{PT}$ [e.g., $(C)\overline{PCT}=\overline{PT}$, $(P)\overline{T}=\overline{PT}$, etc.]. But the $\overline{PT}$ condition is always trivially satisfied for all considered types of field theories.[9]

Therefore, *if, in a field theory satisfying the Wightman axioms (with the possible exception of LC), any first- or second-kind condition θ is violated, then its dual condition $\overline{\theta}$ is violated in such a way that the combined condition $\theta\overline{\theta}=\overline{PT}$ is always an exact symmetry.*

Consider, for instance, a field theory satisfying the requirements of our theorem where the PCT condition on VEV's is violated. Then the $\overline{C}$=WLC condition is violated, too. However, those violations must compensate each other in such a way that the combined condition $(PCT)\overline{C}=\overline{PT}$ is an exact symmetry. This is similarly so for other pairs of dual conditions, such as C and $\overline{PCT}$, P and $\overline{T}$, etc.

As a final remark, let us note that Lagrangian field theories violating any of the above results can easily be constructed. More specifically, it is possible to construct models where, for instance,

(1) a first-kind condition (e.g., P) holds but its dual ($\overline{T}$) is violated, and vice versa;

(2) the PT condition holds, but the VEV's at Jost points are not real valued for nonspinorial field theories;

(3) two dual conditions (e.g., PC and $\overline{CT}$) are violated together with their combined $\overline{PT}$ condition.

However, all our results are based on the assumption that the considered types of field theories satisfy the Wightman axioms with the possible exception of the LC condition. As a consequence, there are significant indications according to which a field theory contradicting any of the above results should violate at least one of the Wightman axioms in such a way that the customary procedure of analytic continuation into the extended tube and/or use of the $L_+(C)$ invariance does not hold.

Therefore, our theorem could ultimately be used as a means for assessing in some instances the violation or the preservation of the Wightman axioms in a given type of field theory from the behavior of the first-kind and second-kind conditions.

APPENDIX: A SIMPLE GENERALIZATION OF THE EDGE OF THE WEDGE THEOREM

The following simple generalization of the edge of the wedge theorem is needed in the proof of our main theorem of Sec. IV.

Theorem. Let Θ be an open set of $\mathfrak{C}^n$ which contains a real environment, E, with E some open set of $\mathfrak{R}^n$. Let $\mathfrak{C}$ be an open convex cone of $\mathfrak{R}^n$. Suppose the function F_1 is holomorphic in

$$D_1=(\mathfrak{R}^n+i\mathfrak{C})\cap\Theta\,, \tag{A1}$$

and F_2 in

$$D_2 = (\Re^n - i\mathcal{C}) \cap \Theta . \quad \text{(A2)}$$

Suppose the limits for $x \in E$

$$\lim_{\substack{y\to 0\\ y\in \mathcal{C}}} F_1(x+iy) = F_1(x) , \quad \text{(A3)}$$

$$\lim_{\substack{y\to 0\\ y\in \mathcal{C}}} F_2(\mathcal{I}(x-iy)) = F_2(\mathcal{I}x) \quad \text{(A4)}$$

exist and are continuous and equal on E, the limit being uniform on E, and $\mathcal{I}$ being an inversion of the real vectors x, y. Then there is a (complex) neighborhood, N, of E and a holomorphic function, G, which coincides with F_1 in D_1 and F_2 in D_2 and is holomorphic in N.

Note: Clearly, the only difference of the above theorem with Theorem 2.15 of Ref. 3 is constituted by the appearance of the inversion $\mathcal{I}$ in (A4).

Proof. Since the function F_2 is holomorphic, it admits an (absolutely) convergent power series expansion. Therefore, one can introduce a new function, F_2', such that

$$F_2'(x-iy) = F_2(\mathcal{I}(x-iy)) , \quad \text{(A5)}$$

namely, in such a way that the signs originated for the $\mathcal{I}$ inversion are incorporated in the coefficients of the F_2' expansion. The theorem can be, then, reformulated in terms of the F_2' function and proved as for Theorem 2.15 of Ref. 3. Q.E.D.

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[1]T. D. Lee and C. N. Yang, Phys. Rev. 104, 254 (1956).

[2]J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, Phys. Rev. Lett. 13, 138 (1964).

[3]R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That* (Benjamin, New York, 1964).

[4]R. Jost, *The General Theory of Quantized Fields* (American Mathematical Society, Providence, Rhode Island, 1965).

[5]We shall use for fields the same notation as used in Ref. 3.

[6]See, for instance, Ref. 3, p. 128.

[7]In relation to the phases see, for instance, G. Feinberg and S. Weinberg, Nuovo Cimento 14, 571 (1959).

[8]We shall denote complex conjugation with an asterisk and Hermitian conjugation with a dagger. Furthermore, we shall assume the unit system for which $\hbar = c = 1$.

[9]Some of the conditions (3.4) are well known. Since $C^2 = 1$ (Ref. 7), the $C = (C)\bar{I} = CC$(WLC) condition is the familiar WLC condition. The $\overline{PT}$ condition is condition (3.40), p. 115 of Ref. 3, which follows from $P_+^{\dagger}$ invariance together with the hypothesis about the mass spectrum of states. For the case of nonspinorial fields, $\bar{I}$ is a reality condition. Consider, for instance, the two-point function of scalar fields $\langle\varphi_1(x_1)\varphi_2(x_2)\rangle_0$. The C condition is given by $\langle\varphi_1(x_1)\varphi_2(x_2)\rangle_0 = \langle\varphi_2(x_2)\varphi_1(x_1)\rangle_0^*$. The $\bar{C}$ = WLC condition in this case coincides with LC. Therefore at Jost points the $\bar{I} = (C)\bar{C}$ condition is given by

$$\langle\varphi_1(x_1)\varphi_2(x_2)\rangle_0 = \langle\varphi_1(x_1)\varphi_2(x_2)\rangle_0^* ,$$

which is a reality condition. Clearly, all conditions (3.4) can be introduced as boundary values of functions holomorphic in τ_n'. Relations (3.4) and all similar relations thereafter must be intended as equalities among different combinations of conditions and not as equalities among operators.

[10]Let us recall that any combination of first-kind conditions produces a first-kind condition [e.g., $(P)(C) = PC$]. It is interesting to remark that any even (odd) combination of second-kind conditions produces a first-kind (second-kind) condition. For instance, $(\bar{P})(\bar{T})$ gives rise to PT and not to $\overline{PT}$, but $(\bar{P})(\bar{C})(\bar{T})$ gives rise to $\overline{PCT}$ and not to PCT. Our second-kind conditions (3.4) can be introduced as a combination of well-known conditions, namely the corresponding first-kind conditions with the C and $\bar{C}$ = WLC conditions. This approach can be reversed in the sense that first-kind conditions too can be introduced as a combination of second-kind conditions. Indeed, the following relations hold:

$$C = (\bar{C})\bar{I} ,$$
$$P = (\bar{P})\bar{I} ,$$
$$T = (\bar{T})\bar{I} ,$$
$$PC = (\overline{PC})\bar{I} = (\bar{P})\bar{C} ,$$
$$CT = (\overline{CT})\bar{I} = (\bar{C})\bar{T} ,$$
$$PT = (\overline{PT})\bar{I} = (\bar{P})T ,$$
$$PCT = (\overline{PCT})\bar{I} = (\bar{P})\overline{CT} = (\bar{C})\overline{PT} = (\overline{PC})\bar{T} .$$

Therefore, we can say that all first-kind conditions can be introduced as the combination of the corresponding second-kind conditions with the C condition and the $\bar{C}$ = WLC condition. Since the WLC condition in this instance occurs twice, this approach is compatible with the validity of the first-kind conditions at any separation. The following relations will also be used later on:

$$\overline{PC} = (P)\bar{C} = (\bar{P})C ,$$
$$CT = (C)\bar{T} = (\bar{C})T ,$$
$$PT = (P)\bar{T} = (\bar{P})T ,$$
$$\overline{PCT} = (PC)\bar{T} = (P)(\bar{C})T = (\bar{P})CT = (\overline{PC})T = (P)\overline{CT} = (C)\overline{PT} .$$

[11]We caution the reader that, as discussed in Sec. III and Ref. 9, relations (4.1) cannot be interpreted as relations among operators, but only as identities of conditions on VEV's obtained through different combinations of first- and/or second-kind conditions.

[12]We shall follow the proof for the $PCT \leftrightarrow \bar{C}$ case as given in Ref. 3, pp. 143 and 144.

[13]Here $\underline{W}'(z_1^*, \ldots, z_n^*)$ could also be written $\underline{W}(z_1^*, \ldots, z_n^*)^*$. The former is considered holomorphic in $\bar{\tau}_n'$ (in line with the theorem of the Appendix), while the latter is considered holomorphic in τ_n'.

[14]Again $\tilde{\underline{W}}(z_1, \ldots, z_n)$ is a different notation for the same function $\underline{W}(z_1^*, \ldots, z_n^*)^*$; both are now holomorphic in τ_n'.

[15]The recently proposed U(3, 1)-invariant analytic extension of the $L^{\dagger}$-invariant VEV's, however, has broader connectivity properties. Indeed, in this case the identity can be continuously connected not only with the total inversion [as for the case of the $L_{+}(C)$ group] but also to the space inversion and to the time inversion, and, therefore, the extension might be suitable for the above direct continuation. For the U(3, 1)-invariant analytic extension of $L^{\dagger}$-invariant VEV's see R. M. Santilli and P. Roman, Nuovo Cimento 2A, 965 (1971); R. M. Santilli, P. Roman, and C. N. Ktorides, Particles and Nuclei 3, 332 (1972); R. M. Santilli and C. N. Ktorides, Phys. Rev. D 7, 2447 (1973). For an extensive review of this topic see C. N. Ktorides, Ph.D. thesis, Boston University, 1973 (unpublished). In relation to the present paper the following remarks are in order. Since this new type of analytic extension with broader connectivity properties holds only for P-conserving VEV's, it can be used only for the P-conserving subcase, of Corollary 1 of Sec. IV (see Table VI). In this subcase, it can be analytically proved that any one condition among the set (PCT, CT, $\overline{C}$, $\overline{PC}$) implies the other three conditions. The same properties hold for the sets (PC, C, $\overline{CT}$, $\overline{PCT}$) and (T, PT, $\overline{P}$, $\overline{I}$). The $\overline{T}$ condition always holds since it is the dual of P. The I and PT conditions trivially hold since they hold for any VEV. This is in full agreement with the results obtained by using the customary Wightman formalism, namely with the P-conserving (or CT-conserving) subcase of Corollary 1. Therefore, no contradiction exists between the consequences of the U(3, 1)-invariant extension versus the $L_{+}(C)$-invariant extension. For the case of arbitrary nonspinorial fields the invariance group of the new extension is the SU(3, 1) group which possesses the same connectivity properties of the $L_{+}(C)$ group (namely only I and I_{st} can be connected).

[16]Notice that the requirements of our theorem coincide with those of the PCT theorem, namely in both cases it is requested that the considered field theory obey the Wightman axioms with the possible exception of local commutativity.

[17]See Table IV. See also the remarks of Ref. 9.

[18]The $\overline{I}$ condition holds only at Jost points; therefore the above property is not generally true at points other than Jost points. For the case of spinorial field theories the $\overline{I}$ condition is not a reality condition in view of the factor i^{F} (see Table III).

[19]This is due to the fact that in the framework of our theorem no direct link can be established between nondual conditions such as PC and T. More specifically, the PC and T conditions cannot be linked through the analytic continuation into τ_n' and use of the connectivity properties of the $L_{+}(C)$ group. In order to derive a T violation from a PC violation the validity of either PCT or $\overline{C}$ = WLC must be assumed. Consider in this respect the cases

$$PC \longleftrightarrow \overline{CT},$$

$$T \longleftrightarrow \overline{P}.$$

of our theorem. Since $CT = (T)\overline{C}$, from the case $PC \longleftrightarrow \overline{CT}$ it follows that if PC is violated and $\overline{C}$ holds, then T must be violated, too; the converse also holds. Indeed, since $\overline{P} = (PC)\overline{C}$, from the case $T \longleftrightarrow \overline{P}$ of our theorem it follows that if T is violated and $\overline{C}$ holds, then PC must be violated too. By using our theorem, then one recovers the known result according to which in a field theory satisfying the Wightman axioms (with the possible exception of LC) where the $\overline{C}$ = WLC condition holds, a PC violation implies a T violation, and vice versa. However, since $\overline{C}$ holds, PCT must hold, too. Therefore, the PC and T violation must compensate each other in such a way that the combined PCT condition holds.

[20]Again, no direct link between possible P and CT (or C and PT) violations can be established. Consider the cases

$$P \longleftrightarrow \overline{T},$$

$$CT \longleftrightarrow \overline{PC}$$

of our theorem. By using arguments similar to those for the PC and T violations (see Ref. 19), it is easy to see that, since $P \longleftrightarrow \overline{T} = (CT)\overline{C}$, if $\overline{C}$ holds, then a P violation implies a CT violation. Vice versa, since $CT \longleftrightarrow \overline{PC} = (P)\overline{C}$, if $\overline{C}$ holds, then a CT violation implies a P violation. However, since $\overline{C}$ holds, the PCT condition must hold, too. Therefore one recovers the known result according to which if $\overline{C}$ holds, possible P and CT violations must compensate each other in such a way that PCT is an exact symmetry. Equivalent results can be obtained for C and PT violations through the use of the cases

$$C \longleftrightarrow \overline{PCT} = (PT)\overline{C},$$

$$PT \longleftrightarrow \overline{I} = (C)\overline{C}$$

of our theorem.