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LIE-ADMISSIBLE APPROACH TO THE HADRONIC STRUCTURE

Volume II

COVERING OF THE GALILEI AND EINSTEIN RELATIVITIES?

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Questo volume e dedicato ai miei figli

LUISA ed ERMANNO SANTILLI

con la speranza che stimoli il loro interesse per il sapere.

Ruggero Maria Santilli

LIE–ADMISSIBLE APPROACH TO THE HADRONIC STRUCTURE

Volume I.

NONAPPLICABILITY OF THE GALILEI AND EINSTEIN RELATIVITIES ?

TABLE OF CONTENTS

Ruggero Maria Santilli

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Volume II.

COVERING OF THE GALILEI AND EINSTEIN RELATIVITIES ?

TABLE OF CONTENTS

PREFACE

The primary objective of this volume is to attempt the construction of a covering of the Galilei relativity which is applicable to nonconservative and Galilei form–noninvariant systems and which is capable of recovering the Galilei relativity identically at the limit of null nonconservative forces. We shall then explore the problem of the possible consequential existence of a relativistic generalization.

Such a task is clearly of a rather delicate nature. In particular, it implies the study of a possible generalization of Galilei's relativity ideas which have remained unchanged for centuries within a Newtonian context.

Almost needless to say, a problem of this nature goes beyond my capabilities as an isolated researcher. As a result, I will have achieved my objective if I succeed in stimulating the awareness of our community of basic studies on the need to reexamine the problem of the relativity laws of Newtonian Mechanics. Equivalently, this volume is an expression of my personal belief that the Galilei relativity is not expected to be the terminal relativity of the systems of our everyday experience, that is, generally nonconservative.

Permit me to begin with the following introductory remarks.

(1) **THE NEED OF A GENERALIZATION OF THE GALILEI RELATIVITY.** The study of this need was the primary objective of Volume I. The starting point was the experimental evidence that the Newtonian forces are generally nonconservative and form–noninvariant under the Galilei transformations. The analysis was conducted by using the methodology of the Inverse Problem in Newtonian Mechanics, that is, the integrability conditions for the existence of a Lagrangian or a Hamiltonian. The study allowed the identification of the following five classes of Galilei symmetry breakings.

I. **Selfadjoint Breaking.** This is the conventional classical breaking of any symmetry, consisting of the addition of a symmetry breaking term to a symmetry–preserving Lagrangian or Hamiltonian. It was called "selfadjoint breaking" because, at the level of the equations of motion, it consists of the addition of form–noninvariant selfadjoint forces to form–invariant systems.

II. **Isotopic Breaking.** This is a new form of breaking induced by the multiplication (rather than the addition) of form–noninvariant terms to the form–invariant equations of motion, in such a way to verify the integrability conditions for the existence of a Lagrangian or a Hamiltonian. As a result, even though the new systems are equivalent to the original ones by construction, they break the Galilean symmetry. It should be recalled here that this breaking is purely formal on relativity grounds in the sense that, when the original systems obey the Galilei relativity, the relativity laws are still applicable to the equivalent system. Nevertheless, the Galilei group is replaced by a nonisomorphic group which leads to the conventional Galilei conservation laws (isotopically mapped Galilei group).

III. **Semicanonical Breaking.** This breaking occurs when some of the acting forces are nonconservative (nonselfadjoint), but they are such to admit an indirect analytic representation within the coordinates of the experimental detection (nonessentially nonselfadjoint systems), and the emerging Lagrangians or Hamiltonians are invariant under the Galilei group. In this case we have the consequential existence of ten first integrals (the generators of the Galilei transformations). However, the physical conservation laws of the Galilei relativity are lost. In particular, this breaking indicated the existence of a dichotomy of generators of Galilei transformations versus the physical conserved quantities which is absent for systems strictly obeying the Galilei relativity.

IV. **Canonical Breaking.** This breaking occurs when some of the acting forces are nonconservative and form–noninvariant under the Galilei transformations, but still such to verify the integrability conditions for the existence of an indirect analytic representation within the coordinates of the experimental detection of the systems. The existence of a Hamiltonian insures the direct applicability of the canonical formalism and, thus, the direct applicability of the Lie algebras as methodological tools. Nevertheless, the canonical formalism of the Galilei relativity is not directly applicable.

V. **Nonselfadjoint Breaking.** This is the most general breaking identified by the methodology of the Inverse Problem, and it is the breaking which more frequently occurs in the Newtonian physical reality of the systems of our everyday experience, such as, particles under drag forces due to friction, damped and forces oscillators, spinning tops with drag and applied torques, etc. The acting forces are not only nonconservative and form–noninvariant under the Galilei transformations, but such to violate the integrability conditions for the existence of an indirect analytic representation within the carrier space of the experimental detection (essentially nonselfadjoint systems). In this case, not only the Lie algebra of the Galilei group cannot be directly introduced, but all Lie algebras loose their direct applicability as a methodological tool in the frame of the observer.

It should be recalled that the Galilei transformations are the (largest possible linear) transformations for the transition from one inertial system to another. This property clearly persists also for breakings I – V. Our analysis is instead centered on the problem of the relativity laws of systems which are nonconservative and form–nonivariant under the Galilei transformations.

Another contribution of the Inverse Problem, which is relevant for relativity considerations, is the indirect universality of a Lagrangian or a Hamiltonian, i.e., the existence of coordinates transformations under which Newtonian systems verify the integrability conditions for an indirect analytic representation. In fact, given a nonconservative and Galilei–noninvariant system, there always exists an equivalence transformation to new coordinates in which the systems reduce to the "free" motion by therefore acquiring a Galilei–invariant form.

When a system in its original form violates the Galilei relativity, a conventional attitude is that of attempting the construction of an equivalent system in new coordinates which is form–invariant under the Galilei transformations. This attitude is undoubtedly consistent on mathematical grounds. Nevertheless, its physical consistency was questioned in Volume I because, for instance,

of the physical inconsistency between the experimentally established nonconservative nature of the system in the coordinate space of its actual occurrence as compared to the conservative nature of the equivalent systemin the mathematical space of the new coordinates. Besides, this new coordinate system is generally nonrealizable in the experimental set up (because it generally involves highly nonlinear and velocity–dependent transformations). Also, the indicated attitude is equivalent to its opposite, namely, the transformation of a system which is experimentally established as obeying the Galilei relativity into a new system in new coordinates which violates this relativity.

It is hoped that the analysis of Volume I has established the need of confronting the problem of the relativity laws which are applicable to nonconservative and Galilei–noninvariant systems in the physical frame of their experimental detection. Once this basic problem has been solved, then we can study the problem of relativity under arbitrary (nonlinear) transformations.

(2) THE COVERING NATURE OF THE INTENDED GENERALIZATION. As stressed in Volume I, new insights in theoretical physics never "destroy" previous accomplishments of proved physical effectiveness. They only implement them in a broader conceptual, physical, and methodological context. A generalization of the Galilei relativity would be inconsistent beginning from its formulation, unless it is a covering of the conventional Galilei relativity. In particular, as recalled in Section I. 1. 2*, the conventional Galilei relativity and its generalization must be compatible, in the sense that there must exist limiting (expansion) procedures of clear physical significance which reduce the new relativity to the old (and viceversa). Also, the new relativity must apply to a non-trivially broader physical context.

Several coverings of the Galilei relativity already exist. The fundamental coverings are those offered by Einstein Special Relativity and Quantum Mechanics. In the former case we have a classical covering of the Galilei relativity for speeds of the order of that of light. In the latter case, we have a covering of the Galilei relativity of quantum mechanical nature while the admissible speed remains nonrelativistic. These two coverings can be considered at the foundations of two corresponding series of coverings, one of classical and one of quantum mechanical nature. The methodological context of the former series is that of (classical) Field Theory or of the General Theory of Gravitation, while that of the second series is Relativistic Quantum Mechanics or Quantum Field Theory.

The covering of the Galilei relativity which is attempted in this volume is according to none of these lines. The intended covering is purely classical by assumption and, thus, quantum mechanical generalizations are excluded. Also, the intended covering is purely nonrelativistic, and, thus, relativistic generalizations are excluded too. As a matter of fact, the novelty of the analysis relies precisely in attempting a covering of the Galilei relativity which, by central assumption, is different

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*All references to sections and formulae of Volume I will be denoted with a prefix I.

than the existing coverings. This objective is made possible by the nature of the acting forces, rather than the value of the action functional (as compared to the Planck's constant) or the value of speed (as compared to the speed of light).

Figure 1 illustrates the objective under consideration, where SA (NSA) stands for variational selfadjointness (nonselfadjointness). Note that the figure refers only to the case of flat spaces, that is, the inclusion of gravitation may imply an additional chain of covering. Note also the preservation of the local–differential character of the geometry, which will be kept throughout our analysis. In fact, the case of the still more general (and more realistic) nonlocal systems call for the identification of suitable analytic, algebraic, and geometric methods. As such, its inclusion may imply a conceivable, further chain of coverings. Finally, note that the figure includes a quantum mechanical generalization. This latter covering will be studied in the next volume, and assumed at the foundation of the structure model of hadrons for which this series of monographs is intended.

To summarize, the covering of the Galilei relativity which is attempted in this monograph is centered on the extension of the acting forces, from the familiar local, selfadjoint, and Galilei form–invariant form of current studies, to the most general possible local, nonselfadjoint, and Galilei form–non–invariant form. This provides the "nontrivially broader physical context" mentioned earlier.

The compatibility of the classical relativistic and the quantum mechanical nonrelativistic coverings with the conventional Galilei relativity are provided by the "limiting procedures of clear physical significance" : $v/c \rightarrow 0$ (Inonu–Wigner contraction) and $(\hbar/\text{action}) \rightarrow 0$ (Correspondence Principle), respectively. The corresponding, but different, limit for the classical nonrelativistic covering is: relativity breaking forces $\rightarrow 0$.

The new covering of the Galilei relativity will therefore be attempted under the uncompromisable condition that it coincides with the conventional Galilei relativity when all relativity breaking forces are null.

(3) THE METHODOLOGICAL TOOLS OF THE INTENDED GENERALIZATION. As stressed in Volume I, Newtonian systems with forces derivable from a potential (i.e., systems which are essentially selfadjoint in the variational sense) can be effectively treated within the context of Lagrange's and Hamilton's equations without external terms. In particular, this implies that the underlying algebraic structure is a Lie algebra while the underlying geometry is a symplectic geometry. These are the well–known and deeply interrelated analytic, algebraic, and geometrical tools of the conventional Galilei relativity.

In the transition to Newtonian systems which are essentially nonselfadjoint (in the variational

sense), the situation is different. First, the integrability conditions for the existence of conventional analytic representations within the frame of the experimental detection are violated. This implies the lack of direct applicability of the conventional canonical tools of the Galilei relativity. In turn, this implies the lack of direct applicability of Lie algebras in canonical (Poisson) form, and of the Galilei Lie algebra in particular. Finally, these occurrences imply the corresponding lack of applicability of the symplectic geometry also when realized via the fundamental (canonical) two–form.

This situation suggests a reinspection of the conventional analytic, algebraic and geometric tools to see whether there exist generalized formulations which are directly applicable to the considered broader class of equations of motion, that is, applicable within the coordinate system of their experimental detection.

On analytic grounds, the most significant possibility of which I am aware is that offered by the analytic equations originally conceived by Lagrange and Hamilton, those with external terms. Their direct applicability (i.e., "universality") to the considered class of equations of motion is self–evident. The Lagrangian or Hamiltonian can represent not only the free motion, but also all the relativity preserving forces, while all the additional, relativity violating forces can be represented with the external terms. The fact that these broader analytic equations are a covering of the conventional equations is also self–evident. And indeed, the former coincide with the latter at the null value of the external terms. As a result, the use of Lagrange's and Hamilton's equations with external terms clearly constitutes a promising analytic context for the intended generalization of the Galilei relativity.

A new perspective also emerges from an algebraic profile. The brackets of the time evolution law characterized by the original analytic equations violate the Lie algebra laws. This literally implies the loss of the Lie algebra as a methodological tool whenever analytic equations with external terms are considered. Rather than considering it a drawback, I believe that this occurrence is most promising on methodological grounds. If the indicated analytic brackets violate the Lie algebra laws, this does not mean that they cannot characterize a more general, well defined, nonassociative algebra. A study of this problem, as we shall review in this volume, reveals that the analytic brackets induced by Hamilton's equations with external terms, when properly written, characterize a **Lie–admissible algebra**. Most intriquingly, this algebra results to be a genuine algebraic covering of the Lie algebra. This is parallel to the fact that Hamilton's equations with external terms characterize an analytic covering of the conventional canonical equations. As a result of this analytic backing, the Lie–admissible algebras clearly constitute a promising algebraic context for the intended covering of the Galilei relativity.

Predictably, a new perspective emerges also from a geometrical profile. In fact, the symplectic

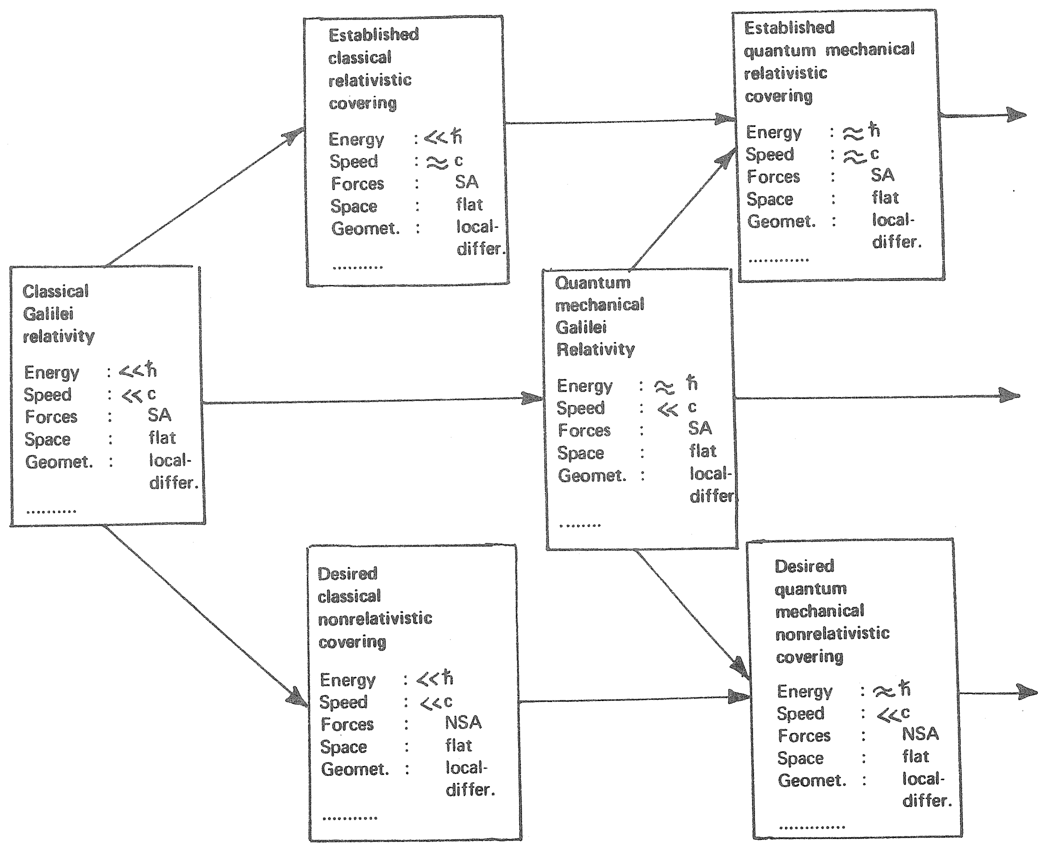

Figure 1: Some conceivable chains of coverings of the Galilei relativity

geometry is deeply linked to the conventional canonical equations and to the antisymmetric character of the Lie algebras. But the Lie–admissible algebras are not antisymmetric.
The need of a suitable covering of the symplectic geometry then emerges. As we shall see, the construction of this covering appears to be feasible.

To put it in different terms, I do not believe that one can achieve a truly novel covering of the Galilei relativity without first identifying the coverings of the central methodological tools of current relativity ideas: the analytic, algebraic, and geometrical tools.

In this monograph, I shall, therefore, first indicate the existence of the coverings of these methodological tools, and then confront the problem of the generalization of the Galilei relativity for arbitrary Newtonian systems.

I cannot close this introduction without indicating the existence of a covering of the Galilei relativity of Lie character, which is presented in Volume II of my *Foundations of Theoretical Mechanics* with Springer–Verlag. I hope that this latter volume will appear in print jointly with this one of Lie–admissible orientation, because the two coverings are deeply inter–related and mutually compatible.

Consider our planet Earth. When isolated from the rest of the Solar System, it verifies all the conventional conservation laws of total quantities. Nevertheless, the internal forces are non–Hamiltonian. Systems of this type were called in Volume I of closed nonselfadjoint type. Stated in different terms, total conservation laws, by no means, demand that the internal forces are necessarily of potential and Galilei–form–invariant type. In fact, they can occur under a substantially more general class of (local) nonselfadjoint forces.

The problem of the relativity which is applicable to closed nonselfadjoint systems is at least two fold.

A. RELATIVITY FOR THE EXTERIOR CLOSED TREATMENT. One can be first interested in identifying the relativity for the characterization of the system as a whole when seen from an outside observer (exterior treatment). In this case, primary emphasis must be given to total conservation laws (closure) and their derivability from suitable symmetries. Technical reasons then suggest that the product of the time evolution is, first of all, antisymmetric (to permit the conservation of the total energy), and, second, it obeys the Jacobi law (to permit integration to a finite time evolution), i.e., it characterizes a Lie algebra. However, and this is the point of departure from current studies in the field, the Lie algebra need not necessarily be realized via the simplest possible product (the Poisson brackets), but can be realized via the most general possible product (the so–called generalized Poisson brackets).

The transition from the conventional to the generalized Poisson brackets within a fixed system of local variables is called a **Lie–isotopy**. Hence, the tools for the exterior treatment of closed nonselfadjoint systems are called of Lie–isotopic type.

The physical meaning of the transition from the simplest possible realization of Lie algebras to the most general possible one is the following. In the former case we have a **closed system of point–like constituents with only action–at–a–distance forces**, while in the latter case, we have a **closed system of extended constituents with action–at–a–distance/potential as well as contact/non-potential forces.** Thus, the Lie–isotopy permits the treatment of the constituents as being extended. This implies the necessary presence of contact interactions for which the notion of potential energy has no physical basis.

As is well known, the Galilei relativity applies only to closed systems of point–like particles with potential and Galilei–form–invariant internal forces. It is therefore clear that the Galilei relativity is physically and mathematically unable to characterize the more general class of closed nonselfadjoint systems.

This latter problem has been studied in detail in my indicated Volume II with Springer–Verlag, resulting in the proposal of a **Lie–isotopic generalization of the Galilei relativity**.

B. RELATIVITY FOR THE OPEN INTERIOR TREATMENT. The study of the relativity according to lines (A) does not exhaust, by far, the relativity problem. In fact, a complementary problem is the identification of the relativity which is applicable to each extended constituent of a closed nonselfadjoint systems, while considering the rest of the system as external (interior problem).

It should be indicated from the outset that no meaningful differentiation between the exterior and the interior relativity exists for conventional conservative systems. As an example, consider the Solar System which, in Newtonian approximation, is closed and selfadjoint. Then the same relativity (the conventional Galilei relativity) applies for both the characterization of the system as a whole (exterior problem) as well as that of the center–of–mass trajectory of each constituent (interior problem).

In the transition to the more general closed nonselfadjoint systems the situation is different, and the exterior relativity is generally insufficient for the characterization of the interior problem. This can be seen in a number of ways, e.g., via the fact that the total value of non–Hamiltonian internal forces must be null to permit closure (Appendix I.1.C). As a result, effects which are ignorable at the exterior level, are not necessarily so at the level of each constituent.

But this is only the beginning. The stability of a closed selfadjoint system is essentially based on that of the orbits of the constituents, as evident for the solar system. The stability of a closed nonselfadjoint system, instead, is achieved under the maximal possible instability of the orbits of the constituents, as evident from the fact that the Earth as a whole is stable, yet trajectories in its atmosphere are unstable.

This latter situation permits the formulation of the relativity problem to be confronted in this volume as consisting of the identification of suitable symmetries of the equations of motion which can effectively characterize nonconservation laws under unrestricted (local) external forces. By comparison, the exterior problem consists of the identification of symmetries capable of representing total conservation laws under non–Hamiltonian internal forces.

But, closed systems are a particular case of the open ones. Thus, the objective of this volume is to achieve a relativity for local Newtonian systems which, not only is a covering of the relativity of the conventional conservative systems, but also of its isotopic generalization for closed nonselfadjoint ones.

The reader should keep in mind that a possible arena of applicability of the covering relativity (which motivated my efforts) is that of the hadronic structure under the assumption that the strong forces are not (entirely) derivable from a potential. The analysis of this volume is therefore an essential prerequisite to that of Volume III of this series in which I shall confront the problems of quantization, of the explicit construction of a structure model of the hadrons, and of the comparison of the predictions of the theory with the experimental data.

A possible arena of applicability of the intended covering relativity is therefore that of the strong interactions in general, of course, upon suitable quantization. As recalled in Volume I, strong and electromagnetic interactions exhibit profound physical differences in the carrier space of their experimental detection (Euclidean or Minkowski space). According to the contemporary approach, both the electromagnetic and the strong interactions are derivable from a potential and, therefore, they belong to the arena of applicability of established relativity. Their differentiation is attempted through additional degrees of freedom of the strong interactions in the mathematical space of the unitary internal groups (which is absent for the electromagnetic interactions). One of the contentions of Volume I is that, perhaps, this is insufficient to achieve a differentiation between these interactions as it occurs in the physical reality. The analysis of this volume opens the possibility of differentiating the electromagnetic and the strong interactions through their relativity laws.

According to established knowledge, the electromagnetic interactions are derivable from a potential; they are strictly Lie in algebraic character; and they obey the established relativity laws. The analysis of this volume opens the possibility of treating the strong interactions as being analytically more general of the electromagnetic interactions (i.e., nonselfadjoint); of being Lie–admissible in algebraic character; and, thus, of obeying Lie–admissible coverings of established laws. This approach is expected to produce a profound differentiation between these interactions in the physical space of their experimental detection.

Once the rudiments of the covering of the Galilei relativity will be identified, I shall touch on the problem of a possible corresponding covering of the Einstein special relativity. The need for this latter covering has also been indicated in Volume I. The problem of the existence of this covering can be reduced to that of the existence of the covering of the Galilei relativity from simple compatibility considerations. Besides, contact nonselfadjoint interactions and their Lie–admissible treatment are incompatible with the physical and mathematical structure of Einstein's special relativity. If a covering of the Galilei relativity of Lie–admissible type exists, a covering of the Einstein special relativity of Lie–admissible type must be expected. As a matter of fact, it is conceivable that the contraction–expansion techniques which interrelate the Galilei and Einstein relativities admit a corresponding formulation at the level of their intended coverings.

Predictably, the problems which I shall leave open are too numerous to suggest an outline. Therefore, the reader should not expect the construction of the intended covering relativity up to the maturity and level of sophistication of the established relativities. In essence, the primary objective of this monograph is to introduce new methodological tools of intriguing possibilities in theoretical physics, and then to indicate that covering relativities within these broader formulations appear to exist.

This limited scope is sufficient for the objectives of Volume III. Irrespective of whether actually constructed or only identified as possible, any new relativity has always proved to have a deep impact in our representation of the physical reality. This was the case, in particular, for the Einstein Special Relativity within the context of the problem of the atomic structure. On grounds of our current experimental and theoretical knowledge, it is conceivable that a similar situation might eventually occur for the problem of the hadronic structure.

Also, the reader should not expect that the term (covering) "relativity" used in this volume has the same meaning as that of the established relativities. As we shall see, the different physical nature of the systems considered implies an inevitable modification of their "relativity" context.

The reader should finally be aware of the conceptual attitude which is implemented during the analysis of this volume. The conventional canonical equations, the Lie algebra structure, and the symplectic geometry are at the foundations of contemporary theoretical physics. All my efforts are centered in indicating the existence of suitable generalizations. This attitude is motivated by my belief that theoretical physics is a science which will never admit terminal descriptions.

December 26, 1977 **Ruggero Maria Santilli**

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NOTE ADDED IN 1982

This volume was written in the last part of 1977, although I released it for printing a number of years later (1982). This delay is due to several reasons and, most particularly, to the desire to have this volume in print jointly with Volume II of my monographs *Foundations of Theoretical Physics* with Springer–Verlag (which is also printed in 1982). The advisability of having the two volumes printed together is suggested by the fact that the Springer–Verlag volume presents topics which are prerequisites for the full understanding of this volume. I am referring, e.g., to the Birkhoffian generalization of the Hamiltonian mechanics and to the Lie–isotopic generalization of the Galilei relativity presented in the Springer–Verlag volume which, in the final analysis, are necessary prerequisites for the Birkhoffian–admissible generalization of the Birkhoffian mechanics and the Lie–admissible generalization of the Lie–isotopic relativity, presented in this volume.

Evidently, several scientific events have occurred in the period 1977–1982, such as: the founding of the *HADRONIC JOURNAL* (1978); the organization of four *Workshops on Lie–admissible Formulations* (1978–1981); and the *First International Conference on Nonpotential Interactions and their Lie–admissible Treatment* (1982). It is also evident that, during the period 1977–1982, there has been the appearance of several independent contributions by mathematicians and physicists which have a direct relevance to the problem of the construction of a Lie–admissible relativity. As a result, these contributions have been invaluable for the achievement of a better maturity of this volume. In turn, this has implied an inevitable rewriting of some of its parts.

In essence, Chapter 1 (containing an elementary introduction to Lie–admissible algebras) has been left unchanged. I have only added, at the end, a list of references of recent studies for a more advanced knowledge of the topic. Chapter 2 (containing the rudiments of a Birkhoffian–admissible generalization of the Birkhoffian mechanics) has been completely rewritten as a consequence of the several writing and rewritings of the corresponding part in the Springer–Verlag monograph dealing with the Birkhoffian mechanics. Chapter 3 (on the Lie–admissible generalization of basic aspects of Lie's theory) has been left essentially unchanged, and I have limited myself to the indication of the most salient contributions in the problem since 1977. Chapter 4 (on the possible existence of a symplectic–admissible generalization of the symplectic geometry) has been rewritten in several parts. Finally, Chapter 5 (presenting the conjecture of the Lie–admissible relativities) has been rewritten in part. The new references have been added to the bibliography of 1977 beginning with call 183.

July 19, 1982 — Ruggero Maria Santilli

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I would like to express my gratitude also to all mathematicians, theoreticians, and experimentalists who participated in the WORKSHOPS ON LIE–ADMISSIBLE FORMULATIONS held from 1978 through 1981 in Cambridge, Massachusetts, and to the FIRST INTERNATIONAL CONFERENCE ON NONPOTENTIAL INTERACTIONS AND THEIR LIE–ADMISSIBLE TREATMENT held at the University of Orléans, France, in January, 1982. It is also a truism to say that without their scientific vision and active involvement, no sufficient maturity in the mathematical, theoretical, and experimental foundations of possible generalized relativities could have been reached.

Finally, I would like to express my gratitude to my wife Carla.

CHAPTER 1

LIE–ADMISSIBLE ALGEBRAS

1.1: STATEMENT OF THE PROBLEM

The fundamental relevance of Lie algebras for contemporary theoretical physics is well known. Within the context of Newtonian systems with forces derivable from a potential, this relevance originates from the fact that the brackets of the time evolution law induced by Hamilton's equations without external terms (the Poisson brackets), satisfy the Lie algebras laws (or identities, sometimes also called axioms)

$$ab + ba = 0, \qquad (1.1.1a)$$

$$(ab)c + (bc)a + (ca)b = 0, \qquad (1.1.1b)$$

here written in their abstract form. This provides a deep interrelation between the analytic and the algebraic approach to the systems considered.

A central objective of this monograph is to attempt the identification of an algebraic covering of Lie algebras consisting of the so-called general Lie-admissible algebras as characterized by the general Lie-admissibility law

$$\begin{aligned}(ab)c &+ (bc)a + (ca)b \\ &+ (cb)a + (ba)c + (ac)b \\ = a(bc) &+ b(ca) + c(ab) \\ &+ c(ba) + b(ac) + a(cb)\end{aligned} \qquad (1.1.2)$$

here also written it its abstract form. The primitive idea that general Lie-admissible algebras constitute an algebraic covering of Lie algebras originates from the properties: (a) general Lie-admissible algebras contain Lie algebras in their classification or, equivalently, the Lie algebras are Lie-admissible, and (b) general Lie-admissible algebras reduce to Lie algebras at the limit when their product becomes anticommutative.

The physical relevance of the general Lie-admissible algebras for Newtonian mechanics will be identified as follows. Newtonian systems with forces not derivable from a potential can be represented with Hamilton's equations with external terms. As we shall see, the generalized brackets of the time evolution law induced by these broader equations violate the Lie algebra laws (1.1.1). But, when properly written, they characterize precisely a general Lie-admissible algebra, i.e., satisfy law (1.1.2). In particular, under certain restrictions, these generalized brackets characterize the subcalss of flexible Lie-admissible algebras with laws

$$(ab)a - a(ba) = 0, \qquad (1.1.3a)$$

$$\begin{aligned}(ab)c &+ (bc)a + (ca)b \\ = a(bc) &+ b(ca) + c(ab),\end{aligned} \qquad (1.1.3b)$$

where Eq. (1.1.3a) is called the flexibility law and Eq. (1.1.3b), called the flexible Lie-admissibility law, is a particular form of Eq. (1.1.2) under the flexibility condition. At the limit of null forces not derivable from a potential, the generalized brackets reduce to the conventional Poisson brackets.

We can therefore say that, for the considered broader class of physical systems, the general or flexible Lie-admissible algebras are related to Hamilton's equations with external terms in precisely the same measure as the Lie algebras are related to Hamilton's equations without external terms. The possible covering nature of the generalized approach is then supported by the property that, at the limit of null forces not derivable from a potential, the analytic and the algebraic approach reacquire their conventional structures.

The above remarks are intended to indicate that Lie-admissible algebras have a fundamental role in our analysis. It is therefore advisable to first present them in their most natural setting, the theory of Abstract Algebras. This is the objective of this first chapter.

The reader should be aware that the brief review of Abstract Algebras presented in this chapter, even though restricted to only the branch of this discipline dealing with nonassociative algebras, is largely insufficient. Our objective is merely that of presenting those aspects of the theory of Abstract Algebras which are essential for a proper characterization of the Lie-admissible algebras as well as for the identification of their relationships with other nonassociative algebras. To assist the interested reader, we shall then quote, during the course of our analysis, a number of relevant references for subsequent study.

The Lie-admissible algebras were identified by A.A. Albert in a paper of 1948,[1] but without a detailed treatment. Two subsequent papers, one by L.M.Weiner of 1957[2] and one by

P.J. Laufer and M.L. Tomber of 1962[3] presented an initial treatment of these algebras. Subsequently, R.M. Santilli identified the significance of these algebras for nonconservative systems as well as continued the study of their properties in papers of 1967,[4] 1968[5], 1969[6] and 1970.[7] In an unpublished note of 1967,[8] R.M. Santilli and G. Soliani studied the possible existence of a Lie-admissible covering of the Bose-Einstein and Fermi-Dirac statistics. In a letter of 1969[9] R.M. Santilli and P. Roman studied the possible statistical implications of Lie-admissible algebras with particular reference to nonconservative plasmas.

Subsequently, new contributions by mathematicians on Lie-admissible algebras have appeared in the specialized literature. H.C. Myung provided additional studies in papers of 1971,[10] 1972,[11,12] and 1976;[13] other properties were identified by A.A. Sagle in a paper of 1971;[14] and D.R. Scribner and H. Strade studied the Lie-admissible algebras with particular reference to their relationship with the nodal noncommutative Jordan algebras in their respective papers of 1971[15] and 1972.[16] The existence of a thesis by W. Coppage of 1963[17] concerning the Pierce decomposition for Lie-admissible algebras has also been lately brought to my attention.

Additional contributions on Lie-admissible algebras have been also made by physicists. M. Kôiv and J. Lôhmus, in a paper of 1972,[18] indicated that a particular class of deformations of Lie algebras is Lie-admissible and worked out the case of a Lie-admissible deformation of the spin ½ Pauli algebra. C.N. Ktorides, in a paper of 1975,[19] achieved a generalization of the Poincaré-

Birkhoff-Witt theorem for Lie-admissible algebras. P.P. Srivastava, in a note of 1976,[20] pointed out that a particular form of graded algebra used in current supersymmetric Bose-Fermi models is Lie-admissible. Finally, R.M. Santilli in forthcoming papers of 1977-1978,[21-25] by using arguments which are mostly Lie-admissible in algebraic character, conjectured a possible nonapplicability for the hadronic constituents of the Pauli esclusion principle and Einstein special relativity as well as the possibility of interpreting unstable hadrons as bound states of suitably selected massive particles produced in their spontaneous decays.

During the course of the analysis of this volume (for the classical profile) and that of Volume III (for the quantum mechanical profile) we shall outline and integrate these contributions with particular reference to their applicability to classical and quantum mechanical nonconservative systems for the primary purpose of attempting a Lie-admissible approach to the hadronic structure.

I would like to take this opportunity to express my appreciation to C.N. Ktorides for calling <u>Santilli algebras</u>[19] the flexible Lie-admissible algebras.

1.2: ELEMENTAL ASPECTS OF ABSTRACT ALGEBRAS

A <u>ring</u> R is a set of elements a,b,c,... equipped with two operations, the addition and the multiplication, satisfying the following properties:

(1) R is an <u>Abelian</u> (i.e., commutative) group under addition,

(2) the multiplication is <u>left and right distributive</u> as well as <u>associative</u>, i.e.,

$$a + b = b + a, \qquad (1.2.1a)$$

$$a(b+c) = ab + ac, \qquad (1.2.1b)$$

$$(a+b)c = ac + bc, \qquad (1.2.1c)$$

$$(ab)c = a(bc), \qquad (1.2.1d)$$

Notice that the multiplication of a ring can be either Abelian (i.e., ab = ba) or not.

A <u>field</u> F is a set of elements $\alpha, \beta, \gamma, \ldots$ equipped with two operations, addition and multiplication, satisfying the following properties:

(a) the addition is associative and commutative, i.e.,

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma, \qquad (1.2.2a)$$

$$\alpha + \beta = \beta + \alpha. \qquad (1.2.2b)$$

There exists an element $0 \in F$, called the <u>zero element</u>, such that $\alpha + 0 = \alpha$ for all $\alpha \in F$. For each $\alpha \in F$ there is an element $-\alpha \in F$ such that $\alpha + (-\alpha) = 0$. There exist elements $\alpha \in F$ which are different than the zero element.

(b) The multiplication is associative and commutative, i.e.,

$$(\alpha\beta)\gamma = \alpha(\beta\gamma), \qquad (1.2.3a)$$

$$\alpha\beta = \beta\alpha, \qquad (1.2.3b)$$

as well as distributive, i.e.,

$$\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma, \qquad (1.2.4$$

The equations $\alpha x = \beta$ and $x\alpha = \beta$ always admit unique solutions for $\alpha \neq 0$.

A field, in essence, is a special case of a ring. A ring R is called a division ring if it has a nonzero element and the equations $ax = b$ and $xa = b$ have unique solutions for all a, $b \in R$ whenever $a \neq 0$. If, in addition, the multiplication is Abelian, we have a field. Thus, a field is a division ring which is Abelian under multiplication.

An algebra U is a vector space of elements a,b,c,... over a field F of elements $\alpha, \beta, \gamma, \ldots$ equipped with an (abstract) product ab satisfying the laws

$$a(b+c) = ab + ac, \qquad (1.2.5a)$$

$$(a+b)c = ac + bc, \qquad (1.2.5b)$$

$$(\alpha a)b = a(\alpha b) = \alpha(ab), \qquad (1.2.5c)$$

for all $a,b,c \in U$ and $\alpha \in F$.

Throughout our analysis the term "algebra" will be referred to as a nonassociative algebra, i.e., an algebra which does not necessarily verifies law (1.2.1d). If this law is verified for all

elements of the algebras, we shall specifically refer to an associative algebra.

The field over which an algebra is defined needs a more detailed characterization. Let F be a field. If there exists a least positive integer p such that $p\alpha = 0$ for all $\alpha \in F$, then we say that F has characteristic p. If no such characteristic exists, we shall say that the field has characteristic zero. For instance, the field of real numbers has characteristic zero but other fields of characteristic different than zero are also admissible. To be properly defined, an algebra U must be referred to a field F of specified characteristic $p = 0,1,2,\ldots$

The only algebras which have a primary physical relevance until now are those over a field of characteristic zero. This is the case of Lie algebras in Classical and Quantum Mechanics and will be the same for the Lie-admissible algebras. Nevertheless, the characteristic of a field cannot be ignored. For instance, the statement that the Cartan classification provides "all" complex simple Lie algebras (see next section) is, on strict ground, erroneous, unless restricted to a field of characteristic zero. And indeed, as we shall indicate later on, there exist simple Lie algebras over a field of characteristic p which are outside Cartan's classification. Notice that the concept of characteristic of a field can be extended to algebras.

Throughout our analysis, when the characteristic of a field is not explicitly stated, it will be tacitly assumed to be zero.

A basis $B = \{b_1, b_2, \ldots, b_n\}$ of a finite-dimensional algebra U over a field F of characteristic p is an independent subset of U which spans U as a vector space. Then every element $a \in U$ can be written

$$a = \alpha^i b_i \; , \quad \alpha^i \in F, \qquad (1.2.6)$$

where the convention on the sum of repeated indices is assumed from here on. The algebra U is then said to have dimension n. The product of U is in this case determined by n^3 structure constants as characterized by the closure rules

$$b_i b_j = C_{ij}^{k} \, b_k . \qquad (1.2.7)$$

A zero algebra U over F is an algebra with nonempty basis where the structure constants are all identically null.

A division algebra U (as for rings) is a (nonempty) algebra for which the equations $ax = b$ and $xa = b$ always admit solutions for $a \neq 0$.

The norm of an element $a \in U$ with respect to a basis B is

$$|a| = \left[\sum_{i=1}^{n} (\alpha^i)^2 \right]^{\frac{1}{2}} . \qquad (1.2.8)$$

A normed algebra is an algebra U with basis B such that

$$|ab| = |a|\,|b| , \qquad (1.2.9)$$

for all, $a, b, \in U$.

It has been proved that the only possible normed algebras U over the field of real numbers are[41]

(I) the complex numbers (dimension n = 2)

$$a = \alpha^i e_i \; , \quad i = 0, 1 \qquad (1.2.10a)$$

$$e_0 = 1 \; , \quad e_1^2 = -1 \; , \qquad (1.2.10b)$$

	e_0	e_1
e_0	1	e_1
e_1	e_1	-1

, (1.2.10c)

(II) the quaternions (dimension n = 4)

$$a = \alpha^i e_i \; , \quad i = 0, 1, 2, 3 \; , \qquad (1.2.11a)$$

$$e_0 = 1 \; , \quad e_i^2 = -1 \; , \qquad (1.2.11b)$$

	e_0	e_1	e_2	e_3
e_0	1	e_1	e_2	e_3
e_1	e_1	-1	e_3	$-e_2$
e_2	e_2	$-e_3$	-1	e_1
e_3	e_3	e_2	$-e_1$	-1

, (1.2.11c)

(III) the octonions, also called Cayley numbers (dimension n = 8)

$$a = \alpha^i e_i \ , \quad i = 0,1,2,3,4,5,6,7, \qquad (1.2.12a)$$

$$e_0 = 1 \ , \quad e_i^2 = -1 \ , \qquad (1.2.12b)$$

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	e_4
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

(1.2.12c)

In particular, the complex numbers and the quaternions are the only possible associative division algebra over the field of real numbers. The octonions, on the contrary, characterize a nonassociative normed algebra.

An alternative algebra U over a field of characteristic p is an algebra satisfying the right and left alternative laws

$$a^2 b = a(ab) \ , \qquad (1.2.13a)$$

$$b a^2 = (ba)a \ , \qquad (1.2.13b)$$

for all $a,b \in U$. All associative algebras are trivially alternative. The only alternative division algebra which is nonassociative is given by the octonions.

An algebra U over a field of characteristic p is called a power-associative algebra when

$$a^n a^m = a^{n+m} \ , \quad n, m = 1,2,3,\dots , \qquad (1.2.14)$$

for all elements $a \in U$. Then the following identities are satisfied

$$a^2 a = a a^2 \ , \qquad (1.2.15a)$$

$$a^2 a^2 = (a^2 a) a \ . \qquad (1.2.15b)$$

Conversely, it is possible to prove that, for fields of characteristic zero, identities (1.2.15) imply power-associativity. It is also possible to prove that alternative algebras are power-associative.

An element a of a power-associative algebra U over a field F of characteristic p is called nilpotent when there exists an integer n such that $a^n = 0$. A nilalgebra is a (power associative) algebra consisting only of nilpotent elements.

An element e of an (arbitrary) algebra U is called idempotent if $e^2 = e$. An idempotent e is called primitive when there exist in U no orthogonal idempotents a,b ($a^2=a$, $b^2= b$, $ab = ba = 0$) such that $e = a + b$. An idempotent e is called principal when there exist no idempotent $a \in U$ ($a^2 = a \neq 0$) which is orthogonal to e (ae = ea = 0). It is possible to prove that any finite-dimensional alternative algebra which is not a nilalgebra contains a principal idempotent.

The commutator of two elements a, and b of an algebra U is the quantity

$$[a,b]_U = ab - ba, \qquad (1.2.16)$$

and it characterizes the amount by which two elements miss obeying the commutativity law of multiplication. The anticommutator is instead given by

$$\{a,b\}_U = ab + ba. \qquad (1.2.17)$$

The associator of three elements a, b and c of an algebra U is the quantity

$$[a,b,c]_U = (ab)c - a(bc), \qquad (1.2.18)$$

and it characterizes the amount by which the elements a, b and c miss obeying the associative law of multiplication.

A number of algebraic laws can be expressed in terms of the associator. For instance, the right and left alternative laws (1.2.13) can be written

$$[a,a,b]_U = 0, \quad [b,a,a]_U = 0 \qquad (1.2.19)$$

The following property

$$a[b,c,d]_U + [a,b,c]_U d \qquad (1.2.20)$$
$$= [ab,c,d]_U - [a,bc,d]_U + [a,b,cd]_U$$

holds for all elements a, b, c and d of an (arbitrary) algebra U.

The reader should keep in mind that the associator is linear in each argument.

An algebra U over a field F of characteristic p is called trace-admissible if there is a bilinear form $\tau(a,b)$ called trace-form, with arguments in U and values in F such that (1) $\tau(a,b) = \tau(b,a)$, (2) $\tau(a\,b,c) = \tau(a,b\,c)$, (3) $\tau(a,b) = 0$ if ab is nilpotent or zero, and $\tau(e,e) \neq 0$ if e is an idempotent of U.

An algebra U is called commutative when

$$[a,b]_U = 0, \qquad (1.2.21)$$

for all elements a,b $\in$ U.

An algebra U is called anticommutative when

$$\{a,b\}_U = 0, \qquad (1.2.22)$$

for all elements a,b $\in$ U.

An algebra U is called flexible when

$$[a,b,a]_U = 0, \qquad (1.2.23)$$

for all elements a,b $\in$ U. The flexibility law can be equivalently formulated (for fields of characteristic $p \neq 2$) in terms of the law

$$[a,b,c]_U + [c,b,a]_U = 0. \qquad (1.2.24)$$

And indeed, from Eq. (1.2.23), we can write

$$((a+c)b)(a+c) - (a+c)(b(a+c))$$
$$= (ab)a + (ab)c + (cb)a + (cb)c \qquad (1.2.25)$$
$$- a(ba) - a(bc) - c(ba) - c(bc)$$
$$= (ab)c - a(bc) + (cb)a - c(ba) = 0.$$

Clearly the flexible law is a covering of both the commutative and the anticommutative law. For the former case we can write

$$(ab)a = (ba)a = a(ba). \qquad (1.2.24)$$

Thus, all commutative algebras are flexible. For the latter case we can write

$$(ab)a = -(ba)a = a(ba). \qquad (1.2.25)$$

Thus, all anticommutative algebras are flexible. The concept of algebraic covering is then completed by noting that there exist flexible algebras which are neither commutative nor anticommutative (e.g., the Lie-admissible algebras, see Section 1.4).

The nucleus N of an algebra U is the set of all elements $g \in U$ which verify the associative law of multiplication for every pair of elements $a, b \in U$

$$[g, a, b]_U = [a, g, b]_U = [a, b, g]_U = 0. \qquad (1.2.26)$$

Since the associator is linear in each argument and, from property (1.2.20),

$$\begin{aligned}[g_1 g_2, a, b]_U &= g_1 [g_2, a, b]_U + [g_1, g_2, a]_U b \qquad (1.2.27)\\ &+ [g_1, g_2 a, b]_U - [g_1, g_2, b]_U, \quad g_1, g_2 \in N\end{aligned}$$

the nucleus N is an associative subalgebra of U. The center C of an algebra U is the set of all elements in the nucleus N of U which commutes with all elements $a \in U$,

$$[c, a]_U = [a, c]_U = 0. \qquad (1.2.28)$$

Thus, the center of an algebra U is the maximal commutative and associative subalgebra of U. This definition also applies to

associative algebras. The field F of an algebra U can also be its center C.

The Kronecker product $P_F = U \otimes_F U'$ of two algebras U and U' over a field F is the tensor product $U \times_F U'$ of the vector spaces U and U' where the product is defined by distributivity and the rule

$$(a \otimes_F a')(b \otimes_F b') = (aa') \otimes_F (bb'), \quad a, b \in U, a', b' \in U' \qquad (1.2.29)$$

If U and U' are finite-dimensional, then $(\dim P_F) = (\dim U) \times (\dim U')$.

The scalar extension U_F of an algebra U over F is the Kronecter product $F \otimes_F U$. Then, any basis of U over F is also a basis of U_F. Also U is a subalgebra of U_F in the sense that it is isomorphic to $1 \otimes_F U$. Notice that the scalar extension U_F of an algebra U may or may not verify the same laws of U.

Let U be an algebra over F with identity e and product ab satisfying a set of laws. Let c admit an inverse c^{-1} $(cc^{-1} = c^{-1}c = e)$. Construct the new algebra U* which is the same vector space as U but equipped with the new product

$$a * b = acb, \qquad (1.2.30)$$

for fixed c, for all elements $a, b \in U$, and for a given association (e.g., $a*b = (ac)b$). U* is called an isotopic extension of U when the new product $a*b$ obeys the same laws of U; otherwise we shall call it a genotopic extension of U. The algebra U* so constructed will be called an isotope or genotope of U, respectively.

The concept of algebraic isotopy is therefore characterized by an invertible law-preserving mapping of the product. The concept of algebraic genotopy is instead characterized by a modification of the product of type (1.2.30) or of the more general type

$$a * b = a c b + b(1-c) a, \quad c = \text{fixed}, \qquad (1.2.31a)$$

$$a * b = a c d + b d e, \quad c, d = \text{fixed}, \qquad (1.2.31a)$$

etc.

Notice also that, when U* is an isotope of U, the algebras U and U* are not necessarily isomorphic. Notice also that the concept of algebraic genotopy can be interpreted as characterized by an invertible law-inducing mapping of the product in the following sense. Let U and U* be two algebras characterized by two non-equivalent sets of laws, and suppose that these algebras are related by a genotopic mapping. Then one can say that the genotopic mapping "induces" the laws of U*. The concept of genotopic mapping will play a crucial role in our attempt to construct a Lie-admissible covering of the Galilei relativity (Chapter 5).

Let A be an associative algebra with product a b over a field F of characteristic p. The λ-mutation algebra A(λ) of A is the same vector space as A but equipped with the new product

$$a * b = \lambda a b + (1-\lambda) b a, \qquad (1.2.32)$$

where λ is a fixed element of the field. The new algebra A(λ) so constructed is generally nonassociative. Nevertheless, it shares several important properties with A. Notice that the concept of λ-mutation is a particular case of that of genotopic mapping, trivially, when the element c of rule (1.2.31a) belongs to the field.

The algebras A(λ) are called mutation algebras because they can reduce to different algebras, depending on the assumed value of the parameter λ. Clearly, A(1) is isomorphic to A; A(0) is antiisomorphic to A; and A(½) is isomorphic to the commutative algebra A^+ with product

$$\{a, b\}_A = a b + b a. \qquad (1.2.33)$$

However, there is no (finite) value of the parameter λ capable of reducing the A(λ) algebra to the anticommutative algebra A^- with product

$$[a, b]_A = a b - b a. \qquad (1.2.34)$$

This lessens the possibilities of physical applications, owing to the fundamental role of Lie algebras in physics, as recalled in Section 1.1.

For this reason I introduced in papers[4-7] the (λ, μ)-mutation algebra A(λ, μ) of an associative algebra A which is the same vector space as A but equipped with the new product

$$a * b = \lambda a b + \mu b a = \rho [a, b]_A + \sigma \{a, b\}_A, \qquad (1.2.35)$$

where $\lambda = \sigma + \rho$ and $\mu = \sigma - \rho$ are fixed elements of the field.

Clearly the algebra A(λ, μ) is a more general realization of the concept of algebraic genotopy, as compared to that of the algebra A(λ). In particular, A(λ, μ) is generally nonassociative. Also, A(1,0) is isomorphic to A; A(0,1) is antiisomorphic to A; A(1,1) is isomorphic to the commutative algebra A^+ with product (1.2.33); A(λ,1-λ) is isomorphic to A(λ); and, last but not least, A(1,-1) is isomorphic to the algebra A^- with product (1.2.34) which is the conventional realization of Lie algebras, e.g., in Quantum Mechanics (see also the next section). Thus, unlike the algebras A(λ), the algebras A(λ,μ) are capable of directly reducing to the Lie algebra. The algebras A(λ,μ) will play a crucial role in our construction of a Lie-admissible covering of the Lie algebra.

Despite the indicated differences, the algebras A(λ) and A(λ,μ) share several algebraic properties which are considered in Section 1.4. This occurrence can be anticipated at this point by noting that the algebras A(λ) and A(λ,μ) are related by an algebraic isotopy. Assume

$$\tau = \lambda + \mu, \quad \lambda = \lambda'\tau, \quad \mu = \mu'\tau, \quad \lambda' + \mu' = 1. \qquad (1.2.36)$$

Then we can write

$$\begin{aligned} a * b &= \lambda a \cdot b + \mu b \cdot a = \lambda'\tau a \cdot b + \mu'\tau b \cdot a \qquad (1.2.37a) \\ &= \lambda' a \circledast b + (1-\lambda') b \circledast a, \\ a \circledast b &= (\lambda + \mu) a \cdot b, \qquad (1.2.37b) \end{aligned}$$

namely, the algebra A(λ,μ) can be interpreted as the isotopic image A*(λ') of A(λ') whenever $\lambda \neq -\mu$. As a result, most of the structure theory of the A( ) mutation algebras (ideals, radicals, representations, etc.) can be extended to the mutation algebra A(λ, μ).

The following variation of the concept of algebraic genotopy

is significant for our analysis. Let U be an algebra over a field of characteristic different than two. Suppose that U, as a vector spece, contains a subspace S which is closed under anti-commutator (1.2.17). Let U^-(S) be the set of all linear combinations of elements of S equipped with the commutator (1.2.16) and suppose that a linear mapping T of U^-(S) into S exists. The algebra U*(U^-,S,T) which is the same vector space as U but equipped with the new product

$$a * b = \frac{1}{2}(ab + ba) + (ab - ba)T \qquad (1.2.38)$$

is called <u>bonded</u> to U. The underlying mapping T is called <u>bonding mapping</u>. Notice that bonding mappings can be extended through the concept of genotopic mapping (1.2.31b), for instance, to products of the type

$$a * b = \lambda(ab + ba) + \mu(ab - ba)T. \qquad (1.2.39)$$

These algebraic structures are clearly more attractive than the mutation algebras A(λ) from a physical viewpoint owing to the presence of the commutator in the product (1.2.38) or (1.2.39). However, unlike the case of the product (1.2.35) of the A(λ, μ) algebras, the commutator enters into the product of algebra $\mathcal{U}$*(U^-,S,T) with a bonding mapping as factor.

Another modification of the concept of bonded algebras can be introduced as follows. Let S be a subspace of an algebra U. Suppose that S is closed under commutator (1.2.16) and let U^+(S) be the set of all linear combinations of elements of S equipped with the anticommutator (1.2.17). Suppose also that a linear mapping T of $U^+(S)$ into S exists. Then the algebra $U^*(U^+,S,T)$ which is the same vector space as U but equipped with the product

$$a * b = [a,b]_U + \{a,b\}T \qquad (1.2.40)$$

is also bonded to U. However, the bonding mapping now appears as a factor to the anticommutator, rather than the commutator, as for algebras $U^*(U^-,S,T)$. The possible significance of algebras $U^*(U^+,S,T)$ for our analysis will be indicated in Section 1.5.

Let U be an algebra over a field F, and let S_1 and S_2 be subspaces of U. We shall denote by S_1S_2 the space spanned by the product a_1a_2 for all elements $a_1 \in S_1$ and $a_2 \in S_2$. A subspace S of U is called a subalgebra of U if $SS \subseteq S$. S is called a right ideal of U if $SU \subseteq S$, a left ideal of U if $US \subseteq S$ and an ideal of U when it is both a right and left ideal. An ideal S of U is called a proper ideal when $S \subset U$ but $S \neq U$. The zero ideal is the ideal consisting of only the zero element. A nilpotent ideal S is an ideal such that $S^m = 0$, where S^n denotes the set of all finite sums of products $a_1 \ldots a_n$ under all possible associations.

Let R be an ideal of an algebra U over a field F. The quotient algegra U/R is the vector space cosets with elements $a+R$, $a \in U$, where addition and multiplication are defined as follows

$$(a+R) + (b+R) = (a+b) + R, \qquad (1.2.41a)$$

$$\alpha(a+R) = \alpha a + R, \qquad (1.2.41b)$$

$$(a+R)(b+R = ab + R, \qquad (1.2.41c)$$

for all $a,b \in U$ and $\alpha \in F$. Clearly the quotient U/R of U so defined is an algebra.

An involution (also called involutory antiauthomorphism of order two) of an algebra U is a linear operator $a \to \bar{a}$ for all $a \in U$ satisfying the laws

$$\overline{ab} = \bar{b}\bar{a}, \quad \bar{\bar{a}} = a. \qquad (1.2.42)$$

When U has an identity, an involution can be written

$$a + \bar{a} \in F, \quad a\bar{a} = \bar{a}a \in F. \qquad (1.2.43)$$

This is the case, for instance, when the elements a are complex number and the involution $a \to \bar{a}$ is the operation of complex conjugation. If the algebra U is an $n \times n$ matrix algebra with elements $a = (a_{ij})$, the standard involution in U is defined by

$$a = (a_{ij}) \to a' = (\bar{a}_{ij})^T, \qquad (1.2.44)$$

where T stands for transpose (e.g., when the a's are matrices with complex elements, a standard involution is the operation of adjoint). Consider now a diagonal matrix of U

$$D = \begin{pmatrix} t_1 & & 0 \\ & t_2 & \\ 0 & & \ddots \; t_m \end{pmatrix}, \qquad (1.2.45)$$

and suppose that (1) the diagonal elements t_i are in the center of U, (2) the t's admits inverses which are also in the center of U and (3) the t's are self-adjoint relative to the involution in U, i.e., $t = \bar{t}$. The mapping

$$a \to D^{-1}a'D \qquad (1.2.46)$$

is called the canonical involution of the (matrix) algebra U relative to T. This notion will be used in the next section in relation to the classification of commutative Jordan algebras.

The literature in Abstract Algebras is rather vast. Without any claim of completeness, we suggest as first readings textbooks [26-31], as second readings monographs [32-42] and as third readings the research monographs [43-52]. The reader with a physics background should be aware that all these references are primarily devoted to the study of algebras other than Lie algebras. Regrettably, no textbook available at this time presents a definition of Lie-admissible algebras, to the best of my knowledge.

On my more specific grounds, the reader can consult references [53-54] for the classification of normed algebras indicated in this section. Paper [55] is instructive for the study of power-associativity, a crucial property for our analysis. For the A(λ) algebras see paper.[1] Bonded algebras are studied, e.g., in papers [56-57]. Their Lie-admissible character is studied in paper [2]. The study of additional papers [58-61] is recommended as an introduction to the algebraic profile of the analysis of this Volume II as well as that of Volume III. Paper is instructive for the axiomatic approach to Abstract Algebras.

As indicated earlier, the concepts of algebraic isotopy and genotopy will play a crucial role in our attempt to construct a covering of the Galilei (and Einstein) relativity. Regrettably, the concept of algebraic isotopy is generally ignored in currently available textbooks in Abstract Algebras, with very few exceptions known to me, such as a monograph by R.H. Bruck of 1958.[63] For

a treatment in the specialized literature on this topic see paper [64].

As R.H. Bruck indicates (loc. cit., Chapter III), the concept of algebraic isotopy is rather old and dates back to the early stages of set theory. And indeed, the concept was apparently identified for the first time within the context of the Latin squares (these are square arrays of n rows and n colums formed from n distinct objects having the property that each row and column of the arrays contains each of the n objects only once). Two Latin squares were called isotopically related if they could be made to coincide by using permutation. The concept of isotopy was then extended to quasigroups (a groupoid is a nonempty set G with of elements a,b ,... equipped with a binary operation ab such that for every pair of distinct elements $a,b \in G$ there is a unique element $c = ab \in G$; a quasigroup is a groupoid such that the equation $ab = c$ uniquely determines one element, once the other two are known). The extension was natural because Latin squares are the multiplication table of (finite) quasigroups. This implies the birth of the concept of isotopic mapping, more generally, of a group as a mutation of its composition law induced by an element of the group. In turn, this implies the existence of a corresponding concept at the level of an algebra, as we shall indicate in Section 1.5. In the final analysis, as R.H. Bruck put it, the concept of algebraic isotopy is "so natural to creep in unnoticed".

The concept of algebraic genotopy does not appear to be treated in the mathematicl literature to the best of my knowledge and it is introduced in my monographs on the Inverse Problem.[65]

See also refs.[66,67] . This concept is, in essence, a generalization of the concept of algebraic isotopy. To indicate the significance of this concept for the analysis of these volumes, it is here appropriate to anticipate that the algebraic-group theoretical structures of the conventional relativities and those of my attempted coverings will result to be genotopically related.

1.3: LIE ALGEBRAS

A Lie algebra L over a field F (of characteristics p) is a vector space over F with elements a,b,c,... equipped with the (abstract) product ab satisfying laws (1.1.1), i.e.,

$$ab + ba = 0, \qquad (1.3.1a)$$

$$(ab)c + (bc)a + (ca)b = 0, \qquad (1.3.1b)$$

where Eq. (1.4.1a) is called anticommutative law and Eq. (1.3.1b) is called the Jacobi law.

A commutative Jordan algebra J over a field F is a vector space over F with elements a,b,c,... equipped with the product ab satisfying the laws

$$ab - ba = 0, \qquad (1.3.2a)$$

$$(a^2 b)a - a^2(ba) = 0, \qquad (1.3.2b)$$

where Eq. (1.3.2a) is called the commutative law and Eq. (1.3.2b) is called the Jordan law.

A noncommutative Jordan algebra J over a field F is a vector space over F with elements a,b,c,... equipped with the product ab satisfying the laws

$$(ab)a - a(ba) = 0, \qquad (1.3.3a)$$

$$(a^2 b)a - a^2(ba) = 0, \qquad (1.3.3b)$$

where Eq. (1.3.3a) is the flexible law (1.2.23) and Eq. (1.3.3b) is, again, the Jordan law.

The fundamental significance of Lie algebras in physics has been recalled in Section 1.1. The amount of physical as well as mathematical literature existing on Lie algebras is so large to discourage even a partial listing. Particularly inspiring is the study of the original work by Sophus Lie, e.g., that of references[68]. In the following we shall make extensive use of a monograph by N. Jacobson.[69]

The commutative Jordan algebras were introduced in a 1933 paper by P. Jordan, J.V. Neuman and E. Wigner. The title of the paper, "Uber Verallgemeinerung-möglichkeiten de Formalismus der Quantenmechanik", indicates the physical inspiration of the study (see the English translation of ref.[70]). Since that time Jordan algebras have been subjected to intensive studies from both a mathematical as well as a physical profile. Within the former context, the commutative Jordan algebras have reached a high degree of sophistication which is comparable to that of Lie algebras (see, for instance, refs.[71-73]). Within the latter context the studies are still in progress (see, for instance, the quantum mechanical treatment of Jordan algebras in ref.[74] , and their application to the hadronic structure in ref.[75]). In any case, it does not appear that the Jordan algebras have reached physical applications comparable to those of Lie algebras, according to the present status of theoretical physics. Perhaps, a different insight is offered by our Lie-admissible approach to physical systems. As we shall see in the next section, a Lie-admissible algebra can have a well defined "content" not only of a Lie algebra, but also of a commutative Jordan algebra. As a

matter of fact, these two algebras sometime appear on equivalent footings within the context of the theory of abstract algebras (not so for the case of classical realizations, as we shall see in Section 1.5). It is therefore tempting to state that the problem of the physical relevance of the commutative Jordan algebras does not appear to be resolved as of today. In any case, the commutative Jordan algebras have stimulated new directions of algebraic studies. Owing to a number of technical reasons to be identified later on, the methodology for the treatment of the Lie-admissible algebras appears to be a symbiosis between that of Lie algebra and that of Jordan algebras, with particular reference to the noncommutative Jordan algebras. The net result is that, while the algebraic profile of the conventional approach to physical systems can be effectively restricted to that of Lie algebras, the algebraic profile of our generalized approach demands the study of Lie algebras as well as Jordan algebras of commutative and noncommutative type.

Of first importance in the study of these algebras is the <u>realization</u> of the product, in terms of associative product. Let A be an associative algebra with elements a,b,c... and product $a \cdot b$ over a field F of characteristic zero. A realization of the Lie algebra product in terms of the associative product is given by

$$[a,b]_A = a \cdot b - b \cdot a \; . \qquad (1.3.4)$$

This yields the Lie algebra A^- which coincides with A as a vector space but is equipped with product (1.3.4). A realization of the commutative Jordan algebra product is given by

$$\{a,b\}^*_A = \frac{1}{2}(a\cdot b + b\cdot a). \qquad (1.3.5)$$

This characterizes a (commutative) Jordan algebra A^+ which coincides with A as vector space but is equipped with product (1.3.5). A realization of the product of the noncommutative Jordan algebra $\tilde{J}$ is given by

$$a*b = \lambda a\cdot b + (1-\lambda) b\cdot a. \qquad (1.3.6)$$

This yields the λ-mutation algebra $A(\lambda)$ of Section 1.2. The interested reader is suggested to verify that products (1.3.4), (1.3.5) and (1.3.6) satisfy not only laws (1.3.1), (1.3.2) and (1.3.3), respectively, but also the fundamental laws (1.2.5) to qualify as products of an algebra. Notice that the algebras L, J and $\tilde{J}$ as well as their realizations A^-, A^+ and $A(\lambda)$ are nonassociative. Notice also the difference between products (1.3.4) and (1.3.5) and the corresponding forms (1.2.16) and (1.2.17). The former are defined in terms of the associative product $a\cdot b$ while the latter are defined in terms of an arbitrary product ab which, as such, is not necessarily associative.

As we shall see in Chapter 3, the Poincaré-Birkhoff-Witt theorem ensures that every Lie algebra L is isomorphic to a subalgebra of some algebra A^-. In nontechnical language this can be restated by saying that every Lie algebra can be represented in terms of product (1.3.4).

The situation for Jordan algebras J and $\tilde{J}$ is different. In these cases there is no analog of the Poincaré-Birkhoff-Witt

theorem. For instance, a commutative Jordan algebra does not necessarily admit a realization in terms of product (1.3.5). When such realization exists, we have the so-called <u>special Jordan algebras</u>. Otherwise we have the so-called <u>exceptional Jordan algebras</u>.

One of the central problems for the study of any algebra is the problem of classification. This demands the identification of the radical, the characterization of the semisimple algebras and their reduction into direct sum of simple algebras. The problem of classification can then be reduced to that of the identification of all the simple algebras. The reader should be aware that these concepts may vary from algebra to algebra and that the definitions of these quantities which are familiar for Lie algebras do not generally extend to other algebras. In the following we shall restrict ourselves to a review of the essential elements of the problem of classification. The problem of the radical of a nonassociative algebra is outlined in more details in Appendix 1.D. The interested reader, however, is urged to study the quoted literature. For simplicity, our presentation is mainly restricted to the case of fields with characteristic zero. The reader should be also aware that the definitions considered, even for the case of Lie algebras, may have to be suitably implemented or modified when the case of characteristic different than zero is considered (and, especially, when the case of characteristic two is studied).

An algebra U is called <u>simple</u> if and only if the only proper ideal of U is the zero ideal and $U^2 \neq 0$. This definition applies to any (nonassociative) algebra and, thus, also to Lie and Jordan algebras. Notice that for any algebra U, U^2 is an ideal of U. However, if U is simple, then $U^2 = U$.

We consider now the notion of radical for an associative finite-dimensional algebra A.

In this case, if S is an ideal of A and the quotient algebra A/S is nilpotent, then A is nilpotent. Also, if S_1 and S_2 are nilpotent ideals, S_1+S_2 is a nilpotent ideal. These properties imply the existence of a unique maximal nilpotent ideal R of A which is called the radical of A. A finite-dimensional associative algebra A is called semisimple when its radical is the zero ideal. Of fundamental importance for the classification of these algebras is the Wedderburn structure theorem

THEOREM 1.3.1: If R is the radical of a finite-dimensional associative algebra A over a field of characteristic zero, the quotient algebra A/R is a semisimple associative algebra. Any semisimple algebra A is uniquely expressible as the direct sum of ideals $S_1, S_2, \ldots, S_n$, i.e.,

$$A = \bigoplus_{i=1}^{n} S_i, \qquad (1.3.7)$$

each of which is a simple associative algebra.

In the transition to a nonassociative algebra the characterization of the radical must be suitably implemented. This is due to the fact that the nonassociative character of the algebra renders ambiguous the concept of nilpotency, unless properly

defined. A (nonassociative as well as not necessarily power-associative) algebra U is called a nilpotent algebra if there exists an integer n such that all possible products of n elements in U are zero. For instance, an algebra U is nilpotent of degree 3 when $a(bc) = (ab)c = 0$, in which case all the following properties hold

$$(ab)c = (bc)a = (ca)b = c(ab) = a(bc) = b(ca)$$
$$= (ba)c = (cb)a = (ac)b = c(ba) = a(cb) = b(ac) = 0. \qquad (1.3.8)$$

The derived series of an algebra U is the iterative sequence

$$U^{(1)} = U, \quad U^{(2)} = U^{(1)}U^{(1)}, \ldots, \quad U^{(s+1)} = U^{(s)}U^{(s)} \qquad (1.3.9)$$

An algebra U is called solvable if $U^{(s)} = o$ for some positive integer s.

The definition of the radical of an associative algebra does not extend to the Lie algebras, because these algebras are all nilalgebras of index two, trivially, from axiom (1.3.12) which can be written

$$a^2 = 0, \text{ for all } a \in L. \qquad (1.3.9)$$

Nevertheless, the concept of solvability of a nonassociative algebra does apply to Lie algebras. And indeed, the radical of a finite-dimensional Lie algebra L is defined as the unique maximal solvable ideal of L. A Lie algebra L is called semisimple when its radical is the zero ideal. Notice that if a Lie algebra L is solvable of index n, i.e., $L^{(n)} = o$, then $L^{(n-1)}$ is Abelian. This is related to another definition of a semisimple Lie algebra also used in the physical literature according to which a Lie algebra L is semisimple when it contains no Abelian ideal except the zero ideal. The following fundamental structure theorem then holds.

THEOREM 1.3.2: If R is the radical of a finite dimensional Lie algebra L over a field of characteristic zero, the quotient algebra L/R is a semismiple Lie algebra. Any semisimple algebra L can be uniquely expressed as the direct sum of ideals $S_1, S_2, \ldots, S_n$, i.e.,

$$L = \bigoplus_{i=1}^{m} S_i \tag{1.3.11}$$

each of which is a simple Lie algebra.

The case of commutative Jordan algebras J can be treated on equivalent grounds. The concept of solvability as defined above applies, and the radical of a commutative Jordan algebra J is the unique maximal solvable ideal of J. A semisimple algebra J is again an algebra whose radical is the zero ideal. The following fundamental structure theorem then holds ($p \neq 2$ is assumed here).

THEOREM 1.3.3: If R is the radical of a finite-dimensional commutative Jordan algebra J over a field of characteristic zero, the quotient algebra J/R is a semisimple commutative Jordan algebra. Any semisimple algebra J is uniquely expressible as the direct sum of ideals $S_1, S_2, \ldots, S_n$, i.e.,

$$J = \bigoplus_{i=1}^{m} S_i, \tag{1.3.12}$$

each of whichis a simple commutative Jordan algebra.

The case of a noncommutative Jordan algebra J is somewhat different. The radical in this case is defined as the unique maximal nil ideal of J. A semisimple algebra J is an algebra, again, whose radical is the zero ideal. The following fundamental structure theorem then holds ($p \neq 2,3$ is assumed here).

THEOREM 1.3.4: If R is the radical of a finite-dimensional noncommutative Jordan algebra $\tilde{J}$ over a field of characteristic zero, the quotient algebra J/R is a semisimple noncommutative Jordan algebra. Any semisimple algebra $\tilde{J}$ is uniquely expressible as the direct sum of ideals $S_1, S_2, \ldots, S_n$, i.e.,

$$\tilde{J} = \bigoplus_{i=1}^{m} S_i, \tag{1.3.13}$$

each of which is a simple noncommutative Jordan algebra.

The reader should keep in mind that the above theorems are formulated, specifically, for finite-dimensional algebras over a field of characteristic zero. Notice that there is no contradiction of the definition of radical for a Lie algebra or a commutative Jordan algebra and that for an associative algebra, because, under the condition of associativity, the concepts of nilpotent ideal and that of solvable ideal coincide. Notice also that the definition of radical of a noncommutative Jordan algebra is more in line with that of an associative algebra than that of a Lie algebra. For more details see Appendix 1.D.

The results of the studies on the problem of classification can be summarized as follows.

I. CLASSIFICATION OF SIMPLE LIE ALGEBRAS

I.A: Algebras over a field of characteristic zero. The Cartan classification of all the complex simple Lie algebras and the construction of the corresponding real forms by means of inner and outer involutive authomorphisms is well known. We have[69]

Classical Algebras: A, B, C, D;

Exceptional Algebras: G_2, F_4, E_6, E_7, E_8.

I.B: Algebras over a field of characteristic $p \neq 0$. The studies on the simple Lie algebras of this type are still in progress. As an indication we quote the following simple Lie algebras:

p-dimensional algebras by Witt;
p^n-dimensional algebras by Zassenhaus;
np^n-dimensional algebras by Jacobson;
rp^n-dimensional algebras by Kaplansky;
$(n-1)(p^n-1)$-dimensional algebras by Frank;
T_n, V_m, L_o and L algebras by Albert and Frank;
L(T, δ , F) algebras by Block.

For a study of Lie algebras over a field of characteristic p see, for instance, ref.[76]. For infinite-dimensional Lie-algebras see, for instance, ref.[77].

II. CLASSIFICATION OF SIMPLE COMMUTATIVE JORDAN ALGEBRAS

II.A: Algebras over a field of characteristic zero. The problem of the classification of simple algebras can in this case be reduced to that of central simple algebras, that is, the simple algebras whose centroid is the base field. A central simple commutative Jordan algebra J is called a reduced Jordan algebra if it contains an identity element $1 = \sum_{i=1}^{n} e_i$, where the e_i are (absolutely) primitive orthogonal idempotents and n is called the degree of J. It can be shown that any reduced simple commutative Jordan algebra J is central simple and that the scalar extension of a central simple commutative Jordan algebra is a reduced simple algebra. Thus, the problem of classification of simple commutative Jordan algebras J can be reduced to that of reduced simple algebras J_n^N of degree n and dimension N over the field F. The following classification then holds.[71-73]

Degree n = 1. In this case the reduced simple algebra is J = eF, where e is the identity element of F. The algebras are special.

Degree n = 2. In this case the reduce simple algebras

J_2^N are those characterized by a symmetric bilinear form (x,y) defined over a vector space V(F) such that: (a) the bilinear form is non - degenerate on V(F), (b) there exists an element $x \in V(F)$ such that (x,x) = 1, and (c) the dimension of V(F) is $N \geq 2$. These algebras are special Jordan algebras.

Degree $n \geq 3$: In this case every reduced simple Jordan algebra J_n^N is isomorphic to a Jordan algebra $J(D_n,T)$ where D_n is an alternative algebra (Section 1.2) which can be associative for $n \geq 4$ and T is a canonical involution (Section 1.2). The algebra $J(D_n,T)$ is the vector space

$$J(D_m,T) = \{x \mid x \in D_m,\ x = T^{-1}x'\,T\} \qquad (1.3.14)$$

where $X \to X'$ is a standard involution in D_n, equipped with the product

$$XY = \frac{1}{2}(X \cdot Y + Y \cdot X). \qquad (1.3.15)$$

The following classification then holds.

(A) D_n is isomorphic to the field F. The involution T is in this case the identity mapping and $J(D_n,T)$ consists of n x n symmetric matrices over F. $N = \frac{1}{2}n(n+1)$.

(B) D_n is isomorphic to the algebra of generalized complex numbers $\mathcal{C}$ over F with basis 1, e, $e^2 = \mu 1$, $\mu \neq 0$ (which contains the algebra of complex numbers as a particular case. The involution T is given in this case by $a + be \to a - be$ (e.g., complex conjugation for the case of complex numbers). $J(D_n,T)$ is given by the algebra of n x n matrices with elements in $\mathcal{C}$ which are T-Hermitians, i.e., $X = T^{-1}X'T$ (self-adjoint for the case of complex numbers). $N = n^2$.

(C) D_n is isomorphic to a four-dimensional algebras of <u>generalized quaternions</u> $\mathcal{Q}(F)$ constructed from that of ordinary quaternion in a way similar to that of case (B). The elements of $J(D_n,T)$ are the 2n x 2n T-Hermitian matrices with generalized quaternions as elements. $N = 2n^2 - n$.

(D) D_n is isomorphic to the eight-dimensional algebras of <u>generalized octonions</u> $\mathcal{O}(F)$. The only possible degree in this case is n = 3 and the elements of $J(D_3,T)$ are 3 x 3 matrices X with the generalized octonions as elements which are such that $X = T^{-1} \cdot X'\,T$, with $X \to X'$ being the standard involution and T being the canonical involution in $\mathcal{O}(F)$. In this case N = 27.

The algebras (A), (B) and (C) are special simple Jordan algebras, while the only algebra (D), i.e., J_3^8, is exceptional. This means that product $X \cdot Y$ of Eq. (1.3.15) is associative for cases (A), (B) and

(C), while it is nonassociative for case (D).

II.B: Algebras over a field of characteristic $p \neq 0$.

Classification II.A extends to the case of a field of characteristic $p \neq 2$ without the appearance of new algebras. See, for instance, ref.[73] and quoted papers.

III. CLASSIFICATION OF SIMPLE NONCOMMUTATIVE JORDAN ALGEBRAS

III.A: Algebras over a field of characteristic zero. The following classification holds.[78,79]

(a) The commutative Jordan algebras of classification II.A;

(b) the flexible quadratic algebras with nondegenerate norm;

(c) the central simple algebras of quasi-associative type (see next section).

III.B: Algebras over a field of characteristic $p \neq 0$.

Classification III.A also extends to the case $p > 0$, $p \neq 2$. New simple algebras, however, now occur. An example is given by the simple nodal noncommutative Jordan algebra $\tilde{J}$[41,80,81] which are such that

(1) every element $a \in \tilde{J}$ can be written in the form $a = \alpha 1 + z$ with $\alpha \in F$ and z nilpotent,

(2) The set N of nilpotent elements z is not a

subalgebra of $\tilde{J}$,

(3) The field is (necessarily) of characteristic $p > 0$.

(4) $\tilde{J}$ can be represented as follows. Let P_n be the truncated polynomial ring in n nilpotent elements $(x_1, \ldots, x_n)$, $x_i^p = 0$, equipped with the partial differential operator $\partial/\partial x_i$. Let $a \cdot b$ be the commutative associative product in P_n. $\tilde{J}$ is the same a P_n as vector space but equipped with the product

$$ab = a \cdot b + \frac{\partial a}{\partial x_i} \cdot \frac{\partial b}{\partial x_j} c_{ij} \qquad (1.3.16)$$

(5) At least one elemnt c_{ij} possesses an inverse, n 2 and

$$c_{ij} = \frac{1}{2} [x_i, x_j]_{P_m} \qquad (1.3.17)$$

Notice that the commutative Jordan algebras appear in the classification of the noncommutative Jordan algebras. This is due to the fact indicated in the Section 1.2, that the flexible law is a covering of the commutative law. Thus, any realization of the product satisfying axioms (1.3.2) also satisfies axioms (1.3.3). However, Lie algebras do not appear in the classification of both the commutative and noncommutative Jordan algebras. This does not prohibit the existence of interrelations between the Lie and Jordan algebras. Most intriguing in this respect is the fact that the exceptional Lie algebras can be related to the exceptional Jordan algebras (see refs.[41,73] and Appendix 1.B).

1.4: LIE-ADMISSIBLE ALGEBRAS

A Lie-admissible algebra[1] U over a field F (of characteristic p) is a vector space over F with elements a, b, c,... equipped withe the (abstract) product ab such that the attached algebra U^-, which is the same vector space as U but equipped with the product

$$[a,b]_U = ab - ba, \qquad (1.4.1)$$

is a Lie algebra. Clearly, if ab is associative, product (1.4.1) characterizes a Lie algebra. Thus, the associative algebras are the fundamental Lie-admissible algebras. The concept of Lie-admissibility, however, indicates that a Lie algebra can also be characterized in terms of a nonassociative product ab, provided that ab-ba is Lie. This transition from an associative to a non-associative product ab in the construction of a Lie algebra will be crucial throughout our analysis of this Volume II and that of Volume III.

From the viewpoint of the defining laws, Lie-admissible algebras can be classified as follows.[4,5]

I. General Lie-admissible algebras. These are all algebras U over a field F of characteristic p satisfying the following law, for $[a,b]_U + [b,a]_U = 0$,

$$[a,b,c]_U + [b,c,a]_U + [c,a,b]_U = [c,b,a]_U + [b,a,c]_U + [a,c,b]_U, \qquad (1.4.2)$$

for all a, b, c $\in$ U. Eq. (1.4.2) is the Jacobi law (1.3.1b) written in terms of the product ab, i.e.,

$$[[a,b]_U,c]_U + [[b,c]_U,a]_U + [[c,a]_U,b]_U = 0, \qquad (1.43)$$

and $[a,b,c]$ is the associator, Eq. (1.2.18). Eq. (1.4.2) will be called the general Lie-admissible law.

II. Flexible Lie-admissible algebras. These are all algebras U over F satisfying the laws

$$[a,b,a]_U = 0, \qquad (1.4.4a)$$

$$[a,b,c]_U + [b,c,a]_U + [c,a,b]_U = 0, \qquad (1.4.4b)$$

for all a, b, c $\in$ U, where Eq. (1.4.4a) is the flexible law (1.2.23) and Eq. (1.4.4b) is the general Lie-admissible condition under the flexible law, as it can be proved by using Eq. (1.2.24). Eq. (1.4.4b) will be called the flexible Lie-admissib law.

III. Lie algebras. Under the condition of the anticommutativity of the product laws (1.4.4) reduce to

$$ab + ba = 0, \qquad (1.4.5a)$$

$$(ab)c + (bc)a + (ca)b = 0, \qquad (1.4.5b)$$

namely, they reduce to the conventional Lie algebra laws.

LEMMA 1.4.1: Any anticommutative Lie-admissible algebra is a Lie algebra.

As we shall see during the course of our analysis, the Lie-admissible algebras constitute an algebraic covering of the Lie algebras. The following properties are significant in this respect.

(A) Lie algebras are Lie-admissible. Consider a Lie algebra L with (abstract) product ab (= -ba). The attached algebra L^- is the same vector space as L but equipped with the new product

$$[a,b]_L = ab - ba = 2ab \qquad (1.4.6)$$

Thus, every Lie algebra is Lie-admissible. This implies the important property that, unlike the case of the commutative and noncommutative Jordan algebras, the Lie algebras are included in the classification of the Lie-admissible algebras.

(B) The Lie-admissible laws are a covering of the Lie algebra laws. The fact that the flexibility law (1.4.4a) is a covering of the anticommutativity law (1.4.5a) has been shown in Section 1.2. On similar grounds, one can see that law (1.4.4b) is a covering of the Jacobi law (1.4.5b). Thus the flexible Lie-admissible laws (1.4.4) are a covering of the Lie algebra laws (1.4.5). On similar grounds one can see that the general Lie-admissibility law (1.4.2) is a covering of the flexible laws (1.4.4) and, thus, of the Lie algebra laws (1.4.5). This implies the important property that the product of a Lie-admissible algebra is, in general, neither commutative, nor anticommutative nor flexible.

(C) Under suitable realizations of the product, the Lie-

admissible algebras can directly reduce to Lie algebras. Consider the (λ , μ) mutation algebras $A(\lambda,\mu)$ of an associative algebra A with product $a \cdot b$. The product in $A(\lambda,\mu)$ is given by Eq. (1.2.35), i.e.,

$$a * b = \lambda a \cdot b + \mu b \cdot a \qquad (1.4.7)$$

Under the condition $\lambda \neq \pm \mu$ product (1.4.7) is neither commutative nor anticommutative, i.e., $A(\lambda,\mu)$ is neither a commutative Jordan algebra nor a Lie algebra. Nevertheless, product (1.4.7) satisfies the general Lie-admissibility condition. Thus, the algebra $A(\lambda,\mu)$ is a Lie-admissible algebra. At a closer analysis one can verify that the product (1.4.7) satisfies the flexible Lie-admissible laws, Eqs. (1.4.4). Thus $A(\lambda,\mu)$ is a flexible Lie-admissible algebra. But

$$\lim_{\lambda \to 1, \mu \to -1} A(\lambda,\mu) = A^- \qquad (1.4.8)$$

and this proves the property considered. A more general example of the product of a Lie-admissible algebra which is not flexible, but satisfies limit (1.4.9), will be given within the context of the classical realizations of the product (Section 1.5 and Chapter 2). Notice that the λ -mutation algebras $A(\lambda)$ of an associative algebra A with product (1.2.32), i.e.,

$$a * b = \lambda a \cdot b + (1-\lambda) b \cdot a \qquad (1.4.9)$$

are also flexible Lie-admissible algebras. Nevertheless, there exists no finite value of the element $\lambda \in F$ capable of satisfying our fundamental limit (1.4.8). It is for this reason that the algebras $A(\lambda, \mu)$ are preferable over the algebras $A(\lambda)$ for our analysis. But the algebras $A(\lambda)$ are a realization of the noncommutative Jordan algebras. A simple inspection reveals that the algebras $A(\lambda, \mu)$ too are a realization of the noncommutative Jordan algebras, i.e., they satisfy laws (1.3.3). This implies the important property that not only the Lie algebras, but also some noncommutative Jordan algebras appear in the classification of the Lie-admissible algebras.

Most of our subsequent efforts will be devoted to an understanding of the physical significance of the algebraic occurrences (A), (B) and (C). In this section we restrict ourselves to an outline of the known properties of the Lie-admissible algebras. In the appendices of this chapter we present certain methodological tools of the theory of Abstract Algebras which can be used for the study of the Lie-admissible algebras.

The following class of algebras is crucial to reach a first understanding of the interplay between the Lie algebras and the commutative Jordan algebras within the context of the Lie-admissible algebras.

A Jordan-admissible algebra[1] U over a field F of characteristic p is a vector space over F with element a, b, c... equipped with the product ab such that the attached algebras U^+, which is the same vector space as U but equipped with the product

$$\{a, b\}_u = \frac{1}{2}(ab + ba), \tag{1.4.10}$$

is a commutative Jordan algebra. Clearly, when the product ab is associative, the above conditions are met for the case of a special commutative Jordan algebras. Thus, the associative algebras are not only Lie-admissible, but also Jordan-admissible. From the viewpoint of the algebraic laws the following classification holds.[4,5]

I'. General Jordan-admissible algebras. These are all algebras U over F satisfying the law ($p \neq 2$ is assumed here)

$$(a^2b)a + a(ba^2) + (ba^2)a + a(a^2b) = a^2(ba) + (ab)a^2 + a^2(ab) + (ba)a^2 \tag{1.4.11}$$

for all $a, b \in U$ which we shall call general Jordan-admissible law. It is obtained by imposing that product (1.4.10) satisfies the Jordan law (1.3.26), i.e.,

$$\{\{\{a,a\}_u, b\}_u, a\}_u = \{\{a,a\}_u, \{b,a\}_u\}_u \tag{1.4.12}$$

II'. Flexible Jordan-admissible algebras. These are all algebras U over F satisfying the laws

$$(ab)a = a(ba), \tag{1.4.13a}$$

$$(a^2b)a + a(a^2b) = a^2(ba) + a^2(ab) \tag{1.4.13b}$$

for all $a, b \in U$, where Eq. (1.4.13a) is the flexible law and Eq. (1.4.13b) is the general Jordan-admissible law under the flexibility condition (see Appendix 1.A). We shall call Eq. (1.4.13b) the flexible Jordan-admissible law.

III'. Commutative Jordan algebras. Under the condition of commutativity of the product, the conditions of Jordan-admissibility reduce to

$$ab = ba, \qquad (1.4.14a)$$

$$(a^2 b)a = a^2(ba), \qquad (1.4.14b)$$

namely, they reduce to the laws of the commutative Jordan algebras.

LEMMA 1.4.2. Any commutative Jordan-admissible algebra is a commutative Jordan algebra. Also, any noncommutative Jordan algebra is a flexible Jordan-admissible algebra.

The theory of the Jordan-admissible algebras is closely related to that of the Lie-admissible algebras. For instance, the Jordan-admissible algebras provide an algebraic covering of the Jordan algebra. Notice that a Lie algebra is trivially Jordan-admissible, in the sense that laws (1.4.11) and (1.4.13) are trivially satisfied because $a^2 = 0$ for all elements of a Lie algebra. Similarly, the commutative Jordan algebras are trivially Lie-admissible in the sense that they trivially satisfy laws (1.4.2) and (1.4.4).

We are now in a position to identify the interplay of the Lie algebras and the commutative Jordan algebras within the

context of the Lie-admissible algebras. We shall call nontrivial Lie-admissible algebra a Lie-admissible algebra which is neither associative nor Lie. The product ab of these algebras is neither commutative nor anticommutative and, as such, it always admits the decomposition

$$ab = \frac{1}{2}[a,b]_U + \{a,b\}_U \qquad (1.4.15)$$

This indicates the important property that a nontrivial Lie-admissible algebra U can jointly be Lie-admissible and Jordan-admissible and we shall symbolically write

$$U = U^- \text{ⓐ} U^+ \qquad (1.4.16)$$

where the symbol ⓐ stands to indicate that the algebras U, U^- and U^+ coincide as vector space and their products are related by Eq. (1.4.15). And indeed, the algebras $A(\lambda, \mu)$ with product (1.4.7), i.e.,

$$ab = \rho[a,b]_A + \sigma\{a,b\}_A \qquad (1.4.17)$$

are precisely of this type, namely, jointly Lie- and Jordan-admissible. The reader is here suggested to verify that product (1.4.17) satisfies not only the Lie-admissibility conditions (1.4.2) and (1.4.4) but also the Jordan-admissibility conditions (1.4.11) and (1.4.13).

In conclusion, a Lie-admissible algebra U can have a nontrivial content not only of a Lie algebra U^- but also of a commutative Jordan algebra U^+. As a result, the study of Lie-admissible algebras demands the use of both, the Lie and the Jordan algebras, and cannot be conducted within the context of the Lie algebra alone.

To elaborate on this aspect, suppose that a nontrivial Lie-admissible algebra U is finite-dimensional with basis $B = \{b_1, \ldots, b_n\}$ and closure relations

$$b_i b_j = {}_U C_{ij}^k b_k , \qquad (1.4.18)$$

where the c's are the structure constants (Section 1.2). It may happen that the same basis B is also closed, individually, under the Lie and commutative Jordan product with corresponding closure conditions and structure constants

$$[b_i, b_j]_U = {}_L C_{ij}^k b_k , \qquad (1.4.19a)$$

$$\{b_i, b_j\}_U = {}_J C_{ij}^k b_k , \qquad (1.4.19b)$$

In this case the structure constants of U can be expressed in terms of the structure constants in U^- and U^+, i.e.,

$${}_U C_{ij}^k = \frac{1}{2} {}_L C_{ij}^k + {}_J C_{ij}^k . \qquad (1.4.20)$$

An example of this occurrence is here useful. As is known, the fundamental representations of the SU(n) Lie algebras (here interpreted as n x n matrices) are closed under both the commutator and the anticommutator. As a result, these representations provides an example of rules (1.4.18), (1.4.19) and (1.4.20). The physical significance of these representations should be recalled. For the case of the SU(2)-spin algebra the fundamental representation is given by the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (1.4.21)$$

and they play a central role for the characterization of the spin ½. For the case of the SU(2)-isospin algebra the fundamental representation is again given by the Pauli matrices, but now acting in an internal space (the isospin space), and it plays a central role for the characterization of isotopic doublets, such as that of the nucleon. Finally, the fundamental representation of the SU(3) algebra is given by the known Gell-Mann λ-matrices and plays a central role for the concept of quark, as currently used for both, for the problem of classification and that of structure of the hadrons (Volume I). The embedding of the SU(n) Lie algebras into their SU(n)-admissible covering is one of the fundamental problems of our analysis and we shall consider it later on in Chapters 3, 4 and 5 as well as in Volume III. At this time let us only point out that, for the case of the Pauli matrices rules (1.4.19) became

$$[\sigma_i, \sigma_j]_A = 2i \varepsilon_{ijk} \sigma_k , \quad \{\sigma_i, \sigma_j\}_A = 2\delta_{ij} \qquad (1.4.22)$$

and rule (1.4.18) can assume the realization

$$\begin{aligned} \sigma_i \sigma_j &= \alpha [\sigma_i, \sigma_j]_A + \beta \{\sigma_i, \sigma_j\} \qquad (1.4.23) \\ &= (2i\alpha \varepsilon_{ijk} + 2\beta \delta_{ij} \delta_{jk}) \sigma_k = {}_U C_{ij}^k \sigma_k \end{aligned}$$

Thus, the $A(\lambda, \mu)$ algebras with basis $B = \{\sigma_i\}$ is a finite-dimensional nontrivial Lie-admissible algebra such that

$$[A(\lambda,\mu)]^- \approx SU(2) , \qquad (1.4.24a)$$

$$\lim_{\lambda \to 1, \mu \to -1} A(\lambda,\mu) \equiv SU(2) , \qquad (1.4.24b)$$

$$[A(\lambda,\mu)]^{+} \approx J(D_3,T), \qquad (1.4.24a)$$

$$\lim_{\lambda;\mu \to \frac{1}{2}} A(\lambda,\mu) \equiv J(D_3,T), \qquad (1.4.24b)$$

where $J(D_3,T)$ is a commutative Jordan algebra of type (B) (Section 1.3) with basis B. The corresponding case of the Gell-Mann λ-matrices will be studied in Chapter 3.

There exist no monograph of which I am aware which is specifically devoted to the theory of Lie-admissible algebras. In the following we shall review the results by L.M. Weiner,[2] P.J. Laufer and M.L. Tomber[3] and R.M. Santilli[4-7] for the case of flexible Lie-admissible algebras. For conciseness, the interested reader is referred to the quoted papers for the proof of all statements. Other contributions will be outlined in the appendices.

Let S be a subspace of an algebra U over a field F which will be assumed of characteristic zero. The commutator space of S, denoted with $U^{(S)}$ is the set of all elements $a \in U$ such that $[a,b]_U = 0$ for all $b \in S$. Let (a) denote the set of all scalar multiples of $a \in U$. The first Weiner's results[2] can be stated as follows.

LEMMA 1.4.3: Let a be an element of a flexible Lie-admissible, power associative algebra U. Then $U^{(a)} \subseteq U^{(a^2)}$.

THEOREM 1.4.1: Let U be a flexible, Lie-admissible,

power-associative algebra of characteristic zero and S a subspace of U. Then $U^{(S)}$ is a subalgebra of U. If S^- is a subalgebra of U^- and $b^2 \in S$, then S is a subalgebra of U.

THEOREM 1.4.2: Let U be a flexible, Lie-admissible, power-associative algebra such that the attached algebra U^- is a direct sum of simple Lie algebras U_i^-

$$U^- = \bigoplus_{i=1}^{m} U_i^- \qquad (1.4.25)$$

where the vector spaces U_i of U_i^- are subspaces of U. Then U is the direct sum of simple algebras U_i

$$U = \bigoplus_{i=1}^{m} U_i \qquad (1.4.26)$$

An element a of a power-associative algebra U generates an associative and commutative subalgebra U_a of U consisting of all polynomials in a. Under the assumption that U is finite-dimensional, U_a is finite-dimensional too. The dimension d of U_a is called the degree of U (with respect to a).

THEOREM 1.4.3: Let U be a Lie-admissible, power-associative algebra and let U^- be a simple Lie algebra of degree three. Then U has degree one.

The above theorem (by Weiner[2]) is significant for the embedding of the SU(2) algebra into an SU(2)-admissible covering. Let

G_i, i=1,2,3, be the generators of $\underline{SU}(2) = U^-$, e.g., the Pauli matrices (1.4.21). A generic element of U can be written

$$a = \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3, \quad \alpha,\beta,\gamma \in F, \qquad (1.4.27)$$

and its square in U is given by

$$a^2 = \lambda\sigma_1 + \mu\sigma_2 + \delta\sigma_3, \quad \lambda,\mu,\delta \in F \qquad (1.4.28)$$

In this case it is possible to prove that

$$a^2 = f(a)a, \quad f(a) = \frac{\delta}{\gamma} \in F, \qquad (1.4.29)$$

and this illustrates Theorem 1.4.3, i.e., that the degree of U is, in this case, one. Weiner also proved that the function f of Eq. (1.4.29) is linear.

THEOREM 1.4.4: If U is a power-associative algebra of degree one, the multiplication in U is given by

$$ab = \frac{1}{2}[f(a)b + f(b)a] + [a,b]_U, \quad f(a), f(b) \in F. \qquad (1.4.30)$$

If, in addition, U is a flexible Lie-admissible algebra and U^- is simple, then $U \approx U^-$.

Notice that a basic assumption by Weiner is that the algebra U is power-associative, besides being flexible Lie-admissible. In this respect let us note that, even though the commutative Jordan algebras are power-associative, and thus the U^+ content of decomposition (1.4.15) is power-associative, this is not necessaily the case for the algebra U. As a matter of fact, the condition that a Lie-admissible algebra is power-associative, is a condition on its U^- content expressible by the rule

$$[\{a,a\}_U, a]_U = 0, \qquad (1.4.31)$$

for all $a \in U$.

In their paper of 1962, Laufer and Tomber[3] begin with the illustration of a property which is significant for the theory of Lie algebras, namely, that an algebra characterized by the Jordan law alone (without the additional condition of anticommutativity of the product) is not necessaily a Lie algebra. Consider an algebra U with basis

$$B = \{b_1, b_2, b_3, b_4\}, \qquad (1.4.32a)$$

$$b_1 b_3 = b_2 b_3 = b_3^2 = b_4 b_3 = b_1, \qquad (1.4.32b)$$

$$b_1 b_4 = b_2 b_4 = b_3 b_4 = b_4^2 = b_2, \qquad (1.4.32c)$$

where all the other products are null. Then, U satisfied the Jacobi law in the form

$$a(bc) + b(ca) + c(ab) = 0, \qquad (1.4.33)$$

but not in its equivalent form

$$(ab)c + (bc)a + (ca)b = 0, \qquad (1.4.34)$$

and, thus, U is not a Lie algebra (laws (1.4.33) and (1.4.34) are equivalent for Lie algebras on account of the anticommutativity of the product). In particular, the Jacobi law (1.4.33) alone is insufficient to characterize a power-associative algebra. However, if laws (1.4.33) and (1.4.34) hold, then

$$a^3 = a^2 a = a a^2 = 0 \qquad (1.4.35a)$$

$$a^2 a^2 = (a^2 a) a = 0 \qquad (1.4.35b)$$

$$a^m = a^{m-1} a = 0, \; m > 1 \qquad (1.4.35c)$$

$$a^m a^n = a^{m+n} = 0, \qquad (1.4.35d)$$

and the algebra is power-associative (Section 1.2). This again confirms the fact that a Lie-admissible algebra is power-associative, provided that certain restrictions on its Lie algebra content are satisfied, because both laws (1.4.33) and (1.4.34) are generally violated for a Lie-admissible algebra.

THEOREM 1.4.5: Let U be a flexible Lie-admissible algebra over an (algebraically closed) field F (of characteristic zero). If U^- is a semisimple Lie algebra, then U is a direct sum of simple flexible Lie-admissible algebras.

The central difference between Theorem 1.4.2 and 1.4.5 is the absence in the latter of the condition of power-associativity. By recalling that a Lie-admissible algebra is not necessarily power-associative, the latter theorem constitutes a significant improvement over the former.

COROLLARY 1.4.5.A: Let U be a flexible Lie-admissible algebra over F. If U^- is the direct sum of central simple Lie algebras, then U is the direct sum of simple flexible Lie-admissible algebras.

Notice that the condition of semisimplicity of the U^- content of U in Theorem 1.4.5 is replaced in Corollary 1.4.5.A by the condition that U^- is a central simple Lie algebra.

The following property[3] is important for our analysis. For the proof see also ref. 10.

THEOREM 1.4.6: Let U be a flexible, Lie-admissible power-associative algebra over F. If U^- is a simple Lie algebra, then $U \approx U^-$.

The above theorem is clearly a gener zation of Theorem 1.4.4 to the case of arbitrary degree. Notice how the condition of power-associativity, under the assumptions considered, forces the Lie-admissible algebra U to be isomorphic to its Lie algebra content U^-.

The reader should keep in mind that for any Lie-admissible algebras U, if U^- is a simple Lie algebra, U is simple. The inverse property, however, does not necessarily hold, namely, U can be simple, but U^- can nevertheless possess proper ideals.

We now continue this section with the extension of known results of the $A(\lambda)$ mutation algebras to the more general $A(\lambda, \mu)$ algebras along a 1967 paper by R.M. Santilli.[4]

The algebras $A(\lambda, \mu)$ can be initially studied within the context of the noncommutative Jordan algebras because, as indicated earlier, these algebras can be nontrivial Lie-admissible algebras. To put this aspect in its proper methodological profile, let us recall that

(a) every noncommutative Jordan algebra is power-associative and trace-admissible,[78]

(b) the radical R of a noncommutative Jordan algebra $\tilde{\mathfrak{J}}$ coincides with the radical of the attached commutative Jordan algebra $\tilde{\mathfrak{J}}^+$, $\tilde{\mathfrak{J}} \ominus \tilde{\mathfrak{J}}^+$ is semisimple and can be expressed as the direct sum of simple algebras,[78] and

(c) the only power-associative, simple and trace-admissible algebras are (1) the noncommutative Jordan algebras, (2) the quasi-associative algebras and (3) the flexible algebras of degree two.[58]

The algebras $A(\lambda, \mu)$ can therefore be studied within the context of the so-called quasi-associative algebras. Let U be an algebra over a field F (of characteristic zero). The algebra U is called a quasiassociative algebra[1] is there exists a scalar extension K of F and a quantity λ in K such that the scalar extension U_k of U is isomorphic to the λ-mutation algebra $A(\lambda)$ of an associative algebra A. We shall call an extended quasiassociative algebra an algebra U over F such that, under the scalar extensions K of F and U_k of U, there exist two quantities λ and μ in K such that U_k is isomorphic to the mutation algebras $A(\lambda, \mu)$ of an associative algebra A.

The transition from a quasiassociative to an extended quasi-associative algebra and vice versa can be performed through our concept of algebraic isotopy over a field (Section 1.2). At the

level of realization of the product this is characterized by mappings (1.2.31) or (1.2.37), e.g.,

$$a * b = \lambda a \cdot b + \mu b \cdot a = \lambda' \tau a \cdot b + (1-\lambda') \tau b \cdot a = \lambda' a \odot b + (1-\lambda') b \odot a \equiv a *' b \qquad (1.4.36a)$$

$$\tau = \lambda + \mu, \quad \lambda = \lambda' \tau, \quad \mu = \mu' \tau, \quad \lambda' + \mu' = 1 \qquad (1.4.36b)$$

Thus, the product of $A(\lambda)$ can be turned into that of $A(\lambda, \mu)$ and vice versa through an isotopic mapping. In particular, the mapping is isotopic in the sense that it is an invertible mapping of the product which preserves the underlying algebraic laws. Thus, the laws obeyed by $A(\lambda)$ and $A(\lambda, \mu)$ coincide (in the sense that both algebras are flexible, Lie-admissible and Jordan admissible). The algebraic isotopy

$$A(\lambda) \xrightarrow{\text{isotopic mapping}} A(\lambda, \mu) \qquad (1.4.37)$$

is, however, nontrivial. For instance, a central property of $A(\lambda)$ is that the powers its elements coincide with the corresponding powers in A, i.e.,

$$aa = a^2\big|_{A(\lambda)} = [\lambda + (1-\lambda)]\, a \cdot a = a^2\big|_A, \text{ etc.}, \qquad (1.4.38)$$

while this is not the case for $A(\lambda, \mu)$, i.e.,

$$aa = a^2\big|_{A(\lambda,\mu)} = (\lambda+\mu)\, a \cdot a = (\lambda+\mu)\, a^2\big|_A, \text{ etc.}, \qquad (1.4.39)$$

unless $\lambda + \mu = 1$, i.e., $A(\lambda, \mu) = A(\lambda)$. Also, the algebras $A(\lambda)$ verify the property

$$[a,b,c]_{A(\lambda)} = (1-\delta)\,[a,b,c]_{[A(\lambda)]^+}, \qquad (1.4.39a)$$

$$\delta = (2\lambda-1)^2, \qquad (1.4.39b)$$

$$[a,b,c]_{[A(\lambda)]^+} = \{\{a,b\}^*_{A(\lambda)},c\}^*_{A(\lambda)} - \{a,\{b,c\}^*_{A(\lambda)}\}^*_{A(\lambda)}, \qquad (1.4.39c)$$

$$\{a,b\}^* = \tfrac{1}{2}(ab+ba), \qquad (1.4.39d)$$

where δ is called the <u>discriminant</u> of A(λ). For A(λ,μ) we have instead[4]

$$[a,b,c]_{A(\lambda,\mu)} = (1-\delta')\,[a,b,c]_{[A(\lambda,\mu)]^+} \qquad (1.4.40a)$$

$$\delta' = \left(\frac{\lambda-\mu}{\lambda+\mu}\right)^2. \qquad (1.4.40b)$$

The above differences between the A(λ) and A(λ,μ) algebras are irrelevant from the viewpoint of the noncommuative Jordan algebras, but they became crucial from the viewpoint of the Lie algebra and, in particular, for our fundamental limit (1.4.8). In essence, the isotopic algebra A* of A charcterized by the product

$$a \circ b = (\lambda+\mu)\, a \cdot b \qquad (1.4.41)$$

is the zero algebra for $\lambda = -\mu$. As result, the limit

$$\lim_{\lambda\to-\mu} A(\lambda,\mu) = \lim_{\lambda\to-\mu} A^*(\lambda') = A^- \qquad (1.4.42)$$

while trivial for the A(λ,μ) algebra, constitutes for the A*(λ') algebra the highly singular limit of the ∞ -mutation of a zero associative algebra A* or, equivalently, of the divergent value of the discriminant (1.4.40b). In turn, this illustrates

the reason why the A(λ,μ) covering of the A(λ) algebras is more effective for limit (1.4.42), otherwise the two algegras are equivalent for $\lambda \neq -\mu$. In conclusion, the theory of noncommutative Jordan algebras can be applied to the flexible Lie-admissible algebras A(λ,μ) provided that $\lambda \neq -\mu$. Most of our subsequent use of the A(λ,μ) algebras will be for a nontrivial covering of a Lie algebra and, thus, for the case $\lambda \neq -\mu$. The theory of noncommutative Jordan algebras is therefore applicable.

The (λ,μ)-mutation algebras can be introduced for an arbitrary algebra U which is not necessarily associative, i.e., according to the product[4]

$$(a,b) = \lambda ab + \mu ba, \quad \lambda,\mu \in F \qquad (1.4.43)$$

where ab is not necessarily associative. This concept of algebraic mutation will play a fundamental role in our analysis, particularly for the problem of the possible existence of a covering of the Galilei and Einstein relativities. It is therefore useful to begin our study of the (λ,μ)-mutation of an arbitrary algebra U, rather than an associative algebra A. For subsequent use, the reader should keep in mind that the mapping

$$U \to U(\lambda,\mu) \qquad (1.4.44)$$

cannot be achieved through an isotopic mapping, because such mapping preserves the algebraic laws by assumption. Nevertheless, the above mapping can be achieved through our notion of <u>algebraic genotopy</u> (Section 1.2) because it violates the algebraic laws of U by construction. And indeed, mapping (1.4.44) is a particular case of mapping (1.2.31b). Thus, the concept of mutation and that of algebraic genotopy are equivalent when restricted to elements of the field, with the latter concept being a covering of the foremr (because, unlike the former, it is applicable also to elements of the algebra).

Let us recall that the power-associativity of an algebra (over a field of characteristic zero) is ensured when the following relations

$$[a, a, a]_U = 0 , \quad [a, a, a^2]_U = 0 \qquad (1.4.45)$$

holds (Section 1.2). But the associator of an element a $U(\lambda, \mu)$ is proportional to that in U according to the relation

$$[a, a, a]_{U(\lambda,\mu)} = (\lambda+\mu)^2 [a, a, a]_U \qquad (1.4.46)$$

We therefore have the following

<u>THEOREM</u> 1.4.7:[4] <u>The mutation algebra $U(\lambda, \mu)$ of an algebra U over a field F (of characteristic zero) is power-associative if and only if U is power-associative.</u>

Notice that for $\lambda = -\mu$, $U(\lambda, \mu)$ is trivially power-associative.

From the relations

$$((a,b), a) = \lambda^2 (ab)a + \lambda\mu (ba)a + \lambda\mu\, a(ab) + \mu^2 a(ba), \qquad (1.4.47a)$$

$$(a, (a,b)) = \lambda^2\, a(ba) + \lambda\mu\, a(ab) + \lambda\mu\, (ba)a + \mu^2 (ab)a, \qquad (1.4.47b)$$

we see that, if the algebra U is flexible, i.e., satisfies Eq. (1.4.13a), then

$$((a,b), a) = (a, (b,a)), \qquad (1.4.48)$$

i.e., $A(\lambda, \mu)$ is flexible too. Also, the algebras U^- and $[U(\lambda, \mu)]^-$ are characterized by the respective product

$$[a, b]_U = ab - ba , \qquad (1.4.49a)$$

$$[a, b]_{U(\lambda,\mu)} = (\lambda-\mu)\, [a, b]_U . \qquad (1.4.49b)$$

Thus, $[U(\lambda, \mu)]^-$ is isomorphic to the isotopic image U^{*-} of U^- with product

$$[a, b]_{U^*} = a*b - b*a , \quad a*b = (\lambda-\mu)ab. \qquad (1.4.50)$$

<u>THEOREM</u> 1.4.8:[4] <u>The mutation algebra $U(\lambda, \mu)$ of an algebra U over F is a flexible Lie-admissible algebra if and only if U is a flexible Lie-admissible algebra.</u>

Notice that for $\lambda = \mu$, $U(\lambda, \mu)$ is trivially flexible and Lie-admissible (in the sense that $[U(\lambda, \mu)]^-$ is the zero Lie algebra).

On similar grounds, U^+ and $[U(\lambda, \mu)]^+$ are characterized by the respective products

$$\{a, b\}^*_U = \frac{1}{2} (ab + ba), \qquad (1.4.51a)$$

$$\{a, b\}^*_{U(\lambda,\mu)} = \frac{\lambda+\mu}{2} (ab + ba). \qquad (1.4.51b)$$

Thus, $[U(\lambda, \mu)]^+$ is isomorphic to the isotopic image U^{*+} of U with product

$$\{a, b\}^*_{U^*} = \frac{1}{2}(a^* b + b^* a), \qquad (1.4.52a)$$

$$a^* b = (\lambda + \mu) ab \qquad (1.4.52b)$$

THEOREM 1.4.9:[4] The mutation algebra $U(\lambda, \mu)$ of an algebra U is a flexible Jordan-admissible algebra if and only if U is a flexible Jordan-admissible algebra.

From Eqs. (1.4.49b) and (1.4.51b) the following identity

$$ab = \frac{\lambda}{\lambda^2 - \mu^2}(a,b) + \frac{\mu}{\mu^2 - \lambda^2}(b,a) \qquad (1.4.53)$$

holds. Thus

THEOREM 1.4.10:[4] If $\hat{U} = U(\lambda, \mu)$ is a mutation algebra of U, then U can be recovered through the mutation $\hat{U}(\alpha, \beta)$ of U with

$$\alpha = \frac{\lambda}{\lambda^2 - \mu^2}, \quad \beta = \frac{\mu}{\mu^2 - \lambda^2} \qquad (1.4.54)$$

The above theorem has the following consequences for the case of an associative algebra U = A. As it is the case for the $A(\lambda)$ algebra,[1] if R is a two-sided ideal of A, (i.e., ab and ba $\in$ R for all b $\in$ R and a $\in$ U), then (a,b) and (b,a) $\in$ R, where (a,b) is product (1.4.43) for ab = a·b. It then follows that the decomposition $A = B \oplus R$ implies $A(\lambda, \mu) = B(\lambda, \mu) \oplus R(\lambda, \mu)$. In particular, $R(\lambda, \mu)$ is solvable (nilpotent) is R is solvable (nilpotent), and the maximal solvable ideal of $A(\lambda, \mu)$ coincides with that of A. Hence, when U is simple, $A(\lambda, \mu)$ is simple too and if A can be written as a direct sum of simple algebras, the same occurs for $A(\lambda, \mu)$. If we define the radical of $A(\lambda, \mu)$ as the (unique) maximal solvable ideal, then Theorems 1.3.2 (for the case $\lambda = -\mu$) and 1.3.4 (for the case $\lambda \neq -\mu$) implies the following

THEOREM 1.4.11:[4] If $R(\lambda, \mu)$ is the radical of a finite-dimensional mutation algebra $A(\lambda, \mu)$ of an associative algebra A over a field F (of characteristic zero), the quotient algebras $A(\lambda, \mu)/R(\lambda, \mu)$ is a semisimple mutation algebra. Any semisimple algebra $A(\lambda, \mu)$ is uniquely expressible as the direct sum of ideas $A_1(\lambda, \mu), A_2(\lambda, \mu), \ldots, A_n(\lambda, \mu)$

$$A(\lambda, \mu) = \bigoplus_{i=1}^{m} A_i(\lambda, \mu), \qquad (1.4.55)$$

each of which is a simple mutation algebra.

It should be stressed that the above definition of radical (and related structure theorem) applies, specifically, for the mutation algebras $A(\lambda, \mu)$ of an associative algebra. For the problem of the definition of the radical for a general Lie-admissible algebra see Appendies 1.D and 1.E.

1.5: CLASSICAL REALIZATIONS OF THE LIE-ADMISSIBLE ALGEBRAS

The algebraic structures considered until now are abstract in the sense that they are dealing with an algebra U as a linear vector space of unspecified elements a, b,..., product ab and given algebraic laws (that is, the algebra can be associative,nonassociative, alternative, flexible, Lie, Lie-admissible, etc.).

A primary objective of this monograph is to conduct an algebraic study of arbitrary Newtonian systems of N particles with masses m_k, k=1,2,...,N, in the three-dimensional Euclidean space of their experimental detection with Cartesian coordinates $\{\underline{r}^k\} = \{r^{kx}, r^{ky}, r^{kz}\}$.

This objective demands the study of the so-called classical realizations of abstract algebras, namely, the realizations of the elements of the algebras in terms of functions and the realization of the product in terms of suitably selected brackets. More specifically, the vector space U of abstract elements a,b,c, ..., can be realized in terms of the space

$$\mathcal{U} = \{A(t,a), B(t,a), C(t,a), \ldots\} \qquad (1.5.1a)$$

$$A, B, C, \ldots \in \mathcal{C}^\infty(R_{t,a}) \qquad (1.5.1b)$$

of functions in time t and the variables

$$\{a^\mu\} = \{r^{ka}, p_{ka}\} \qquad (1.5.2)$$

$$\mu = 1,2,\ldots,6N, \quad k=1,2,\ldots,N, \quad a = x,y,z$$

where the variables p_{ka} can at this time be conceived as

representing the linear momentum, $p_{ka} = m_k \dot{r}_{ka}$ [as we shall see in Chapter 4, upon introduction of a suitable topology, the variables a^μ can be interpreted as local charts of a manifold given, for instance, by the cotangent bundle T*M equipped with a suitable two-form]. The ground field F will be assumed to have characteristic zero throughout this section, as well as the remaining chapters of this volume.

We now equip the function space $\mathcal{U}$ with the brackets (or bilinear composition law)

$$A \circ B = \frac{\partial A}{\partial a^\mu} S^{\mu\nu}(t,a) \frac{\partial B}{\partial a^\nu}, \qquad (1.5.3a)$$

$$S^{\mu\nu} \in \mathcal{C}^\infty(R_{t,a}), \qquad (1.5.3b)$$

$$|S^{\mu\nu}|(R_{t,a}) \neq 0, \qquad (1.5.3c)$$

where condition (1.5.3c) ensures the regularity (and, thus, the invertibility) of the matrix $(S^{\mu\nu})$ everywhere in the considered region $R_{t,a}$ of the (local) variables (t, a^μ). Continuity conditions (1.5.1b) and (1.5.3b) are assumed for simplicity (as well as for geometrical considerations, see Chapter 4). We shall then refer to brackets (1.5.3) as being of class C^∞ and regular. We shall also call brackets (1.5.3) nontrivial when the $S^{\mu\nu}$ tensor possesses an essential dependence on at least some of the a^μ variables.

By keeping into accout that brackets (1.5.3) satisfy rules (1.2.5) identically, i.e.,

$$A \circ (B + C) = A \circ B + A \circ C, \qquad (1.5.4a)$$

$$(A + B) \circ C = A \circ C + B \circ C, \qquad (1.5.4b)$$

$$(\alpha \circ A) \circ B = A \circ (\alpha \circ B) = \alpha \circ (A \circ B) \tag{1.5.4c}$$
$$= (A \circ \alpha) \circ B = A \circ (B \circ \alpha) = (A \circ B) \circ \alpha = 0$$

the function space $\mathcal{U}$, when equipped with brackets (1.3.5), constitutes a classical realization of an abstract algebra U. Let us recall from Section 1.2 that Eqs. (1.5.4a) and (1.5.4b) are the right and left distributive laws. Strictly speaking, both these laws must be satisfied by the composition law of a function space to induce an algebra. If only one of these laws is satisfied, then the emerging structures are sometimes called left distributive algebra or right distributive algebra. Clearly, in order not to lessen its possible physical significance, the classical realizations of abstract algebras must be both, left and right distributive.

Irrespective of the explicit form of the $S^{\mu\nu}$ tensor, brackets (1.3.5) also satisfy the right and left differential laws

$$A \circ BC = (A \circ B)C + B(A \circ C), \tag{1.5.5a}$$
$$AB \circ C = (A \circ C)B + A(B \circ C), \tag{1.5.5b}$$

where BC is the ordinary (associative) product of functions, and the right and left scalar rule

$$A \circ \alpha = 0, \tag{1.5.6a}$$
$$\alpha \circ A = 0, \quad \alpha \in F \tag{1.5.6b}$$

Differential rules (1.5.5) are significant on both, algebraic as well as physical grounds. However, scalar rules (1.5.6) are algebraically restrictive. This is an indication that brackets

(1.5.3) are not the most general classical realization of the abstract product of an algebra, and more general realizations which satisfy the basic algebraic rules (1.5.4) as well as the differential rules (1.5.5), but violate the scalar rules (1.5.6) are, at least in principle, conceivable (see, for instance, Weiner's product (1.4.30) for algebras of degree one). In any case, most of our subsequent analysis will be restricted to classical realizations of abstract algebras in terms of brackets (1.5.3). It is therefore significant to identify the necessary and sufficient conditions for these brackets to characterize the most important algebras of our analysis, the Lie-admissible algebra.

General Lie-admissibility condition (1.4.2), expressed in terms of brackets (1.5.3), becomes

$$\begin{aligned}(A \circ B) \circ C &- A \circ (B \circ C) + (B \circ C) \circ A - B \circ (C \circ A) \\ &+ (C \circ A) \circ B - C \circ (A \circ B) = (C \circ B) \circ A - C \circ (B \circ A) \\ &+ (B \circ A) \circ C - B \circ (A \circ C) + (A \circ C) \circ B - A \circ (C \circ B).\end{aligned} \tag{1.5.7}$$

This condition must hold for arbitrary elements A,B,C,... of U. Thus, it must hold also for $A = a^{\mu}$, $B = a^{\nu}$ and $C = a^{\tau}$. The lack of derivative with respect to time in brackets (1.5.3) then yields the following

THEOREM 1.5.1: A necessary and sufficient condition for nontrivial brackets (1.5.3) to satisfy the general Lie-admissible condition (1.5.7) is that all the following equations

$$(S^{\tau\rho} - S^{\rho\tau}) \frac{\partial}{\partial a^\rho} (S^{\mu\nu} - S^{\nu\mu})$$
$$+ (S^{\mu\rho} - S^{\rho\mu}) \frac{\partial}{\partial a^\rho} (S^{\nu\tau} - S^{\tau\nu}) \qquad (1.5.8)$$
$$+ (S^{\nu\rho} - S^{\rho\nu}) \frac{\partial}{\partial a^\rho} (S^{\tau\mu} - S^{\mu\tau}) \equiv 0$$

are identically verified by the $S^{\mu\nu}$ tensor everywhere in the considered region of the (local) variables.

The above theorem in essence translates the general Lie-admissibility law into a system of conditions on the $S^{\mu\nu}$ tensor. When Eqs. (1.5.8) are verified, the function space $\mathcal{U}$ equipped with brackets (1.5.3) is a classical realization of a general Lie-admissible algebra. Notice that Theorem 1.5.1 transforms the algebraic law (1.5.7) into a (quasilinear) system of first-order partial differential equations in the unknown functions $S^{\mu\nu}$. Thus, any (class C^∞ and regular) solution $S^{\mu\nu}$ of Eqs. (1.5.8) characterizes brackets (1.5.3) which are Lie-admissible in the general sense. Notice also that, unlike the case of ordinary differential equations, (consistent) systems of partial differential equations often admit solutions with functional degrees of freedom. As a result, if equations (1.5.8) are consistent, the explicit form of the tensor $S^{\mu\nu}$ of the general Lie-admissible brackets is not expected to be unique or, equivalently, a family of general Lie-admissible brackets is expected to exist. As we shall see in Chapter 2, this is precisely the case for Theorem 1.5.1.

Flexible Lie-admissible conditions (1.4.4), expressed in terms of brackets (1.5.3), became

$$(A \circ B) \circ C - A \circ (B \circ C) + (C \circ B) \circ A - C \circ (B \circ A) = 0, \qquad (1.5.9a)$$

$$(A \circ B) \circ C - A \circ (B \circ C) + (B \circ C) \circ A - B \circ (C \circ A) + (C \circ A) \circ B - C \circ (A \circ B) = 0. \qquad (1.5.9b)$$

By repeating the same argument as that for Theorem 1.5.1 we reach the following

THEOREM 1.5.2: A necessary and sufficient condition for nontrivial brackets (1.5.3) to satisfy the flexible Lie-admissible conditions (1.5.9) is that all the following equations

$$S^{\mu\rho} \frac{\partial S^{\tau\nu}}{\partial a^\rho} + S^{\tau\rho} \frac{\partial S^{\nu\mu}}{\partial a^\rho} - \frac{\partial S^{\mu\nu}}{\partial a^\rho} S^{\rho\tau} - \frac{\partial S^{\tau\nu}}{\partial a^\rho} S^{\rho\mu} \equiv 0 \qquad (1.5.10a)$$

$$(S^{\tau\rho} - S^{\rho\tau}) \frac{\partial S^{\mu\nu}}{\partial a^\rho} + (S^{\mu\rho} - S^{\rho\mu}) \frac{\partial S^{\nu\tau}}{\partial a^\rho} + (S^{\nu\rho} - S^{\rho\nu}) \frac{\partial S^{\tau\mu}}{\partial a^\rho} \equiv 0, \qquad (1.5.10b)$$

are identically verified by the tensor $S^{\mu\nu}$ everywhere in the considered region of the (local) variables.

Notice that we have selected the flexible law in the linearized form (1.2.24) rather than in the form (1.4.4a). This is useful to reach a condition in three generally different indeces, (e.g., μ, ν and τ) rather than in two indeces (e.g., μ and ν). The study of the classical realization of law (1.4.4a) is left as an exercise for the interested reader.

Eqs. (1.5.8), (1.5.10a) and (1.5.10b) will be called a <u>classical realization of the general Lie-admissible law, the flexible law and the flexible Lie-admissible law</u>, respectively.

It is an instructive exercise for the interested reader to prove the following property. In a way fully parallel to the fact that Eq. (1.4.4b) is a special case of Eq. (1.4.2) under Eq. (1.4.4a), the flexible Lie-admissible law (1.5.10b) is a particular case of the general Lie-admissible law (1.5.8) under the flexible condition (1.5.10a).

Finally the Lie algebra laws, Eqs. (1.4.5), expressed in terms of brackets (1.5.3), became

$$A \circ B + B \circ A = 0, \qquad (1.5.11a)$$

$$(A \circ B) \circ C + (B \circ C) \circ A + (C \circ A) \circ B = 0. \qquad (1.5.11b)$$

By repeating the same argument as that for the preceding theorem, we reach the following

<u>THEOREM</u> 1.5.3: <u>A necessary and sufficient condition for nontrivial brackets (1.5.3) to satisfy the Lie algebra laws (1.5.11) is that all the following equations</u>

$$S^{\mu\nu} + S^{\nu\mu} \equiv 0, \qquad (1.5.12a)$$

$$S^{\tau\rho}\frac{\partial S^{\mu\nu}}{\partial a^\rho} + S^{\mu\rho}\frac{\partial S^{\nu\tau}}{\partial a^\rho} + S^{\nu\rho}\frac{\partial S^{\tau\mu}}{\partial a^\rho} \equiv 0, \qquad (1.5.12b)$$

<u>are identically verified by the tensor $S^{\mu\nu}$ everywhere in the considered region of the (local variables</u>.

The above property is well known.[83] It is here only presented in a different algebraic perspective. When conditions (1.5.12) are verified, brackets (1.5.3) are generally denoted with the symbol

$$A \circ B = [A, B]^{*} \quad , \quad S^{\mu\nu}(t, a) = \Omega^{\mu\nu}(t, a) \qquad (1.5.13)$$

and called the <u>generalized Poisson brackets</u>. For the case of lack of explicit dependence of time and under a number of technical implementations (see Chapter 4), a solution $\Omega^{\mu\nu}(a)$ of Eqs. (1.5.12) is called a <u>co-symplectic</u> or <u>Lie form</u>.

It should be here indicated that, in line with our definition of nontrivial brackets, when the $S^{\mu\nu}$ elements are constants, the above theorems do not apply and the use of the algebraic laws is requested. The familiar context of conventional canonical formulations is then recovered with the following

<u>COROLLARY</u> 1.5.3.A: <u>A form of the $S^{\mu\nu}$ tensor with constant elements which is admitted by the Lie algebra laws (1.5.11) is the fundamental co-symplectic form</u>

$$(\omega^{\mu\nu}) = \begin{pmatrix} ([r^{ia}, r^{jb}]) & ([r^{ia}, p_{jb}]) \\ ([p_{ia}, r^{jb}]) & ([p_{ia}, p_{jb}]) \end{pmatrix} = \begin{pmatrix} 0_{3N\times 3N} & 1_{3N\times 3N} \\ -1_{3N\times 3N} & 0_{3N\times 3N} \end{pmatrix}, \qquad (1.5.14)$$

in which case brackets (1.5.13) are the conventional Poisson brackets

$$[A,B] = \frac{\partial A}{\partial a^\mu}\,\omega^{\mu\nu}\,\frac{\partial B}{\partial a^\nu} = \frac{\partial A}{\partial r^{ka}}\frac{\partial B}{\partial p_{ka}} - \frac{\partial B}{\partial r^{ka}}\frac{\partial A}{\partial p_{ka}} \qquad (1.5.15)$$

A few remarks are here in order. Theorem 1.5.3, 1.5.2, and 1.5.1 essentially provide a hierarchy of classical realizations of algebraic structures which is fully parallel to the abstract hierarchy of Lie-admissible algebras of class III, II and I of Section 1.4. The conventional Poisson brackets emerge as the simplest conceivable Lie-admissible brackets in a hierarchy of brackets of increasing methodological needs. As we shall see in Chapter 2, this hierarchy can be interpreted as the algebraic counterpart of a corresponding hierarchy of Newtonian forces.

In this section, we are interested in identifying the general solution of integrability conditions (1.5.12) and (1.5.8). In this way, we shall reach the algebraic foundations of our subsequent studies.

For this purpose, it is recommendable to rewrite the fundamental cosymplectic tensor in a way more suitable for its generalization to an arbitrary cosymplectic form. Introduce the notation

$$R^\circ_\mu = \begin{cases} p_{ka}, & \mu = 1,2,\ldots,3N \\ 0, & \mu = 3N+1,\ldots,6N \end{cases} \qquad (1.5.16)$$

Then, the inverse of matrix ($\omega^{\mu\nu}$)

$$(\omega_{\mu\nu}) = \left(\|\omega^{\alpha\beta}\|^{-1}\right)_{\mu\nu} = \begin{pmatrix} 0_{3N\times3N} & -1_{3N\times3N} \\ 1_{3N\times3N} & 0_{3N\times3N} \end{pmatrix} \qquad (1.5.17)$$

called fundamental symplectic form, can be explicitly written

$$\omega_{\mu\nu} = \frac{\partial R^\circ_\nu}{\partial a^\mu} - \frac{\partial R^\circ_\mu}{\partial a^\nu} \qquad (1.5.18)$$

The desired expression of the fundamental symplectic/Lie tensor is then given by

$$(\omega^{\mu\nu}) = \left(\left\| \frac{\partial R^\circ_\alpha}{\partial a^\beta} - \frac{\partial R^\circ_\beta}{\partial a^\alpha} \right\|^{-1}\right)^{\mu\nu} \qquad (1.5.19)$$

The use of geometric arguments based on the most general possible closed and exact two forms, then permit the proof of the following

COROLLARY 1.5.3B: Under sufficient topological conditions (e.g., those of the symplectic geometry) the general solution of the Lie algebra laws (1.5.12a) and (1.5.12b), called the general cosympletic/Lie form, is given by

$$\Omega^{\mu\nu}(a) = \left(\| \Omega_{\alpha\beta}(a) \|^{-1} \right)^{\mu\nu} \quad (1.5.20a)$$

$$\Omega_{\alpha\beta} = \frac{\partial R_\beta(a)}{\partial a^\alpha} - \frac{\partial R_\alpha(a)}{\partial a^\beta} \quad (1.5.20b)$$

where the covariant form ($\Omega_{\alpha\beta}$) is the general symplectic form, and the R's are 6N independent, arbitrary functions of the local variables verifying regularity condition (1.5.3c). The generalized Poisson brackets then admit the explicit form

$$[A,B]^* = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu}(a) \frac{\partial B}{\partial a^\nu}$$

$$= \frac{\partial A}{\partial a^\mu} \left(\left\| \frac{\partial R_\alpha}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\alpha} \right\|^{-1} \right)^{\mu\nu} \frac{\partial B}{\partial a^\nu} \quad (1.5.21)$$

The interested reader is encouraged to prove that tensor (1.5.20) is indeed a solution of Equations (1.5.12) [the proof that we have the general solution is much more involved because of subtle geometric arguments, as indicated earlier].

The first step of our algebraic hierarchy is now clear. It is given by the Lie-algebra isotopy

$$[A,B] \Longrightarrow [A,B]^* \quad (1.5.22)$$

which can be explicitly written, by recalling the regularity of both the fundamental and the general cosymplectic tensors,

$$\omega^{\mu\nu} \Longrightarrow \Omega^{\mu\nu}(a) = g^\mu{}_\alpha(a)\, \omega^{\alpha\nu} \quad (1.5.23)$$

$$g^\mu{}_\alpha = \Omega^\mu{}_\rho\, \omega_{\rho\alpha}$$

We now pass to the study, first, of a simple realization of Lie-admissible laws (1.5.8), and, second, of their general solution. It is easy to see that the conditions considered restrict only the antisymmetric part of the tensor $S^{\mu\nu}$. We then have the following

COROLLARY 1.5.1A: A particular solution of the Lie-admissible laws (1.5.8), whose antisymmetric part has constant elements, is given by

$$S^{\mu\nu}(a) = \omega^{\mu\nu} + t^{\mu\nu}(a) \qquad (1.5.24)$$

where $\omega^{\mu\nu}$ is the fundamental cosymplectic/Lie tensor and $t^{\mu\nu}$ (a) is an arbitrary (not necessarily regular) symmetric tensor

$$t^{\mu\nu}(a) = t^{\nu\mu}(a) \qquad (1.5.25)$$

The Lie-admissible brackets whose antisymmetric part is given by the Poisson brackets can then be written

$$(A,B) = \frac{\partial A}{\partial a^\mu} S^{\mu\nu} \frac{\partial B}{\partial a^\nu} = \qquad (1.5.26)$$

$$= \frac{\partial A}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial B}{\partial a^\nu} + \frac{\partial A}{\partial a^\mu} t^{\mu\nu} \frac{\partial B}{\partial a^\nu} \stackrel{\text{def}}{=} [A,B] + \{A,B\}$$

and shall be called fundamental Lie-admissible brackets.

The second step of our algebraic hierarchy has now been identified. It consists of the Lie-admissible genotopy

$$[A,B] \Longrightarrow (A,B) = [A,B] + \{A,B\} \qquad (1.5.27)$$

that is, of the alteration of the Lie algebras characterized by the Poisson brackets into the structurally more general Lie-admissible algebras characterized by brackets (1.5.26).

The use of the preceding results of this section then permits the following

COROLLARY 1.5.1B: Under sufficient topological conditions (e.g., those of the symplectic-admissible geometry to be introduced in Chapter 4), the general solution of the Lie-admissible laws (1.5.8) is given by

$$S^{\mu\nu}(a) = \Omega^{\mu\nu}(a) + T^{\mu\nu}(a) \qquad (1.5.28)$$

where $\Omega^{\mu\nu}$ (a) is the general cosymplectic/Lie tensor and $T^{\mu\nu}$ (a) is an arbitrary (not necessarily regular) symmetric tensor

$$T^{\mu\nu} = T^{\nu\mu} \qquad (1.5.29)$$

The corresponding brackets, called general Lie-admissible brackets, can then be written

$$(A,B)^* = \frac{\partial A}{\partial a^\mu} S^{\mu\nu}(a) \frac{\partial B}{\partial a^\nu} \qquad (1.5.30)$$

$$= \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu}(a) \frac{\partial B}{\partial a^\nu} + \frac{\partial A}{\partial a^\mu} T^{\mu\nu} \frac{\partial B}{\partial a^\nu} \stackrel{\text{def}}{=}$$

$$= [A,B]^* + \{A,B\}^*$$

We reach in this way, the last step of our algebraic hierarchy, which is given by the Lie-admissible isotopy

$$\begin{pmatrix} (A,B) \\ = [A,B] + \{A,B\} \end{pmatrix} \Longrightarrow \begin{pmatrix} (A,B)^* \\ = [A,B]^* + \{A,B\}^* \end{pmatrix} \qquad (1.5.31)$$

Table 1.5.1 summarizes the essential steps. Note that the intermediary layer characterized by the realizations of the flexible Lie-admissible laws, is not included. Its study is left to the interested reader.

We have therefore the following

THEOREM 1.5.4: For given brackets (1.5.3) which are either Lie (Type III), or flexible Lie-admissible (Type II) or general Lie-admissible (Type I), the class of all possible isotopic mappings exhausts the class of all possible solutions of Eqs. (1.5.8), (1.5.10), or (1.5.12), respectively.

For instance, starting from the conventional Poisson brackets (1.5.15) the class of all possible isotopic mappings (1.5.22) char-

acterizes all possible classical realizations of the Lie algebra product. The same situation occurs for the flexible and the general Lie-admissible algebras.

On similar grounds we have the following

THEOREM 1.5.5: Starting from given Lie-admissible brackets (1.5.3) either of Type III, or of Type II, or of Type I, the class of all possible Lie-admissible genotopic mappings exhausts the class of all possible Lie-admissible brackets.

In conclusion, the analysis of this chapter indicates the existence of a hierarchy of three classes of Lie-admissible algebras with enclosure properties

$$\begin{pmatrix} \text{Type III} \\ \text{Lie algebras} \end{pmatrix} \subset \begin{pmatrix} \text{Type II} \\ \text{Flexible} \\ \text{Lie-admissible} \\ \text{algebras} \end{pmatrix} \subset \begin{pmatrix} \text{Type I} \\ \text{General} \\ \text{Lie-admissible} \\ \text{algebras} \end{pmatrix} \qquad (1.5.32)$$

To put it differently, the "degrees of freedom" of each classical realizations of the product are characterized by the isotopic mappings. The transition from one type to another is instead characterized by the genotopic mappings. Notice that both, the isotopic and the genotopic mappings we are here referring to are outside the context of the transformation theory because they occur within a fixed system of local variables by construction. The enclosure properties (1.5.31) can be trivially proved by noting that any solution of eith Eqs. (1.5.12), or

(1.5.10), or (1.5.8) is a solution of the subsequent equations. In particular, this is one reason why the conventional Poisson brackets are flexible and Lie-admissible.

To avoid possible misunderstanding, it should be indicated that Corollary 1.5.3B provides the general solution of Lie-admissible laws (1.5.8), but not necessarily the general solution of the classical realization of regular, bilinear, Lie-admissible brackets.

This is due to the assumption of bilinear form (1.5.3a) as the brackets of the theory, as conventionally used in contemporary mechanics, and which is sufficient for the analysis of this monograph.

However, the reader should keep in mind that more general bilinear brackets are conceivable. One example is readily given by the brackets

$$A \times B = \frac{\partial A}{\partial a^\mu} S^{\mu\nu} \frac{\partial B}{\partial a^\nu} + AB \qquad (1.5.33)$$

(where, again, AB is the usual product of functions) which, as the reader can easily see, characterize an algebra, i.e., verify both the right and left distributive laws as well as the scalar law.

The study of these more general brackets will be left to the interested reader. Our preference for brackets (1.5.3) is given by the fact that they admit an exponential form, with consequential achievement of a (generalized) group structure which is clearly important for our relativity objectives. The possibility of constructing a corresponding exponentiation for more general brackets, e.g., for brackets (1.5.32), has not been studied until now, to my best knowledge, and it will not be considered in this volume.

As incidential notes, we give the conditions on the tensor $S^{\mu\nu}$ of (non-trivial) brackets to satisfy other relevant types of algebras.

The right and left alternative laws (1.2.15) imply that

$$\frac{\partial S^{\mu\mu}}{\partial a^\rho} S^{\rho\nu} - S^{\mu\rho} \frac{\partial S^{\mu\nu}}{\partial a^\rho} = 0, \qquad (1.5.33a)$$

$$S^{\nu\rho} \frac{\partial S^{\mu\mu}}{\partial a^\rho} - \frac{\partial S^{\nu\mu}}{\partial a^\rho} S^{\rho\mu} = 0. \qquad (1.5.33b)$$

The reader should keep in mind that the Lie brackets do not satisfy these laws.

The power-associativity laws (1.2.15) imply that

$$\left(S^{\mu\rho} - S^{\rho\mu}\right) \frac{\partial S^{\mu\mu}}{\partial a^\rho} = 0. \qquad (1.5.34a)$$

$$\frac{\partial S^{\mu\mu}}{\partial a^\alpha} S^{\alpha\beta} \frac{\partial S^{\mu\mu}}{\partial a^\beta} - \frac{\partial}{\partial a^\alpha}\left(\frac{\partial S^{\mu\mu}}{\partial a^\rho} S^{\rho\mu}\right) S^{\alpha\mu} = 0. \qquad (1.5.34b)$$

The Poisson brackets trivially satisfy these laws because $[a^\mu, a^\mu] \equiv 0$.

The general Jordan-admissible law (1.4.11) implies that

$$\frac{\partial}{\partial a^\alpha}\left[\frac{\partial S^{\mu\mu}}{\partial a^\rho}\left(S^{\rho\nu} + S^{\nu\rho}\right)\right]\left(S^{\alpha\mu} + S^{\mu\alpha}\right)$$
$$= \frac{\partial S^{\mu\mu}}{\partial a^\alpha}\left(S^{\alpha\beta} + S^{\beta\alpha}\right)\frac{\partial}{\partial a^\beta}\left(S^{\nu\mu} + S^{\mu\nu}\right) \qquad (1.5.35)$$

The flexible Jordan-admissible laws (1.4.13) imply that

$$S^{\mu\rho} \frac{\partial S^{\nu\mu}}{\partial a^\rho} - \frac{\partial S^{\mu\nu}}{\partial a^\rho} S^{\rho\mu} = 0, \qquad (1.5.36a)$$

$$\frac{\partial}{\partial a^\alpha}\left(\frac{\partial S^{\mu\mu}}{\partial a^\rho} S^{\rho\nu}\right) S^{\alpha\mu} + S^{\mu\alpha}\frac{\partial}{\partial a^\alpha}\left(\frac{\partial S^{\mu\mu}}{\partial a^\rho} S^{\rho\nu}\right)$$
$$= \frac{\partial S^{\mu\mu}}{\partial a^\alpha} S^{\alpha\beta}\frac{\partial}{\partial a^\beta}\left(S^{\nu\mu} + S^{\mu\nu}\right). \qquad (1.5.36b)$$

Finally, the commutative Jordan algebra laws (1.4.14) imply that

$$S^{\mu\nu} - S^{\nu\mu} = 0, \qquad (1.5.37a)$$

$$\frac{\partial}{\partial a^\alpha}\left(\frac{\partial S^{\mu\mu}}{\partial a^\alpha} S^{\rho\nu}\right) S^{\alpha\mu} = \frac{\partial S^{\mu\mu}}{\partial a^\alpha} S^{\alpha\beta}\frac{\partial S^{\nu\mu}}{\partial a^\beta}. \qquad (1.5.37b)$$

As indicated in Section 1.4, the joint study of Lie and Jordan algebras is essential for the abstract treatment of Lie-admissible algebras. In Section 1.4 we also indicated the existence of algebras which where jointly Lie-admissible and Jordan admissible, *e.g.*, the mutation algebras A(λ , μ). A fundamental question of our analysis is whether this property has a counterpart at the level of the classical realizations of the brackets. This problem can be initially studied by demanding that all the following equations

$$\begin{cases} \left(S^{\tau\rho} - S^{\rho\tau}\right)\dfrac{\partial}{\partial a^\rho}\left(S^{\mu\nu} - S^{\nu\mu}\right) + \left(S^{\mu\rho} - S^{\rho\mu}\right)\dfrac{\partial}{\partial a^\rho}\left(S^{\nu\tau} - S^{\tau\nu}\right) \\ \quad + \left(S^{\nu\rho} - S^{\rho\nu}\right)\dfrac{\partial}{\partial a^\rho}\left(S^{\tau\mu} - S^{\mu\tau}\right) = 0, & (1.5.38a) \\ \dfrac{\partial}{\partial a^\alpha}\left[\dfrac{\partial S^{\mu\mu}}{\partial a^\rho}\left(S^{\rho\nu} + S^{\nu\rho}\right)\right]\left(S^{\alpha\mu} + S^{\mu\alpha}\right) \\ = \dfrac{\partial S^{\mu\mu}}{\partial a^\alpha}\left(S^{\alpha\beta} + S^{\beta\alpha}\right)\dfrac{\partial}{\partial a^\beta}\left(S^{\nu\mu} + S^{\mu\nu}\right) & (1.5.38b) \end{cases}$$

are identically verified by the $S^{\mu\nu}$ tensor everywhere in the considered region of the local variables.

As we shall see, while (Eqs. (1.5.38a) may have nontrivial solutions (i.e., solutions with non-constant elements), the joint sets of equations (1.5.38) severely restrict the possibility for the existence of nontrivial solutions.

On similar grounds, to search for nontrivial brackets (1.5.3) which are jointly flexible, Lie- and Jordan-admissible, one can impose that all the equations

$$\begin{cases} S^{\mu\rho}\dfrac{\partial S^{\tau\nu}}{\partial a^\rho} - S^{\tau\rho}\dfrac{\partial S^{\nu\mu}}{\partial a^\rho} - S^{\mu\rho}\dfrac{\partial S^{\nu\rho}}{\partial a^\rho} + S^{\rho\tau}\dfrac{\partial S^{\mu\nu}}{\partial a^\rho} = 0, & (1.5.39a) \\ \left(S^{\tau\rho} - S^{\rho\tau}\right)\dfrac{\partial S^{\mu\nu}}{\partial a^\rho} + \left(S^{\mu\rho} - S^{\rho\mu}\right)\dfrac{\partial S^{\nu\tau}}{\partial a^\rho} + \left(S^{\nu\rho} - S^{\rho\nu}\right)\dfrac{\partial S^{\tau\mu}}{\partial a^\rho} = 0, & (1.5.39b) \\ \dfrac{\partial}{\partial a^\alpha}\left(\dfrac{\partial S^{\mu\mu}}{\partial a^\rho} S^{\rho\nu}\right) S^{\alpha\mu} + S^{\mu\alpha}\dfrac{\partial}{\partial a^\alpha}\left(\dfrac{\partial S^{\mu\mu}}{\partial a^\rho} S^{\rho\nu}\right) \\ = \dfrac{\partial S^{\mu\mu}}{\partial a^\alpha} S^{\alpha\beta}\dfrac{\partial}{\partial a^\beta}\left(S^{\nu\mu} + S^{\mu\nu}\right), & (1.5.39c) \end{cases}$$

are identically verified by the $S^{\mu\nu}$ tensor everywhere in the considered region of the local variables.

Notice that the flexible condition appears only once and in the linearized form (1.5.10a). Eqs. (1.5.39b) and (1.5.39c) are the Lie- and Jordan admissibility conditions, respectively, under the flexible law. The problem can therefore be formulated by studying whether, in the class of flexible brackets (1.5.3), there exists a subclass which satisfied both conditions (1.5.39b) and (1.5.39c).

As we shall see better during the course of our analysis, while the classical realizations of the Lie-admissible algebras can be introduced in a way parallel to the abstract treatment of Section 1.4, the situation is different for the Jordan-admissible algebras and, thus, for the problem of classical realizations of Lie-admissible algebras which are also Jordan-admissible. An initial understanding of this situation can be reached as follows. Considered the realization of a Lie algebra L with (abstract) elements a,b,c,... in terms of the product

$$[a,b]_A = a\cdot b - b\cdot a, \qquad (1.5.40)$$

where a·b is associative. The "symmetrized" version of this product, i.e.,

$$\{a,b\}_A = a\cdot b + b\cdot a, \qquad (1.5.41)$$

characterizes a special commutative Jordan algebra J. Consider now the classical realization of the product $[a,b]_A$, e.g., the conventional Poisson brackets

$$[A,B]_{\mathcal{A}} = \frac{\partial A}{\partial r^{ka}}\frac{\partial B}{\partial p_{ka}} - \frac{\partial A}{\partial p_{ka}}\frac{\partial B}{\partial r^{ka}}. \qquad (1.5.42)$$

A "symmetrized" version of these brackets would read

$$\{A,B\}_{\mathcal{A}} = \frac{\partial A}{\partial r^{ka}}\frac{\partial B}{\partial p_{ka}} + \frac{\partial A}{\partial p_{ka}}\frac{\partial B}{\partial r^{ka}}. \qquad (1.5.43)$$

These brackets, however, even though satisfying the commutative law, do not constitute a classical realization of the commutative Jordan algebras because they violate the Jordan law, i.e.,

$$\{\{A,A\}_{\mathcal{A}},B\}_{\mathcal{A}},A\}_{\mathcal{A}} \neq \{\{A,A\}_{\mathcal{A}},\{B,A\}_{\mathcal{A}}\}_{\mathcal{A}}. \qquad (1.5.44)$$

This occurrence is, in essence, at the basis of the indicated difficulties of constructing a classical realization of the Lie-admissible algebras which is also Jordan-admissible.

Consider, from a broader algebraic profile, the mutation algebra $A(\lambda,\mu)$ with product (1.2.35), i.e.,

$$a * b = \lambda a\cdot b + \mu b\cdot a. \qquad (1.5.45)$$

As shown in Section 1.4, these algebras are jointly, flexible Lie- and Jordan-admissible. A "similar" classical version of product (1.5.45) in one space dimension was proposed in ref.[6]

$$A * B = \lambda \frac{\partial A}{\partial r}\frac{\partial B}{\partial p} + \mu\frac{\partial A}{\partial p}\frac{\partial B}{\partial r}. \qquad (1.5.46)$$

These brackets are Lie-admissible, i.e., satisfy laws (1.5.7) as the reader can verify by inspection . However, brackets (1.5.46) are not Jordan-admissible because they violate laws (1.4.13b). In essence, the algebra U^+ attached to the algebra U with product (1.5.4) is characterized by the brackets

$$\{A,B\}_U = (\lambda+\mu)\left(\frac{\partial A}{\partial r}\frac{\partial B}{\partial p} + \frac{\partial A}{\partial p}\frac{\partial B}{\partial r}\right), \qquad (1.5.47)$$

which violate the Jordan laws, as it is the case for brackets (1.5.43 . For the case of the algebra U^- attached to U we have instead the brackets

$$[A,B]_U = (\lambda-\mu)\left(\frac{\partial A}{\partial r}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial r}\right), \qquad (1.5.48)$$

which do satisfy the Lie algebra laws. Thus, U is Lie-admissible but not jointly Jordan-admissible.

Notice that brackets (1.5.46) are not flexible, as the reader can verify by inspection. Also, the structure of brackets (1.5.46), when extended to more than one space dimension, i.e., the λ and μ- terms become matrices, is no longer Lie-admissible. In other words, the Lie-admissibility of these brackets is dependent on the one-dimensionality of the space component.

Therefore, the classical realizations of commutative Jordan algebras, or of the Jordan-admissible algebras, appear to be a non-trivial task which may demand new insights, e.g., the use of the bonded algebras U* (U,S,T) with product (1.2.40). This aspect is here left to the interested reader.

As we shall see in Volume III, the situation is different at a quantum mechanical level. And indeed, by recalling that product (1.5.40) is a quantum mechanical realization of the Lie product, it is conceivable to expect that a quantum mechanical realization of a Lie-admissible product could be of type (1.5.45) and, as such, it characterizes a joint Lie- and Jordan-admissible algebra.

We now close this section with the remark that <u>the notion of Lie-admissible algebras is at the very foundation of Lie's theory</u>. In fact, the fundamental Lie rule (see Chapter 3 for detail) is precisely based on the notion of Lie-admissibility, only expressed in its simplest possible form, the associative form

$$[X_i, X_j] = X_i X_j - X_j X_i = C_{ij}^{\ k} X_k \qquad (1.5.49a)$$

$$X_i = U_i^{\mu}(a) \frac{\partial}{\partial a^{\mu}} \qquad (1.5.49b)$$

$$X_i X_j = \text{Associative Lie-admissible Algebra} \qquad (1.5.49c)$$

However, and quite intriguingly, when considering the realization of Lie algebras via the Poisson brackets, the underlying algebra turns out to be a <u>general, nonassociative, Lie-admissible algebra</u>, as evident from the property that the brackets

$$A \times B = \frac{\partial A}{\partial r^{ka}} \frac{\partial B}{\partial p_{ka}} \qquad (1.5.50)$$

verify all conditions for the characterization of an algebra; the algebra is nonassociative because of the property

$$(A \times B) \times C \neq A \times (B \times C) \qquad (1.5.51)$$

and, finally, the algebra is Lie-admissible because of the familiar form

$$[A, B] = A \times B - B \times A = \frac{\partial A}{\partial r^{ka}} \frac{\partial B}{\partial p_{ka}} - \frac{\partial B}{\partial r^{ka}} \frac{\partial A}{\partial p_{ka}} \qquad (1.5.52)$$

It is not inconceivable that the difficulties for identifying a classical realization of Jordan and Jordan-admissible algebras may be due to the nonassociativity of brackets (1.5.50). Also, the non-associativity of the algebra underlying Poisson brackets, when compared to the associativity of the algebra of operators underlying Heisenberg's equations, is one of the reasons illustrating the lack of achievement until now of a fully consistent quantization, as we shall see in Volume III.

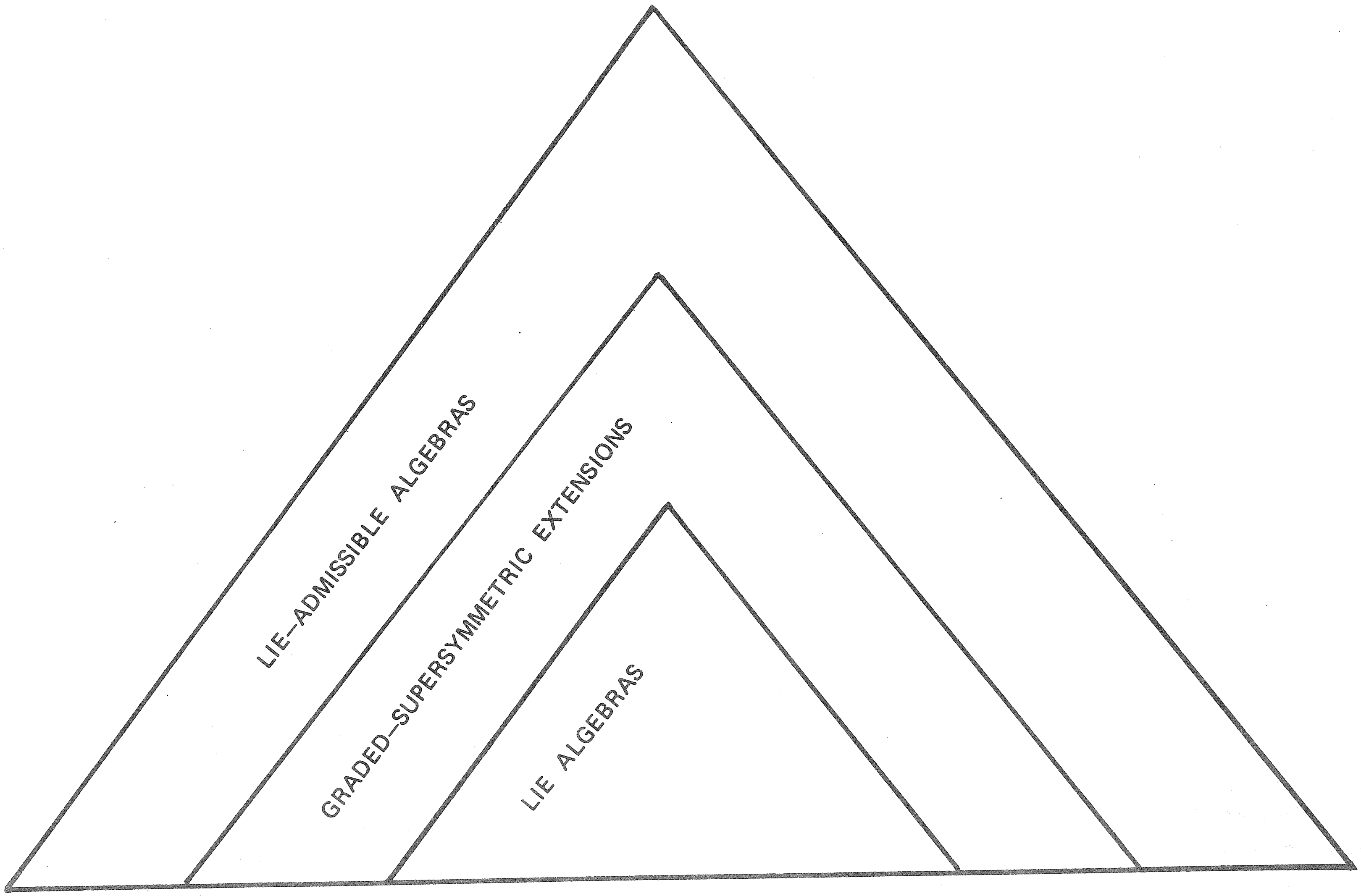


TABLE 1.5.1: A pictorial view of the generalization of Lie algebras offered by the Lie-admissible algebras. Chapter 1 essentially presents the generalization at the level of the axioms and of the primary algebraic properties. The Lie-admissible generalization of Lie's theory is discussed in Chapter 3. The inclusion of graded-supersymmetric extensions is presented in Appendix 3.E. The transition from contemporary physical models based on Lie algebras or their graded-supersymmetric extensions, to the generalized Lie-admissible models, essentially permits the treatment of particles as being extended, by therefore admitting additional, contact/nonpotential/non-Hamiltonian interactions. A classible generalization of Galilei and Einstein (special) relativities for extended particles is submitted in Chapter 5. The operator version on a (suitable formulation of the) Hilbert space is presented in the forthcoming Volume III.

NOTE ADDED in 1982

A considerable number of mathematical and physical studies have been conducted on Lie-admissible algebras since the time of writing this chapter. All these studies are directly relevant, not only for the classical analysis of this volume, but also for the quantum mechanical profiles of Volume III, as we shall see. These contributions are scattered throughout the five volumes of the HADRONIC JOURNAL [Volume 1, (1978) through Volume 5, (1982), as well as in other Journals. We refer the reader interested in these mathematical studies to:

- the Bibliography and Index of studies on Lie-admissible algebras by Tomber et al[183-185];
- the Proceedings of the Second (1979) and Third (1980) Workshops on Lie-admissible Formulations[186-187]; and
- the Proceedings of the First International Conference on Nonpotential Interactions and their Lie-admissible Treatment (1982)[188].

with particular reference to the studies by the following

- Mathematicians: G. M. Benkart (Univ. of Wisconsin), D. J. Britten (Univ. of Windsor, Canada), Y. Ilamed (Soreq Nuclear Research Center, Israel), M. Kôiv and J. Lôhmus (Acad. Sciences Estonian USSR), H. C. Myung (Univ. of Northern Iowa), R. H. Oehmke (Univ. of Iowa),

S. Okubo (Univ. of Rochester), J. M. Osborn (Univ. of Wisconsin), L. Sorgsepp (Acad. of Sciences Estonian USSR), M. L. Tomber (Michigan State Univ.), G. P. Wene (Univ. of Texas at San Antonio), and others.

- Theoretical physicists: G. Eder (Atominstitut, Austria), R. Mignani (Universitá di Roma, Italy), E. Kapuscik (I.N.F. Warsaw, Poland), Chun-Xuan Jiang (Peking, China), A. Schober (I.B.R., Cambridge), J. Kobussen (Universität Zürich, CH), R. Trostel (Technische Universität, Berlin, W. Germany), D. P. K. Ghikas (University of Patras, Greece), J. Fronteau and A. Tellez-Arenas (Université d'Orléans, France), S. Guiaşu (Université de Québec), J. Salmon (Conservatoire Nationale Paris), R. M. Santilli (I.B.R., Cambridge), T. L. Gill (Howard University), and others.

- Experimental physics (team leaders): R. J. Slobodrian (Université Laval, Québec), H. E. Conzett (Lawrence Berkeley Laboratory), H. Rauch (Atominstitut, Austria), G. Matone (Ital. Nat. Lab., Italy), and others.

APPENDIX 1.A: ASSOCIATIVE MULTIPLICATION ALGEBRA OF A LIE-ADMISSIBLE ALGEBRA

In these appendices the symbol U designates a (nonassociative, abstract) algebra U with elements a,b,c,... and product ab over a field F of elements $\alpha, \beta, \gamma, \ldots$ and characteristic zero. The symbols U^- and U^+ designate the algebras which coincide with U as vector spaces, but are equipped with the products

$$[a,b]_U = ab - ba, \qquad (1.A.1a)$$

$$\{a,b\}_U = \tfrac{1}{2}(ab + ba), \qquad (1.A.1b)$$

respectively.

Given an algebra U, we are here interested to construct an endomorphism of U (as vector space) which characterizes an associative algebra $\mathcal{A}(U)$ of linear operators in U. The realization of U we are interested in is that offered by right and left multiplications

$$R_b : a \rightarrow ab \quad \text{or} \quad aR_b = ab, \qquad (1.A.2a)$$

$$L_b : a \rightarrow ba \quad \text{or} \quad aL_b = ba, \qquad (1.A.2b)$$

for all $a \in U$. If $\mathcal{R}(U)$ and $\ell(u)$ are the sets of all right and left multiplications on U, respectively, $\mathcal{A}(U)$ can be defined as the enveloping associative algebra of $\mathcal{R}(U) \cup \ell(U)$. A generic element of $\mathcal{A}(U)$ can then be written $\alpha S_1 S_2 \ldots S_n$ where the S's are either left or right multiplications and $\alpha \in F$. The product in $\mathcal{A}(U)$, e.g., $R_a L_b$, is associative. The algebra $\mathcal{A}(U)$ so constructed is called the associative multiplication algebra of U.[41,71,73]

Note that

$$(a\alpha)R_b = (a\alpha)b = a(\alpha b) = aR_{\alpha b} = aR_{b\alpha}, \quad \alpha R_b = R_{b\alpha}, \qquad (1.A.3a)$$

$$aR_{b+c} = a(b+c) = ab + ac = aR_b + aR_c$$
$$= a(R_c + R_b), \quad R_{b+c} = R_b + R_c \qquad (1.A.3b)$$

$$a(ab) = aR_{ab} = aR_{aR_b}, \quad R_{ab} = R_{aR_b}, \qquad (1.A.3c)$$

and similar relations hold for the left multiplication.

Consider first the case when U is an associative algebra A. Then the associative law (1.2.1c), i.e.,

$$[a,b,c]_A = 0, \qquad (1.A.4)$$

can be written

$$R_{bc} = R_b R_c. \qquad (1.A.5)$$

Similarly, the equivalent version of the associative law

$$[c,b,a]_A = 0, \qquad (1.A.6)$$

can be written

$$L_{bc} = L_b L_c. \qquad (1.A.7)$$

The mapping $a \rightarrow R_a$ ($a \rightarrow L_a$) is a homomorphism (antihomomorphism) of the associative algebra A into the multiplication algebra $\mathcal{A}(A)$ of all linear transformations of A. Thus, it constitutes a representation of A. If A possesses an identity element,

$a \rightarrow R_a$ is a faithful representation because $R_a = R_b$ implies $a = b$. This indicates that the enveloping algebras $\mathcal{A}(A)$ of all linear operations in A provides an effective tool for the study of associative algebras.

For a nonassociative algebra U the mapping $a \rightarrow R_a$ ($a \rightarrow L_a$) is not a homomorphism (antihomomorphism). Nevertheless, the algebra $\mathcal{A}(U)$ is equally useful because of its associative nature.

If U is a Lie algebra L, then laws (1.3.1), i.e.,

$$ab + ba = 0, \qquad (1.A.8a)$$

$$(ab)c + (bc)a + (ca)b = 0, \qquad (1.A.8b)$$

can be written

$$R_a = -L_a, \qquad (1.A.9a)$$

$$R_{bc} = [R_b, R_c]_{\mathcal{A}}. \qquad (1.A.9b)$$

If U is a commutative Jordan algebra J, laws (1.3.2), i.e.,

$$ab - ba = 0, \qquad (1.A.10a)$$

$$(a^2 b) a = a^2 (ba), \qquad (1.A.10b)$$

became

$$R_a = L_a, \qquad (1.A.11a)$$

$$[R_a, R_{a^2}]_{\mathcal{A}} = 0, \qquad (1.A.11b)$$

and $\mathcal{A}(J)$ can be expressed in terms of either the right or the left multiplication only. Law (1.A.10b) can also be written, after

some manipultations,[71]

$$[R_a, R_{bc}]_{\mathcal{A}} + [R_b, R_{ca}]_{\mathcal{A}} + [R_c, R_{ab}]_{\mathcal{A}} = 0. \qquad (1.A.12)$$

If U is a power-associative algebra, law (1.2.15a), i.e.,

$$a^2 a = a a^2, \qquad (1.A.13)$$

can be linearized according to the form[1]

$$[\{a,b\}_U, c]_U + [\{b,c\}_U, a]_U + [\{c,a\}_U, b]_U = 0, \qquad (1.A.14)$$

and expressed as follows

$$R_{ab+ba} - L_{ab+ba} \qquad (1.A.15)$$

$$= (R_a + L_a)(R_b - L_b) + (R_b + L_b)(R_a - L_a).$$

Similarly, law (1.2.15b), i.e.,

$$a^2 a^2 = a^3 a, \qquad (1.A.16)$$

can be written

$$L_{a^2 a} = (R_a + L_a)(R_{a^2} + L_{a^2} - R_a R_a) - L_{a^2} R_a, \qquad (1.A.17)$$

Proceeding along the same lines, if U is an alternative algebra, then laws (1.2.13), i.e.,

$$[a,a,b]_U = 0, \quad [b,a,a]_U = 0 \qquad (1.A.18)$$

can be written

$$L_{a^2} = L_a^2, \quad R_{a^2} = R_a^2 \qquad (1.A.19)$$

If U is a flexible algebra, then law (1.2.23), i.e.,

$$[a, b, a]_U = 0 \qquad (1.A.20)$$

can be written

$$[R_a, L_a]_{\mathcal{A}} = 0 \qquad (1.A.21)$$

Similarly, the linearized flexibility law (1.2.24), i.e.,

$$[a, b, c]_U + [c, b, a]_U = 0, \qquad (1.A.22)$$

becomes

$$L_{ab} - L_b L_a = R_{ab} - R_b R_a. \qquad (1.A.23)$$

If $b = a^2$ we have in particular

$$L_{a^2} - L_a^2 = R_{a^2} - R_a^2. \qquad (1.A.24)$$

Thus, an alternative algebra is flexible.

A necessary and sufficient condition for an algebra U to be a general Lie-admissible algebra, from law (1.4.2), i.e.,

$$[a,b,c]_U + [b,c,a]_U + [c,a,b]_U = [c,b,a]_U + [b,a,c]_U + [a,c,b]_U, \qquad (1.A.25)$$

is given by[1]

$$R_{ab-ba} - L_{ab-ba} = (R_a - L_a)(R_b - L_b) - (R_b - L_b)(R_a - L_a) \qquad (1.A.26)$$

A necessary and sufficient condition for an algebra U to be a flexible Lie-admissible algebras, from laws (1.4.4), i.e.,

$$[a, b, a]_U = 0, \qquad (1.A.27a)$$

$$[c, a, b]_U + [b, c, a]_U - [c, b, a]_U = 0 \qquad (1.A.27b)$$

is given by[5]

$$[R_a, L_a]_{\mathcal{A}} = 0 \qquad (1.A.28a)$$

$$R_{ab-ba} = [R_a, (R_b - L_b)]. \qquad (1.A.28b)$$

If laws (1.A.28) are written in the equivalent form

$$[a, b, a]_U = 0, \qquad (1.A.29a)$$

$$[a,b,c]_U - [b,a,c]_U - [a,c,b]_U = 0, \qquad (1.A.29b)$$

then we have the equivalent multiplication rules

$$[R_a, L_a]_{\mathcal{A}} = 0, \qquad (1.A.30a)$$

$$L_{ab-ba} = [L_a, (R_b - L_b)]. \qquad (1.A.30b)$$

Associative algebras are general and flexible Lie-admissible algebras. Commutative Jordan algebras are trivially Lie-admissible in the sense that, Eqs. (1.A.26) or (1.A.28) are trivially satisfied by Eqs. (1.A.11).

A necessary and sufficient condition for an algebra U to be a general Jordan-admissible algebra, from law (1.4.11), i.e.,

$$(a^2 b)a + a(ba^2) + (ba^2)a + a(a^2 b) = a^2(ba) + (ab)a^2 + a^2(ab) + (ba)a^2 \qquad (1.A.31)$$

is given by[1]

$$[(R_{a^2} + L_{a^2}), (R_a + L_a)]_{\mathcal{A}} = 0 \qquad (1.A.32)$$

We now use the right and left multiplications to prove that the flexible Jordan -admissibility law (1.4.13b), i.e.,

$$(a^2b)a + a(a^2b) = a^2(ba) + a^2(ab) \qquad (1.A.33)$$

is a subcase of law (1.A.31) under the flexibility condition.[5] Law (1.A.23) can be written

$$L_{a^2}R_a - R_aL_{a^2} = R_{a^2}L_a - L_aR_{a^2}, \qquad (1.A.34a)$$

$$L_{a^2}L_a - L_{aa^2} = R_{a^2}R_a - R_{a^2a}, \qquad (1.A.34b)$$

$$L_aL_{a^2} - L_{a^2a} = R_aR_{a^2} - R_{aa^2}. \qquad (1.A.34c)$$

By summing up, Eq. (1.A.32) becomes, after use of Eqs. (1.A.13),

$$2(L_{a^2}R_a + L_{a^2}L_a - R_aL_{a^2} - L_aL_{a^2}) = 0, \quad (1.A.35)$$

and it coincides with Eq. (1.A.33) when rewritten in terms of the ordinary multiplication, up to the numerical factor two. A necesary and sufficient condition for an algebra to be a <u>flexible Jordan-admissible algebra</u> is therefore given by[5]

$$[R_a, L_a]_A = 0, \qquad (1.A.36a)$$

$$L_{a^2}R_a + L_{a^2}L_a - R_aL_{a^2} - L_aL_{a^2} = 0. \quad (1.A.36b)$$

Associative algebras are general and flexible Jordan-admissible algebras. Lie algebras are trivially Jordan-admissible in the sense that Eqs. (1.A.32) and (1.A.36) are trivially satisfied by laws (1.A.9). However, the commutative Jordan algebras are nontrivial Jordan-admissible.

Without proof we quote in the following certain properties which can be established through the use of the associative multiplication algebra. The <u>characteristic</u> of U is <u>prime to n</u> where the equation nx = 0 holds if and only if a = 0. U is said of <u>characteristic prime to zero</u> if it is prime to n for all integers $n > 1$.

<u>THEOREM</u> 1.A.1:[1] <u>Let U be a flexible algebra of characteristic prime to 30. Then U is power-associative if and only if $a^2a^2 = (a^2a)a$ for all $a \in U$.</u>

<u>THEOREM</u> 1.A.2:[1] <u>An alternative algebra of characteristic prime to six is Lie-admissible if and only if it is associative</u>.

<u>THEOREM</u> 1.A.3:[71] <u>Every element of a commutative Jordan algebra generates an associative subalgebra</u>.

<u>THEOREM</u> 1.A.4:[41] <u>An ideal S of an algebra U is nilpotent if and only if the associative multiplication subalgebra A(S) of A(U) is nilpotent</u>.

<u>THEOREM</u> 1.A.5:[41] <u>The center C of a simple algebra U is either zero or a field. In the latter case U is a central simple algebra over C</u>.

Notice that Theorems 1.A.4 and 1.A.5 apply to Lie-admissible algebras.

APPENDIX 1.B: LIE MULTIPLICATION ALGEBRA OF A LIE-ADMISSIBLE ALGEBRA

An effective approach in the study of an unknown algebra is that offered by the use of a known algebra. In Appendix 1.A we presented the main ideas in the use of an associative algebra, i.e., the multiplication algebra, for the study of Lie-admissible algebras. Clearly, Lie algebras constitutes another significant alternative for the study of Lie-admissible algebras. As a matter of act, Lie algebras can be used from more than one profile within such a context. In particular, the following approaches are significant for the analysis of these monographs.

(1) The study of the "Lie algebra content" of a Lie-admissible algebra U, that is, the attached algebra U^-. For an outline of this approach see Sections 1.4 and 1.5, Appendices 1.D and 1.E and Chapter 3.

(2) The study of the embedding of a Lie algebra L into a Lie admissible algebra U such that $U^- \approx L$. This approach is studied in Chapter 3.

(3) The study of the Lie multiplication algebra of a Lie-admissible algebra. This approach is outlined in this appendix.

A <u>derivation</u> D in any algebra U is a linear mapping of U into U satisfying the rule[41]

$$(ab)D = (aD)b + a(bD) \qquad (1.B.1)$$

The set $\mathcal{D}(U)$ of all derivations of U forms an algebra. This can be seen from the properties

$$(ab)(D_1 + D_2) = (ab)D_1 + (ab)D_2, \qquad (1.B.2a)$$

$$(\alpha ab)D = \alpha((ab)D) = (ab)(\alpha D). \qquad (1.B.2b)$$

In particular,

$$(ab)[D_1, D_2] = (a[D_1, D_2])b + a(b[D_1, D_2]), \qquad (1.B.3a)$$

$$[D_1, D_2] = D_1 D_2 - D_2 D_1. \qquad (1.B.3b)$$

Thus, the algebra $[\mathcal{D}(U)]^-$ with the product (1.B.3b), where $D_1 D_2$ is associative, is a Lie algebra.

More generally, let $\mathcal{D}$ be a subspace of the associative algebra $\mathcal{A}$ of all linear operators of U (as a vector space). The <u>Lie enveloping algebra</u>[28] L(D) of D is the intersection of all Lie subalgebras $\mathcal{A}^-$ containing D. By using the iterative rules

$$\mathcal{D}_1 = \mathcal{D}, \quad \ldots \quad \mathcal{D}_{i+1} = [\mathcal{D}_1, \mathcal{D}_i] \qquad (1.B.4)$$

the Lie enveloping algebra of D is

$$LD = \bigoplus_{i=1}^{\infty} \mathcal{D}_i \qquad (1.B.5)$$

Notice that

$$[\mathcal{D}_i, \mathcal{D}_j] \subseteq \mathcal{D}_{i+j} \qquad (1.B.6a)$$

$$[[\mathcal{D}_1, \mathcal{D}_2], \mathcal{D}_3] \subseteq [[\mathcal{D}_2, \mathcal{D}_3], \mathcal{D}_1] + [[\mathcal{D}_3, \mathcal{D}_1], \mathcal{D}_2] \qquad (1.B.6b)$$

where the last property follows from the Jacobi law. Thus

$$[\mathcal{D}_{i+1}, \mathcal{D}_j] = [[\mathcal{D}_1, \mathcal{D}_i], \mathcal{D}_j]$$
$$\subseteq [[\mathcal{D}_i, \mathcal{D}_j], \mathcal{D}_1] + [[\mathcal{D}_j, \mathcal{D}_1], \mathcal{D}_i] \qquad (1.B.6)$$
$$\subseteq [\mathcal{D}_{i+j}, \mathcal{D}_1] + [\mathcal{D}_{j+1}, \mathcal{D}_i]$$
$$\subseteq \mathcal{D}_{i+j+1} \subseteq \mathcal{D}_1$$

and L(D) is a Lie subalgebra of $\mathcal{A}^-$.

The realization L(U) of the derivative algebra $\mathcal{D}$(U) we are interested in is that in terms of the right and left multiplications (Appendix 1.A). $\mathcal{D}$ is a derivation of U if and only if, from Eq. (1.B.1), one of the following properties

$$[R_a, \mathcal{D}] = R_{a\mathcal{D}}, \qquad (1.B.7a)$$

$$[L_a, \mathcal{D}] = L_{a\mathcal{D}}, \qquad (1.B.7b)$$

hold for all $a \in U$. Thus, L(U) is a subalgebra of the Lie algebra $[\mathcal{A}(U)]^-$ of the associative multiplication algebra of U. The Lie algebra L(U) so constructed is called the Lie derivative algebra[41] of U.

The explicit form of the derivative $\mathcal{D}$ in terms of the right and left multiplications depends on the algebraic laws of U.

If U is an associative algebra A, from Eqs. (1.A.5) and (1.A.7), i.e.,

$$R_a R_b = R_{ab}, [R_a, L_b] = 0, L_a L_b = L_{ba} \qquad (1.B.8)$$

we have[56]

$$\mathcal{D} = R_a - L_a \qquad (1.B.9)$$

i.e.,

$$\mathcal{L}(A) = R(\mathcal{A}) \ominus L(\mathcal{A}) \qquad (1.B.10)$$

If U is a Lie algebra L, then in view of Eqs. (1.A.9), i.e.,

$$R_a = -L_a, [R_a, R_b] = R_{ab} \qquad (1.B.11)$$

we have[28]

$$\mathcal{D} = R_a \qquad (1.B.12)$$

i.e.,

$$\mathcal{L}(L) = R(\mathcal{A}) \qquad (1.B.13)$$

In this case the mapping $a \to R_a$ is called the adjoint mapping[69] of a

$$R_a = ad(a) \qquad (1.B.14)$$

If U is a commutative Jordan algebra J, in view of Eq. (1.A.12), i.e.,

$$[R_a, [R_b, R_c]] = R_{a[R_b, R_c]} \qquad (1.B.15)$$

we have[41]

$$\mathcal{D} = [R_b, R_c] \qquad (1.B.16)$$

i.e.,

$$\mathcal{L}(J) = [R(\mathcal{A})]^- \qquad (1.B.17)$$

If U is a flexible Lie-admissible algebra (with identity), in view of Eqs. (1.A.23), i.e.,

$$[R_a, (R_b - L_b)] = R_{a(R_b - L_b)} \qquad (1.B.18a)$$

$$[L_a, (R_b - L_b)] = L_{a(R_b - L_b)} \qquad (1.B.18b)$$

we have

$$\mathcal{D} = R_b - L_b \qquad (1.B.19)$$

i.e.,

$$L(U) = R(A) \ominus L(A) \qquad (1.B.20)$$

Thus, the realization of the derivatives of the associative and the flexible Lie-admissible algebras coincides. The reader should keep this property in mind for the analysis of Chapter 3 and, particularly, for the generalization of the Poincare-Birkhoff-Witt theorem to Lie-admissible algebras.

To the best of my knowledge, the realization of the derivative in terms of left and right multiplications for the case of a general Lie-admissible algebra has not been investigated until now. Notice that Eqs. (1.A.26) can be written

$$[(R_a - L_a), (R_b - L_b)] = R_{a(R_b - L_b)} - L_{a(R_b - L_b)} \qquad (1.B.21)$$

By putting $\mathcal{D} = R_a - L_a$, Eq. (1.B.21) can be interpreted as the difference of the two expressions

$$[R_a, \mathcal{D}] = R_{a\mathcal{D}} \quad , \quad [L_a, \mathcal{D}] = L_{a\mathcal{D}} \qquad (1.B.22)$$

However, in general, these expressions are not individually verified for a general Lie-admissible algebra. Thus, $\mathcal{D} = R_a - L_a$ is not a realization of the derivation for the algebra considered

according to the conventional approach.

Lie multiplication algebras L(U) have several important applications for the study of Lie algebras, Jordan algebras and their relationship. See in this respect reference[28,58,60]. For instance, it is possible to prove that the Lie exception algebra G_2 (Section 1.3) is the derivation algebra $L(J_3^8)$ of the exceptional Jordan algebra (octonions or Cayley numbers).

A derivation $\mathcal{D}$ of U is called an inner derivation when it is an element of L(U). The quadratic form

$$K = Tr\ (ad\,a)(ad\,b) \qquad (1.B.23)$$

is called the Killing form.[69] The following theorem on Lie algebras are well known.

THEOREM 1.B.1:[69] Let L be a finite-dimensional Lie algebra over a field F of characteristic zero possessing a nondegenerate Killing form. Then every derivation $\mathcal{D}$ of L is inner.

THEOREM 1.B.2:[69] Let L be a finite-dimensional Lie algebra over a field F of characteristic zero, R its radical and $\hat{R}$ its nilpotent radical. Then any derivation $\mathcal{D}$ of L maps R into $\hat{R}$.

The radical of L was defined in Section 1.3. The nilpotent radical of L is the radical which contains every nilpotent ideal of L. The extension of Theorem 1.B.2 to Lie-admissible algebras has not been worked out until now, to my knowledge. Nevertheless, an extension of Theorem 1.B.1 exists and reads

THEOREM 1.B.3:[41,73] Let U be a finite-dimensional algebra over a field F of characteristic zero with a left (or right) identity and suppose that U is the direct sum of simple ideals S_i, i.e.,

$$U = \bigoplus_{i=1}^{m} S_i \qquad (1.B.24)$$

Then every derivation of U is inner.

The above theorem is applicable to flexible Lie-admissible algebras. Nevertheless, its extension to the general Lie-admissible algebra is problematic owing to the lack of knowledge of the realization of a derivation in terms of the right and left multiplications, as indicated earlier. In general, the study of the Lie derivative algebra L(U) of a flexible Lie-admissible algebra can be conducted through a judicious extension of the Lie derivative algebra of a noncommutative Jordan algebra. However, the study of the Lie derivative algebra of a genuine (i.e., nonflexible) general Lie-admissible algebra demands a specific study which is here left to the interested reader.

APPENDIX 1.C: PIERCE DECOMPOSITION OF A LIE-ADMISSIBLE ALGEBRA

The use of associative and Lie algebras for the study of Lie-admissible algebras has been outlined in the preceding appendices. Clearly, other algebras which can be effectively used for the study of the Lie-admissible algebras are the commutative Jordan algebras. In this appendix we shall outline the main ideas for this approach. The interested reader is however urged to consult the quoted literature for technical details, as well as for further developments.

The use of idempotents $e(e^2 = e)$ plays a central role in the study of the associative algebras A[1] as well as the commutative Jordan algebras J[71] and their noncommutative generalization $\tilde{J}$.[82] The same approach is inapplicable to the Lie algebras L (because $a^2 = o$ for all $a \in L$). Nevertheless, the use of idempotents becomes again applicable as well as significant for nontrivial Lie-admissible algebras, i.e., those other than (associative and) Lie. Let U be an algebra of this type over a field F (of characteristic zero). The product ab in U is neither commutative nor anticommutative (Section 1.4). Thus, it admi decomposition (1.4.15), i.e.,

$$ab = \frac{1}{2}[a,b]_U + \{a,b\}_U \qquad (1.C.1a)$$

$$[a,b]_U = ab - ba, \quad \{a,b\}_U = \frac{1}{2}(ab + ba) \qquad (1.C.1b)$$

where the product $[a,b]_U$ characterizes the Lie algebra content U^- of U, while the product $\{a,b\}_U$ characterizes the Jordan algebra content U^+ of U. It is then evident that the study of U by using idempotents puts the emphasis in its commutative Jordan algebra content. And indeed, all idempotents of U are idempotents of U^+. This approach should be compared to that of Appendix 1.B where the use of the derivations puts the emphasis on the study of U through a Lie algebra structure.

In this appendix we shall first reinterpret known results for commutative Jordan algebras J as the attached algebra U^+ of a Lie-admissible algebra U and then touch on the problem of their extension to U.

Identities (1.A.12) in U^+, i.e.,

$$[R_a, R_{bc}]_{\mathcal{R}} + [R_b, R_{ca}]_{\mathcal{R}} + [R_c, R_{ab}]_{\mathcal{R}} = 0 \qquad (1.C.2)$$

can be written for ab = $\{a,b\}_{U^+}$

$$\begin{aligned} &(da)(bc) + (db)(ca) + (dc)(ab) \\ &= [d(bc)]a + [d(ca)]b + [d(ab)]c \end{aligned} \qquad (1.C.3)$$

yielding, after some manipulations

$$\begin{aligned} &R_a R_{bc} + R_b R_{ca} + R_c R_{ab} \\ &= R_{a(bc)} + R_c R_a R_b + R_b R_a R_c \end{aligned} \qquad (1.C.4)$$

Assume that a = b = c = e is an idempotent in U^+. Then we can write[71]

$$2R_e^3 - 3R_e^2 + R_e = R_e(2R_e - 1)(R_e - 1) \qquad (1.C.5)$$

The solutions (0,½,1) of this equation are called the <u>characteristic roots</u> of R_a. Notice tha, since $eR_e = e$, one of the characteristic roots of R_a must be 1.

The sets

$$U^+_i(e) = \{a \,/\, a \in U^+, aR_e = ia, \; i = 0, \tfrac{1}{2}, 1\}, \qquad (1.C.6)$$

are invariant subspaces of U^+ under R_e. The direct sum decomposition which then emerges

$$U^+ = U^+_0(e) \oplus U^+_{\frac{1}{2}}(e) \oplus U^+_1(e), \qquad (1.C.7)$$

is called the <u>Pierce decomposition</u>[41,71,73] of U relative to the idempotent e.

As a simple example, consider the case when U^+ is the commutative Jordan algebra induced by the <u>Pauli matrices</u> (Section 1.4)

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (1.C.8)$$

A generic element of U^+ can be written

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \qquad (1.C.9)$$

Introduce now the idempotent

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (1.C.10)$$

Then the element a admits the Pierce decomposition

$$a = \begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix} \oplus \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} \oplus \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix} \qquad (1.C.11)$$

More generally, Lemma 9.1, page 173 of ref. 71 for commutative Jordan algebras J can be easily reinterpreted as characterizing the U^+ content of U.

<u>LEMMA</u> 1.C.1: <u>The subspaces $U^+_i(e)$, i = 0,½,1, of the</u>

Pierce decomposition of the commutative Jordan algebra U^+ of a Lie-admissible algebra U with respect to the dempotent e satisfy the properties

$$(U_1^+)^2 \subseteq U_1^+, \quad (U_0^+)^2 \subseteq U_0^+, \qquad (1.C.12a)$$

$$U_1^+ U_{\frac{1}{2}}^+ \subseteq U_{\frac{1}{2}}^+, \quad U_0^+ U_{\frac{1}{2}}^+ \subseteq U_{\frac{1}{2}}^+, \qquad (1.C.12b)$$

$$U_0^+ U_1^+ = 0; \quad (U_{\frac{1}{2}}^+)^2 \subseteq U_0 \oplus U_1. \qquad (1.C.12c)$$

Notice, from the above theorem, that the idempotent element e is the identity element of U_1^+ and that U_0^+ and $U_{\frac{1}{2}}^+$ are Jordan subalgebras of U^+.

Suppose now that U^+ has a unit element 1 admitting a decomposition in terms of pairwise orthogonal idempotents, i.e.,

$$1 = \bigoplus_{i=1}^{n} e_i, \quad \{e_i, e_j\}_U = 0, \quad \{e_i, e_i\}_U = e_i, \qquad (1.C.13)$$

and introduce the subspaces of U^+

$$U_{ii}^+ = \{a \mid a \in U^+, \ a e_i = a\}, \qquad (1.C.14a)$$

$$U_{ij}^+ = \{a \mid a \in U^+, \ a e_i = a e_j = \tfrac{1}{2} a\}. \qquad (1.C.14b)$$

The Pierce decomposition of U^+ with respect to the idempotents e_i is then given by[71]

$$U^+ = \bigoplus_{i=1}^{n} U_{ii}^+ \bigoplus_{\substack{i,j=1 \\ i<j}}^{n} U_{ij}^+, \qquad (1.C.15a)$$

$$U_1^+(e_i) = U_{ii}^+, \quad U_{\frac{1}{2}}^+(e_i) = \bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} U_{ij}^+, \quad U_0^+(e) = \bigoplus_{\substack{j,k=1 \\ j,k \neq i}}^{n} U_{jk}^+. \qquad (1.C.15b)$$

Let us here recall that any finite-dimensional semisimple commutative Jordan algebra over a field F (of characteristic zero) has a unit element 1.[71,73] Thus, decomposition (1.C.15) apply when U^+ is semisimple.

If U^+ is of degree one (Section 1.3), then the unit element of U_1^+ is the only idempotent e. It is possible to prove in this case that $U_0^+(e) \oplus U_1^+(e)$ is contained in the radical of U^+. Thus, when U^+ is simple, $U^+ = U_1^+(e)$, i.e., $U^+ = Fe$. We recover in this way the central simple commutative Jordan algebras of degree 1 of Section 1.3.

If U^+ is of degree two, then

$$1 = e_1 \oplus e_2, \quad \{e_1, e_2\}_U = 0, \quad \{e_i, e_i\}_U = 0, \ i = 1,2 \qquad (1.C.16)$$

In this case $U_{11}^+ = Fe_1, U_{22}^+ = Fe_2$ and U^+ is simple if $n \geqslant 2$. This yields for U^+ the special commutative Jordan algebra which can be realized as a subalgebra C^+ attached to a Clifford algebra with symmetric bilinear forms $(x,y) = F \oplus V(F)$.[71,73]

The case of algebras U^+ of degree 3 can be studied along the same lines and, when U^+ is simple, yields the classification of Section 1.3. For details see references[71,73] and quoted papers.

The extension of the Pierce decomposition to noncommutative Jordan algebras has been studied in the existing literature. See, for instance, in this respect, reference[82]. But a noncommutative Jordan algebra can be a flexible Lie-admissible algebra. Thus, the results can be reinterpreted for our context.

Suppose that U is a flexible Lie-admissible algebra which is not Lie nor associative. Since idempotents in U coincide with the idempotents in U^+, the Pierce decomposition (1.C.7) can be extended to U according to the rules[82]

$$\mathcal{U} = \mathcal{U}_0 \oplus \mathcal{U}_1 \oplus \mathcal{U}_2 \qquad (1.C.17a)$$

$$\mathcal{U}_i = \{a \mid ea + ae = ia,\ i = 0, 1, 2\} \qquad (1.C.17b)$$

Instead of enclosures (1.C.12) we now have the weaker forms

$$\mathcal{U}_i^2 \subset \mathcal{U}_i\,, \quad \mathcal{U}_i\mathcal{U}_1 + \mathcal{U}_1\mathcal{U}_i \subset \mathcal{U}_1 \qquad (1.C.18)$$

Similarly, decomposition (1.C.15) extends to U according to

$$\mathcal{U} = \bigoplus_{i,j=0}^{m} \mathcal{U}_{ij} \qquad (1.C.19a)$$

$$\mathcal{U}_{00} = \{a \mid ea + ae = 0\}, \qquad (1.C.19b)$$

$$\mathcal{U}_{ii} = \{a \mid e_i a + a e_i = ea + ea = 2a\}, \qquad (1.C.19c)$$

$$\mathcal{U}_{i0} = \{a \mid e_i a + a e_i = ea + ea = a\} = \mathcal{U}_{0i}, \qquad (1.C.19d)$$

$$\mathcal{U}_{ij} = \{a \mid e_i a + a e_i = e_j a + a e_j = a\} = \mathcal{U}_{ji} \qquad (1.C.19e)$$

$$a \in \mathcal{U}_{ij} \Rightarrow e_i a = a e_j \qquad (1.C.19f)$$

The extensions of the Pierce decomposition to a general (nonflexible) Lie-admissible algebras has not been investigated until now, to my knowledge. One difficulty for this extension is due to the

fact that for a general Lie-admissible algebra the notion of characteristic roots of the right andleft multiplications must be subject to a suitable technical implementation. First of all, at least two identities are needed in the hope of identifying these characteristic roots. The identity (1.A.26) for the general Lie-admissibility, i.e.,

$$R_{a_i a_j - a_j a_i} - L_{a_i a_j - a_j a_i} = (R_{a_i} - L_{a_i})(R_{a_j} - L_{a_j}) - (R_{a_j} - L_{a_j})(R_{a_i} - L_{a_i}) \qquad (1.C.20)$$

is therefore insufficient when considered alone. A candidate for the second identity is given by power-associativity (notice that general Lie-admissible algebras are not, in general, power-associative). In this case Eq. (1.A.16) can be linearized to the form

$$\sum_6 \mathrm{Perm}\,(a_i a_j + a_j a_i)(a_2 a_s + a_s a_2) = \sum_4 \mathrm{Perm}\left[\sum_3 \mathrm{Perm}\,(a_i a_j + a_j a_i)\,a_2\right] a_s \qquad (1.C.21)$$

that is

$$\begin{aligned} & L_{(a_i a_j + a_j a_i)a_2} + L_{(a_j a_2 + a_2 a_j)a_i} + L_{(a_2 a_i + a_i a_2)a_j} \\ &= (R_{a_i} + L_{a_i})(R_{a_j a_2 + a_2 a_j} + L_{a_j a_2 + a_2 a_j} - R_{a_j} R_{a_2} - R_{a_2} R_{a_j}) \\ &+ (R_{a_j} + L_{a_j})(R_{a_i a_2 + a_2 a_i} + L_{a_i a_2 + a_2 a_i} - R_{a_i} R_{a_2} - R_{a_2} R_{a_i}) \\ &- R_{a_2} L_{a_i a_j + a_j a_i} + R_{a_j} L_{a_i a_2 + a_2 a_i} - R_{a_i} L_{a_j a_2 + a_2 a_j} \end{aligned} \qquad (1.C.22)$$

By putting $a_i = a_j$ in Eq. (1.C.20) and $a_i = a_j = a_2$ in Eq. (1.C.22) we have

$$R_{a_i a_i} + L_{a_i a_i} = (R_{a_i} + L_{a_i})(R_{a_i} - L_{a_i}), \qquad (1.C.22a)$$

$$L_{(a_i a_i) a_i} = (R_{a_i} + L_{a_i})(R_{a_i a_i} + L_{a_i a_i} - R_{a_i} R_{a_i}) - L_{a_i a_i} R_{a_i}. \qquad (1.C.22b)$$

Finally, by putting $a_i = e$ we reach the system[5]

$$\begin{cases} (R_e - L_e)(R_e + L_e - 1) = 0, & (1.C.23a) \\ (R_e + L_e)^2 - (R_e + L_e) R_e^2 - L_e (R_e + 1) = 0. & (1.C.23b) \end{cases}$$

The comparison of the above system with the simple equation (1.C.5) indicates the increase in complexity for the extension of the Pierce decomposition to general Lie-admissible and power-associative algebras. Studies along these lines are encouraged.

For a specific study on the Pierce decomposition for Lie-admissible algebras see ref[17]. For the relation between these algebras and the nodal noncommutative Jordan algebras see refs.[15,16]

APPENDIX 1.D: RADICAL OF A LIE-ADMISSIBLE ALGEBRA

A central problem in the study of an algebra is that of classification. The use of the concept of radical has proved to be particularly effective in this respect. The first major breakthrough in the study of algebras by means of radicals was achieved by E.Cartan in 1894.[84] The result was a characterization of Lie algebras, as recalled in Section 1.3 (see also Appendix 1.E). The second major breakthrough was achieved by J.H.M. Wedderburn in the early part of this century.[85] This resulted in the first structure theorem for associative algebras, as recalled in Section 1.3. These studies were then extended by several authors to a number of nonassociative algebras, such as commutative and noncommutative Jordan algebras, flexible algebras, alternative algebras, etc. Nevertheless, the structure problem of rather large classes of algebras remained still open. A third major breakthrough was finally achieved by N. Jacobson in 1945[86] who identified a concept of radical which is applicable to a rather large class of rings and algebras.

This appendix is devoted to a brief outline of the problem of the radical of a nonassociative algebra. Since most of the literature available is for rings, we shall first review the definitions of radical for rings and then touch on the problem of their extension to algebras. For textbooks and monographs with specific emphasis on the problem of the radical see references[43-52]

See, however also monographs[32-42]. For initial contributions on the problem see, for instnace, papers[87-97]. For more recent contributions, see, for instance, paper[98-109].

In a way similar to the corresponding definitions for algebras (Sections 1.2, 1.3 and 1.4), we shall say that a *ring* R (Section 1.2) is a *zero ring* when ab = 0 for all elements a,b $\in$ R. Several other definitions for algebras, particularly those for left or right *ideals*, apply to rings too. Similarly, a ring R is called a *simple ring* when the only proper ideal of R is the zero ideal and $R^2 \neq 0$. An element a $\in$ R is *nilpotent* if $a^n = 0$ for some integer n. An ideal I of R is a *nil ideal* if all its elements are nilpotent and is a *nilpotent ideal* if $I^n = 0$ for some integer n.

Consider an associative ring R, and let S be a non-empty subset of R. The right ideal of R

$$\mathcal{R}(S) = \{a \mid a \in R,\ ba = 0 \text{ for all } a \in S\} \qquad (1.D.1)$$

is called the *right annihilator* of S. If $\mathcal{R}$(S) consists of only one element a, the ideal generated by a is called the *principal ideal*. A *principle ideal ring* R is a ring (commutative under addition) with identity, such that every ideal is a principal ideal (e.g., the integers). A *maximal right ideal* of a ring R is a right ideal M $\neq$ R such that for any right ideal I of R either I = M or I = R. A ring R is called *right Artinian* when there is an ideal I in the set $\mathfrak{F}$ of all right ideals of R such that if I' $\in \mathfrak{F}$ and I' $\subset$ I, then I' = I. When the term "right" (or "left") is omitted, the above definitions imply both right and left properties.

The *nilpotent radical* (also called the *classical radical* or the *Wedderburn-Artin radical*) of a ring R is the unique maximal

nilpotent ideal of R. This definition originates from a number of properties, among which the following theorem (where R is associative).

THEOREM 1.D.1: *Every proper ideal of a ring R with identity is contained in a maximal ideal of R.*

THEOREM 1.D.2: *The sum of a finite number of nilpotent right (left) ideals of a ring R is a nilpotent right (left) ideal of R.*

A ring R is called *semisimple* when it is at least right (or left) Artinian and the only nilpotent ideal is the zero ideal. Notice that this definition does not coincide with the conventional definition of semisimplicity of a ring (where the condition on the ring of being Artinian is often omitted). Under the above definition, the following structure theorem holds.

THEOREM 1.D.3: *Every semisimple ring R is the direct sum of a finite number of simple rings.*

Wedderburn Structure Theorem 1.3.1 can be more technically formulated within such a context.

Let R be a ring. A (nonempty) set M is called a *left R-module* if M is an Abelian group under addition; for every element a $\in$ R and b $\in$ M, ab $\in$ M, and

$$a_1(b_1+b_2) = a_1b_1 + a_1b_2, \quad (a_1+a_2)b_1 = a_1b_1 + a_2b_1, \qquad (1.D.2a)$$

$$(a_1a_2)b_1 = a_1(a_2b_1), \qquad (1.D.2b)$$

for all $a_1, a_2 \in R$ and $b_1, b_2 \in M$. A module M of a ring is irreducible when $RM \neq 0$ and M has no proper submodule.

The Jacobson radical of a ring R can be defined as the quantity[44]

$$J(R) = \cap \mathcal{A}(M) \qquad (1.D.3)$$

where M is an irreducible R-module and $\mathcal{A}(M)$ is the (right and left) annihilator of M. Thus, the Jacobson radical is not the nilpotent radical. Instead, it is a broader radical which, as such, is applicable to a broader class of rings. For instance, the Jacobson radical of a quadratic Jordan ring[110] J (i.e., the Jordan ring of qudratic operators) can be interpreted as the maximal ideal of quasi-invertible elements (an element $a \in R$ is quasi-invertible if 1-a is invertible). Thus, if R does not contain the unit element, the Jacobson radical is clearly preferable over the nilpotent radical.

To better identify the difference between the Jacobson and the nilpotent radical, let us indicate that any definition of radical must satisfy a number of requirements, among which, most importantly,

(1) the radical must exist for all rings of the class admitted;

(2) if W is the radical of a ring R, the radical of R/W must be the zero ideal, and

(3) the radical of an ideal of a ring R must be the intersection of the ideal and the radical of R.

The nilpotent radical satisfies properties (2) and (3) but not necessarily property (1). The Jacobson radical, instead,

satisfies all properties (1), (2) and (3).

It should be here indicated that a rather large number of radicals have been identified in the specialized literature (see papers[87-110]), such as the Levitzki, McCoy, Bauer, Brown, Amitsur, Nagata radicals, etc. For instance, the Levitski radical[95] of a ring R is the sum of all semi-nilpotent ideals of R (an ideal is called semi-nilpotent if every ring generated by a finite set of its elements is nilpotent). The McCoy radical[96] I of a ring R is the set of all elements $a \in R$ such that every M-system contains an element of I (an M-system of a ring R is the subset of elements of R such that for all pair $c, d \in M$, there exist an element $a \in R$ such that $cad \in M$). Under certain conditions (called minimality conditions) the Levitski and the Jacobson radicals coincide. However, the McCoy and the Jacobson radicals are generally different. The selection of one radical versus another depends on several elements, including the aspect of the structure theory under study.

The reader should be aware of the care needed in the extension of the definitions of radical from rings to algebras. The following approach is effective in this respect. (See, for instance, reference 44).

Let U be an algebra over a ring K (with identity) which is commutative under addition. Let M be an Abelian ground under addition. M is an algebra-U-module if and only if it is a K-module (as defined above for rings) such that

$$\alpha(a m) = (\alpha a) m = a(\alpha m) \qquad (1.D.4)$$

for all $\alpha \in K$, $a \in U$ and $m \in M$.

THEOREM 1.D.4: If an algebra U is a K-algebra over a commutative ring K with identity, the nilpotent and Jacobson radicals of U as algebra and as ring coincide.

In conclusion, the theory of modules (originally developed outside the area of Abstract Algebras) has resulted to be valuable for the study of the transition from rings to algebras. In the absence of this theory (or of some equivalent approach), caution must be exercised for the extension of the definitions and results from rings to algebras. This is precisely the case of the general Lie-admissible algebras, as we shall comment below.

Let us recall from Section 1.3 that the solvable radical of a finite-dimensional (nonassociative) algebra U is the unique maximal solvable ideal of U, while the nilpotent radical of U is the unique maximal nilpotent ideal of U. From the definitions of nilpotency and solvability (Section 1.3) we see that the latter imply the former. Despite its broader character, the nilpotent radical is often insufficient to provide a structure theorem, and additional properties must be included.

As an indication, let us recall that a symmetric bilinear form $\langle a,b\rangle$, is called an associative form of an algebra U when

$$\langle a, bc\rangle = \langle ab, c\rangle \tag{1.D.5}$$

holds for all $a,b,c \in U$. The following theorem then holds.[44]

THEOREM 1.D.5: Suppose that a finite-dimensional (nonassociative) algebra over a field of characteristic zero satisfies the following properties:

(a) U possesses a nondegenerate associative form $\langle a,b\rangle$;

(b) U contains no nonzero nilpotent ideal.

Then U is uniquely expressible as a direct sum of a finite number of ideals U_i

$$U = \bigoplus_{i=1}^{n} U_i \tag{1.D.6}$$

each of which is a simple ideal.

In conclusion, the concept of nilpotent radical produces a structure theorem, according to the above information, when complemented with the concept of nondegenerate associative form.

Finite-dimensional associative algebras U over F do possess a nondegenerate associative form and, thus, the nilpotent radical is sufficient for the Structure Theorem 1.3.1.

For the case of commutative Jordan algebras J over F we have

$$\langle a, b\rangle = tr\,(R_{ab}) \tag{1.D.7}$$

where R_{ab} is the right multiplication of ab (Appendix 1.A). In this case it is possible to prove that the associative forms of semisimple algebras J is always nondegenerate.[28,58,60] Also, for commutative algebras the notions of solvability and that of nilpotency coincide. Thus, the solvable radical is sufficient for the Structure Theorem 1.3.3.

For the case of Lie algebras L over F the associative form is the Killing form (Appendix 1.B), i.e.,

$$\langle a,b\rangle = tr\,(ad\,a)(ad\,b) = tr\,(R_a)(R_b). \qquad (1.D.8)$$

In this case it is possible to prove, by using Cartan's criterion (see Appendix 1.E), that the Killing form of a semisimple algebra L is also always nondegenerate.[69] Therefore, the concept of solvable ideal is again sufficient for Structure Theorem 1.3.2.

For the case of noncommutative Jordan algebras $\tilde{J}$ the preferred notion of radical is that of nilpotent radical owing to the property that the notion of nilpotency implies that of solvability, as recalled earlier. Technical argu ments then lead to the sufficiency of the nilpotent radical for Struccure Theorem 1.3.4.[82]

For the case of the mutation algebras $A(\lambda, \mu)$ the definition of radical assumed in Section 1.4 is that of the solvable radical (i.e., the same as that for the $A(\lambda)$ algebras). We leave it as an exercise for the interested reader to prove that the symmetric form $\langle a,b\rangle$ is always nondegenerate for semisimple algebras $A(\lambda, \mu)$. Thus, the solvable radical is sufficient for Structure Theorem 1.4.11. Notice that this notion of radical allows the identification of both Lie as well as commutative Jordan algebras in the classification of the $A(\lambda, \mu)$ algebras.

In the case of a flexible Lie-admissible algebra U a more suitable notion of radical is that of nilpotent radical, in line with that of noncommutative Jordan algebras (owing to the relationship between these two algebras indicated in Section 1.4). A structure theory is then offered by Theorem 1.D.5. It is expected that a more restrictive structure theorem can be formulated for these algebras, in line with the Structure Theorem 1.3.4 for

noncommutative Jordan algebras. For more details see Appendix 1.E.

This completes our identification of the applicability of known insights on the radical to the Lie-admissible algebras.

The radical of a general (nonflexible) Lie-admissible algebra U will not be defined in this monograph. This is due to several technical reasons whose inspection would bring us outside the scope of this work (e.g., because of the possible inclusion of the category theory[50] and the sheaf theory[52]). In essence, the general Lie-admissible algebras are rather broad algebras indeed. In particular, they are not necessarily power-associative, and they do not appear to necessarily admit a nondegenerate associative form (for the semisimple case). Clearly, the Jacobson radical is one of the best candidates for these algebras. However, the problem whether this radical (or some other radical) is effective for a structure theory demands a specific study which is here left to the interested reader.

APPENDIX 1.E. CARTAN APPROACH TO LIE-ADMISSIBLE ALGEBRAS

As recalled in Section 1.3 and Appendix 1.D, the Cartan approach provides a complete classification of all simple finite-dimensional Lie algebras L over a field F of characteristic zero. For a methodological treatment of this approach within the context of the theory of Abstract Algebras see reference[69].

Particularly significant for our analysis is the fact that N. Jacobson[111] succeeded in 1966 in extending most of this methodological context to commutative Jordan algebras J. For an additional presentation by Jacobson see the monograph.[60] More recently, there have been attempts in extending the Cartan approach to arbitrary (nonassociative) algebras (e.g., D.M.Foster[112]). Finally, the studies by H.C.Myung[10-13] are particularly significant for our analysis.

These studies open the possibility of extending Cartan approach to Lie-admissible algebras U, owning to the general decomposition of their product

$$ab = [a,b]^*_U + \{a,b\}^*_U , \qquad (1.E.1a)$$

$$[a,b]^*_U = a*b - b*a , \qquad (1.E.1b)$$

$$\{a,b\}^*_U = a*b + b*a , \quad a*b = \tfrac{1}{2}ab , \qquad (1.E.1c)$$

where $[a,b]*_U$ characterizes the Lie algebra content U^- of U and $\{a,b\}*_U$ characterizes the commutative Jordan algebra content U^+ of U.

In this appendix we shall (1) reinterpret known results[69] on the Cartan approach for the Lie algebras L as the attached algebra U^- of a Lie-admissible algebra U, (2) reinterpret the results by N. Jacobson on the Cartan approach to commutative Jordan algebras J as the attached algebra U^+ of a Jordan-admissible algebra U, and (3) reinterpret Foster's results for flexible Lie-admissible algebras and outline the results by H.C. Myung. This should provide sufficient material for further investigations on the Cartan approach to Lie-admissible algebras by interested researchers.

In this appendix we shall only consider finite-dimensional algebras with unit over a field of characteristic zero.

For a given Lie algebra content U^- of a Lie-admissible algebra U one can construct

(a) the <u>derived series</u> $U^{-(n)}$

$$U^- \supseteq U^{-(1)} = U^-U^- \supseteq U^{-(2)} = U^{-(1)}U^{-(1)} \supseteq \dots \supseteq U^{-(m)} = U^{-(m-1)}U^{-(m-1)} \qquad (1.E.2)$$

(b) the <u>central series</u> U^{-n}

$$U^- \quad U^{-1} \supseteq U^{-2} = U^{-1}U^- \supseteq U^{-3} = U^{-2}U^- \supseteq \dots \supseteq U^{-m} \qquad (1.E.3)$$

where AB stands for the product $[a,b]_U$ of all elements $a \in A$ and $b \in B$, with $A,B = U^{-(i)}$ or $= U^{-i}$, $i = 1,2,\dots,n$.

The Lie algebra content U^- of U is called <u>solvable</u> (<u>nilpotent</u>) if $U^{-(n)} = 0$ ($U^{-n} = 0$) for some positive integer n. In particular, the sum of two solvable (nilpotent) ideads of U^- is solvable (nilpotent). Notice that a nilpotent algebra U^- is also solvable, but the converse is not necessarily true (e.g., when U^- is a two-dimensional non-Abelian Lie algebra). Thus, two radicals of U^- are significant. The <u>solvable radical</u> (or <u>radical</u> for short) of a Lie algebra U^- is the (unique) maximal solvable ideal of U^-. The <u>nil radical</u> of U^- is the (unique) maximal nilpotent ideal of U^-. It is possible to prove that the nil radical of a Lie algebra U^- is contained in the (solvable) radical of U^-. Thus the classification of U^- must be conducted in terms of the (solvable) radical.

Of particular importance for such classification is <u>Engel's theorem</u>.

<u>THEOREM</u> 1.E.1:[69] <u>If the attached algebra U^- of a Lie-admissible algebra U is a (finite-dimensional) Lie algebra, then U^- is nilpotent if and only if ad(a) (= R_a) is nilpotent for every a $\in U^-$.</u>

If the attached algebra U^- admits the interpretation of a Lie algebra of linear transformations on a vector space V, the <u>Fitting decomposition</u> of V relative to an element T of U^- is given by

$$V = V_0 \oplus V_1 \quad , \qquad (1.E.4a)$$

$$V_0 = \{a \mid a \in U^-, \; aT^m = 0\} , \qquad (1.E.4b)$$

$$V_1 = \bigcap_{i=1}^{\infty} VT^i , \qquad (1.E.4c)$$

where V_0 (V_1) is called the <u>Fitting null (one) one commponent</u> of V relative to T. <u>Fittings' lemma</u> states that V_0 and V_1 are invariant with respect to any element T of U^-. Fitting decomposition (1.E.4) holds for several algebras other than a Lie algebra

as well as when V = U (as vector space).

Let S^- be a subalgebra of a Lie algebra U^- attached to a Lie-admissible algebra U. The <u>normalizer</u> N^- of S^- is the set of all a $\in U^-$ such that $[a,b]_U \in S^-$ for all b $\in S^-$. Then N^- is a subalgebra of U^- containing S^- and S^- is an ideal of N^-.

A subalgebra S^- of a (finite-dimensional) Lie algebra U^- of a Lie-admissible algebra U (over a field F of characteristic zero) is called a <u>Cartan subalgebra</u>[69] if S^- is nilpotent and S^- is its own normalizer.

<u>THEOREM</u> 1.E.2:[69] <u>Let S^- be a nilpotent subalgebra of a (finite-dimensional) Lie algebra U^- attached to a Lie-admissible algebra U over a field F (of characteristic zero). Then S^- is a Cartan subalgebra of U^- if and only if it coincides with the Fitting component U_0^- of U^- relative to ad(S^-).</u>

An element a $\in U^-$ is called <u>regular</u> if the dimension m of the Fitting null component of U^- relative to ad(a) (= R_a) is minimal. If n = dim(U^-), n-m is called the <u>rank</u> of a $\in$ U.

<u>THEOREM</u> 1.E.3:[69] <u>If the attached algebra U^- of a Lie-admissible algebra U (over an infinite field F of characteristic zero) is a (finite-dimensional) Lie algebra and a is a regular element of U^-, then the Fitting null component U_0^- of U^- relative to ad(a) is a Cartan subalgebra of U^-.</u>

These properties lead to the known Cartan's criterion for semisimplicity: if the Killing form (1.D.7) of a (finite-dimensional) Lie algebra U^- of a Lie-admissible algebra U (over a field F of characteristic zero) is non-degenerate, then U^- is semisimple. Under suitable technical implementations this methodological context leads to Structure Theorem 1.3.2 (easily extendable to the U^- algebra of a Lie-admissible algebra U) and the Cartan classification of simple Lie algebras (Section 1.3). For brevity, we here refer the reader to monograph.[69]

We now turn our attention to the commutative Jordan algebra content U^+ of a Jordan-admissible algebra U. The notions of derived series (1.E.2) and central series (1.E.3) can be extended to U^+. The same occurs for the notions of solvability and nilpotency. Again, the radical of U^+ is the maximal solvable ideal of U^+. Under the assumption that U^+ is finite-dimensional, the following three conditions are then equivalent:

(a) U^+ is solvable;

(b) U^+ is nilpotent;

(c) U^+ is a nil algebra.

A generalization of the associator, Eq. (1.2.18), which is here relevant is the so-called associator of order n, as defined by the iterative rules

$$\begin{aligned} &A_n[a_1, a_2, \ldots, a_n] \\ &= [A_{n_1}[a_1, \ldots, a_{n_1}], A_{n_2}[a_{n_1+1}, \ldots, a_{n_1+n_2}], \\ &\quad A_{n_3}[a_{n_1+n_2+1}, \ldots, a_{n_1+n_2+n_3}]] \\ &n_1 + n_2 + n_3 = n \end{aligned} \qquad (1.E.5)$$

where A_{n_i} is the associator of order n_i. Any algebra U is called associator nilpotent if there exists a minimum positive odd integer n, called the index of associator nilpotency, such that every associator of order n formed with elements of U is null. Subalgebras and direct sum of associator nilpotent algebras are associator nilpotent.

The above definition of associator nilpotency has been used by N. Jacobson[111] in his extension of the Cartan's approach to the commutative Jordan algebras. An algebra of this type is called almost nil commutative Jordan algebra if U^+ (has an identity) and admits the decomposition $U^+ = F1 \oplus R^+$, where R^+ is a nil ideal. The first result by Jacobson can be reinterpreted for the U^+ algebras of Jordan-admissible algebras as follows.

THEOREM 1.E.5: A (finite-dimensional) commutative Jordan algebra U^+ attached to a Jordan-admissible algebra U over an (algebraically closed) field F is associator nilpotent if and only if it is a direct sum of ideals which are almost nil algebras.

Let us note, from Section 1.A, that

$$c\,(R_a R_b - R_{ab}) = [c, a, b] = c\,R_{a,b} \qquad (1.E.6)$$

Thus, associator nilpotency of index n implies that

$$R_{a_1,b_1} \cdots R_{a_n,b_n} = 0 \qquad (1.E.7)$$

for all $a_i, b_i \in U^+$. If

$$(R_{a_i^r, a^s})^m = 0, \quad r, s = 0, 1, 2, \ldots \qquad (1.E.8)$$

then $a \in U^+$ is called an associator nilpotent element of U^+.

The analogue of Engel's theorem reads

THEOREM 1.E.4: A (finite-dimensional) commutative Jordan algebra U^+ attached to a Jordan-admissible algebra U (over an infinite field F) is associator nilpotent if and only if every element $a \in U^+$ is associator nilpotent relative to U^+.

Suppose that S^+ is an associator nilpotent subalgebra of U^+. The Fitting decomposition of U^+ can be introduced as follows. The set $L(U^+)$ of linear transformations in U^+

$$\hat{R}_{a,b} = R_a R_b - R_{\{a,b\}} \qquad (1.E.9)$$

forms a Lie algebra. If a and b belong to an associator nilpotent subalgebra S^+ of U^+, then $L(U^+)$ is a nilpotent Lie algebra. This allows the Fitting decomposition of U^+

$$U^+ = U^+_0 + U^+_1, \qquad (1.E.10a)$$

$$U^+_b = \{a \mid a \in U^+,\ a\, U^{+N}_b = 0,\ U^{+N}_b = (\hat{R}_{b_i b_i})^N,\ N = \dim(U^+)\}, \qquad (1.E.10b)$$

$$U^+_0 = \cap_{b \in S^+} U^+_b, \qquad (1.E.10c)$$

which is now computed with respect to $L(U^+)$.

A Cartan subalgebra[111] S^+ of a (finite-dimensional) commutative Jordan algebra U^+ attached to a Jordan-admissible algebra U (over a field of characteristic zero) is a subalgebra of the Fitting null component U^+_0 of U relative to the Lie algebra $L(U^+)$ of linear transformations (1.E.9).

In the locally quoted paper, Jacobson indicated a significant relationship between a Cartan subalgebra of a commutative Jordan algebra U^+ and its Pierce decomposition (Appendix 1.C). In particular he proved that a subalgebra S^+ of U^+ is a Cartan subalgebra if and only if

$$S^+ = \bigoplus_{i=1}^{m} U^+_{ii} \qquad (1.E.11)$$

where U^+_{ii} is given by Eqs. (1C.14a).

An element $a \in U^+$ is associator regular if Dim (U^+_a) is minimal. A property corresponding to that of Theorem 1.E.3 can be formulated as follows

THEOREM 1.E.6: If a is an associator regular element of a commutative Jordan algebra U^+ attached to a Jordan-admissible algebra U, then U^+_a is a Cartan subalgebra of U^+.

This completes our review of Cartan's approach to the U^+ algebras of Jordan-admissible algebras U. Notice that these algebras U can also be jointly Jordan- and Lie-admissible without affecting the applicability of the methodological tools. For further elaboration see references 73, 111.

We now reinterpret the results by D.M. Foster[112] for (finite-dimensional) flexible Lie-admissible algebras U (over a field F of characteristic zero) with elements a,b,c,... and abstract product ab admitting decomposition (1.E.1). The first notions which are here needed are those of solvability and nilpotency. Let us recall that the definitions of these notions for Lie algebras extend to commutative Jordan algebras with a simple implementation of the product. Nevertheless, a generalization of these notions resulted to be needed to introduce the Cartan approach for the latter algebras, i.e., the notion of associator nilpotency. In the transition to an algebra whose product is neither anticommutative nor commutative, additional technical difficulties arise. They are due, for instance, to the fact that, for Lie algebras, the notions of nilpotency and right (or left) nilpotency coincide while this is not the case for the broader algebras under consideration.

The definition of nilpotency for an arbitrary algebra U assumed by D.M. Foster can be specialized for a flexbile Lie-admissible algebra U as follows. Let $f(a_1,a_2,\ldots,a_n)$ be a fixed, multiplinear (i.e., linear in each argument) but otherwise arbitrary element of U. Let S be a subalgebra of U. Instead of the notion of central and derived series we introduce the following iterative series

$$f^k(S) = \{a \mid a \in U \text{ is a finite sum of elements of the form } f(a_1,a_2,\ldots,a_n),\ a_1 \in f^{k-1}(S),\ a_i \in S,\ 1 < i \leq n\} \quad (1.E.12a)$$

$$f^{(k)}(S) = \{a \mid a \in U \text{ is a finite sum of elements of the for } f(a_1,a_2,\ldots,a_n),\ a_i \in f^{(k-1)}(S),\ 1 < i \leq n\} \quad (1.E.12b)$$

The subalgebra S of U is called f-nilpotent (f-solvable) if $f^n(S) = 0$ ($f^{(n)}(S) = 0$) for some positive integer n. Notice that, as it was the case for the ordinary notions of solvability and nilpotency, f-nilpotent subalgebras are f-solvable.

In essence, the notion of f-nilpotency (f-solvability) is a generalization of the ordinary notion of right (or left) nilpotency (solvability), i.e., that defined in terms of central (derived) series. At the same time, the notion of f-nilpotency (f-solvability) is a generalization of that of associator nilpotency (solvability), i.e., that defined in terms of central (derived) series of associators. As such, it is suitable for algebras whose product is neither anticommutative nor commutative. The radical R of a (finite-dimensional) flexible Lie-admissible algebra U over a field F (of characteristic zero), can be thus effectively defined as the maximal f-solvable ideal of U. It is possible to prove that this ideal is unique and such that the radical of U/R is the null ideal.

A second notion which is here needed is that of a nilpotent element of U. The definition that $f^n(a,a,\ldots,a) = 0$ for some positive integer n must be excluded because of difficulties in recovering the corresponding definition for the Lie algebra subclass of U. In turn, this is linked to the need of a suitable definition of ad (a) of an element $a \in U$. In this latter respect, we define as the adjoint map of an element $a \in U$ the linear transformation in U

$$a \to T(a,a_1,\ldots,a_{n-1}) = f(a,a_1,\ldots,a_{n-1}) \quad (1.E.13)$$

for fixed elements $a_1,a_2,\ldots,a_{n-1}$ in U. An element $a \in U$ is called f-nilpotent if $T(a,a_1,\ldots,a_{n-1})$ is a nilpotent linear transformation of U. The algebra U is an f-nil algebra if every element is f-nilpotent. In the following we shall use the notation

$$T_{U,f}(S) = \{T(a_1, \ldots, a_{m-1}) \mid a_i \in S\}, \qquad (1.E.14a)$$

$$B_a = \{b \mid b \in U,\ b(T(a, a, \ldots, a))^n = 0\}. \qquad (1.E.14b)$$

THEOREM 1.E.7: Let U be a (finite-dimensional) flexible Lie-admissible algebra over a field F (of characteristic zero). If there exists an element $a \in U$ such that:

(1) $a \in B_a$, (2) B_a is a subalgebra of U and

3) $U(S(a,a,\ldots,a))^n \leq M$ for some positive integer n where M is a maximal subalgebra of U, then $B_a = U$.

Let us recall that some central properties of the flexible Lie-admissible algebras are that (a) they contain in their classification Lie algebras and (b) they can reduce directly into Lie algebras at the limit when the product becomes anticommutative. An indication of the recovering of the conventional Lie algebra notions from Theorem 1.E.7 is then instructive. Suppose that the algebra U of Theorem L.E.7 is a Lie algebra L. Assume then f = ab (= -ba). Rule (1.E.13) then recovers the conventional definition of ad (a) as R_a, property (3) of Theorem 1.E.7 recovers the conventional property that $L(ad(a))^n \in M$ where M is a maximal subalgebra of L, and the theorem results to be a generalization of the property of Lie algebras according to which if every maximal subalgebra of a Lie algebra L is an ideal, L is nilpotent.[56]

Cartan's approach to Lie algebra is essentially based on the

following properties:

(L-1) A Lie algebra L is nil if and only if it is nilpotent.

(L-2) For all $a \in L$, $B_a = \{b \in L \mid b(ad(a))^n = 0\}$, for some positive integer $n > 0$, is a subalgebra of L containing a,

(L-3) If S is a nilpotent subalgebra of L, ad(S) is nilpotent and the Fitting null component of L relative to ad(S) is a subalgebra of L containing S.

A result by D.M. Foster which is particularly significant for our analysis is that properties (L-1), (L-2) and (L-3) can be generalized to a rather large class of algebras. An inspection indicates that they can be generalized to the flexible Lie-admissible algebras U as well. A function f on U is called an Engel function for U when

(U-1) U is f -nil if and only if it is f-nilpotent,

(U-2) For all $a \in U$, $B_{u,f}$ is a subalgebra of U containing a, and

(U-3) if S is an f-nilpotent subalgebra of U, $L_{U,f}(S)$ is a subalgebra of U containing S.

We are now equipped to introduce a crucial notion. A subalgebra S of a (finite-dimensional) flexible Lie-admissible algebra U over a field F (of characteristic zero) is an f-Cartan subalgebra[99] of U if S is f-nilpotent with respect to an Engel function f and coincides with the Fitting null component of U relative to $L_U(S)$. The quantity B_a, $a \in U$, is called an f-Engel subalgebra of U. B_a is also called minimal f-Engel in U if $B_b = B_a$ for all $b \in U$.

An analog of the Existence Theorem 1.E.3 can be formulated as follows (from Theorem 5.2 of reference 112)

THEOREM 1.E.8: If, for an element a of a (finite-dimensional) flexible Lie-admissible algebra U (over a field F of characteristic zero), B_a is a minimal f-Engel function, then B_a is an f-Cartan subalgebra of U.

For an extension of the concept of regular (or associator regular) element as well as for additional results also extendable to flexible Lie-admissible algebras see reference 99.

Notice that in the last part of this appendix we have not assumed that the algebra U is Jordan-admissible. This assumption is suggested by the fact indicated in Section 1.5 that classical realization of Lie-admissible algebras U are not, in general, Jordan admissible. If, however, the algebras U are, in addition, Jordan-admissible, the indicated properties apply and can be complemented with additional properties originating from the second part of this appendix, i.e., that for U^+ .

If the algebra U is a mutation algebra $A(\lambda, \mu)$ of an associative algebra A (Section 1.4), then the notions of f-solvable radical and solvable radical (as assumed in Section 1.4) are equivalent. This is a consequence of Theorem 1.4.10 (the explicit proof is left as an exercise for the interested reader). As a result, the notion of radical of a flexible Lie-admissible algebra U as the uniqeu maximal f-solvable ideal of U is a generalization of the notion of solvable radical of the mutation algebras $A(\lambda, \mu)$.

Such generalization is needed because the product of a flexible Lie-admissible algebra need not necessarily be that of $A(\lambda, \mu)$ (e.g., it can be that of a genotopic mapping of a nonassociative algebra).

We now outline some of the studies by H.C. Myung[11-13] on flexible Lie-admissible algebras. Let us recall that a mapping $a \rightarrow \alpha(a)$ of a Lie algebra U^- into the base field F is called a weight[69] if there exists a nonzero element x_α such that

$$x_\alpha (a^R - \alpha(a))^k = 0 \qquad (1.E.15)$$

for some suitable k. The weights associated with $ad_{U^-}H^-$, where H^- is a nilpotent subalgebra of U^-, are called roots[69] of H^- in U^-. Then U^- admits the vector space decomposition

$$U^- = U^-_\alpha \oplus U^-_\beta \oplus \cdots \oplus U^-_\delta \qquad (1.E.16)$$

where U^-_α is the root space corresponding to the root α and consists of the set of elements x_α in U^- such that

$$x_\alpha (D_h - \alpha(h))^{k(h)} = 0 \qquad (1.E.17)$$

for all h in the (split) Cartan subalgebra H^- of U^-, where D_h is a derivation in U^-. We also have that

$$U^-_\alpha U^-_\beta \subseteq 0 \quad \text{if } \alpha+\beta \text{ is not a root} \qquad (1.E.18a)$$

$$U^-_\alpha U^-_\beta \subseteq U^-_{\alpha+\beta} \quad \text{if } \alpha+\beta \text{ is a root} \qquad (1.E.18b)$$

and H^- is the root space corresponding to the root 0.

THEOREM 1.E.9:[11] Let U be a finite-dimensional, flexible, Lie-admissible algebra over a field F (of characteristic $\neq 2$). Suppose that U^- has a (split) Abelian Cartan subalgebra H^- which is nil in U. Then, if $\dim U_\alpha = 1$ for a root $\alpha \neq 0$ and U^- has the center 0, $U^- \approx U$.

Let us recall that a Lie algebra U^- is called classical when (a) the center of U^- is 0; (b) $U^- = [U^-,U^-]_U$; (c) U^- has an Abelian Cartan subalgebra H^-, called classical Cartan subalgebra, such that

(1) $U^- = \bigoplus_\alpha U^-_\alpha$;

(2) if $\alpha \neq 0$ is a root, $\dim [U^-_\alpha, U^-_{-\alpha}] = 1$

(3) if α and β, $\beta \neq 0$, are roots, not all $\alpha + \beta$ are roots.

Then Theorem 1.E.9 implies the following

COROLLARY 1.E.9.A:[11] If U^- is a classical Lie algebra having a classical Cartan subalgebra which is nil in U, then $U^- \approx U$.

Recall that the Levi factor T^- of a Lie algebra U^- is a semisimple subalgebra of U^- such that $U^- = T^- \oplus R^-$, where R^- is the (solvable) radical of U^-. Even though T^- may be an ideal of U^-, it is not, in general, an ideal of U even when U is a nil algebra. The following theorem studies certain conditions under which T^- is an ideal of U.

THEOREM 1.E.10:[11] Consider a finite-dimensional, flexible, Lie-admissible algebra U over a field F (which is algebraically closed and of characteristic zero) such that the radical R^- of U^- is nilpotent in U. Then a Levi factor T^- of U^- is an ideal of U if and only if T^- has a Cartan subalgebra H^- which is a nil subalgebra of U and $[R^-,H^-]_U = 0$. Furthermore, if U is simple, then either U^- is nilpotent or U is a Lie algebra.

A finite-dimensional algebra U^- is called a reductive Lie algebra when it is a completely reducible Lie algebra. This is the case when the quotient algebra U^-/C^-, where C^- is the center of U^-, is semisimple.

THEOREM 1.E.11:[11] Consider a finite-dimensional, flexible, Lie-admissible algebras U over a field F (of characteristic zero), such that U^- is a reductive Lie algebra. If $[U,U]_U$ has a (split) Cartan subalgebra that is power-associative, nil subalgebra of U, then $[U,U]_U$ is an ideal of U. Furthermore, if U is simple, then U is either commutative or a Lie algebra.

The following property is also significant.

THEOREM 1.E.12:[11] Consider a finite-dimensional, flexible, Lie-admissible algebra U over a field F (of characteristic zero) such that U^- is a reductive Lie algebra. Then, if there exists a (split) Cartan subalgebra H^- of U^- with $h^3 = 0$ for all $h \in H^-$, the center C^- of U^- is an ideal of U. Furthermore, if U is simple, then it is a Lie algebra.

The above results are relevant for the problem of a generalization of Lie algebras into nontrivial Lie-admissible algebras. They in essence indicate that any simple, flexible, Lie-admissible nil algebra of dimension $n \leq 3$ is in actuality a Lie algebra. As a result, the nilpotency is a restriction on the intended generalization under the indicated conditions. Their extension to arbitrary dimension n appears to be unknown at this time.

Along the same lines we have the following properties

THEOREM 1.E.13:[12] Consider a finite-dimensional, power-associative, flexible, Lie-admissible algebra over a field F (of characteristic zero). Then U is a nilalgebra if and only if there exists a Cartan subalgebra of U^- which is nil in U.

THEOREM 1.E.14:[13] Consider a finite-dimensional, flexible, Lie-admissible algebra U over a field F (of characteristic $\neq 2$). Suppose that S^- is a subalgebra of U^- and H^- a Cartan subalgebra of S^-. Then S^- is a subalgebra of U if and only if $H^-H^- \subseteq S^-$.

The following examples worked out by H.C. Myung[11] are useful to illustrate the above properties.

Example 1: Let U be of dimension 3, elements x,y,h and multiplication table

$$\begin{cases} xh = x, \quad yh = \frac{1}{2}(\alpha+1)y, \quad h^2 = h \\ hy = \frac{1}{2}(1-\alpha)y, \quad \alpha \in F, \quad \alpha \neq 0,1 \end{cases} \qquad (1.E.19)$$

with all other products being null. Then U is a flexible Lie-admissible algebra (the explicit proof is an instructive exercise). The attached algebra U^- is a solvable Lie algebra. $H^- = Fh$ is a Cartan subalgebra which induces the Cartan decomposition $U = Fh \oplus Fx = Fy$ for roots 0, 1 and α. The center of U^- is 0 but H^- is not nil in U.

Example 2: Let U be of dimension 4, elements x,y,z,h and multiplication table

$$\begin{cases} xy = z + \frac{1}{2}h, \quad yx = z - \frac{1}{2}h, \quad h^2 = -z \\ xh = -hx = \frac{1}{2}x, \quad yh = -hy = -\frac{1}{2}y \end{cases} \qquad (1.E.20)$$

with all other products being null. Then, U is a flexible Lie-admissible algebra. An Abelian Cartan subalgebra of U^- is $H^- = Fh \oplus Fz$. Then the Cartan decomposition of U^- with respect to the roots 0, 1 and -1 is $U^- = H^- \oplus Fx \oplus Fy$. In this case H^- is a nil subalgebra of U; $u^3 = 0$ for all elements of H^-; the center C^- of U^- is Fz; and U is not a Lie algebra nor an associative algebra.

Example 3: Let T^- be of dimension 3, elements x,y,h and multiplication table

$$xh = x\,, \quad yh = -y\,, \quad xy = h \qquad (1.E.21)$$

and let $U_\alpha = T^- \oplus Fz$ be the alebra defined by

$$zs, sz = \alpha s, \quad s \in T^-, z^2 = z, \alpha \in F \qquad (1.E.22)$$

The U_α is a flexible Lie-admissible algebra;
$[U_\alpha, U_\alpha]_U = T^-$ admits the Cartan subalgebra $H^- = Fh$
with $h^2 = 0$; $u^3u = u^2u^2$ for all $u \in U_\alpha$ and the algebra
is power-associative (we tacitly assume characteristic zero)
if and only if

$$2\alpha^3 - 3\alpha^2 + 2 = 0 \qquad (1.E.23)$$

namely, if and only if the roots are 0, ½ and 1.

Notice that is Example 3 the flexible Lie-admissible algebra is constructed from a Lie algebra. This construction is studied in details by A.A. Sagle[14] for the case when the given Lie algebra is reductive. For brevity we here refer the interested reader to the quoted paper and related references.

By keeping into account the content of Section 1.4, we can conclude by saying that Lie-admissible algebras offer a genuine possibility of constructing a nontrivial covering of Lie algebras along lines which will be elaborated in Chapter 3. The case of flexible Lie-admissible algebras appear to be effectively treatable with established methods, such as the use of the Lie algebra content and a generalization of the Cartan decomposition. The extensions of the results for the case of general Lie-admissible algebra

appears to be unknown at this time, owing to the need of a prior effective identification of notions such as that of the radical. This study is here left to the interested researcher.

CHAPTER 2

BIRKHOFF–ADMISSIBLE COVERING OF HAMILTON'S AND OF BIRKHOFF'S EQUATIONS

2.1: STATEMENT OF THE PROBLEM.

The body of methodological tools currently known under the name Analytic Mechanics was originally conceived for the study of unconstrained, conservative, Newtonian systems of N particles in a three-dimensional Euclidean space with Carthesian coordinates $r^k = r^{ka}$ $\{r_{ka}\}$ $k = 1,2,\ldots,N$, $a = x,y,z$,

$$m_k \ddot{r}_{ka} - f_{ka}(r) = 0, \quad f_{ka} = -\frac{\partial V}{\partial r^{ka}} \qquad (2.1.1)$$
$$k = 1,2,\ldots,N, \quad a = x,y,z$$

The part of the mechanics which is essential for joint analytic, algebraic, and geometric studies is that based on the transformation of the systems to the equivalent, first-order form

$$\begin{cases} \dot{r}_{ka} = p_{ka}/m_k \\ \dot{p}_{ka} = f_{ka}(r) \end{cases} \qquad (2.1.2)$$

and their representation via the conventional Hamilton's equations[115,122] of the contemporary literature, those without external terms

$$\begin{cases} \dot{r}_{ka} = \dfrac{\partial H}{\partial p_{ka}}, \\ \dot{p}_{ka} = -\dfrac{\partial H}{\partial r^{ka}}, \end{cases} \qquad H = T(p) + V(r) \qquad (2.1.3)$$

which we write in the unified notation of Eqs. (1.5.2) and (1.5.19)

$$\dot{a}^\mu = \omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}, \quad \mu = 1,2,\ldots,6N \qquad (2.1.4a)$$

$$\begin{pmatrix} \dot{r}_{ka} \\ \dot{p}_{ka} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial r^{ka} \\ \partial H/\partial p_{ka} \end{pmatrix} = \begin{pmatrix} \partial H/\partial p_{ka} \\ -\partial H/\partial r^{ka} \end{pmatrix} \qquad (2.1.4b)$$

$$\omega^{\mu\nu} = \left(\left\| \frac{\partial R^\circ_\alpha}{\partial a^\beta} - \frac{\partial R^\circ_\beta}{\partial a^\alpha} \right\|^{-1} \right)^{\mu\nu} \qquad (2.1.4c)$$

$$(a^\mu) = \begin{pmatrix} r \\ p \end{pmatrix}, \quad (R^\circ_\mu) = (p, 0) \qquad (2.1.4d)$$

It is well known that conservative systems do not exhaust, by far, the systems of our Newtonian reality. In fact, to avoid the validity of the perpetual motion in our environment, one has to acknowledge the existence of a virtually endless variety of forces, such as

(A) Newtonian and non-Newtonian forces (i.e., forces independent and dependent on the acceleration, respectively);

(B) Local and non-local forces (i.e., forces which occur at a collection of isolated points, or at all points of a surface or a volume, respectively, in the latter case the forces demand an integro-differential representation); and

(C) Self-adjoint and non-self-adjoint forces (i.e., forces which verify or do not verify, respectively, the integrability conditions for the existence of a potential function).

For these aspects we refer the reader to monographs[65, 189] and quoted references (see, in particular, the Introduction of monograph[189]). A very brief outline of the theorems of variational self-adjointness is presented in Chapter 2 of the preceding volume of this series.

In this volume we shall study the most general possible, local, Newtonian, and non-self-adjoint systems which we write in the form

$$m_k \ddot{r}_{ka} - f^{SA}_{ka}(t, r, \dot{r}) - f^{NSA}_{ka}(t, r, \dot{r}) = 0 \qquad (4.1.5)$$

where SA (NSA) denotes the verification (violation) of the integrability conditions for the existence of a potential.

The reader familiar with the quoted literature will recall that the symbols SA and NSA ultimately represent action-at-a-distance/potential and contact/nonpotential forces, respectively. By recalling that points can only interact at a distance, the admission of non-self-adjoint forces therefore constitutes a first representation of extended particles. The understanding is that a more adequate treatment is provided by non-local/integro-differencial equations of motion.

In different terms, the primary physical arena of the conventional Hamilton's equations is constituted by systems of massive points. The point-like character of the particles then implies, first, the locality of the theory, and, second, the existence of only potential forces.

The primary physical arena of study of this (and of the next) volume is constituted instead by systems of extended particles. The extended character of the constituents then implies the additional existence of contact forces for which the notion of potential energy has no physical basis. These forces are generally non-local. Nevertheless, they can be approximated via power series expansions in the velocities[189], by therefore reaching, as a first treatment, systems (2.1.5).

The most natural equations for the representation of systems (2.1.5) are given by the true Hamilton's equations, those originally conceived by Hamilton with external terms,

$$\begin{cases} \dot{r}_{ka} = \dfrac{\partial H}{\partial p_{ka}} \\ \dot{p}_{ka} = -\dfrac{\partial H}{\partial r^{ka}} + F_{ka}(t, r, \dot{r}) \end{cases} \quad (2.1.6)$$

which we write in the unified form

$$\dot{a}^{\mu} = \omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}} + F^{\mu}(t, a) \quad (2.1.7a)$$

$$F^{\mu} = \begin{pmatrix} 0 \\ F^{NSA}_{\mu} \end{pmatrix} \quad (2.1.7b)$$

In essence, it appears that (unlike his followers of this century) Hamilton was fully aware of the fact that the forces of our macroscopic environment are generally of non-potential type. He, therefore, conceived his equations in such a way that the Hamiltonian represents the total energy, while all nonpotential forces are represented by the external terms.

For a presentation of the historical legacy of Hamilton and of its relationship to numerous other legacies of Lagrange, Liouville, Jordan, Fermi, Einstein, and others, we refer the interested reader to Section 2.1 of memoir[190].

This volume is devoted to the study of local, Newtonian, and non-self-adjoint systems via the true Hamilton's equations.

In the actuation of this program, a first difficulty soon emerges. As shown in detail in the next section, in the transition from Eqs. (2.1.4) to the historical true form (2.1.7) we have the loss of the

Lie algebra in the brackets of the time evolution, as well as the lack of any algebraic structure.

The implications of this finding are rather deep. In fact, while Eqs. (2.1.4) can be exponentiated to (the canonical realization of) a continuous Lie transformation group, this exponentiation does not appear to be possible for Eqs. (2.1.7).

By recalling that the exponentiated form of the canonical equations (for conservative systems) constitutes the "time-compenent" of Galilei's relativity, we immediately see the inability to reach a covering relativity for systems (2.1.5) if Eqs. (2.1.7) are assumed in their historical form.

Thus, one of our objectives is to identify a suitable reformulation of the true Hamilton's equations which permits their exponentiation to a continuous transformation group. As we shall see, the use of the Lie-admissible algebras permits the achievement of this objective in a rather natural way.

For completeness, we indicate here that a generalization of Hamiltonian mechanics has been recently achieved and called Birkhoffian mechanics.[189] It is based on the following generalization of Eqs. (2.1.4) studied in detail by G. D. Birkhoff[124]

$$\dot{a}^\mu = \Omega^{\mu\nu}(a)\,\frac{\partial B(t,a)}{\partial a^\nu} \qquad (2.1.8a)$$

$$\Omega^{\mu\nu} = \left(\left\|\frac{\partial R_\alpha}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\alpha}\right\|^{-1}\right)^{\mu\nu}, \quad R \neq R^0 \qquad (2.1.8b)$$

where the function B(t,a) is called the Birkhoffian (owing to considerable differences with the Hamiltonian), and where the tensor $\Omega^{\mu\nu}(a)$ is that of Eqs. (1.5.20). The time evolution of Birkhoff's equations is therefore of Lie character, although of a form structurally more general than the conventional Hamiltonian form.

In fact, the time evolution of a given function A(a) in the a = $(\underset{\sim}{r},\underset{\sim}{p})$ variables is given, for the conventional Hamilton's equations, by

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu}\dot{a}^\mu = \frac{\partial A}{\partial a^\mu}\,\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} = [A, H] \qquad (2.1.9)$$

and for Birkhoff's equations by[125-128]

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu}\,\Omega^{\mu\nu}(a)\,\frac{\partial B}{\partial a^\nu} = [A, B]^* \qquad (2.1.10)$$

where [A,H] and [A,H]* are the conventional Poisson brackets (1.5.15) and its generalized form (1.5.21), respectively. The transition from Hamiltonian to Birkhoffian mechanics is therefore a form of Lie algebra isotopy.[189]

The time evolution of the same function A(a) for Eqs. (2.1.7) is instead given by

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu}\left(\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} + F^\mu\right) = [A, H] + \frac{\partial A}{\partial a^\mu}F^\mu \overset{\text{def}}{=} A \times H \qquad (2.1.11)$$

The loss of the Lie algebra character is then self-evident, e.g., from the lack of totally antisymmetric character of "A x H".

It is important to identify the different role of Eqs. (2.1.7) and (2.1.8) in our analysis beginning from these preliminary considerations.

Consider a local and closed system with non-Hamiltonian internal forces, and assume that its time evolution is characterized by brackets "A x H" where H is the total energy E_{tot}. A necessary condition for consistency is therefore that these brackets are totally antisymmetric. In fact, the total energy can be conserved,

$$\dot{H} = H \times H \equiv 0 \quad , \quad H = E_{tot} \qquad (2.1.12)$$

if and only if "A x H" is antisymmetric. The use of additional arguments regarding the integrability of the(local) time evolution to a (finite) exponentiated form then identifies the Jacobi law as an additional condition, thus reaching the Lie algebra laws.

The brackets themselves need not necessarily be the conventional Poisson brackets. In actuality, their selection would lead to contradictions because the systems are non-Hamiltonian by assumption. In fact, the Lie brackets result to be the most general possible ones, i.e., the Birkhoffian/generalized Poisson brackets

$$A \times H \equiv [A, H]^* \qquad (2.1.13)$$

One sees in this way that Birkhoff's equations are rather naturally set for the representation of closed, local, and non-Hamiltonian systems (even though their use for different systems is possible). This is the spirit of the presentation of Birkhoff's equations of monograph[189] which will be preserved in this volume.

We pass now to the study of open, local, and non-Hamiltonian systems, i.e., to systems which cannot be considered as isolated from their environment or from the rest of the universe. Suppose again that the time evolution is represented by brackets "A x H" where H is

now the energy of the (sub) system considered. It is easy to see that a necessary condition for consistency is now that the brackets "A x H" are not Lie. In fact, the time rate of variation of the energy

$$\dot{H} = H \times H = f(t) \neq 0 \quad , \quad H = E \qquad (2.1.14)$$

can now be non-null if and only if the brackets "A x H" are not totally antisymmetric.

It is easy to see that the true Hamilton's equations permit a consistent and direct treatment of the case. In fact, for Eqs. (2.1.7), we have

$$\dot{H} = [H, H] + \frac{\partial H}{\partial a^\mu} F^\mu = \frac{\partial H}{\partial a^\mu} F^\mu = \dot{r}^{k a} F^{NSA}_{k a} \qquad (2.1.15)$$

yielding the time rate of variation of the energy exactly as occurring in the physical reality.

As a result, the historical equations originally conceived by Hamilton's will be preferred to Birkhoff's equations for the treatment of open, local, and non-Hamiltonian systems throughout the analysis of this and of the subsequent volume of this series.

In actuality, we shall identify not only a reformulation of Eqs. (2.1.7) permitting exponentiation, but we shall also seek the form which more properly generalizes Birkhoff's equations into a Lie-admissible form. We hope in this way to continue the analysis of monograph[189], by putting the basis for a possible generalization of the Birkhoffian mechanics we shall call Birkhoffian-admissible mechanics.

The continuity and regularity assumptions which will be implemented throughout our analysis are the following. Newtonian systems will be defined in a region R of their variables, where the term "region" is referred to an open and connected set. All systems will be assumed to be of class $\mathcal{C}^\infty$ in R. This implies, in particular, that the acting forces satisfy this continuity property. Also, all systems will be assumed to be regular in R, i.e., their functional determinant

$$\mathcal{H}(R) = \left|\frac{\partial \varphi_{ia}}{\partial \ddot{r}^{jb}}\right|(R) \neq 0, \quad \varphi_{ia} = \varphi_{ia}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}, \ddot{\underset{\sim}{r}}, \dddot{\underset{\sim}{r}}) = 0 \qquad (2.1.16)$$

will be assumed to be non-null in R (except at a finite number of isolated zeros). On practical grounds, it will be sufficient to select, for the region of definition of the systems considered, a point of the local variables and its neighborhood such that the functional determinant is non-null in it. All Hamiltonians considered, irrespective of whether for Eqs. (2.1.3), or (2.1.7), will be assumed to be regular, i.e., the determinant

$$\left|\frac{\partial^2 H}{\partial p_{ia} \partial p_{jb}}\right|(R) \neq 0 \qquad (2.1.17)$$

is non-null at least at one point of the local variables $(t, \underset{\sim}{r}, \underset{\sim}{p})$ and in its neighborhood.

The terminology used in Volume I will be preserved during this work. In addition, we shall also tacitly imply a knowledge of the terminology of Brikhoffian mechanics[189]. For instance, a "Birkhoffian representation of a Newtonian system" is referred to the knowledge of the (6N + 1)-functions R_μ and B. Similarly, the "autonomous", "semiautonomous", and "nonautonomous" Birkhoff's equations occur when R = R(a)/B = B(a), R = R(a)/B=B(t,a), and R=R(t,a)/B=B(t,a), respectively. Also, we shall sometime use the word "vector field" for the Newtonian vector Ξ^μ, with the understanding that the geometric quantity

is given by the familiar form (for the autonomous case)

$$\Xi = \Xi^\mu \frac{\partial}{\partial a^\mu} = \Xi^{ka}\frac{\partial}{\partial r^{ka}} + \Xi_{ka}\frac{\partial}{\partial p_{ka}} \qquad (2.1.18)$$

$$(\Xi^\mu) = \begin{pmatrix} p_{ka}/m \\ f^{SA}_{ka} + F^{NSA}_{ka} \end{pmatrix}$$

2.2: THE LACK OF ALGEBRAIC CHARACTER OF HAMILTON'S EQUATIONS WITH EXTERNAL TERMS.

Consider Eqs. (2.1.7), with time evolution (2.1.11). As indicated in the preceding section, the lack of totally antisymmetric character of the brackets

$$A \times B \stackrel{\text{def}}{=} [A, B] + \frac{\partial A}{\partial a^\mu} F^\mu \qquad (2.1.1)$$

implies their lack of Lie character.

It is easy to see that the brackets are <u>not</u> Lie-admissible. In fact, the attached brackets are given in this case by

$$A \times B - B \times A = \left(\frac{\partial A}{\partial a^\mu} - \frac{\partial B}{\partial a^\mu} \right) F^\mu \qquad (2.2.2)$$

and, evidently, they are not Lie.

At a deeper inspection, it is possible to show that <u>brackets (2.2.1) do not possess a consistent algebraic structure</u>, i.e., they do not form an algebra. In fact, for brackets to characterize the product of any algebra, they must first verify the right and left distributive law, and the scalar law (Section 1.5). One can readily see that brackets (2.2.1) do verify the left distributive law, but they violate the right distributive law

$$\begin{aligned} (A+B) \times C &= A \times C + B \times C \\ A \times (B+C) &\neq A \times B + A \times C \end{aligned} \qquad (2.2.3)$$

Similarly, they verify the left part of the scalar law, but violate its right part, i.e.,

$$\begin{aligned} \alpha \times (A \times B) &= A \times (\alpha * B) = (\alpha \times A) \times B \\ (A \times B) \times \alpha &\neq A \times (B \times \alpha) \neq (A \times \alpha) \times B \end{aligned} \qquad (2.2.4)$$

It is also easy to see that the brackets satisfy the left differential law (1.5.5a), but violate the right version of the same law, Eq. (1.5.5b), i.e.,

$$\begin{aligned} (AB) \times C &= (A \times C) B + A (B \times C) \\ A \times (BC) &\neq (A \times B) C + B (A \times C) \end{aligned} \qquad (2.1.5)$$

We can, therefore, conclude by saying that the familiar form of Hamilton's equations with additive external terms leads to brackets which do not characterize a consistent algebraic structure. This situation indicates that, despite their preservation for over one century, Hamilton's equations with external terms must be suitably modified to yield an acceptable algebraic structure .

2.3: HAMILTON-ADMISSIBLE GENERALIZATION OF HAMILTON'S EQUATIONS AS THE ANALYTIC ORIGIN OF LIE-ADMISSIBLE ALGEBRAS.

In this section, we shall present a reformulation of Eqs. (2.1.7) admitting brackets of the time evolution which, first of all, characterize a consistent algebraic structure, and, secondly, that algegra results to be Lie-admissible.

The origin of the lack of consistent algebra for brackets (2.2.1) is given by their lack of bilinear character, as manifest in the additive part due to the external term.

The reformulation of Hamilton's equations with external terms which resolves this insufficiency was proposed by Santilli in ref.s[191,192] and it can be written

$$\dot{a}^\mu = S^{\mu\nu}(t,a)\frac{\partial H(t,a)}{\partial a^\nu}, \quad \mu=1,2,\dots,6N \qquad (2.3.1a)$$

$$S^{\mu\nu} = \omega^{\mu\nu} + t^{\mu\nu}(t,a), \quad t^{\mu\nu}=t^{\nu\mu} \qquad (2.3.1b)$$

$$\omega^{\mu\nu} = \left(\left\|\frac{\partial R^0_\alpha}{\partial a^\beta} - \frac{\partial R^0_\beta}{\partial a^\alpha}\right\|^{-1}\right)^{\mu\nu}, \quad R^0 = (p, 0), \qquad (2.3.1c)$$

$$(F^\mu) = \left(t^{\mu\nu}\frac{\partial H}{\partial a^\nu}\right) = \begin{pmatrix} 0 & 0 \\ 0 & F^{NSA}/(\partial H/\partial p)\end{pmatrix}\begin{pmatrix}\partial H/\partial r \\ \partial H/\partial p\end{pmatrix}, \qquad (2.3.1d)$$

$$\det \omega^{\mu\nu} \neq 0\ (=1);\ \det(S^{\mu\nu}) \neq 0\ (=1);\ \det(t^{\mu\nu}) = 0 \qquad (2.3.1e)$$

with time evolution

$$\dot{A} = \frac{\partial A}{\partial a^\mu} S^{\mu\nu}(t,a)\frac{\partial H}{\partial a^\nu} \stackrel{\text{def}}{=} (A, H) \qquad (2.3.2)$$
$$= \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} + \frac{\partial A}{\partial a^\mu} t^{\mu\nu}(t,a)\frac{\partial H}{\partial a^\nu} = [A,H] + \{A,H\}$$

The new brackets (A,H) are manifestly bilinear and, thus, they do verify the right and left distributive laws as well as the scalar law as in Eqs. (1.5.4).

Furthermore, it is equally easy to see that brackets (A,H) are Lie-admissible. In fact, unlike case (2.2.2), the attached brackets are now given by

$$(A,H) - (H,A) = 2\,[A,H] \qquad (2.3.3)$$

and they are manifestly Lie.

Equations (2.3.1) were called <u>Hamilton-admissible equations</u> in ref.s[191,192] in the dual sense:

(1) The generalized equations admit the <u>conventional</u> Hamilton's equations (2.1.4) when all non-self-adjoint forces are null; and

(2) The generalized equations possess a Lie-admissible algebraic structure.

The same terminology will be preserved in this volume.

It should be indicated that a modification of the canonical equations of the type

$$\dot{r} = \frac{\partial H}{\partial p}, \quad \dot{p} = \varepsilon\frac{\partial H}{\partial r}, \quad r \in \mathbb{R}_1, \qquad (2.3.4)$$

where ε is a constant, was proposed by Duffin in 1962[162].

The Lie-admissible algebraic character of Eqs. (2.3.4) was identified by Santilli in 1969[6].

The subsequent contributions of 1978, ref.s[191,192], presented a number of generalized equations possessing a Lie-admissible algebraic structure. Form (2.3.1) is the simplest and most direct one, and it will be assumed as the starting grounds of our analysis. The underlying brackets (A,H), as characterized by Eqs. (2.3.2), are called the fundamental Lie-admissible brackets (Section 1.5).

By recalling that the fundamental cosymplectic/Lie tensor represents in a unified way all fundamental canonical commutation rules according to Eqs. (1.5.14), the fundamental Lie-admissible tensor $S^{\mu\nu}(t,a)$ also represents the fundamental brackets of the theory according to the structure

$$((a^\mu, a^\nu)) = (S^{\mu\nu}(t,a)) = \begin{pmatrix} (r^{ia}, r^{ib}) & (r^{ia}, p_{ib}) \\ (p_{ia}, \quad) & (p_{ia}, p_{ib}) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & (F^{NSA}/(p/m)) \end{pmatrix} \quad (2.3.5)$$

in which one sees that different components of the linear momentum do not generally "commute" [we shall see in Section 2.5 that this is also the case for the components of the coordinates, which is somewhat reminiscent of gravitational/curvature effects].

As a result of the occurrence above, the variables r^{ka} and p_{ka} of Eqs. (2.3.1) do not span a phase space. They span instead a generalized space we shall call dynamical space, and which will be more technically identified in Chapter 4 as a symplectic-admissible manifold. The brackets (a^μ, a^ν) will be called fundamental dynamical brackets.

Eqs. (2.3.1) are written in their contravariant form because it is the form exhibiting the algebraic structure in a direct way. From regularity properties (2.3.1e), we see that the equations admit also the covariant form

$$S_{\mu\nu}(t,a)\,\dot a^\nu = \frac{\partial H(t,a)}{\partial a^\mu}, \quad \mu = 1,2,\ldots,6N \quad (2.3.6a)$$

$$S_{\mu\nu} = (\| S^{\alpha\beta} \|^{-1})_{\mu\nu}, \quad (2.3.6b)$$

$$(S_{\mu\nu}) = (\omega_{\mu\nu}) + (t_{\mu\nu}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -F^{NSA}/(p/m) & 0 \\ 0 & 0 \end{pmatrix} \quad (2.3.6c)$$

$$\omega_{\mu\nu} = (\| \omega^{\alpha\beta} \|^{-1})_{\mu\nu}, \quad t_{\mu\nu} \neq (\| t^{\alpha\beta} \|^{-1})_{\mu\nu} \quad (2.3.6d)$$

which is preferable for geometric considerations, as we shall see.

Recall that the covariant tensor of Hamiltonian mechanics

$$\omega_{\mu\nu} = \frac{\partial R^\circ_\nu}{\partial a^\mu} - \frac{\partial R^\circ_\mu}{\partial a^\nu}, \quad R^\circ = (p_{ka}, 0) \quad (2.3.7)$$

characterizes the Lagrange's brackets

$$\{A,B\} = \frac{\partial a^\mu}{\partial A}\,\omega_{\mu\nu}\,\frac{\partial a^\nu}{\partial B} \quad (2.3.8)$$

which are related to Poisson brackets by the "inverse property" characterized by 6N independent functions $A_k(a)$

$$\sum_{k=1}^{6N} [A_i, A_k]\,\{A_k, A_j\} = \delta_i^j \quad (2.3.9)$$

Along fully equivalent, yet generalized lines, the covariant tensor $s_{\mu\nu}(t,a)$ characterized the generalized brackets

$$\widetilde{\overbrace{A,B}} \stackrel{\text{def}}{=} \frac{\partial a^\mu}{\partial A} s_{\mu\nu}(t,a) \frac{\partial a^\nu}{\partial B} \qquad (2.3.10)$$

which verify the rules

$$\sum_{k=1}^{6N} (A_i, A_k) \widetilde{\overbrace{A_k, A_j}} = \delta_i^j \qquad (2.3.11)$$

As we shall see in Chapter 4, the tensor $s_{\mu\nu}(t,a)$ geometrizes the fundamental Lie-admissible brackets, and will be called the fundamental symplectic-admissible form.

2.4: DIRECT UNIVERSALITY OF HAMILTON-ADMISSIBLE EQUATIONS IN NEWTONIAN MECHANICS.

After having identified the basic equations of our analysis, it is important to probe their representational capabilities.

Consider the most general possible, open, local, non-self-adjoint Newtonian systems of the class admitted, which, in their second-order form, can be written

$$m_k \ddot{r}_{ka} - f^{SA}_{ka}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) - F^{NSA}_{ka}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) = 0$$
$$k = 1, 2, \ldots, N, \quad a = x, y, z \qquad (2.4.1)$$

The energy, in this case, is the total energy of the maximal self-adjoint subsystem (the sum of the kinetic energy and the potential energy of all self-adjoint forces),

$$E = \sum_{k=1}^{N} \frac{1}{2} m_k \dot{\underset{\sim}{r}}_k \cdot \dot{\underset{\sim}{r}}_k + U(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) \qquad (2.4.2)$$

From the theorems of the inverse problem[65,189], we know that the potential function $U(t,\underset{\sim}{r},\dot{\underset{\sim}{r}})$ can be at most linear in the velocities, i.e., it is of the type,

$$U = \alpha_{ka}(t, \underset{\sim}{r}) \dot{r}^{ka} + \beta(t, \underset{\sim}{r}) \qquad (2.4.3)$$

Let L be the Lagrangian of the maximal self-adjoint subsystem, i.e.,

$$L = \sum_{k=1}^{N} \frac{1}{2} m_k \dot{\mathbf{r}}_k \cdot \dot{\mathbf{r}}_k - U(t, \mathbf{r}, \dot{\mathbf{r}}) \qquad (2.4.4)$$

and introduce the generalized momenta

$$p_{ka} = \frac{\partial L}{\partial \dot{r}^{ka}} = m_k \dot{r}_{ka} - \alpha_{ka}(t, \mathbf{r}) \qquad (2.4.5)$$

with implicit form in the velocity

$$\dot{r}_{ka} = \frac{1}{m_k}\left(p_{ka} + \alpha_{ka}\right) \qquad (2.4.6)$$

Then, system (2.4.1) can be turned into the equivalent first-order form

$$\dot{a}^\mu = \Xi^\mu(t,a) = \Xi_1^\mu(t,a) + \Xi_2^\mu(t,a), \qquad (2.4.7a)$$

$$(a^\mu) = \begin{pmatrix} r_{ka} \\ p_{ka} \end{pmatrix}, \quad \left(\Xi_1^\mu\right) = \begin{pmatrix} \frac{1}{m_k}(p_{ka} + \alpha_{ka}) \\ f_{ka}^{SA}(t,a) \end{pmatrix}, \qquad (2.4.7b)$$

$$\left(\Xi_2^\mu\right) = \begin{pmatrix} 0 \\ F_{ka}^{NSA}(t,a) \end{pmatrix}. \qquad (2.4.7c)$$

It is easy to see that the component Ξ_1^μ is Hamiltonian, i.e., there exists a Hamiltonian function

$$H = \sum_{k=1}^{N} \frac{1}{2m_k} \mathbf{p}_k \cdot \mathbf{p}_k + \tilde{U}(t, \mathbf{r}, \mathbf{p}) = T(\mathbf{p}) + \tilde{U}(t, \mathbf{r}, \mathbf{p}) \qquad (2.4.8)$$

under which we have the identities

$$\Xi_1^\mu \equiv \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu} \qquad (2.4.9)$$

However, it is equally easy to see that the complete system Ξ^μ is not Hamiltonian in the variables considered.

Nevertheless, all possible systems (2.4.7) are <u>Hamilton-admissible</u>, i.e., they always admit a representation in terms of Hamilton-admissible equations (2.3.1). Equivalently, for all possible systems (2.4.7), there always exists a Hamiltonian H and a fundamental Lie-admissible tensor s (t,a), verifying the identities

$$\Xi^\mu = s^{\mu\nu} \frac{\partial H}{\partial a^\nu} \qquad (2.4.10)$$

follow from the separate properties

$$\Xi_1^\mu = \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu}, \quad \Xi_2^\mu = t^{\mu\nu} \frac{\partial H}{\partial a^\nu}, \quad H = T + U \qquad (2.4.11)$$

where we have made use of rule (2.3.1d).

In fact, the needed general solution is provided by rules (2.3.11).

We have proved in this simple way the direct universality of the Lie-admissible algebras in mechanics, i.e., their capability to represent all systems of the class admited (universality), in the time and Cartesian coordinates of the experimenter (direct universality).

THEOREM 2.4.1: All possible local, class $\mathcal{C}^{\infty}$, unconstrained, generally nonconservative/non-self-adjoint, Newtonian systems of N particles in a three-dimensional Euclidean space with local coordinates r^{ka}, k=1,...N, a = x,y,z, in their first-order form

$$\dot{a}^{\mu} = \Xi^{\mu}(t,a), \quad \mu = 1,2,\ldots,6N \tag{2.4.12a}$$

$$\begin{pmatrix} \dot{r}_{ka} \\ \dot{p}_{ka} \end{pmatrix} = \begin{pmatrix} \frac{1}{m_k}(p_{ka} + \alpha_{ka}) \\ f^{SA}_{ka}(t,a) + F^{NSA}_{ka}(t,a) \end{pmatrix}, \tag{2.4.12b}$$

$$f^{SA}_{ka} = -\frac{\partial U}{\partial r^{ka}} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}^{ka}}, \tag{2.4.12c}$$

$$U = \alpha_{ka}(t,r)\,\dot{r}^{ka} + \beta_{ka}(t,r), \tag{2.4.12d}$$

always admit a representation in terms of Hamilton-admissible equations

$$\dot{a}^{\mu} = S^{\mu\nu}(t,a)\frac{\partial H(t,a)}{\partial a^{\nu}}, \tag{2.4.13a}$$

$$S^{\mu\nu} = \omega^{\mu\nu} + t^{\mu\nu}(t,a), \quad t^{\mu\nu} = t^{\nu\mu} \tag{2.4.13b}$$

$$\omega^{\mu\nu} = \left(\left\| \frac{\partial R^{0}_{\alpha}}{\partial a^{\beta}} - \frac{\partial R^{0}_{\beta}}{\partial a^{\alpha}} \right\|^{-1}\right)^{\mu\nu}, \quad (t^{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & F^{NSA}/(p/m) \end{pmatrix} \tag{2.4.13c}$$

$$H = \sum_{k=1}^{N} \frac{1}{2m_k}\, p_k \cdot p_k + \hat{U}(t,r,p) \tag{2.4.13d}$$

whose brackets of the time evolution for functions A(a) in the space of the a-variables

$$\begin{aligned} \dot{A}(a) &= \frac{\partial A}{\partial a^{\mu}} S^{\mu\nu}(t,a)\frac{\partial H}{\partial a^{\nu}} = (A,H) \\ &= [A,H] + \frac{\partial A}{\partial p} \cdot \left(\frac{F^{NSA}}{(p/m)}\right) \cdot \frac{\partial H}{\partial p} \end{aligned} \tag{2.4.14}$$

characterize a Lie-admissible algebra, i.e., verify the Lie-admissible law

$$\begin{aligned} &[A,B,C] + [B,C,A] + [C,A,B] \\ &= [C,B,A] + [B,A,C] + [A,C,B] \end{aligned} \tag{2.4.15}$$

where

$$[A,B,C] = ((A,B),C) - (A,(B,C)) \tag{2.4.16}$$

COROLLARY 2.4.1A: The Hamilton-admissible representations of Theorem 2.4.1 can always be selected in such a way that:

(1) the variables t and r^{ka} represent time and Cartesian coordinates of the observer;

(2) the function $H(t,\underline{r},\underline{p})$ represnets the (generally nonconserved) energy of the system, i.e., the total energy of the maximal self-adjoint subsystem; and

(3) all nonpotential forces are represented by the symmetric part of the Lie-admissible tensor $s^{\mu\nu}(t,\underline{r},\underline{p})$.

We learn in this way a property that will be crucial throughout our analysis. Consider a conservative, Hamiltonian, system represented in terms of the conventional canonical equations

$$\dot{a}^{\mu} = \omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}} \qquad (2.4.16)$$

$$H = T(\underline{p}) + \tilde{U}(t,\underline{r},\underline{p})$$

with exponential law (see next chapter for more detail).

$$a^{\mu}(t) = e^{t\,\omega^{\alpha\beta}\frac{\partial H}{\partial a^{\beta}}\frac{\partial}{\partial a^{\alpha}}}\, a^{\mu}(0) \qquad (2.4.17)$$

The additional presence of nonpotential forces due to the extended size of the particles can be simply represented by adding a symmetric tensor $t^{\mu\nu}$ to the totally antisymmetric Lie tensor $\omega^{\mu\nu}$,

while preserving the local variables and the Hamiltonian. This yields the Hamilton-admissible equations with corresponding generalized transformation groups (see next chapter)

$$\dot{a}^{\mu} = \left(\omega^{\mu\nu} + t^{\mu\nu}\right)\frac{\partial H}{\partial a^{\nu}}, \qquad (2.4.18a)$$

$$t^{\mu\nu} = t^{\nu\mu}, \quad \det(t^{\mu\nu}) = 0 \qquad (2.4.18b)$$

$$a^{\mu}(t) = e^{t\,(\omega^{\alpha\beta} + t^{\alpha\beta})\frac{\partial H}{\partial a^{\beta}}\frac{\partial}{\partial a^{\alpha}}}\, a^{\mu}(0) \qquad (2.4.18c)$$

In fact, the nonpotential forces are represented precisely by the departure of the theory from conventional Lie structures, according to the rule

$$\left(s^{\mu\nu} - \omega^{\mu\nu}\right)\frac{\partial H}{\partial a^{\nu}} = F^{\mu}, \quad (F^{\mu}) = \begin{pmatrix} 0 \\ \underline{F}^{NC} \end{pmatrix} \qquad (2.4.19)$$

The transition from the conventional Poisson brackets [A,B] to the fundamental Lie-admissible brackets (A,H) was called a Lie-admissible genotopy in Section 1.5. We learn here its physical meaning. In fact, we can say that the genotopy of a Lie algebra into a Lie-admissible algebra is an algebraic representation of the extended character of particle-constituents via the admission of contact/

nonpotential forces.

The explicit verification of the following property is instructive.

COROLLARY 2.4.1B: Brackets (2.4.14) characterize a general Lie-admissible algebra which, in general, is non-flexible and non-Jordan-admissible.

The covering property of the Hamilton-admissible-equations over the conventional ones is expressed by the following

COROLLARY 2.4.1C: The conventional Hamilton equations are admitted by their Hamilton-admissible generalization in the following two-fold way:

(A) The conventional equations are recovered identically when all nonpotential forces are null; and

(B) The conventional equations are preserved in the broader context as the attached equations, i.e., as the equations characterized by the antisymmetric component of the Lie-admissible product (A,H).

Appendix 2.A provides a number of examples of representations of nonconservative systems with Hamilton-admissible equations.

Theorem 2.4.1 has been formulated for the class of systems of primary interest for this monograph. However, it should be indicated that the Lie-admissible algebras appear to have representational capabilities substantially more general than those of Theorem 2.4.1. We mention, in particular, the representation of the following more

general systems.

(I) Direct universality of Hamilton-admissible equations for non-Newtonian systems. Suppose that the forces depend explicitly on the accaleration, i.e., the system is non-Newtonian. Then it can be written in the general second order form

$$F_{ka}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}, \ddot{\underset{\sim}{r}}) = 0, \quad k = 1, 2, \ldots, N, \quad a = x, y, z \tag{2.4.20}$$

Its reduction to an equivalent first-order form can be done via arbitrary prescriptions for the characterization of the noncanonical momenta, e.g., by assuming $\underset{\sim}{p} = m\dot{\underset{\sim}{r}}$. The decomposition of the resulting vector field into a Hamiltonian a non-Hamiltonian component then permits representation (2.4.11). A similar result applies for systems of order higher than the second, i.e.

$$F_i(t, z^{(0)}, z^{(1)}, \ldots, z^{(m)}) = 0, \tag{2.4.21}$$

$$i = 1, 2, \ldots, n, \quad z \in \mathbb{R}_n, \quad z^{(k)} = \frac{d^k z}{dt^k}$$

which can always be reduced to an equivalent, even-dimensional, first-order form under sufficient regularity and continuity conditions (see Chart 4.3 of monograph[189] for the explicit construction).

We can, therefore, conclude by saying that the direct universality of Theorem 2.4.1 can be readily extended to non-Newtonian systems of arbitrary (finite) dimensionality.

(II) Direct universality of Hamilton-admissible equations for impulsive-discontinuous forces. Theorem 2.4.1 has been formulated under continuity conditions recommendable for a conventional geometric study of Lie-admissible algebras (Class C^{∞}). Nevertheless, it should be indicated that, at least formally, the equations are capable of representing also impulsive-discontinuous forces. In fact, all non-Hamiltonian forces are represented by the tensor

$$(t^{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & F^{NSA}/(p/m) \end{pmatrix} \qquad (2.4.22)$$

The possible discontinuities, therefore, occur in the numerator of the diagonal terms $F^{NSA}/(\partial H/\partial p)$. Needless to say, the incorporation of discontinuous forces raises a number of delicate technical aspects which need suitable study. Here, we simply limit ourself to the remark that the admission of discontinuous forces does not alter the Lie/symplectic contents of the theory. In fact, the discontinuities appear only in the totally symmetric part of the Lie-admissible

product

$$(A,B) = [A,B] + \frac{\partial A}{\partial p} \frac{F^{DISCONT.}}{(p/m)} \frac{\partial B}{\partial p} \qquad (2.4.23)$$

while the antisymmetric/Lie part continues to enjoy conventional topological properties.

(III) Direct universality of Hamilton-admissible equations for integro-differential/nonlocal systems. As recalled in Section 2.1, systems (2.4.1) constitute a crude approximation of the substantially complex physical reality of extended particles. Under the condition that the motion of their center-of-mass is treated via local/differential equations, the forces are of action-at-a-distance/local/potential as well as of contact/nonlocal/integral type. We reach in this way the most general (unconstrained) systems known at this time, the so-called integro-differential, variationally non-self-adjoint system, which we write in the form

$$m_k \ddot{r}_{\mu k} - f^{SA}_{\mu k\,LOCAL} - F^{NSA}_{\mu k\,LOCAL} - \mathcal{F}^{NSA}_{\mu k\,NONLOCAL} \qquad (2.4.24)$$

It is easy to see that, at least on formal grounds, Hamilton-admissible equations are directly universal for all systems (2.4.24). In fact, the nonlocal terms are represented again, by the symmetric part of the Lie-admissible tensor

$$(t^{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{F^{NSA}_{LOCAL}}{(p/m)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{\mathcal{F}^{NSA}_{NONLOCAL}}{(p/m)} \end{pmatrix} \qquad (2.4.25)$$

and of the Lie-admissible brackets

$$(A, B) = [A, B] + \frac{\partial A}{\partial p} \frac{F^{NSA}_{LOCAL}}{(p/m)} \frac{\partial B}{\partial p} + \frac{\partial A}{\partial p} \frac{\mathcal{F}^{NSA}_{NONLOCAL}}{(p/m)} \frac{\partial B}{\partial p} \qquad (2.4.26)$$

Additional generalizations of the representational capabilities of Hamilton-admissible equations are possible, e.g., for constrained systems , but they will not be considered at this time.

An example of degenerate Lie-admissible brackets is presented in Appendix 2.B

2.5: <u>BIRKHOFFIAN-ADMISSIBLE GENERALIZATION OF BIRKHOFF'S EQUATIONS</u>

We shall soon see that the Hamilton-admissible equations are insufficient for our program, despite their basic character originating from the simplicity of the Lie-admissible structure as well as from their direct universality.

By recalling the general solution of Theorem 1.5.1 (Corollary 1.5.1B), the equations we are interested in can be written in the form

$$\dot{a}^\mu = S^{\mu\nu}(t,a) \frac{\partial B(t,a)}{\partial a^\nu} = 0, \mu = 1, 2, \ldots, 6N \qquad (2.5.1a)$$

$$S^{\mu\nu} = \left(\left\| \frac{\partial R_\alpha}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\alpha} \right\|^{-1} \right)^{\mu\nu} + T^{\mu\nu}(t,a) \qquad (2.5.1b)$$

$$(2.5.1c)$$

$$\det (S^{\mu\nu})(R) \neq 0$$

$$\det (S^{\mu\nu} - S^{\nu\mu})(R) \neq 0 \qquad (2.5.1d)$$

$$T^{\mu\nu} = T^{\nu\mu}, \quad S^{\mu\nu} - S^{\nu\mu} = \Omega^{\mu\nu} \qquad (2.5.1e)$$

and they will be called <u>Birkhoff-admissible equations</u> in the sense of admitting Birkhoff's equations when the symmetric tensor $T^{\mu\nu}$ is null.

It will be soon evident that, while the function H(t,a) of Hamilton-admissible equations represents the energy, this is not necessarily the case for the function B(t,a) of Birkhoff-admissible equations. This occurrence, which is present already in the Birkhoffian mechanics[189], has suggested the name of <u>Birkhoffian</u> for the function B(t,a).

Eqs. (2.5.1) are contravariant and, as such, are directly set for the identification of the algebraic property that the brackets of the time evolution

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu} S^{\mu\nu}(t,a) \frac{\partial B}{\partial a^\nu}$$

$$= \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu}(t,a) \frac{\partial B}{\partial a^\nu} + \frac{\partial A}{\partial a^\mu} T^{\mu\nu}(t,a) \frac{\partial B}{\partial a^\nu} \quad (2.5.2)$$

$$\stackrel{\text{def}}{=} (A,B)^* = [A,B]^* + \{A,B\}^*$$

constitute <u>general Lie-admissible brackets</u> (Section 1.5).

The covariant form of Birkhoff-admissible equations will be written

$$S_{\mu\nu}(t,a)\,\dot{a}^\nu - \frac{\partial B(t,a)}{\partial a^\mu} = 0 \quad (2.5.3a)$$

$$[\Omega_{\mu\nu}(t,a) + T_{\mu\nu}(t,a)]\,\dot{a}^\nu - \frac{\partial B(t,a)}{\partial a^\mu} = 0 \quad (2.5.3b)$$

$$\det(S_{\mu\nu})(R) \neq 0 \qquad \det(\Omega_{\mu\nu})(R) \neq 0 \quad (2.5.3c)$$

The following comments are in order here. Consider the fundamental Lie-admissible tensor (2.3.1b) and its covariant form constructed via the rule

$$S_{\mu\nu} = \left(\| S^{\alpha\beta} \|^{-1}\right)_{\mu\nu} \quad (2.5.4)$$

where

$$S_{\mu\nu} = \omega_{\mu\nu} + t_{\mu\nu}, \qquad S^{\mu\nu} = \omega^{\mu\nu} + t^{\mu\nu} \quad (2.5.5)$$

In this case, the antisymmetric matrix ($\omega_{\mu\nu}$) is the inverse of the antisymmetric part of the matrix ($s^{\mu\nu}$), i.e., we can write, jointly with Eqs. (2.5.5),

$$\omega_{\mu\nu} = \left(\| S^{\alpha\beta} - S^{\beta\alpha} \|^{-1}\right)_{\mu\nu} \quad (2.5.6)$$

$$= \left(\| \omega^{\alpha\beta} \|^{-1}\right)_{\mu\nu}$$

while, as the reader is urged to verify,

$$t_{\mu\nu} \neq \left(\| t^{\alpha\beta} \|^{-1}\right)_{\mu\nu} \quad (2.5.7)$$

In the transition to an arbitrary (regular) matrix $(M^{\mu\nu})$, properties (2.5.6) are generally violated, i.e., the antisymmetric matrix $(M^{\mu\nu} - M^{\nu\mu})$ <u>is not</u>, in general, equal to the inverse of the antisymmetric component of matrix $(M_{\mu\nu})$, and a similar result occurs for the symmetric part. We, therefore, have in general

$$M_{\mu\nu} = \left(\| M^{\alpha\beta} \|^{-1}\right)_{\mu\nu} \tag{2.5.8a}$$

$$M_{\mu\nu} - M_{\nu\mu} \neq \left(\| M^{\alpha\beta} - M^{\beta\alpha} \|^{-1}\right)_{\mu\nu} \tag{2.5.8b}$$

$$M_{\mu\nu} + M_{\nu\mu} \neq \left(\| M^{\alpha\beta} + M^{\beta\alpha} \|^{-1}\right)_{\mu\nu} \tag{2.5.8c}$$

However, it is possible to prove that, for the case of Eqs. (2.5.1) one can always select the general Lie-admissible tensor $S^{\mu\nu}$ in such a way to preserve properties (2.5.6), i.e.,

$$S_{\mu\nu} = \left(\| S^{\alpha\beta} \|^{-1}\right)_{\mu\nu} \tag{2.5.9a}$$

$$S_{\mu\nu} = \Omega_{\mu\nu} + T_{\mu\nu} \; ; \; S^{\mu\nu} = \Omega^{\mu\nu} + T^{\mu\nu} \tag{2.5.9b}$$

$$\Omega_{\mu\nu} = \left(\| \Omega^{\alpha\beta} \|^{-1}\right)_{\mu\nu} \tag{2.5.9c}$$

$$T_{\mu\nu} \neq \left(\| T^{\alpha\beta} \|^{-1}\right)_{\mu\nu} \tag{2.5.9d}$$

$$\det \left(S^{\mu\nu}\right)(R) \neq 0 \, , \; \det \left(\Omega^{\mu\nu}\right)(R) \neq 0 \tag{2.5.9e}$$

To show this property in the simplest possible way (prior to the introduction of any geometric argument), we shall show how to turn the contravariant and covariant Birkhoff's equations into Equations (2.5.1) and (2.5.4), respectively. The equivalence of the latter equations, with underlying properties (2.5.9) will then be a mere consequence.

Consider the nonautonomous Birkhoff's equations (ref.[189], Section 4.2) in the following covariant and contravariant forms

$$\left[\frac{\partial R_\nu(t,a)}{\partial a^\mu} - \frac{\partial R_\mu(t,a)}{\partial a^\nu}\right]\dot{a}^\nu - \left[\frac{\partial B(t,a)}{\partial a^\mu} + \frac{\partial R_\mu(t,a)}{\partial t}\right] = 0, \tag{2.5.10a}$$

$$\dot{a}^\mu = \Omega^{\mu\nu}(t,a)\left[\frac{\partial B(t,a)}{\partial a^\nu} + \frac{\partial R_\nu(t,a)}{\partial t}\right], \tag{2.5.10b}$$

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} = \left(\| \Omega^{\alpha\beta} \|^{-1}\right)_{\mu\nu}, \tag{2.5.10c}$$

$$\det \left(\Omega_{\mu\nu}\right)(R) \neq 0 \tag{2.5.10d}$$

As is well know, the autonomous tensor $\Omega^{\mu\nu}(a)$ constitutes the most general possible (regular) form of a Lie tensor. However, the Lie algebra is lost for the general, nonautonomous case for reasons rather similar to those occurring in the transition from the conventional Hamilton's equations to those with external terms.

Consider the time evolution for Eqs. (2.5.10)

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}\frac{\partial B}{\partial a^\nu} + \frac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}\frac{\partial R_\nu}{\partial t} \stackrel{def}{=} A \times B \qquad (2.5.11)$$

It is easy to see that the underlying brackets "A x B" do not characterize a consistent algebra (let alone a Lie algebra) because of the lack of bilinear character.

As done for Eqs. (2.3.1), the regaining of an algebraic structure calls for a reformulation of the last term of Eqs. (2.5.11) which recovers the bilinearity. This yields the (unique) rules

$$\Omega^{\mu\nu}\frac{\partial R_\nu}{\partial t} = T^{\mu\nu}\frac{\partial B}{\partial a^\nu}, \quad T^{\mu\nu} = T^{\nu\mu} \qquad (2.5.12)$$

$$\mu = 1, 2, \ldots, 6N$$

under which the non-algebraic, nonautonomous Birkhoff's equations are turned into the algebraically consistent, Birkhoff-admissible equations (2.5.1). The similarity of rule (2.5.12) with the corresponding Hamilton-admissible one, Eqs. (2.3.1d) is remarkable.

In short, the nonautonomous Birkhoffian mechanics can be turned into a form admitting a consistent algebra in the brackets of the time evolution. However, in the process, the algebras occurring in the autonomous case (Lie algebras) are lost in favor of covering algebras (Lie-admissible algebras).

By recalling that the transition from autonomous to nonautonomous systems is one representative of nonconservation, we recover in this way the result of Section 2.3, to the effect that the mapping from Lie algebras to Lie-admissible algebras is a representative of the non-conservative character of the systems, via the most general possible Lie-admissible algebraic genotopy

$$\dot{a}^\mu = [A, B]^* \Longrightarrow \dot{a}^\mu = (A, B)^* \qquad (2.5.13)$$

We pass now to the case of the covariant equations. In this case we have the rules

$$\begin{aligned} &\Omega_{\mu\nu}\dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right) \\ &\equiv (\Omega_{\mu\nu} + T_{\mu\nu})\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} \\ &T_{\mu\nu} = T_{\mu\nu}, \quad T_{\mu\nu} \neq (\|T^{\alpha\beta}\|^{-1})_{\mu\nu} \end{aligned} \qquad (2.5.14)$$

which clearly show the preservation of properties (2.5.9a) and (2.5.9b). The construction of the symmetric tensor can be done via the equations

$$\frac{\partial R_\mu}{\partial t} = T_{\mu\nu}\dot{a}^\nu \qquad (2.5.15)$$

$$= T_{\mu\alpha}\Omega^{\alpha\beta}\left(\frac{\partial B}{\partial a^\beta} + \frac{\partial R_\beta}{\partial t}\right)$$

The validity of properties (2.5.9d) is then self-evident. This concludes our proof of the equivalence of Eqs. (2.5.1) and (2.5.4) under rules (2.5.9).

It is possible to show that the reformulation of the nonautonomous Birkhoff's equations into the Birkhoff-admissible form is always possible under the assumed topological conditions. This can be shown by nothing that rules (2.5.12) always admit a solution in the unknown elements of the symmetric tensor $T^{\mu\nu}$ for given 6N functions $R_\mu(t,a)$ and a Birkhoffian B. In fact, one can always assume that the matrix $(T^{\mu\nu})$ is diagonal. In this case, rule (2.5.4) characterizes 6N inhomogeneous algebraic equations in 6N unknowns which, under the assumed conditions, are always consistent.

Consider, as an example, the case of only one space dimension.

Then, rule (2.5.4) becomes

$$\begin{pmatrix} 0 & \Omega_1 \\ -\Omega_1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial R_1}{\partial t} \\ \frac{\partial R_2}{\partial t} \end{pmatrix} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}\begin{pmatrix} \frac{\partial B}{\partial z} \\ \frac{\partial B}{\partial p} \end{pmatrix}, \qquad (2.5.16)$$

yielding the solution

$$T_1 = \Omega_1 \frac{\partial R_2}{\partial t}\left(\frac{\partial B}{\partial z}\right)^{-1}, \qquad (2.5.17a)$$

$$T_2 = -\Omega_1 \frac{\partial R_1}{\partial t}\left(\frac{\partial B}{\partial p}\right)^{-1}. \qquad (2.5.17b)$$

The case for space dimensions higher than one can be solved along similar lines. The study of the corresponding covariant case, according to rule (2.5.15), will be left to the interested reader. Some examples of Birkhoff-admissible representations are presented in Appendix 2.A.

In monograph[189], we have proved the universality of the following representation of Hamilton's equations via Birkhoff's equations (Section 6.1, p. 65).

$$\Omega_{\mu\nu}\dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right) \qquad (2.5.18)$$

$$= h_\mu{}^\alpha\left[\omega_{\alpha\beta}\dot{a}^\beta - \frac{\partial H}{\partial a^\alpha}\right], \quad \det(h)(R) \neq 0$$

that is, for each given Hamiltonian H(t,a), there always exists a regular matrix of multiplicative functions $(h_\mu{}^\alpha(t,a))$ under which Birkhoffian representation (2.5.18) holds. The proof is based on variational self-adjointness of both Hamilton's and Birkhoff's equations, when joint with the universality of the isotopic transformations of first-order systems.

The use of the same argument then leads to the following

PROPOSITION 2.5.1: Under sufficient topological conditions, Hamilton-admissible equations for all possible Hamiltonians H(t,a) and symmetric tensors $t_{\mu\nu}(t,a)$ always admit an indirect representation in terms of the Birkhoff-admissible equations according to the rule

$$[\Omega_{\mu\nu}(t,a) + T_{\mu\nu}(t,a)]\dot{a}^\nu - \frac{\partial B(t,a)}{\partial a^\mu} \qquad (2.5.19a)$$

$$\equiv h_\mu{}^\alpha(t,a)\left\{[\omega_{\alpha\nu} + t_{\alpha\nu}(t,a)]\dot{a}^\nu - \frac{\partial H(t,a)}{\partial a^\alpha}\right\}$$

$$\Omega_{\mu\nu} + T_{\mu\nu} = h_\mu{}^\alpha(\omega_{\alpha\nu} + t_{\alpha\nu}) \qquad (2.5.19b)$$

$$\frac{\partial B}{\partial a^\mu} = h_\mu{}^\alpha \frac{\partial H}{\partial a^\alpha} \qquad (2.5.19c)$$

$$\det(h_\mu{}^\alpha)(R) \neq 0 \qquad (2.5.19d)$$

By recalling the universality of Hamilton-admissible equations, each Newtonian system then admits a representation in terms of both Hamilton-admissible and Birkhoff-admissible equations. To put it differently, the passage from the simple Lie-admissible structure of Hamilton-admissible equations to the most general possible Lie-admissible form turns out to be a degree of freedom of the representation.

In the passage, the Hamiltonian (read, the total energy) is replaced by the Birkhoffian functions B(t,a), i.e., a quantity which does not necessarily represent the total energy.

Recall, also from ref.[189], that the transition from the (conventional) Hamilton's equations to (the semiautonomous) Birkhoff's equations is a form of Lie algebra isotopy.

$$\begin{pmatrix} \dot{a}^\mu = \omega^{\mu\nu}\dfrac{\partial H}{\partial a^\nu} \\[2ex] [A,B] \\ = \dfrac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\dfrac{\partial B}{\partial a^\nu} \end{pmatrix} \Longrightarrow \begin{pmatrix} \dot{a}^\mu = \Omega^{\mu\nu}(a)\dfrac{\partial B}{\partial a^\nu} \\[2ex] [A,B]^* \\ = \dfrac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}\dfrac{\partial B}{\partial a^\nu} \end{pmatrix} \qquad (2.5.20)$$

We learn here that the transition from Hamilton-admissible to Birkhoff-admissible equations

$$\begin{pmatrix} \dot a^{\mu} = S^{\mu\nu}(t,a)\dfrac{\partial H}{\partial a^{\nu}} \\ \\ (A,B) \\ = \dfrac{\partial A}{\partial a^{\mu}} S^{\mu\nu} \dfrac{\partial B}{\partial a^{\nu}} \end{pmatrix} \Longrightarrow \begin{pmatrix} \dot a^{\mu} = S^{\mu\nu}(t,a)\dfrac{\partial B}{\partial a^{\nu}} \\ \\ (A,B)^{*} \\ = \dfrac{\partial A}{\partial a^{\mu}} S^{\mu\nu} \dfrac{\partial B}{\partial a^{\nu}} \end{pmatrix} \qquad (2.5.21)$$

is a form, this time, of Lie-admissible isotopy.

Furthermore, we learn that the latter isotopy can be an identity Lie-admissible isotopy, i.e., each time evolution can be identically written either in terms of the fundamental Lie-admissible brackets with respect to the Hamilton H(t,a) or in terms of the general Lie-admissible brackets with respect to a (suitably selected) Birkhoffian B(t,a), according to the rule

$$\dot A(a) = [A, H) \equiv (A, B)^{*} \qquad (2.5.22)$$

which are based on the reformulation

$$S^{\mu\nu}\frac{\partial H}{\partial a^{\nu}} \equiv S^{\mu\beta}\frac{\partial B}{\partial a^{\beta}} \qquad (2.5.23)$$

In conclusion, the fundamental Lie-admissible structure (2.3.1) provides one of the simplest possible representation of Newtonian systems. The structure which is ultimately permitted on algebraic grounds for each given system is the most general possible Lie-admissible form.

The implications of these findings for the transformation theory will be identified in Section 2.8. The analytic aspects will be studied in Section 2.7. Finally, the geometrical interpretation will be provided in Chapter 4.

In closing this section, it may be useful to recall that Hamilton's and Birkhoff's equations are variationally self-adjoint, that is, they verify the conditions of self-adjointness[65,189]

$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0, \qquad (2.5.24a)$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial a^{\tau}} + \frac{\partial \Omega_{\nu\tau}}{\partial a^{\mu}} + \frac{\partial \Omega_{\tau\mu}}{\partial a^{\nu}} = 0, \qquad (2.5.24b)$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial t} = \frac{\partial \Gamma_{\mu}}{\partial a^{\nu}} - \frac{\partial \Gamma_{\nu}}{\partial a^{\mu}} \qquad (2.5.24c)$$

here expressed for the general, first-order, form

$$\Omega_{\mu\nu}(t,a)\dot a^{\nu} + \Gamma_{\mu}(t,a) = 0 \qquad (2.5.25)$$

We should recall, in particular, that the above notion of self-adjointness applies specifically to the covariant version of the equations considered, and we shall write

$$\left\{ \omega_{\mu\nu}\, \dot{a}^{\nu} - \frac{\partial H}{\partial a^{\mu}} \right\}_{SA} = 0 \qquad (2.5.26a)$$

$$\left\{ \Omega_{\mu\nu}(t,a)\, \dot{a}^{\nu} - \left(\frac{\partial B}{\partial a^{\mu}} + \frac{\partial R_{\mu}}{\partial t} \right) \right\}_{SA} = 0 \qquad (2.5.26b)$$

In fact, the conditions (2.5.24) are the integrability conditions for the existence of a (first-order) variational principle, which, as known, characterize analytic equations in their covariant form.

It is easy to see that Hamilton-admissible and Birkhoff-admissible equations are variationally non-self-adjoint, and we shall write

$$\left\{ s_{\mu\nu}\, \dot{a}^{\nu} - \frac{\partial H}{\partial a^{\mu}} \right\}_{NSA} = 0, \qquad (2.5.27a)$$

$$\left\{ S_{\mu\nu}\, \dot{a}^{\nu} - \frac{\partial B}{\partial a^{\mu}} \right\}_{NSA} = 0, \qquad (2.5.27b)$$

as the reader can verify via the use of conditions (2.5.24).

Finally, we shall refer to Hamilton-admissible and Birkhoff-admissible equations as "coverings" of Hamilton's and Birkhoff's equations, respectively, in the sense that:

(1) the generalized equations apply to a physical arena broader than that of the conventional ones;

(2) the generalized equations are based on analytic, algebraic, and geometric methods structurally more general than those of the conventional ones; and

(3) the conventional equations (and their methods) can be recovered as particular cases of the generalized theory.

As we shall see, the same notion of covering will be used in the proposal of the generalized relativity.

As an illustration of the intended notion of covering, consider the fundamental brackets for the case of the general, nonautonomous Birkhoff-admissible equations.

$$\left((a^{\mu}, a^{\nu})^{*} \right) = \begin{pmatrix} (z^{ia}, z^{jb})^{*} & (z^{ia}, p_{jb})^{*} \\ (p_{ia}, z^{jb})^{*} & (p_{ia}, p_{jb})^{*} \end{pmatrix} = \left(S^{\mu\nu}(t,a) \right) = \begin{pmatrix} S^{ia\,jb} & S^{ia}{}_{jb} \\ S_{ia}{}^{jb} & S_{ia\,jb} \end{pmatrix} \qquad (2.5.28)$$

The above brackets constitute a "two-fold" covering of conventional brackets (1.5.14) in the following way . First, brackets (2.5.28) recover the fundamental brackets of Birkhoffian mechanics

$$
\left([a^\mu, a^\nu]^* \right) = \begin{pmatrix} [z^{ia}, z^{jb}]^* & [z^{ia}, p_{jb}]^* \\ [p_{ia}, z^{jb}]^* & [p_{ia}, p_{jb}]^* \end{pmatrix} = \left(\Omega^{\mu\nu}(t,a) \right) = \begin{pmatrix} \Omega^{ia\,;\,jb} & \Omega^{ia}{}_{;\,jb} \\ \Omega_{ia}{}^{jb} & \Omega_{ia\,;\,jb} \end{pmatrix} \tag{2.5.29}
$$

for null values of the symmetric tensor $T^{\mu\nu}$. Second, they recover the conventional, canonical brackets under the additional restriction that the cosymplectic tensor is the fundamental one.

2.6: GAUGE DEGREES OF FREEDOM OF BIRKHOFF-ADMISSIBLE EQUATIONS.

In the preceding section we have identified the isotopic degrees of freedom of Birkhoff-admissible equations. In this section, we shall show the existence of additional degrees of freedom of the theory which also occur within a fixed system of local variables.

Birkhoff's equations remain unchanged under the following gauge transformations[189]

$$
R_\mu(t,a) \longrightarrow R^\dagger_\mu(t,a) = R_\mu(t,a) + \frac{\partial G(t,a)}{\partial a^\mu} \tag{2.6.1a}
$$

$$
B(t,a) \longrightarrow B^\dagger(t,a) = B(t,a) - \frac{\partial G}{\partial t} \tag{2.6.1b}
$$

where the gauge function G(t,a) verifies the assumed continuity conditions.

In fact, under transformation (2.5.1a), Birkhoff's tensor remains unchanged

$$
\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \equiv \frac{\partial R^\dagger_\nu}{\partial a^\mu} - \frac{\partial R^\dagger_\mu}{\partial a^\nu} \stackrel{\text{def}}{=} \Omega^\dagger_{\mu\nu} \tag{2.6.2}
$$

while under joint transformations (2.5.1a) and (2.5.1b) we have the property

$$\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t} \equiv \frac{\partial B^+}{\partial a^\mu} + \frac{\partial R^+_\mu}{\partial t} \qquad (2.6.3)$$

It is easy to see that these gauge transformations have the following counterpart for Birkhoff-admissible equations

$$\begin{aligned} &\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right) \\ \equiv\ &\Omega^{+\mu\nu}\left(\frac{\partial B^+}{\partial a^\nu} + \frac{\partial R^+_\nu}{\partial t}\right) \\ =\ &\left(\Omega^{\mu\nu} + T^{\mu\nu}\right)\frac{\partial B}{\partial a^\nu} = S^{\mu\nu}\frac{\partial B}{\partial a^\nu} \\ \equiv\ &\left(\Omega^{+\mu\nu} + T^{+\mu\nu}\right)\frac{\partial B^+}{\partial a^\nu} = S^{+\mu\nu}\frac{\partial B^+}{\partial a^\nu} \end{aligned} \qquad (2.6.4a)$$

$$\Omega^{\mu\nu} \equiv \Omega^{+\mu\nu} = \left(\|\Omega^+_{\alpha\beta}\|^{-1}\right)^{\mu\nu}, \qquad (2.6.4b)$$

$$T^+_{\mu\nu} = \frac{1}{2}\left(S^+_{\mu\nu} + S^+_{\nu\mu}\right) \qquad (2.6.4c)$$

where

$$T^{+\mu\nu}\frac{\partial B^+}{\partial a^\nu} \equiv T^{\mu\nu}\frac{\partial B}{\partial a^\nu} \qquad (2.6.5)$$

By recalling that Hamilton-admissible equations are a particular case of the Birkhoff-admissible ones under the particularizations

$$\left\{\left(\left\|\frac{\partial R_\alpha}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\alpha}\right\|^{-1}\right)^{\mu\nu} + T^{\mu\nu}\right\}\frac{\partial B}{\partial a^\nu}\Bigg|_{\substack{R_\mu = R^\circ_\mu \\ B = H \\ T^{\mu\nu} = t^{\mu\nu}}} = \left(\omega^{\mu\nu} + t^{\mu\nu}\right)\frac{\partial H}{\partial a^\nu} \qquad (2.6.6)$$

it is easy to see that gauge transformation (2.6.4) also apply to Eqs. (2.3.1). For subsequent notational need, we shall then write

$$R^\circ_\mu \longrightarrow R^{+\circ}_\mu = R^\circ_\mu + \frac{\partial G}{\partial a^\mu}, \qquad (2.6.7a)$$

$$H \longrightarrow H^+ = H - \frac{\partial G}{\partial t}, \qquad (2.6.7b)$$

$$t^{\mu\nu}\frac{\partial H}{\partial a^\nu} = t^{+\mu\nu}\frac{\partial H^+}{\partial a^\nu} \qquad (2.6.7c)$$

$$\left(\omega^{\mu\nu} + t^{\mu\nu}\right)\frac{\partial H}{\partial a^{\nu}} \equiv \left(\omega^{\mu\nu} + t^{+\mu\nu}\right)\frac{\partial H^{+}}{\partial a^{\nu}} \qquad (2.6.7d)$$

It is clear that the gauge transformations considered here are <u>different</u> than the isotopic transformation of the preceding session. In fact, Birkhoff's tensor $\Omega^{\mu\nu}$ remains unchanged in the former case, while it varies in the latter.

The physical implications of gauge transformations are rather intriguing and they deserve a comment. Consider the Hamilton-admissible representation of nonpotential systems according to the rules

$$\begin{pmatrix} \frac{1}{m}(p+\alpha) \\ f^{SA} + F^{NSA} \end{pmatrix} = \left(\left(\omega^{\mu\nu} + t^{\mu\nu}\right)\frac{\partial H}{\partial a^{\nu}} \right) \qquad (2.6.8)$$

As stressed in Section 2.3 and 2.4, in this case all potential forces are represented by the Hamiltonian, while all nonpotential forces are represented by the algebraic tensor $t^{\mu\nu}$.

When representation (2.5.8) is subjected to gauge transformations, this distinction in the representation of potential and nonpotential

forces is lost. In fact, we now have a new Hamiltonian H^{+} (which cannot be the total energy any longer), and a new symmetric tensor $t^{+\mu\nu}$, according to rules (2.5.7).

Intriguingly, <u>the gauge transformations can remove all forces from the Hamiltonian and transfer them in the algebraic tensor</u>. In fact, under the assumption of autonomous potential forces and of dimensionless local coordinates, one can assume the gauge function

$$G = t\,U(a) \qquad (2.6.9)$$

where U is the total potential. Then the new Hamiltonian represents only the kinetic energy,

$$H^{+} = H - \frac{\partial G}{\partial t} = H - U = T(p) \qquad (2.6.10)$$

while all acting forces, whether of potential or nonpotential type, are represented in by the Lie-admissible tensor, and, more specifically, by its symmetric part

$$\left(t^{+\mu\nu}\right) = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{f^{SA} + F^{NSA}}{(p/m)} \end{pmatrix} \qquad (2.6.11)$$

This result was expected because our basic equations (2.3.1) also apply under the assumption of considering all forces as external, thus giving raise to representation (2.5.11). We have learned, however, that the shift of forces from the Hamiltonian to the external terms is ultimately a manifestation of the degrees of freedom of the Lie-admissible algebras.

The novelty of our findings is self-evident. In fact, the degrees of freedom considered here are impossible for Hamilton's equations owing to the impossibility of representing forces via the Lie tensor $\omega^{\mu\nu}$.

The reader is encouraged to work out the extension of the results of this section with the inclusion of the isotopic degrees of freedom of the Lie-admissible product. He will discover in this way additional, equally novel ways of shifting the representation of interactions from the Hamiltonian to, this time, the Lie/Birkhoffian component of the Lie-admissible tensor (rather than only its symmetric part).

2.7: DERIVATION OF BIRKHOFF-ADMISSIBLE EQUATIONS FROM VARIATIONAL PRINCIPLES.

My first exposure to variational principles for nonconservative systems was that by Goldstein (ref.[129] pages 38-40)

$$\delta \int_{t_1}^{t_2} dt\,(L + W) = - \int_{t_1}^{t_2} dt \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{z}^k} - \frac{\partial L}{\partial z^k} - F_k \right) \delta z^k = 0 \qquad (2.7.1)$$

The principle above is not a conventional variational principle. In fact, it is in actuality a variational principle with the subsidiary constraint

$$\delta W = -F_k\, \delta z^k \qquad (2.7.2)$$

without which Eqs. (2.7.1) would evidently not hold.

Numerous advances have occurred since the above approach (conceived by Goldstein in 1950). In fact, the Inverse Problem of the calculus of variations[65,189] has permitted the representation of all possible nonconservative Newtonian systems via the conventional, first-order, Pfaffian variational principle

$$\delta A = \delta \int_{t_1}^{t_2} dt \left[R_\mu(t,a)\,\dot{a}^\mu - B(t,a) \right] = 0 \qquad (2.7.3)$$

where the variations are the conventional ones (first-order variations with fixed end-points), and only the integrand of the action functional is generalized, from the restrictive, canonical form

$$\delta \int_{t_1}^{t_2} dt \left[R^{0}_{\mu}(a)\, \dot{a}^{\mu} - H(t,a) \right] = 0, \quad R^{0} = (p_{\mu}, 0) \quad (2.7.4)$$

to the most general possible first-order form.

These new achievements permit the derivation of Birkhoff-admissible equations from a variational principle in more than one way.

The first is via the Birkhoffian expression of principle (2.7.3)

$$\delta A = \delta \int_{t_1}^{t_2} dt \left(R_{\mu}\, \dot{a}^{\mu} - B \right) \quad (2.7.5)$$

$$= \int_{t_1}^{t_2} dt \left\{ \left[\frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}} \right] \dot{a}^{\nu} - \left[\frac{\partial B}{\partial a^{\mu}} + \frac{\partial R_{\mu}}{\partial t} \right] \right\}_{SA} \delta a^{\mu} = 0$$

and then its reformulation into Birkhoff-admissible equations via rules (2.5.12),

$$\delta A \equiv \int_{t_1}^{t_2} dt \cdot \left\{ \left(\Omega_{\mu\nu} + T_{\mu\nu} \right) \dot{a}^{\nu} - \frac{\partial B}{\partial a^{\mu}} \right\}_{NSA} \delta a^{\mu} = 0 \quad (2.7.6)$$

A second, and perhaps more significant approach is permitted by degrees of freedom of the variations. A generalization of conventional variational principles was proposed in ref.[189], Chart 5.7. It is based on the transition from the conventional variations

$$\delta a^{\mu} = \varepsilon\, f^{\mu}(t), \quad \varepsilon \approx 0 \quad (2.7.7)$$

to the generalized form

$$\delta^{*} a^{\mu} = g_{\alpha}{}^{\mu}(t,a)\, \delta a^{\alpha} \quad (2.7.8a)$$

$$\det\left(g_{\alpha}{}^{\mu} \right)(R) \neq 0 \quad (2.7.8b)$$

under which principle (2.7.3) becomes

$$\delta^{*} A \quad (2.7.9)$$

$$= \int_{t_1}^{t_2} dt \left\{ g_{\mu}{}^{\alpha} \left[\left(\frac{\partial R_{\beta}}{\partial a^{\alpha}} - \frac{\partial R_{\alpha}}{\partial a^{\beta}} \right) \dot{a}^{\beta} - \left(\frac{\partial B}{\partial a^{\alpha}} + \frac{\partial R_{\alpha}}{\partial t} \right) \right]_{SA} \right\}_{NSA} \delta a^{\mu} = 0$$

As a result, the variational self-adjointness of principle (2.7.5) can be "broken" by therefore permitting the direct representation of structurally more general equations.

In fact, by recalling that Birkhoff-admissible equations are variationally non-self-adjoint (Section 2.5), there always exists a matrix of integrating functions $g_\alpha{}^\mu(t,a)$ such that

$$g_\mu{}^\alpha \left(\Omega_{\alpha\nu} \dot{a}^\nu - \frac{\partial B}{\partial a^\alpha} - \frac{\partial R_\alpha}{\partial t} \right) \equiv \left(\Omega'_{\mu\nu} + T'_{\mu\nu} \right) \dot{a}^\nu - \frac{\partial B'}{\partial a} \qquad (2.7.10)$$

thus yielding the following derivation of Birkhoff-admissible equations from the first-order Pfaffian variational principle.

$$\delta^* A = \delta^* \int_{t_1}^{t_2} dt \left[R_\mu(t,a)\, \dot{a}^\mu - B(t,a) \right] \qquad (2.7.11)$$

$$= \int_{t_1}^{t_2} \left\{ \left[\left(\frac{\partial R'_\alpha}{\partial a^\mu} - \frac{\partial R'_\mu}{\partial a^\alpha} \right) + T'_{\mu\alpha} \right] \dot{a}^\alpha - \frac{\partial B'}{\partial a^\mu} \right\} \delta a^\mu = 0$$

If noncontemporaneous variations with variable end-points are used, we have the generalized variational principle with end-points contributions [189]

$$\hat{\delta}^* \int_{t_1}^{t_2} \left[R_\mu(t,a)\, da^\mu - B(t,a)\, dt \right] = \left| R_\mu(t,a)\, \hat{\delta}^* a^\mu - B(t,a)\, \hat{\delta}^* t \right|_{t_1}^{t_2} \qquad (2.7.12)$$

The reader will recall that the Birkhoffian particularization of the above principle permitted the construction of the following Birkhoffian generalization of Hamilton-Jacobi equations [189]

$$\frac{\partial A}{\partial t} + B(t,a) = 0 \qquad (2.7.13)$$

$$R_\mu(a) = \frac{\partial A}{\partial a^\mu}$$

under regularity property

$$\det \left(\frac{\partial R_\mu}{\partial a^\nu} \right) (R) \neq 0 \qquad (2.7.14)$$

which can always be verified via the use of gauges (2.6.1) [see ref.[189], Section 6.1 for detail].

Eqs. (2.7.13) are at the basis of Lie-isotopic generalization of Schrödinger's mechanics which will be used in Volume III of this series. The Birkhoffian-admissible principle (2.7.12) will then be used in Volume III for the identification of the more general Lie-admissible operator form.

Note that the gauge degrees of freedom within the context of variational principles imply the transformation

$$A = \int_{t_1}^{t_2} dt \left[R_\mu \dot{a}^\mu - B \right] \Rightarrow \tag{2.7.15}$$

$$\Rightarrow A^{+} = \int_{t_1}^{t_2} dt \left[\left(R_\mu + \frac{\partial G}{\partial a^\mu} \right) \dot{a}^\mu - \left(B - \frac{\partial G}{\partial t} \right) \right]$$

2.8: INVARIANCE OF BIRKHOFF-ADMISSIBLE EQUATIONS UNDER ARBITRARY TRANSFORMATIONS.

In this chapter, we have identified our basic Lie-admissible equations; we have established their direct universality in mechanics and derivability from a variational principle; and finally, we have identified their isotopic and gauge degrees of freedom. The entire analysis has been conducted until now, by specific intent, not only without the transformation theory, but actually within one, single, fixed system of local variables: those of the experimenter.

We study now the behaviour of the theory under the regular (invertible) transformations

$$\begin{aligned} t &\to t' \equiv t \\ a^\mu &\to a'^\mu = a'^\mu(a) \end{aligned} \tag{2.8.1}$$

verifying all needed topological conditions (class C^∞, one-to-one, etc.).

Recall that Hamilton's equations do not preserve their form under arbitrary transformation. This has been the historical reason for restricting the Hamiltonian transformation theory to the subclass of transformations (2.7.1) given by the canonical transformations

$$\omega^{\mu\nu} \to \omega'^{\mu\nu} = \frac{\partial a'^\mu}{\partial a^\alpha} \omega^{\alpha\beta} \frac{\partial a'^\nu}{\partial a^\beta} \equiv \omega^{\mu\nu} \tag{2.8.2}$$

Arbitrary transformations (2.7.1) essentially transform Hamilton's equations into Birkhoff's equations, according to the rule for the autonomous case[189]

$$\omega_{\mu\nu}\dot a^{\nu} - \frac{\partial H(a)}{\partial a^{\mu}} = \frac{\partial a'^{\alpha}}{\partial a^{\mu}}\left\{ \Omega'_{\alpha\beta}(a')\dot a'^{\beta} - \frac{\partial B'(a')}{\partial a'^{\alpha}}\right\} \qquad (2.8.3)$$

o-, equivalently,

$$R^{\circ}_{\mu}(a)\{=(p,0)\} \rightarrow R'_{\mu}(a') = \left(R_{\alpha}\frac{\partial a^{\alpha}}{\partial a'^{\mu}}\right)(a(a')) \qquad (2.8.4)$$
$$H(a) \rightarrow B'(a') = H(a(a'))$$

Once Hamiltonian mechanics is lifted to the covering Birkhoffian mechanics, one reaches the preservation of the structure of the theory under arbitrary transformations. In fact, Birkhoff's equations are the most general possible equations possessing a Lie algebra structure, and this feature is preserved under arbitrary transformations, according to the rules

$$\left[\frac{\partial R_{\nu}(a)}{\partial a^{\mu}} - \frac{\partial R_{\mu}(a)}{\partial a^{\nu}}\right]\dot a^{\nu} - \frac{\partial B(a)}{\partial a^{\mu}} \qquad (2.8.5)$$
$$= \frac{\partial a'^{\alpha}}{\partial a^{\mu}}\left\{\left[\frac{\partial R'_{\beta}(a')}{\partial a'^{\alpha}} - \frac{\partial R'_{\alpha}(a')}{\partial a'^{\beta}}\right]\dot a'^{\beta} - \frac{\partial B'(a')}{\partial a'^{\alpha}}\right\}$$

$$\frac{\partial a^{\alpha}}{\partial a'^{\mu}}\left(\frac{\partial R_{\beta}}{\partial a^{\alpha}} - \frac{\partial R_{\alpha}}{\partial a^{\beta}}\right)\frac{\partial a^{\beta}}{\partial a'^{\nu}} = \frac{\partial R'_{\nu}}{\partial a'^{\mu}} - \frac{\partial R'_{\mu}}{\partial a'^{\nu}}, \qquad (2.8.5b)$$

$$\frac{\partial a'^{\mu}}{\partial a^{\alpha}}\left(\left\|\frac{\partial R_{\sigma}}{\partial a^{\rho}} - \frac{\partial R_{\rho}}{\partial a^{\sigma}}\right\|^{-1}\right)^{\alpha\beta}\frac{\partial a'^{\nu}}{\partial a^{\beta}} = \left(\left\|\frac{\partial R'_{\sigma}}{\partial a'^{\rho}} - \frac{\partial R'_{\rho}}{\partial a'^{\sigma}}\right\|^{-1}\right)^{\mu\nu}. \qquad (2.8.5c)$$

This is a well-known property of Lie algebras (or of the symplectic geometry) according to which the Lie character of the tensor $\Omega^{\mu\nu}$(a) is independent from the selected local coordinates, by therefore setting the basis for coordinate-free globalizations.

It is important to show that an equivalent property holds for the more general Birkhoff-admissible equations. It is at this point that the insufficiency of the Hamilton-admissible equations emerge in their full light. In fact, properties (2.8.3) establish that the fundamental Lie component $\omega^{\mu\nu}$ of the representations is not preserved by arbitrary transformations, but it is turned instead into the Birkhoffian tensor. This lack of preservation of the algebraic structure is removed by the isotopic degrees of freedom (Section 2.5), under which one can assume from the outset a Birkhoff-admissible representation. The preservation of the structure of this representation under arbitrary transformations is then established by the following property.

LEMMA 2.8.1: Under sufficient regularity and continuity conditions, a Lie-admissible tensor $S^{\mu\nu}$(a) remains Lie-admissible under all possible transformations of the local variables $a \rightarrow a'=a'(a)$.

PROOF. Suppose that the rank-two tensor $S^{\mu\nu}(a)$ is regular, in the local sense

$$\det(S^{\mu\nu})(R) \neq 0 \qquad \det(S^{\mu\nu} - S^{\nu\mu})(R) \neq 0 \qquad (2.8.6)$$

and it is Lie-admissible, in the sense of verifying the necessary and sufficient conditions of Theorem 1.5.1, i.e.,

$$(S^{\mu\rho} - S^{\rho\mu}) \frac{\partial}{\partial a^\rho} (S^{\nu\tau} - S^{\tau\nu}) + (S^{\nu\rho} - S^{\rho\nu}) \frac{\partial}{\partial a^\rho} (S^{\tau\mu} - S^{\mu\tau}) + (S^{\tau\rho} - S^{\rho\tau}) \frac{\partial}{\partial a^\rho} (S^{\mu\nu} - S^{\nu\mu}) = 0 \qquad (2.8.7)$$

Then, under all possible transformations which are regular and of the same continuity class of $S^{\mu\nu}$

$$a^\mu \to a'^\mu(a), \qquad \det\left(\frac{\partial a'^\mu}{\partial a^\nu}\right)(R) \neq 0 \qquad (2.8.8)$$

the transformed tensor

$$S'^{\mu\nu}(a') = \frac{\partial a'^\mu}{\partial a^\alpha} S^{\alpha\beta}(a(a')) \frac{\partial a'^\nu}{\partial a^\beta} \qquad (2.8.9)$$

preserves regularity properties (2.8.6) in view of the rules

$$\det(S'^{\mu\nu}) = \det\left(\frac{\partial a'^\mu}{\partial a^\alpha}\right) \det(S^{\alpha\beta}) \det\left(\frac{\partial a'^\nu}{\partial a^\beta}\right) \qquad (2.8.10a)$$

$$\det(S'^{\mu\nu} - S'^{\nu\mu}) = \det\left(\frac{\partial a'^\mu}{\partial a^\alpha}\right) \det(S^{\alpha\beta} - S^{\beta\alpha}) \det\left(\frac{\partial a'^\nu}{\partial a^\alpha}\right) \qquad (2.8.10b)$$

and it is still Lie-admissible, that is, it verifies conditions (2.8.7) in the new coordinate systems on account of the properties

$$\begin{aligned} &\Omega'^{\mu\rho} \frac{\partial \Omega'^{\nu\tau}}{\partial a'^\rho} + \Omega'^{\nu\rho} \frac{\partial \Omega'^{\tau\mu}}{\partial a'^\rho} + \Omega'^{\tau\rho} \frac{\partial \Omega'^{\mu\nu}}{\partial a'^\rho} \\ &= \Big[\frac{\partial a'^\mu}{\partial a^\alpha} \frac{\partial}{\partial a^\beta} \Big(\frac{\partial a'^\nu}{\partial a^\gamma} \frac{\partial a'^\tau}{\partial a^\delta} \Big) + \frac{\partial a'^\nu}{\partial a^\alpha} \frac{\partial}{\partial a^\beta} \Big(\frac{\partial a'^\tau}{\partial a^\gamma} \frac{\partial a'^\mu}{\partial a^\delta} \Big) \\ &\quad + \frac{\partial a'^\tau}{\partial a^\alpha} \frac{\partial}{\partial a^\beta} \Big(\frac{\partial a'^\mu}{\partial a^\gamma} \frac{\partial a'^\nu}{\partial a^\delta} \Big) \Big] \Omega^{\alpha\beta} \Omega^{\gamma\delta} \\ &\quad + \Big(\frac{\partial a'^\mu}{\partial a^\alpha} \frac{\partial a'^\nu}{\partial a^\gamma} \frac{\partial a'^\tau}{\partial a^\delta} + \frac{\partial a'^\nu}{\partial a^\alpha} \frac{\partial a'^\tau}{\partial a^\gamma} \frac{\partial a'^\mu}{\partial a^\delta} + \frac{\partial a'^\tau}{\partial a^\alpha} \frac{\partial a'^\mu}{\partial a^\gamma} \frac{\partial a'^\nu}{\partial a^\delta} \Big) \Omega^{\alpha\beta} \frac{\partial \Omega^{\gamma\delta}}{\partial a^\beta} \equiv 0 \end{aligned} \qquad (2.8.11)$$

$$\Omega'^{\mu\nu} = S'^{\mu\nu} - S'^{\nu\mu},$$

$$\Omega^{\mu\nu} = S^{\mu\nu} - S^{\nu\mu}$$

which are ensured by their original form (2.8.7). Q.E.D.

We reach in this way the following transformation rule of the Birkhoff-admissible equations

$$\left[\left(\frac{\partial R_\nu(a)}{\partial a^\mu} - \frac{\partial R_\mu(a)}{\partial a^\nu}\right) + T_{\mu\nu}(a)\right]\dot{a}^\nu - \frac{\partial B(a)}{\partial a^\mu} \tag{2.8.12a}$$

$$= \frac{\partial a'^\alpha}{\partial a^\mu}\left\{\left[\left(\frac{\partial R'_\beta(a')}{\partial a'^\alpha} - \frac{\partial R'_\alpha(a')}{\partial a'^\beta}\right) + T'_{\alpha\beta}(a')\right]\dot{a}'^\beta - \frac{\partial B'(a')}{\partial a'^\alpha}\right\}$$

$$R'_\alpha(a') = \left(R_\mu \frac{\partial a^\mu}{\partial a'^\alpha}\right)(a(a')) \tag{2.8.12b}$$

$$T'_{\alpha\beta}(a') = \frac{\partial a^\rho}{\partial a'^\alpha} T_{\rho\sigma}(a(a')) \frac{\partial a^\sigma}{\partial a'^\beta} \tag{2.8.12c}$$

$$B'(a') = B(a(a')) \tag{2.8.12d}$$

The property expressed by Lemma 2.7.1 was identified by Santilli (ref.[191], pp. 1316-1321), apparently for the first time within the broader context of the Lie-admissible algebras. Some of its implications are the following.

First, the property implies the possibility of reaching a global, coordinate-free treatment of Lie-admissible algebras, evidently, from the frame independence of their algebraic structure which is exactly equivalent to the corresponding Lie case.

Second, the property permits the ignorance of any essential distinction between Hamilton-admissible and the Birkhoff-admissible equations, trivially, because they are expected to be indistinguishable at the coordinate-free level, in exactly the same way as it occurs for Hamilton's and Birkhoff's equations[189].

Third, and perhaps more importantly for the objective of this work, Lemma 2.8.1 sets for foundations for a possible globalization of the generalization of the Galilei relativity for nonconservative systems to be presented later on, in a way fully parallel, although generalized, to the contemporary globalization of the conventional Galilei relativity for conservative systems.

It should be kept in mind that the conventional property [the frame-independence of the Lie character of Birkhoff's tensor] is a particular case of Lemma 2.8.1, trivially, because Lie tensors are Lie-admissible.

Finally, the contents of this section illustrates the novelty of the approach. In fact, Lie-admissible equations cannot be reached from Lie-type equations (Hamilton or Birkhoff) via the transformations theory.

We pass now to the study of the transformation properties of Birkhoff-admissible equations under the most general possible, non-contemporaneous transformations in time, space, and momenta, which we write in the unified notation

$$(\hat{a}^\mu) = (t, a^\mu) = (t, r, p) \Longrightarrow \tag{2.8.13}$$

$$\Longrightarrow (\hat{a}'^\mu) = (t', a'^\mu) = (t'(t,r,p), r'(t,r,p), p'(t,r,p))$$

$$\mu = 0, 1, 2, \ldots, 6N, \quad \hat{a}^0 = t$$

It should be indicated at this point that the carrier space of Lemma 2.8.1 is the Kronecker product $E_{3N}(\underset{\sim}{r}) \times E_{3N}(\underset{\sim}{p})$ equipped with the tensor $S_{\mu\nu}$ (a) [to be reinterpreted in Chapter 4 as a symplectic-admissible manifold, that is, the cotange bundle T*M with local charts $a = (\underset{\sim}{r}, \underset{\sim}{p})$ equipped with the symplectic-admissible structure $S_{\mu\nu}$ (a)].

The study of transformations (2.7.13) demand the enlargement of the carrier space to the odd-dimensional Kronecker product $E_1(t) \times E_{3N}(\underset{\sim}{r}) \times E_{3N}(\underset{\sim}{p})$ equipped with a suitably selected (6N+1)-dimensional form we shall denote with the symbol $\hat{S}_{\mu\nu}(a)$ [to be identified in Chapter 4 as a contact-admissible manifold, i.e., the space $\mathbb{R} \times T^*M$ with local charts $\hat{a} = (t,a)$ equipped with the structure $\hat{S}^{\mu\nu}(a)$].

The proper study of the transformations under consideration, therefore, demands the prior identification of the tensor $\hat{S}^{\mu\nu}(a)$. With the understanding that the selection is, by far, non-unique, the form suggested is characterized by the following <u>unified formulation of the Birkhoff-admissible equations</u>

$$\hat{S}_{\mu\nu}(\hat{a})\, d\hat{a}^{\nu} = 0 \,, \quad \mu = 0,1,2,\ldots,6N \qquad (2.8.14)$$

where

$$\left(\hat{S}^{\mu\nu}\right) = \left(\begin{array}{c|c} \hat{S}_{00} & \hat{S}_{0\nu} \\ \hline \hat{S}_{\mu 0} & S_{\mu\nu} \end{array}\right) = \left(\begin{array}{c|c} \frac{\partial B}{\partial a^{\alpha}} T^{\alpha\beta} \frac{\partial B}{\partial a^{\beta}} & -\frac{\partial B}{\partial a^{\nu}} \\ \hline \frac{\partial B}{\partial a^{\mu}} & (\Omega_{\mu\nu} + T_{\mu\nu}) \end{array}\right)$$

$$T_{\mu\nu} = \tfrac{1}{2}\left(S_{\mu\nu} + S_{\nu\mu}\right) \neq \left(\|T^{\alpha\beta}\|^{-1}\right)_{\mu\nu} = \left(\|\tfrac{1}{2}(S^{\alpha\beta} + S^{\beta\alpha})\|^{-1}\right)_{\mu\nu} \qquad (2.8.15)$$

It is easy to see that the first term of Eqs. (2.8.14) is an identity, while the last 6N terms reproduce Eqs. (2.5.9) identically.

Structure (2.8.15) has been suggested on mere grounds of being a covering of the corresponding Birkhoffian form (see ref.[189], Section 5.3, Eqs. (5.3.38) in particular), i.e., of being capable of recovering the latter (contact) formulation identically when the symmetric tensors T and T are identically null.

At the risk of being repetitive, it should be indicated that form (2.8.14) is not the only possible unified form of Birkhoff-admissible equations which is a covering of the corresponding unified treatment of Birkhoff's equations in (6N+1)-dimension. In fact, several additional forms are possible owing to the degrees of freedom offered by the symmetric terms (which is absent in the conventional contact case).

After having identified the proper form of the equations, their behaviour under arbitrary transformations (2.8.13) is characterized by the general transformation laws of rank two tensors

$$\hat{S}_{\mu\nu}(\hat{a})\, d\hat{a}^{\nu} = \frac{\partial \hat{a}'^{\alpha}}{\partial \hat{a}^{\mu}} \hat{S}'_{\alpha\beta}(\hat{a}')\, d\hat{a}'^{\beta} = 0 \qquad (2.8.16a)$$

$$\hat{S}'_{\alpha\beta}(\hat{a}') = \left(\frac{\partial \hat{a}^{\rho}}{\partial \hat{a}'^{\alpha}} \hat{S}_{\rho\sigma} \frac{\partial \hat{a}^{\sigma}}{\partial \hat{a}'^{\beta}}\right)(\hat{a}(\hat{a}')) \qquad (2.8.16b)$$

It is easy to see that the Birkhoffian-admissible character of the original equations is preserved. We, therefore, have the following

LEMMA 2.8.2: Under sufficient topological conditions, Birkhoff-admissible equations preserve their Birkhoff-admissible character under the most general possible (nonlinear) transformations of time, space coordinates, and momentum components.

An intriguing consequence of the above results is that the "new time" (i.e., the component of $\hat{a}'$ which can be consistently considered to be the image of $\hat{a}^o = t$) can be any component of $\hat{a}'$ and not necessarily $\hat{a}'^o$. This is due to the fact that, in order to have a consistent geometric structure, "time" must be associated to the comatrix of $(S_{\mu\nu})$, i.e., to the diagonal term $\hat{S}_{oo}$, as correctly done in Eqs. (2.8.14). Now, transformations (2.8.13) ensure that the new form $(\hat{S}'_{\mu\nu})$ in (6N+1)-dimension does contain a 6N x 6N submatrix having the same geometric-algebraic properties of $(S_{\mu\nu})$. The point is that this image can occur via any 6N x 6N submatrix of $(\hat{S}'_{\mu\nu})$, and not necessarily as the comatrix of $\hat{S}'_{oo}$. For a detailed analysis of this occurrence (which is present already at the level of Birkhoffian mechanics), the interested reader may consult ref.[189], Section 5.3, in particular Lemma 5.3.3 and its corollaries).

Stated explicitly, our findings imply that, when the restriction of the analytic treatment to conservative/Hamiltonian forms is lifted, and more general structures are admitted, space and time coordinates achieve equivalence already in Newtonian mechanics, and prior to any relativistic extensions.

In addition, one can see that the type of space-time equivalence identified here is considerably more general than that of special relativity.

This is a first illustration of the objectives stated in the Preface, to the effect that the Newtonian generalization of Galilei's relativity we are seeking will inevitably imply, for consistency, a corresponding generalization of Einstein's special relativity.

2.9: SYMMETRIES AND NONCONSERVATION LAWS

The analysis conducted until now has been conceived to reach a direct and self-evident generalization of Hamiltonian/Birkhoffian settings. The study of the problem of symmetries of nonconservative systems demands instead a rather radical departure from traditional patterns of the contemporary literature. In fact, on one side:

- The historical motivation for the introduction of symmetries in physics, which is still in force today, is the representation of conservation law; while, on the other side,
- A central objective of our studies is the use of symmetries for the representation of nonconservation laws (time rates of variation of physical quantities).

To put it differently, recall that Galilei's relativity essentially reduces the familiar, ten, conservation laws of total physical quantities to a ten-parameter Lie symmetry group (Galilei's group). Since we are interested in constructing a relativity for nonconservative systems, the preservation of the same mental attitude would inevitably lead to inconsistencies (e.g., a possible distorsion of nonconservative systems into a form treatable by methods and insights of conservative mechanics). Instead, the research attitude which appears recommendable is to insist in the preservation of the physical structure of the systems, that is, their nonconservative character, and search for a symmetry approach capable of its treatment.

At a deeper look, the novel way of studying symmetries we are

referring here also results to be a covering of the conventional way. In fact, conservation laws are a particular case of nonconservation laws occurring for null time rate of variations. As a result, the representation of the time rate of variations via symmetries will be automatically a covering of the conventional use for conservation laws.

Our studies will be presented according to the following steps. In this section, we shall identify the class of symmetries which is suitable for our program, and point out other types of symmetries which, even though mathematically conceivable, are potentially misleading for nonconservative systems, unless properly treated.

In the next chapter, we shall then identify the group structure of the symmetries considered.

The characterization of the time rate of variations will be done in the last chapter jointly with the formulation of the proposed generalized relativity.

It is recommendable to begin with a very brief outline of the conventional approaches to symmetries.

SYMMETRIES OF HAMILTONIAN EQUATIONS. Consider a conservative Newtonian system in first-order form

$$\left(\dot{a}^\mu\right) = \begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p/m \\ f^{SA}(r) \end{pmatrix} \overset{\text{def}}{=} \left(\Gamma'^\mu(a)\right), \quad r \in \mathbb{R}_n \qquad (2.9.1)$$

which is manifestly Hamiltonian, i.e.,

$$\Gamma^{\mu}(a) \equiv \omega^{\mu\nu}\frac{\partial H(a)}{\partial a^{\nu}} \qquad (2.9.2)$$

As is well-known, a reason for the restriction of the transformation theory to canonical transformation (2.8.1),

$$\omega^{\mu\nu} \rightarrow \omega'^{\mu\nu} = \frac{\partial a'^{\alpha}}{\partial a^{\mu}}\omega_{\alpha\beta}\frac{\partial a^{\beta}}{\partial a^{\nu}} \equiv \omega^{\mu\nu} \qquad (2.9.3)$$

is deeply linked to symmetries and conservation laws. In fact, once the class of transformation admitted preserves the values of the Lie tensor, the symmetries are simply characterized by the subclass of canonical transformations which preserve the Hamiltonian.

$$H(a) \rightarrow H'(a') = H(a(a')) \equiv H(a') \qquad (2.9.4)$$

We reach in this way the following definition of <u>symmetries of Hamilton's equations</u>, as the transformations which first, are canoni-

ical and, second, leave the Hamiltonian form-invariant

$$\begin{aligned} &\omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial H(a)}{\partial a^{\mu}} \\ &= \frac{\partial a'^{\alpha}}{\partial a^{\mu}}\left[\omega'_{\alpha\beta}\dot{a}'^{\beta} - \frac{\partial H'(a')}{\partial a'^{\alpha}}\right] \\ &\equiv \frac{\partial a'^{\alpha}}{\partial a^{\mu}}\left[\omega_{\alpha\beta}\dot{a}'^{\beta} - \frac{\partial H'(a')}{\partial a'^{\alpha}}\right] \\ &\equiv \frac{\partial a'^{\alpha}}{\partial a^{\mu}}\left[\omega_{\alpha\beta}\dot{a}'^{\beta} - \frac{\partial H(a')}{\partial a'^{\alpha}}\right] \end{aligned} \qquad (2.9.5)$$

Evidently, the form-invariance of Hamilton's equations implies that of the equations represented

$$\begin{aligned} \Gamma^{\mu}(a) \longrightarrow \Gamma'^{\mu}(a') &= \left(\Gamma^{\alpha}\frac{\partial a'^{\mu}}{\partial a^{\alpha}}\right)(a(a')) \\ &\equiv \Gamma^{\mu}(a') \end{aligned} \qquad (2.9.6)$$

Under the condition that the transformations constitute an n-parameter continuous, Lie, transformation group, Noether's theorem applies, yielding the conservation law of n physical quantities, i.e.,

$$\dot{X}_k(a) = \frac{\partial X_k}{\partial a^{\mu}}\Gamma^{\mu} \equiv 0 \qquad (2.9.7)$$

$$k = 1, 2, \ldots, n$$

SYMMETRIES OF BIRKHOFFIAN MECHANICS. Consider now the generalization of closed conservative (self-adjoint) systems (2.9.1) into a nonconservative (non-self-adjoint) form

$$(\dot{a}^\mu) = \begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p/m \\ f^{SA}(r) + F^{NSA}(r,p) \end{pmatrix} \stackrel{def}{=} \left(\tilde{\Gamma}^\mu(a)\right) \quad (2.9.8)$$

which is still closed, i.e., which verifies the properties (Volume I, Appendix 1.C)

$$\sum_k p_k \cdot F_k^{NSA} = 0 \, , \quad (2.9.9a)$$

$$\sum_k F_k^{NSA} \equiv 0 \, , \quad (2.9.9b)$$

$$\sum_k r_k \times F_k \equiv 0 \, , \quad (2.9.9c)$$

The system is now no longer Hamiltonian in the local variables considered. However, it is always Birkhoffian, i.e.,

$$\tilde{\Gamma}^\mu(a) \equiv \Omega^{\mu\nu}(a) \frac{\partial B(a)}{\partial a^\nu} \quad (2.9.10)$$

As indicated in the preface, the Birkhoffian mechanics has been presented in ref.[189] for the primary objective of permitting the study of systems which are structurally more general than the

Hamiltonian ones, yet are still closed. The closed character, therefore, preserves the null time rate of variations of physical quantities. In turn, this implies a preservation of the conceptual lines of Hamiltonian symmetries, although expressed in the most general possible form.

Owing to this occurrence, the transformations of Birkhoff's equations which have resulted to be particularly relevant are the generalized canonical transformations

$$\Omega_{\mu\nu}(a) \to \Omega'_{\mu\nu}(a') = \frac{\partial a^\alpha}{\partial a'^\mu} \Omega_{\alpha\beta} \frac{\partial a^\beta}{\partial a'^\nu} \equiv \Omega_{\mu\nu}(a') \quad (2.9.11)$$

The symmetries of Birkhoff's equations (for the authonomous case) are then given by the subclass of generalized canonical transformations which leave the Birkhoffian form-invariant, i.e.,

$$\begin{aligned} &\Omega_{\mu\nu}(a)\dot{a}^\nu - \frac{\partial B(a)}{\partial a^\mu} \\ &= \frac{\partial a'^\alpha}{\partial a^\mu}\left[\Omega'_{\alpha\beta}(a')\dot{a}'^\beta - \frac{\partial B'(a')}{\partial a'^\alpha}\right] \\ &\equiv \frac{\partial a'^\alpha}{\partial a^\mu}\left[\Omega_{\alpha\beta}(a')\dot{a}'^\beta - \frac{\partial B'(a')}{\partial a'^\alpha}\right] \\ &\equiv \frac{\partial a'^\alpha}{\partial a^\mu}\left[\Omega_{\alpha\beta}(a')\dot{a}'^\beta - \frac{\partial B(a')}{\partial a'^\alpha}\right] \end{aligned} \quad (2.9.12)$$

The form-invariance of the systems is then consequential,

$$\tilde{\Gamma}^{\mu}(a) \Longrightarrow \tilde{\Gamma}'^{\mu}(a') = \left(\tilde{\Gamma}^{\alpha}\frac{\partial a'^{\mu}}{\partial a^{\alpha}}\right)(a(a')) \equiv \tilde{\Gamma}^{\mu}(a') \qquad (2.9.13)$$

As shown in ref.[189] (section 6.3 in particular), when the symmetries are characterized by an n-parameter continuous group of transformations, there exist n conserved quantities in a way fully similar to case (2.9.7). However, the groups resulted to be more general than the canonical ones, and were called Lie-isotopic groups (Chapter 3).

What is important for our analysis here is that, apart technical generalizations dictated by the non-Hamiltonian character of the systems, the conceptual structure of symmetries and conservation laws of Hamiltonian mechanics is preserved in the transition to the covering Birkhoffian mechanics.

SYMMETRIES OF HAMILTON-ADMISSIBLE EQUATIONS. In the transition to the Hamilton-admissible equations, the above pattern is profoundly altered.

Consider an open, nonsonservative, first-order system [e.g., a subset of systems (2.9.8)]

$$\left(\Xi^{\mu}(a)\right) = \begin{pmatrix} p/m \\ f^{SA}(r) + F^{NSA}(r,p) \end{pmatrix} \qquad (2.9.14)$$

and let $E(\underset{\sim}{r},\underset{\sim}{p}) = E(a)$ be the nonconserved energy. Suppose that a transformation $a \rightarrow a'(a)$ is a symmetry,

$$\Xi^{\mu}(a) \Longrightarrow \Xi'^{\mu}(a') = \left(\Xi^{\alpha}\frac{\partial a'^{\mu}}{\partial a^{\alpha}}\right)(a(a')) \equiv \Xi^{\mu}(a') \qquad (2.9.15)$$

From universality theorem 2.4.1, we know that the system is Hamiltonian-admissible,

$$\begin{aligned} \Xi^{\mu} &\equiv S^{\mu\nu}(a)\frac{\partial H(a)}{\partial a^{\nu}} \\ &= (\omega^{\mu\nu} + t^{\mu\nu})\frac{\partial H}{\partial a^{\nu}} \end{aligned} \qquad (2.9.16)$$

$$H = E = T(\underset{\sim}{p}) + \quad (\underset{\sim}{r})$$

Now, the energy E is not conserved by assumption. Thus, a necessary condition for the symmetry to be physically acceptable is that it DOES NOT leave the Hamiltonian form-invariant, i.e.,

$$H(a) \Longrightarrow H'(a') = H(a(a')) \neq H(a') \qquad (2.9.17)$$

In turn, this necessarily implies the FORM NON-INVARIANCE of the

Lie-admissible tensor

$$S^{\mu\nu}(a) \Rightarrow S'^{\mu\nu}(a') = \frac{\partial a'^{\mu}}{\partial a^{\alpha}} S^{\alpha\beta} \frac{\partial a'^{\nu}}{\partial a^{\beta}} \neq S^{\mu\nu}(a') \tag{2.9.18}$$

But the transformation is a symmetry of the equations of motion by assumption. Therefore, the form-noninvariances of the Hamiltonian and of the algebraic tensor must be such to"conpensate each other", according to the general rules

$$\Xi^{\mu}(a) = S^{\mu\nu}(a) \frac{\partial H(a)}{\partial a^{\nu}} \Rightarrow$$

$$\Rightarrow S'^{\mu\nu}(a') \frac{\partial H'(a')}{\partial a'^{\nu}} = \Xi^{\mu}(a') \tag{2.9.19}$$

The departures from the conventional Hamiltonian settings are now eviden . In fact, in the Hamiltonian case, a necessary condition of consistency is the form invariance of the Hamiltonian (the form invariance of the Lie tensor is then a consequence) while in the covering Hamilton-admissible case, a necessary condition of consistency is the LACK of form-invariance of the Hamiltonian (the form noninvariance of the Lie-admissible tensor is then also a consequence).

Owing to the novelty of the occurrence, it is important to review

here an example worked out by Santilli (ref.[190], p. 1679).

Consider the nonconservative particle with quadratic (nonlinear) damping in one-dimension

$$m\ddot{r} + \gamma \dot{r}^{2} = 0, \quad m = 1 \tag{2.9.20}$$

which can be written in first-order form

$$(\Xi^{\mu}) = \begin{pmatrix} p \\ -\gamma p^{2} \end{pmatrix}, \quad p = m\dot{r} = \dot{r} \tag{2.9.21}$$

Its Hamilton-admissible representation is then given by (Appendis 2.1)

$$(S^{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ -1 & \gamma p \end{pmatrix}, \quad H = \frac{1}{2} p^{2} = \text{Kin. En.} \tag{2.9.22}$$

Consider now the transformations

$$(a') = \begin{pmatrix} r' \\ p' \end{pmatrix} = \begin{pmatrix} r + \frac{1}{\gamma} \ln(1 + \gamma t' p) \\ p/(1 + \gamma t' p) \end{pmatrix} \tag{2.9.23}$$

It is easy to see that they constitute a symmetry of the equations of motion, owing to the verification of the form-invariance condition

$$(\dot{a}'^{\mu}) = \begin{pmatrix} \frac{dr'}{dt'} \\ \frac{dp'}{dt'} \end{pmatrix} = \begin{pmatrix} p' \\ -\gamma p'^2 \end{pmatrix} = \left(\Xi^{\mu}(a') \right) \quad (2.9.24)$$

Nevertheless, it is equally easy to see that transformations (2.9.23) do not leave the Hamiltonian form-invariant

$$H(p) = H(p(p')) = \frac{p'^2}{2(1+\gamma t' p)^2} \neq \frac{1}{2} p'^2 = H(p') \quad (2.9.25)$$

It is then predictable that, for consistency, the Lie-admissible tensor must not be preserved. And in fact, we have

$$\left(S'^{\mu\nu}(a') \right) = \begin{pmatrix} \left(\frac{\gamma t'^2 p}{(1+\gamma t' p)} \right) & \left(\frac{1}{(1+\gamma t' p)^2} + \frac{\gamma t' p}{(1+\gamma t p)^3} \right) \\ \left(\frac{-1}{(1+\gamma t' p)^2} + \frac{\gamma t' p}{(1+\gamma t' p)^3} \right) & \left(\frac{\gamma p}{(1+\gamma t' p)^4} \right) \end{pmatrix} \quad (2.9.26)$$

Yet, the two form-noninvariances (2.9.25) and (2.9.26) are such to

permit the form-invariance of the underlying vector field, property (2.9.23).

The example above is, by far, non-accidental. In fact, we shall learn later on that transformations (2.9.22) characterize one of the most important dynamical symmetries, the time evolution of the system. We shall also learn that the transformations actually possess a group structure more general than the Lie-isotopic one.

At this time, we limit ourselves to the remarks that the example above establishes the following property.

LEMMA 2.9.1: Suppose that an m-dimensional system of first-order differential equations on a manifold M(x) with local coordinates $x \in R_m$ admits the representation in terms of a suitable tensor $X^{\mu\nu}(x)$ and a function $Y(x)$,

$$\dot{x}^{\mu} = \Xi^{\mu}(x) = X^{\mu\nu}(x) \frac{\partial Y(x)}{\partial x^{\nu}} \quad (2.9.27)$$

Then, the class of all possible symmetry transformations of the quantities $X^{\mu\nu}$ and Y

$$x \rightarrow x' = x'(x) \quad (2.9.28a)$$

$$X^{\mu\nu}(x) \rightarrow X'^{\mu\nu}(x') \equiv X^{\mu\nu}(x') \quad (2.9.28b)$$

$$Y(x) \rightarrow Y'(x') \equiv Y(x) \quad (2.9.28c)$$

does not necessarily exhaust the class of all possible symmetries of the system

$$\Xi^{\mu}(x) \to \Xi'(x') \equiv \Xi^{\mu}(x') \qquad (2.9.29)$$

Note that the property above holds for all possible systems, whether Hamiltonian or not. The emphasis, however, depends on each case considered. When the system is Hamiltonian (conservative) the class of all possible symmetries of the tensor $X^{\mu\nu} = \omega^{\mu\nu}$ and of the function $Y = H$ turn out to be the physically most important one, with a straightforward generalization to the (closed) Birkhoffian case. However, when the system is nonconservative (non-Hamiltonian), the physically most important class of symmetries result to be that which does not leave form-invariant the algebraic tensor $X^{\mu\nu} = S^{\mu\nu}$ and the energy $Y = H$. In fact, this is the case for the time-evolution of the system as well as other basic transformations, as we shall see.

The following definition is now self-evident.

DEFINITION 2.9.1: Let

$$\begin{aligned}\hat{\Xi} &= \hat{\Xi}^{\mu}\frac{\partial}{\partial \hat{a}^{\mu}} \\ &= \Xi^{\mu}\frac{\partial}{\partial a^{\mu}} + \frac{\partial}{\partial t} \\ &= \Xi^{ka}(r,p)\frac{\partial}{\partial r^{ka}} + \Xi_{ka}(r,p)\frac{\partial}{\partial p_{ka}} + \frac{\partial}{\partial t}\end{aligned} \qquad (2.9.30)$$

be a Newtonian vector field in unified notation, represented via Birkhoff-admissible equations (2.7.14)

$$\hat{S}_{\mu\nu}(\hat{a})\, d\hat{a}^{\nu} = 0, \quad \mu = 1, 2, \ldots, 6N \qquad (2.9.31)$$

or their Hamilton-admissible particularization. The (smoothness and regularity preserving) transformations

$$\hat{a}^{\mu} \to \hat{a}'^{\mu} = \hat{a}'^{\mu}(\hat{a}) \qquad (2.9.32)$$

are called symmetries when they leave form-invariant the vector field

$$\begin{aligned}\hat{\Xi}(\hat{a}) &= \hat{\Xi}^{\mu}(\hat{a})\frac{\partial}{\partial \hat{a}^{\mu}} \\ = \hat{\Xi}'(\hat{a}') &= \hat{\Xi}^{\mu}(\hat{a}(\hat{a}'))\frac{\partial \hat{a}'^{\alpha}}{\partial \hat{a}^{\alpha}}\frac{\partial}{\partial \hat{a}'^{\alpha}} = \hat{\Xi}'^{\alpha}(\hat{a}')\frac{\partial}{\partial \hat{a}'^{\alpha}} \\ \equiv \hat{\Xi}(\hat{a}') &= \hat{\Xi}^{\alpha}(\hat{a}')\frac{\partial}{\partial \hat{a}'^{\alpha}}\end{aligned} \qquad (2.9.33)$$

but not necessarily the tensor $\hat{S}^{\mu\nu}(a)$, i.e.,

$$\hat{S}^{\mu\nu}(\hat{a}) \to \hat{S}'^{\mu\nu}(\hat{a}') = \frac{\partial \hat{a}'^{\mu}}{\partial \hat{a}^{\alpha}}\hat{S}^{\alpha\beta}\frac{\partial \hat{a}'^{\nu}}{\partial \hat{a}^{\beta}} \neq \hat{S}^{\mu\nu}(\hat{a}') \qquad (2.9.34)$$

A symmetry will be called:

- Contemporaneous or noncontemporaneous, depending on whether it does not or it does imply transformation of time;
- Manifest of nonmanifest, depending on whether it can or it cannot be identified via simple means (usually, a visual inspection);
- Discrete or continuous, depending on whether it constitutes an inversion of space-time coordinates, or it can be continuously connected to the identity transformation;
- Of pure space, time, or space-time character, depending on whether it implies transformations of space only, of time only, or of mixed space-time variables.

The reader should keep in mind the result of Section 2.8, to the effect that the "new time" t' of Eqs. (2.9.32) is not necessarily the image of the physical time t under the transformations, but it can be any component of the transformed variables a'.

APPENDIX 2.A: EXAMPLES OF HAMILTON-ADMISSIBLE AND OF BIRKHOFF-ADMISSIBLE REPRESENTATIONS OF NONCONSERVATIVE NEWTONIAN SYSTEMS

For the reader's convenience, we illustrate in this Appeddix the "mechanics" of the representation of nonconservative, generally non-Hamiltonian Newtonian systems via Hamilton-admissible and Birkhoff-admissible equations.

The case of Hamilton-admissible equations is trivial. In fact, Theorem 2.4.1 of direct universality provides the explicit solution of the structure of the algebraic tensor and of the Hamiltonian for all possible systems of the class admitted.

To illustrate this point, consider the case, for simplicity, when the potential force if null and we have only one space dimension. Then, for the linear velocity dependent drag force, we have the Newtonian vector field

$$(\dot{a}^\mu) = \begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} = (\Xi^\mu) = \begin{pmatrix} p \\ -\gamma p \end{pmatrix}, \quad m = 1 \qquad (2.A.1)$$

Its Hamilton-admissible representation is trivially given by

$$(R^\circ_\mu) = (p, 0), \quad H = \tfrac{1}{2} p^2, \quad (t^{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & -\gamma \end{pmatrix} \qquad (2.A.2)$$

or, more explicitly

$$(\dot{a}^{\mu}) = \begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p \\ -\gamma p \end{pmatrix} \qquad (2\text{-A}.3)$$

$$= \left[(\omega^{\mu\nu}) + (t^{\mu\nu}) \right] \begin{pmatrix} \dfrac{\partial H}{\partial a^{\nu}} \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\gamma \end{pmatrix} \right] \begin{pmatrix} \dfrac{\partial H}{\partial p} \\ 0 \end{pmatrix}$$

If the drag force is quadratic in the velocity, we have the vector field

$$(\Xi^{\mu}) = \begin{pmatrix} p \\ -\gamma p^2 \end{pmatrix} \qquad (2\text{-A}.4)$$

The Hamilton-admissible representation is then characterized by

$$(R^{0}_{\mu}) = (p, 0), \quad H = \frac{1}{2} p^2, \quad (t^{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & -\gamma p \end{pmatrix} \qquad (2\text{-A}.5)$$

Similarly, for an arbitrary nonpotential force, we have the representation

$$(R^{0}_{\mu}) = (p, 0), \quad H = \frac{1}{2} p^2, \quad (t^{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & F^{NSA}/p \end{pmatrix} \qquad (2\text{-A}.6)$$

which is clearly a particular case of that of Theorem 2.4.1 for $f^{SA} = 0$. The generalization to the case with non-null potential forces is trivially given by the addition of the potential function in H.

A comparative analysis with the conventional Hamiltonian theory is instructive, particularly for our quantum mechanical objectives of Volume III. The existence theory of the Inverse Problem[65,189] establishes that, under sufficient topological conditions, all systems in one space-dimension admit a Lagrangian or a Hamiltonian (this property fails to hold beginning with two space-dimensions, as proved by Davis in 1948). Since all systems considered here have one-space dimension, they can all be represented via the conventional Hamilton's equations. However, necessary conditions for the existence of the representation are that:

(a) the canonical momentum p_{can} DOES NOT represent the physical linear momentum $p_{phys} = m\dot{r}$; and

(b) the canonical Hamiltonina H_{can} DOES NOT represent the physical total energy $H = E_{phys.}$;

(with an additional condition for the angular momentum when the systems have more than one space dimension).

For instance, a conventional Hamiltonian representation of systems (2.A.1) is given by

$$(R^{0}_{\mu}) = (p, 0), \quad H_{can} = e^{\gamma t} \frac{1}{2} p^2_{can}, \quad p_{can} \neq p_{phys.} \qquad (2\text{-A}.7)$$

in which one can see an illustration of conditions (a) and (b) above.

By comparison, the mathematical algorithms "p" and "H" of Eqs. (2.A.2) represent the physical linear momentum and the energy.

The reasons for the insistence in mathematical symbols of direct physical meaning will be evident in Volume III, when we shall show that fundamental physical laws, principles, and insights of quantum mechanics for conservative forces need a necessary generalization for nonconservative forces, provided, again, that the operators at hand have a direct physical meaning.

We pass now to the illustration of Birkhoff-admissible representations of Section 2.5. Consider the one-dimensional damped oscillator

$$\left[(\ddot{r} + r)_{SA} + \gamma \dot{r} \right]_{NSA} = 0 , \quad m = 1, \ k = 1 \qquad (2.A.8)$$

in the first-order form

$$(\Xi^{\mu}) = \begin{pmatrix} p \\ -r - \gamma p \end{pmatrix} \qquad (2.A.9)$$

Then, a Birkhoff-admissible representation is provided by the functions

$$(R_{\mu}) = (e^{\gamma t} p, 0), \quad B = \frac{1}{2}(p^2 + r^2)e^{\gamma t},$$
$$(T^{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & -\gamma e^{-\gamma t} \end{pmatrix} \qquad (2.A.10)$$

In fact, for the contravariant form, we have the equations

$$(\dot{a}^{\mu}) = \begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p \\ -r - \gamma p \end{pmatrix} \qquad (2.A.11)$$
$$= (\Omega^{\mu\nu} + T^{\mu\nu})\left(\frac{\partial B}{\partial a^{\nu}}\right) = \left[\begin{pmatrix} 0 & e^{-\gamma t} \\ -e^{-\gamma t} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\gamma e^{-\gamma t} \end{pmatrix} \right] \begin{pmatrix} e^{\gamma t} r \\ e^{\gamma t} p \end{pmatrix}$$

with underlying use of rule (2.5.12). For the covariant form we have the equations

$$(\Omega_{\mu\nu} + T_{\mu\nu})(\dot{a}^{\nu}) - \left(\frac{\partial B}{\partial a^{\mu}}\right) \qquad (2.A.12)$$
$$= \left[\begin{pmatrix} 0 & -e^{\gamma t} \\ e^{\gamma t} & 0 \end{pmatrix} + \begin{pmatrix} -\gamma e^{\gamma t} & 0 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} - \begin{pmatrix} e^{\gamma t} r \\ e^{\gamma t} p \end{pmatrix} = e^{\gamma t} \begin{pmatrix} -\dot{p} - \gamma p \\ \dot{r} - p \end{pmatrix} = 0$$

where now the underlying rule is given (2.5.15).

Note the verification of all regularity properties of Eqs. (2.5.9). Note also that, for null value of the damping force (i.e., for $\gamma = 0$) the representation recovers the conventional Hamiltonian one.

The interested reader can construct additional representations, e.g., by converting the Birkhoffian representations of monograph[189] (examples at the end of each chapter) into the Birkhoff-admissible ones.

APPENDIX 2.B: AN EXAMPLE OF DEGENERATE LIE-ADMISSIBLE EQUATIONS

All analytic equations with a Lie-admissible structure considered in the main text of this volume are regular, i.e., the Lie-admissible tensor $S^{\mu\nu}$ verifies the regularity conditions

$$\det (S^{\mu\nu})(a) \neq 0 \qquad (2.B.1)$$

(besides other regularity properties).

In ref.[192], I showed the existence of Lie-admissible equations of degenerate nature, i.e., such that

$$\det (S^{\mu\nu})(a) \equiv 0 \qquad (2.B.2)$$

A review of the analysis is the following. Suppose that a nonconservative system is assigned in the $(a) = (r,p)$ coordinates and performs the (calss C^∞, regular) transformations

$$r_k \rightarrow r'_k \equiv r_k, \quad p_k \rightarrow w_k = w_k(t, r, p) \qquad (2.B.3)$$

The problem under consideration is that of constructing a representation of the system considered in the new coordinates (r,w), Hamiltonian

$\bar{H} = \bar{H}(t,r,w) - H(r,p(t,r,w))$, and generalized equations

$$\begin{cases} \dot{r}^{ia} = A^{ia\,jb}\dfrac{\partial \bar{H}}{\partial r^{jb}} + B^{ia}{}_{jb}\dfrac{\partial \bar{H}}{\partial w_{jb}} \\ \dot{p}_{ia} = C_{ia}{}^{jb}\dfrac{\partial \bar{H}}{\partial r^{jb}} + D_{ia\,jb}\dfrac{\partial \bar{H}}{\partial w_{jb}} \end{cases} \qquad (2.B.4)$$

under the conditions that the brackets of the time evolution law

$$\dot{\bar{A}} \stackrel{\text{def}}{=} (\bar{A}, \bar{H})^*_{(r,w)}$$

$$= \frac{\partial \bar{A}}{\partial r^{ia}} A^{ia\,jb}\frac{\partial \bar{H}}{\partial r^{jb}} + \frac{\partial \bar{A}}{\partial r^{ia}} B^{ia}{}_{jb}\frac{\partial \bar{H}}{\partial w_{jb}} \qquad (2.B.5)$$

$$+ \frac{\partial \bar{A}}{\partial p_{ia}} C_{ia}{}^{jb}\frac{\partial \bar{H}}{\partial r^{jb}} + \frac{\partial \bar{A}}{\partial p_{ia}} D_{ia\,jb}\frac{\partial \bar{H}}{\partial w_{jb}}$$

are Lie-admissible and singular.

A solution of the problem is given by the following generalized equations (where upper bar denotes computation in the $(\underset{\sim}{r},\underset{\sim}{w})$-variables)

$$\dot{\bar{a}}^{\mu} = \bar{S}^{\mu\nu}(\bar{a})\frac{\partial \bar{H}}{\partial \bar{a}^{\nu}} \qquad (2.B.6a)$$

$$\begin{pmatrix} \dot{z}^{ka} \\ \\ \dot{w}_{ka} \end{pmatrix} = \begin{pmatrix} \overline{\dfrac{\partial w_{jb}}{\partial p_{ia}}}\,\dfrac{\partial \bar{H}}{\partial w_{jb}} \\ \\ -\dfrac{\partial w_{ia}}{\partial p_{ck}}\dfrac{\partial w_{jb}}{\partial z^{ck}} \end{pmatrix} \qquad (2.B.6b)$$

which where called singular Lie-admissible generalization of Hamilton's equations. The singular character of these equations is selfevident because the matrix of their characteristic tensor is singular,

$$(\bar{S}^{\mu\nu}) = \begin{pmatrix} 0 & \overline{\dfrac{\partial w_{ib}}{\partial p_{ia}}} \\ \\ 0 & -\overline{\dfrac{\partial w_{ia}}{\partial p_{ck}}\dfrac{\partial w_{ib}}{\partial z^{ck}}} \end{pmatrix} \qquad (2.B.7)$$

The Lie-admissible character can be proved in the following way. The generalized brackets are now given by

$$\dot{\bar{A}} = (\bar{A},\bar{H})^{*}_{(\underset{\sim}{z},\underset{\sim}{w})} = \frac{\partial \bar{A}}{\partial z^{ia}}\overline{\frac{\partial w_{jb}}{\partial p_{ia}}}\frac{\partial \bar{H}}{\partial w_{jb}} - \frac{\partial \bar{A}}{\partial w_{ia}}\overline{\frac{\partial w_{ia}}{\partial p_{kc}}\frac{\partial w_{jb}}{\partial z^{kc}}}\frac{\partial \bar{H}}{\partial w_{jb}} \qquad (2.B.8)$$

The Lie character of the attached brackets can then be seen by performing an inverse transform to the original variables, i.e.,

$$(\bar{A},\bar{H})^{*}_{(\underset{\sim}{z},\underset{\sim}{w})} - (\bar{H},\bar{A})^{*}_{(\underset{\sim}{z},\underset{\sim}{w})}$$

$$= \frac{\partial \bar{A}}{\partial z^{ia}}\overline{\frac{\partial w_{jb}}{\partial p_{ia}}}\frac{\partial \bar{H}}{\partial w_{jb}} - \frac{\partial \bar{A}}{\partial w_{ia}}\overline{\frac{\partial w_{ia}}{\partial p_{jb}}}\frac{\partial \bar{H}}{\partial z^{jb}}$$

$$-\frac{\partial \bar{A}}{\partial w_{ia}}\left(\overline{\frac{\partial w_{ia}}{\partial p_{kc}}\frac{\partial w_{jb}}{\partial z^{kc}}} - \overline{\frac{\partial w_{ia}}{\partial z^{kc}}\frac{\partial w_{jb}}{\partial p_{kc}}}\right)\frac{\partial \bar{H}}{\partial w_{jb}} \qquad (2.B.9)$$

$$= \overline{\frac{\partial \bar{A}}{\partial \bar{a}}}\left(\frac{\partial \bar{a}}{\partial a}\right)(\omega^{\mu\nu})\left(\frac{\partial \bar{a}}{\partial a}\right)^{T}\left(\frac{\partial \bar{H}}{\partial \bar{a}}\right)$$

$$\equiv \frac{\partial A}{\partial z^{ka}}\frac{\partial H}{\partial p_{ka}} - \frac{\partial A}{\partial p_{ka}}\frac{\partial H}{\partial z^{ka}} = [A,H]_{(\underset{\sim}{z},\underset{\sim}{p})}$$

namely, the attached brackets coincide with the conventional Poisson brackets.

As a simple example, the nonconservative system

$$m\ddot{r} + \gamma r \dot{r}^2 = 0 \qquad (2.B.10)$$

can be represented with Esq. (2.B.6) via the functions

$$\bar{H} = \frac{1}{2} W^2 , \quad W = \alpha p r \qquad (2.B.11)$$
$$p = m\dot{r} , \quad m^2 \alpha^2 (1+\alpha) = \gamma$$

Almost needless to say, regular Hamilton-admissible equations are preferable over the singular form (2.B.6) for numerous reasons, such as the fact that these latter equations lose a consistent geometrical backing.

Nevertheless, singular equations (2.B.6) are useful to illustrate the variety of realizations of Lie-admissible algebras which are possible in Newtonian Mechanics.

APPENDIX 2C: THE DUAL LIE-ISOTOPIC/LIE-ADMISSIBLE TREATMENT OF ELECTROMAGNETIC AND STRONG INTERACTIONS.

A primary objective of these monographs is to attempt a representation of the strong interactions which is structurally more general than contemporary approaches, in the hope of achieveing a representation of hadrons and of their constitutents as extended charge distributions under conditions of mutual penetration.

In this appendix, we shall recall the classical form of the electromagnetic and strong interactions of Volume I. We shall then identify the analytic representations permitted by the advances of this volume. The particle version will be studied in Volume III as an operator image of the classical treatment.

Recall from Volume I that non-Hamiltonian systems considered in the main text of this volume

$$m\ddot{r}_{ka} - f^{SA}_{ka}(\underline{r}, \dot{\underline{r}}) - F^{NSA}_{ka}(t, \underline{r}, \dot{\underline{r}}) = 0 \qquad (2.C.1)$$
$$k = 1, 2, \ldots, N; \quad a = x, y, z;$$

are assumed as the Newtonian limit of the electromagnetic and strong interactions according to the following specifications:

(I) The variationally self-adjoint forces are assumed as representing the Lorents force $F_{Lor.}$ plus any potential term of conventional forms of strong interactions that persists at the classical limit under the correspondence principle; and

(II) The variationally non-self-adjoint forces are interpreted as corrections to the self-adjoint terms due to the extended character of the principle under short range interactions.

$$m \ddot{r}_{ka} - F^{SA}_{LORENTZ}(t, r, \dot{r}) - F^{SA}_{strong}(t, r, \dot{r}) - F^{NSA}_{strong}(t, r, \dot{r}) = 0 \qquad (2.C.2)$$

Thus, the terms F^{NSA}_{strong} should be initially conceived as being small when compared to the other terms, e.g., as admitting the factorization

$$F^{NSA}_{strong} \cong \varepsilon Z^{NSA}, \quad \varepsilon \approx 0 \qquad (2.C.3)$$

although the case of finite values should be kept in mind for further developments.

In our analysis, non-Hamiltonian systems (2.C.2) are subjected to a dual treatment.

BIRKHOFFIAN TREATMENT OF THE CLOSED-EXTERIOR CASE. In Volume I, we pointed out that systems which are closed, i.e., verify the conventional Galilean conservation laws (of total energy, total linear momentum, total angular momentum, and uniform motion of the center of mass)

$$\dot{X}_i(t, r, \dot{r}) = 0, \quad \{X_i\} = \{E_{tot}, P_{tot}, M_{tot}, G\} \qquad (2.C.4)$$
$$i = 1, 2, \ldots, 10$$

can also admit nonpotential internal forces. In fact, laws (2.C.4) are verified identically when the nonpotential forces satisfy the identities

$$\sum_{k=1}^{N} \dot{r}_{k} \cdot F^{NSA}_{k\,strong} \equiv 0, \qquad (2.C.5a)$$

$$\sum_{k=1}^{N} F^{NSA}_{k} \equiv 0, \qquad (2.C.5b)$$

$$\sum_{k=1}^{N} r_{k} \times F^{NSA}_{strong} \equiv 0. \qquad (2.C.5b)$$

Note that the above identities constitute a system of seven algebraic equations in 3N unknowns (the components of the corces F^{NSA}_{strong}). Thus, closed non-Hamiltonian systems can be trivially constructed for $N \geq 3$. For $N = 2$, as well as for less trivial cases, see Appendix 1.C of Volume I, and Section 6.3 of monograph[189].

As stressed in Section 2.1, when the closed (exterior) treatment of a system is desired, it is recommendable to select formulations possessing a Lie algebraic character. The non-Hamiltonian nature of the systems then necessarily (and uniquely) implies the representation of the systems via Birkhoff's equations

$$\dot{a}^\mu = [a^\mu, B]^* = \frac{\partial a^\mu}{\partial a^\alpha}\Omega^{\alpha\beta}(a)\frac{\partial B(t,a)}{\partial a^\beta} \quad (2.C.6a)$$

$$[a^\mu, a^\nu]^* = \Omega^{\mu\nu}(a) = \left(\left\|\frac{\partial R_\alpha}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\alpha}\right\|^{-1}\right)^{\mu\nu} \quad (2.C.6b)$$

$$(a^\mu) = \begin{pmatrix} r \\ p \end{pmatrix}, \quad \mu = 1, 2, \ldots, 6N, \quad \det(\Omega^{\mu\nu})(a) \neq 0 \quad (2.C.6c)$$

It should be recalled from ref.[189] that the forces are represented in this case by the structure of the brackets themselves. Under the condition that the total (conserved) energy is represented by the Birkhoffian, the nonpotential forces are represented by the Lie isotopy. To put it differently, the Newtonian image of the extended character of hadrons is represented by the replacement of the conventional Poisson brackets with the generalized ones

$$\begin{pmatrix} [a^\mu, a^\nu] = \omega^{\mu\nu} \\ \text{point-like} \\ \text{particles} \end{pmatrix} \Longrightarrow \begin{pmatrix} [a^\mu, a^\nu]^* = \Omega^{\mu\nu} \\ \text{extended} \\ \text{particles} \end{pmatrix} \quad (2.C.7)$$

If, in addition, the nonpotential forces are small, e.g., in the sense of Eqs. (2.C.3), then isotopy (2.C.7) is essentially characterized by the transition

$$(R^o_\mu) = (p, 0) \Longrightarrow (R_\mu) = \left(e^{\varepsilon\alpha(t,r,p)}\, p + \varepsilon'\beta(t,r,p);\ \varepsilon''\gamma(t,r,p)\right) \quad (2.C.8)$$

$$\varepsilon, \varepsilon', \varepsilon'' \approx 0$$

BIRKHOFFIAN-ADMISSIBLE TREATMENT OF THE OPEN-INTERIOR CASE. As stressed in Volume I, conservative systems can be effectively treated via one single formulation, whether one considers a system as a whole (closed treatment), or one of its parts (open treatment). As a result, Galilei's relativity is effective in this case, not only for the description of the entire system, but also for one of its constituents, while considering the rest as external.

In the transition to non-Hamiltonian systems, the above situation is altered. In fact, dynamical effects which are essential at the level of the constituents may "cancel out" at the collective description, as evidentiated by Eqs. (2.C.5B). Besides, while the emphasis for the exterior treatment is in the conservation laws, that for the interior case becomes shifted to the nonconservation law, as a necessary condition to maximize the interactions (recall that when all its physical characteristics are conserved, a particle is free).

As a result of this occurrence, analytic formulations and relativities which hold for the exterior treatment, are not necessarily valid for the open interior description, and new, complementary approaches must be identified, of course, under the condition of achieving compatibility with the exterior case.

This is the reason why the generalization of Galilei's relativity proposed in monograph[189] is insufficient for our program, unless complemented with a still more general relativity capable of characterizing time rates of variation.

The representation of open systems proposed in this volume is now know, and it is that in terms of the Birkhoff-admissible equations (Section 2.5)

$$\dot{a}^\mu = (a^\mu, B)^* = \frac{\partial a^\mu}{\partial a^\alpha} S^{\alpha\beta}(t,a) \frac{\partial B(t,a)}{\partial a^\beta} \quad (2.C.9a)$$

$$(a^\mu, a^\nu)^* = S^{\mu\nu} = \left(\left\| \frac{\partial R_\alpha}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\alpha} \right\|^{-1}\right)^{\mu\nu} + T^{\mu\nu} \quad (2.C.9b)$$

$$(a^\mu) = \begin{pmatrix} r \\ p \end{pmatrix}, \quad T^{\mu\nu} = T^{\nu\mu}, \quad \det (S^{\mu\nu})(R) \neq 0, \quad \det (S^{\mu\nu} - S^{\nu\mu})(R) \neq 0 \quad (2.C.9c)$$

The acting forces can be now represented via the algebraic tensor and the Birkhoffian, in any desired combination, thanks to the gauge transformations of Section 2.6.

It is important to illustrate the representation with a specific example. It is sufficient to consider the case of only ONE particle under the Lorentz force due to an external system of charged particles under the assumption that all F^{SA}_{strong} are identically null,

$$m \ddot{r}_i \quad (2.C.10)$$

$$+ e\left[\left(\frac{\partial \varphi}{\partial r^i} - \frac{\partial A_i}{\partial t}\right) - \delta^{mn}_{ij} \frac{\partial A_n}{\partial r^m} \dot{r}^j\right]^{SA}_{Lorentz} - F^{NSA}_{strong} = 0$$

$$i = x, y, z$$

In this case, the Birkhoffian representation of the Newtonian electromagnetic interactions is given in (monograph[189], Example 4.1, page 98)

$$p = m\dot{r}$$

$$B = \frac{1}{2m} p^2 + e\,\varphi(t,r) \quad (2.C.11)$$

$$(R_\mu) = (p + eA, \; 0)$$

The additional terms F^{NSA}_{strong} are represented by

$$(T^{\mu\nu}) = \begin{pmatrix} 0 & \dfrac{F^{NSA}_{strong} + \dfrac{\partial A}{\partial t}}{(p/m)} \\ 0 & \end{pmatrix} \quad (2.C.12)$$

This results in the structure

$$(\dot{a}^\mu) = \begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix}$$

$$= \left(S^{\mu\nu} \frac{\partial B}{\partial a^\nu}\right) = \left[(\Omega^{\mu\nu}) + (T^{\mu\nu})\right]\left(\frac{\partial B}{\partial a^\nu}\right) \quad (2.C.13)$$

$$= \left[\begin{pmatrix} 0 & 1 \\ -1 & \left[e\left(\dfrac{\partial A_i}{\partial r^j} - \dfrac{\partial A_j}{\partial r^i}\right)\right] \end{pmatrix} + \begin{pmatrix} 0 & \\ 0 & \dfrac{F^{NSA}_{strong} + \dfrac{\partial A}{\partial t}}{(p/m)} \end{pmatrix}\right] \begin{pmatrix} e\dfrac{\partial \varphi}{\partial r} \\ p/m \end{pmatrix}$$

which is here assumed as a primitive Newtonian image of the Lie-admissible treatment of electromagnetic and strong interactions.

It may be recommendable to indicate at this point the basic operator form of Eqs. (2.C.6) and (2.C.9) which will be at the foundation of the particle treatment of Volume III.

LIE-ISOTOPIC GENERALIZATION OF HEISENBERG'S EQUATIONS. It can be constructed via an operator Lie-isopy, that is, as the transition from Heisenberg's equations (where $\tilde{a}$, $\tilde{H}$, etc, are now operators).

$$i\dot{\tilde{a}}^{\mu} = [\tilde{a}^{\mu}, \tilde{H}] = \tilde{a}^{\mu}\tilde{H} - \tilde{H}\tilde{a}^{\mu}, \quad \hbar = 1 \tag{2.C.14a}$$

$$[\tilde{a}^{\mu}, \tilde{a}^{\nu}] = i\tilde{\omega}^{\mu\nu} = \left(\left\| \frac{\partial R^{o}_{\alpha}}{\partial a^{\beta}} - \frac{\partial R^{o}_{\beta}}{\partial a^{\alpha}} \right\|^{-1}\right)^{\mu\nu} \tag{2.C.14b}$$

$$(\tilde{a}^{\mu}) = \begin{pmatrix} \tilde{r} \\ \tilde{p} \end{pmatrix}, (\tilde{R}^{o}_{\mu}) = (\tilde{p}, 0), \quad \mu = 1, 2, \ldots, 6N \tag{2.C.14c}$$

$$\det(\omega^{\mu\nu}) = 1 \tag{2.C.14d}$$

to the generalized form

$$i\dot{\tilde{a}}^{\mu} = [\tilde{a}^{\mu}, \tilde{B}]^{*} = \tilde{a}^{\mu}\tilde{T}\tilde{B} - \tilde{B}\tilde{T}\tilde{a}^{\mu}, \tag{2.C.15a}$$

$$[\tilde{a}^{\mu}, \tilde{a}^{\nu}]^{*} = i\tilde{\Omega}^{\mu\nu}(a) = \left(\left\| \frac{\partial R_{\alpha}}{\partial a^{\beta}} - \frac{\partial R_{\beta}}{\partial a^{\alpha}} \right\|^{-1}\right)^{\mu\nu} \tag{2.C.15b}$$

$$(\tilde{a}^{\mu}) = \begin{pmatrix} \tilde{r} \\ \tilde{p} \end{pmatrix}, \quad \tilde{R}_{\mu} \neq \tilde{R}_{o}, \quad \mu = 1, 2, \ldots, 6N \tag{2.C.15c}$$

$$\det(\Omega^{\mu\nu}) \neq 0 \tag{2.C.15d}$$

which does preserve the Lie character of the algebra, as the reader can see.

Under certain technical precautions ensuring that the operator $\tilde{B}$ represents the total energy, the extended character of hadrons will therefore be represented in our analysis by suitable operators T .

LIE-ADMISSIBLE GENERALIZATION OF HEISENBERG'S EQUATIONS. It can be constructed via an operator Lie-admissible genotopy, i.e., via the alteration of the Lie-isotopic generalization into the structurally more general form

$$i\dot{\tilde{a}}^{\mu} = (\tilde{a}^{\mu}, \tilde{B})^{*} = \tilde{a}^{\mu}\tilde{R}\tilde{B} - \tilde{B}\tilde{S}\tilde{a}^{\mu} \tag{2.C.16a}$$

$$(\tilde{a}^{\mu}, \tilde{a}^{\nu})^{*} = \tilde{S}^{\mu\nu} = \tilde{\Omega}^{\mu\nu} + \tilde{T}^{\mu\nu} \tag{2.C.16b}$$

$$(\tilde{a}^{\mu}) = \begin{pmatrix} \tilde{r} \\ \tilde{p} \end{pmatrix}, \ \det(S^{\mu\nu}) \neq 0, \ \det(\Omega^{\mu\nu}) \neq 0; \tilde{T}^{\mu\nu} = \tilde{T}^{\nu\mu} \tag{2.C.16c}$$

$$(\tilde{A}, \tilde{B})^{*} = \tfrac{1}{2}[\tilde{A}, \tilde{B}]^{*} + \tfrac{1}{2}\{\tilde{A}, \tilde{B}\} = \tfrac{1}{2}(\tilde{A}\tilde{T}\tilde{B} - \tilde{B}\tilde{T}\tilde{A}) + \tfrac{1}{2}(\tilde{A}\tilde{Q}\tilde{B} + \tilde{B}\tilde{Q}\tilde{A}), \quad \tilde{T} = \tilde{R} - \tilde{S}, \ \tilde{Q} = \tilde{R} + \tilde{S}$$

Again, under the assumption that the operator $\tilde{B}$ represents the total energy, the extended character of hadrons will be represented by the deviations of the left and right operators R and S from 1 . The nonconservative character of the particle, while considering the rest of the strong system as external, is characterized by the sum $\frac{1}{2}(\tilde{R} + \tilde{S})$.

Both Lie-isotopic and Lie-admissible generalizations of Heisenberg's equations were proposed by Santilli in the same memoir[194], pages 752 , and 746 respectively.

As a concluding remark illustrating the departures of our approach from conventional settings, we indicate here that the Lagrangian image of Birkhoff-admissible representations is characterized by second-order Lagrangians, that is, Lagrangians depending on the accelerations.

This point can be readily seen by turning Birkhoff-admissible equations into their equivalent second-order form. This can be done (under the so-called conditions of strict regularity[189], in addition to the regularity conditions of this volume) by:

(A) Constructing the normal first-order form of the equations,

$$\left(\dot a^{\mu}\right) = \begin{pmatrix}\dot r \\ \dot p\end{pmatrix} = \left(S^{\mu\nu}\frac{\partial B}{\partial a^{\nu}}\right) \overset{def}{=} \begin{pmatrix} W(t,r,p) \\ Z(t,r,p)\end{pmatrix} \qquad (2.C.17)$$

(B) Identifying the implicit form of the momentum in the velocities from the first set of Eqs. (2.C.17)

$$\dot r = W(t,r,p) \quad \Longrightarrow \quad p = Y(t,r,\dot r) \qquad (2.C.18)$$

and then

(C) Reducing the generalized, Pfaffian action of the theory to the equivalent form in configuration space

$$A = \int_{t_1}^{t_2} dt \left[R_{\mu}(t,a)\, a^{\mu} - B(t,a)\right] \qquad (2.C.19)$$

$$\overset{def}{=} \int_{t_1}^{t_2} dt \left[P_k(t,r,p)\,\dot r^{k} - Q^{k}(t,r,p)\,\dot p_k - B(t,r,p)\right] =$$

$$= \int_{t_1}^{t_2} dt \left[P_k(t,r,Y)\,\dot r^{k} - Q^{k}(t,r,Y)\,\dot Y_k - B(t,r,Y)\right]$$

$$\overset{def}{=} \int_{t_1}^{t_2} dt\, L(t, r, \dot r, \ddot r)$$

with underlying equations

$$-\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot r^{k}} + \frac{d}{dt}\frac{\partial L}{\partial \dot r^{k}} - \frac{\partial L}{\partial r^{k}} = 0 \qquad (2.C.20)$$

Note that the second-order Lagrangians are totally degenerate (i.e., linear in the accelerations). Thus, Esq. (2.C.20) are actually of <u>second</u>-order and not of higher order.

By comparison, all contemporary, high energy Lagrangian models, including those in quantum field theory, are of first-order type. As we shall see in Volume III, our isotopic lifting of the Hilbert space (which is essentially induced by the generalized associative product A*B = ATB), when applied to a first-order Lagrangian theory (e.g., as currently used, say, in chromodynamics), permits corrections and improvements whose classical origin are the generalization of the couplings into a non-self-adjoint form including accelerations.

CHAPTER 3

LIE–ADMISSIBLE COVERING OF LIE'S THEORY

3.1: STATEMENT OF THE PROBLEM

We are now sufficiently equipped to begin the identification of a central objective of this monograph. It essentially consists of attempting a Lie-admissible algebraic-group theoretic characterization of symmetry breakings and nonconservation laws as a covering of the conventional Lie characterization of exact symmetries and conservation laws.

By using the analysis of the preceding chapters, the following five aspects can now be identified.

(1) Exact symmetries and conservation laws. Our starting ground is the established ground: a conservative Newtonian system possessing an n-dimensional continuous Lie group G as an exact symmetry (ES) which is representative of physical conservation laws. The system is represented with the conventional Hamilton's equations without external terms

$$\left(\omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial H}{\partial a^{\mu}}\right)^{ES}_{SA} = 0, \quad (\omega_{\mu\nu}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \left(\frac{\partial R^{0}_{\nu}}{\partial a^{\mu}} - \frac{\partial R^{0}_{\mu}}{\partial a^{\nu}}\right) \quad (3.1.1)$$

$$a = (\underset{\sim}{r}, \underset{\sim}{p}); R^{0} = (\underset{\sim}{p}, 0); \mu = 1, 2, \ldots, 6N$$

while the conservation laws can be expressed in the conventional canonical form

$$\dot{X}_i = [X_i, H] = \frac{\partial X_i}{\partial a^{\mu}} \omega^{\mu\nu} \frac{\partial H}{\partial a^{\nu}} \equiv 0, \quad \omega^{\mu\nu} = (\|\omega_{\alpha\beta}\|^{-1})^{\mu\nu} \quad (3.1.2)$$

where X_i are the generators of G. It should be stressed that, in this context, all mathematical algorithms possess a direct physical significance. The phase space variables $\{a^{\mu}\} = \{r^{ka}, p_{ka}\}$ represent the Cartesian coordinates of the Euclidean space of the experimental detection of the system and the physical linear momenta $p_{ka} = m_k \dot{r}_{ka}$, respectively; the Hamiltonian H represents the physical total energy; and the generators X_i directly characterize physical quantities, such as the physical total linear momentum or angular momentum.

(2) Symmetry breaking and nonconservative laws. We now suppose that the system is subjected to additional forces which produce a G-symmetry breaking (SB). This broader system can be represented with the conventional Hamilton's equations with external terms

$$\left[\left(\omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial H}{\partial a^{\mu}}\right)^{ES}_{SA} - F_{\mu}\right]^{SB}_{NSA}, \quad \{F_{\mu}\} = \begin{Bmatrix} -\underset{\sim}{F}^{NSA} \\ \underset{\sim}{0} \end{Bmatrix} \quad (3.1.3)$$

This implies, in particular, that the forces $\underset{\sim}{F}$ are form-noninvariant under the G-transformations. Under the additional requirements that they are nonconservative, i.e., nonself-adjoint (NSA), this implies the nonconservation laws of the physical quantities X_i which can be written in the form

$$\dot{X}_i = [X_i, H] + \frac{\partial X_i}{\partial a^{\mu}} F^{\mu} = \frac{\partial X_i}{\partial \underset{\sim}{p}} \cdot \underset{\sim}{F}^{NSA} \neq 0 \quad (3.1.4)$$

Notice that in the transition from Eqs. (3.1.1) and (3.1.2) to Eqs. (3.1.3) and (3.1.4), respectively, the algorithms have been selected in such a way to preserve their direct physical significance. The variables $\{a^{\mu}\} = \{r^{ka}, p_{ka}\}$ now do not span a phase space (Section 2.3). Nevertheless, they preserve the direct physical significance of the conservative case. The Hamiltonian H is still the total physical energy of the system as the sum of the kinetic energy and the potential energy of all forces derivable from a potential function $U(t, \underset{\sim}{r}, \underset{\sim}{p})$. Finally, the quantities

X_i preserve their original direct physical significance.

Concrete examples of breakings (3.1.3) occur in the systems of our everyday experience. Consider, for instance, the conservative approximation of the one-dimensional harmonic oscillator. It possesses an exact symmetry under the one-parameter group of translations in time $T_1(t)$. This exact symmetry, however, is grossly broken by the oscillators which occur in the physical reality, owing to the presence of dissipative forces, $F = F^d(t, r, \dot{r})$. By including applied forces $F^a(t)$ (which are needed to preserve the motion for the desired period of time), we then have the case of (generally) nonself-adjoint breaking of the symmetry under translations in time

$$T_1(t):\ (m\ddot{r} + K r)^{ES}_{SA} = 0 \rightarrow \left[(m\ddot{r} + K r)^{ES}_{SA} - F^d - F^a\right]^{BS}_{NSA} = 0 \qquad (3.1.5)$$

As a second example, consider the conservative abstraction of the physical spinning top under gravity. It exhibits an exact symmetry under the group of rotations. However, if this symmetry was actually realized, it would literally imply the existence of the "perpetual motion" in our environment. It is an experimental evidence that the angular momentum of the spinning top decays in time. For the case of one degree of rotational freedom, we then have the (generally) nonself-adjoint breaking of the SO(2) symmetry due to drag torques

$$SO(2):\ (I\ddot{\theta})^{ES}_{SA} = 0 \longrightarrow \left[(I\ddot{\theta})^{ES}_{SA} - T\right]^{BS}_{NSA} = 0 \qquad (3.1.6)$$

where the possible inclusion of an applied torque is tacitly assumed.

As a third example, consider the two-body Coulomb system in

vacuum. It exhibits a symmtry under the full Galilei group G(3.1). Suppose now that the charged particles move in a physical medium which, as such, produces dissipative forces. Then, in general, the entire symmetry G(3.1) is broken by these additive forces in the sense that we now have the "nonconservation laws" of the energy, linear momentum and angular momentum, as well as the lack of uniform motion of the center of mass (the system tending to rest in a finite period of time). We shall then write

$$G(3.1):\ (m_k\ddot{r}_k - f^{Coul}_k)^{ES}_{SA} = 0 \rightarrow \left[(m_k\ddot{r}_k - f^{Coul}_k)^{ES}_{SA} - F_k\right]^{BS}_{NSA} = 0 \qquad (3.1.7)$$

An aspect of these symmetry breakings which is of central physical relevance is that in the transition from the case of exact symmetry, to the more general case of broken symmetry, the physical quantities are unaffected in their definition. For instance, in breaking (3.1.7) the total physical energy, i.e., the Hamiltonian $H^{ES} = H = T + V$, remains unchanged under the presence of the additive dissipative forces and/or applied forces. And indeed, these forces are only responsible for the variation of H in time.

Similarly, the definition of the <u>physical</u> (not canonical) angular momentum of the spinning top with one degree of rotational freedom is unique and cannot be affected by the presence of dissipative and/or applied torques. Thus, while the SO(2) symmetry is broken in example (3.1.6), this does not imply a modification of the definition of angular momentum, but only its acquisition of the broader nonconserved character.

The reader can then see the corresponding case of the ten physical generators G_i, i=1,2,...,10, of the Galilei symmetry in breaking (3.1.7). Again, they are unaffected in their definition by the breaking, although their character is now modified into a nonconservative form.

The reader should be aware that we have tacitly implemented the assumption of Appendix I.2.B, that is, the self-adjoint system within the inner brackets of the right hand side of Eqs. (3.1.7) is the maximal associated self-adjoint system. This essentially ensures that the original potential energy is not modified by additive forces also derivable from a potential, i.e., all the additive forces F_k^{NSA} are genuinely nonconservative.

We should recall, as an incidental note, that breakings (3.1.3) were called nonself-adjoint breakings in Volume I, to distinguish them from the selfadjoint breakings, i.e. the conventional (classical) breakings of the contemporary literature via an additive term in the Hamiltonian

$$H^{ES} \Longrightarrow H^{BS} = H^{ES} + H_1^{BS} \tag{3.1.8}$$

The reader should also keep in mind that the nonself-adjoint breakings are inclusive of the simpler semicanonical and canonical breakings and that breakings (3.1.3) are a local approximation of a still more general class of symmetry breaking, that via nonlocal nonself-adjoint forces, as expected due to the extended character of systems moving in resistive media. Even though Hamilton-admissible equations can formally represent these latter breakings, our analysis will be restricted to local self-adjoint breakings.

(3) Hamilton/Birkhoff-admissible characterization of symmetry breaking and nonconservation laws. As now familiar, Hamilton's equations with external terms do not allow a consistent algebra in the time evolution (Section 2.2). To bypass this difficulty, we represent Eqs. (3.1.3) via our Hamilton-admissible equations (Section 2.3)

$$\left(s_{\mu\nu}\, \dot{a}^\nu - \frac{\partial H}{\partial a^\mu} \right)^{BS}_{NSA} = 0\,; \quad \det (s_{\mu\nu})(R) \neq 0\,; \quad \det (s_{\mu\nu} - s_{\nu\mu})(R) \neq 0$$
$$(s_{\mu\nu}) = (\omega_{\mu\nu}) + (t_{\mu\nu}) = \left(\frac{\partial R^\circ_\nu}{\partial a^\mu} - \frac{\partial R^\circ_\mu}{\partial a^\nu} \right) + \begin{pmatrix} -F^{NSA}/(p/m) & 0 \\ 0 & 0 \end{pmatrix} \tag{3.1.9}$$

or our still more general Birkhoff-admissible equations (Section 2.5)

$$\left(S_{\mu\nu}\, \dot{a}^\nu - \frac{\partial B}{\partial a^\mu} \right)^{BS}_{NSA} = 0\,, \quad \det (S_{\mu\nu})(R) \neq 0\,; \quad \det (S_{\mu\nu} - S_{\nu\mu})(R) \neq 0 \tag{3.1.10}$$
$$S_{\mu\nu} = \Omega_{\mu\nu} + T_{\mu\nu}\,; \quad \Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\,, \quad T^{\mu\nu} = T^{\nu\mu}\,, \quad R \neq R^\circ$$

where the reader should recall that H (B) represents the Hamiltonian (Birkhoffian).

Nonconservation laws can then be equivalently and identically represented either via the former or the latter equations, according to the structures (Section 2.5)

$$\dot{X}_i = (X_i, H) = \frac{\partial X_i}{\partial a^\mu}\, s^{\mu\nu} \frac{\partial H}{\partial a^\nu} \tag{3.1.11}$$
$$\equiv (X_i, B)^* = \frac{\partial X_i}{\partial a^\mu}\, S^{\mu\nu} \frac{\partial B}{\partial a^\nu} = \frac{\partial X_i}{\partial p} \cdot F^{NSA}$$
$$s^{\mu\nu} = (\| s_{\alpha\beta} \|^{-1})^{\mu\nu}\,, \quad S^{\mu\nu} = (\| S_{\alpha\beta} \|^{-1})^{\mu\nu}$$

Hereon we shall alternatively use Equations (3.1.9) or (3.1.10) depending on the objectives at hand. Note that the Hamiltonian in (3.1.9) is invariant under the original symmetry G. The breaking of this symmetry and the nonconservation laws are represented with a modification of the structure of the canonical equation which is representative of the symmetry breaking forces, i.e.,

$$\left(s^{\mu\nu} - \omega^{\mu\nu}\right)\frac{\partial H}{\partial a^{\nu}} \equiv F^{\mu} \qquad (3.1.12)$$

Alternatively, we can say that, while the tensor $\omega_{\mu\nu}$ of Eqs. (3.1.1) contains no representative of the acting forces, the tensor $s_{\mu\nu}$ of Eqs. (3.1.9) has the primary function of representing the symmetry breaking forces.

(4) Lie-admissible characterization of symmetry breaking and nonconservation laws. The central methodological aspect of the Hamilton-admissible equations is that the brackets (A,B) of the time evolution law (3.1.11) violate the Lie algebra identities, but characterize a fully admissible algebra which we have identified as a Lie-admissible algebra. In Chapter 1 we have indicated that the Lie-admissible identities constitute an algebraic covering of the Lie identities. In Chapter 2 we have indicated that the Lie-admissible algebras possess an analytic origin which is fully parallel to that of the Lie algebras; and the transformation theory of the broader analytic context is a covering of the conventional canonical theory. We therefore expect that this broader transformation theory is such to admit a Lie-admissible structure for the infinitesimal transformations while preserving a group structure for the finite transformations. These latter possibilities clearly deserve a more detailed inspection. We reach in this way the objective of this chapter. It consists of the study of the problem whether the Lie-admissible algebras might provide a covering of Lie's theory. More specifically, the objective of this chapter consists of the study of the problem whether central methodological aspects of Lie's theory (such as, Lie's first, second and third theorem, the universal enveloping associative algebras, etc.) admit a covering which, in the neighborhood of the origin, is characterized by a Lie-admissible algebra. If the answer is affirmative, we clearly open the possibility of providing a methodological characterization of symmetry breaking and nonconservation laws which is fully parallel to, although a covering of the conventional characterization of exact symmetries and conservation laws. The reader should be aware in this respect that, to the best of my knowledge, broken symmetries remain algebraically and group theoretically undefined in the available treatments.

(5) Symplectic-admissible characterization of symmetry breakings and nonconservation laws. The deep relationship of the symplectic geometry with the canonical equations and the Lie algebras has been recalled in Volume I and associated to the variational self-adjointness of these methodological tools. The embedding of the canonical equations into a nonself-adjoint form and of the Lie algebras into the Lie-admissible algebras inevitably imply the lack of direct applicability of the symplectic geometry as currently known. We reach in this way the objective of Chapter 4. It consists of the study of the problem whether there exists a covering of the symplectic geometry associated to the Hamilton-admissible equations and to the Lie-admissible algebras. As we shall see, the answer appears to be affirmative and the emerging broader geometry will be called symplectic-admissible. This latter aspect will conclude the identification of the classical methdological tools which are needed

for our study of the hadronic structure. On classical grounds, our hope is therefore that of indicating the existence of interrelated and compatible analytic, algebraic and geometrical coverings of conventional formulation for the characterization of symmetry breakings and nonconservation laws.

The analysis of this chapter, which is devoted to aspect (4), will be conducted along the following three sequential steps.

Step 1: Review of certain aspects of the Lie's theory. This is intended to avoid excessive reference consultation; to establish our notation; and to provide a readily accessible ground of comparison.

Step 2: Identification of a Lie covering of Lie's theory. With these terms we intend to refer to the problem of the possible existence of a generalization of Lie's theory which is fully Lie in algebraic character, but such to be realized in terms of the broadest possible (regular) product which is admitted by the Lie algebra identities, the generalized Poisson Brackets

$$[A,B]^* = \frac{\partial A}{\partial a^\mu}\, \Omega^{\mu\nu}(a) \frac{\partial B}{\partial a^\nu}, \qquad (3.1.13)$$

as originating from the time evolution law of Birkhoff's equations

$$\left[\Omega_{\mu\nu}(a)\, \dot{a}^\nu - \frac{\partial B}{\partial a^\nu}\right]_{SA} = 0, \quad (\Omega_{\mu\nu}) = (\Omega^{\mu\nu})^{-1}$$

$$\Omega_{\mu\nu} = \frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial R_\nu}{\partial a^\mu}. \qquad (3.1.14)$$

Step 3: Identification of a Lie-admissible covering of Lie's theory. With these terms we refer to the problem of the possible existence of a generalization of Lie's theory which is non-Lie in algebraic character, but Lie-admissible, i.e., realized in terms of the broadest possible (regular) product which is admitted by the general Lie-admissible conditions.

The crucial role of intermediary Step 2 is now evident. In fact, the Lie algebra which is generally attached to a Lie-admissible algebra, is not characterized by the conventional Poisson brackets, and instead by the most general possible Lie brackets.

Our basic assumption in the studies of Steps 2 and 3 will be the following ones, which are evident from the preceding remarks of this section,

A - the crucial preservation of the generators of the original exact symmetry, while the presence of symmetry breaking forces is represented with a modification of the Lie methodology into a Lie-admissible form;

B - the equally crucial preservation of the Euclidiean space of the experimental detection of the system (to avoid mathematical characterizations which are experimentally unrealizable) which corresponds to the preservation of the base manifold of the original exact symmetry; and

C - the preservation of the parameters of the original exact symmetry, that is, time for case (3.1.5), angle for case (3.1.6) and the conventional ten parameters of the Galilei group for case (3.1.7).

As a result, the building blocks of our Lie-admissible characterization of symmetry breakings and nonconservation laws are, by central assumption, the same as those of the original exact symmetry. Only the structure of the algebra is altered from Lie to Lie-admissible forms.

To have an introductory idea of the assumptions above, recall the canonical realization of Lie transformation groups (reviewed in Section 3.2)

$$G: \quad a' = e^{\theta \omega^{\alpha\beta} \frac{\partial X}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}} a \tag{3.1.15}$$

with infinitesimal form in the neighborhood of the identity transformations (Section 3.2)

$$\frac{da}{d\theta} \cong \frac{a' - a}{d\theta} \cong [a, X]_{\text{Poisson}} \tag{3.1.16}$$

and underlying universal enveloping associative algebra $\mathcal{A}(\underline{G})$ of the Lie algebra $\underline{G}$ of Lie group (3.1.15) [see Section 3.4 for details].

$$\mathcal{A}(\underline{G}) = {}^{\alpha}T(\underline{G}) / \mathcal{R}(\underline{G}) \tag{3.1.17}$$

$$[\mathcal{A}]^- \approx \underline{G}$$

which provide the conventional characterization of symmetries in Newtonian mechanics.

The characterization of conservation laws via a Lie transformation group is provided by the familiar expression for the case when the generator is the Hamiltonian, and the parameter is an infinitesimal time interval dt

$$\begin{aligned} \frac{dX_k}{dt} &= \frac{X_k' - X_k}{dt} \\ &= \left\{ e^{dt\, \omega^{\alpha\beta} \frac{\partial H}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}} - 1 \right\} X_k / dt \\ &= [X_k, H]_{\text{Poisson}} \equiv 0 \end{aligned} \tag{3.1.18}$$

In this chapter we shall show the existence of generalized groups of continuous transformations possessing a structure of the type

$$\hat{G}: \quad \hat{a} = e^{\theta S^{\alpha\beta}(a) \frac{\partial X}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}} a \tag{3.1.19}$$

which is evidently characterized by the replacement of Lie's tensor $\omega^{\mu\nu}$ with Lie-admissible tensor $S^{\mu\nu}$, while preserving the base manifold (that with local a-coordinates), the parameter θ, and the generator X of the original symmetry (3.1.15).

The intriguing profile is that, while structure (3.1.19) does constitute a continuous group, its algebra in the neighborhood of the identity transformation <u>is not</u> a Lie algebra, but instead a Lie-admissible algebra with brackets

$$\frac{da}{d\theta} \cong \frac{\hat{a} - a}{d\theta} \cong (a, X)^* = \text{NONLIE, LIEADMISSIBLE} \tag{3.1.20}$$

As we shall see, these are not isolated occurrences. In fact, they can be interpreted as manifestation of the apparent existence of a Lie-admissible generalization of the entire Lie theory, including a generalization of enveloping associative algebra (3.1.17) into a <u>non-associative</u> Lie-admissible form of the type (Section 3.5)

$$\mathcal{U}(G) = {}^{\alpha}\hat{T}(G)/\hat{R} \qquad (3.1.21)$$
$$[\mathcal{U}]^- \approx G^* \not\approx G$$

where, as we shall see, the attached algebra G^* <u>is not</u> homomorphic to G.

It is evident that the Lie-admissible structure (3.1.19) can provide a <u>covering-breaking of conventional Lie symmetries</u>. In fact, the covering nature is self-evident from the fact that structure (3.1.15) is a particular case of (3.1.19). The breaking character is also self-evident. In fact, when the broader symmetry (3.1.19) holds, the simpler symmetry (3.1.15) is expectedly broken.

This illustrates our central idea of not leaving the broken symmetry complitely undefined, but of replacing it with a structurally more general symmetry.

The characterization of the time-rates of variation is equally straightforward, and occurs when, again, the "generator" is the Hamiltonian, and the parameter is an infinitesimal time interval, according to the Lie-admissible rules

$$\frac{dX_k}{dt} \cong \frac{X'_k - X_k}{dt} \cong \left\{ e^{dt\, S^{\alpha\beta}\frac{\partial H}{\partial a^\beta}\frac{\partial}{\partial a^\alpha}} - 1 \right\} X_k/dt$$
$$= (X_k, H) = \frac{\partial X_k}{\partial p_\mu} \cdot F^{NSA}_\mu \neq 0 \qquad (3.1.22)$$

where we have used Hamilton-admissible equations [otherwise the Birkhoffian would have been the "generator"].

The study of the problems related to Steps 2 and 3 can be apparently conducted by using more than one methodological approach. The approach which I have selected is one of the simplest possible to the best of my knowledge at this time. It consists of a realization in terms of the transformation theory of the notion of algebraic isotopy and genotopy identified in Chapter 1 on abstract grounds. More specifically, the study of Step 2 will be conducted by identifying the class of Lie algebraic isotopies of Lie's theory, that is, the invertible mappings which preserve the Lie character of the underlying algebra. The study of the problem of Step 3 will be conducted instead by identifying the class of Lie-admissible algebraic genotopies of Lie's theory, that is, the invertible mappings which, this time, violate the Lie algebraic character by assumption, but induce a Lie-admissible algebraic structure. The reason for the preference of this methodological approach to other conceivable approaches is the following. A central aspect of our objective is to attempt the construction of an algebraic-group theoretic characterization of symmetry breakings and nonconservation laws as a generalized formulation which preserves the base manifold, the parameters, and the generators of the conventional treatment of exact symmetries and conservation laws. An objective of this nature demands not only the identification of the methodoloyies of Step 2 and 3, but also, and most importantly, their relationship with the conventional Lie's theory. The notions of algebraic isotopy and genotopy serve precisely this purpose because they map the conventional Lie's treatment into the desired broader formulation without affecting, by construction, the base manifold, the parameters and the generators.

Perhaps, the novelty of our analysis, besides the broader methodological context under consideration, relies in its intended use. The existence of a number of mechanisms of symmetry breakings has been recalled earlier. To the best of my knowledge, all these mechanisms have been generally conceived for and applied to the breaking of the internal symmetries while leaving the fundamental space-time symmetries unaffected. The intended use of our approach to symmetry breaking is somewhat the opposite of that. The primary intended use of the methods under consideration is for the study of the breaking of the fundamental space-time symmetries according to the lines discussed earlier. My hope is to be able to confront in this way the problem of the relativity laws which are applicable to systems with forces not derivable from a potential. This relativity aspect will be considered in Chapter 5, while the analysis of this chapter is solely devoted to the algebraic-group theoretic profile.

Almost needless to say, this chapter is grossly deficient in content, treatment, and mathematical rigour. This is not the place to recall the vastity and degree of sophistication achieved by the contemporary treatments of Lie's theory. The sole purpose of this chapter is the indication that a Lie-admissible covering of Lie's thoery exists and, in due time, can be constructed. For this purpose, I shall provide my best efforts in presenting the main ideas in a way as elementary as possible and leave to the interested reader the task of their reinspection and possible reformulation in terms of more rigorous (but more abstract) mathematical languages.

NOTE ADDED IN 1982

Since the time of writing this chapter, the "Lie-isotopic generalization of Lie's theory" has appeared in paper[191] and in monograph[189]. The original presentation written here has been kept in this volume for self-consistency.

The "Lie-admissible generalization of Lie's theory" of Sections 3.3 and 3.5 has appeared in print in paper[189] only, without any additional contribution by independent researchers, to my knowledge. I am referring, more specifically, to the Lie-admissible generalization of Lie's first, second, and third theorem, and to the construction of an enveloping algebra which is nonassociative, nonflexible, and general Lie-admissible.

The reader, however, should keep in mind that the virtual totality of the mathematical studies appeared since 1977 on Lie-admissible algebras do complement the foundations of the Lie-admissicle generalization of Lie's theory, although from different perspectives. For references, see the "Note Added in 1982" at the end of Section 1.5.

3.2: LIE'S FIRST, SECOND AND THIRD THEOREMS

Let us begin by recalling the notion of Lie's transformation group. Since we are primarily interested in application to Newtonian mechanics, we shall consider transformations acting in the variables

$$\{a^\mu\} = \{r^{ka}, p_{ka}\}, \quad \mu = 1,2,\dots,6N, \tag{3.2.1}$$

where r^{ka} and p_{ka}, for the purpose of this section, space the phase space of a Newtonian system. A generic transformation of these variables depending on n (independent) parameters θ^i, i=1,2,...n, can be written

$$a^\mu \to a'^\mu = f^\mu(a^1,\dots,a^{6N};\theta^1,\dots,\theta^n), \tag{3.2.2}$$

or, in symbolic notation

$$a \to a' = T(\theta)a = f(a;\theta). \tag{3.2.3}$$

Eqs. (3.2.2) characterize a <u>Lie group of transformations</u> when the following conditions are verified:

(1) the functions f^μ are analytic in their variables,

(2) for any given two transformations

$$a' = f(a;\theta), \quad a'' = f(a';\theta'), \tag{3.2.4}$$

there exists a set of parameters

$$\theta''^i = g^i(\theta,\theta') \tag{3.2.5}$$

characterized by analytic functions g^i called <u>group composition laws</u>, such that

$$a'' = f(a;\theta''), \tag{3.2.6}$$

(3) the transformations recover the identity transformations at the null value of the parameters, i.e.,

$$a = f(a;0), \tag{3.2.7}$$

(4) in correspondence to each transformation (3.7.2) there exist a (unique) inverse transformation

$$a = f(a';\theta^{-1}), \tag{3.2.8}$$

and, thus, the transformations are regular, i.e., their Jacobian is nonnull

$$\left|\frac{\partial f}{\partial a}\right| \neq 0 \tag{3.2.9}$$

(5) the combination (product) of any transformation (3.2.2) with its inverse (3.2.8) yields the identity transformation.

As we shall see in the next chapter, the space of the a-variables can be more properly identified as a Hausdorff, second countable, infinite-differentiable, 6N-dimensional manifold. But these technical implementations are not essential for the rudimentary treatment of this chapter.

A central property of the groups under consideration is that they constitute <u>n-parameter connected Lie groups</u>, that is, groups of transformations depending on n (independent) parameters which are continuously connected to the identity.* In turn, this implies the possibility of studying these groups through their infinitesimal <u>transformations, because a finite</u> connected transformation can be

(*) The meaning of the term "connected to the identity" used in this volume is more generally referred to in the existing literature as "continuous". The former terminology has been preferred here for historical reasons.

recovered as an infinite succession of infinitesimal transformations. This is, in essence, the primitive methodological context of Lie's theorems which we shall here briefly outline.

Consider the transformations

$$a' = f(a;\theta) \quad , \quad a = f(a;0), \tag{3.2.10}$$

and perform the infinitesimal variations

$$a' = a + da = f(a;\theta + d\theta) \; , \; a + da = f(a;\delta\theta), \tag{3.2.11}$$

where $d\theta$ and $\delta\theta$ represent the variation of the parameters. This yields

$$da = \frac{\partial f(a;\theta)}{\partial\theta}\, d\theta \; , \tag{3.2.12a}$$

$$da = \left| \frac{\partial f(a;\theta)}{\partial\theta} \right|_{\theta=0} \delta\theta . \tag{3.2.12b}$$

The transformation $\theta + d\theta$ can then be interpreted as the product of transformations relative to θ and $\theta + \delta\theta$, i.e.,

$$\theta^i + d\theta^i = \phi^i(\theta, \delta\theta) \tag{3.2.13}$$

for which

$$\theta^i + d\theta^i = \phi^i(\theta, 0) + \left[\frac{\partial\phi^i(\theta,\alpha)}{\partial\alpha^j}\right]_{\alpha=0} \delta\theta^j + \ldots$$

$$\cong \theta^i + \mu^i{}_j(\theta)\,\delta\theta^j \tag{3.2.14}$$

Thus,

$$d\theta^i = \mu^i{}_j(\theta)\,\delta\theta^j \tag{3.2.15}$$

The above formula represents a relation between $d\theta$ and $\delta\theta$ which can also be written

$$\delta\theta^j = \lambda^j{}_i(\theta)\, d\theta , \tag{3.2.16}$$

where

$$\lambda^j{}_k\, \mu^k{}_i = \mu^j{}_k\, \lambda^k{}_i = \delta^j{}_i \tag{3.2.17}$$

By putting

$$u^\mu{}_i(a) = \left[\frac{\partial f^\mu(a;\theta)}{\partial\theta^i}\right]_{\theta=0} \tag{3.2.18}$$

and by using Eqs. (3.7.16), Eqs. (3.2.12b) become

$$da^\mu = u^\mu{}_k(a)\, \lambda^k{}_j(\theta)\, d\theta^j \tag{3.2.19}$$

We reach in this way <u>Lie's First Theorem</u>.

<u>THEOREM 3.2.1: If the transformations</u>

$$a'^\mu(\theta) = f^\mu(a;\theta) \tag{3.2.20}$$

<u>form an n-dimensional (connected) Lie group, then</u>

$$\frac{\partial a'^\mu}{\partial\theta^j} = u^\mu{}_k(a)\, \lambda^k{}_j(\theta) \tag{3.2.21}$$

<u>where the functions $u^\mu{}_k(a)$ are analytic.</u>

Let A(a) be a class C^∞ function of the a-variables. The infinitesimal (Lie's) transformation $a \to a+da$ induces a variation of A(a) which can be written

$$dA(a) = \frac{\partial A}{\partial a^\mu}\, da^\mu = \frac{\partial A}{\partial a^\mu}\, u^\mu_i\, \delta\theta^i \qquad (3.2.22)$$

$$= \delta\theta^k\, u^\mu_k \frac{\partial}{\partial a^\mu} A(a) = \delta\theta^k X_k A(a).$$

The n independent quantities

$$X_k(a) = u^\mu_k(a)\frac{\partial}{\partial a^\mu} = \left[\frac{\partial f^\mu(a;\theta)}{\partial\theta^k}\right]_{\theta=0}\frac{\partial}{\partial a^\mu}, \qquad (3.2.23)$$

are called the <u>infinitesimal generators</u> of the transformations (or of the group). For our later need, we shall refer to X_i as the <u>standard generators</u> of G.

We are now interested in the (necessary and sufficient) conditions for transformations (3.2.2) to constitute a group. By using the converse of the Poincaré lemma they can be written

$$\frac{\partial^2 a'^\mu}{\partial\theta^i\partial\theta^j} = \frac{\partial^2 a'^\mu}{\partial\theta^j\partial\theta^i}, \qquad (3.2.24)$$

that is,

$$\frac{\partial u^\mu_k}{\partial\theta^i}\lambda^k_j + u^\mu_k\frac{\partial\lambda^k_j}{\partial\theta^i} = \frac{\partial u^\mu_k}{\partial\theta^j}\lambda^k_i + u^\mu_k\frac{\partial\lambda^k_i}{\partial\theta^j}$$

Thus,

$$u^\mu_k\left\{\frac{\partial\lambda^k_j}{\partial\theta^i} - \frac{\partial\lambda^k_i}{\partial\theta^j}\right\} = \lambda^k_j\frac{\partial u^\mu_k}{\partial\theta^i} - \lambda^k_i\frac{\partial u^\mu_k}{\partial\theta^j}$$

$$= \lambda^k_j\frac{\partial u^\mu_k}{\partial a^\nu}\frac{\partial a^\nu}{\partial\theta^i} - \lambda^k_i\frac{\partial u^\mu_k}{\partial a^\nu}\frac{\partial a^\nu}{\partial\theta^j}$$

$$= \lambda^r_j u^\nu_l \lambda^l_i\frac{\partial u^\mu_r}{\partial a^\nu} - \lambda^k_i u^\nu_l \lambda^l_j\frac{\partial u^\mu_k}{\partial a^\nu}$$

$$= \lambda^r_j\lambda^k_i\left\{u^\nu_k\frac{\partial u^\mu_r}{\partial a^\nu} - u^\nu_r\frac{\partial u^\mu_k}{\partial a^\nu}\right\}. \qquad (3.2.26)$$

Therefore

$$u^\nu_i\frac{\partial u^\mu_j}{\partial a^\nu} - u^\nu_j\frac{\partial u^\mu_i}{\partial a^\nu} = C^k_{ij}\, u^\mu_k \qquad (3.2.27)$$

where

$$C^k_{ij} = \mu^r_i\,\mu^s_j\left\{\frac{\partial\lambda^k_r}{\partial\theta^s} - \frac{\partial\lambda^k_s}{\partial\theta^r}\right\}. \qquad (3.2.28)$$

The n^3 quantities C^k_{ij} are independent from θ. And indeed, by differentiating Eqs. (3.2.27) with respect to θ and after simple manipulation we have

$$\partial C^k_{ij} / \partial\theta^l \equiv 0; \; i,j,k,l = 1,2,\ldots,n. \qquad (3.2.29)$$

We reach in this way <u>Lie's Second Theorem</u>.

<u>THEOREM</u> 3.2.2: <u>If X_i, i=1,2,...,n are the generators of an n-dimensional Lie group, they satisfy the relations</u>

$$[X_i, X_j]_A = X_iX_j - X_jX_i = C^k_{ij}X_k \qquad (3.2.30)$$

<u>where the quantities C^k_{ij} are constants (Lie structure constants)</u>.

Eqs. (3.2.30) can be explicitly written

$$[X_i, X_j]_A = \left[u^\mu_i \frac{\partial}{\partial a^\mu}, u^\nu_j \frac{\partial}{\partial a^\nu}\right]$$
$$= u^\mu_i u^\nu_j \frac{\partial^2}{\partial a^\mu \partial a^\nu} + u^\mu_i \frac{\partial u^\nu_j}{\partial a^\mu}\frac{\partial}{\partial a^\nu} \qquad (3.2.31)$$
$$- u^\mu_i u^\nu_j \frac{\partial^2}{\partial a^\mu \partial a^\nu} - u^\nu_j \frac{\partial u^\mu_i}{\partial a^\nu}\frac{\partial}{\partial a^\mu}$$

Therefore

$$\left[u^\mu_i \frac{\partial}{\partial a^\mu}, u^\nu_j \frac{\partial}{\partial a^\nu}\right]_A = C^k_{ij}\, u^\mu_k \frac{\partial}{\partial a^\mu} \qquad (3.2.32)$$

and the product $[X_i, X_j]_A$ is Lie, that is, it satisfies the identities

$$[X_i, X_j]_A + [X_j, X_i]_A = 0 \qquad (3.2.33a)$$

$$[[X_i, X_j]_A, X_k]_A + [[X_j, X_k]_A, X_i]_A + [[X_k, X_i]_A, X_j]_A = 0 \qquad (3.2.33b)$$

We reach in this way Lie's Third Theorem.

THEOREM 3.2.3: The structure constants of a Lie group satisfy the identities

$$C^k_{ij} + C^k_{ji} = 0, \qquad (3.2.34a)$$

$$C^k_{ij} C^r_{kl} + C^k_{jl} C^r_{ki} + C^k_{li} C^r_{kj} = 0. \qquad (3.2.34b)$$

Lie's first, second and third theorem essentially provide the

one-to-one correspondence between a given (connected) Lie group G and its Lie algebra G. In particular, the group can be characterized in the neighborhood of the identity via the structure constants. We here tacitly imply that there may exist different Lie groups all having the same Lie algebra, that is, the same behavior in the neighborhood of the identity. However, among all the Lie groups with the same Lie algebra, there is only one which is simply connected and which is called the universal covering group. As a notable case, the group SU(2)(SL(2.C)) is the universal covering group of the group of rotations (SO(3) (the Lorentz group SO(3.1)).

The inverse transition from a Lie algebra to a corresponding Lie group is characterized by the inverse of Lie's first, second and third theorem. For brevity, we here refer the reader to the existing literature, e.g., reference.[131]

In essence, different Lie groups are characterized by different composition laws (3.2.5), say, g^i and g'^i. However composition laws which are globally different may be equivalent or coincide in the neighborhood of the identity in which case the composition laws are called analytically isomorphic. The Lie algebra then remains the same up to possible change of basis, that is, algebraic isomorphisms.

The simplest way to perform the transition from a Lie algebra to a corresponding Lie group is through the exponential mapping. Write Eqs. (3.2.19) in the form

$$\frac{\partial a^\mu}{\partial \theta^i} = u^\mu_k(a)\,\lambda^k_i(\theta) = \lambda^k_i(\theta)\, X_k(a)\, a^\mu, \qquad (3.2.35)$$

and introduce the one-dimensional parametrization

$$\theta^k = \tau \alpha^k , \quad a'^{\mu} = a'^{\mu}(\theta(\tau)) = a''^{\mu}(\tau). \qquad (3.2.36)$$

Then we can write

$$a''^{\mu}(\tau) = T^{\mu}{}_{\nu}(\tau)\, a^{\nu}, \qquad (3.2.37a)$$

$$a^{\nu} = a''^{\nu}(\tau)\Big|_{\tau=0}. \qquad (3.2.37b)$$

To compute the elements $T^{\mu}{}_{\nu}(\tau)$, consider the equations

$$\frac{da^{\mu}}{d\tau} = \frac{\partial a^{\mu}}{\partial \theta^i}\frac{d\theta^i}{d\tau} = \alpha^k \lambda^r_k(\theta)\, X_r(a)\, a''^{\mu}(0) \quad (3.2.38a)$$

$$\frac{d}{d\tau} T^{\mu}{}_{\nu}(\tau)\, a''^{\nu}(0) = \alpha^k \lambda^r_k(\theta)\, X_r(a)\, T^{\mu}{}_{\nu}(\tau)\, a''^{\nu}(0). \quad (3.2.38b)$$

But $a^{\mu}(0)$ are arbitrary initial values. Thus, the solutions of the total differential equations

$$\frac{d}{d\tau} T^{\mu}{}_{\nu}(\tau) = \alpha^k \lambda^r_k(\theta)\, X_r(a(\tau))\, T^{\mu}{}_{\nu}(\tau) \quad (3.2.39)$$

with initial conditions

$$T^{\mu}{}_{\nu}(0) = \delta^{\mu}{}_{\nu}, \qquad (3.2.40a)$$

$$\frac{d}{d\tau} T^{\mu}{}_{\nu}(\tau)\Big|_{\tau=0} = \alpha^k \lambda^r_k(\theta)\, X_r(a(0))\, \delta^{\mu}{}_{\nu}, \qquad (3.2.40b)$$

can be written

$$T^{\mu}{}_{\nu}(\tau) = \sum_{m=0}^{\infty} \frac{1}{m!}\left[\theta^k X_k(a(0))\, \delta^{\mu}_{\nu}\right]^m \quad (3.2.41)$$

yielding the exponential mapping

$$a'^{\mu} = e^{\theta^k X_k(a)}\, a^{\mu} = T_a(\theta)\, a^{\mu} \quad (3.2.42)$$

If, instead of the variables of the base manifold, we have a function of the same variables, the procedure also applies and we can write

$$A(a') = e^{\theta^k X_k(a)}\, A(a) \qquad (3.2.43)$$

The infinitesimal generators can then be recovered through the rule

$$X_k = \left[\frac{\partial T_a(\theta)}{\partial \theta^k}\right]_{\theta=0}. \qquad (3.2.44)$$

In particular, realization (3.2.42) of the group transformation $f^{\mu}(a;\theta)$ is manifestly connected and its verification of the remaining properties to qualify as a Lie group is trivial.

The product of two elements of the group can be written in abstract notation

$$e^{\beta} e^{\alpha} = e^{\rho} \qquad (3.2.45)$$

where the element ρ is given by the so-called Baker-Campbell-Hausdorff formula

$$\rho = \alpha + \beta + \frac{1}{2}[\alpha,\beta] + \frac{1}{12}\left[(\alpha-\beta),[\alpha,\beta]\right] + \dots \quad (3.2.46)$$

In a more explicit form, Eqs. (3.2.46) can be written

$$e^{X_\beta} e^{X_\alpha} = e^{X_\rho} \qquad (3.2.47)$$

where

$$X_\rho = X_\alpha + X_\beta + \frac{1}{2}[X_\alpha, X_\beta]_A + \frac{1}{12}[(X_\alpha - X_\beta), [X_\alpha, X_\beta]_A]_A + \cdots \qquad (3.2.48)$$

hold, for sufficiently small values of the parameters.

As a final remark, let us recall that Lie's transformation groups trivially satisfy the associativity law of composition, e.g., in one of the forms

$$(ab)c = a(bc) \quad , \qquad (3.2.49a)$$

$$(T_a T_b) T_c = T_a (T_b T_c), \qquad (3.2.49b)$$

$$g(ab, c) = g(a, bc). \qquad (3.2.49c)$$

where the (abstract) group elements are denoted with the symbols a, b, and c.

For completeness, let us indicate that in our review of the exponential mapping we have tacitly implemented all convergence conditions, also called integrability conditions for the existence of a (this time, finite) group. For brevity we here refer the reader to the specialized literature on this subject, such as reference[132]. For a study of the case when these integrability conditions can be violated, see reference[133].

As indicated earlier in this section, we are primarily interested in the application of Lie's theory to Hamiltonian systems. This calls for a review of the canonical realization of (connected) Lie groups. In turn, this implies the restriction to those transformations in phase space which preserve the Hamiltonian form of the equations of motion. Such a restriction is particularly significant for relativity considerations, as we shall recall in Chapter 5.

Let us review the canonical realizations of infinitesimal Lie transformations. They are given by the infinitesimal canonical transformations reviewed in Section 2.7. And indeed, Eqs. (2.7.17), i.e.,

$$da^\mu \to \delta a^\mu = \delta\theta^k \, \omega^{\mu\nu} \frac{\partial G_k}{\partial a^\nu} \quad , \qquad (3.2.50)$$

yield a canonical realization of Eqs. (3.2.18), i.e.,

$$U^\mu_{\ k}(a) \longrightarrow \omega^{\mu\nu} \frac{\partial G_k(a)}{\partial a^\nu} \qquad (3.2.51)$$

In turn, this yields a canonical realization of the infinitesimal generators (3.2.23), i.e.,

$$X_k = U^\mu_{\ k}(a) \frac{\partial}{\partial a^\mu} \longrightarrow \omega^{\mu\nu} \frac{\partial G_k(a)}{\partial a^\nu} \frac{\partial}{\partial a^\mu} \qquad (3.2.52)$$

The fundamental Lie rule (3.2.20) is then realized in terms of the conventional Poisson brackets (up to additive neutral elements of the universal enveloping associative algebra, see Section 3.4 in this respect)

$$[X_i, X_j]_A = C_{ij}^{\ k} X_k \longrightarrow [G_i, G_j]_{Poisson} = C_{ij}^{\ k} G_k \qquad (3.2.53)$$

Finally, the canonical realization of the exponential mapping (3.2.42) can be reached as follows. Write Eqs. (3.2.5)) in the form

$$\frac{\partial a'^{\mu}}{\partial \theta^i} = [a'^{\mu}, G_i]_{(a')} \qquad (3.2.55)$$

By performing an identity isotopic transformation (i.e., an ordinary canonical transformation) we can write

$$\frac{\partial a'^{\mu}}{\partial \theta} = [a'^{\mu}(a;\theta), G(\theta)]_{(a)} \qquad (3.2.56)$$

The above expressions can be interpreted as a system of partial differential equations in the unknown (new) phase space variables a'^{μ} subject to the initial conditions

$$a'^{\mu}(a;\theta)\Big|_{\theta=0} = a^{\mu} \qquad (3.2.57)$$

A solution can be identified via a (formal) power series expansion for sufficiently small values of the parameters and we write

$$a'^{\mu} = a^{\mu} + \frac{\theta^k}{1!}[a^{\mu}, G_k] + \frac{\theta^i\theta^j}{2!}[[a^{\mu}, G_i], G_j] + \dots$$

$$= e^{\theta^k \omega^{\alpha\beta} \frac{\partial G_k}{\partial a^{\beta}} \frac{\partial}{\partial a^{\alpha}}} a^{\mu} \qquad (3.2.58)$$

For a function of the phase space variables we similarly have

$$A(a') = e^{\theta^k \omega^{\alpha\beta} \frac{\partial G_k}{\partial a^{\beta}} \frac{\partial}{\partial a^{\alpha}}} A(a) \qquad (3.2.59)$$

The composition of two transformations is then given by

$$\exp\left\{\theta'^k \omega^{\mu\nu} \frac{\partial G_k}{\partial a^{\nu}} \frac{\partial}{\partial a^{\mu}}\right\} \exp\left\{\theta^k \omega^{\alpha\beta} \frac{\partial G_k}{\partial a^{\beta}} \frac{\partial}{\partial a^{\alpha}}\right\}$$

$$= \exp\left\{(\theta'^k + \theta^k)\omega^{\rho\sigma} \frac{\partial G_k}{\partial a^{\sigma}} \frac{\partial}{\partial a^{\rho}} + \frac{1}{2}[\theta^i G_i, \theta'^j G_j]\right.$$

$$\left. + \frac{1}{12}\left[(\theta^i G_i - \theta'^j G_j), [\theta^i G_i, \theta'^j G_j]\right] + \dots\right\} \qquad (3.2.59)$$

where the brackets $[A,B]$ are now the conventional Poisson brackets.

Group (3.2.57) constitutes a canonical realization of a connected Lie group. We recover in this way a central aspect of the conventional transformation theory of Hamilton's equations without external terms: finite canonical transformations which are connected to the identity form a Lie group. This includes familiar transformations such as space-time translations, rotations and Galilei boosts.

3.3: LIE-ISOTOPIC AND LIE-ADMISSIBLE COVERINGS OF LIE'S FIRST, SECOND, AND THIRD THEOREM

In this section we shall study the problem whether Lie's first, second and third theorem admit a Lie-admissible covering (Step 3 of Section 3.1). More explicitly, this objective consists of the study whether there exist generalized transformations which exhibit a Lie-admissible characterization in the neighborhood of the identity while preserving a global, connected, Lie group.

Before entering into the study of this problem, it is advisable to identify the intermediate step of the Lie covering of Lie's first, second and third theorem (Step 2 of Section 3.1). We are here referring to a possible generalized form of the transformation theory which preserves a global, connected, Lie group while in the neighborhood of the identity demands the use of the generalized Poisson brackets.

An application of this latter framework has been identified in Section I.2.11 within the context of exact symmetries and conservation laws in a way independent from the Lie-admissible analysis for nonconservative systems considered in this volume. We are here referring to the notion of <u>isotopically related Lie groups</u> (<u>algebras</u>), that is, generally nonisomorphic symmetry groups (algebras) which lead to the same conservation laws.

In order to assist the reader in a better identification of this latter objective, let us review a specific example of isotopically related Lie groups (and algebras).

Consider the three-dimensional isotropic harmonic oscillator

in the familiar canonical representation

$$\dot{a}^\mu - \omega^{\mu\nu}\frac{\partial H^c}{\partial a^\nu} = 0, \quad (\omega^{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (3.3.1a)$$

$$\{a^\mu\} = \{r, p\}, \quad H^c = \frac{1}{2}\left[(p_x^2+p_y^2+p_z^2) - (r_x^2+r_y^2+r_z^2)\right], \quad m=k=1. \qquad (3.3.1b)$$

It exhibits an exact symmetry under the group of rotation SO(3). The generators of the Lie algebra <u>SO</u>(3) are the components of the conserved angular momentum

$$M = r \times p = \text{cons.} \qquad (3.3.2)$$

with the familiar commutation rules

$$[M_1, M_2] = M_3, \quad [M_2, M_3] = M_1, \quad [M_3, M_1] = M_2. \qquad (3.3.3)$$

A canonical realization of the group SO(3), by using Eqs. (3.2.57), is then given by

$$a'^\mu = e^{\theta^k \omega^{\alpha\beta}\frac{\partial M_k}{\partial a^\beta}\frac{\partial}{\partial a^\alpha}} a^\mu \qquad (3.3.4)$$

where the θ's are the <u>Euler angles</u>.

The notion of isotopically related Lie groups originates from the fact that, by no means, the groups SO(3) is the only symmetry group which leads (via Noether's theorem) to the conservation laws (3.2.2). To identify a nonisomorphic (Lie) symmetry group which leads to the <u>same</u> conservation laws consider the equivalent Hamiltonian[134]

$$H^{*c} = \frac{1}{2}\left[(p_x^2 - p_y^2 + p_z^2) - (r_x^2 - r_y^2 + r_z^2)\right] \qquad (3.3.5)$$

Clearly, this Hamiltonian violates the SO(3)-symmetry. Nevertheless, the conservation laws (3.3.2) are unaffected because they hold in virtue of the equations of motion, while the systems represented by Hamiltonians (3.3.1b) and (3.3.5) are trivially equivalent. This implies that the equivalent Hamiltonian (3.3.5) is expected to possess a new symmetry group which leads to the original conservation laws (3.3.2).

A study of this problem (Sect. I.2.1.11) indicates that the new manifest symmetry of Hamiltonian (3.3.5) which leads to the conservation laws of the angular momentum is the Lorentz group in (2+1)-dimension, the group SO(2.1).

Now, the groups SO(3) and SO(2.1) are nonisomorphic. The conventional attitude in the realization of the Lie algebras SO(3) and SO(2.1) is that of using nonequivalent generators which lead to the correct form of the closure rules: the realization of SO(3) in terms of the angular momentum components and closure rules (3.3.3), and the realization of SO(2.1) in terms of different generators and closure rules, i.e.,

$$[J_1, J_2] = J_3\,, \quad [J_2, J_3] = -J_1\,, \quad [J_3, J_1] = J_2, \qquad (3.3.6)$$

In our case a somewhat opposite attitude is needed. The physical relevance of the SO(2.1) symmetry of Hamiltonians (3.3.5) is that of representing the conservation of the angular momentum. Therefore, both algebras SO(3) and SO(2.1) must be realized in terms of the same generators. But, in order to produce a nonisomorphic Lie algebra , the only possibility which remains is

that of modifying the brackets.

We reach in this way the canonical realization of the SO(2.1) algebra in terms of the angular momentum components

$$[M_1, M_2]^* = M_3\,, \quad [M_2, M_3]^* = -M_1\,, \quad [M_3, M_1]^* = M_2 \qquad (3.3.7a)$$

$$[A, B]^* = \frac{\partial A}{\partial a^\mu}\, \Omega^{\mu\nu}\, \frac{\partial B}{\partial a^\nu} \qquad (3.3.7b)$$

$$(\Omega^{\mu\nu}) = \begin{pmatrix} O_{3\times 3} & \begin{pmatrix} +1 & & 0 \\ & -1 & \\ 0 & & +1 \end{pmatrix} \\ \begin{pmatrix} -1 & & 0 \\ & +1 & \\ 0 & & -1 \end{pmatrix} & O_{3\times 3} \end{pmatrix} \qquad (3.3.7c)$$

The canonical realization of the group SO(2.1) is then given by

$$a^{*\mu} = e^{\theta^k \Omega^{\alpha\beta} \frac{\partial M_k}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}}\, a^\mu \qquad (3.3.8)$$

To summarize, the base manifold with local coordinates a^μ, the generators $M_i(a)$ and the parameters θ^k remain unchanged in the transition from the SO(3) structure (3.3.4) to the SO(2.1) image (3.3.8), while the (contravariant form of) the fundamental symplectic tensor $\omega^{\mu\nu}$ is modified into a different form $\Omega^{\mu\nu}$ which, as we shall see in Chapter 4, is still Lie. In particular, brackets (3.3.7b), even though generalized, are still Lie. Finally, even though the parameters θ^k are unchanged in the mapping from structure (3.3.4) to (3.3.8), their range is not preserved. Thus, in the latter structure they do not constitute, strictly speaking, Euler's angles.

The Lie algebras (3.3.3) and (3.3.7a) (groups (3.3.4) and (3.3.8)) so constructed have been called isotopically related Lie algebras (groups).[65,189]

The reader should be aware that in the simple example here considered the Lie tensor $\Omega^{\mu\nu}$ still possess constant elements, while this is not necessarily the case in general. More specifically, the transition from a Lie algebra G to one of its isotopic images G* generally demands the use of Lie tensors with an arbitrary (but class C^{∞}) dependence on the (local) variables. For the relativity analysis of Chapter 5 the reader should also recall from Chapter I.2 that this mechanism of symmetry breaking is also applicable, in principle, to relativity groups, yielding what we have called isotopically related Galilei and Poincaré groups. Nevertheless, the breakings are purely formal on relativity grounds because the original, relativity conservation laws are unaffected by construction. Therefore, these isotopic images of the relativity groups (when they exist) have no direct bearing on the problem of the breaking of Galilei and Einstein relativities due to nonconservative and form-nonvariant forces. Finally, the reader should keep in mind that the notion of algebraic isotopy was presented in Sections 1.2 and 1.5 on mainly algebraic grounds, but without consideration to the possible existence of a group image.

The objective of the first part of this section (Step 2) can now be better identified. Realization (3.3.8) of the Lie group SO(2.1) represents a genuine generalization of Lie's theory which is outside its conventional treatment, as reviewed in Section 3.2. This is primarily due to the appearance of the generalized symplectic form $\Omega^{\mu\nu}$ which is not realizable in terms of the Lie

fundamental rule (3.2.30). To preserve a group structure which is isomorphic to the conventional treatment of the SO(2.1) group, structure (3.3.8) also demands a joint reformulation of the generators which, again, is not realizable in terms of their standard form (3.2.23), as in Eqs. (3.2.52).

Step 2 of our analysis of this section therefore consists in the identification of a generalization of given Lie transformations which is such to: (a) preserve the base manifold, the generators and the parameters of the original, connected, Lie group, and (b) produce a generally nonisomorphic but equally connected Lie group whose behaviour in the neighborhood of the identity is still Lie, although of a generalized nature. This objective simply constitutes a realization within the context of the transformation theory of the abstract notion of algebraic isotopy of Chapter 1.

DEFINITION 3.3.1: Suppose that the transformations

$$a'^{\mu} = f^{\mu}(a;\theta) \qquad (3.3.9)$$

constitute an n-dimensional (connected) Lie group G. A Lie isotopic image (or, more simply, an isotope) of G is an n-dimensional connected Lie group G* which is representable in terms of the original transformations (3.3.9) via $36N^2$ multiplicative functions $g^{\mu}{}_{\nu}(a;\theta)$, called isotopic functions,

$$a^{*\mu} = f^{*\mu}(a;\theta) \equiv g^{\mu}{}_{\nu}(a;\theta)\, f^{\nu}(a;\theta) \qquad (3.3.10)$$

<u>in such a way to preserve the Lie character of the infinitesimal transformations when expressed in terms of the generators of G.</u>

Notice that, by assumption, both sets of transformations $f^\mu(a;\theta)$ and $f^{*\mu}(a;\theta)$ constitute connected Lie groups of the same dimensionality. This implies that properties (1) through (5) of Section 3.2 (for the transformations $f^{*\mu}(a;\theta)$ to be Lie) holds. However, mapping (3.3.10) of the original transformations (3.3.9) implies that the groups G and G* are generally nonisomorphic. This is due to the fact that composition law (3.2.5) is modified into a new law, say,

$$\theta''^i = g^{*i}(\theta,\theta'), \qquad (3.3.11)$$

which is not, in general, analytically isomorphic to the original law, as requested by the nontrivial functional dependence of the isotopic functions. In essence, as can be seen later on in this section, the <u>analytic isomorphy</u> of the composition law implies possible change of basis within the context of the same product, the same algebra and, thus, the same structure constants; while the <u>algebraic isotopy</u> of the composition law implies the preservation of the same basis, but the change of the product, yielding generally different structure constants and, thus, generally nonisomorphic algebras.

Since the isotopically mapped transformations $f^{*\mu}(a;\theta)$ are connected, they can also be studied in terms of their behaviour in the neighborhood of the identity. The repetition of the

analysis of Section 3.2, from Eqs. (3.2.10) to (3.2.19), is trivial, yielding the infinitesimal transformations

$$da^\mu = u^{*\mu}{}_k(a)\, \lambda^k{}_i(\theta)\, d\theta^i , \qquad (3.3.12a)$$

$$u^{*\mu}{}_k(a) = \left| \frac{\partial}{\partial\theta^k} g^\mu{}_\nu(a;\theta)\, f^\nu(a;\theta) \right|_{\theta=0} . \qquad (3.3.12b)$$

In order to identify the conditions under which these transformations constitute an isotopic image of transformations (3.2.19) (that is, are Lie in algebraic character) we introduce the following factorizations

$$u^{*\mu}{}_k(a) = g_k{}^i(a)\, u^\mu{}_i(a) \qquad (3.3.13)$$

which, as the reader can verify, are always possible. In particular, they are such that

$$\lim_{g^\mu{}_\nu \to \delta^\mu{}_\nu} u^{*\mu}{}_k(a) = u^\mu{}_k(a), \qquad (3.3.14)$$

in which case we have the <u>identity isotopic transformations</u>.

The following Lie covering of <u>Lie's first theorem</u> then holds.

<u>THEOREM</u> 3.3.1: <u>If the transformations</u>

$$a^{*\mu} = f^{*\mu}(a;\theta) = g^\mu{}_\nu(a;\theta)\, f^\nu(a;\theta) \qquad (3.3.15)$$

<u>characterize an isotopic image G* of a Lie group G of transformations $f^\mu(a;\theta)$, then</u>

$$\frac{\partial a^{*\mu}}{\partial \theta_j} = g_i{}^k(a)\, u^\mu_k(a)\, \lambda^i{}_j(\theta) \qquad (3.3.16)$$

<u>where the functions $g_j{}^k(a)$ and $u_k{}^\mu(a)$ are analytic.</u>

It should be here indicated that the analytic character of the functions $g_j^{\ k}(a)$ and $u_k^{\ \mu}(a)$ could, in principle, be differentiated as far as the region of respective analyticity is concerned.

Theorem 3.3.1 is clearly a simple generalization of Theorem 3.2.1. Nevertheless, the algebraic implications of the former theorem are nontrivial because it implies a modification of the behavior of the group in the neighborhood of the identity as can be seen by comparing the first-order expansion of the isotopically mapped context

$$a^{*\mu} = a^{\mu} + \theta^{i} g_i^{\ j}(a) u^{\mu}_{\ j}(a) + \cdots \qquad (3.3.17)$$

with the corresponding expansion of the original structure

$$a'^{\mu} = a^{\mu} + \theta^{i} u^{\mu}_{\ i}(a) + \cdots \qquad (3.3.18)$$

We are now interested in identifying the integrability conditions on the $g_i^{\ k}(a)$ functions to characterize a (Lie) algebraic isotopy. First of all, let us recall that Eqs. (3.2.27), (3.2.28) and (3.2.30) hold for the original group by assumption, i.e.,

$$u^{\nu}_{\ i} \frac{\partial}{\partial a^{\nu}} u^{\mu}_{\ j} - u^{\nu}_{\ j} \frac{\partial}{\partial a^{\nu}} u^{\mu}_{\ i} = C_{ij}^{\ \ k} u^{\mu}_{\ k}, \qquad (3.3.19a)$$

$$C_{ij}^{\ \ k} = \mu^{r}_{\ i} \mu^{s}_{\ j} \left\{ \frac{\partial \lambda_r^{\ k}}{\partial \theta^{s}} - \frac{\partial \lambda_s^{\ k}}{\partial \theta^{r}} \right\}, \qquad (3.3.19b)$$

$$[X_i, X_j]_A = X_i X_j - X_j X_i = C_{ij}^{\ \ k} X_k, \qquad (3.3.19c)$$

$$X_i = u^{\mu}_{\ i}(a) \frac{\partial}{\partial a^{\mu}}. \qquad (3.3.19d)$$

Secondly, the group G* with transformations $f^*(a;\theta)$ is also Lie and, thus, can be subjected to the same standard realization as that of G, i.e.,

$$u^{*\nu}_{\ \ i} \frac{\partial}{\partial a^{\nu}} u^{*\mu}_{\ \ j} - u^{*\nu}_{\ \ j} \frac{\partial}{\partial a^{\nu}} u^{\mu}_{\ i} = C^{*\ k}_{ij} u^{*\mu}_{\ \ k}, \qquad (3.3.20a)$$

$$C^{*\ k}_{ij} = \mu^{*r}_{\ \ i} \mu^{*s}_{\ \ j} \left\{ \frac{\partial \lambda^{*k}_{r}}{\partial \theta^{s}} - \frac{\partial \lambda^{*k}_{s}}{\partial \theta^{r}} \right\}, \qquad (3.3.20b)$$

$$[X^*_i, X^*_j]_A = X^*_i X^*_j - X^*_j X^*_i = C^{*\ k}_{ij} X^*_k, \qquad (3.3.20c)$$

$$X^*_i = u^{*\mu}_{\ \ i}(a) \frac{\partial}{\partial a^{\mu}}. \qquad (3.3.20d)$$

However, this conventional realization of the isotope G* has only a secondary significance for our approach because it implies a change of the generators, i.e.,

$$X_i = u^{\mu}_{\ i} \frac{\partial}{\partial a^{\mu}} \longrightarrow X^*_i = u^{*\mu}_{\ \ i} \frac{\partial}{\partial a^{\mu}} \qquad (3.3.21)$$

while, by central requirement, the groups G and G* must be realized in terms of the same generators X_i.

In order to achieve this objective, we inspect Lie's fundamental rule (3.3.19c) for the original group G and image (3.3.20c) for the isotope G*. In essence, in the transition from G to G* we have the preservation of the associative algebra and a (generally nonisomorphic) change of the generators, i.e.,

$$A: \quad X_i X_j \longrightarrow X_i^* X_j^* \qquad (3.3.22)$$

Our contention is that rule (3.3.20c) can be equivalently achieved by preserving the generators X_i of G, and by changing instead the associative product $X_i X_j$ into a new product, say $X_i * X_j$, which is still associative. This is precisely a realization of the notion of <u>isotopic mapping of the associative product</u> (Section 1.2).

For the case at hand we assume that the associative algebra A with basis X_i and product $X_i X_j$ is mapped into an isotopic image A* with the same basis X_i, but new product

$$A: X_i X_j \Rightarrow A^*: X_i * X_j = g_i^{\ r} X_r \, g_j^{\ s} X_s \qquad (3.3.23)$$

$$X_r = u^\mu_{\ r} \frac{\partial}{\partial a^\mu}$$

The fact that this broader product satisfy the associativity law, i.e.,

$$(g_i^{\ r} X_r \, g_j^{\ s} X_s) \, g_k^{\ t} X_t = g_i^{\ r} X_r (g_j^{\ s} X_s \, g_k^{\ t} X_t) \qquad (3.3.24)$$

ensures that A* is an isotope of A. The implications of this mechanism for the universal envolping associative algebras will be pointed out in Section 3.5.

By using product (3.3.23), the fundamental Lie's rule (3.3.20c) can now be rewritten

$$u^\nu_{\ i} \frac{\partial}{\partial a^\nu} * u^\mu_{\ j} - u^\nu_{\ j} \frac{\partial}{\partial a^\nu} * u^\mu_{\ i} = C'^{\,k}_{ij} \, u^\mu_{\ k}, \qquad (3.3.25a)$$

$$C'^{\,k}_{ij} = C^{*\,r}_{ij} \, g_r^{\ k}(a) \qquad (3.3.25b)$$

The necessary and sufficient conditions for the functions $g_j^{\ k}(a)$ to be isotopic, i.e., to characterize a Lie algebraic isotopy of rule (3.3.19c), can then be written

$$g_i^{\ k} u^\nu_{\ k} \frac{\partial}{\partial a^\nu} g_j^{\ \ell} - g_j^{\ k} u^\nu_{\ k} \frac{\partial}{\partial a^\nu} g_i^{\ \ell} \qquad (3.3.26)$$

$$= g_j^{\ r} g_i^{\ s} C_{rs}^{\ \ell} + C^{*\,r}_{ij} g_r^{\ \ell}$$

We reach in this way the following <u>Lie's covering of Lie's second theorem</u>.

<u>THEOREM 3.3.2: The generators $X_i(a)$, i=1,2,...,n, of an isotope G* of a Lie group G satisfy the relations</u>

$$[X_i, X_j]_{A^*} = X_i * X_j - X_j * X_i = C'^{\,k}_{ij}(a) \, X_k, \qquad (3.3.27)$$

<u>where the quantities $C'^{\,k}_{ij}(a)$ (here called Lie structure functions) are generally dependent on the (local) coordinates of the manifold of the original group G.</u>

Under conditions (3.3.26), the left hand side of Eqs. (3.3.27) characterizes a Lie product, although in an isotopically mapped version of the conventional form. This implies that Lie's identities (3.2.33) still hold, i.e.,

$$[X_i, X_j]_{A*} + [X_j, X_i]_{A*} = 0, \qquad (3.3.29a)$$

$$[[X_i, X_j]_{A*}, X_k]_{A*} + [[X_j, X_k]_{A*}, X_i]_{A*} + [[X_k, X_i]_{A*}, X_j]_{A*} = 0. \qquad (3.3.29b)$$

We reach in this way the following Lie covering of Lie's third theorem.

THEOREM 3.3.3: The structure functions of an isotope G* of a Lie group G satisfy the identities

$$C'^{\,k}_{ij} + C'^{\,k}_{ji} = 0, \qquad (3.3.30a)$$

$$C'^{\,k}_{ij} C'^{\,r}_{ke} + C'^{\,k}_{je} C'^{\,r}_{ki} + C'^{\,k}_{ei} C'^{\,r}_{kj} + [C'^{\,r}_{ij}, X_e]_{A*} + [C'^{\,r}_{je}, X_i]_{A*} + [C'^{\,r}_{ei}, X_j]_{A*} = 0 \qquad (3.3.30b)$$

In essence, the above theorems indicate that the constancy of the structure quantities of a Lie group is a direct consequence of the use of the standard realization of Section 3.2. If a different realization of the same group is assumed, these structure quantities can acquire a dependence on the local coordinates, but in such a way to still yield closure rule and to preserve the Lie identities.

The reason for the use of the terms "Lie covering" of Lie's theorems can now be better indicated. Even though the group G* is realized in terms of the generators of another group, and the structure quantities are no longer constants and the behaviour in the neighborhood of the identity is modified, the generalized framework is still Lie in algebraic character, as guaranteed by the integrability conditions (3.3.26).

We shall leave to the interested reader the study of topics such as the implications of Theorem 3.3.2 for the representation theory (particularly for the adjoint representations) and the construction of the inverse theorems.

Under the tacit assumption of the validity of all necessary convergence conditions, the generalization of the exponential mapping (3.2.42) is straightforward. This mapping holds for the isotope G* in its conventional realization, i.e.,

$$a^{*\mu} = e^{\theta^k X^*_k(a)} a^\mu = T^*_a(\theta)\, a^\mu \qquad (3.3.30)$$

Redefinition $X^*_k = g^i_k(a)\, X_i(a)$ then implies the isotopic image of the exponential mapping

$$a^{*\mu} = e^{\theta^k g_k{}^i X_i(a)}\, a^\mu. \tag{3.3.31}$$

The composition law, Eq. (3.2.47) then reads

$$e^{X^*_\beta}\, e^{X^*_\alpha} = e^{X^*_\gamma}, \tag{3.3.32}$$

yielding the isotopic image of the Baker-Campbell-Hausdorff formula

$$g^\alpha X_\alpha = g^\rho_\alpha X_\rho + g^\sigma_\beta X_\sigma + \frac{1}{2}[X_\alpha, X_\beta]_{A^*} + \frac{1}{12}\left[(X_\alpha - X_\beta), [X_\alpha, X_\beta]_{A^*}\right]_{A^*} + \cdots . \tag{3.3.33}$$

This ensures that the isotope G* is a finite group of connected transformations also in reformulation (3.3.31).

As predictably, the realization of the Lie isotopic mapping in Newtonian mechanics demands the use of a Lie covering of the conventional Hamilton's equations. Such a covering is provided by Birkhoff's equations (2.1.8). The transformation theory of these broader equations (that of generalized canonical transformations) is therefore expected to provide a classical realization of the isotopic context here considered.

In fact, the infinitesimal generalized transformations

$$da^\mu \rightarrow \delta a^\mu = \delta\theta^k\, \Omega^{\mu\nu}(a)\, \frac{\partial G_k}{\partial a^\nu} \tag{3.3.34}$$

provide a realization of the isotopic transformations (3.3.12), i.e.,

$$U^{*\mu}{}_k = g_k{}^i(a)\, U^\mu_i(a) \rightarrow \Omega^{\mu\nu}(a)\, \frac{\partial G_k}{\partial a^\nu}. \tag{3.3.35}$$

Brackets (3.3.27) can then be realized in terms of the generalized Poisson brackets (again, up to neutral elements of the center of the enveloping algebra), i.e.,

$$[X_i, X_j]_{A^*} \rightarrow [G_i, G_j]^* = \frac{\partial G_i}{\partial a^\mu}\, \Omega^{\mu\nu}(a)\, \frac{\partial G_j}{\partial a^\nu}. \tag{3.3.36}$$

Finally, the generalized canonical realization of the isotopically mapped exponential law, Eq. (3.3.31), can be reached in a way fully parallel to that of Eqs. (3.2.57). Write Eqs. (3.3.34) in the form

$$\frac{\partial a^{*\mu}}{\partial \theta^k} = [a^{*\mu}, G_k]^*_{(a^*)} \tag{3.3.37}$$

and perform an identity isotopic transformation (i.e., a generalized canonical transformation) to fixed coordinates a^μ, i.e.,

$$\frac{\partial a^{*\mu}(a;\theta)}{\partial \theta^k} = [a^{*\mu}(a;\theta), G_k(\theta)]^*_{(a)}. \tag{3.3.38}$$

These equations can again be interpreted as a system of partial differential equations in the unknown coordinates $a^{*\mu}$ subject to the initial conditions

$$a^{*\mu}(a;\theta)\big|_{\theta=0} = a^\mu \tag{3.3.39}$$

A solution can be expressed in terms of a (formal) power series expansion (for sufficiently small values of the parameters)

$$a^{*\mu} = a^\mu + \frac{\theta^i}{1!}[a^\mu, G_i]^* + \frac{\theta^i\theta^j}{2!}[[a^\mu, G_i]^*, G_j]^* + \cdots$$

$$= e^{\theta^k \Omega_{(k)}^{\mu\gamma}(a) \frac{\partial G_k}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}} a^\mu \qquad (3.3.40)$$

which yields the desired realization of Eqs. (3.3.31).

To summarize, what we have studied in this section until now is simply an application of the notion of algebraic isotopy within the context of Lie theory. The isotopy acts on the fundamental rule of Lie's theory, rule (3.3.19c), and more specifically, on the associative algebra A which characterize this rule. Furthermore, this algebraic isotopy is complemented with a corresponding group isotopy. Finally, the isotopically mapped context exhibits a direct analytic realization in terms of the transformation theory of the self-adjointness preserving (and, thus, Lie preserving) generalization of Hamilton's equations (Birkhoff's equations).[189]

As we shall see in Section 3.5, the true origin of the mechanism does not rest within the context of the Lie algebras "per sé". Instead, it originates at the level of the universal enveloping associative algebras which do admit consistent isotopic images induced by mapping (3.3.23). This, in turn, implies both an infinitesimal and a global Lie isotopy and, as such, can be considered as the primitive origin of the notion of Lie isotopy.

We are now equipped to study the possible exitence of a Lie-admissible covering of Lie's theory (Step 3 of Section 3.1). The objective is now that of attempting a mapping of Lie's theory which does not preserve the Lie algebraic character in the neighborhood of the identity, but which induces instead a Lie-admissible character. This is precisely in line with the concept of algebraic

genotopy of Section 1.2 as a natural generalization of that of algebraic isotopy.

DEFINITION 3.3.2: Suppose that the transformations

$$a'^\mu = f^\mu(a;\theta) \qquad (3.3.41)$$

constitute an n-dimensional (connected) Lie group G. A Lie-admissible genotopic image (or, more simply, a genotope) of G is an n-dimensional connected Lie group $\hat{G}$ which is representable in terms of the original transformations (3.3.41) via $36N^2$ multiplicative functions $h^\mu{}_\nu(a;\theta)$ called genotopic functions

$$\hat{a}^\mu = \hat{f}^\mu(a;\theta) = h^\mu{}_\nu(a;\theta)\, f^\nu(a;\theta) \qquad (3.3.42)$$

in such a way to induce a Lie-admissible character of the infinitesimal transformations when expressed in terms of the generators of G.

To reach a first identification of the meaning of the last part of this definition, let us recall that for the original transformations (3.3.41) we have a conventional Lie infinitesimal behaviour, i.e.,

$$a'^\mu \cong a^\mu + \theta^i \left[\frac{\partial f^\mu}{\partial \theta^i}\right]_{\theta^i} = a^\mu + \theta^i u^\mu_i(a) = a^\mu + \theta^i [x_i, a^\mu]_A \qquad (3.3.43)$$

where $[x_i, a^\mu]_A$ is the conventional product in the attached algebra A^- of A. In the transition to the isotopically mapped context we have a modified infinitesimal character which, however, is still Lie, i.e.,

$$a^{*\mu} \approx a^\mu + \left|\frac{\partial}{\partial\theta^i} g^\mu_\nu f^\nu\right|_{\theta=0} \theta^i = a^\mu + g_i^{\ k} u^\mu_k \theta^i$$
$$= a^\mu + [X_i, a^\mu]_{A^*}, \quad X_i = u^\mu_k \frac{\partial}{\partial a^\mu} \tag{3.3.44}$$

where $[X_i, a^\mu]_{A^*}$ is now a generalized Lie product in the attached

algebra A^{*-} of the isotope A* of A. In the transition to a Lie-admissible genotopically mapped context, Definition 3.3.2 is intended to express the condition that the infinitesimal behaviour is further generalized to the point of losing the Lie character, but acquiring a Lie-admissible character, again, when expressed in terms of the generators of the original group. We shall then write

$$\hat{a}^\mu \cong a^\mu + \theta^i \left[\frac{\partial}{\partial\theta^i} h^\mu_{\ \nu}(a;\theta) f^\nu(a;\theta)\right]_{\theta=0}$$
$$= a^\mu + \theta^i (X_i, a^\mu) \tag{3.3.45}$$

where the product (X_i, a^μ) is not Lie by central requirement, but must be such to satisfy the Lie-admissibility condition (1.4.1), i.e.,

$$(X_i, a^\mu) - (a^\mu, X_i) = [X_i, a^\mu]_{A^*} \tag{3.3.46}$$

Our objective is that of studying the possible existence of realizations (3.3.45) of the infinitesimal behaviour of the Lie group $\hat{G}$ which satisfy the Lie-admissible condition (3.3.46). The reader should keep in mind from the outset that the Lie structure associated to the Lie-admissible context is of isotopic rather than conventional nature, as indicated by rule (3.3.46). This confirms the need for our analysis of the Lie covering of Lie's theory considered earlier in this section.

According to Definition 3.3.2, the genotope $\hat{G}$ of G is Lie and, thus, verifies properties (1) through (5) of Section 3.2 when subjected to the conventional realization in terms of transformations $\hat{f}^{\mu}(a;\theta)$. However, the composition law (3.2.5) is subjected to a broader generalization than that of law (3.3.11), and we shall write

$$\theta''^{i} = \hat{g}^{i}(\theta,\theta') \qquad (3.3.47)$$

The relation of this new composition law with the original law (3.2.5) is, in general, neither an analytic isomorphy nor an algebraic isotopy. This implies the following

THEOREM 3.3.4: A (connected) Lie group G, its isotope G* and its genotope $\hat{G}$ are generally nonisomorphic among themselves. In particular, both the isotopic and the genotopic mappings do not, in general, preserve, the Abelian or non-Abelian, compact or noncompact and semisimple or nonsemisimple character of the original group.

To put it in simpler terms, besides connectivity, the only character of the original group which is generally preserved in the isotopic and genotopic mappings is the dimensionality. The terms "generally nonisomorphic" of Theorem 3.3.4 are intended to take into account the fact that, both, the isotopic and the genotopic mappings include, as a particular case, the identity mapping. The study of the conditions under which nonidentity isotopic and genotopic mappings induce nonisomorphic groups is left to the interested reader.

Since the genotope $\hat{G}$ is a connected Lie group, it can also be studied in the neighborhood of the identity transformations. Eqs.

(3.2.10) through (3.2.19) still hold and we write

$$da^{\mu} = \hat{u}^{\mu}_{k}(a)\,\lambda^{k}_{i}\,d\theta^{i}, \qquad (3.3.48a)$$

$$\hat{u}^{\mu}_{k}(a) = \left[\frac{\partial}{\partial\theta^{k}} h^{\mu}_{\nu}(a;\theta)\,f^{\nu}(a;\theta)\right]_{\theta=0} \qquad (3.3.48b)$$

Our problem is now reduced to the study of an explicit form of the functions $\hat{u}_{i}^{\mu}(a)$ which is such to allow the preservation of the generators of the original group while admitting a Lie-admissible algebraic structure. More specifically, our problem is that of turning the integrability conditions for the conventional realization of the $\hat{G}$ group

$$\hat{u}^{\nu}_{i}\frac{\partial}{\partial a^{\nu}}\hat{u}^{\mu}_{j} - \hat{u}^{\nu}_{j}\frac{\partial}{\partial a^{\nu}}\hat{u}^{\mu}_{i} = \hat{C}^{k}_{ij}\,\hat{u}_{k}, \qquad (3.3.49a)$$

$$\hat{C}^{k}_{ij} = \hat{\mu}^{r}_{i}\,\hat{\mu}^{s}_{j}\left\{\frac{\partial\hat{\lambda}^{k}_{r}}{\partial\theta^{s}} - \frac{\partial\hat{\lambda}^{k}_{s}}{\partial\theta^{r}}\right\}, \qquad (3.3.49b)$$

$$[\hat{X}_{i},\hat{X}_{j}]_{A} = \hat{X}_{i}\hat{X}_{j} - \hat{X}_{j}\hat{X}_{i} = \hat{C}^{k}_{ij}\,\hat{X}_{k}, \qquad (3.3.49c)$$

$$\hat{X}_{k} = \hat{u}^{\mu}_{k}(a)\frac{\partial}{\partial a^{\mu}}, \qquad (3.3.49d)$$

in a form which is Lie-admissible when realized in terms of the generators X_i of the original group G.

A study of this problem indicates the necessity, for such an objective, of turning the associative algebra A with basis X_i into a nonassociative but Lie-admissible algebra U with basis X_i. This is precisely a realization of the notion of algebraic genotopy of an associative algebra (Section 1.2). On comparative grounds, the mapping from G to its isotope G* can be realized in terms of the isotopy (3.3.23) of the algebra A which, by construction, is still associative, while the mapping from G to its genotope $\hat{G}$ demands a mapping of the algebra A which, this time, does not preserve the associative law, also by construction, although it is Lie-admissible.

In conclusion, our objective is that of identifying a realization of the functions $\hat{u}_i^\mu(a)$ which is such to allow the mapping of the associative product $\hat{X}_i\hat{X}_j$ in terms of the standard generators $\hat{X}_i$ of $\hat{G}$ into a nonassociative but Lie-admissible product, say $X_i \circ X_j$, in terms of the standard generators X_i of the original group G, and we shall write

$$A : \hat{X}_i \hat{X}_j \Longrightarrow U : X_i \circ X_j, \qquad (3.3.50a)$$

$$[X_i, X_j]_U = X_i \circ X_j - X_j \circ X_i \equiv [X_i, X_j]_{A^*}, \qquad (3.3.50b)$$

where Eqs. (3.3.50b) should be interpreted as a condition on the product $X_i \circ X_j$.

To achieve this objective we introduce the following realization of the $\hat{u}_i^\mu(a)$ functions

$$\hat{u}^\mu_i(a) = [\alpha_i^r(a) + \beta_i^r(a)]\, u_r^\mu(a) \qquad (3.3.51)$$

which, as it was the case for realization (3.3.13) it is always possible. However, unlike the latter realization, Eqs. (3.3.51) exhibit the presence of at least n^2 redundant functions out of the set of $2n^2$ functions α_i^r and β_i^r.

To eliminate these n^2 redundant functions we introduce the n^2

subsidiary conditions on realization (3.3.51)

$$\alpha_i^r u_r^\nu \frac{\partial}{\partial a^\nu}(\beta_j^s u_s^\mu) + \beta_i^r u_r^\nu \frac{\partial}{\partial a^\nu}(\alpha_j^s u_s^\mu) - \alpha_j^r u_r^\nu \frac{\partial}{\partial a^\nu}(\beta_i^s u_s^\mu) - \beta_j^r u_r^\nu \frac{\partial}{\partial a^\nu}(\alpha_i^s u_s^\mu) = 0 \qquad (3.3.52)$$

under which rule (3.3.49a) becomes

$$\left[\alpha_i^r u_r^\nu \frac{\partial}{\partial a^\nu}(\alpha_j^s u_s^\mu) - \beta_j^r u_r^\nu \frac{\partial}{\partial a^\nu}(\beta_i^s u_s^\mu)\right] - \left[\alpha_j^r u_r^\nu \frac{\partial}{\partial a^\nu}(\alpha_i^s u_s^\mu) - \beta_i^r u_r^\nu \frac{\partial}{\partial a^\nu}(\beta_j^s u_s^\mu)\right] = \hat{C}_{ij}^{\ k}(\alpha_k^r + \beta_k^r)\, u_r^\mu . \qquad (3.3.53)$$

It is easy to see that the product

$$X_i \circ X_j \equiv \alpha_i^r u_r^\nu \frac{\partial}{\partial a^\nu}\left(\alpha_j^s u_s^\mu \frac{\partial}{\partial a^\mu}\right) - \beta_j^r u_r^\nu \frac{\partial}{\partial a^\nu}\left(\beta_i^s u_s^\mu \frac{\partial}{\partial a^\mu}\right) \qquad (3.3.54)$$

is a realization of the nonassociative, Lie-admissible, genotopic mapping (3.3.50). And indeed, product (3.3.54) is nonassociative on account of the properties

$$\begin{aligned}(X_i \circ X_j) \circ X_k = {} & \alpha_i^r u_r^\nu \frac{\partial}{\partial a^\nu}\left(\alpha_j^s u_s^\mu \frac{\partial}{\partial a^\mu}\right)\alpha_k^t u_t^\rho \frac{\partial}{\partial a^\rho} \\ & - \beta_k^r u_r^\nu \frac{\partial}{\partial a^\nu}\left[\alpha_i^s u_s^\mu \frac{\partial}{\partial a^\mu}\left(\alpha_j^t u_t^\rho \frac{\partial}{\partial a^\rho}\right)\right] \\ & - \beta_i^r u_r^\nu \frac{\partial}{\partial a^\nu}\left(\alpha_i^s u_s^\mu \frac{\partial}{\partial a^\mu}\right)\alpha_k^t u_t^\rho \frac{\partial}{\partial a^\rho} \\ & + \beta^r u_r^\nu \frac{\partial}{\partial a^\nu}\left[\beta_i^r u_r^\alpha \frac{\partial}{\partial a^\alpha}\left(u_i^s u_s^\mu \frac{\partial}{\partial a^\mu}\right)\right] \\ \neq {} & X_i \circ (X_j \circ X_k)\end{aligned} \qquad (3.3.55)$$

The fact that product (3.3.54) is Lie-admissible is ensured by Eqs. (3.3.53).

In conclusion, product (3.3.54) allows the generalization of the fundamental Lie rule (3.2.30)

$$[X_i, X_j]_U = X_i \circ X_j - X_j \circ X_i = \hat{C}^{*k}_{ij} X_k , \qquad (3.3.56a)$$

$$\hat{C}^{*k}_{ij} = C_{ij}^{r} [\alpha_r^{k}(a) + \beta_r^{k}(a)] \qquad (3.3.56b)$$

which holds for the attached algebra U^- of a nonassociative algebra U. This is precisely a realization within the context of the transformation theory of the primitive notion of Lie-admissible algebra, Eq. (1.4.1).

The reader should be aware that form (3.3.56) is the broadest possible generalization of the fundamental rule of Lie's theory which preserves a Lie algebra content. And indeed, any further generalization of the nonassociative product $X_i \circ X_j$, that is, a generalization which is not Lie-admissible, would imply the loss of fundamental rule (3.3.56). In turn, as we shall illustrate in due time, this would imply the lack of a group structure and the consequential general nonapplicability to the theory of finite, connected, transformations.

To restate our findings in different terms, the notion of Lie-admissibility is at the very foundation of Lie's theory, only expressed in its simplest possible form, the attached form A^- of an associative algebra A. When seen from this profile, the

existence of a generalization of Lie's theory in terms of the attached algebra U^- of a nonassociative Lie-admissible algebra U appears rather natural. In turn this provides a first understanding of the rather peculiar situation of the Lie-admissible theory where the Lie algebra is lost by central requirement, nevertheless a connected Lie group structure persists!

The remaining part of the analysis of this section is straightforward. The proof of the following Lie-admissible covering of Lie's first theorem is left to the interested reader.

THEOREM 3.3.5: If the transformations

$$\hat{a}^{\mu} = h^{\mu}{}_{\nu}(a;\theta) f^{\nu}(a;\theta) = \hat{f}^{\mu}(a;\theta) \qquad (3.3.57)$$

characterize a Lie-admissible genotopic image $\hat{G}$ of a connected Lie group G of transformations $f^{\mu}(a;\theta)$, then

$$\frac{\partial \hat{a}^{\mu}}{\partial \theta^{i}} = [\alpha_i^{r}(a) + \beta_i^{r}(a)] u^{\mu}_{r}(a) \qquad (3.3.58a)$$

$$\alpha_i^{r} u_r^{\nu} \frac{\partial}{\partial a^{\nu}}(\beta_j^{s} u_s^{\mu}) + \beta_i^{r} u_r^{\nu} \frac{\partial}{\partial a^{\nu}}(\alpha_j^{s} u_s^{\mu}) - \alpha_j^{r} u_r^{\nu} \frac{\partial}{\partial a^{\nu}}(\beta_i^{s} u_s^{\mu}) - \beta_j^{r} u_r^{\nu} \frac{\partial}{\partial a^{\nu}}(\alpha_i^{s} u_s^{\mu}) = 0 \qquad (3.3.58b)$$

where the functions $\alpha_i^{r}(a)$, $\beta_i^{r}(a)$ and $u_i^{\mu}(a)$ are analytic.

The integrability conditions for the functions α_i^{r}, β_i^{r} and u_r^{μ} to characterize a genotope $\hat{G}$ of G, that is, to be genotopic functions, can be written

$$\alpha_i^{\,\ell} U_\ell^\nu \frac{\partial \alpha_j^k}{\partial a^\nu} - \alpha_j^{\,\ell} U_\ell^\nu \frac{\partial \alpha_i^k}{\partial a^\nu} - \beta_j^{\,\ell} U_\ell^\nu \frac{\partial \beta_i}{\partial a^\nu} + \beta_i^{\,\ell} U_\ell^\nu \frac{\partial \beta_j^k}{\partial a^\nu} = \hat{C}^{*\,k}_{ij} - (\alpha_i^\ell \alpha_j^s + \beta_i^\ell \beta_j^s)\, C_{\ell s}^k \quad (3.3.59)$$

where the C's are the structure constants of the original group G and the $\hat{C}$*'s are the structure constants of the isotope $\hat{G}$* of $\hat{G}$ in form (3.3.56). Notice that such isotope $\hat{G}$* is not isomorphic, in general, to the genotope $\hat{G}$ even when reformulated in terms of the standard generators $\hat{X}_i^*$, i.e.,

$$\hat{C}^{*\,k}_{ij} \neq \hat{C}_{ij}^k \quad (3.3.60)$$

This situation indicates that the analysis of $\hat{G}$ shoud be conducted in terms of the product $X_i \circ X_j$, rather than the product X_i, X_j $_U$. In particular, we can write

$$X_i \circ X_j = [(\alpha_i^\ell \alpha_j^s + \beta_i^\ell \beta_j^s) C_{\ell s}^k X_k - (\alpha_i^\ell \alpha_j^k - \beta_i^s \beta_j^k)] X_k \stackrel{\text{def}}{=} \hat{U}_{ij}^k (a) X_k \quad (3.3.61)$$

We reach in this way the following Lie-admissible covering of Lie's second theorem.

THEOREM 3.3.6: The generators X_i, i=1,2,...,n, of a genotope $\hat{G}$ of a Lie group G satisfy the relations

$$X_i \circ X_j = \hat{U}_{ij}^k(a,X) X_k \; ; \; \hat{U}_{ij}^k = C^{*\,k}_{ij} + D^{*\,ke}_{ij} X_e, \quad C^{*\,k}_{ij} = -C^{*\,k}_{ji}, \; D^{*\,ke}_{ij} = D^{*\,ke}_{ji} = D^{*\,ek}_{ij} \quad (3.3.62)$$

where the quantities U_{ij}^k (here called Lie-admissible structure quantities) are generally dependent on the (local) coordinates of the manifold of the original group G.

The covering nature of the above theorem can be expressed in terms of the following corollaries.

COROLLARY 3.3.6.A: The Lie-admissible structure quantities satisfy the identities

$$\frac{1}{2}\left(\hat{U}_{ij}^k - \hat{U}_{ji}^k\right) = \hat{C}^{*\,k}_{ij}(a) \quad (3.3.63)$$

Notice that the knowledge of the functions C^{*k}_{ij}(a) does not uniquely determine the functions U_{ij}^k(a), but only their antisymmetric part. Note also the preservation of the a-dependence in antisymmetrization (3.3.63).

COROLLARY 3.3.6.B: Under the values

$$\alpha_i^j = \delta_i^j \quad , \beta_i^j = 0 \quad (3.3.64)$$

the product $X_i \circ X_j$ becomes associative and the genotope $\hat{G}$ recovers the original group G identically.

The above corollary gives another hint to the fact that the primitive origin of the notion of Lie-admissible genotopic mapping of a Lie group lies within the context of the universal enveloping algebra. And indeed, as we shall see in Section 3.5, mapping (3.3.50) allows the formulation of a consistent Lie-admissible covering of the universal enveloping associative algebra of a Lie algebra. In turn, it is such a covering which allows a Lie-admissible infinitesimal behaviour while preserving the global structure of a Lie group.

Limit (3.3.64), however, does not exhaust all possible cases. Another case which is significant can be formulated as follows.

COROLLARY 3.3.6.C: Under the values

$$\alpha_i^j, \beta_i^j \rightarrow \delta_i^j \qquad (3.3.65)$$

the Lie-admissible structure quantities $\hat{U}_{ij}^i(a)$ reduce to the Lie structure constants C_{ij}^k of the original group G in which case the structure quantities of the genotope G* of G originating from rule (3.3.56) reduce to twice the structure constants C_{ij}^k.

Notice that in this case the isotope G* can be made to formally coincide with the original group via the trivial isotopy

$$A: X_i X_j \Rightarrow A^*: X_i * X_j = \frac{1}{2} X_i X_j \qquad (3.3.66)$$

The alternative of Corollaries 3.3.6.B and 3.3.6.C have

nontrivial implications for our analysis. As we shall see in Section 3.5, the latter alternative exhibits a number of problematic aspects for the construction of an enveloping nonassociative algebra whose product is Lie. These problematic aspects are absent when the enveloping nonassociative algebra is Lie-admissible. In turn, this indicates that in the transition from a Lie to a Lie-admissible context, what we actually have is the transition from a universal enveloping associative algebra to a universal enveloping Lie-admissible algebra. This feature will have crucial implications for the relativity analysis of Chapter 5.

As a further property, products (3.3.54) satisfy, by construction, the general Lie-admissibility condition (1.4.2), i.e.,

$$[X_i, X_j, X_k]_U + [X_j, X_k, X_i]_U + [X_k, X_i, X_j]_U \qquad (3.3.67a)$$
$$= [X_k, X_j, X_i]_U + [X_j, X_i, X_k]_U + [X_i, X_k, X_j]_U$$
$$[X_i, X_j, X_k]_U = (X_i \circ X_j) \circ X_k - X_i \circ (X_j \circ X_k) \qquad (3.3.67b)$$

This implies the following Lie-admissible covering of Lie's third theorem.

THEOREM 3.3.7: The structure quantities of a genotope $\hat{G}$ of a Lie group G satisfy the identities

$$(\hat{U}_{ij}^k - \hat{U}_{ji}^k)(\hat{U}_{ke}^r - \hat{U}_{ek}^r)$$
$$+ (\hat{U}_{je}^k - \hat{U}_{ej}^k)(\hat{U}_{ki}^r - \hat{U}_{ik}^r)$$
$$+ (\hat{U}_{ei}^k - \hat{U}_{ie}^k)(\hat{U}_{kj}^r - \hat{U}_{jk}^r) \quad +$$

$$+[(\hat{U}^{z}_{i;j}-\hat{U}^{z}_{j;i}),X_e]_U$$

$$+[(\hat{U}^{z}_{i;e}-\hat{U}^{z}_{e;i}),X_i]_U+[(\hat{U}^{z}_{e;i}-\hat{U}^{z}_{i;e}),X_j]_U=0 \qquad (3.3.68)$$

The property expressed by the above theorem is, in the final analysis, the direct expression of the Lie-admissiblity within the context of the transformation theory.

The covering nature of Theorem 3.3.7 is indicated by the following

COROLLARY 3.3.7.A: Under values (3.3.64), Lie-admissible identities (3.3.68) recover Lie identities (3.2.34).

Alternatively, we have

COROLLARY 3.3.7.B: Under values (3.3.65), Lie-admissible identities (3.3.68) become twice Lie identities (3.2.34).

The crucial preservation of the exponential mapping is ensured by the existence of mapping (3.2.42) for the standard realization of $\hat{G}$. Our problem is to see whether such expontial mapping still characterizes a connected Lie group under redefinition

$$\hat{X}_k(a) \longrightarrow [\alpha^{z}_{k}(a)+\beta^{z}_{k}(a)]\,X_z(a) \qquad (3.3.69)$$

We reach in this way the notion of a Lie-admissible genotopically mapped exponential law

$$\hat{a}^{\mu} = e^{\theta^k[\alpha^{z}_{k}(a)+\beta^{z}_{k}(a)]X_z(a)}\,a^{\mu} \qquad (3.3.70)$$

$$\equiv \overrightarrow{e}^{\,\theta^k\alpha^{z}_{k}(a)X_z(a)}\,a^{\mu}\,\overleftarrow{e}^{\,X_s(a)\beta^{s}_{k}(a)\theta^k}$$

We begin to see the reasons for the dual isotopy characterized by functions α^{z}_{k} and β^{s}_{k}, although the true reasons will appear clear only in Volume III. In fact, the last expression of Eqs. (3.3.70) characterizes a bimodule. The Lie-admissible structure originates simply from the differenziation of the left and right action. Realization of this structure in a suitable formulation of Hilbert spaces, then yields the Lie-admissible generalization (2.C.16) of Heisenberg's equations.

Eqs. (3.3.70) clearly characterize a Lie group of connected, finite transformations. Its composition law, however, is non-standard, and it is expressible as

$$e^{\hat{X}_\beta}\,e^{\hat{X}_\alpha} = e^{\hat{X}_\gamma} \qquad (3.3.71a)$$

$$\hat{X}_\gamma = \hat{X}_\alpha + \hat{X}_\beta + \frac{1}{2}[\hat{X}_\alpha,\hat{X}_\beta]_A + \frac{1}{12}[(\hat{X}_\alpha-\hat{X}_\beta),[\hat{X}_\alpha,\hat{X}_\beta]_A]_A + \dots \qquad (3.3.71b)$$

again, expressed in abstract form. When the standard generators $\hat{X}_k$ of $\hat{G}$ are replaced with the standard generators X_i of G we have, from rule (3.3.56),

$$[\hat{X}_\alpha,\hat{X}_\beta]_{\hat{A}} \equiv [X_\alpha,X_\beta]_U \equiv [X_\alpha,X_\beta]_{A^*} \qquad (3.3.72)$$

Thus, the composition law of the genotope $\hat{G}$ in its "natural realization" (that is, that in terms of the generators of G) is given by

$$e^{x' X_\beta} e^{x X_\alpha} = e^{x'' X_\gamma}, \tag{3.3.73a}$$

$$x'' X_\gamma = x X_\alpha + x' X_\beta + \frac{1}{2} [X_\alpha, X_\beta]_{A^*} + \frac{1}{12} [(X_\alpha - X_\beta), [X_\alpha, X_\beta]_{A^*}]_{A^*} + \cdots, \tag{3.3.73b}$$

where the abstract factors x and x' are representative of the genotopic functions and the Baker-Campbell-Hausdorff formula is that of isotope $\hat{G}^*$ associated to $\hat{G}$ via rule (3.3.56). The fact that x" is still genotopic is left as a study for the interested reader.

We reach in this way a central concept of our analysis, which is introduced at this time as a reformulation of Definition 3.3.2.

DEFINITION 3.3.3: A Lie-admissible group of transformations is a set G of transformations

$$\hat{a}^\mu = h^\mu_{\ \nu}(a; a) f^\nu(a; \theta), \tag{3.3.74}$$

$$\mu = 1, 2, \ldots, 6N,$$

depending on n independent parameters $\theta = \{\theta^1, \theta^2, \ldots, \theta^m\}$ and on the transformation of f^μ of a Lie group G in the same parameter which possesses:

(a) a Lie-admissible structure in the neighborhood of the identity according to rule (3.3.54),

(b) a genotopically mapped exponential structure according to rule (3.3.70), and

(c) an isotopically mapped composition law according to rule (3.3.73).

With the above definition we here attempt the identification of a group image of the algebraic notion of Lie-admissibility for subsequent verification as well as for possible finalization by independent researchers. In essence, properties (a), (b) and (c) of Definition 3.3.3 allow, through an inverse procedure, the existence of a standard realization of $\hat{G}$. This is the first intended meaning of the notion of group-theoretic Lie-admissibility. The second intended meaning is a relation between two nonisomorphic groups G and $\hat{G}$ of the same dimensionality, as inherent in the same concept of genotopic mapping. The third, and perhaps most important meaning which is intended with Definition 3.3.3 is the capability of the group $\hat{G}$ of directly reducing to the nonisomorphic group G through limit (3.3.64). As we shall see in Chapter 5, this latter property will play a crucial role in our attempt to construct a covering of conventional relativities.

We must now insepct the problem of a classical realization of the notion of Lie-admissible group. This brings into focus in a natural way the nonconservative Newtonian systems and their analytic presentation in terms of our canonical-admissible equations. Therefore, we expect that the transformation theory of, this time, the Lie-admissible generalization of Hamilton's equations constitutes a classical realization of the considered context.

And indeed, the infinitesimal transformations

$$\delta a^{\mu} = \delta\theta^{k}\, S^{\mu\nu}(a)\, \frac{\partial G_k(a)}{\partial a^{\nu}} \qquad (3.3.75)$$

provide a realization of the genotopic transformations (3.3.58), i.e.,

$$\hat{U}^{\mu}_{k} = [\alpha^{r}_{k}(a)+\beta^{r}_{k}(a)]\,U^{\mu}_{r}(a) \rightarrow S^{\mu\nu}(a)\frac{\partial G_k(a)}{\partial a^{\nu}} = (\Omega^{\mu\nu}+T^{\mu\nu})\,\partial G_k/\partial a^{\nu} \qquad (3.3.76)$$

The generators X_i are realized in terms of the structure

$$\hat{X}_{k} = [\alpha^{r}_{k}(a)+\beta^{r}_{k}(a)]\,U^{\mu}_{r}(a)\frac{\partial}{\partial a^{\mu}} \rightarrow S^{\mu\nu}(a)\,\frac{\partial G_k(a)}{\partial a^{\nu}}\,\frac{\partial}{\partial a^{\mu}}. \qquad (3.3.77)$$

The Lie-admissible abstract product (3.3.54) can then be realized in terms of the Lie-admissible classical product (2.5.2) (again, up to neutral elements of the center of the universal enveloping Lie-admissible algebra, see in this respect Section 3.5), i.e.,

$$X_i \circ X_j \rightarrow (G_i, G_j)^{*} = \frac{\partial G_i}{\partial a^{\mu}}\, S^{\mu\nu}\, \frac{\partial G_j}{\partial a^{\nu}}. \qquad (3.3.78)$$

This yields the desired Lie-admissible character in the neighborhood of the identity

$$\hat{a}^{\mu} \cong a^{\mu} + \theta\, (a^{\mu}, G)^{*} \qquad (3.3.79)$$

by therefore reaching a realization of rule (3.3.45) under the Lie-admissible condition (3.3.46).

Furthermore, a Lie-admissible realization of the exponential mapping (3.3.70) can be achieved as follows. Write Eqs. (3.3.75) in the form

$$\frac{\partial \hat{a}^{\mu}}{\partial \theta} = (\hat{a}^{\mu}, G)^{*}_{(\hat{a})}, \qquad (3.3.79)$$

and performs the reduction of the brackets to the a-variables

$$\frac{\partial \hat{a}^{\mu}}{\partial \theta} = (\hat{a}^{\mu}(a;\theta), G(\theta))^{*}_{(a)} \qquad (3.3.80)$$

This equation can also be interpreted as a system of partial differential equations in the unknown coordinates $\hat{a}^{\mu}$ subject to the initial conditions

$$\hat{a}^{\mu}(a;\theta)^{*}\big|_{\theta=0} = a^{\mu} \qquad (3.3.81)$$

A solution can be also expressed in terms of a (formal) power series expansion

$$\hat{a}^{\mu} = a^{\mu} + \frac{\theta}{1!}(a^{\mu}, G)^{*} + \frac{\theta\theta}{2!}\left((a^{\mu}, G)^{*}, G\right)^{*} + \dots \qquad (3.3.82)$$

which (under the necessary convergence conditions) yields the finite, connected, group structure

$$\hat{a}^{\mu} = e^{\theta^{k} S^{\alpha\beta}_{(k)}(a)\frac{\partial G_k(a)}{\partial a^{\beta}}\frac{\partial}{\partial a^{\alpha}}}\Big|_{k=\text{fixed}}\; a^{\mu} \qquad (3.3.83)$$

which is the desired classical realization of the notion of Lie-admissible group.

A few comments are now in order. The attentif reader will have noted the restriction of group (3.3.83) to only one dimension (i.e., per each value of k). This is due to the need of additional studies on the composition law of Lie-admissible groups.

Second, Eqs. (3.3.83) have been written with a differentiation of the Lie-admissible tensor per each generator, i.e., $S^{\mu\nu} = S^{\mu\nu}_{(k)}$. This is clearly a consequence of the property that, if one tensor, say, $S^{\mu\nu}_{(o)}$ occurs for the time evolution according to Eqs. (2.5.1) (when the generator is the Birkhoffian), a different tensor is needed, in general, for different generators as per Eqs. (3.3.79).

This feature is crucial for the concept of Lie-admissible covering of a Lie algebra or group. It essentially implies that given a conventional canonical realization of a Lie group G

$$a'^{\mu} = e^{\theta^1 \omega^{\alpha\beta} \frac{\partial G_1}{\partial a^\beta} \frac{\partial}{\partial a^\alpha} + \theta^2 \omega^{\alpha\beta} \frac{\partial G_2}{\partial a^\beta} \frac{\partial}{\partial a^\alpha} + \ldots + \theta^m \omega^{\alpha\beta} \frac{\partial G_m}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}} a^{\mu}, \tag{3.3.84}$$

a Lie-admissible covering can be achieved with the genotopic mapping of only one element, while leaving all the other unchanged, i.e.,

$$\hat{a}'^{\mu} = e^{\theta^1 S^{\alpha\beta}_{(1)}(a) \frac{\partial G_1}{\partial a^\beta} \frac{\partial}{\partial a^\alpha} + \theta^2 \omega^{\alpha\beta} \frac{\partial G_2}{\partial a^\beta} \frac{\partial}{\partial a^\alpha} + \ldots + \theta^m \omega^{\alpha\beta} \frac{\partial G_m}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}} a^{\mu}, \tag{3.3.85}$$

provided, of course, that the structure is consistent with the integrability conditions (3.3.52) and (3.3.59). As we shall see, this can be the case for semidirect products, such as the Euclidean group in two dimension

$$G = SO(2) \times T(2) \tag{3.3.86}$$

where, say, only the generator of rotation is embedded into a Lie-admissible context, while the generators of translations are left unaffected (or vice versa). However, for a structure of type (3.3.86) we should expect, in general, a Lie-admissible embedding

via different tensors $S^{\mu\nu}_{(k)}$ for each of the three generators.

This feature is clearly absent in the conventional structure (3.3.84) owing to the applicability of the same Lie tensor $\omega^{\mu\nu}$ to all generators. It is, however, of some significance to note that the feature already appears at the level of Lie isotope G* of G. And indeed, the isotopic functions in structure (3.3.31) can also vary, at least in principle, from index to index, yielding different symplectic (tensor) forms $\Omega^{\mu\nu}_{(k)}$, as in structure (3.3.40).

As indicated in Section 3.1, this feature is crucial to attempt a partial Lie-admissible covering of a symmetry group. We are here referring to the case when the original exact symmetry is, say, the semidirect product of two semisimple groups, $G = G_1 \times G_2$, and only one of them is broken, say, G_1. In this case a consistent Lie-admissible covering is $\hat{G} = \hat{G}_1 \times G_2$ where G_2 is unchanged. Such consistency ultimately originates from the central property of Lie-admissible algebras according to which they can be Lie as a particular case.

One of the several topics which we shall leave open is the study of the representation theory of Lie-admissible algebras and groups. For rudimentary remarks in this respect see Appendix 3.B.

Another peculiar feature of the Lie-admissible groups (3.3.83) is that their action on the (topological) manifold is nongeodesic, contrary to the corresponding occurrence for the canonical structure (3.3.84). Rather than being a drawback, this is interpreted as one of the central features which is needed to identify a consistent relativity for systems with forces not derivable from a potential. In fact, the trajectories of these systems in the physical space of their experimental detection is always nongeodesic, as indicated in Appendix I.3.C. For further remarks on this nongeodesic aspect see Appendix 3.C.

Predictably, this nongeodesic character has deep geometrical implications , which are reflected in the fact that the geometry characterized by the two-forms $S^{\mu\nu}_{(k)}$ is nonsymplectic. A preliminary study of the geometry underlying the Lie-admissible structures (3.3.83) is conducted in Chapter 4.

As a final remark, realizations (3.3.83) exhibit in a direct way the covering nature of the Lie-admissible groups over the conventional Lie groups. In essence, besides being of generalized nature and representing the symmetry breaking forces in the S-tensor, the Lie-admissible groups (3.3.83) recover the conventional canonical groups (3.3.84) identically under the limit of null value of the forces not derivable from a potential, i.e.,

$$\lim_{\underset{\mu}{F}^{NSA}\to 0} e^{\theta^k S^{\mu\nu}_{(k)} \frac{\partial G_k}{\partial a^\nu}\frac{\partial}{\partial a^\mu}} = e^{\theta^k \omega^{\mu\nu}\frac{\partial G_k}{\partial a^\nu}\frac{\partial}{\partial a^\mu}} \tag{3.3.87}$$

In the final analysis, this occurence can be considered as an expression of the deep interrelation between our algebraic and analytic treatments of nonconservative systems.

For initial examples see Appendix 3.A. For an outline see Tables 3.3.1 and 3.3.2.

Conservative Newtonian systems

$$\left(m_k \ddot{\underset{m}{r}}_k - \underset{m}{f}_k\right)_{SA} = 0$$

Representation in terms of Hamilton's equations		Representation in terms of the Lie covering of Hamilton's equations
$\left(\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right)_{SA} = 0$		$\left(\Omega_{\mu\nu}(a)\dot{a}^\nu - \frac{\partial B}{\partial a^\mu}\right)_{SA} = 0$
Canonical realization of Lie groups	Lie covering →	Lie covering of the canonical realization of Lie groups
$a'^\mu = e^{\theta^k \omega^{\alpha\beta}\frac{\partial G_k}{\partial a^\beta}\frac{\partial}{\partial a^\alpha}} a^\mu$		$a^{*\mu} = e^{\theta^k \Omega^{\alpha\beta}_{(k)}(a)\frac{\partial G_k}{\partial a^\beta}\frac{\partial}{\partial a^\alpha}} a^\mu$
Lie's first, second and third theorem		Lie covering of Lie's first, second and third theorem
$\frac{\partial a^\mu}{\partial \theta^i} = u^\mu_k(a)\lambda^k_i,$		$\frac{\partial a^\mu}{\partial \theta^i} = g_k^{\ell}(a) u^\mu_\ell \lambda^k_i(\theta),$
$[X_i, X_j]_A = X_i X_j - X_j X_i = C^k_{ij} X_k, \quad \partial C^k_{ij}/\partial a^\mu = 0,$		$[X_i, X_j]_{A^*} = X_i * X_j - X_j * X_i = C^{*k}_{ij}(a) X_k, \quad \partial C^{*k}_{ij}/\partial a^\mu \neq 0,$
$C^k_{ij} + C^k_{ji} = 0,$		$C^{*k}_{ij} + C^{*k}_{ji} = 0,$
$C^k_{ij} C^r_{k\ell} + C^k_{j\ell} C^r_{ki} + C^k_{\ell i} C^r_{kj} = 0.$		$C^{*k}_{ij} C^{*r}_{k\ell} + C^{*k}_{j\ell} C^{*r}_{ki} + C^{*k}_{\ell i} C^{*r}_{kj} = 0.$

Table 3.3.1: Lie covering of Lie's theory. It consists of a generalization of Lie's theory realized in terms of the most general (nondegenerate) form of the Lie product, rather than the conventional Poisson brackets. The generalization is inessential for the conventional Lie treatment of exact symmetries and conservation laws because the preservation of the Lie character renders the reduction of the generalized into the conventional formulations always possible (via the Darboux's charts of Section 4.4). However, the generalization becomes

crucial when studying the Lie-admissible covering of Lie's theory (Table 3.3.2). The starting ground is the generalization of Hamilton's equations into the most general form which preserves the Lie character of the brackets of the time evolution law. This form is provided by what we have called the Birkhoff's equations. The Lie's covering of Lie's theory is then provided by the transformation theory of these broader equations. On algebraic grounds the covering is offered by the notion of Lie isotopy, that is, the use of invertible mappings of the product which preserve the Lie character. This approach yields a natural generalization of Lie's first, second and third theorem. However, the structure constants now acquire a dependence on the coordinates of the base manifold. This algebraic isotopy results to possess a group theoretic image yielding, under integrability conditions, a connected Lie group of finite transformations called isotopically mapped Lie group. The structure (or realization) of this group, however, is nonstandard. In particular, it is realized in terms of the generator of a generally nonisomorphic group with the same dimensionality. At the level of the transformations, the relationship between those of the original group and those of the isotopic image is generally nonlinear and representable in terms of the isotopic functions $g^{\mu}{}_{\nu}(a; \vartheta)$. Each Lie algebra can, at least in principle, admit a family of nonisomorphic isotopes. The Lie's covering of Lie's theorems then allow the construction of a corresponding family of generally nonisomorphic groups of the same dimensionality which are all isotopically related to the original group.

Nonconservative Newtonian Systems

$$\left[(m_k \ddot{x}_k - f_k)_{SA} - F_k\right]_{NSA} = 0$$

Representation in terms of the Lie covering of Hamilton's equations $\left[\Omega_{\mu\nu}(a)\dot{a}^{\nu} - \frac{\partial B}{\partial a^{\mu}}\right]_{SA} = 0$		**Representation in terms of the Lie-admissible covering of Hamilton's equations** $\left[S_{\mu\nu}(a)\dot{a}^{\nu} - \frac{\partial B}{\partial a^{\mu}}\right]_{NSA} = 0$
Lie covering of the canonical realization of Lie groups $a^{*\mu} = e^{\theta^k \Omega^{\alpha\beta}_{(k)} \frac{\partial G_k}{\partial a^{\beta}} \frac{\partial}{\partial a^{\alpha}}} a^{\mu}$	Lie-admissible covering →	**Lie-admissible covering of the canonical realization of Lie groups** $\hat{a}^{\mu} = e^{\theta^k S^{\alpha\beta}_{(k)} \frac{\partial G_k}{\partial a^{\beta}} \frac{\partial}{\partial a^{\alpha}}} a^{\mu}$
Lie's covering of Lie's first, second and third theorem $\frac{\partial a^{\mu}}{\partial \theta^i} = g_k{}^{\ell}$; $[X_i, X_j]_{A^*} = X_i * X_j - X_j * X_i = C^{*k}_{ij}(a) X_k$; $C^*_{ij} + C^*_{ji} = 0$; $C^{*k}_{ij} C^{*r}_{k\ell} + C^{*k}_{j\ell} C^{*r}_{ki} + C^{*k}_{\ell i} C^{*r}_{ki} = 0$		**Lie-admissible covering of Lie's first, second and third theorem** $\frac{\partial a^{\mu}}{\partial \theta^i} = [\alpha_k{}^r(a) + \beta_k{}^r(a)] u_r^{\mu}(a) \lambda^k_i(\theta)$; $X_i \circ X_j = \hat{u}^k_{ij}(a;x) X_k$; $[X_i, X_j]_U = X_i \circ X_j - X_j X_i = \hat{C}^{*k}_{ij}(a) X_k$; $\hat{u}^k_{ij} - \hat{u}^k_{ji} = \hat{C}^{*k}_{ij}(a)$; $\hat{u}^k_{rs}\hat{u}^{\ell}_{kt} + \hat{u}^k_{st} u^{\ell}_{kr} + \hat{u}^k_{tr}\hat{u}^{\ell}_{ks}$ $\hat{u}^k_{sr}\hat{u}^{\ell}_{tk} + \hat{u}^k_{ts}\hat{u}^{\ell}_{rk} + \hat{u}^k_{rt}\hat{u}^{\ell}_{sk}$ $= \hat{u}^k_{sr}\hat{u}^{\ell}_{kt} + \hat{u}^k_{ts}\hat{u}^{\ell}_{kr} + \hat{u}^k_{rt}\hat{u}^{\ell}_{ks}$ $+ \hat{u}^k_{rs}\hat{u}^{\ell}_{tk} + \hat{u}^k_{st}\hat{u}^{\ell}_{rk} + \hat{u}^k_{tr}\hat{u}^{\ell}_{sk}$

Table 3.3.2: Lie-admissible covering of Lie's theory. It consists of a generalization of Lie's theory in terms of the most general form of the (nondegenerate)

Lie-admissible product. The starting ground is the Lie-admissible generalization of Hamilton's equations, i.e., our Birkhoff -admissible equations of Section 2.5. The Lie-admissible covering of Lie's theory is then provided by the transformation theory of these broader equations. On algebraic grounds, the covering can be characterized with the notion of algebraic genotopy, that is, the use of invertible mappings of the product which induce a Lie-admissible structure. The mapping, however, rather than acting at the level of the Lie algebra "per sé," acts at the level of the associative algebra of the fundamental Lie rule, by opening the possibility of a Lie-admissible covering of the universal enveloping associative algebras of a Lie algebra (Table 3.5.1). This yields a nontrivial generalization of Lie's theorems. First of all, Lie's structure constants are generalized into quantities generally dependent on the generators as well as the coordinates of the base manifold, by confirming the infinite-dimensional nature of the algebra and, thus, its generalized enveloping character. Despite this generalization, the structure quantities obey covering laws of Lie's second and third theorem. The proper treatment of this Lie-admissible covering demands the use of the isotopic covering of Lie's theorems (Table 3.3.1) because the Lie algebra content is that of an isotopically mapped algebra rather than an algebra in its standard realization. Intriguingly, despite the loss of a Lie algebra in the neighborhood of the origin, the approach exhibits, under integrability conditions, the structure of a connected Lie group of finite transformations. The realization of the group, however, is highly nonstandard. And indeed, it is realized in terms of the generators, parameters and base manifold of a nonisomorphic group in such a way that the underlying envelope is nonassociative (Table 3.5.1). This renders allowable only nonlinear representations (Appendix 3.B). In turn, this nonlinearity of the transformation theory will play a crucial role in our attempt of differentiating the electromagnetic and strong interactions in the physical space of their experimental verification, rather than in unitary internal spaces, as we shall see.

3.4: UNIVERSAL ENVELOPING ASSOCIATIVE ALGEBRAS

The physical and mathematical significance of the universal enveloping associative algebra , say A(G), of a Lie algebra G is well known.

On physical grounds the need of the algebra A(G) is due to the fact that all Lie algebras are nilalgebras (Section 1.2). This means that the powers X^n, $n \geq 2$ of an elements $X \in G$ are identically null in G (trivially, because the Lie product is anticommutative). The net effect is that a number of physical quantities (such as the square of the angular momentum or the Gell-Mann-Okubo mass formula) cannot be defined within the context of the Lie algebras (SO(3) and SU(3), respectively), by therefore necessarily demanding the use of the universal enveloping associative algebras A(G). In fact, since the product of A(G) is associative, quantities such as X^n, $n \geq 2$, can now be defined in A(G) as the ordinary product XXX···X (n times).

On mathematical grounds the algebras A(G) are crucial for a number of technical aspects and, in particular, for the transition from the Lie algebra G to a corresponding Lie group G. In fact, exponential mapping (3.2.42), i.e.,

$$1 + \frac{\theta^k}{1!} X_k + \frac{\theta^i \theta^j}{2!} X_i X_j + \cdots \qquad (3.4.1)$$

makes sense if and only if it is defined within the context of A(G) because, from the second-order term on, all elements of this expansion are outside of G. Furthermore, the algebra A(G) is constructed in terms of the generators of G, contains all the Casimir invariants (which, again, are outside of G), and plays a crucial role for the representation theory (see Appendix 3.B in this latter respect).

On account of these occurrences, we can state in simplistic terms that the universal enveloping associative algebras are the true representatives of the dual algebraic-group theoretic aspects of Lie's theory.

The role of these algebras for our analysis can therefore be anticipated. In essence, the dual algebraic-group theoretic structure of the Galilei (Einstein) relativities can be characterized with the universal enveloping associative algebras A(G(3.1)) (A(P(3.1))) of the Galilei algebra G(3.1) and (the Poincare algebra P(3.1)). The identification of a possibly Lie-admissible covering of these algebras for systems with forces not derivable from a potential will clearly be valuable for the problem of the relativity laws which are applicable to the systems considered, as we shall see in Chapters 5.

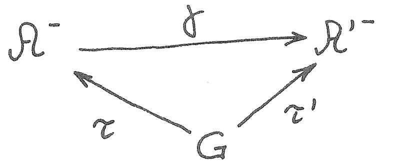
The universal enveloping algebras will also play a crucial role for the problem of quantization. In essence, the quantization of conserved quantities, such as the components of the angular momentum M_i, i=1,2,3, as generators of SO(3), demands the use of the enveloping algebra A(SO(3)) in order to properly identify the eigenvalues of $\underset{\sim}{M}^2$ (besides those of M_3). If a Lie-admissible covering of A(SO(3)) can be identified at a classical level to account for the broader case of nonconservation of the generators M_i, such a structure will clearly be

valuable for the problem of quantization of nonconservative systems, as we shall see in Volume III.

We are now in a position to better identify a central objective of this monograph, as a complement to its initial identification in Section 3.1. It essentially consists of attempting a Lie-admissible algebraic-group theoretic characterization of symmetry breakings and nonconservation laws via a covering of the universal enveloping associative algebra of the exact symmetry algebra. Then, the transition from the associative to the nonassociative character of the envelop is representative of the symmetry breaking forces.

In this section we shall review certain basic elements of the universal enveloping associative algebra of a Lie algebra, here tacitly assumed to be the Lie algebra of an exact symmetry of a Newtonian system. By closely following the treatment by N. Jacobson,[69] we shall restrict ourself only to the abstract aspect of the topic considered. The realization of the considered notions in terms of the canonical formulations of a Newtonian system will be left to the interested reader. All algebras and field will be assumed to be of characteric zero. The problem of the possible existence of a Lie-admissible covering will be considered in Section 3.5.

DEFINITION 3.4.1: The universal enveloping associative algebra of a Lie algebra G is the set $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is an associative algebra and τ a homomorphism of $\underline{G}$ into the attached algebra $\mathcal{A}^-$ of $\mathcal{A}$ (Section 1.2) satisfying the following property. If $\mathcal{A}'$ is another associative algebra and τ' a homomorphism of $\underline{G}$ into $\mathcal{A}'^-$, there exists a unique homomorphism γ of $\mathcal{A}$ into $\mathcal{A}'$ such that $\tau = \gamma\tau'$ i.e., the following diagram is commutative.



(3.4.2)

The construction of a universal enveloping associative algebra $\mathcal{A}$ of $\underline{G}$ (which we shall write $\mathcal{A}(\underline{G})$, $\underline{G} \approx [\mathcal{A}(\underline{G})]^-$) can be conducted as follows. Consider the algebra $\underline{G}$ as a vector space whose basis is the set of generators X_i, i=1,2,.. ..,n, in their standard realization (3.2.23). The tensorial product $\underline{G} \times \underline{G}$ is the ordinary Kronecker (or direct) product (see also Section 1.2) of $\underline{G}$ with itself as a vector space. As such, it constitutes an algebra, that is, it satisfies the right and left distributive and scalar laws

$$X_i \otimes (X_j + X_k) = X_i \otimes X_k + X_i \otimes X_k , \quad (3.4.3a)$$

$$(X_i + X_j) \otimes X_k = X_i \otimes X_k + X_j \otimes X_k , \quad (3.4.3b)$$

$$(\alpha X_i) \otimes X_j = X_i \otimes \alpha X_j = \alpha (X_i \otimes X_j) \quad (3.4.3c)$$
$$= (X_i \alpha) \otimes X_j = X_i \otimes X_j \alpha = (X_i \otimes X_j)\alpha$$

In addition, and most importantly for our analysis, it satisfies the associative law

$$(X_i \otimes X_j) \otimes X_k = X_i \otimes (X_j \otimes X_k) \quad (3.4.4)$$

The ordinary product of two elements $X_i \otimes X_j$ and $X_r \otimes X_s$ is then defined by

$$(X_i \otimes X_j)(X_r \otimes X_s) = X_i X_r \otimes X_j X_s \quad (3.4.5)$$

where $X_i X_s$ is the ordinary associative product on $\underline{G}$ as vector space. Since this product is also associative, the associative law is verified also for product of type (3.4.5) (if $X_i X_s$ is non-associative, then product (3.4.5) is nonassociative too).

The tensorial product of two elements $X_i \otimes X_j$ and $X_r \otimes X_s$ is instead defined by

$$(X_i \otimes X_j) \otimes (X_r \otimes X_s) = X_i \otimes X_j \otimes X_r \otimes X_s \quad (3.4.6)$$

where no product ordering ambiguities arise because the product $\otimes$ is associative.

Finally, the direct sum of two elements X_i and X_j of $\underline{G}$ is the ordinary sum $X_i \oplus X_j$ on vector spaces. As such it is distributive, associative, as well as, this time, commutative.

We now define the tensorial products

$$\underline{G}^{(1)} = \underline{G} , \quad \underline{G}^{(2)} = \underline{G} \otimes \underline{G} , \quad \underline{G}^{(3)} = \underline{G} \otimes \underline{G} \otimes \underline{G} , \text{etc.} \quad (3.4.7)$$

The most general associative tensor algebra which can be constructed on $\underline{G}$ as vector space can then be written

$$\mathcal{T} = F1 \oplus \underline{G}^{(1)} \oplus \underline{G}^{(2)} \oplus \underline{G}^{(3)} \oplus \cdots \tag{3.4.8}$$
$$= F1 \oplus \underline{G} \oplus \underline{G} \otimes \underline{G} \oplus \underline{G} \otimes \underline{G} \otimes \underline{G} \oplus \cdots ,$$

where F is the base field.

Let $\mathcal{R}$ be the ideal of $\mathcal{T}$ generated by all elements of the form

$$[X_i, X_j]_A - (X_i \otimes X_j - X_j \otimes X_i), \tag{3.4.9}$$

where $[X_i, X_j]_A$ is the ordinary Lie product, that is, Eq. (3.2.30).

The <u>universal enveloping associative algebra</u> $\mathcal{A}(\underline{G})$ of $\underline{G}$ can then be expressed as the quotient

$$\mathcal{A}(\underline{G}) = \mathcal{T}(X) / \mathcal{R}(X) . \tag{3.4.10}$$

It is possible to prove that this algebra satisfies all the requirements of Definition 3.4.1. Thus, structure (3.4.10) can equivalently be assumed as the definition of universal enveloping associative algebra of $\underline{G}$.

Our next objective is that of identifying a basis of $\mathcal{A}(\underline{G})$, as well as its reduction to A($\underline{G}$), that is, the enveloping algebra expressed in terms of the ordinary associative product (rather than the tensorial product), of primary use in physical applications.

The quantities

$$M_m = X_{i_1} \otimes X_{i_2} \otimes \cdots \otimes X_{i_m} \tag{3.4.11}$$

are called <u>monomials of degree m</u>. They are the basis of the tensor algebra

$$\underline{G}^{(m)} = \underline{G} \otimes \underline{G} \otimes \cdots \otimes \underline{G} \quad (m\text{-times}) \tag{3.4.12}$$

The <u>index</u> of a monomial M_m is defined by

$$\text{index } M_m = \sum_{r<s} \eta_{rs} , \quad \eta_{rs} = \begin{cases} 0 & \text{if } i_r \le i_s \\ 1 & \text{if } i_r > i_s \end{cases} \tag{3.4.13}$$

The <u>standard monomials</u> are given by

$$M^s_m = X_{i_1} \otimes X_{i_2} \otimes \cdots \otimes X_{i_m} \tag{3.4.14}$$
$$i_1 \le i_2 \le \cdots \le i_m$$

that is, they are the monomials of index zero.

The construction of a basis of $\mathcal{A}(\underline{G})$ essentially demands the comparison of different, generally nonstandard, monomials of the same degree and their reduction to a standard form. By using Eqs. (3.4.9) we can write

$$\begin{aligned} &X_{i_1} \otimes \cdots \otimes X_{i_k} \otimes X_{i_{k+1}} \otimes \cdots \otimes X_{i_m} \\ &= X_{i_1} \otimes \cdots \otimes X_{i_{k+1}} \otimes X_{i_k} \otimes \cdots \otimes X_{i_m} \\ &+ X_{i_1} \otimes \cdots \otimes [X_{i_k}, X_{i_{k+1}}] \otimes \cdots \otimes X_{i_m} \text{ Mod } \mathcal{R} \end{aligned} \tag{3.4.15}$$

Thus, any nonstandard monomial of given degree and index can be decomposed into a monomial of the same degree but lower index, plus a linear combination modulo $\mathcal{R}$ of monomials of lower degree with coefficient in the base field, called F-linear combination.

It then follows that any nonstandard monomial can be reduced, modulo $\mathcal{R}$ to an F-linear combination of standard monomials.

LEMMA 3.4.1: Every element of the tensor algebra $\mathcal{T}$ is congruent, modulo $\mathcal{R}$, to an F-linear combination of 1 and standard monomials.

Alternatively, we can say that every coset in $\mathcal{T}$ is a linear combination of $1 + \mathcal{R}$ plus the cosets of the standard monomials.

In order to reduce the envelope $\mathcal{A}(\underline{G})$ of $\underline{G}$ to its conventionally used form $A(\underline{G})$ we introduce the vector space $A^{(m)}$ with elements

$$A^{(m)} = X_{i_1} X_{i_2} \cdots X_{i_m} \tag{3.4.16}$$

expressed in terms of the conventional associative product and write

$$A(\underline{G}) = F1 \oplus A^{(1)} \oplus A^{(2)} \oplus A^{(3)} \oplus \cdots \tag{3.4.17}$$

The following property can be proved.

LEMMA 3.4.2: There exists a linear mapping ℓ of $\mathcal{A}(\underline{G})$ into $A(G)$ such that

$$1\ell = 1, \tag{3.4.18a}$$

$$(X_{i_1} \otimes X_{i_2} \otimes \cdots \otimes X_{i_m})\ell = X_{i_1} X_{i_2} \cdots X_{i_m}, \tag{3.4.18b}$$

$$\begin{aligned} &\big(X_{i_1} \otimes \cdots \otimes X_{i_k} \otimes X_{i_{k+1}} \otimes \cdots \otimes X_{i_m} \\ &- X_{i_1} \otimes \cdots \otimes X_{i_{k+1}} \otimes X_{i_k} \otimes \cdots \otimes X_{i_m}\big)\ell \\ &= \big(X_{i_1} \otimes \cdots \otimes [X_{i_k}, X_{i_{k+1}}]_A \otimes \cdots \otimes X_{i_m}\big)\ell \end{aligned} \tag{3.4.18c}$$

Now, the elements of the ideal $\mathcal{R}$ of $\mathcal{T}$ can be expressed as F-linear combinations of the elements

$$\begin{aligned} &X_{i_1} \otimes \cdots \otimes X_{i_k} \otimes X_{i_{k+1}} \otimes \cdots \otimes X_{i_m} \\ &- X_{i_1} \otimes \cdots \otimes X_{i_{k+1}} \otimes X_{i_k} \otimes \cdots \otimes X_{i_m} \\ &- X_{i_1} \otimes \cdots \otimes [X_{i_k}, X_{i_{k+1}}]_A \otimes \cdots \otimes X_{i_m} \end{aligned} \tag{3.4.19}$$

Lemma 3.4.2 then implies that the images of these elements under the ℓ-mapping are the zero element $0 \in F$. Thus

$$\mathcal{R}\ell = 0 \tag{3.4.20}$$

and ℓ provides a linear mapping of $\mathcal{A}(\underline{G})$ into $A(\underline{G})$. In particular, from Eqs. (3.4.18), ℓ maps cosets of 1 and standard monomials of $\mathcal{A}(\underline{G})$ into 1 and the elements of $A(\underline{G})$. Since these images are linearly independent, they constitute a basis of $A(\underline{G})$.

We reach in this way the fundamental property of universal enveloping associative algebras, the Poincaré-Birkhoff-Witt theorem, which can be formulated as follows.

THEOREM 3.4.1: The cosets of 1 and the standard monomials form a basis of the universal enveloping associative algebra $\mathcal{A}$ (G) of a Lie algebra G.

The mapping of Lemma 3.4.2 then allows the more semplicistic use of A(G) instead of $\mathcal{A}$ (G). The emerging structure is ∞ -dimensioanl with basis

$$1\ ,\ X_i\ ,\ \underset{i_1 \le i_2}{X_{i_1} X_{i_2}}\ ,\ \underset{i_1 \le i_2 \le i_3}{X_{i_1} X_{i_2} X_{i_3}}\ ,\ \dots \qquad (3.4.21)$$

and element

$$X_{i_1}^{k_1} X_{i_2}^{k_2} \dots X_{i_m}^{k_m}\ ,\ k_1, k_2, \dots, k_m = 0, 1, 2, \dots \qquad (3.4.22)$$

where the X's are the generators of G and n its dimensionality.

It can be proved that the mapping of G into A(G) is one-to-one. This has crucial implications for the representation theory (Appendix 3.B). Also, the Casimir invariants of G, i.e., the quantities

$$[C, X_i]_A = 0\ ,\ i = 1, 2, \dots, m\ ,\ C = C(X) \qquad (3.4.23)$$

result to be elements of the center (Section 1.2) of the universal enveloping associative algebras. This includes the trivial case when the invariants are constants, in which case they are often called the neutral elements of the center of A(G), as they appear, for instance, in the canonical realization of the Lie product, Eqs. (3.2.53).

3.5: LIE-ISOTOPIC AND LIE-ADMISSIBLE COVERINGS OF UNIVERSAL ENVELOPING ASSOCIATIVE ALGEBRAS

In Section 3.3 we indicated that the notion of Lie-admissibility is at the very foundation of Lie's theory, only expressed in its simplest possible form, the associative form (3.2.30), i.e.,

$$[X_i, X_j]_A = X_i X_j - X_j X_i = C_{ij}^{k} X_k \qquad (3.5.1)$$

This concept was then expressed in more rigorous terms in Section 3.4. It essentially emerged that the notion of universal enveloping associative algebra A(G) of a Lie algebra G is the true expression of that of associative Lie-admissibility.

In particular, the following features acquired a central role.

(1) All Lie algebras G are isomorphic to some subalgebra of the attached algebra $[A(G)]^-$ of A(G), and the associative envelope A(G) of G is unique, up to isomorphisms.

(2) For a given finite-dimensional Lie algebra G with basis

$$G : X_1, X_2, \dots, X_m \qquad (3.5.2)$$

the associative envelope A(G) is infinite dimensional with basis

$$1\ ,\ X_i\ ,\ \underset{i_1 \le i_2}{X_{i_1} X_{i_2}}\ ,\ \underset{i_1 \le i_2 \le i_3}{X_{i_1} X_{i_2} X_{i_3}}\ ,\ \dots \qquad (3.5.3)$$

and elements

$$X_{i_1}^{k_1} X_{i_2}^{k_2} \dots X_{i_m}^{k_m}\ ,\ k_1, k_2, \dots, k_m = 0, 1, 2, \dots \qquad (3.5.4)$$

(3) The associative envelop $A(\underline{G})$ is a symbiotic characterization of the dual algebraic-group theoretic profile of Lie's theory in the sense that (under suitable integrability conditions) it ensures both an infinitesimal Lie behaviour in the neighborhood of the origin, via rule (3.5.1), and a global Lie group structure via the exponential mapping

$$e^{\theta^i X_i} = 1 + \frac{\theta^i}{1!} X_i + \frac{\theta^i \theta^j}{2!} X_i X_j + \cdots \qquad (3.5.5)$$

The above features, in turn, are at the foundation of the algebraic-group theoretic characterization of exact symmetries and conservation laws, in general, and of the Galilei and Einstein relativities, in particular.

As by now familiar, the central objective of this monograph is to attempt the construction of a Lie-admissible covering of the Lie algebraic-group theoretic characterization of exact symmetries and conservation laws, with the hope of achieving some insights for the problem of the relativity laws which are applicable to systems with forces not derivable from a potential.

In Chapter 2 we indicated the existence of a covering of the analytic formulations whose algebraic structure is nonassociative-Lie-admissible. In Section 3.3 we indicated the existence of a nonassociative-Lie-admissible covering of Lie's theory which is fully parallel to the generalized analytic context. In order to complete the identification of the rudiments of the Lie-admissible approach, we remain with the problem whether the notion of universal enveloping associative algebra $A(\underline{G})$ admits a consistent

nonassociative but Lie-admissible covering. This is the objective of this section.

The reader should keep in mind that our objective is to attempt the construction of a Lie-admissible covering of established formulations <u>without changing the generators of the original exact symmetry algebra</u>.

The physical motivation of this central requirement is by now selfevident. A physical quantity, such as the total angular momentum, is unique for given dimensionality, masses and coordinate space. As such, it cannot be modified when additional nonconservative forces are admited. This is the reason why all our efforts have been and will be centered in preserving the conventional physical form of the generators, and modifying instead their methodology to account for nonconservative forces.

The mathematical motivation of the requirement considered is also, by now, selfevident. When the transition from a Lie context to a Lie-admissible covering is performed by preserving the generators of the original Lie algebra, the covering nature of the latter over the former context can be identified. In particular, the Lie-admissible context recovers the conventional Lie context identically under a clear limiting procedure: that of null value of the symmetry breaking forces.

The mathematical methods we have used to characterize this generator preserving transition are centered on the notion of algebraic genotopy, that is, an invertible mapping of the product which induces a Lie-admissible structure.

We reach in this way the identification of the methods we shall use to attempt the identification of a nonassociative Lie-admissible covering of the universal enveloping associative algebra of a Lie algebra. It essentially consists of the algebraic genotopy already identified in Section 3.3, that is, the invertible mapping of the associative product $X_i \otimes X_j$ of $\mathcal{A}(\underline{G})$ into the nonassociative Lie-admissible product $X_i \circ X_j$ according to Eqs. (3.3.54), i.e.,

$$\mathcal{A} : X_i \otimes X_j \Rightarrow \mathcal{U} : X_i \circ X_j \tag{3.5.6}$$

$$= \alpha_i^{r} U_r^{\nu} \frac{\partial}{\partial a^\nu} \otimes \alpha_j^{s} U_s^{\nu} \frac{\partial}{\partial a^\nu} - \beta_j^{r} U_s^{\nu} \frac{\partial}{\partial a^\nu} \otimes \beta_i^{s} U_s^{\mu} \frac{\partial}{\partial a^\mu}$$

A problem, however, immediately arises. Denote with $\mathcal{U}$ the algebra emerging from the generator preserving mapping (3.5.6) of $\mathcal{A}(\underline{G})$. From rule (3.5.1) we know that the "Lie content" of $\mathcal{A}(\underline{G})$ (that is, the attached algebra $[\mathcal{A}(\underline{G})]^-$) is homomorphic to $\underline{G}$. The problem consists in identifying whether such "Lie content" is preserved within the context of a possible $\mathcal{U}$-covering of $\mathcal{A}(\underline{G})$. From the analysis of Section 3.3 we can already conclude that this is not necessarily the case, because the attached algebra $\mathcal{U}^-$ of $\mathcal{U}$ characterizes an isotope $\hat{\underline{G}}^*$ of $\hat{\underline{G}}$ according to rule (3.3.27), i.e.,

$$[X_i, X_j]_{\mathcal{U}} = [X_i, X_j]_{A^*} = \hat{C}^{*\,k}_{ij}(a)\, X_k \tag{3.5.7a}$$

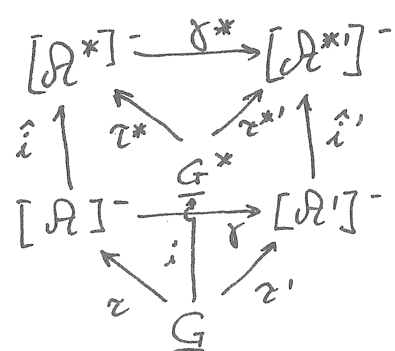
$$X_i \circ X_j - X_j \circ X_i \equiv X_i * X_j - X_j * X_i \tag{3.5.7b}$$

Theorem 3.3.4 then confirms that such isotope $\hat{\underline{G}}^*$ is generally

nonisomorphic to $\underline{G}$.

We reach in this way the need of first identifying the notion of isotopically mapped enveloping algebra (Step 2 of Section 3.1), that is, an invertible mapping which preserves the associative character of the envelop, and then proceeding to the construction of a nonassociative Lie-admissible extension (Step 3 of Section 3.1).

DEFINITION 3.5.1: An isotopically mapped universal enveloping associative algebra of a Lie algebra $\underline{G}$ is the set $[(\mathcal{A}, \tau)\mathcal{A}^*, \hat{i}, \tau^*]$ where: $(\mathcal{A}, \tau)$ is the universal enveloping associative algebra of $\underline{G}$ according to Definition 3.4.1; $\hat{i}$ is an isotopic mapping of $\underline{G}$, $\hat{i}\underline{G} = \underline{G}^*$; $\mathcal{A}^*$ is an associative algebra generally nonisomorphic to $\mathcal{A}$; and τ^* is a homomorphism of $\underline{G}^*$ into $[\mathcal{A}^*]^-$, satisfying the following property. If $\mathcal{A}^{*\prime}$ is still another associative algebra and $\tau^{*\prime}$ a homomorphism of $\underline{G}^*$ into $[\mathcal{A}^{*\prime}]^-$, there exist a unique homomorphism γ^* of $\mathcal{A}^*$ into $\mathcal{A}^{*\prime}$, $\tau^* = \gamma^* \tau^{*\prime}$, and a unique isotopy of $\mathcal{A}$ into $\mathcal{A}^*$, $\hat{i}\mathcal{A} = \mathcal{A}^*$, such that the following diagram is commutative.



(3.5.8)

In essence the above definition is intended to characterize the fact that, for a given Lie isotopy $\underline{G} \xrightarrow{i} \underline{G}^*$, there exists a unique isotopy $\mathcal{A} \xrightarrow{\hat{i}} \mathcal{A}^*$ at the level of their universal associative envelopes.

The construction of an isotope $\mathcal{A}^*$ of $\mathcal{A}$ can be conducted in the following way. Perform the <u>isotopic mapping of the tensorial product</u> $X_i \otimes X_j$ of $\mathcal{A}$,

$$X_i \otimes X_j \longrightarrow X_i * X_j, \qquad (3.5.9)$$

that is, an invertible modification of the product $\otimes$ via elements of the base manifold and/or of the field which preserves the associative law, i.e.,

$$(X_i * X_j) * X_k = X_i * (X_j * X_k), \qquad (3.5.10)$$

as well as, of course, the right and left distributive law and the scalar law (to qualify as the product of an algebra), i.e.,

$$X_i * (X_j + X_k) = X_i * X_j + X_i * X_k, \qquad (3.5.11a)$$

$$(X_i + X_j) * X_k = X_i * X_k + X_j * X_k, \qquad (3.5.11b)$$

$$(\alpha * X_i) * X_j = X_i * (\alpha X_j) = \alpha * (X_i * X_j). \qquad (3.5.11c)$$

The product of the elements $X_i * X_j$ and $X_r * X_s$ is then given by

$$(X_i * X_j) * (X_r * X_s) = X_i * X_j * X_r * X_s, \qquad (3.5.12)$$

and, as it was the case for product (3.4.6), no ordering ambiguity arises because the product $*$ is associative.

The construction of $\mathcal{A}^*$ is then straightforward. Define the isotopes $\underline{G}^*_1$, $\underline{G}_2^*$, ..., of product (3.4.7) as follows

$$\underline{G}_1^* = \underline{G}, \quad \underline{G}_2^* = \underline{G} * \underline{G}, \quad \underline{G}_3^* = \underline{G} * \underline{G} * \underline{G}, \quad \text{etc.} \qquad (3.5.13)$$

where, again, $\underline{G}$ is a vector space with basis X_i. Notice within such a context that $\underline{G}_1^* \equiv \underline{G}$, that is, the generators are unchanged and only the product is changed (in an isotopic way).

The most general <u>isotope of the associative tensorial algebra</u> (3.4.8) can then be written

$$\mathcal{T}^*(X) = F1 \oplus \underline{G}^{(1)*} \oplus \underline{G}^{(2)*} \oplus \underline{G}^{(3)*} \oplus \cdots \tag{3.5.14}$$

$$= F1 \oplus \underline{G} \oplus \underline{G} * \underline{G} \oplus \underline{G} * \underline{G} * \underline{G} \oplus \cdots$$

The isotope of the ideal $\mathcal{R}$ of $\mathcal{T}$ is then given by elements of the type

$$[X_i, X_j]_{A*} - (X_i * X_j - X_j * X_i) \tag{3.5.15}$$

and results to be the ideal $\mathcal{R}^*$ of $\mathcal{T}^*$.

An isotopically mapped universal enveloping associative algebra of a Lie algebra $\underline{G}$ can then be written

$$\mathcal{A}^*(\underline{G}) = \mathcal{T}^*(X) / \mathcal{R}^*(X) \tag{3.5.16}$$

It is possible to prove that this algebra satisfies the requirements of Definition 3.5.1. Thus, structure (3.5.16), besides being an isotope of $\mathcal{A}$, is the universal enveloping associative algebra of $\underline{G}^*$ and, as such, can be equivalently assumed as the definition of $\mathcal{A}^*$.

The identification of the basis of $\mathcal{A}^*$ is also straightforward. Define the isotopes of monomials (3.4.11) as follows

$$M^*_m = X_{i_1} * X_{i_2} * \cdots X_{i_m} . \tag{3.5.17}$$

The notion of degree and index trivially extend to this isotopically mapped context. Similarly, the notion of standard monomial can be extended too, and have the standard isotopically mapped monomials

$$M^{*s}_m = X_{i_1} * X_{i_2} * \cdots X_{i_m} \qquad i_1 \leq i_2 \leq \cdots \leq i_m \tag{3.5.18}$$

In the reduction of a nonstandard monomial to a standard form, however, a new feature now emerges. Consider the isotope of Eqs. (3.4.15), i.e.,

$$\begin{aligned} X_{i_1} * \cdots * X_{i_k} * X_{i_{k+1}} * \cdots * X_{i_m} \\ = X_{i_1} * \cdots * X_{i_{k+1}} * X_{i_k} * \cdots * X_{i_m} \\ + X_{i_1} * \cdots * [X_{i_k}, X_{i_{k+1}}]_{A*} * \cdots * X_{i_m} \end{aligned} \tag{3.5.19}$$

But Eqs. (3.5.7) imply that the isotopically mapped product $[X_i, X_j]_{A*}$ reduces to a linear combination of the generators via functions on the base manifold, the structure functions. This feature was absent in Eqs. (3.4.15) because the conventional product $[X_i, X_j]_A$ yields a linear combination of the generators via elements of the field, the structure constants.

For case (3.5.1) we had what is called an F-linear combination of generators. The combination originating from rule (3.5.7) will then be called F*-linear combination. Notice that the coefficients of such combinations are not necessarily elements of the ideal $\mathcal{R}^*$. This, however, does not affect the generalization of Lemma 3.4.1 which reads

LEMMA 3.5.1: Every element of the isotope $\mathcal{T}^*$ of the tensor algebra $\mathcal{T}$ is congruent, modulo $\mathcal{R}^*$, to an F*-linear combination of 1 and standard isotopically mapped monomials.

The reduction of $\mathcal{A}^*$ to the isotope A* of the envelop A primarily used in physical applications (i.e., structure (3.4.17) with conventional associative product $X_i X_j$) is also straightforward. Hoping not to create notational confusion, we shall preserve the symbol * for such image product too. The reason is trivial. The differentiation between the tensorial and the associative product becomes much weaker at the isotopic level because the transition from one to the other is precisely a case of identity algebraic isotopy.

The generalizations of Lemma 3.4.2 is then left to the interested reader. We reach in this way the following Lie covering of the Poincaré-Birkhoff-Whitt theorem.

THEOREM 3.5.1: The cosets of 1 and the standard isotopically mapped monomials form a basis of the isotopically mapped universal enveloping associative algebra A*(G) of a Lie algebra G.

Notice that is this theorem we have used the notation A*(G) rather than A*(G*). The former symbol stands to emphasize that the generators of G are unchanged in the isotopic mapping by construction. This imples that G cannot be necessarily recovered as a subalgebra of the attached algebra $[A^*(G)]^-$, i.e., we have in general

$$G \not\approx [A^*(G)]^- \qquad (3.5.20)$$

This is an indication of the nontriviality of the generalization considered. And indeed, what can be constructed via the attached

algebra $[A^*(G)]^-$ of A*(G) is the isotope G* of G, i.e.,

$$[\mathcal{A}^*(G)]^- \approx G^* \not\approx G \qquad (3.5.21)$$

Again, A*(G) is ∞-dimensional with basis

$$1,\ X_i,\ X_{i_1} * X_{i_2},\ X_{i_1} * X_{i_2} * X_{i_3}, \ldots \qquad (3.5.22)$$
$$i_1 \leq i_2 \qquad i_1 \leq i_2 \leq i_3$$

and elements

$$X_{i_1}^{k_1} * X_{i_2}^{k_2} * \ldots * X_{i_m}^{k_m},\ k_1, k_2, \ldots, k_m = 0, 1, 2, \ldots \qquad (3.5.21)$$

where the powers are now in A*(G), that is

$$X_{i_2}^{k_2} = X_{i_2} * X_{i_2} * \ldots * X_{i_2}\ (k_2\text{-times}) \qquad (3.5.23)$$

A simple example of A*(G) envelops will be given later on in this section. More elaborate examples will be provided during the course of our analysis.

The reader should be aware of the fact that the isotope A*(G) of A(G) is not unique in the sense that there may exist several nonisomorphic associative algebras, say, A', A", A'", etc., with corresponding nonisomorphic attached algebras $G' \approx [A']^-$, $G'' \approx [A'']^-$, $G''' \approx [A''']^-$, etc., which are all isotopically related to the same algebra A. This is due to the fact that one given Lie algebra G can be isotopically related to several nonisomorphic Lie algebras (of the same dimensionality).

As a final note, the set of all isotopic mappings of a given Lie algebra can be construed as to form an (abstract) group. Since this notion will not be needed for our analysis, we shall not indulge in its study at this time.

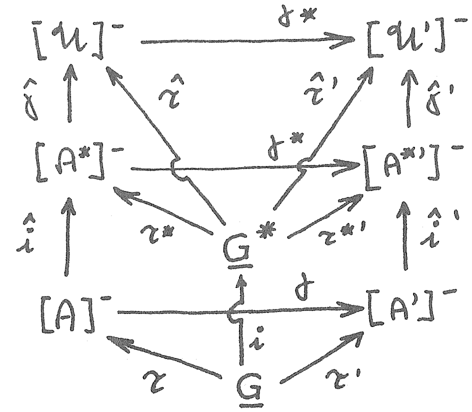
We are now equipped to study the nonassociative Lie-admissible covering of A(G).

DEFINITION 3.5.2: A Lie-admissible genotopically mapped universal enveloping associative algebra of a Lie algebra G is the set

$\{[(\mathcal{A},\tau),\mathcal{A}^*,\hat{i},\tau^*],\mathcal{U},\hat{\tau},\delta\}$ where:

- $(\mathcal{A},\tau)$ is the universal enveloping associative algebras of G according to Definition 3.4.1;
- $[(\mathcal{A},\tau),\mathcal{A}^*,\hat{i},\tau^*]$ is the isotopically mapped universal enveloping associative algebra according to Definition 3.5.1;
- $\mathcal{U}$ is a Lie-admissible algebra;
- $\hat{\tau}$ is a homomorphism of G* into $\mathcal{U}^-$; and
- δ is an isomorphism of $[\mathcal{A}^*]^-$ into $\mathcal{U}^-$;

such that the following property holds. If $\mathcal{U}'$ is another Lie-admissible algebra and $\hat{\tau}'$ a homomorphism of G* into $\mathcal{U}'^-$, there exists a homomorphism γ^* of $\mathcal{U}^-$ into $\mathcal{U}'^-$, i.e., the following diagram is commutative.



(3.5.25)

The central idea which is intended with the above definition is that the enveloping algebra of a Lie algebra is not unique. This is the very essence of the notion of Lie-admissibility. There is here no contradiction with Definitions 3.4.1 and 3.5.1. The enveloping algebras of these definitions are indeed unique and, thus, "universal". However, this property holds if and only if the condition of associativity of the product is assumed. In the transition from the former to the latter definition the "Lie algebra content" changes according to Eqs. (3.5.1) and (3.5.7). Explicitly, a nontrivial isotopic mapping $A(G) \xrightarrow{\hat{i}} A^*(G)$ (that is, an associativity preserving mapping of the product which yields nonisomorphic algebras) implies that

$$G \approx [\mathcal{A}(G)]^- \not\approx [\mathcal{A}^*(G)]^- \approx G^* \qquad (3.5.26)$$

Therefore, A(G) is "universal" for G while A*(G) is "universal" for G*, and no contradiction arises.

The uniqueness of the envelop ceases to hold when the restriction of associativity of the product is lifted. As a result, at least in principle, there exist more than one Lie-admissible algebra which can be constructed as the envelop of a Lie algebra.

This occurrence can be presented with the following lemmas.

LEMMA 3.5.2: When the Lie-admissible algebra of Definition 3.5.2 is associative, diagrams (3.4.2) and (3.5.25) coincide.

Explicitly, in this case $\mathcal{U} \equiv \mathcal{A}$ and the only admissible isotope is the identity, i.e., $\mathcal{U} \equiv \mathcal{A}^* \equiv \mathcal{A}$.

However, Lie-admissible algebras can also be nonassociative. We then have the following first property.

LEMMA 3.5.3: When the Lie-admissible algebra $\mathcal{U}$ of Definition 3.5.2 is the Lie algebra G, diagram (3.5.25) reduces to (3.5.8).

This essentially indicates that the product of a genuine nonassociative envelop of a Lie algebra cannot be antisymmetric because in this case the envelop becomes trivally identical to the algebra itself. On similar ground one can see that the product of the envelop of a Lie algebra cannot be totally symmetric either. Thus, commutative Jordan algebras (Section 1.3) are excluded.

In essence, we are here referring to the true notion of Lie-admissibility according to rule (1.4.1), whereby the same Lie algebra G can be constructed as the attached algebras U^-, U'^-, U''^-, etc. of different algebras U, U', U", etc. where the term "different" is referred to algebras characterized by different laws.

The classification of these possibility has been done in Chapter 1 (see references [4-7]). The first distinction is provided by

A. Associative algebras;

B. Nonassociative algebras.

All associative algebras are Lie-admissible. This is not the case for the nonassociative algebras. The most meaningful algebras of class B are those of Section 1.4, i.e.:

B.I. General Lie-admissible algebras;

B.II. Flexible Lie-admissible algebras; and

B.III. Lie algebras.

For the construction of a nontrivial envelop only the algebras of type A, B.I and B.II are significant because those of type B.III yield the identity of the envelop with the algebra itself (Lemma 3.5.3).

We remain with two additional possibilities: the flexible and the general Lie-admissible algebras, i.e., the nontrivial Lie-admissible algebras according to Section 1.4 From now on the term "Lie-admissible" will be tacitly referred to these algebras. Similarly, the term "genotope" will be referred to a "nontrival

Lie-admissible genotope". To identify the significance of these notations the reader should be aware that a genotopic mapping, unless to a specified new algebra, can induce an arbitrary algebra. Thus, the "genotope of a universal enveloping associative algebra" implies, at least in principle, a mapping to an algebra which is not necessarily Lie-admissible.

The construction of a genotope of $\mathcal{A}(\underline{G})$ according to Definition 3.5.2 can be done as follows. Perform a <u>genotopic mapping of the tensorial product</u> of $\mathcal{A}(\underline{G})$ which verifies the following properties:

(1) it is, by central construction, nonassociative, i.e.,

$$\mathcal{A}: X_i \otimes X_j \Rightarrow \mathcal{U}: X_i \circ X_j \qquad (3.5.27)$$
$$(X_i \circ X_j) \circ X_k \neq X_i \circ (X_j \circ X_k)$$

nevertheless,

(2) it characterizes an algebra (in the sense of Section 1.2), i.e., it satisfies the right and left distributive and scalar rules

$$X_i \circ (X_j + X_k) = X_i \circ X_j + X_i \circ X_k , \qquad (3.5.28a)$$
$$(X_i + X_j) \circ X_k = X_i \circ X_k + X_j \circ X_k , \qquad (3.5.28b)$$
$$(\alpha X_i) \circ X_k = X_i \circ (\alpha X_k) = \alpha (X_i \circ X_k)$$
$$= (X_i \alpha) \circ X_k = X_i \circ (X_k \alpha) = (X_i \circ X_k)\alpha \qquad (3.5.28c)$$

(3) it is Lie-admissible, i.e.,

$$\mathcal{U}^- : [X_i , X_j]_{\mathcal{U}} = X_i \circ X_j - X_j \circ X_i \equiv \qquad (3.5.29)$$
$$\equiv [X_i , X_j]_{\mathcal{A}^*}$$

A realization of this mapping is given by Eqs. (3.5.6). This realization will be tacitly assumed here on.

For the case of the product of two elements there is no ordering ambiguity. However, beginning with the product of three elements on, an ordering ambiguity arises due to the nonassociative nature of the algebra. And indeed, in this case we have two elements

$$(X_i \circ X_j) \circ X_k \quad , \quad X_i \circ (X_j \circ X_k) \qquad (3.5.30)$$

which are generally different for the product $\circ$, while coincide with the associative product as well as all its isotopic images. The number of different elements then increases with the degree of the product. In fact, for degree four we have

$$((X_{i_1} \circ X_{i_2}) \circ X_{i_3}) \circ X_{i_4} , \quad (X_{i_1} \circ (X_{i_2} \circ X_{i_3})) \circ X_{i_4} \qquad (3.5.31)$$
$$X_{i_1} \circ ((X_{i_2} \circ X_{i_3}) \circ X_{i_4}) , \quad X_{i_1} \circ (X_{i_2} \circ (X_{i_3} \circ X_{i_4}))$$

By keeping into account this feature, the <u>genotope of the tensorial products</u> (3.4.7) will be symbolically written

$$\hat{\underline{G}}^{(1)} = \underline{G} \quad , \quad \hat{\underline{G}}^{(2)} = \underline{G} \otimes \underline{G} \quad , \quad \hat{\underline{G}}^{(3)} = \underline{G} \otimes \underline{G} \otimes \underline{G} \quad , \ldots \qquad (3.5.32)$$

with the understanding that the product yields all possible different associations per each given degree.

The most general <u>genotope of the associative tensor algebra</u> $\mathcal{T}$, Eq. (3.4.8), can then be written

$$\hat{\mathcal{A}}_T(X) = F1 \oplus \hat{G}^{(1)} \oplus \hat{G}^{(2)} \oplus \hat{G}^{(3)} \oplus \cdots \qquad (3.5.33)$$
$$= F1 \oplus \underline{G} \oplus \underline{G} \odot \underline{G} \oplus \underline{G} \odot \underline{G} \odot \underline{G} \oplus \cdots,$$

where the symbol $\odot$ now represents all possible different associations indicated earlier.

In line with Definitions (3.4.9) and (3.5.15), the genotope $\hat{\mathcal{R}}$ of the ideal $\mathcal{R}$, of $\mathcal{A}_T$ can be defined, again, as the set of elements

$$[X_i, X_j]_{\mathcal{U}} - (X_i \circ X_j - X_j \circ X_i) \qquad (3.5.34)$$
$$= [X_i, X_j]_{A^*} - (X_i * X_j - X_j * X_i),$$

This trivially implies the following

LEMMA 3.5.4: The genotope $\hat{\mathcal{R}}$ and the isotope $\mathcal{R}^*$ of $\mathcal{R}$ coincide.

The reader should here keep in mind that the ideals $\mathcal{R}^*$ and $\mathcal{R}$ are, in general, different, e.g., they include different Casimir invariants (3.4.23) for nonisomorphic algebras $\underline{G}$ and $\underline{G}^*$.

We now consider the genotope of the universal enveloping algebra of a Lie algebra

$$\mathcal{U}(\underline{G}) = \hat{\mathcal{A}}_T(\underline{G}) / \hat{\mathcal{R}}(\underline{G}) \qquad (3.5.35)$$

To see that this can indeed characterize an envelop of $\underline{G}^*$ and, thus, satisfy Definition 3.5.2, let $\hat{\tau}$ be the mapping of

$\underline{G}^*$ into $\mathcal{U}^-$ and let $\hat{\tau}'$ be a mapping such that

$$1\hat{\tau}' = 1, \qquad (3.5.36a)$$

$$(X_{i_1} \circ \cdots \circ X_{i_m})\hat{\tau}' = (X_{i_1}\hat{\tau}) \circ \cdots \circ (X_{i_m}\hat{\tau}), \qquad (3.5.36b)$$

for a given ordering. Then we have

$$X_i\hat{\tau}' = X_i\hat{\tau}, \qquad (3.5.37a)$$

$$[X_i, X_j]_{\mathcal{U}}\hat{\tau}' - ((X_i\hat{\tau}') \circ (X_j\hat{\tau}') - (X_j\hat{\tau}') \circ (X_i\hat{\tau}'))$$
$$= [X_i, X_j]_{A^*}\tau - ((X_i\hat{\tau}) \circ (X_j\hat{\tau}) - (X_j\hat{\tau}) \circ (X_i\hat{\tau})) \qquad (3.5.37b)$$

in which case the elements of the ideal $\hat{\mathcal{R}}$ belong to the kernel of $\hat{\tau}'$. This induces a mapping of $\mathcal{U}$ into a new algebra, say $\mathcal{U}'$, which is such to preserve the attached algebras, i.e., it is an isomorphism when restricted to $\mathcal{U}^-$ and $\mathcal{U}'^-$. This yields the context of Definition 3.5.2. Notice that the above argument does not ensure that $\mathcal{U}$ and $\mathcal{U}'$ are isomorphic. This is a desired feature within a Lie-admissible context, in line with Definition 3.5.2.

To restate this situation in different terms, we have the following

LEMMA 3.5.5: The genotope $\mathcal{U}(\underline{G})$ of $\mathcal{A}(\underline{G})$ is Lie-admissible.

Explicitly, under the notation

$$\hat{X}_i = X_i + \hat{R}\ , \quad \hat{X}_i \circ \hat{X}_j = X_i \circ X_j + \hat{R} \qquad (3.5.38)$$

we have

$$[\mathcal{U}(G)]^- : [\hat{X}_i, \hat{X}_j]_{\mathcal{U}} = \hat{X}_i \circ \hat{X}_j - \hat{X}_j \circ \hat{X}_i \qquad (3.5.39)$$

but we can write

$$\begin{aligned} [\hat{X}_i, \hat{X}_j]_{\mathcal{U}} &= X_i \circ X_j - X_j \circ X_i + \hat{R} \\ &= X_i * X_j - X_j * X_i + R^* \qquad (3.5.40) \\ &\equiv [\hat{X}_i, \hat{X}_j]_{A^*}\ , \end{aligned}$$

thus $\mathcal{U}(\underline{G})$ is Lie-admissible. Again, the "Lie content" of $\mathcal{U}(\underline{G})$ is not $\underline{G}$ but instead its isotope $\underline{G}^*$ according to Eqs. (3.5.40).

We now pass to the study of the basis of $\mathcal{U}(\underline{G})$. We shall here closely follow the analysis by C.N. Ktorides.[19] A first difficulty is the identification of the genotope of the standard monomials under the ordering ambiguities created by the nonassociative nature of the product. In principle, beginning from the case of degree three on, different monomials of the same degree are not linearly independent in view of the structure (3.3.61) of the Lie-admissible quantities. However, the possibility that a linear dependence occurs for given indices should not be excluded, owing to the dependence of the genotopic functions in the indices (Section 3.3). We reach in this way the following

DEFINITION 3.5.3: The standard genotopically mapped monomials of degree m are the union of all F*-linearly independent monomials of the same degree, different associations and index zero.

Explicitly, the monomials under consideration are the collection of all elements of the type

$$\hat{M}^s_m = \left\{ \begin{matrix} X_{i_1} \circ X_{i_2} \circ \cdots \circ X_{i_m} \\ i_1 \le i_2 \le \cdots \le i_m \end{matrix} \right\}, \qquad (3.5.41)$$

for all possible different associations which are "F*-linearly independent", that is, cannot be reduced to a linear combinations of the other monomials with coefficients in the field and in the base manifold.

For the case of degree two we have only the possibility

$$\hat{M}^s_2 = \underset{i_1 \le i_2}{X_{i_1} \circ X_{i_2}} \qquad (3.5.42)$$

For degree three we have in general two possibilities, i.e.,

$$\hat{M}^s_3 = \left\{ \begin{matrix} (X_{i_1} \circ X_{i_2}) \circ X_{i_3}\ , & X_{i_1} \circ (X_{i_2} \circ X_{i_3}) \\ i_1 \le i_2 \le i_3 & i_1 \le i_2 \le i_3 \end{matrix} \right\} \qquad (3.5.43)$$

However, Definition 3.5.3 excludes those which are F*-linearly dependent for given values of the indeces. Similar cases occur for higher degrees.

To construct a base we must now work out the reduction of arbitrary monomials $\hat{M}_m$ to a standard form.

For the case of degree two we simply have

$$[X_{i_2} \circ X_{i_1} = X_{i_1} \circ X_{i_2} + [X_{i_2}, X_{i_1}]_{\mathcal{U}} \quad \mathrm{Mod}\ \hat{R}. \tag{3.5.44}$$

Therefore, a nonstandard monomial $\hat{M}_2$ can be reduced to a standard monomial $\hat{M}_2^s$ plus an F*-linear combination of $\hat{M}_1^s$ monomials, i.e.,

$$\hat{M}_2 = \hat{M}_2^s + \hat{M}_1^s \tag{3.5.45}$$

For the case of degree three a number of possibilities occur. First, the following possibilities

$$(X_{i_2} \circ X_{i_1}) \circ X_{i_3} = (X_{i_1} \circ X_{i_2}) \circ X_{i_3} + [X_{i_2}, X_{i_1}]_{\mathcal{U}} \circ X_{i_3} \quad \mathrm{Mod}\ \hat{R} \tag{3.5.46a}$$

$$X_{i_1} \circ (X_{i_3} \circ X_{i_2}) = X_{i_1} \circ (X_{i_2} \circ X_{i_3}) + X_{i_1} \circ [X_{i_3}, X_{i_2}]_{\mathcal{U}} \quad \mathrm{Mod}\ \hat{R} \tag{3.5.46b}$$

are trivial. However, the case

$$\begin{aligned}(X_{i_3} \circ X_{i_2}) \circ X_{i_1} &= X_{i_1} \circ (X_{i_2} \circ X_{i_3}) \\ &+ X_{i_1} \circ [X_{i_3}, X_{i_2}]_{\mathcal{U}} \quad \mathrm{Mod}\ \hat{R} \\ &+ [(X_{i_3} \circ X_{i_2}), X_{i_1}]_{\mathcal{U}} \quad \mathrm{Mod}\ \hat{R}\end{aligned} \tag{3.5.47}$$

is not trivial because, in general,

$$\begin{aligned}&[(X_{i_3} \circ X_{i_2}), X_{i_1}]_{\mathcal{U}} \neq \\ &\neq X_{i_3} \circ [X_{i_2}, X_{i_1}]_{\mathcal{U}} + [X_{i_3}, X_{i_1}]_{\mathcal{U}} \circ X_{i_2}\end{aligned} \tag{3.5.48}$$

But, under realization (3.5.6) of the genotopically mapped product we can write

$$\begin{aligned}&[(X_{i_3} \circ X_{i_2}), X_{i_1}]_{\mathcal{U}} = \\ &= [\alpha^r_{i_3} X_r, X_{i_1}]_{A^*} \otimes \alpha^s_{i_2} X_s + \alpha^r_{i_3} X_r \otimes [\alpha^s_{i_2} X_s, X_{i_1}]_{A^*} \\ &- [\beta^r_{i_2} X_r, X_{i_1}]_{A^*} \otimes \beta^s_{i_3} X_s + \beta^r_{i_2} X_r \otimes [\beta^s_{i_3} X_s, X_{i_1}]_{A^*}\end{aligned} \tag{3.5.49}$$

Also, the redefinition $\otimes \rightarrow *$ is always possible owing to the freedom in the F*-linear mapping. Thus, a nonstandard monomial $\hat{M}_3$ can always be reduced to an F*-linear combination of standard genotopically mapped and standard isotopically mapped monomials of decreasing degrees and we write

$$\hat{M}_3 = \hat{M}_3^s + \hat{M}_2^s + \hat{M}_1^s + M^{*s}_3 + M^{*s}_2 \tag{3.5.50}$$

The generalization of the approach to an arbitary degree then yields the reduction

$$\hat{M}_m = \hat{M}_1^s + \sum_{i=2}^{m} \hat{M}_i^s + \sum_{i=2}^{m} M^{*s}_i \tag{3.5.51a}$$

$$\hat{M}_1^s \equiv M^{*s}_1 \equiv M^s_1 \tag{3.5.51b}$$

<u>LEMMA</u> 3.5.6: <u>A genotopically mapped monomial of degree m can always be expressed as an F*-linear combination, modulo $\hat{R}$, of 1, standard genotopically mapped monomials and standard isotopically mapped monomials of degree $\leq$ m.</u>

To restate it in different terms, in the transition from the isotope $\mathcal{A}^*(\underline{G})$ to the genotope $\mathcal{U}(\underline{G})$ a new feature appears. Within the former context all monomials can be reduced to standard monomials of the same algebra without recursion to the standard monomials of the original algebra $\mathcal{A}(\underline{G})$. Within the latter context, the monomials cannot be all reduced to standard from without recursion to the standard monomials of $\mathcal{A}^*(\underline{G})$.

This is, after all, predicable from the nonassociativity of the genotope as well as the symbiotic nature of $\mathcal{U}(\underline{G})$ and $\mathcal{A}^*(\underline{G})$ of characterizing, by construction, the same algebra $\underline{G}$*.

In the final analysis, this occurrence can also be anticipated from the structure (3.3.73) of the product of elements of the Lie-admissible group which is precisely of isotopic, rather than genotopic, nature. In other words, from this composition law of Lie-admissible groups we expect that the basis of the genotope $\mathcal{U}(\underline{G})$ is composed of elements of the genotope itself plus elements of the isotope $\mathcal{A}^*(\underline{G})$.

On equivalent grounds, the occurrence can be anticipated from Lemma 3.5.4 on the identity of the genotope $\hat{\mathcal{R}}$ and the isotope $\mathcal{R}^*$ of the ideal $\mathcal{R}$. As a matter of fact, the occurrence here considered can be interpreted as an indication that the genotope $\mathcal{U}(\underline{G})$ can be considered as a sort of "prolongation" of the isotope $\mathcal{A}^*(\underline{G})$ with elements whose underlying product is nonassociative, but such not to alter the Lie algebra content, i.e., $[\mathcal{U}(\underline{G})]^- \equiv [\mathcal{A}^*(\underline{G})]^-$. This is fully in line with Definition 3.5.2.

We now remain with the question whether the standard monomials

$$\hat{M}^s_m = \left\{ \begin{matrix} X_{i_1} \circ X_{i_2} \circ \cdots \circ X_{i_m} \\ i_1 \le i_2 \le \cdots \le i_m \end{matrix} \right\}, \quad M^{*s}_m = \begin{matrix} X_{i_1} * X_{i_2} * \cdots * X_{i_m} \\ i_1 \le i_2 \le \cdots \le i_m \end{matrix} \tag{3.5.52}$$

are independent among themselves. In general, this is not the case because, again, of the possible variations of the genotopic functions with the indeces.

The analysis above, even though insufficient to provide a proof for the case of general Lie-admissible algebras, is nevetheless sufficient to render plausible the following conjecture on the general Lie-admissible covering of the Poincaré-Birkhoff-Witt theorem

CONJECTURE 3.5.1: The cosets of 1 and the union of F*-linearly independent standard genotopically mapped monomials and standard isotopically mapped monomials form a basis of a nontrivial Lie-admissible genotope $\mathcal{U}(\underline{G})$ of the universal enveloping associative algebra $\mathcal{A}(\underline{G})$ of a Lie algebra $\underline{G}$.

As we shall indicate below, the property expressed by the above conjecture has been proved for the case of flexible Lie-admissible algebras by Ktorides[19]. For the more general case of unrestricted Lie-admissible algebras, the conjecture appears to be true for all cases of practical interest in which we are involved, such as the exponentiation of general Lie-admissible brackets. This is essentially based on the following rewriting of monomials in $\mathcal{U}(\underline{G})$ for one given association (here ignored)

$$\hat{X}_{i_1}\circ\hat{X}_{i_2}\circ\cdots\circ\hat{X}_{i_m}$$
$$= \alpha^*\hat{X}_i + \beta^*(\hat{X}_{i_1}\circ\hat{X}_{i_2}) + \gamma^*(\hat{X}_{i_1} * \hat{X}_{i_2}) \qquad (3.5.53a)$$
$$+ \delta^*(\hat{X}_{i_1}\circ\hat{X}_{i_2})\circ\hat{X}_{i_3} + \rho^*(\hat{X}_{i_1}\circ(\hat{X}_{i_2}\circ\hat{X}_{i_3}))$$
$$+ \sigma^*(\hat{X}_{i_1} * \hat{X}_{i_2} * \hat{X}_{i_3}) + \cdots$$
$$i_1 \le i_2 \le i_3 \le \cdots, \quad \hat{X}_i = X_i + \hat{R} \qquad (3.5.53b)$$

where $\alpha^*, \beta^*, \gamma^*$, etc., are the elements of the F*-linear combination, that is, besides including elements of the field, they generally depend on the base manifold.

The reduction of the right hand side of Eqs. (3.5.53) to elements of the basis, i.e., F*-linearly independent elements, demands the identification of the nonassociative Lie-admissible product $X_i \circ X_j$. As a result, we content ourselves of illustrating this aspect with explicit examples. The study of the general methods for the construction of such basis is here left to the interested reader.

The analysis by C.N. Ktorides was based on what this author called <u>Santilli algebras</u>[19] (i.e., the flexible Lie-admissible algebras) for the particular case of the <u>Santilli-Soliani product</u>[19] in tensorial form

$$X_i \circ X_j = \lambda X_i \otimes X_j + \mu X_j \otimes X_i \qquad (3.5.54)$$
$$\lambda, \mu \in F$$

Conjecture 3.5.1 is a simple generalization of Theorem 2.1 by C.N. Ktorides[19] to general Lie-admissible algebras. Ktorides case, besides its significance for explicit examples

(Appendix 3.F) as well as for quantization (Volume III), provides a significant illustration of our conjecture. It is therefore significant to briefly consider it.

Product (3.5.54) is a simple but genuine example of nontrivial Lie-admissible genotopic mapping of the tensorial product $\otimes$. In fact, product (3.5.54), first of all characterizes an algebra, the tensorial form $\mathcal{A}(\lambda,\mu)$ of the mutation algebra $A(\lambda,\mu)$ of Section 1.4. Secondly, such algebra is nonassociative, by therefore satisfying a central requirement of our analysis. Finally, product (3.5.54) is Lie-admissible. However, since the elements λ and μ belong to the field, the isotope $\underline{G}^*$ of $\underline{G}$ is trivial, and it is given by

$$[X_i, X_j]_{\mathcal{A}(\lambda,\mu)} = (\lambda-\mu)[X_i, X_j]_{\mathcal{A}}, \quad \lambda \neq \mu \qquad (3.5.55)$$

This can be written in a way which formally coincides with $\underline{G}$, via the isotope of $\mathcal{A}(\underline{G})$

$$\mathcal{A}: X_i X_j \Rightarrow \mathcal{A}^{*\prime}: X_i * X_j = (\lambda-\mu) X_i X_j \qquad (3.5.56)$$

The analysis of this section, from Eqs. (3.5.27) to (3.5.35), trivially apply for product (3.5.54), yielding the tensorial algebra

$$\hat{\mathcal{L}}(\lambda,\mu) = F1 \oplus \underline{G} \oplus \underline{G}\circ\underline{G} \oplus \cdots \qquad (3.5.57)$$

The ideal $\hat{\mathcal{R}}(\lambda,\mu)$ is then given by elements of the type

$$[X_i, X_j]_{A(\lambda,\mu)} - (X_i \circ X_j - X_j \circ X_i) \qquad (3.5.58)$$

The flexible Lie-admissible genotopic mapping of the universal enveloping associative algebra of a Lie algebra constructed, apparently for the first time, by C.N. Ktorides, was then given by

$$\mathcal{U}(\lambda,\mu) = \hat{\mathcal{T}}(\lambda,\mu) / \hat{\mathcal{R}}(\lambda,\mu) \qquad (3.5.59)$$

and called <u>universal enveloping mutation algebra</u> of a Lie algebra in line with the terminology of references[4-7].

Instead of Lemma 3.5.6 we now have the following

<u>LEMMA</u> 3.5.7: <u>A monomial of degree m in $\hat{\mathcal{T}}(\lambda,\mu)$ can always be expressed as an F-linear combination, modulo $\hat{\mathcal{R}}(\lambda,\mu)$, of 1, standard monomials in $\hat{\mathcal{T}}(\lambda,\mu)$, and standard monomials in $\mathcal{T}^*(\lambda,\mu)$ of degree $\leq$ (m-2)</u>.

In this lemma $\mathcal{T}^*(\lambda,\mu)$ is trivial because proportional to $\mathcal{T}$ via elements of the field. The major difference between Lemmas 3.5.6 and 3.5.7 is given by the fact that the standard monomials in $\mathcal{T}^*$ occur in the former with degree $\leq$ m and in the latter with degree $\leq$ (m-2). This is simply due to the fact that the F*-linear combination of Lemma 3.5.6 reduces to the ordinary F-linear combination for Lemma 3.5.7, that is, the constancy of the coefficients allows the reduction of the degree by two, as it can be seen, e.g., in Eqs. (3.5.49).

For practical applications the ordinary associative product $X_i X_j$ rather than the tensorial product $X_i \otimes X_j$ is used in structure

(3.5.54). This reduction is allowed by the following

<u>LEMMA</u> 3.5.8[19]: <u>For a given universal enveloping mutation algebra $\mathcal{U}(\lambda,\mu)$ there always exists a homomorphism</u>

$$\mathcal{U}(\lambda,\mu) \xrightarrow{\sigma} A(\lambda,\mu) \qquad (3.5.60)$$

<u>such that</u>

$$(\lambda X_i \otimes X_j + \mu X_j \otimes X_i)\sigma = \lambda\, \sigma(X_i)\,\sigma(X_j) + \mu\, \sigma(X_j)\,\sigma(X_i) = \lambda X'_i X'_j + \mu X'_j X'_i \qquad (3.5.61)$$

Whenever no ambiguity arises, we shall use the identifications

$$\mathcal{U}(\lambda,\mu) \equiv A(\lambda,\mu), \quad X'_i \equiv X_i \qquad (3.5.62)$$

In conclusion, pending verifications by independent researchers, it appears that the true characterization of the notion of associative Lie-admissibility (i.e., via the universal enveloping associative algebras of a Lie algebra) admits a consistent nonassociative but Lie-admissible covering.

This is, perhaps, the most crucial point of the entire analysis of these volumes, both classically and quantum mechanically.

On classical grounds, the notion of nonassociative-Lie-admissiblity (i.e., via a genotope of the associative enveloping algebra) appears to confirm the existence of a dual algebraic-group theoretic profile of our Lie-admissible approach in the sense of admitting a Lie-admissible algebraic characterization in the neighborhood of the origin, while allowing a generalization of the exponential law which results, under suitable integrability conditions, into a finite, connected, group of transformations (the Lie-admissible group of Definition 3.3.3). Clearly, without a Lie-admissible envelop, such a context is questionable.

The primary intended use of these Lie-admissible covering approaches, as pointed out in Section 3.1, is for attempting a breaking of the fundamental space-time Lie symmetries under forces not derivable from a potential (represented in our approach precisely by the genotopic mapping). The ultimate hope, as we shall see in Volume III is to attempt a profound differentiation between the electromagnetic and the strong interactions as it occurs in the physical reality, via a differentiation of the corresponding relativity laws.

On quantum mechanical grounds, the analysis of this chapter will also be crucial. In fact, the attempt of quantizing systems with forces not derivable from a potential of Volume III is largely based on the transition from an associative envelop of operators in a Hilbert space to a nonassociative but Lie-admissible form.

NOTE ADDED IN 1982.

Conjecture 3.5.1 has remained unproved until now. Ktorides, Myung and Santilli[195] published in 1980 a refinement of Ktorides proof for the flexible case, which can be expressed as follows.

THEOREM 3.5.3: Let $\underline{G}$ be a Lie algebra over a field F, and A any associative algebra with unit element over F. If f is a homomorphism of $\underline{G}$ into A^-, then there exists a unique homomorphism g of the quotient algebra

$$\mathcal{U}_{\lambda,\mu} \equiv \mathcal{U}(G)_{\lambda,\mu} = T(G)_{\lambda,\mu}/R$$
$$T(G)_{\lambda,\mu} = F1 \oplus G_1' \oplus G_2' \oplus \cdots, \quad \lambda, \mu \in F$$
$$G_n' = \sum_{i=1}^{n-1} G_i' \circ G_{n-i}', \quad a \circ b = \lambda ab + \mu ba \qquad (3.5.63)$$

into the mutation algebra $A(\lambda, \mu)$ of A such that

$$gi = \frac{1}{\lambda - \mu} f \qquad (3.5.64)$$

The following advances in the understanding of exponentiation in nonassociative Lie-admissible enveloping algebras have occurred since 1977.

Santilli achieved in 1978[194] a realization of Theorem 3.5.1 on the Lie-isotopic generalization of the enveloping algebra which is based on the following isotopic product of vector fields

$$X_i * X_j = X_i T X_j \quad , \quad T = \text{fixed} \tag{3.5.65}$$

$$X_s = U^{\mu}_{s}(a)\partial_\mu \; , \quad T = t^{\mu}(a)\partial_\mu \; , \quad s = i, j, \; \partial_\mu = \frac{\partial}{\partial a^\mu}$$

with basis

$$1, \; X_{i_1} T X_{i_2} \; , \; X_{i_1} T X_{i_2} T X_{i_3} , \ldots \tag{3.5.66}$$

and exponentiation

$$\left\{1 + \frac{\theta^k}{1!} T X_k + \frac{\theta^i \theta^j}{2!} T X_i T X_j + \ldots \right\} A = e^{\theta^k T X_k} A \tag{3.5.67}$$

It is an instructive exercise for the intersted reader to prove that Theorem 3.5.1 for product (3.5.65) and basis (3.5.66) is indeed correct.

In the same paper[194], Santilli proposed the following realization of the general Lie-admissible product (today called <u>fundamental realization of Lie-admissible algebras</u>)

$$X_i \circ X_j = X_i R X_j + X_j S X_i \; , \quad R, S = \text{fixed} \tag{3.5.68}$$

where the elements can be vector fields

$$X_s = U^{\mu}_{s}(a)\partial_\mu \; , s = i, j \; , \quad R = R^\mu(a)\partial_\mu \; , \quad S = S^\mu(a)\partial_\mu \tag{3.5.69}$$

or, more generally, operators on a (suitable formulation of) Hilbert space. (A flexible particularizetion of product (3.5.68) was proposed in ref.[196]).

Product (3.5.68) supports the existence of a general Lie-admissible envelope for the following reasons. First of all, product (3.5.68) permit the exponentiation to the finite form

$$A(\theta) = \left(1 + X_k R \frac{\theta^k}{1} + X_i R X_j R \frac{\theta^i \theta^j}{2!} + \ldots \right) \times A(0) \times$$
$$\times \left(1 + \frac{(-\theta^k)}{1!} S X_k + \frac{(-\theta^i)(-\theta^j)}{2!} S X_i S X_j + \ldots \right)$$
$$= e^{X^k R \theta_k} A(0) \, e^{-\theta^k S X_k} \tag{3.5.70}$$

In turn, the existence of the exponential form is, per se, sufficient to render plausible the existence of a basis and, thus, the validity of Conjecture 3.5.1.

The reader should however note that exponentiation (3.5.70) is of two-sided character and it is isotopic associative in each size. By recalling the validity of isotopic theorem 3.5.1, exponentiation (3.5.70) therefore appears to admit a rigorous mathematical structure. To express this important point in different terms, the infinite convergent series needed to reach an exponential form are not expressed in terms of an infinite series of recurring products (3.5.68). Instead, they are expressed via two separate, associative isotopic products, one per each side. The general Lie-admissible structure emerges from the mere difference of the left and right isotopic operators.

The crucial reduction (3.5.49) which is needed to reach basis (3.5.53) appears possible for product (3.5.68). In fact, by recalling that, from one side, third-order terms in the generators are a combination of terms of decreasing order

$$X_i * X_j = X_i T X_j = \alpha^{\mu}_{(a)} \partial_\mu$$
$$= \alpha^{\mu}(a) \partial_\mu + \beta^{\mu\nu}(a) \partial_\mu \partial_\nu \qquad (3.5.71)$$
$$+ \gamma^{\mu\nu\tau}(a) \partial_\mu \partial_\nu \partial_\tau$$

and, on the other side, the F*-linear combinations are of the type

$$F^* (X_i; X_i X_j; X_i X_j X_k) = \qquad (3.5.72)$$
$$= A^i_k(a) X_i + B^{ij}_{rs}(a) X_i X_j + C^{ijk}_{\ell mn}(a) X_i X_j X_k)$$

we can indeed have decompositions of the type

$$X_i \circ X_j = F^* (X_i ; X_i * X_j) \qquad (3.5.73)$$

under which basis (3.5.53) would follows because of the rules

$$[X_i, X_j]_U = X_i \circ X_j - X_j \circ X_i$$
$$\equiv X_i * X_j - X_j * X_i = [X_i, X_j]_{A^*}$$
$$(3.5.74)$$
$$[X_i * X_j, X_k]_{A^*} = [X_i, X_j]_{A^*} * X_j + X_i * [X_j, X_k]_{A^*}$$
$$T = R - S$$

which confirm the crucial role plaid by the associative isotopy in our study.

For a recent, comprehensive study on exponentiation, the reader is referred to Myung[197] and quoted references.

It is easy to predict that generalized exponentiations of the type (3.5.70) will be at the foundation of the generalized time evolutions and physical laws in the interior of hadrons, to be presented in Volume III.

Nonassociative Lie-admissible covering of the universal enveloping associative algebra

$$\mathcal{U}(G) = \hat{\mathcal{T}}(G)/\hat{\mathcal{R}}(G)$$

$$\hat{\mathcal{T}}(G) = F1 \oplus G \oplus G \circ G \oplus \cdots$$

$$\hat{\mathcal{R}}(G) : [X_i, X_j]_{\mathcal{U}} - (X_i \circ X_j - X_j \circ X_i)$$

$$[\mathcal{U}(G)]^- \approx [\mathcal{A}^*(G)]^- \approx G^* \not\approx G$$

Isotopic image of the universal enveloping associative algebra

$$\mathcal{A}^*(G) = \mathcal{T}^*(G)/\mathcal{R}^*(G)$$

$$\mathcal{T}^*(G) = F1 \oplus G \oplus G * G \oplus \cdots$$

$$\mathcal{R}^*(G) : [X_i, X_j]_{A*} - (X_i * X_j - X_j * X_i)$$

$$G^* \approx [\mathcal{A}^*(G)]^- \not\approx G$$

Universal enveloping associative algebra

$$\mathcal{A}(G) = \mathcal{T}(G)/\mathcal{R}(G)$$

$$\mathcal{T}(G) = F1 \oplus G \oplus G \otimes G \oplus \cdots$$

$$\mathcal{R}(G) : [X_i, X_j]_A - (X_i \otimes X_j - X_j \otimes X_i)$$

$$G \approx [\mathcal{A}(G)]^-$$

Table 3.5.1: The concept of Lie-admissibility. The concept is at the very foundation of Lie's theory, but only expressed in its simplest possible form, the associative form. In particular, the associative Lie-admissible algebra of the conventional Lie's theory results to be the universal enveloping associative algebra $\mathcal{A}(G)$ of a Lie algebra G. The concept of associative Lie-admissibility is then simply expressible by the property that G is homomorphic to the attached algebra $[\mathcal{A}(G)]^-$ of $\mathcal{A}(G)$. The existence of the associative envelop has crucial implications for the conventional Lie's characterization of exact symmetries and conservation laws. First of all, it guarantees the existence of a Lie algebra in the neighborhood of the identity, the symmetry algebra. Secondly, it allows the construction, under integrability conditions, of a connected Lie group of finite transformations, the symmetry group. Thirdly, it allows the proper construction of the representation theory. In particular, the representations can be either linear or nonlinear. The canonical realization of the approach then yields transformations which leave form-invariant the equations of motion while characterizing the conservation laws via the generators of the algebra. This produces the reduction of physical laws to a symbiotic algebraic-group theoretic characterization. The fundamental aspect of the analysis of these volumes rests on the attempt of constructing a nonassociative generalization of the conventional Lie-admissibility here outlined. A central aspect of Lie-admissibility is that a Lie algebra G can be recovered not only as the attached algebra A^- of an associative algebra, but also via the attached algebra U^- of a nonassociative algebra U (Section 1.4), provided that U is Lie-admissible. The classification of the possible nonassociative algebras yields: I. the general Lie-admissible algebra, II. the flexible Lie-admissible algebra, and III. the Lie algebras themselves (Section 1.4). This algebraic characterization, however, is per se insufficient for the application to the transformation theory. We reach in this way the ultimate characterization of the concept of Lie-admissibility in our analysis. It consists of a nonassociative Lie-admissible algebra realized as the nonassociative envelop of a Lie algebra. It is this realization which allows the Lie-admissible covering of Lie's theory (Table 3.3.2). And indeed, such nonassociative envelop first allows a Lie-admissible algebraic behaviour in the neighborhood of the identity. Secondly, it allows the construction, under integrability conditions, of a connected group of finite transformations, the Lie-admissible group of Section 3.3. The covering nature of the nonassociative over the associative Lie-admissible envelop is then generated by its construction via the genotopic mapping, that is, via a nonassociative modification of the product while preserving the parameters, the base manifold and the generators. Via the use of the Birkhoff-admissible equations, this modification is directly representative of the symmetry breaking forces. It therefore appears that, pending verifications by independent researchers, our Lie-admissible approach provides a covering of the conventional Lie approach capable of characterizing symmetry breakings and nonconservation laws. The physical generators are preserved in the covering. Their nonconserved character is then expressed by the nonassociative generalization of their envelop. The original exact symmetry group is then replaced by the exponential form of the nonassociative envelop, the Lie-admissible group. As we shall see in Chapter 5, once realized with analytic means, (i.e., our Birkhoff-admissible transformations), this allows a fundamental result: the identification of a covering group under which nonconservative equations of motion are form-invariant. The representation of this covering group, however, are nonlinear (Appendix 3.B), owing to the nonassociative character of the envelop. In conclusion, the Lie-admissible covering

of Lie's theory tentatively identified in this chapter results to be a promising methodological context for the study of the relativity laws which are applicable to systems with forces not derivable from a potential. The potential applicability of the approach, however, goes beyond these relativity problems. For instance, the Lie-admissible approach is a covering of the deformations theory of Lie-algebras (Appendix 3.D) as well as of the recent sypersymmetric approaches in particle physics (Appendix 3.E).

APPENDIX 3.A: SIMPLE EXAMPLE OF LIE-ISOTOPIC AND LIE-ADMISSIBLE ALGEBRAS AND GROUPS

One of the simplest examples of one-dimensional, connected, Lie transformation groups is that of dilations on a line, D(1),

$$x' = f(x;\theta) = e^{\theta} x \qquad (3.A.1)$$

where the f-symbol stands to represent Lie transformations (3.2.20). The standard generator for this group, Eqs. (3.2.23), is given by

$$X = x\frac{\partial}{\partial x} \qquad (3.A.2)$$

In fact, exponential mapping (3.2.42), i.e.,

$$e^{\theta x \frac{\partial}{\partial x}} x = \left[1 + \frac{\theta}{1!}\left(x\frac{\partial}{\partial x}\right) + \frac{\theta^2}{2!}\left(x\frac{\partial}{\partial x}\right)^2 + \cdots\right] x$$
$$= x + \frac{\theta}{1!} x + \frac{\theta^2}{2!} x + \cdots = e^{\theta} x \qquad (3.A.3)$$

reproduces transformation (3.A.1) identically. The composition law (3.2.5) is trivial and reads

$$\theta'' = \theta' + \theta \qquad (3.A.4)$$

as induced by the product

$$x'' = f(x';\theta') = e^{\theta'+\theta} x \qquad (3.A.5)$$

We are now interested in constructing a <u>Lie isotopic mapping</u> $D^*(1)$ of D(1), that is, a Lie group according to Definition 3.3.1. This is, in essence, a new group constructed with the generator and parameter of D(1).

For this purpose we select the following transformations of $D^*(1)$

$$x^* = h(x;\theta)\, f(x;\theta) = f^*(x;\theta) = \frac{x}{1-\theta x} \qquad (3.A.6a)$$

$$h = e^{\theta}/(1-\theta x) \qquad (3.A.6b)$$

where the function h is that of transformations (3.3.10). It is easy to see that the functions $g_i^{\ j}$ of the Lie isotopic exponential mapping (3.3.31) are in this simple case, given by

$$g = x \qquad (3.A.7)$$

In fact, law (3.3.31), reproduces transformation (3.G.6) identically

$$e^{\theta x^2 \frac{\partial}{\partial x}} x = \left[1 + \frac{\theta}{1!}\left(x^2 \frac{\partial}{\partial x}\right) + \frac{\theta^2}{2!}\left(x^2\frac{\partial}{\partial x}\right)^2 + \ldots\right] x$$

$$= x + \theta x^2 + \theta^2 x^3 + \ldots \qquad (3.A.8)$$

$$= \frac{x}{1-\theta x}$$

The isotopically mapped transformation law now coincides with law (3.6.4), as it can be seen from the composition law

$$(x^*)^* = \frac{x^*}{1-\theta' x^*} = \frac{\frac{x}{1-\theta x}}{1-\theta' \frac{x}{1-\theta x}} = \frac{x}{1-(\theta+\theta')x} \qquad (3.A.9)$$

This is not surpising, owing to the local isomorphic character of one-dimensional groups . This property, however, is expected to be generally lost for groups of higher dimensionality.

Notice that the original transformation (3.A.1) is linear in x while its Lie-admissible image (3.A.6) is nonlinear. This is fully in line with the nonlinear nature of the Lie-admissible approach, as indicated in Appendix 3.B. Notice also that transformation (3.A.1) is analytic for all values θ , while this is not the case for transformation (3.A.6) at a fixed value of x. This is also typical of the Lie-admissible approach.

As a second simple example, consider a conservative Newtonian system in a two-dimensional Euclidean space $E_2(\underset{\sim}{r})$ with coordinates r_x, r_y, physical linear momenta p_x, p_y and Hamiltonian $H = T + V$ which exhibits an exact symmetry under the Lie group of rotations SO(2). Thus, H is invariant under SO(2) and the physical angular momentum

$$J = r_x p_y - r_y p_x \qquad (3.A.10)$$

is conserved.

The familiar canonical realization of SO(2) can be written

$$a'^{\mu} = e^{\theta \omega^{\alpha\beta} \frac{\partial J}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}} a^\mu, \qquad (3.A.11a)$$

$$(a^\mu) = \begin{pmatrix} r_x \\ r_y \\ p_x \\ p_y \end{pmatrix}, \quad (\omega^{\mu\nu}) = \begin{pmatrix} 0_{2\times 2} & 1_{2\times 2} \\ -1_{2\times 2} & 0_{2\times 2} \end{pmatrix}. \qquad (3.A.11b)$$

For the space component, Eqs. (3.A.11a) read

$$\begin{pmatrix} r'_x \\ r'_y \end{pmatrix} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} + \frac{\theta}{1!}\begin{pmatrix} [r_x, J] \\ [r_y, J] \end{pmatrix} + \frac{\theta^2}{2!}\begin{pmatrix} [[r_x, J], J] \\ [[r_y, J], J] \end{pmatrix} + \dots$$

$$= \begin{pmatrix} r_x \\ r_y \end{pmatrix} + \frac{\theta}{1!}\begin{pmatrix} -r_y \\ r_x \end{pmatrix} + \frac{\theta^2}{2!}\begin{pmatrix} -r_x \\ -r_y \end{pmatrix} + \dots \qquad (3.A.12)$$

$$= \begin{pmatrix} r_x \cos\theta - r_y \sin\theta \\ r_x \sin\theta + r_y \cos\theta \end{pmatrix},$$

while for the momentum component we similarly have

$$\begin{pmatrix} p'_x \\ p'_y \end{pmatrix} = \begin{pmatrix} p_x \cos\theta - p_y \sin\theta \\ p_x \sin\theta + p_y \cos\theta \end{pmatrix}. \qquad (3.A.13)$$

We now suppose that this SO(2) symmetry is broken due to nonconservative forces and we implement the Lie-admissible approach along the lines of Section 3.1. For simplicity, we assume that the nonconservative forces are such to yield a flexible Lie-admissible algebra characterized by the tensor

$$S_J^{\mu\nu}(t) = \begin{pmatrix} 0_{2\times 2} & \lambda(t)\, 1_{2\times 2} \\ -\mu(t)\, 1_{2\times 2} & 0_{2\times 2} \end{pmatrix}. \qquad (3.A.14)$$

This implies that, by central assumption, the physical angular momentum (3.A.10) is now nonconserved, i.e.,

$$\frac{d}{dt} J \neq 0 . \qquad (3.A.15)$$

Tensor (3.A.14), according to the analysis of Section 3.3, characterizes a Lie-admissible genotopic mapping of SO(2), that is, a <u>Lie-admissible group $\hat{SO}(2)$ of "rotations" in the Euclidean space</u> $E_2(r)$

$$\hat{a}^\mu = e^{\theta S_J^{\alpha\beta}(t) \frac{\partial J}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}} a^\mu . \qquad (3.A.16)$$

Simple calculations for the space components yield the transformation laws

$$\begin{pmatrix} \hat{r}_x \\ \hat{r}_y \end{pmatrix} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} + \frac{\theta}{1!}\begin{pmatrix} (r_x, J) \\ (r_y, J) \end{pmatrix} + \frac{\theta^2}{2!}\begin{pmatrix} ((r_x, J), J) \\ ((r_y, J), J) \end{pmatrix} + \dots$$

$$= \begin{pmatrix} r_x \\ r_y \end{pmatrix} + \frac{\theta}{1!}\begin{pmatrix} -\lambda r_y \\ \lambda r_x \end{pmatrix} + \frac{\theta^2}{2!}\begin{pmatrix} -\lambda^2 r_x \\ -\lambda^2 r_y \end{pmatrix} + \dots \qquad (3.A.17)$$

$$= \begin{pmatrix} r_x \cos(\lambda(t)\theta) - r_y \sin(\lambda(t)\theta) \\ r_x \sin(\lambda(t)\theta) + r_y \cos(\lambda(t)\theta) \end{pmatrix},$$

where the brackets (A,B) are the Lie-admissible brackets for tensor (3.A.14).

Similarly, for the momentum components we have

$$\begin{pmatrix} \hat{p}_x \\ \hat{p}_y \end{pmatrix} = \begin{pmatrix} p_x \cos(\mu(t)\theta) - p_y \sin(\mu(t)\theta) \\ p_x \sin(\mu(t)\theta) + p_y \cos(\mu(t)\theta) \end{pmatrix}. \qquad (3.A.18)$$

The composition law is again given by

$$\theta' = \theta + \theta' \qquad (3.A.19)$$

Thus $\hat{SO}(2) \approx SO(2)$. Despite that, the relationship between the original transformations (3.A.12) and their Lie-admissible image (3.A.17) is highly nonlinear, as it was already the case for the relationship between $\hat{D}(1)$ and D(1).

The group $\hat{SO}(2)$ is called "Lie-admissible" to indicate that it is a covering of the conventional SO(2) group in the following sense:

(1) transformations (3.A.17) are more general than transformations (3.G.12) which are recovered under the limit

$$\lambda, \mu \to 1 \qquad (3.A.19)$$

(2) at this limit the tensor (3.A.14) recover the conventional canonical form $\omega^{\mu\nu}$,

$$\lim_{\lambda} S_J^{\mu\nu} = \lim_{\dot{J}=0} S_J^{\mu\nu} = \omega^{\mu\nu} \qquad (3.A.20)$$

namely, the Lie approach to the original exact symmetry is recovered in full; and

(3) the departure of the Lie from the Lie-admissible group, e.g.,

$$\vartheta^{\mu\nu} = S_J^{\mu\nu} - \omega^{\mu\nu} \qquad (3.A.21)$$

is a representative of the symmetry breaking forces.

Thus, the $\hat{SO}(2)$ group provides an algebraic-group theoretical characterization of the broken SO(2) context. The Lie-admissible character in the neighborhood of the identity can be easily seen from the first-order expansion

$$\hat{a}^{\mu} \simeq a^{\mu} + \theta(a^{\mu}, J) \qquad (3.A.22)$$

This illustrates one of the central points of our analysis, the fact that a departure from the Lie algebra of the exact symmetry can produce an algebraic characterization of the broken symmetry.

Notice that transformations (3.A.17), even though different, are close to rotations (3.A.12). The interested reader is here urged to work out other cases of Lie-admissible tensor (3.A.14). He will then see that, in general, the transformations of the Lie-admissible group $\hat{SO}(2)$ lose any connection with the conventional rotations. Notice also that the group SO(2) admits a family of Lie-admissible coverings $\hat{SO}(2)$, that is, there exist a family of Lie-admissible transformations which can be all constructed from SO(2). Each element of this family is characterized by the explicit form of the SO(2) breaking forces. All the elements of this family, however, share properties (1), (2) and (3).

The extension to a Lie-admissible covering $\hat{SO}(3)$ of SO(3) along the same lines is here left to the interested reader. He will then see that now, in general, $\hat{SO}(3)$ is no longer locally isomorphic to SO(3).

To conclude this Appendix we shall consider instead the case of the Lie-admissible covering of the SU(2)-spin for the most significant case of the Spin 1/2 representations, the Pauli matrices σ_i, i=1,2,3. Rather than using only the methods of Sections 3.3 and 3.5, we shall here review the treatment by M. Kôiv and J. Lôhmus[158] via a flexible Lie-admissible mutation, and jointly indicate the implementation of the results within the context of Sections 3.3. and 3.5. This will also be useful to indicate the link between the approach to Lie-admissible structures as a generalization of the deformation theory and their treatment as a generalization of Lie's theory.

To attempt a finite mutation, it is essential to start from the associative algebra of Pauli matrices

$$\sigma_\mu \sigma_\nu = C^\rho_{\mu\nu}\sigma_\rho , \qquad C^\rho_{\mu\nu} = \begin{cases} i\varepsilon_{\mu\nu\rho} & , \mu,\nu,\rho < 4 \\ \delta_{\mu\nu} & , \rho = 4 \\ \delta_{\mu\rho} & , \nu = 4 \\ \delta_{\nu\rho} & , \mu = 4 \end{cases} \qquad (3.A.23)$$

$$\mu,\nu = 1,2,3,4 , \quad \sigma_4 = 1$$

rather than their Lie algebra. This introduces the Lie-admissible concept since the very beginning. The problem is then turned into that of modifying structure (3.A.23) into a nonassociative, but flexible Lie-admissible form.

We have two equivalent alternatives, one is provided by a redefinition of the basis $\sigma_\mu \to \hat{\sigma}_\mu$ which preserves the associative character of the algebra

$$\sigma_\mu \sigma_\nu \longrightarrow \hat{\sigma}_\mu \hat{\sigma}_\nu \qquad (3.A.24)$$

and the other is to preserve the basis σ_μ and change the product into a flexible Lie-admissible form

$$\sigma_\mu \sigma_\nu \longrightarrow \sigma_\mu \circ \sigma_\nu \qquad (3.A.25)$$

For case (3.A.25) we have new "structure constants", i.e.,

$$\sigma_\mu \circ \sigma_\nu = \left(C^\rho_{\mu\nu} + M^\rho_{\mu\nu} \right) \sigma_\rho \qquad (3.A.26)$$

where the C's are the original structure constants as in Eqs. (3.A.23) and the M's are new quantities here called mutation coefficients which must be restricted by the flexible Lie-admissibility

conditions. Such conditions are explicitly given by

$$M^4_{44} = \bar{e} , \quad M^4_{i4} = M^4_{4i} = e = M^i_{44} , \quad M^4_{ik} = \rho_{ik}$$

$$M^i_{ii} = -e + i\varepsilon_{ikj}\hat{\rho}_{ki} , \quad \hat{\rho}_{kj} = \rho_{kj} - \rho_{jk} ,$$

$$M^i_{kk} = -f_i \ (i \neq k), \quad M^i_{ik} = i\varepsilon_{kij}\rho_{ij} \ (i \neq k), \qquad (3.A.27)$$

$$M^i_{ki} = i\varepsilon_{ikj}\rho_{ji} \ (i \neq k),$$

$$M^i_{kj} + M^k_{ij} = i\varepsilon_{ijk}(\rho_{ii} - \rho_{kk}), \quad M^k_{ij} = -M^k_{ji} ,$$

where there is no summation over repeated indices.

In conclusion, we have (at this stage) the emergence of the following independent quantities

$$\{ e, \bar{e} , f_i , \rho_{ik} , \text{one of the } M^k_{ij} \} . \qquad (3.A.28)$$

The conditions for a finite deformation (with finite mutation coefficients $M^\rho_{\mu\nu}$) can be written

$$M^\rho_{\mu\nu} M^\gamma_{\rho\tau} - M^\rho_{\nu\tau} M^\gamma_{\mu\rho} + M^\rho_{\tau\nu} M^\gamma_{\rho\mu} - M^\rho_{\nu\mu} M^\gamma_{\tau\rho} = 0 \quad (3.A.29)$$

This implies a reduction of the independent quantities (3. .28). The classification of meaningful possibilities yields the following classes of algebras.

(I) $\hat{SU}_1(2)$ characterized by

$$f_i = i\varepsilon_{ijk}\rho_{jk} , \quad \rho_{ik} = \rho_{ki} , \quad e = \rho_{kk} \qquad (3.A.30)$$

in which case the mutation coefficients are

$$M^{\ k}_{ij} = i\,e\,\varepsilon_{ijk} \qquad (3.A.31)$$

(II) $\hat{SU}_2(2)$ characterized by the parameters $e, \tilde{e}$ and $\rho_{ik}=\rho_{ki}$ connected by

$$\rho_{ii}-\rho_{jj} = \frac{\rho_{ij}\rho_{ki}}{\rho_{kj}} - \frac{\rho_{ji}\rho_{kj}}{\rho_{ki}} \qquad (3.A.32)$$

The mutation coefficients are

$$M^{\ k}_{ij} = i\,\varepsilon_{ijk}\left(\rho_{ii} + \frac{\rho_{ij}\rho_{jk}}{\rho_{ik}}\right) \qquad (3.A.33)$$

(III) $\hat{SU}_3(2)$ characterized by the parameters e and ρ_{ik} connected by

$$\rho_{ii}-\rho_{jj} = \frac{{}^{+}A^{\ i}_{kj}}{\bar{\rho}_{ij}} - \frac{{}^{+}A^{\ i}_{ki}}{\bar{\rho}_{ik}},$$

$$e-\rho_{jj} = \frac{{}^{+}A^{\ k}_{ij}}{\bar{\rho}_{ij}} + \frac{{}^{-}A^{\ i}_{jk}}{\bar{\rho}_{jk}}, \qquad (3.A.34)$$

$$(\rho_{ij}\rho_{ki}\rho_{jk} - \rho_{ik}\rho_{ji}\rho_{kj})(\bar{\rho}_{ji}\rho_{ji}+\bar{\rho}_{jk}\bar{\rho}_{jk})=0,$$

$${}^{\pm}A^{\ k}_{ij} = \rho_{kj}\rho_{ji} \pm \rho_{ik}\rho_{ij}, \quad \bar{\rho}_{ik} = \rho_{ik}+\rho_{ki}$$

where there is also no summation on repeated indices.

The mutation coefficients are

$$M^{\ k}_{ij} = i\,\varepsilon_{ijk}\left(e + \frac{{}^{-}A^{i}}{\bar{\rho}_{ki}}\right). \qquad (3.A.35)$$

At the level of first-order mutation we can put

$$a^{(1)\ k}_{\ \ ij} = e + \rho_{kk} - \rho_{ii} - \rho_{jj} - 2\,\frac{\rho_{kj}\rho_{ik}}{\rho_{ii}}$$

$$a^{(2)\ k}_{\ \ ij} = e + \rho_{kk} - \rho_{ii} - \rho_{jj} - 2\,\frac{{}^{+}A^{\ k}_{ii}}{\bar{\rho}_{ij}} \qquad (3.A.36)$$

which, under Eqs. (3.A.32)-(3.A.34), satisfy the properties

$$a^{(1,2)k}_{\ \ \ ij} = \qquad (3.A.37)$$
$$= a^{(1,2)\ i}_{\ \ \ jk} = \mu^{(1,2)}$$

It then follows that all nonisomorphic first-order mutation of the associative algebra in the Pauli matrices can be determined in terms of the parameters

$$\hat{SU}_1(2): \ e, \rho_{ii}$$

$$\hat{SU}_2(2): \ \lambda = \tilde{e} - e, \ \mu = -\frac{\mu^{(2)}}{2}(\varphi) \qquad (3.A.38)$$

$$\hat{SU}_3(2): \ \rho = -\frac{\mu^{(1)}}{2}, \quad \hat{SU}_2(2) \subset \hat{SU}_3(2)$$

The image $\hat{\sigma}_\mu$ of the basis under mutation (3.A.38) can be written

$$\hat{\sigma}_\mu = e_\mu + b_{\mu\nu}e_\nu$$

$$b_{44} = e, \ b_{4i} = 0, \ b_{i4} = \frac{1}{2}\varepsilon_{ijk}\bar{\rho}_{jk} \qquad (3.A.39)$$

$$b_{ik} + b_{ki} = \frac{\bar{\rho}_{jk}}{2}, \quad b_{ii} = \frac{e+\rho_{ii}}{2}$$

in which case we have the closure rules

$$e_\mu \circ e_\nu = \hat{C}_{\mu\nu}^{\lambda} e_\lambda \tag{3.A.40}$$

$$\hat{C}_{ij}^{k} = i\varepsilon_{ijk}(1+\psi), \quad \hat{C}_{44}^{4} = 1+\alpha, \quad \hat{C}_{ij}^{k} = C_{ki}^{k}$$

expressible with the table

	e_1	e_2	e_3	e_4
e_1	e_4	$i(1+\psi)e_3$	$-i(1+\psi)e_2$	e_1
e_2	$-i(1+\psi)e_3$	e_4	$i(1+\psi)e_1$	e_2
e_3	$i(1+\psi)e_2$	$-i(1+\psi)e_1$	e_4	e_3
e_4	e_1	e_2	e_3	$(1+\alpha)e_4$

(3.A.41)

Product (3.A.40) is Lie-admissible because

$$e_\mu \circ e_\nu - e_\nu \circ e_\mu = [e_\mu, e_\nu]^* \tag{3.A.42}$$

and, jointly, it is Jordan-admissible because

$$e_\mu \circ e_\nu + e_\nu \circ e_\mu = 2\delta_{\mu\nu} \tag{3.A.43}$$

The analysis by M. Kôiv and J. Lôhmus can therefore be summarized with the following

LEMMA 3.A.1: The associative algebra $A(\sigma)$ of the Pauli matrices σ_μ admits a nontrival flexible, Lie-admissible, finite, mutation $U_{\alpha,\psi}(\sigma)$ depending on

two arbitrary parameters. This latter algebra is not, in general, power-associative. However it becomes power-associative at the limit $\alpha = 0$.

To complement this analysis, we now need the identification of the product of Eqs. (3.A.40). It is easy to see that this product characterizes the λ-mutation (Section 1.4) of the associative algebra $A(\sigma)$. And indeed, under the identification

$$e_\mu = \sigma_\mu, \tag{3.A.44a}$$

$$\sigma_\mu \circ \sigma_\nu = \lambda \sigma_\mu \sigma_\nu + (1-\lambda)\sigma_\nu \sigma_\mu, \tag{3.A.44b}$$

$$1+\psi = 2\lambda+1, \tag{3.A.44c}$$

the closure rules

$$\sigma_\mu \circ \sigma_\nu = \hat{C}_{\mu\nu}^{\rho} \sigma_\rho \tag{3.A.45}$$

coincide with rules (3.A.40), i.e., with diagrams (3.A.41).

We can summarize this additional finding with the following

LEMMA 3.A.2: The only nontrivial, finite, power-associative, flexible, Lie-admissible mutation of the associative algebra $A(\sigma)$ of the Pauli matrices is constituted by the λ-mutation (3.A.44a) of the associative product.

Notice that this result can be directly obtained with the review of λ-mutation algebra of Section 1.4 and closure rules

(3.A.23), without use of the analysis by M. Kôiv and J. Lôhmus. Nevertheless, the analysis by these authors indicates that the λ-mutation can be derived via the theory of mutation of the associative algebra as a generalization of that of the deformation theory. Notice that this generalization is here crucial. In fact, since the algebra $\mathcal{A}(\sigma)$ is simple, there exist no consistent nontrivial deformation (that is, a nontrivial modification of the product which preserves its associative character). Alternatively, the only consistent deformations are the isotopic mappings

$$\mathcal{A}(\sigma) \to \mathcal{A}^*(\sigma) \Rightarrow \sigma_\mu * \sigma_\nu = \gamma\, \sigma_\mu \sigma_\nu, \quad \gamma \in F \qquad (3.A.46)$$

which are trivially equivalent to $\mathcal{A}(\sigma)$. It is precisely the transition from a deformation to a Lie-admissible mutation which has rendered indentifiable nontrivially different algebras.

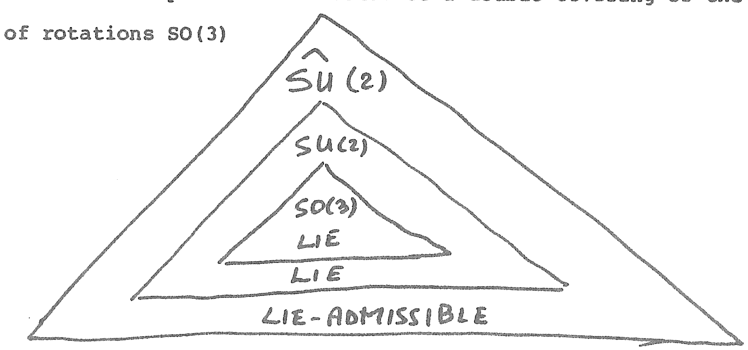
In conclusion, the algebraic approach to mutation of Section 1.4 and the approach to mutation of Appendix 3.D yield the same result under the condition of power-associativity. It is possible to prove that if this condition is relaxed, the two approaches also yields the same results, up to possible isotopic mappings of the associative product.

Notice that an equally significant mutation of Pauli's algebra is provided by the (λ, μ) mutation with product

$$\sigma_\mu \circ \sigma_\nu = \lambda\, \sigma_\mu * \sigma_\nu + \mu\, \sigma_\nu * \sigma_\mu \qquad (3.A.47)$$

The implementation of this approach with the methods of

Section 3.5 is here left to the interested reader. In essence, the product of the λ and (λ, μ) mutations can be assumed as the product of the enveloping mutation algebra of the spin 1/2 algebra. At the limit $\lambda \to 1$ (or $\lambda \to 1, \mu \to 0$) one recovers the conventional Pauli algebra. The use of the generalized exponential map (3.3.70) is then expected to yield a Lie-admissible group as a covering of the Pauli realization of SU(2). This indicates the possible existence of a "double covering" of the group of rotations SO(3)



The first covering, SU(2), is the conventional, simply connected, Lie covering (yielding the additional spinorial representations). The second covering, $\hat{SU}(2)$, is in actuality a family of Lie-admissible coverings which, in the simplest case, is characterizable via the free parameter λ, although a considerably more complex structure is expected, in general, to occur. The important aspect is that the nature of the covering $\hat{SU}(2)$ over SU(2) is different than that of SU(2) over SO(3). In fact, unlike the case of the latter, all consistent SU(2)-coverings are such to recover SU(2) identically under the limit to the conventional associative structure of the envelop.

The reader is by now aware that the indicated λ- and (λ, μ)- mutations of the Pauli algebra are examples of broken spin 1/2 algebra. As we shall see in Volume III, this implies a mutation of the concept of fixed, conventionally quantized spin into a more general nonconserved form which will be crucial for our model on the structure of the hadrons.

APPENDIX 3.B: THE NONLINEAR CHARACTER OF THE LIE-ADMISSIBLE COVERING-BREAKING OF LIE SYMMETRIES.

As recalled earlier, the algebraic-group theoretic characterization of the conventional Lie symmetries can be done via the universal enveloping associative algebra A(G) (Section 3.4), in the sense that such algebra guarantees a Lie algebra character in the neighborhood of the identity as well as, under integrability conditions, a global connected group structure of finite transformations. It then follows that the associative envelop A(G) plays a crucial role for the representation theory.

As is well known, one of the most familiar uses of Lie's theory for the treatment of exact symmetries and conservation laws is via linear transformations. A notable case is that of the Lorentz covariance in field theory

$$\psi'(x') = S(\Lambda)\psi(x), \quad x' = \Lambda x, \; \Lambda \in L_+^{\uparrow} \qquad (3.B.1)$$

which is manifestly linear in the fields.

By following N. Jacobson[69], it is significant for our analysis to identify the algebraic origin of this linearity of the transformation theory. This will then be useful to identify the corresponding properties for the Lie-admissible approach to symmetry breaking.

A linear representation of the associative envelop A(G) of a Lie algebra G over a field F (of characteristic zero) is a homomorphism of A(G) into an algebra $\mathcal{T}$ of linear transformations of a vector space M over F. Suppose that the elements X, Y $\in$ A(G) are represented with x, y, respectively. Then

$$X+Y \to x+y\ ,\ \alpha X \to \alpha x\ ,\ XY \to xy\ ,\ \alpha \in F \quad (3.B.2)$$

A <u>right A-module</u> is a vector space M over F together with a binary product of M x A($\underline{G}$) into M which maps elements (x, X), $x \in M$, $X \in A(\underline{G})$ into elements $xX \in M$ such that

$$(x_1 + x_2)X = x_1X + x_2X\ ,\ X(x_1 + x_2) = Xx_1 + Xx_2\ , \quad (3.B.3a)$$

$$\alpha(xX) = (\alpha x)X = x(\alpha X)\ , \quad (3.B.3b)$$
$$= (xX)\alpha = (x\alpha)X = x(X\alpha)\ ,$$

$$x(XY) = (xX)Y \quad (3.B.3c)$$

If $X \to x$ is a representation of A($\underline{G}$), then the base space M can be turned into a right A-module via the definition $xY = xy$ in which case

$$(x_1 + x_2)Y = (x_1 + x_2)y = x_1y + x_2y\ , \quad (3.B.4a)$$

$$\alpha(xY) = \alpha(xy) = (\alpha x)y = x(\alpha y)\ , \quad (3.B.4b)$$

$$x(YZ) = x(yz) = (xy)z = (xY)z = (xy)Z\ . \quad (3.B.4c)$$

Conversely, let M be an A-module and for all $X \in A(\underline{G})$ denote with R the mapping

$$X \overset{R}{\to} x \quad (3.B.5)$$

It is then possible to prove that Eqs. (3.B.3) imply that R is a <u>linear</u> representation of A($\underline{G}$). Also, any linear representation of A($\underline{G}$), when restricted to $\underline{G}$, via Lie's rule (3.2.30), yields a representation of $\underline{G}$. As a result, the representation theory of Lie algebras (and groups) can be constructed from that of their enveloping associative algebras.

These properties have a number of consequences.

THEOREM 3.B.1: <u>Any Lie algebra has a faithful representation by linear transformations.</u>

where the term "faithful" is referred to the one-to-one nature of the representation. The dimensionality of this representation, however, is unspecified. Suppose now that $\underline{G}$ is finite-dimensional. It is then possible to prove that a necessary and sufficient condition for a faithful representation of $\underline{G}$ to be finite-dimensional is that the envelop A($\underline{G}$) of $\underline{G}$ contains an ideal R of finite (co-) dimension such that $R \cap \underline{G} = 0$. A number of technical implementations then lead to the following property, known as <u>Ado's theorem</u>.

THEOREM 3.B.2: <u>Every finite-dimensional Lie algebra has a faithful finite-dimensional linear representation.</u>

As these properties indicate, the notion of enveloping associative algebra A($\underline{G}$) therefore plays a fundamental role for that of $\underline{G}$. In the transition to the Lie-admissible context, it is then natural to expect that the Lie-admissible genotope U($\underline{G}$) of A($\underline{G}$) (Section 3.5) will play a fundamental role for the representation theory of Lie-admissible algebras. A new feature, however, occurs. It can be expressed with the following

THEOREM 3.B.3: Lie-admissible genotopical mappings of universal enveloping associative algebras generally admit only nonlinear representations.

In fact, let

$$X \to x, \ Y \to y, \ etc. \tag{3.B.6}$$

be a representation of U(G), and set

$$yZ = Z, \ x(yX) = xZ \tag{3.B.7}$$

Then

$$(xy)X \neq x(yX) \tag{3.B.8}$$

because the product in U is generally nonassociative.

In conclusion, one of the direct implications of the nonassociative nature of the envelop U(G) is the general loss of the linear representations. This occurrence has been identified, apparently for the firs time, by C.N. Ktorides.[19] Notice that the term "generally" has been included in the formulation of Theorem 3.B.3 to stress the fact that the algebra U(G) can be, as a particular case, associative, in which case the conventional Lie context is recovered in full.

The nonlinear nature of the representation of the nonassociative Lie-admissible envelop U(G) will have fundamental implications

for our analysis, particularly from a relativity profile. As we shall see in Volume III, it will essentially imply the possibility of a profound differentiations between the electromagnetic and the strong interactions in the physical space of their experimental detections.

In essence, the electromagnetic interactions are known to be fully compatible with Einstein relativity. This implies that the underlying representations of the Poincaré algebra can indeed be linear. In turn, this implies the linear structure (3.B.1) of the transformation properties of the fields under the Lorentz transformations. If the strong interactions are assumed as derivable from a potential, a fully equivalent situation holds, that is, the fields representative of the electromagnetic and of the strong interactions exhibit fully equivalent, linear, relativity transformations. The need for their differentiation in the mathematical space of the internal unitary degrees of freedom then follows.

Suppose, instead, that the strong interactions are not derivable from a potential according to the conjecture of Volume I. Then we have a self-adjoint breaking of the Poincaré symmetry, fully in line with breakings (3.1.3). In this case, if a nontrivial Lie-admissible characterization holds, the fields representative of the strong interactions can only obey a nonlinear relativity transformation law. This produces the desired profound differentiations between the electromagnetic and the strong interactions in the space of their experimental detection.

In closing, it is here relevant to recall that considerable progress has recently been done on the study of the nonlinear representations of Lie groups.[135-143] These studies are particularly significant for our Lie-admissible approach because, (at least in principle and under a number of technical implementations) could be extended to the study of the representation theory of the Lie-admissible groups.

APPENDIX 3.C: NONGEODESIC CHARACTER OF THE LIE-ADMISSIBLE COVERING-BREAKING OF LIE SYMMETRIES

In Appendix I.3.C we indicated, by using the primitive context of Maupertius's principle, that nonconservative systems are nongeodesic in character. In this appendix we shall begin a more adequate characterization of this aspect for subsequent refinements. We shall also indicate a rather intriguing connection between nonconservative systems and broken symmetries in the sense that they share a nongenodesic character.

The geometric approach to the concept of geodesic can be conducted in a variety of ways. See, for instnace, in this respect references [144-151]. The definition of geodesic[147,150,151] we shall use is that of a trajectory $\alpha(\theta)$, $-\infty \leq c_1 \leq \theta \leq c_2 \leq +\infty$ of at least class C^1 in a manifold $\mathcal{M}$ (Section 4.2) such that the family of tangent vectors $\dot{\alpha}(\theta) = \frac{d}{d\theta}\alpha(\theta)$ is parallel to $\alpha(\theta)$. The manifold $\mathcal{M}$ is here tacitly assumed to be equipped with a linear connection. Topological considerations then indicates that the geodesic is in actuality of class C^∞.

The following property is relevant for our analysis.

> LEMMA 3.C.1: If the independent parameters θ of a geodesic $\alpha(\theta)$ are subjected to a (class C^∞, invertible) transformation $\theta' = f(\theta)$, the new trajectory $\alpha'(\theta') = \alpha(\theta(\theta'))$ is a geodesic if and only if the functions f are linear.

Suppose that $\mathcal{M}$ is a manifold (with affine connection).

Denote with m a generic point of $\mathcal{M}$ and with X the family of tangent vectors at m. Then there exists a unique geodesic $\alpha(\theta)$ in M such that

$$\mathring{\alpha}(0) = X, \quad \alpha(0) = m \tag{3.C.1}$$

This geodesic is usually denoted with $\alpha_x(\theta)$. For X = 0 we have $\alpha_0(\theta) = m$ for all $\theta = \mathbb{R}$.

The family of tangent vectors X can be more properly interpreted as a vector field on $\mathcal{M}$ (Section 4.3). This vector field is called complete when it generates a global, connected, (one-parameter) group of transformations of $\mathcal{M}$, i.e., the vector fields X are the standard generator (Section 3.2) of a connected Lie group G of transformations.

This leads in a natural way to the property that the exponential map (3.2.42) is a geodesic of the (topological) manifold $\mathcal{M}$ of G, i.e.,

$$e^{\theta X} = \alpha_x(\theta) \tag{3.C.2}$$

In particular, the exponential mapping

$$X \to e^{X} = \alpha_x(1) \tag{3.C.3}$$

is a geodesic at a fixed point θ = 1 of $\mathcal{M}$. Also, the inverse mapping

$$e^{\theta X} \longrightarrow \theta \tag{3.C.4}$$

yields the so-called normal coordinates of $\mathcal{M}$ at m.

LEMMA 3.C.2: The action of a connected Lie group, realized according to the standard construction of Section 3.2, on its (topological manifold is geodesic.

The reader should be aware that this is a crucial property of the currently available relativities (Galilei and Einstein) in the sense that the action of the relativity groups (the Galilei and Poincaré groups in their standard construction) on their topological manifolds is geodesic.

It is useful for our analysis to identify a realization of this notion of geodesic within the context of the canonical formulations. Le M be the manifold spanned by the phase space coordinates $\{a^\mu\} = \{r^{ka}, p_{ka}\}$ of a conservative Newtonian system with Hamiltonian H (a) = T + V. Then Eqs. (3.C.1) can be subjected to the canonical interpretation

$$\mathring{a}^\mu(t) = \Xi^\mu(a) = \omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}, \quad \mathring{a}^\mu(0) = a^\mu_0 \tag{3.C.5}$$

namely, they are Hamilton's equations (without external terms) with initial conditions.

LEMMA 3.C.3: Hamilton's equations (3.C.5) are the geodesic equations in the (phase) space of the variables a^μ.

The reader should be aware that this is, on strict grounds, a reinterpretation of geodesic (3.C.1). And indeed, the canonical realization of the vector field X is given by

$$X \to \omega^{\mu\nu} \frac{\partial H}{\partial a^{\nu}} \frac{\partial}{\partial a^{\mu}} \qquad (3.C.6)$$

This is the canonical generator of the one-dimensional group of translations in time with parameter $\theta = t$. The exponential mapping then yields the true geodesic

$$e^{t\omega^{\mu\nu} \frac{\partial H}{\partial a^{\nu}} \frac{\partial}{\partial a^{\mu}}} = \alpha(t) \qquad (3.C.7)$$

Therefore, on strict grounds, it is the geodesic $\alpha(t)$ which is tangent to $\omega^{\mu\nu}(\partial H/\partial a^{\nu})(\partial/\partial a^{\mu})$.

The extension to the canonical realization of an n-dimensional connected Lie group with generators G_i and parameters θ^i, i=1,2, ...,n, is straightforward and reads

$$e^{\theta^i \omega^{\mu\nu} \frac{\partial G_i}{\partial a^{\nu}} \frac{\partial}{\partial a^{\mu}}} = \alpha(\theta^1, \ldots, \theta^n), \qquad (3.C.8)$$

which is the canonical realization of the geodesic characterized by the exponential mapping of the generators G_i.

Notice the crucial role of the co-symplectic structure $\omega^{\mu\nu}$ for the existence of a geodesic motion in the group manifold.

These rudimentary notions are sufficient for our objectives at this time. We therefore pass to the identification of the corresponding properties of our Lie-admissible approach to nonconservative systems and/or broken symmetries.

One of the central requirements according to which we have (tacitly) constructed our Hamilton-admissible equations

is to alter the geodesic character of Hamilton's equations. This has been intentional and motivated by the need of attempting a mathematical characterization of the physical differences between conservative and nonconservative systems in the physical space of their experimental detection, that is, $\{a^{\mu}\} = \{r^{ka}, p_{ka}\}$, where r^{ka} are the Cartesian components of the Euclidean space of the experimental set up and p_{ka} are the components of the physical linear momentum.

As stressed throughout our analysis, this latter requirement is intended to avoid the characterization of nonconservative systems in mathematically equivalent systems of coordinates, which are unrealizable in actual experiments owing to the generally nonlinear dependence of the new coordinates in the old as well as their derivatives.

Again, the attitude which we have implemented in the transition from conservative to nonconservative systems is to preserve the direct physical significance of the algorithms at hand and alter, instead, the methodology for their treatment. In this way, it is the departure from conventional formulations which is representative of the nonconservative nature of the systems. In relation to the subject of this appendix, we expect that the departure from a geodesic structure of the canonical and Lie formulations is a representative of nonconservative forces.

But our Hamilton-admissible and Lie-admissible formulations are also applicable to symmetry breaking (Section 3.1). This indicates a possible nongeodesic similarity between nonconservative systems and broken symmetries.

Suppose that system (3.C.5) is implemented with additional nonconservative or symmetry breaking forces $\{F^\mu\} = \{0, F_{ka}\}$. Then, under the assumption of the preservation of the a^μ variables, the new system can be written

$$\dot{a}^\mu(t) = \hat{\Xi}^\mu(t,a) = \Xi^\mu(a) + F^\mu(t,a) \qquad (3.C.9a)$$
$$= \omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} + F^\mu \qquad (3.C.9b)$$
$$a^\mu(0) = a^\mu_0$$

The use of the Hamilton-admissible equations then yields

$$\dot{a}^\mu(t) = \hat{g}^\mu{}_\nu(t,a)\,\Xi^\nu(a) = S^{\mu\nu}(t,a)\frac{\partial H}{\partial a^\nu} \qquad (3.C.10a)$$
$$a^\mu(0) = a^\mu_0 \qquad (3.C.10b)$$
$$S^{\mu\nu}(t,a) = \hat{g}^\mu{}_\rho(t,a)\,\omega^{\rho\nu}, \quad S^{\mu\nu}\frac{\partial H}{\partial a^\nu} = \omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} + F^\mu \qquad (3.C.10c)$$

namely, $\dot{a}^\mu$ is no longer tangent to $\hat{\Xi}^\mu$. As a matter of fact, since the a^μ and Ξ^μ quantities of the original conservative system have remained unchanged in the transition to its nonconservative form, this is a central requirement to ensure the presence of nonconservative forces.

<u>LEMMA</u> 3.C.4: <u>Hamilton-admissible equations (3.C.10) are not geodesic equations in the (dynamical) space of the variables a^μ, unless they reduce to a Hamiltonian form (i.e., all nonconservative or symmetry breaking forces are identically null).</u>

But the Hamiltonian H of the conservative system is also the generator of a Lie-admissible group of translations in time (Section 3.3). Instead of Eqs. (3.C.7) we then have

$$e^{t S^{\mu\nu}\frac{\partial H}{\partial a^\nu}\frac{\partial}{\partial a^\mu}} = \beta(t). \qquad (3.C.11)$$

It is easy to see that this is no longer, in general, a geodesic because it is not tangent to $\hat{\Xi}$.

The generalization to a Hamilton-admissible realization of an n-dimensonal Lie-admissible group is straightforward and reads

$$e^{\theta^i S^{\mu\nu}_{(i)}\frac{\partial G_i}{\partial a^\nu}\frac{\partial}{\partial a^\mu}} = \beta(\theta^1, \theta^2, \dots, \theta^n) \qquad (3.C.12)$$

again yielding a nongeodesic action of the Lie-admissible group in its manifold.

In abstract notations, this occurrence can be seen as follows. Let $\hat{G}$ be an n-dimensional connected Lie group in the standard realization of Section 3.3 with generators $\hat{X}_i$ and parameters $\hat{\theta}^i$. As such its action on the manifold $\mathcal{M}$ with normal coordinates $\hat{\theta}^i$ is geodesic and we shall write

$$e^{\hat{\theta}^i \hat{X}_i} = \alpha(\hat{\theta}). \qquad (3.C.13)$$

In the reformulation of $\hat{G}$ as a Lie-admissible group we have (Section 3.3)

$$\hat{X}_i \rightarrow (\alpha_i^2 + \beta_i^2)X_2 \qquad (3.C.14a)$$
$$\hat{\theta}^i \rightarrow \theta^i(\hat{\theta}) \qquad (3.C.14b)$$

Thus,

$$e^{\theta^i (\alpha_i^j + \beta_i^j) X_j} = \alpha'(\theta) \qquad (3.C.15)$$

The generally nongeodesic nature of this reformulation then follows either from Lemma 3.C.1 or because of the presence of the α and β functions.

LEMMA 3.C.5: The action of a connected Lie-admissible group, realized according to the natural construction of Section 3.3, on its (topological) manifold is generally nongeodesic.

The term "generally" is here representative of the covering nature of the property of Lemma 3.C.5 over that of Lemma 3.C.2. In fact, by construction, the Lie-admissible group $\hat{G}$ can reduce or coincide, not with its standard realization, but instead with the group G in its standard realization (that is, with generators X_i and parameter θ^i) at the limit of null nonconservative or symmetry breaking forces. The reader should here keep in mind that $\hat{G}$ and G are nonisomorphic for all nontrivial additive forces F^μ and dimensionality greater than one.

In essence, the property of Lemma 3.C.5 can be expected from the fact that, by construction, the group $\hat{G}$ acts on the (topological) manifold of a generally nonisomorphic group G.

Notice that the notion of geodesic is not "destroyed" by our Lie-admissible approach, but simply embedded in a more general

notion. Its identification demands the prior study of the geometry which characterizes the Lie-admissible tensor $S^{\mu\nu}$ (Chapter 4). As such, it will not be considered at this time. We can however expect that this broader notion is a covering of that of geodesic because the latter notion is recovered identically at the limit of null nonconservative or symmetry breaking forces.

As we shall see during the course of our analysis, the property identified in this appendix will have crucial implications for our model of the structure of hadrons, on classical, quantum mechanical, as well as gravitational grounds.

On classical grounds it will imply that the relativity laws which are apparently applicable to nonconservative systems are nongeodesic in nature, contrary to the geodesic character of the Galilei and Einstein relativities.

On quantum mechanical grounds it will imply that the trajectories of our hadronic constituents (for the case of several constituents) are nonstationary, contrary to the strictly stationary character of the orbits of the atomic (and nuclear) structure. In turn, this will apparently result to be crucial for the identification of the hadronic constituents with physical particles.

Finally, on gravitational grounds the property under consideration will apparently imply a departure from Einstein's theory of graviation, in line with the analysis of Volume I, for the case of the interior problem. We are here referring to our

APPENDIX 3.D: LIE-ADMISSIBLE COVERING-BREAKING OF THE DEFORMATION THEORY

As is known, the theory of deformation of an algebra in general, and of a Lie algebra in particular, has lately achieved a rather high degree of sophistication since the initial studies by M. Gerstenhaber[152]. See, for instance, references[153-158]. On physical grounds, the deformation theory is significant for a number of topics of particle physics (such as contraction-expansion problems from one relativity algebra to another), although there are indications of physical relevance also in other branches of physics (e.g., nuclear theory).

Let us here briefly outline the main idea of the deformation theory of a Lie algebra. Let $\underline{G}$ be an n-dimensional Lie algebra with basis X_i, i=1,2,...,n. A deformation $\underline{G}_\varepsilon$ of $\underline{G}$ is a family of Lie algebra defined on the same vector space $\underline{G}$ (that is, in terms of the same generators X_i) but with a new product, say $[X_i, X_j]_\varepsilon$, depending on a parameter ε. Such a product is generally assumed to admit a (unique) expansion in the neighborhood of the value $\varepsilon = 0$ and we write

$$[X_i, X_j]_\varepsilon = [X_i, X_j]_A + \varepsilon F_1(X_i, X_j) + \varepsilon^2 F_2(X_i, X_j) + \cdots \tag{3.D.1}$$

where the functions F are subjected to the subsidiary conditions

$$F_r(X_i, X_j) = -F(X_j, X_i), \tag{3.D.2a}$$

$$\sum_{\text{Cyclic }(i,j,k)} F_r(X_i, F_s(X_j, X_k)) = 0. \tag{3.D.2b}$$

$$r+s = k = 0, 1, 2, \ldots, \infty,$$

which ensure the preservation of the Lie character of the product $[X_i, X_j]_\varepsilon$. When only first-order terms in ε are considered, one has the so-called first-order deformation $\underline{G}_\varepsilon^{(1)}$ of $\underline{G}$. Otherwise, one has higher-order deformations. If the right hand side of Eqs. (3.D.1) satisfies suitable integrability conditions, one has a "finite deformation" of $\underline{G}$.

As one of the many examples which have been studied in the literature, the commutation rules[156]

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(g_{\mu\sigma}L_{\nu\rho} - g_{\mu\rho}L_{\nu\sigma} + g_{\nu\rho}L_{\mu\sigma} - g_{\nu\sigma}L_{\mu\rho}), \tag{3.D.3a}$$

$$[L_{\mu\nu}, P'_\rho] = i(g_{\nu\rho}P'_\mu - g_{\mu\rho}P'_\nu), \tag{3.D.3b}$$

$$[P'_\mu, P'_\nu] = \alpha i \varepsilon^2 L_{\mu\nu} \tag{3.D.3c}$$

where

$$P'_\mu = \mathscr{C}_\varepsilon(P_\mu) \tag{3.D.4}$$

is the "image" of P_μ under the deformation, characterize a deformation of the Poincaré algebra $\underline{P}$(3.1). In particular, for $\alpha = +1$ the deformation produces the $\underline{SO}$(3.2) algebra, while for $\alpha = -1$ it produces the $\underline{SO}$(4.1) algebra. Thus, the deformation theory is useful to study the relationship between Einstein's and De Sitter's "universes".

Upon suitable implementations, the deformation theory is expected to be an effective approach for the study of our Lie-admissible covering-breaking of Lie symmetries. The reason is due to the same construction of the theory, the preservation of the basis of the original algebra to induce a generally nonisomorphic algebra, which is at the very foundation of our Lie-admissible approach. The principal difference is due to the fact that the Lie algebra structure is preserved in the deformation theory by central assumption, while it is generally lost in the Lie-admissible theory, also by central assumption. However, Lie-admissible algebras are a covering of a Lie algebra and, thus, they can be Lie as a particular case. The covering nature of the Lie-admissible approach over the deformation approach then follows.

As a matter of fact, the term mutation often used within the context of Lie-admissible algebras[4-7] (see Chapter 1) is intended to express a covering of the term deformation.

Intriguingly, the theory of deformation of Lie algebras is a true realization of our intermediate step prior to the construction of a Lie-admissible covering-breaking of a Lie algebra. We are here referring to the notion of isotope G^* of G (Section 3.3). In fact, rule (3.D.1) can be interpreted as a Lie algebra preserving mapping of the product (the central requirement of algebraic isotopy). The Lie-admissible genotope $\hat{G}$ of G (Section 3.3) is then the covering notion.

To summarize, when seen from the algebraic profile of Chapter 1, the deformation theory belongs to the class of algebraic isotopy, while the Lie-admissible theory belongs to the covering class of algebraic genotopy. The point remains that the main idea of the deformation theory (the construction of a generally nonisomorphic algebra from a given Lie algebra) is also common to that of Lie-admissible algebras. It is therefore conceiveable that a suitable

implementation of the methods of the deformation theory can prove to be effective for the study of the Lie-admissible algebras.

The notion of Lie-admissible theory as a covering of the deformation theory has been formulated, apparently for the first time, by M. Kôiv and J. Lôhmus[158], following the argument by R.M. Santilli[4-6] according to which, in the study of non-Lie algebras for possible physical applications, the Lie algebras should not be abandoned, but preserved in an embedded form.

By following the analysis by M. Kôiv and J. Lôhmus, we shall consider again a Lie algebra G with generators X_i, $i=1,2,\ldots,n$ and product $[X_i,X_j]_A$. A Lie-admissible mutation $\hat{G}_\varepsilon$ of G is a family of Lie-admissible algebras defined on the same vector space G, but expressed in terms of a new product, the Lie-admissible product, say $(X_i \circ X_j)_\varepsilon$ depending on a parameter ε. Such a product is again assumed to admit a (unique) expansion in the neighborhood of the value $\varepsilon \approx 0$ and we write

$$(X_i \circ X_j)_\varepsilon = [X_i, X_j]_A + \varepsilon F_1(X_i,X_j) + \varepsilon^2 F_2(X_i,X_j) + \cdots \tag{3.D.5}$$

The conditions on the F_i functions can then be identified by selecting a general or a flexbile Lie-admissible structure. For instance, the flexibility condition (1.4.4a) for expansion (3.D.5) reads

$$\sum_{\substack{r+s=k=0,1,2,\ldots \\ i,j}} \left[F_r(F_s(X_i,X_j),X_i) - F_r(X_i,F_s(X_j,X_i))\right] = 0, \tag{3.D.6}$$

and a similar additional set of conditions holds for the flexible Lie-admissibility (1.4.4b).

The following result by M. Kôiv and J. Lôhmus[158] is significant.

THEOREM 3.D.1: For mutations (3.D.5) the Lie algebras must form a subclass to the class of the deformed algebras.

If mutations (3.D.5) are applied to an arbitrary algebra U (rather than, specifically, to a Lie algebra), the above theorem implies that associative, alternative, Lie, commutative and non-commutative Jordan algebras can be deformed into Lie-admissible algebras. A most significant case is when flexibility is preserved. Then the notion of mutation yields the mapping from one algebra to a different algebra of the same class. In fact, the associative, alternative, Lie, commutative Jordan, noncommutative Jordan and flexible Lie-admissible algebras are all flexible algebras.

One of the most significant possibilities of this deformation-type approach to Lie-admissible algebras is offered by the notion of "infinitesimal departure" from a Lie-algebra structure. Such an approach is mathematically and physically significant both per sé, as well as intermediate step prior to a "finite departure" from a Lie algebra structure, as indicated in Appendix 3.A and Volume III.

For inst ce, product (1.4.43) of the mutation algebras $A(\lambda, \mu)$, i.e.,

$$X_i \circ X_j = \lambda X_i X_j + \mu X_j X_i = \rho [X_i, X_j]_A + \sigma \{X_i, X_j\}_A \quad (3.D.7)$$

can be written for $\rho = 1$, $\varepsilon = \sigma \cdot \rho$,

$$X_i \circ X_j = [X_i, X_j]_A + \varepsilon \{X_i, X_j\} \quad (3.D.8)$$

in which case it characterizes a flexible first-order mutation $\hat{G}^{(1)}$ of a Lie algebra G with generators X_i.

It is here significant to indicate that the study of these first-order mutations of Lie algebras, particularly for the case of the SU(2)-spin, are relevant for the study whether infinitesimal deviations from the validity of Pauli exclusion principle within the nuclear structure are consistent with the experimental data.

We are here referring to the argument by R.M. Santilli[21] according to which we do not know at this moment whether the established validity of the Pauli principle within the nuclear structure is quantitatively comparable to our current knowledge of the validity of the PCT symmetry in particle physics or is at a stage prior to the discovery of the parity violation.

By keeping into account that Pauli principle produces an excellent agreement with experimental data at the nuclear level, the only conceivable deviations must be of infinitesimal character. This is precisely in line with the notion of first-order mutation of the SU(2)-spin algebra. For more details see Appendix 3.A and Volume III.

The reader should be aware that this first-order mutation of the SU(2)-spin algebra is grossly insufficient for our model on the hadronic structure.

APPENDIX 3.E: LIE-ADMISSIBLE COVERING-BREAKING OF SUPERSYMMETRIES

One of the generalizations of Lie algebras which is currently studied in a number of contexts is given by the so-called graded Lie algebras whose product involves both, commutators and anti-commutators. Rather significantly for our analysis, one of the first uses of graded Lie algebras has been in relation to the deformation theory[152,153]. Subsequently, these algebras were studied in relation to the second quantization of Fermi-Dirac systems[159] as well as for dual models[160-161]. More recently, graded Lie algebras have been used by D.V. Volkov and V.P. Akulov,[162] J. Wess and B. Zumino[163] as well as A. Salam and J. Strathdee[164] to attempt a generalization of the notion of Lie symmetry. Following the study by these latter authors as well as several others (see, for instance, references[165-170]) these broader symmetries have been called supersymmetries, and they have been extended to include gauge theories as well as gravitational theories.

In this appendix we shall indicate that graded Lie algebras are a particular case of Lie-admissible algebras. Thus, the notion of supersymmetry is a particular case of our notion of Lie-admissible symmetry. Furthermore, like many other symmetries, supersymmetries must be often broken to extract phsyically significant informations. Once broken, supersymmetries become algebraically undefined (in the sense of Section 3.1. The second objective of this appendix is to indicate that the generalization of supersymmetries into Lie-admissible symmetries provides a methodological context for the treatment of the broken graded framework. In conclusion,

Lie-admissible algebras provide a dual covering-breaking of supersymmetries which is fully along the lines of Section 3.1 but only referred to graded rather than conventional Lie algebras. The reader should be aware that the objective of this appendix is simply that of indicating the existence of these occurrences. Their detailed study, as well as the possible construction of explicit models, will be left to the interested reader.

Let G_m be an m-dimensional vector space with basis

$$\{Z_\mu\} = \{X_1, \dots, X_m; Y_1, \dots, Y_m\}, \quad \mu = 1, 2, \dots, m+m. \tag{3.E.1}$$

If the following closure relations

$$[X_i, X_j]_A = C_{ij}^k X_k, \tag{3.E.2a}$$

$$[X_i, Y_j]_A = D_{ij}^k Y_k \tag{3.E.2b}$$

$$\{Y_i, Y_j\}_A = E_{ij}^k X_k \tag{3.E.2c}$$

hold, G_m is called a graded Lie algebra.

Structure (3.E.2) of the closure properties has a number of algebraic (and geometrical) consequences. In particular, it demands the generalization of a number of familiar notions (such as Lie derivative). For our needs it is here sufficient to indicate that the Jacobi identity is now generalized into the following four relations

$$[X_i, [X_j, X_k]_A] + \text{Cycl. perm.} = 0, \tag{3.E.3a}$$

$$[X_i, [X_j, Y_k]] + \text{Cycl. perm.} = 0, \tag{3.E.3b}$$

$$[X_i, \{Y_j, Y_k\}] + \text{cycl. perm.} = 0, \qquad (3.E.3c)$$

$$[Y_i, \{Y_j, Y_k\}] + \text{cycl. perm.} = 0. \qquad (3.E.3d)$$

To put it in nontechnical terms, a central features of the graded Lie algebras is that their product varies, as in Eqs. (3.E.2), from the commutator to the anticommutator, or vice versa, depending on the generators considered.

This indicates a first connection with the Lie-admissible algebras because, as stressed in Section 3.3, one of their significant properties is precisely that their product may vary from generator to generator.

A second connection between graded Lie algebras and Lie-admissible algebras is indicated by the fact that, by central requirement, the product of the latter algebras is neither totally antisymmetric nor totally symmetric. As such, it can be decomposed into a combination of commutators and anticommutators yielding a generalization of structure (3.E.2).

To put this aspect in a more explicit form, consider the following product on basis (3.E.1).

$$Z_\mu \circ Z_\nu = a_{\mu\nu}^{\rho\sigma}\,[Z_\rho, Z_\sigma] + b_{\mu\nu}^{\rho\sigma}\,\{Z_\rho, Z_\sigma\} \qquad (3.E.4)$$

where $a_{\mu\nu}^{\rho\sigma}$ and $b_{\mu\nu}^{\rho\sigma}$ are tensors on the base field. Under the conditions

$$a_{\mu\nu}^{\rho\sigma} = -a_{\nu\mu}^{\rho\sigma}, \quad b_{\mu\nu}^{\rho\sigma} = b_{\nu\mu}^{\rho\sigma} \qquad (3.E.5)$$

product (3.E.4) is Lie-admissible. For instance, the following explicit form of the a- and b-tensor

$$a_{\mu\nu}^{\rho\sigma} = \rho\left(\delta_\mu^\rho \delta_\nu^\sigma - \delta_\mu^\sigma \delta_\nu^\rho\right), \quad b_{\mu\nu}^{\rho\sigma} = \sigma\, \delta_\mu^\rho \delta_\nu^\sigma \qquad (3.E.6)$$

recovers the product of the mutation algebras of Section 1.4, i.e.,

$$Z_\mu \circ Z_\nu = \rho\,[Z_\mu, Z_\nu] + \sigma\,\{Z_\mu, Z_\nu\} \qquad (3.E.7)$$

This, however, is a particular case of the more general Lie-admissible product of type (3.E.4).

It should be here stressed that product (3.E.7) does not yield a graded Lie algebra because, trivially, it implies a mixture of commutators and anticommutators for all generators, contrary to structure (3.E.2). It has been here recalled merely to indicate the mixing of commutators and anticommutators in the Lie-admissible product.

It is easy to see that graded Lie algebras are a particular case of the Lie-admissible algebras with product of type (3.E.4). And indeed, by assuming again partition (3.E.1), we can write

$$(Z_\mu \circ Z_\nu) = \begin{pmatrix} \left(\alpha_{ij}^{rs}[X_r, X_s] + \beta_{ij}^{rs}\{X_r, X_s\}\right) & \left(\gamma_{ij}^{rs}[X_r, Y_s] + \delta_{ij}^{rs}\{X_r, Y_s\}\right) \\ \left(\rho_{ij}^{rs}[Y_r, X_s] + \sigma_{ij}^{rs}\{Y_r, X_s\}\right) & \left(\lambda_{ij}^{rs}[Y_r, Y_s] + \mu_{ij}^{rs}\{Y_r, Y_s\}\right) \end{pmatrix} \qquad (3.E.8)$$

The particular values

$$\alpha_{ij}^{rs} = \gamma_{ij}^{rs} = -\rho_{ij}^{rs} = \delta_i^r\delta_j^s - \delta_j^r\delta_i^s$$
$$\beta_{ij}^{rs} = \delta_{ij}^{rs} = \sigma_{ij}^{rs} = \lambda_{ij}^{rs} = 0 \qquad (3.E.9)$$
$$\mu_{ij}^{rs} = \delta_i^r\delta_j^s$$

then yield structure (3.E.2), up to a numerical constant which, if needed, can be formally removed via the isotopy $z_\mu z_\nu \rightarrow z_\mu * z_\nu = 2(z_i z_j)$.

It is in this sense that the Lie-admissible algebras can be seen as a covering of the graded Lie algebras. The breaking of the latter algebras through their embedding into the former algebras can then be conducted along the lines of Appendix 3.A. Notice that when a full Lie-admissible structure is used on the original basis, the breaking of the graded context is ensured.

The connection between graded Lie algebras and Lie-admissible algebras was identified, apparently for the first time, by P.P. Srivastava[170] in relation to a sypersymmetric model with product of the type

$$M_1 \circ M_2 = M_1 M_2 - (-1)^{m_1 m_2} M_2 M_1 \qquad (3.E.10)$$

where $m_1 (m_2)$ is the total number of Fermi indices in the operator $M_1 (M_2)$. It is easy to see that product (3.E.10), to begin with, characterizes an algebra (in the sense of Section 1.2). And indeed, we have

$$M_i \circ (M_j + M_k) = M_i \circ M_j + M_i \circ M_k ,$$
$$(M_i + M_j) \circ M_k = M_i \circ M_k + M_j \circ M_k , \qquad (3.E.11)$$
$$(\alpha M_i) \circ M_j = \alpha (M_i \circ M_j) = M_i \circ (\alpha M_j)$$
$$i, j, k = 1, 2$$

Secondly, product (3.E.10) is nonassociative, i.e.,

$$(M_i \circ M_j) \circ M_k \neq M_i \circ (M_j \circ M_k) \qquad (3.E.12)$$

Thirdly, it is Lie-admissible because the product

$$[M_i, M_j]_u = (1 - (-1)^{m_i m_j}) [M_i, M_j]_A \qquad (3.E.13)$$

is Lie. In conclusion, supersymmetric model (3.E.10) exhibit its Lie-admissible nature in a form even more direct than that of models (3.E.2).

In closing, the reader should be alerted that the Lie-admissible covering-breaking of graded Lie algebras may have nontrivial physical implications, particularly from a statistical profile. In essence, in the transition from a Lie model to a graded Lie model there is no loss of conventional statistics (or parastatistics) owing to the structure (3.E.2) of the grading. This does not appear to be necessarily the case in the transition from a graded Lie algebra to a Lie-admissible algebra, particularly when the latter algebras are used to produce a breaking of the fundamental space-time symmetries.

Within the context of conventional approaches to particle physics this can be an unwanted feature. A sort of opposite situation occurs for our approach to the hadronic structure. In fact, one of our central objectives, to be investigated in

Volume III, is precisely that of breaking the conventional statistical (or parastatistical) character of the hadronic constituents (our eletons of Section I.1.3) in such a way that conventional statistics are recovered only for the hadron as a whole. As a matter of fact, this appears to be the truly fundamental requirement to attempt the identification of the constituents of the (unstable) hadrons with suitably selected massive particles produced in their <u>spontaneous</u> decays, along the assumptions of Chapter I.1.

Thus, contrary to current trends, our primary intended use of the Lie-admissible covering-breaking of supersymmetries is precisely the attempt of breaking conventional statistics in the hope of identifying the existence of covering statistics for the hadronic constituents.

CHAPTER 4

SYMPLECTIC–ADMISSIBLE COVERING OF SYMPLECTIC GEOMETRY

4.1: STATEMENT OF THE PROBLEM

As is well known, the geometry which underlies the conventional canonical formulations and Lie algebraic approaches is the symplectic geometry. This provides a geometrical characterization of Newtonian systems in general, and of exact symmetries in particular, as a complement to the analytic and algebraic formulations. In particcular, these analytic, algebraic and geometrical formulations result to be not only compatible among themselves, but deeply interrelated.

In Chapter 2 we have indicated the existence of a consistent covering of Hamilton's equations we have called Birkhoff -admissible, for the treatment of nonconservative Newtonian systems in the coordinate space of their experimental detection. In Chapter 3 we have indicated the existence of a consistent covering of Lie algebraic formulations, called Lie-admissible, for the study of the same systems, which results to be not only compatible with the Birkhoff-admissible formulations, but also deeply related to them. Our analysis would be incomplete without the indication of the existence of a consistent covering of the symplectic geometry, here called symplectic-admissible, which is compatible with, as well as deeply interrelated to the Birkhoff -admissible and the Lie-admissible formulations. This is the objective of this chapter.

Almost needless to say, our objective is simply that of indicating the existence of such a covering geometry. Its actual construction will predictably demand time.

For the reader's convenience we shall first review certain basic aspects of the conventional symplectic geometry and then

consider the problem of their symplectic-admissible covering. The reader should be aware that our analysis will be as elementary as possible.

A primary arena of applicability of the intended covering geometry is that of the Newtonian systems of our everyday experience (genuinely nonconservative) in the Euclidean space of their experimental detection. Moreover, the intended arena of applicability which motivated this study is for the quantization of forces not derivable from a potential in the hope of gaining some insight for the problem of the hadronic structure, along the epistemological lines of Volume I. This aspect will be considered in Volume III. An additional arena of intended applicability is that for classical and quantum mechanical relativistic extensions. Also, the covering geometry here considered could be of some assistance for the study of the possible existence of a "nongeodesic" generalization of the available theories of gravitation for the interior case only, along the conjectural remarks of Section I.3.4. Finally, the intended geometry is proposed in the hope of being relevant for the problem of a unified theory of weak, electromagnetic and strong interactions, as we shall indicate in Volume III.

The reader should be aware that in our rudimentary treatment we shall often sacrifice the beauty of the coordinate-free formulation of geometric notions for their treatment in local variables. This is essentially motivated by our primary objective of indicating the relationship of the symplectic-admissible geometry with the Birkhoff -admissible and Lie-admissible formulations.

In fact, this can be best done in terms of local variables, rather than of abstract geometrical notions.

We have, however, other motivations for this attitude. There is no doubt that the coordinate-free formulation of geometric notions is fully consistent on mathematical grounds. However, it is not immune of problematic aspects on physical grounds, unless properly identified and treated.

In essence, the coordinate-free formulation of symplectic geometry allows the unified treatment of fundamental geometric notions for all cordinate systems which preserve the symplectic structure (the Darboux's charts of Section 4.4). Suppose now that among all these geometrically equivalent systems of coordinates there is one which coincides with that of the experimental verification of the system. Then, no problem of physical consistency arises.

This is typically the case for conservative systems in the Euclidean space $E_3(\underline{r})$ of their experimental detection with Cartesian coordinates r^{ka}. The phase space variables $\{r^{ka}, p_{ka}\}$, $p_{ka} = m_k \dot{r}_{ka}$, of their canonical formulation then possess a direct physical significance. As we shall review in Section 4.4, this space can be geometrically characterized as a <u>symplectic manifold</u>. The use of a coordinate-free characterization of these manifolds is then consistent on both mathematical and physical grounds.

The situation for nonconservative systems is different. In essence, for these systems, particularly when of the essentially nonself-adjoint type, a Hamiltonian for their representation in the physical coordinates $\{r^{ka}, p_{ka}\}$ does not exist. Therefore, there

is the lack of existence of a symplectic-Hamiltonian characterization of the systems considered in the coordinates of the experimenter. Established treatments, confirmed by the "Thoerem of the Indirect Universality" of the Inverse Problem (Section I.2.8) ensure the existence of a new system of coordinates, $\{r'^{ka}, p'_{ka}\}$ under which the systems considered can be equivalently represented in terms of a Hamiltonian. This allows the geometric characterization of the systems when not explicity dependent on time, in terms of the symplectic manifolds. The use of the coordinate-free characterization of these manifolds is then mathematically consistent but not immune from physical problems because no element of the family of admissible local coordinates (i.e., all coordinate systems obtainable with Darboux's charts) coincides with the indicated physical coordinates. As a matter of fact, these coordinate systems are generally unrealizable with experiments*, because related to the original physical coordinates via nonlinear, velocity-dependent transformations

$$r'^{ka} = r'^{ka}(\underline{r}, \underline{p}), \qquad (4.1.1)$$

where, as stressed in Section I.2.8, the "velocity-dependence" is essential. If one keeps into account that our primary intended use of the geometrical notions is for the trajectories of the hadronic constituents (under the assumption that the strong hadronic forces are not drivable from a potential), the difficulties for an experimental realization of the canonical coordinates appear in their full light.

We reach in this way the central property of the intended

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* as well as generally noninertial, irrespective of whether the system in the r'^{ka} coordinates is inertial or not.

covering geometry: its initial construction in the coordinate space of the experimental detection of the considered nonconservative system, in a way fully parallel to the initial construction of the symplectic geometry for conservative systems. Once this crucial property is ensured, then the problem of the coordinate-free formulation will be confronted. As we shall see, the coordinate-free formulation of the symplectic geometry appears to possess a symplectic-admissible covering.

In conclusion, the symplectic- Hamiltonian characterization of conservative systems is mathematically and physically consistent because there always exists one element of the family of admissible coordinates whose space component coincides with that of the experimental verification and whose momentum component coincides with the physical momentum. On the contrary, the symplectic - Hamiltonian characterization of nonconservative (essentially nonself-adjoint) systems is fully consistent on mathematical grounds, but not immune from problematic aspects on physical grounds because there never exist one element of the family of admissible coordinates whose space component coincides with that of the experimental verification and whose momentum component coincides with the physical momentum.* The covering symplectic-admissible geometry is precisely intended to remove the problematic aspects of the latter case, that is, to ensure that the geometric, Birkhoff-admissible characterization of nonconservative systems is such that there always exist one element of the family of admissible coordinates whose space component coincides with that of the experimental verification and whose momentum component coincides

* It should be here indicated for clarity that the statement is referred to the case of globally Hamiltonian vector fields (see Section 4.4). The proof of the property is then a simple consequence of the methods of the Inverse Problem.

with the physical momentum. It is hoped that in this way the covering geometry is consistent on both mathematical and physical grounds.

There are, however, additional reasons which motivated my efforts at the identification of a covering geometry. One of the fundamental problems of the relativity laws which are applicable in Newtonian Mechanics is the identification of a transformation group which leaves form-invariant nonconservative (nonlinear) systems. The use of the symplectic geometry has proved to be effective for the study of this problem but for the subclass of conservative systems. Despite repeated efforts, I have been unable to confront this problem by using the conventional symplectic geometry, but now for the broader class of nonconservative systems. As we shall see in Chapter 5, if the covering symplectic-admissible geometry is instead assumed, it appears that a solution of the problem considered can indeed be reached.

In conclusion, it appears that the indicated problematic aspects of the use of the symplectic geometry for the characterization of nonconservative systems have direct problematic images within the context of relativity considerations. This, however, does not exclude the possibility that, once the applicable relativity laws have been identified at the level of the symplectic-admissible geometry, they admit a reformulation in terms of the conventional symplectic geometry. The point made above, however, persists. To avoid insidious physical aspects, the problem of the relativity laws which are applicable to nonconservative systems must be first confronted in the physical space of their experimental detection. Once this basic objective has been successfully achieved, then the problem of the reformulation of these relativity laws in equivalent coordinate systems can be confronted.

The insistence on the preservation of the direct physical significance of the momentum component of the geometric coordinates is also motivated by quantum mechanical considerations. In fact, the expectation values of the conventional quantum mechanical operator

$$\underset{\sim}{p} \rightarrow \frac{\hbar}{i} \underset{\sim}{\nabla} \qquad (4.1.2)$$

even though mathematically consistent, have a questionable physical meaning when the symbol "$\underset{\sim}{p}$" represents a (classical) mathematical entity of the type

$$\underset{\sim}{p} = \underset{\sim}{\gamma} \tanh^{-1} \beta \dot{\underset{\sim}{z}}^2 , \quad \underset{\sim}{\gamma}, \beta = \text{const}. \quad (4.1.3).$$

This aspect will be elaborated in detail in Volume III.

But, besides these technical aspects, there is an epistemological aspect which motivated my efforts at the identification of a covering geometry and which should not be overlooked. Nonconservative systems are known to be profoundly different than conservative systems on conceptual, technical and physical grounds. The best way to attempt a methodological characterization of these differences is by attempting the construction of a different geometry. This leads in a natural way to the search of a new geometry specifically conceived for nonconservative mechanics. In addition, it virtually ensures the covering nature of the new over the conventional geometry, because nonconservative systems

trivially include the subclass of conservative systems for all null nonconservative forces.

Intriguingly, it appears that, perhaps, greater consistency problems arise in the attempt of characterizing Lie-admissible formulations via the Riemannian geometry, as currently known. In essence, the difficulties for the symplectic geometry rest in the inability to technically characterize the symmetric part of the Lie-admissible tensor. The difficulties with the Riemannian geometry are somewhat opposite, in the sense that they arise from the inability to characterize the antisymmetric part of the Lie-admissible tensor. In turn, this situation opens intriguing problems concerning the compatibility of gravitational theories as currently known under the assumption of non-potential hadronic forces.

Permit me to stress that I am not an expert of differential geometry. Therefore, the entire content of this chapter demands the inspection and possible technical finalization by interested experts. I have been simply forced into a research [which, strictly speaking, is a topic for pure mathematicians] because of the lack of existence of geometrical studies on Lie-admissible algebras, to the best of my knowledge, as well as the crucial need to inspect this aspect at least in a rudimentary way in order to make any progress on the problem of the relativity laws which are applicable to systems with forces not derivable from a potential.

NOTE ADDED IN 1982

Since the time of writing this chapter (Fall 1977), the following studies on the geometry of the Lie-admissible algebras have appeared in print:

- a paper by W. Sarlet of 1979[198] on the conditions for rank two-tensors to be jointly Lie-admissible and symplectic-admissible;
- a paper by R.H. Oehmke on the geometry of the Lie-admissible algebra recently appeared in print[199]; and
- a paper by A.A.Sagle on geometric means to construct Lie-admissible algebras from given Lie algebras which also appeared in print quite recently[200].

The study of these papers is recommended here as a pre-requisite for any researcher interested in reaching an in depth understanding of the generalized analytic, algebraic, and geometric methods proposed in these volumes.

Needless to say, there exist a considerable number of contributions in the mathematical and physical literature which are potentially relevant to the identification of the geometry of the Lie-admissible algebras, although this aspect has not been investigated as yet. As an example, we mention here the studies by A. Schober on the use of non-Desarguesian geometries, which have also recently appeared in print[201].

Finally, the reader should keep in mind that I have established in monograph[189] the "direct universality" of the symplectic geometry (actually of the contact geometry)for all local, closed, non-Hamiltonian systems satisfying certain topological conditions.In this way, the closed character of Hamiltonian formulations is preserved in the covering Birkhoffian formulations, although the restrictions of the forces to be only of potential type is relaxed.

The analysis of this chapter can therefore be considered as a complement of the geometric studies of monograph[189], this time, for the class of open non-Hamiltonian systems.

The reader will recall (Section 2.1 and Volume I) that the former approach is intended for the exterior representation of non-Hamiltonian systems, while the latter approach is intended for the complementary interior representation of the same systems.

Upon quantization (Volume III), I hope to achieve a dual representation of a hadron with an interior dynamics structurally more general than that of the electromagnetic interactions: first, a closed, Lie-isotopic representation of a hadron as a whole, and, second, a complementary, open, Lie-admissible representation of the hadronic constituents.

4.2: THE CONCEPT OF HAUSDORFF, SECOND COUNTABLE, ∞-DIFFERENTIABLE MANIFOLD

A geometric approach to Newtonian systems can be formulated by transforming the systems of n second-order equations of motion into equivalent systems of 2n first-order equations, by interpreting these latter systems as vector fields on manifolds and then by identifying the applicable geometry.

In this section we shall recall, for the reader's convenience, certain basic aspects of point set topology and identify the notion of manifold which is needed for the approach. The interpretation of Newtonian systems as vector fields on manifold will be outlined in Section 4.2. The geometrical aspect will be considered thereafter.

For the content of this section the reader is referred, for instance, to S. Sternberg[151] and to R. Abraham and J.E. Marsden[171] (particularly Appendix A and quoted references).

A topological space is a set M together with a collection of subsets $\mathcal{O}$ called open sets such that (a) $M \in \mathcal{O}$, (b) if $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{O}$ then $\mathcal{O}_1 \cap \mathcal{O}_2 \in \mathcal{O}$, and (c) the union of any collection of open sets is open. The closed sets of M are the elements $\{S | CS \in \mathcal{O}\}$ where C denotes the complement $CS = \hat{S} \setminus S = \{m \in \hat{S}, m \notin S\}$.

The relative topology on a subset M_1 of a topological space M is given by

$$\mathcal{O}_{M_1} = \{ \mathcal{O}_1 \cap M_1 \,|\, \mathcal{O}_1 \in \mathcal{O} \} \tag{4.2.1}$$

The basis for the topology is a collection B of open sets such that every open set of M_1 is a union of elements of M_1. This

topology is called first countable if and only if for each point $m \in M$ there is a countable collection $N_i(m)$ of neighborhoods of m such that for any neighborhood N(m) of m there is an $\hat{N}$ such that $N_{\hat{N}} \in N$. The topology is called second countable if and only if it has a countable basis. A second countable topology is, of course, also first countable. This is the case, for example, of the union of open intervals of the real line $\mathbb{R}$.

A topological space M is called a Hausdorff space if and only if each two distinct points have disjoint neighborhoods. Let us also recall that a first countable space is Hausdorff if and only if all sequences have at most one limit point.

A topological space M is called compact if and only if for every covering of M by open sets M_i there is a finite subcovering. A topological space is called locally compact if and only if each point $m \in M$ has a neighborhood whose closure is compact. A topological space M is called paracompact if and only if it is Hausdorff and every open covering of M has a locally finite refinement of open sets. The refinement $\{\hat{M}_i\}$ of a covering $\{M_i\}$ occurs when for every $\hat{M}_i$ there is an M_i such that $\hat{M}_i \in M_i$. A locally finite covering $\{\hat{M}_i\}$ of M occurs when each point $m \in M$ admits a neighborhood N(m) which intersect only a finite number of $\hat{M}_i$.

For a first countable and compact topological space, every sequence has a convergent subsequence (Bolzano-Weierstrass theorem). Also, every compact subset of a Hausdorff space is closed, and every second countable, locally compact, Hausdorff space is paracompact.

A mapping $\varphi : M \to M'$ is continuous at $m \in M$ if and only if for evey neighborhood $N(\mathscr{C}(m))$ there is a neighborhood $N(m)$ such that $\mathscr{C}(N) \in N$. A mapping $\mathscr{C}$ is continuous if and only if it is continuous at all points $m \in M$.

A mapping $\mathscr{C}: M \rightarrow M'$ is a bijection when it is one-to-one and onto, and it is a homeomorphism when $\mathscr{C}$ and $\mathscr{C}^{-1}$ are continuous. Then M and M' are said to be homeomorphic. A local chart $(M_1, \mathscr{C})$ is a bijection from a subset $M_1 \in M$ to an open subset Θ. M_1 is then called the domain of M. An atlas on M is a family of charts $(M_i, \mathscr{C}_i)$ such that $M = \bigcup M_i$. Two atlases are equivalent if and only if their union is an atlas.

We are primarily interested in topological spaces which can be locally interpreted as Euclidean spaces E_n. By following S. Sternberg[151] we shall then define an n-manifold as a Hausdorff space with a countable basis which is locally homeomorphic to an n-dimensional Euclidean space E_n.

An r-differentiable structure on an n-manifold M is a collection Ω of (real valued) functions of class C^r, $r=0,1,2,\ldots,\infty$, all defined on an open subset of M such that

(a) if $M_1 \subset M_2 \subset M$ and $\omega \in \Omega$ is defined on M_2, then the restriction of ω to M_1, $\omega | M_1$, is in Ω;

(b) if $M_1 = \bigcup M_{1i}$ and ω is defined in on M_1 with $\omega | M_{1i}$ for all i, then $\omega \in \Omega$; and

(c) every point $m \in M$ has a neighborhood $N(m)$ and an homeomorphism $h: N(m) \rightarrow \Theta(E_n)$ such that $\omega(N(m)) \in \Omega$ if and only if $\omega \cdot h^{-1}$ is a function of n (real) variables of class C^r.

The local coordinates of an n-manifold are then given by $h(m) = (x^1(m),\ldots,x^n(m))$.

An r-differentiable manifold M is an n-manifold M equipped with an r-differentiable structure Ω in the local coordinates $h(m) = \{x^i(m)\} = x(m) = x$ and we shall write $M(x,\Omega)$. It should be indicated here that local coordinates are generally omitted in the symbol representing a differentiable manifold. Nevertheless, we shall use them for the intent of avoiding possible confusions in the transition from one system of local coordinates to another.

The notion of manifold which we shall use for the geometric approach to Newtonian systems is that of a Hausdorff, second countable, ∞-differentiable manifold M or manifold, for short. In essence, this notion provides the needed topological characterization of the carrier space of Newtonian mechanics, the Euclidean space of the Cartesian coordinates $\underset{\sim}{r}^k$ for systems without subsidiary constraints (or the configuration space of the generalized coordinates q^k for systems with holonomic subsidiary constraints).

4.3: NEWTONIAN SYSTEMS AS VECTOR FIELDS ON MANIFOLDS

Let M be a (Hausdorff, second countable, ∞-differentiable) manifold which is locally homeomorphic to an n-dimensional Euclidean space E_n. This manifold is insufficient to characterize Newtonian systems because, for instance, nonequivalent trajectories pass through each point $m \in M$. In this section we shall review the additional notions which are needed to achieve a proper characterization of Newtonian systems. For technical details the reader is referred, for instance, to S. Sternberg,[151] and R. Abraham and J.E. Marsden,[171] R. Hermann,[172] and G. Caratu, G. Marmo, E.J. Saletan, A. Simoni and B. Vitale.[173]

Consider two Euclidiean spaces, say, E_n and E'_n and let $\mathcal{O}$ be an open subset of E_n. The Cartesian product $\mathcal{O} \times E'_n$ is called a local vector bundle with base space $\mathcal{O} \times \{0\}$, also called the zero section. Let $m \in \mathcal{O}$. The fiber over m is given by $\{m\} \times E'_n$. The map $\pi : \mathcal{O} \times E'_n \to \mathcal{O}$ is called the projection of $\mathcal{O} \times E'_n$ onto $\mathcal{O}$.

In essence, the space $\mathcal{O} \times E'_n$ can be implemented, at least locally, into a manifold with a vector space attached to each of its points. Its significance for Newtonian systems is due to the possibility of associating to each trajectory $x(t) = \{x^1(t),\ldots,{}^n(t)\} \in E_n$ the tangent vector at each value t, i.e., $\dot{x}(t)$. Therefore, the approach allows the geometrical treatment of the Cartesian coordinates and velocities of a Newtonian system. This objective, however, demands additional notions.

A map $\varphi : \mathcal{O} \times E'_n \to \hat{\mathcal{O}} \times \hat{E}'_m$ of local vector bundles is called a local vector bundle isomorphism if and only if φ is a C^∞ diffeomorphism and $\varphi(m,f) = (\varphi_1(m), \varphi_2(m)\cdot f)$, where $\varphi_2(m)$ is a linear isomorphism for all $m \in \mathcal{O}$.

A local bundle chart of E_n is a pair $(\mathcal{O}, \varphi)$ with $\mathcal{O} \subset E_n$ and $\varphi : \mathcal{O} \subset E_n \to \hat{\mathcal{O}} \times \hat{E}'_m$ is a bijection onto a local bundle $\hat{\mathcal{O}} \times \hat{E}'_m$. A vector bundle atlas is a family of local bundle charts $\{\mathcal{O}_i, \varphi_i\}$ such that $E_n = \bigcup \mathcal{O}_i$ and the overlap maps between members of $\{\mathcal{O}_i, \varphi_i\}$ are C^∞ diffeomorphisms. A <u>vector bundle structure</u> C on E_n is an equivalence class of vector bundle atlases. A <u>vector bundle</u> can now be defined as the pair $A = (E_n, C)$. The <u>zero section</u> of E_n is a surjective (onto) projection subspace $E_n^o \in E_n$. A C^r <u>section</u> of π is a map $\eta : E_n \to E_n'$ of class C^r such that for each element $a \in E_m$, $\pi(\eta(a)) = a$. The set of all C^r sections of π is denoted with $\Gamma^r(\pi)$.

To proceed further, we now replace the Euclidean space E_n with a manifold M. A trajectory (also called <u>curve</u>) at $m \in M$ is a map $x: I \to M$ of at least class C^1 (no <u>corner points</u>, i.e., discontinuity in the first order derivatives) from an open interval I of the real line R into M such that for $x = x(t)$, t and $t = 0 \in I$, $x(0) = m$. Let x and x' be trajectories at m and $(\mathcal{O}, \varphi)$ a local chart. If $x \cdot \varphi$ and $x' \cdot \varphi$ coincide at $t = 0$, x and x' are tangent at $t = o$.

The <u>tangent space</u> $T_m M$ of M at $m \in M$ is the set of all equivalence classes of curves at m. The <u>tangent bundle</u> TM of M is the set of all tangent spaces at all points of M. The <u>tangent bundle projection</u> is the mapping $\tau : TM \to M$, i.e., the restriction of the elements of TM to those of the base manifold M. The dimension of TM is 2n. Let $a = \{a^\mu\}$, $\mu = 1, 2, \ldots, 2n$, be local coordinates of TM. Then $\tau a = x \in M$. A <u>basis</u> on TM can be given in terms of the derivatives $(\partial / \partial a^\mu)_m$. Introduce the notation[151]

$$\left.\frac{d f(\varphi(t))}{dt}\right|_{t=0} = \langle \Xi_m, (df)_m \rangle \qquad (4.3.1)$$

$$\Xi \in TM$$

Then, in correspondence with $(\partial / \partial a^\mu)_m$ we have the quantities $(da^\mu)_m$ verifying the properties

$$\left\langle \left(\frac{\partial}{\partial a^\mu}\right)_m, (da^\nu)_m \right\rangle = \delta_\mu^\nu \qquad (4.3.2)$$

Thus $(da^\mu)_m$ can be interpreted as the basis of a 2n-space T^*M dual to TM called the <u>cotangent bundle</u>. The elements $\Xi \in TM$ and $\Xi^* \in T^*M$ are called <u>contravariant vector fields</u> and <u>covariant vector fields</u>, respectively. In local coordinates a^μ of TM and $a^{*\mu}$ of T^*M we shall write $\Xi = \Xi(a(t))$ and $\Xi^* = \Xi^*(a^*(t))$.

These latter notions can be generalized in terms of tensors. The <u>vector bundle of tensors of contravariant order r and covariant order s</u> on M will be denoted with $T_s^r(M)$. Then, the tangent bundle is $T_o^1(M) = TM$ while the cotangent bundle is $T^*M = T_1^o(M)$. A <u>tensor field</u> on M is an element of the C^∞ section of $T_s^r(M)$, $\Gamma^\infty(T_s^r(M))$. A contravariant vector field is then an element of $\Gamma^\infty(T_o^1(M))$ while a covariant vector field is an element of $\Gamma^\infty(T_1^o(M))$.

We now equip the spaces TM and T^*M with an ∞-differentiable structure which, in the respective local coordinates a and a* can be interpreted as a set of functions $C(a(t))$ and $C^*(a^*(t))$. This allows the reinterpretation of TM and T^*M as Hausdorff, second-countable, ∞-differtiable, 2n- manifolds. The transition from M to TM or to T^*M is often called <u>lifting</u> of the base manifold.*

The vector fields $\Xi(a(b))$ and $\Xi^*(a^*(b))$ with corresponding structures in TM and T^*M, i.e., $C(a(b))$ and $C^*(a^*(b))$ do not depend explicitly on time t. Thus, they can only characterize <u>autonomous systems</u>. For <u>nonautonomous systems</u> (i.e., systems

* Notice that this is, in actuality, an enlargment of M.

with an explicit dependence of time) the vector fields assume the form in local coordinates $\Xi(t,a(t))$ and $\Xi^*(t,a^*(t))$. Similarly, for the differentiable structures we have $C(t,a(t))$ and $C^*(t,a^*(t))$. These broader systems demand an extension of the tangent and cotangent bundles into the 2n+1 dimensional spaces $TM \times \mathbb{R}$ and $T^*M \times \mathbb{R}$, respectively, where the real line $\mathbb{R}$ is representative of the time variable. These latter spaces are turned into (2n+1)-manifolds by their differentiable structure because these notions are independent of the dimensionality of the space considered.

Consider now a Newtonian system of N particles in a three-dimensional Euclidean space $E_3(\underline{r})$ with Cartesian coordinates $\{\underline{r}^k\} = \{r^{ka}\}$, $k=1,2,\ldots,N$, $a=x,y,z$, characterized by the systems of second-order ordinary differential equations

$$Q_{ka}(\underline{r}) = \left[A_{ka\,jb}(t,\underline{r},\dot{\underline{r}})\ddot{r}^{jb} + B_{ka}(t,\underline{r},\dot{\underline{r}}) \right]_{ENSA} = 0 \qquad (4.3.3)$$

which we assume to be of class C^{∞}, regular and, in general, essentially nonself-adjoint (ENSA). This system can be characterized in terms of either contravariant or covariant vector fields. However, each of these two characterizations can be realized in an infinite variety of ways, all admitting the same bundle projection, i.e., all characterizing the same trajectory in $E_3(\underline{r})$.

Let us consider first the characterization of systems (4.3.3) in terms of contravariant vector fields. The simplest possibility can be implemented as follows. Since Eqs. (4.3.3) satisfy all the requirements for the theorem on implicit functions, they can be

equivalently written

$$\ddot{x}^k - f^k(t,x,\dot{x}) = 0, \quad k=1,2,\ldots,n=3N,$$
$$(f^i) = -(A)^{-1}(B) = -(A^{ki}B_i) \qquad (4.3.4)$$

where $x = \{x^1,\ldots,x^n\}$ represents the Cartesian coordinates in a given ordering and the f's are the implicit functions of the system. Associate now the (contrvariant) velocities $y = \dot{x}$ to x and introduce the unified notation

$$\Xi^\mu = \begin{cases} y^\mu, \\ f^{\mu-n}, \end{cases} \quad a^\mu = \begin{cases} x^\mu, & \mu=1,2,\ldots,n \\ y^{\mu-n}, & \mu=n+1,n+2,\ldots,2n \end{cases} \qquad (4.3.5)$$

Then the system (4.3.3) of n = 3N second-order differential equations is turned into the equivalent system of 2n = 6N first-order equations

$$\dot{a}^\mu - \Xi^\mu(t,a) = 0 \qquad (4.3.6)$$

This yields the desired characterization. And indeed, the functions Ξ^μ can be interpreted as components of vector fields on $TM \times \mathbb{R}$ (or on TM when there is no explicit dependence on time). Here M is the topological characterization of the carrier space E_3 indicated in Section 4.2. The projection

$$\tau\, a = x \qquad (4.3.7)$$

assures that the projection of trajectories to M yields those in $E_3(\underline{r})$, i.e., the physical trajectories of the particles.

Identifications (4.3.5), however, are by no means unique.

In fact, any other association to x of contravariant quantities other than the velocities is equally acceptable, provided that it carries the needed continuity and regularity conditions to yield the same projection (4.3.7). This situation can be realized by introducing arbitrary prescriptions for the characterization of new contravariant quantities $\hat{y}^i$, i.e., functions

$$\rho^k(t,x,\dot{x},\hat{y}) = 0 \qquad (4.3.8)$$

which satisfy all the conditions of the implicit function theorem for the existence, uniqueness and single-valuedness of the implicit functions

$$\dot{x}^i = g^k(t,x,\hat{y})$$

The substitution of these functions in Eqs. (4.3.3) yields the system

$$\begin{aligned} Q_k(x) &= A_{ki}(t,x,g)\,\dot{g}^i + B_k(t,x,g) \\ &\equiv \alpha_{ij}(t,x,\hat{y})\,\dot{\hat{y}}^j + \beta_i(t,x,\hat{y}) \end{aligned} \qquad (4.3.9)$$

which, under the identifications

$$\Xi'^{\mu} = \begin{cases} g^{\mu}, \\ f^{\mu-n}, \end{cases} \quad a'^{\mu} = \begin{cases} x^{\mu}, & \mu = 1,2,\ldots,n \\ \hat{y}^{\mu-n}, & \mu = n+1, n+2, \ldots, 2n \end{cases} \qquad (4.3.10)$$

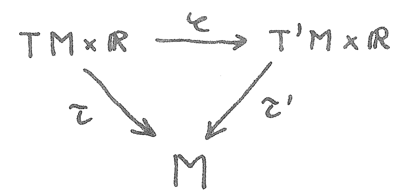
$$(f) = -(\alpha)^{-1}(\beta)$$

can be written in the form

$$\dot{a}'^{\mu} - \Xi'^{\mu} = 0 \qquad (4.3.11)$$

In conclusion, there exists an infinite number of liftings

to TM x ℝ (or to TM) all satisfying projection (4.3.7). For any two elements of this family we have the invertible diagram



(4.3.13)

where φ is a (fibre preserving) diffeomorphism.*

One of the simplest possibilities for lifting system (4.3.3) in terms of covariant vector fields is the following. Suppose that the system is either essentially self-adjoint or nonessentially nonself-adjoint. Then a Lagrangian $L(t,x,\dot{x})$, for its analytic representation in the coordinates x of its experimental detection exists (Chapter I.2). Introduce the conventional canonical prescriptions for the association to the (contravariant) coordinates x^i of the (covariant) momenta p_i

$$p_i = \frac{\partial L}{\partial \dot{x}^i} \qquad (4.3.14)$$

The identifications

$$\Xi^*_{\mu} = \begin{cases} f_{\mu}, \\ \hat{y}^{\mu-n}, \end{cases} \quad a^*_{\mu} = \begin{cases} p_{\mu}, & \mu = 1,2,\ldots,n, \\ x^{\mu-n}, & \mu = n+1, n+2, \ldots, 2n, \end{cases} \qquad (4.3.15)$$

then yields the system

$$a^*_{\mu} - \Xi^*_{\mu}(t,a^*) = 0 \qquad (4.3.16)$$

---

* The notation T'M is here intended to express the fact that the only admitted degrees of freedom of the coordinates are those in the contravariant quantities y^k (that is, the local coordinates on M are kept fixed).

with the cotangent projection

$$\tau^* a^* = x \tag{4.3.17}$$

System (4.3.16) can then be characterized in terms of the covariant vector field Ξ^* in T*M x $\mathbb{R}$ (or in T*M when there is no explicit time dependence). This characterization, however, is not unique, Irrespective of whether a Lagrangian for system (4.3.3) exists or not, one can consider an arbitrary function $G(t,x,\dot{x})$ and introduce the covariant quantities

$$\hat{p}_i = \frac{\partial G}{\partial \dot{x}^i} \Rightarrow \dot{x}^i = \hat{g}^i(t,x,\hat{p}) \tag{4.3.18}$$

Then the identifications

$$\Xi'^*_\mu = \begin{cases} \hat{f}_\mu \\ \hat{g}^{\mu-n} \end{cases}, \quad a'^*_\mu = \begin{cases} \hat{p}_\mu, & \mu = 1,2,\ldots,n \\ x^{\mu-n}, & \mu = n+1, n+2, \ldots, 2n \end{cases} \tag{4.3.19}$$

yields the system

$$\dot{a}'^*_\mu - \Xi'^*_\mu(t,a) = 0 \tag{4.3.20}$$

The quantities Ξ'^* can then be interpreted as covariant vector fields in T'*M x $\mathbb{R}$ (or (T'*M). A projection of type (4.3.17) still holds.

As a result, there exists a family of liftings to T*M x $\mathbb{R}$ (or to T*M), all satisfying projection (4.3.17). For any two elements of this family we have the invertible diagram

$$\begin{array}{ccc} T^*M \times \mathbb{R} & \xrightarrow{\ \varphi^*\ } & T'^*M \times \mathbb{R} \\ & \tau^* \searrow \quad \swarrow \tau'^* & \\ & M & \end{array} \tag{4.3.21}$$

where φ^* is, again a (fibre preserving) diffeomorphism.

It should be here indicated that, for any given pair of liftings, one to TM x $\mathbb{R}$ and one to T*M x $\mathbb{R}$ there always exists a fibre-preserving (bijective) diffeomorphism η: TM x $\mathbb{R} \to$ T*M x $\mathbb{R}$ and we shall write

$$\begin{array}{ccc} TM \times \mathbb{R} & \xrightarrow{\ \eta\ } & T^*M \times \mathbb{R} \\ & \tau \searrow \quad \swarrow \tau^* & \\ & M & \end{array} \tag{4.3.22}$$

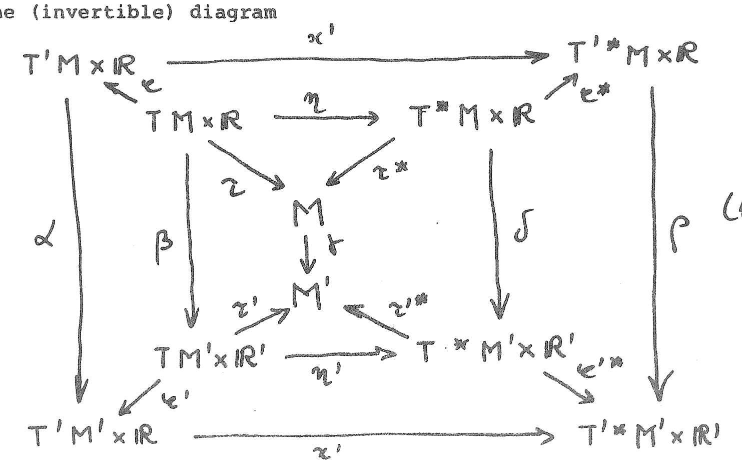
A third family of representations of system (4.3.3) as vector fields on manifolds can be constructed as follows. By assumption, the space $E_3(\underset{\sim}{r})$ of the system is that of its experimental identification. Nevertheless, the system can be transformed into an equivalent system of second-order differential equations in new variables $t' \in E'_1, x' \in E'_n$. We shall assume that the mapping $t \to t'$, $x \to x'$, besides being single-valued, of class C^∞ and everywhere invertible, is realized in terms of the coordinates transformations

$$\begin{aligned} t &\to t' = t(t,x,\dot{x}) \\ x &\to x' = x'(t,x,\dot{x}) \end{aligned} \tag{4.3.23}$$

Here, a possible explicit dependence of the transformations in the velocities is essential for our program. In fact, when system

(4.3.3) is essentially nonself-adjoint, a Lagrangain for its analytic representation in E_n does not exist. Nevertheless, there always exists a new space E'_m in which the system admits an analytic representation, <u>provided</u> that the transformations laws (4.3.23) are explicitly dependent on the velocities. This is the Universality Theorem for Indirect Analytic Representations of Section I.2.8 (related technical restrictions are here tacitly assumed).

Clearly, system (4.3.3) can be written, as an equivalent form in t', x', in an infinite variety of ways characterized by all possible transformations (4.2.23). For each of these possibilities there exists an infinite number of liftings to $TM' \times \mathbb{R}'$ and a similar number of $T^*M' \times \mathbb{R}'$. This situation can be symbolically represented with the (invertible) diagram*



(4.3.24)

The cases considered until now still do not exhaust all the possible characterizations of Newtonian systems as vector fields.

* The notations TM'x R' is intended to express the fact that only the degrees of freedom of trajectories is taken into account, while the notation T'M'x R' is intended to indicate the inclusion of both, the degrees of freedom of the trajectories and of their liftings.

This is due to the fact that all liftings (4.3.13), (4.3.21) and (4.3.24) occur with the conventional technical specification of the differentiable structure (see, for instance, ref. [17]). As such, this structure is not explicitly related to the vector fields.

We reach in this way a crucial point of our analysis, particularly from a geometrical profile. It consists of the characterization of Newtonian systems as vector fields on <u>manifolds</u>, that is, their characterization as vector fields on spaces TM or T*M or TM x $\mathbb{R}'$ or T*M x $\mathbb{R}$ via the joint characterization of an ∞-differentiable geometrical structure.

This objective can be realized as follows. Under the prescriptions

$$\hat{\alpha}_{ij}(t,x,y)\,\dot{x}^j + \hat{\beta}_i(t,x,y) = 0 \qquad (4.3.25)$$
$$\hat{\alpha}, \hat{\beta} \in \mathcal{C}^\infty, \quad |\hat{\alpha}| \neq 0$$

system (4.3.3) can be written in the equivalent first order form

$$\hat{\alpha}_{ij}(t,x,y)\,\dot{x}^j + \hat{\beta}_i(t,x,y) = 0 \qquad (4.3.26)$$
$$\alpha_{ij}(t,x,y)\,\dot{y}^j + \beta_i(t,x,y) = 0$$

By introducing the notations

$$(C_{\mu\nu}) = \begin{pmatrix} (\hat{\alpha}_{ij}) & 0 \\ 0 & (\alpha_{ij}) \end{pmatrix}, \quad (D_\mu) = \begin{pmatrix} \hat{\beta} \\ \beta \end{pmatrix} \qquad (4.3.27)$$
$$\{a^\mu\} = \{x, y\}$$

Eqs. (4.3.27) can be written in the covariant general form

$$\left[C_{\mu\nu}(t,a)\,\dot a^{\nu} + D_{\mu}(t,a)\right]^{C^\infty,R} = 0, \qquad (4.3.28)$$

which is the general covariant form of a (quasilinear) system of first-order ordinary differential equations.

The matrix $(C_{\mu\nu})$ is regular by construction and, thus, it is everywhere invertible. As a result, Eqs. (4.3.28) can always be written in the equivalent form

$$\dot a^{\mu} - \Xi^{\mu}(t,a) = 0 \qquad (4.3.29a)$$

$$(\Xi^{\mu}) = -(C_{\mu\nu})^{-1}(D_{\mu}) = -(C^{\mu\nu}D_{\nu}) \qquad (4.3.29b)$$

which is called a contravariant normal form in the literature of ordinary differential equations. Vice-versa, any normal form (4.3.29) can always be written in an equivalent general form (4.3.28). In particular, for any given normal form, there exists an infinite variety of equivalent general forms within the same set of (local) variables, i.e., all admitting the same implicit functions Ξ^{μ}.

A significance of the two possibilities (4.3.28) and (4.3.29) for the geometric characterization of Newtonian systems is the following. The use of the normal form essentially yields vector fields with a differentiable structure of conventional technical characterization. The situation for the general form is different. In fact, the tensor $C_{\mu\nu}$ characterize by construction, an ∞-differentiable nowhere degenerate two-form

$$C_2 = C_{\mu\nu}\, da^{\mu} \otimes da^{\nu} \qquad (4.3.30)$$

where the symbol $\otimes$ represents the tensorial product used in Sec. 3.4. As such, this form can be assumed as a geometric form on TM x $\mathbb{R}$ (or TM). This turns TM x $\mathbb{R}$ (TM) into a (2n+1)-dimensional (2n-dimensional) manifold $M(a, C_2)$. The degees of freedom in the differentiable structure for fixed local variables and implicit functions can be characterized in terms of the multiplication of a class C^{∞} nowhere degenerate matrix of element $h_{\mu}{}^{\nu}(t,a)$ and we shall symbolically write

$$\left\{h_{\mu}{}^{\nu}\left[C_{\nu\rho}\,\dot a^{\rho} + D_{\nu}\right]^{C^\infty,R}\right\}^{C^\infty,R} = \qquad (4.3.31)$$

$$= C'_{\mu\rho}\,\dot a^{\rho} + D'_{\mu} = 0, \quad C'_{\mu\rho} = h_{\mu}{}^{\nu}C_{\nu\rho}, \; D'_{\mu} = h_{\mu}{}^{\nu}D_{\nu}$$

where C^{∞} and R stand for the assumed continuity and regularity conditions.

In conclusion, for each given lifting to TM there exist an infinite number of subsequent implementation to $M(a,C_2)$ characterized by the transition from the normal form to all possible general forms on $M(a,C_2)$ and we shall symbolically write

$$\begin{array}{ccc} M(a,C_2) & \xrightarrow{\;h\;} & M(a,C'_2) \\ & \searrow^{\alpha} \quad {}^{\beta}\swarrow & \\ & TM\times\mathbb{R} & \\ & \downarrow \tau & \\ & M & \end{array} \qquad (4.3.32)$$

where $M(a,C_2)$ and $M(a,C'_2)$ should be more properly interpreted as atlases on TM for the geometric study of the structures c_2 and C'_2/

and h is a mapping which preserves the local coordinates and the implicit functions of the form.

To summarize, given a Newtonian system (4.3.3), the following five classes of different possibilities for its characterization as vector fields are relevant for our analysis:

I. lifting to $T M \times \mathbb{R}$ via a contravariant normal form;

II. liftings to $T^*M \times \mathbb{R}$ via a covariant normal form;

III. liftings to $M(a, C_2)$ through a covariant general form;

IV. liftings to $M(a^*, C^2)$ through a contravariant general form; and

V. liftings of classes I, II, III and IV induced by mappings $M \rightarrow M'$ of the base manifold M as per Eqs. (4.3.23).

The contravariant or covariant projections of all liftings of class I, II, III and IV yield, by construction, the same trajectory $x = \{r^{1x}, r^{1y}, r^{1z}, \ldots\ldots r^{Nz}, r^{Ny}, r^{Nz}\}$ as experimentally detected. The corresponding projections of the liftings of class V yield a trajectory x' in a space different than that of the experimental set up. Nevertheless, by construction, the trajectory x' is equivalent to x in the sense that it is uniquely reducible to it through the inverse of transformation laws (4.3.23).

4.4: SYMPLECTIC GEOMETRY

In this section we shall review the rudiments of the geometric characterization of the canonical formulations of contemporary Analytic Mechanics which, as is well known, is given by the symplectic manifolds (or, more generally, the symplectic geometry) for the autonomous case, and by the contact manifolds for the nonautonomous case. The reader should be aware that, rather than emphasizing the coordinate-free, abstract, geometrical approach, we shall provide an effort in indicating as much as possible the connection of the geometric ideas with Analytic Mechanics. This will often demand the use of local coordinates.

For the full geometrical treatment as well as for all technical details the reader is referred to the existing literature on the subject such as P. Dedecker (1953),[174] S. Kobayashi and K. Nomizu (1963),[150] R.L. Bishop and R.J. Crittenden (1964),[175] R. Jost (1964),[176] C.L. Siegel (1964),[177] S. Sternberg (1964),[151] R. Abraham and J.E. Marsden (1967),[171] R. Hermann (1968), L.H. Loomis and S. Sternberg (1968),[178] J.M. Souriau (1970),[179] M. Spivak (1970),[180] P.L. Garcia (1974),[181] A.L. Lichnerowicz (1976),[182] G. Marmo, E. J. Saletan, A. Simoni, and B. Vitale (1977)[173], et al.

Let us begin by considering the case of autonomous systems. For conciseness, we assume that the reader is familiar with the calculus of differential forms. See, for instance, in this latter respect, H. Flanders (1963),[121] D. Lovelock and H. Rund (1975)[120] and R.M. Santilli (1978, Vol. I).[65]

The tensorial product of Eq. (4.3.30) is restricted to the exterior (wedge) product "$\wedge$" throughout this section.

Let M be a (Hausdorff, second countable, ∞-differentiable, N-dimensional), manifold. A covariant two-form $\Omega_2 \in T_2^0(M)$ pn M can be realized according to the following possibilities.

(1) Tensorial realizations. Let a^μ, $\mu = 1,2,3,..,2N$, be local coordinates of $T_2^0(M)$. Then the form Ω_2 can be locally realized in terms of a covariant tensor of rank two and we shall write

$$\Omega_2 : \Omega_{\mu\nu}(a) \qquad (4.4.1)$$

(2) Matrix realization. Let $\{e^\mu\}$ be an (ordered basis) of TM. Then the values of Ω_2 along such basis can be interpreted as the elements of an NxN matrix and we shall write

$$\Omega_2 : \Omega_2 = \Omega_2(e^\mu, e^\nu) = (\Omega_{\mu\nu})_e \qquad (4.4.2)$$

(3) Differential realization. Let a^μ be local variables of TM. Then Ω_2 can be locally interpreted as the two-form

$$\Omega_2 : \Omega_2 = \Omega_{\mu\nu}(a)\, da^\mu \wedge da^\nu \qquad (4.4.3)$$
$$= \frac{1}{2} \delta^{\mu_1\mu_2}_{\nu_1\nu_2} \Omega_{\mu_1\mu_2} da^{\nu_1} \wedge da^{\nu_2}$$

where $\wedge$ is the (antisymmetric) exterior product and $\delta^{\mu_1\mu_2}_{\nu_1\nu_2}$ is the generalized Kronecker delta defined as

$$\delta^{\mu_1\mu_2\cdots\mu_p}_{\nu_1\nu_2\cdots\nu_p} = \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \cdots & \delta^{\mu_1}_{\nu_p} \\ \vdots & & \\ \delta^{\mu_p}_{\nu_1} & \cdots & \delta^{\mu_p}_{\nu_p} \end{vmatrix} \qquad (4.4.4)$$

$$p \leq 2n$$

A two-form Ω_2 is said to be non-degenerate, when the matrix $(\Omega_{\mu\nu})$ is nonsingular for all possible bases.

The Poincaré lemma states that every exact (covariant) two-form on M is closed. A two-form is closed when its exterior derivative is identically null, i.e., (locally)

$$d\Omega_2 = \frac{1}{3!} \delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3} \frac{\partial \Omega_{\mu_1\mu_2}}{\partial a^{\mu_3}} da^{\nu_1} \wedge da^{\nu_2} \wedge da^{\nu_3} \equiv 0 \qquad (4.4.5)$$

The necessary and sufficient conditions for a covariant (skewsymmetric) two-form to be closed can than be locally expressed for realizations (1)

$$\Omega_{\mu_1\mu_2} + \Omega_{\mu_2\mu_1} = 0, \qquad (4.4.6a)$$

$$\frac{\partial \Omega_{\mu_1\mu_2}}{\partial a^{\mu_3}} + \frac{\partial \Omega_{\mu_2\mu_3}}{\partial a^{\mu_1}} + \frac{\partial \Omega_{\mu_3\mu_1}}{\partial a^{\mu_2}} = 0. \qquad (4.4.6b)$$

Corresponding conditions exist for realizations (2) and (3).
A covariant two-form Ω_2 is exact when there exist a one-form

$$\Omega_1 = R_\mu da^\mu \qquad (4.4.7)$$

sometimes called primitive form, such that

$$\mathcal{R}_2 = d\,\mathcal{R}_1 = \frac{1}{2!}\delta^{\mu_1\mu_2}_{\nu_1\nu_2}\frac{\partial R_{\mu_1}}{\partial a^{\mu_2}}\,da^{\nu_1}\wedge da^{\nu_2} = \frac{1}{2}\left(\frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial R_\nu}{\partial a^\mu}\right)da^\mu \wedge da^\nu \qquad (4.4.8)$$

The proof of the Poincaré lemma is then trivial because

$$d\mathcal{R}_2 = d(d\mathcal{R}_1) \qquad (4.4.9)$$

$$= \frac{1}{2!\,3!}\delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3}\frac{\partial^2 R_{\mu_1}}{\partial a^{\mu_2}\partial a^{\mu_3}}\,da^{\nu_1}\wedge da^{\nu_2}\wedge da^{\nu_3} \equiv 0$$

where the last identity occurs owing to the totally antisymmetric nature of the generalized Kronecker symbol.

A necessary condition for a skewsymmetric covariant two-form on M to be nondegenerate is that dim(M) is even.

The reader should keep in mind that all nondegenerate skewsymmetric forms (on finite-dimensional spaces) are equivalent up to linear transformations. See in this respect S. Sternberg[151] Chapter 1. For the coordinate-free characterization of the concept of closure we refer the reader to the available literature on differential geometry.

A nondegenerate, covariant, two-form $\mathcal{R}_2$ on M uniquely characterizes a contravariant two-form* $\mathcal{R}^2$ according to the following possibilities.

(1') Tensorial realization.

$$\mathcal{R}^2: \quad \mathcal{R}^{\mu\nu}(a), \quad (\mathcal{R}^{\mu\nu}) \equiv (\mathcal{R}_{\mu\nu})^{-1} \qquad (4.4.10)$$

(2') Matrix realization.

$$\mathcal{R}^2: \quad \mathcal{R}^2 = \{\mathcal{R}_2(e,e)\}^{-1} \qquad (4.4.11)$$

* This form is also called the co-form.

(3') Differential characterization

$$\mathcal{R}^2: \quad \mathcal{R}^2 = \mathcal{R}^{\mu\nu}(a)\,\frac{\partial}{\partial a^\mu}\wedge\frac{\partial}{\partial a^\nu}, \qquad (4.4.12)$$

where $\frac{\partial}{\partial a^\mu}$ is the dual basis of da^μ (Section 4.2).

When a nondegenerate two-form $\mathcal{R}_2$ is closed, there exists (unique) properties of the corresponding contravariant form $\mathcal{R}^2$. For instance, properties (4.4.6) for $\mathcal{R}_2$ yield the properties for $\mathcal{R}^2$.

$$\mathcal{R}^{\mu_1\mu_2} + \mathcal{R}^{\mu_2\mu_1} = 0, \qquad (4.4.13a)$$

$$\mathcal{R}^{\mu_1\rho}\frac{\partial\mathcal{R}^{\mu_2\mu_3}}{\partial a^\rho} + \mathcal{R}^{\mu_2\rho}\frac{\partial\mathcal{R}^{\mu_3\mu_1}}{\partial a^\rho} + \mathcal{R}^{\mu_3\rho}\frac{\partial\mathcal{R}^{\mu_1\mu_2}}{\partial a^\rho} = 0. \qquad (4.4.13b)$$

Under definition (4.4.10), Eqs. (4.4.6) and (4.4.13) are equivalent (Theorem I.2.7.4).

A first geometrical use of (nondegenerate) two-forms is that they allow the mapping from the tangent bundle TM to the cotangent bundle T*M. For a given nondegenerate $\mathcal{R}_2$ on TM there exists a mapping $\mathcal{R}_\flat$: TM → T*M given by

$$\mathcal{R}_\flat(e)\cdot e' = \mathcal{R}(e,e') \qquad (4.4.14)$$

where e, e' ∈ TM.

A second application is the lifting to the cotangent bundle. Let Ξ be a vector field on TM, then we have the mapping

$$\mathcal{R}_\flat\cdot\Xi = \Xi^* \in T^*M \qquad (4.4.15)$$

It is possible to prove that the mapping $\Omega_\flat$:TM → T*M is a vector bundle isomorphism. The inverse mapping is then given by $\Omega_\# = \Omega_\flat^{-1}$.

A third application of (nondegenerate) two-forms is for the raising and lowering of the indices. Let a and a* be local coordinates* for TM and T*M, respectively

Then we can write

$$\Omega_\flat a = a^* \quad , \quad \Omega_\# a^* = a \qquad (4.4.16)$$

and, more generally, we can write for a vector field

$$\Xi^*_\mu = \Omega_{\mu\nu}\Xi^\nu \quad , \quad \Xi^\mu = \Omega^{\mu\nu}\Xi^*_\nu \qquad (4.4.17)$$

If we omit the star, we imply that a^μ and a_μ (Ξ^μ and Ξ_μ) are related by a two-form.

We are now equipped to recall a crucial concept for our analysis, that of symplectic manifold. In line with our objective of keeping in touch with Analytic Mechanics as much as possible, we shall first review a definition which is mostly analytic in nature and then review a more geometric definition.

I. Analytic Approach. When the cotangent bundle T*M is equipped with an ∞-differentiable, nowhere degenerate two-form Ω^2, it is turned into a (Hausdorff, second-countable, ∞-differentiable, 2n-dimensional) manifold M(a, Ω^2). When restricted to exact differentials of functions on M(a, Ω^2), the two-form Ω^2 induces a bilinear composition law which, in local coordinates, can be written[176]

$$\Omega^2(df, dg) = f \,\square\, g = \frac{\partial f}{\partial a^\mu}\Omega^{\mu\nu}(a)\frac{\partial g}{\partial a^\nu} \qquad (4.4.18)$$

* Strictly speaking, a and a* are local coordinates for the fibers on TM and T*M, respectively.

and verifies properties (1.5.4), i.e.,

$$\begin{aligned} f_1 \,\square\, (g_1 + g_2) &= f_1 \,\square\, g_1 + f_1 \,\square\, g_2 \\ (f_1 + f_2) \,\square\, g_1 &= f_1 \,\square\, g_1 + f_2 \,\square\, g_2 \\ (\alpha f_1) \,\square\, g_1 &= f_1 \,\square\, (\alpha g_1) = \alpha (f_1 \,\square\, g_1), \text{etc.} \end{aligned} \qquad (4.4.19)$$

Again, Ω^2 is nowhere degenerate if and only if it is nondegenerate at all points of TM. By writing

$$\Omega^2(f, g) = \lambda_f(g) \qquad (4.4.20)$$

the (unique) covariant two-form Ω_2 which is associated to a nowhere degenerate form Ω^2 is characterized by

$$\Omega_2(\lambda_f, \lambda_g) = \Omega^2(f, g) \qquad (4.4.21a)$$

and induces the local bilinear composition law

$$f \circ g = \Omega_2(df, dg) = \frac{\partial a^\mu}{\partial f}\Omega_{\mu\nu}(a)\frac{\partial a^\nu}{\partial g} \qquad (4.4.21b)$$

as well as the (fiber preserving) transition to the cotangent bundle T*M equipped with Ω_2. The emerging manifold M(a, Ω_2) is called a symplectic manifold (sometimes also called canonical manifold[176]) when structure (4.4.18) satisfies the additional properties

$$J(f, g) = f \,\square\, g + g \,\square\, f = 0 \qquad (4.4.22a)$$

$$J(f, g, h) = (f \,\square\, g) \,\square\, h + (g \,\square\, h) \,\square\, f + (h \,\square\, f) \,\square\, g = 0 \qquad (4.4.22b)$$

The form Ω_2 is then called a symplectic form or symplectic structure (or canonical structure), while the corresponding contravariant form Ω^2 is called cosymplectic form or structure

Conditions (4.4.23) essentially ensure that brackets (4.4.18) satisfy the Lie algebra identities, i.e., they are the <u>generalized Poisson brackets</u>,* and we shall write

$$f \,\square\, g = [f, g]^* \tag{4.4.24}$$

Equivalently, we can say that the manifold M(a, Ω_2) is a symplectic manifold when the elements of the matrix ($\Omega^{\mu\nu}$) = ($\Omega_{\mu\nu}$)$^{-1}$ satisfy all conditions (4.4.13) (Theorem 1.5.3).

It then follows that brackets (4.4.22) are the <u>generalized Lagrange brackets</u> and we shall write

$$f \circ g = \{f, g\}^* \tag{4.4.25}$$

Rule (4.4.10) for the transition from Ω^2 to Ω_2 is then reflected in the properties that the generalized Poisson and Lagrange brackets are each the "inverse" of the other, i.e., the following property

$$\sum_{k=1}^{6N} [f_i, f_k]^* \{f_k, f_j\} = \delta_{ij} \tag{4.4.26}$$

identically holds.

II. <u>Geometrical Approach</u>. Equip, again, TM with an ∞-differentiable, covariant, nowhere degenerate two-form Ω_2. The emerging manifold M(a, Ω_2) is called a <u>symplectic manifold</u> when the two-form Ω_2 is closed.

Clearly, approaches I and II are (locally) equivalent. In fact, the necessary and sufficient conditions for the two-form Ω_2 to be closed in local coordinates, Eqs. (4.4.6), are equivalent to the necessary and sufficient conditions for brackets

* The distinction between the generalized and conventional Poisson and (see below) Lagrange brackets is here introduced mainly for compatibility with ou preceding analytic treatment. It should be indicated that such a distinction is not, in general, made in the full, coordinate-free, geometric treatment of the topic.

(4.4.18) to satisfy identities (4.4.23)(Theorem I.2.7.2). Nevertheless, the reader should be aware that a full geometrical treatment can be achieved within the context of approach II, owing to its coordinate-free nature.

A central application of symplectic manifolds in Analytic Mechanics is that of providing a geometric characterization of the conventional phase space.* This characterization does not, in general, occur within the above given definitions of symplectic manifolds because, for instance, their local variables a^μ $\mu = 1,2,\ldots,2n$ when expressed in terms of n covariant variables y_k and n contravariant variables x^k, do not necessarily characterize a phase space (i.e., y_k and x^k are not necessaily related, via the existence of a Lagrangian, by the rule $y_k = \partial L/\partial x^k$). As a result, the symplectic structure must be reduced to a suitable form to establish the contact with the conventional canonical formulations. Again, we shall review this reduction along two approaches, one analytic and one geometrical.

I'. <u>Analytic Approach</u>. The brackets which occur in the customary treatment of Analytic Mechanics are not the generalized Poisson brackets, but instead the conventional Poisson and Lagrange brackets

$$[f, g] = \frac{\partial f}{\partial x^k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial x^k}, \tag{4.4.27a}$$

$$\{f, g\} = \frac{\partial p_k}{\partial f}\frac{\partial x^k}{\partial g} - \frac{\partial x^k}{\partial f}\frac{\partial p_k}{\partial g}, \tag{4.4.27b}$$

* We here tacitly imply that the manifold M(a, Ω_2) is diffeomorphic to a cotangent bundle.

where the variables x^k and p_k are canonically conjugated. An inspection indicates that these brackets can be written in the form

$$[f, g] = \frac{\partial f}{\partial a^\mu}\,\omega^{\mu\nu}\,\frac{\partial g}{\partial a^\nu}, \qquad (4.4.28a)$$

$$\{f, g\} = \frac{\partial a^\mu}{\partial f}\,\omega_{\mu\nu}\,\frac{\partial g}{\partial a^\nu}, \qquad (4.4.28b)$$

where the underlying tensor $\omega^{\mu\nu}$ and $\omega_{\mu\nu}$ can be represented in terms of the matrices

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix} = (\omega_{\mu\nu})^{-1} = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ 1_{n\times n} & 0_{n\times n} \end{pmatrix}^{-1} \qquad (4.4.29)$$

The corresponding two-forms ω_2 and ω^2 characterized by $(\omega_{\mu\nu})$ and $(\omega^{\mu\nu})$, respectively, are often called the fundamental symplectic and cosymplectic forms, respectively. The fact that the form ω_2 is symplectic is established by the fact that, for instance, brackets (4.4.28) are a particular case of brackets (4.4.24), i.e., they are Lie brackets.

The geometrical characterization of the conventional phase space with symplectic manifolds demands the reduction of an arbitrary symplectic form $\mathcal{R}_2$ the fundamental form ω_2. This can be achieved with the Pauli Theorem (in the terminology of ref.[176]).

THEOREM 4.4.1: Given a symplectic form $\mathcal{R}_2$ on a 2n-dimensional manifold $M(a, \mathcal{R}_2)$ with local coordinates a^μ, $\mu = 1,2,\ldots,2n$, there always exists a diffeomorphism $\mathscr{C}: M(a, \mathcal{R}_2) \to M(a', \omega_2)$ realizable through class C^∞, everywhere invertible

transformations

$$a^\mu \to a'^\mu = a'^\mu(a) \qquad (4.4.30)$$

under which the form $\mathcal{R}_2$ reduces to the fundamental symplectic form ω_2,

$$\mathscr{C}: \mathcal{R}_2 \to \omega_2 \qquad (4.4.31)$$

To see this property, recall that the matrix $(\mathcal{R}_{\mu\nu})$ is everywhere regular and, thus, it always admits the factorization

$$(\mathcal{R}_{\mu\nu}(a)) = (h_\mu{}^\rho(a)\,\omega_{\rho\sigma}\,h_{\sigma\nu}(a)) = (h(a))\,(\omega)\,(h(a))^{-1} \qquad (4.4.32)$$

Its elements transform under (4.4.30) according to

$$\mathcal{R}_{\mu\nu} \to \mathcal{R}'_{\mu\nu} = \frac{\partial a^\rho}{\partial a'^\mu}\,\mathcal{R}_{\rho\sigma}(a(a'))\,\frac{\partial a^\sigma}{\partial a'^\nu} \qquad (4.4.33)$$

Reduction (4.4.31) then simply occurs by selecting transformations (4.4.30) in such a way that

$$\left(\frac{\partial a}{\partial a'}\right) = (h)^{-1} \qquad (4.4.34)$$

In conclusion, every symplectic manifold can always be reduced through suitably selected diffeomorphisms to a form $M(a, \omega_2)$ in which: (a) the local coordinates $a = \{x, p\}$ are canonically conjugated, (b) the symplectic structure is the fundamental form (4.4.29), and (c) the underlying brackets are the conventional

Poisson brackets.

II. Geometrical Approach. The reduction of an arbitrary symplectic form Ω_2 to the fundamental form ω_2 is given by Darboux's Theorem[171].

THEOREM 4.4.2: Suppose that Ω is a non-degenerate two-form on a manifold. Then the form Ω is symplectic, i.e., $d\Omega = 0$, if and only if there exists a chart (U, φ) at each point $m \in M$ such that $\varphi(m) = 0$ and with $\varphi(u) = \{x^1(u), x^2(u), \ldots, x^n(u), p_1(u), p_2(u), \ldots, p_n(u)\}$ the form Ω can be reduced to the canonical form $\omega = dp_k \wedge dx^k$.

Alternatively, we can say that in a symplectic manifold $M(a, \Omega_2)$ there always exists a chart (ϑ, φ) at each point $m \in M$ such that for

$$\varphi(m) = 0 , \quad \varphi(t) = \{x(t), p(t)\} \tag{4.4.35}$$

we have

$$(\vartheta, \varphi) : \Omega_2 \rightarrow \Omega_2 \big| \vartheta = dp_k \wedge dx^k \tag{4.4.36}$$

Once written in matrix notation, Eqs. (4.4.36) imply

$$(\omega_{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ 1_{n\times n} & 0_{n\times n} \end{pmatrix} . \tag{4.4.37}$$

We reach in this way the inverse of matrix (4.4.29), in accordance with rules (4.4.2) or (4.4.3). In actuality the matrix (4.4.29) is unimodular, antisymmetric and orthogonal. Thus, its inverse coincides with its transpose.

The use of a Darboux chart for the generalized Lagrange brackets yields their reduction to the conventional Lagrange brackets

$$(\vartheta, \varphi) : \{f, g\}^* \rightarrow \{f, g\} \tag{4.4.38}$$

and a similar situation occurs for the Poisson brackets. In particular, property (4.4.26) is invariant under a Darboux chart, i.e.,

$$(\vartheta, \varphi) : \sum_{k=1}^{m} [f_i, f_k]^* \{f_k, g_j\}^* \rightarrow \rightarrow \sum_{k=1}^{m} [f_i, f_k] \{f_k, g_j\} = \delta_{ij} \tag{4.4.39}$$

In essence, Darboux's Theorem constitutes the geometric counterpart of local property (4.4.31), which, according to Pauli's Theorem, can be proved with ordinary transformation theory.

The geometric characterization of the phase space which is offered by Darboux's or Pauli's Theorem is still insufficient for the canonical characterization of Newtonian systems. This latter step demands the geometrical interpretations of the analytic equations.

I". Analytic Approach. Consider a Newtonian system in a general covariant first-order form in $M(a, C_2)$, Eqs. (4.3.28), i.e.,

$$\left[C_{\mu\nu}(a)\,\dot{a}^{\nu} + D_{\mu}(a)\right]^{C^{\infty},R} = 0 \qquad (4.4.40)$$

The tensor $C_{\mu\nu}$ of this system does not necessarily characterize a closed two-form. However, the family of equivalent forms

$$\left\{h_{\mu}{}^{\nu}\left[C_{\nu\rho}\,\dot{a}^{\rho} + D_{\mu}\right]^{C^{\infty},R}\right\}^{C^{\infty},R} = \left[C'_{\mu\rho}\,\dot{a}^{\rho} + D'_{\mu}\right]^{C^{\infty},R} = 0 \qquad (4.4.41)$$

always admits at least one element $C'_{\mu\nu}$ such that the related two-form C'_2 is closed, i.e., satisfy Eqs. (4.4.6).

This property can be proved by interpreting Eqs. (4.4.6) as a quasilinear system of partial differential equations in the unknown functions $h_{\mu}{}^{\nu}$ and prescriptions for the y's. A study of this system then indicates that it is consistent, namely, a solution always exists. For details see P. Havas,[123] Appendix B, W. Sarlet and F. Cantrijn,[119] and R.M. Santilli.[25,65,189]

It then follows that Eqs. (4.4.40) always admit an indirect analytic representation in terms of Birkhoff's equations

$$\Omega_{\mu\nu}(a)\,\dot{a}^{\nu} - \frac{\partial B}{\partial a^{\mu}} = 0, \quad \Omega_{\mu\nu} = \frac{\partial R_{\mu}}{\partial a^{\nu}} - \frac{\partial R_{\nu}}{\partial a^{\mu}} \qquad (4.4.42)$$

This is the Step 2 of the proof of the Theorem of Indirect Universality of the Inverse Problem as outlined in Section I.2.8.

The geometrical significance of Birkhoff's equations is now self-evident. They characterize the most general form of the analytic equations (for autonomous systems) which is embodied in

a symplectic manifold. In fact, by central assumption, the tensor $\Omega_{\mu\nu}$ of Birkhoff's equations satisfy Eqs. (4.4.6). Thus, Ω_2 is closed.

The geometric significance of Pauli's Theorem can now be identified. Theorem 4.4.1 essentially indicates that there always exists a transformation (4.4.30) under which Birkhoff's equations reduce to Hamilton's equations, i.e.,[119]

$$\Omega_{\mu\nu}(a)\,\dot{a}^{\nu} - \frac{\partial B(a)}{\partial a^{\mu}} = \frac{\partial a'^{\rho}}{\partial a^{\mu}}\left[\omega_{\rho\sigma}\,\dot{a}'^{\sigma} - \frac{\partial H(a')}{\partial a'^{\rho}}\right], \quad H(a') = B(a) \qquad (4.4.43)$$

This is the third and last step of the proof of the Theorem of Indirect Universality of the Inverse Problem (Section I.2.8). [*] In addition, it is also possible to prove that there exists a transformation (4.4.30) such that the Hamiltonian is $H = T = \frac{1}{2m_k} p'_k \cdot p'_k$. This is a reduction of a Newtonian system of particles which are interacting in the space of its experimental detection to an equivalent system of free particles in a new representation space.

In conclusion, all class C^{∞}, regular Newtonian systems can be embodied into a symplectic manifold $M(a, \Omega_2)$ whose underlying general analytic equations are Birkhoff's equations. The local variables, however, are not necessarily canonically conjugate and, thus, they do not necessarily span a phase space. Nevertheless, there always exists a diffeomorphism $\mathcal{C}: M(a, \Omega_2) \to M(a', \omega_2)$ under which Birkhoff's equations reduce to Hamilton's equations in which case the local variables span a phase space.

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[*] NOTE ADDED IN 1982: a detailed proof of the theorem is presented in ref.[189].

If the system is conservative, the phase space variables $\{a^\mu\}$ can be the physical variables $\{r^{ka}, p_{ka}\}$, $p_{ka} = m_k \dot{r}_{ka}$. If the system is nonconservative the phase space variables never coincide with the physical variables.

II". Geometric Approach. Let $M(a, \Omega_2)$ be a 2n-manifold with a nowhere degenerate, closed two-form Ω_2. Let A_p be a covariant p-form in $M(a, \Omega_2)$, i.e.,

$$A_p = A_{\mu_1 \mu_2 \cdots \mu_p}(a)\, da^{\mu_1} \wedge da^{\mu_2} \wedge \cdots \wedge da^{\mu_p} \quad (4.4.44)$$

$$A_{\mu_1 \mu_2 \cdots \mu_p} \in C^\infty$$

A 0-form is then a scalar, i.e., $A_0 = \rho(a)$. The inner product of A_p with a contravariant vector field $\Xi^\mu(a)$ is defined as the (p-1)-form[120]

$$\Xi \lrcorner A_p = A_{(p-1)} \quad (4.4.45)$$

$$= \frac{1}{(p-1)!} \delta^{\nu_1 \nu_2 \cdots \nu_p}_{\mu_1 \mu_2 \cdots \mu_p} A_{\nu_1 \nu_2 \cdots \nu_p} \Xi^{\mu_p} da^{\mu_1} \wedge da^{\mu_2} \wedge \cdots \wedge da^{\mu_{p-1}}$$

with the particular case

$$\Xi \lrcorner A_0 = \Xi \lrcorner \rho(a) \equiv 0 \quad (4.4.46)$$

Suppose now that the A_p form is the symplectic form Ω_2. Then its inner product with Ξ^μ yields

$$\Xi_1^{\Omega_2} = \Xi \lrcorner \Omega_2 = \Omega_{\mu\nu} \Xi^\nu da^\mu = \Xi_\mu da^\mu \quad (4.4.47)$$

and it is often written in the alternative notation[171]

$$i_\Xi \Omega_2 \equiv \Omega_\Xi \equiv \Xi_1^{\Omega_2} \equiv \Xi \lrcorner \Omega_2 \quad (4.4.48)$$

We shall say that a vector field Ξ is Birkhoffian, when the 1-form $\Xi_1^{\Omega_2}$ is exact, i.e., at a point $m \in M(a, \Omega_2)$ there exists a neighborhood $N(m)$ and a function $B(a)$ on $N(m)$ such that

$$\Xi_1^{\Omega} = \Omega_{\mu\nu} \Xi^\nu da^\mu = \Xi_\mu da^\mu = dB = \frac{\partial B}{\partial a} da. \quad (4.4.49)$$

where $\Omega_{\mu\nu}$ is an exact , symplectic, noncanonical (Birkhoffian) form. Thus, the vector field Ξ^μ, under condition (4.4.49), admits the representation in terms of Birkhoff's equations

$$\Xi^\mu \equiv \Omega^{\mu\nu}(a) \frac{\partial B(a)}{\partial a^\nu}, \quad (4.4.50)$$

or, more generally,

$$\dot{a}^\mu - \Xi^\mu(a) \equiv \dot{a}^\mu - \Omega^{\mu\nu}(a) \frac{\partial B(a)}{\partial a^\nu}. \quad (4.4.51)$$

Consider now the symplectic manifold $M(a, \omega_2)$. Then for the inner product of a vector field Ξ^μ on this manifold and its exact, symplectic, canonical (fundamental) form, we have

$$\Xi_1^{\omega_2} = \Xi \lrcorner \omega_2 = \omega_{\mu\nu} \Xi^\nu da^\mu = \Xi_\mu da^\mu. \quad (4.4.52)$$

The vector field Ξ^μ on $M(a, \omega_2)$ is called Hamiltonian when the 1-form $\Xi_1^{\omega_2}$ is exact, i.e.,

$$\Xi_1^{\omega_2} = \Xi_\mu da^\mu = dH = \frac{\partial H}{\partial a^\mu} da^\mu. \quad (4.4.53)$$

In this case the vector field Ξ^μ admits a representation in terms of Hamilton's equations, i.e.,

$$\Xi^\mu = \omega^{\mu\nu} \frac{\partial H^H}{\partial a^\nu}, \qquad (4.4.54)$$

or, more generally,

$$\dot{a}^\mu - \Xi^\mu(a) \equiv \dot{a}^\mu - \omega^{\mu\nu} \frac{\partial H^H(a)}{\partial a^\nu}. \qquad (4.4.55)$$

Notice the similarities as well as the differences between Birkhoffian and Hamiltonian vector fields. In both cases the symplectic form plays the fundamental role of lowering the index of the vector field $\Xi^\mu(a)$. Thus, the two cases are equivalent within the context of the calculus of exterior form. The only difference is given by the fact that, for the Birkhoffian case, the form Ω_2 is a general (exact) symplectic form, while for the Hamiltonian case the symplectic form is the fundamental form ω_2.

Our unified vectorial notation $\{a^\mu\}$ for the phase space variables r^{ka} and p_{ka} does not appear to be used in currently available treatements of symplectic geometry, to the best of my knowledge. It has been selected because it allows the study of the geometric equivalence of Birkhoff's and Hamilton's equations in a more transparent way. in fact, , while the latter equations can be easily written in the separate variables r^{ka} and p_{ka} the corresponding formulation for the former equations is considerable more tedius.

Another reason for the selection of the unified notation $\{a^\mu\}$ is that it allows the study of the self-adjointness of Hamilton's and Birkhoff's equations which is of rather nontrasparent nature when written in the separate r^{ka} and p_{ka} variables. In turn, this allows

the study of the relationship between the analytic, algebraic and geometrical aspects in local variables, as we shall indicate later on.

But there is an additional reason for the preference of the unified $\{a^\mu\}$ notation. It is related to the equivalence of Birkhoff's and Hamilton's equations on geometric grounds, that is, the fact that both Hamilton's form ω_2 and Birkhoff's form Ω_2 are exact. Let us first review the case in conventional notation and then indicate the equivalence with our notation. The one-form

$$\theta_1 = p_{ka}\, dr^{ka} \qquad (4.4.56)$$

is called <u>canonical form</u>. It possesses a fundamental significance in symplectic geometry because its exterior derivative yields the fundamental symplectic form as appearing in Darboux's Theorem, Eq. (4.4.36), i.e.,

$$\theta_2 = d\theta_1 = dp_{ka} \wedge dr^{ka} \qquad (4.4.57)$$

Thus, the fundamental symplectic form θ_2 is exact.

The same results can be easily expressed in terms of the vectorial notation $\{a^\mu\} = \{r^{ka}, p_{ka}\}$. Introduce the one-form

$$\omega_1 = R^o_\mu\, da^\mu, \quad (R^o_\mu) = (\underset{\sim}{p}, \underset{\sim}{0}) \qquad (4.4.58)$$

This form is evidently the canonical one form

$$\omega_1 = p_{ka}\, dr^{ka} \equiv \theta_1 \qquad (4.4.59)$$

Thus, the form ω_1 is such that its exterior derivative coincides with the fundamental symplectic form

$$d\omega_1 = \frac{1}{2}\left(\frac{\partial R^o_\nu}{\partial a^\mu} - \frac{\partial R^o_\mu}{\partial a^\nu}\right) da^\mu \wedge da^\nu = dp_{k\alpha} \wedge dr^{k\alpha} \qquad (4.4.60)$$

The equivalence of the above context with the corresponding generalization for Birkhoff s equations can now be easily indicated. Introduce the one-form

$$\Omega_1 = R_\mu(a)\, da^\mu \; ; R_\mu \neq R^o_\mu \qquad (4.4.61)$$

where the R's are the functions of Birkhoff's tensor

$$\Omega_{\mu\nu} = \frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial R_\nu}{\partial a^\mu} \qquad (4.4.62)$$

Then the exterior derivative of Ω_1 yields the two-form

$$\Omega_2 = d\Omega_1 = \frac{\partial R_\mu}{\partial a^\nu} da^\nu \wedge da^\mu = \frac{1}{2}\left(\frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial R_\nu}{\partial a^\mu}\right) da^\mu \wedge da^\nu \qquad (4.4.63)$$

which is precisely the exact symplectic form of Birkhoff's equations. Thus, the more general Ω_2 form is exact in a way fully parallel to the case of the ω_2 form.*

The fact that Birkhoff's equations constitute a symplectic covering of Hamilton s equations is then established by the property that, under the particular values

$$R_\mu(a) = R^o_\mu(a), \qquad (4.4.64)$$

the general symplectic form Ω_2 reduces to the fundamental form ω_2.

* Again, the reader should be aware that we are here not treating the general, coordinate-free symplectic context, but only that part with a direct connection with our analytic treatment of dynamics, that is, that related to Hamilton's and Birkhoff's equations.

In conclusion, the transition from a Hamiltonian to a Birkhoffian vector field is characterized by the transition of the lowering tensor from the fundamental tensor $\omega_{\mu\nu}$ to the general symplectic tensor $\Omega_{\mu\nu}$. If the contracted form $\Xi_1^{\omega_2}$ is exact, one recovers the necessary and sufficient condition for the existence of a Hamiltonian as identified by the Inverse Problem, Eqs. (I.2.6.11); i.e.,

$$\frac{\partial \Xi_\mu}{\partial a^\nu} - \frac{\partial \Xi_\nu}{\partial a^\mu} = 0 \qquad (4.4.65)$$

Suppose now that these conditions (under the $\omega_{\mu\nu}$ tensor) are violated. This does not preclude the existence of a general symplectic tensor $\Omega_{\mu\nu}$ under which the same conditions holds. In this case the vector fields are only locally Hamiltonian (Birkhoffian in our terminology). The meaning of the terms "locally Hamiltonian" can now be better identified. In essence they express the property guaranteed by the Darboux's charts that any locally Hamiltonian vector field can always be transformed into a Hamiltonian vector field in a new set of local coordinates.

The geometrical significance of the conditions of variational selfadjointness of first-order, ordinary, differential equations can now be easily identified. These conditions for systems (4.4.40) are given by Eqs. (I.2.7.2), i.e.,

$$\Omega_{\mu_1\mu_2} + \Omega_{\mu_2\mu_1} = 0, \qquad (4.4.66$$

$$\frac{\partial \Omega_{\mu_1\mu_2}}{\partial a^{\mu_3}} + \frac{\partial \Omega_{\mu_2\mu_3}}{\partial a^{\mu_1}} + \frac{\partial \Omega_{\mu_3\mu_1}}{\partial a^{\mu_2}} = 0, \qquad (4.4.66b)$$

$$\frac{\partial D_{\mu_1}}{\partial a^{\mu_2}} - \frac{\partial D_{\mu_2}}{\partial a^{\mu_1}} = 0. \qquad (4.4.66c)$$

The following three equivalent meanings (in local coordinates) then holds.

(A) Analytic significance of the conditions of variational selfadjointness. They guarantee the existence of an analytic representation of system (4.4.44) in terms of Birkhoff's equations in general (Theorem 2.B.1), i.e.,

$$\left[\Omega_{\mu\nu}\dot{a}^{\nu} + D_{\mu}\right]_{SA} \equiv \left[\Omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial B}{\partial a^{\mu}}\right]_{SA}, \quad (4.4.67)$$

or in terms of Hamilton s equations, in particular, for $\Omega_{\mu\nu} \equiv \omega_{\mu\nu}$ (Theorem I.2.6.3), i.e.,

$$\left[\omega_{\mu\nu}\dot{a}^{\nu} + D_{\mu}\right]_{SA} \equiv \left[\omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial H}{\partial a^{\mu}}\right]_{SA}, \quad (4.4.68)$$

that is, they are the integrability conditions for canonical formulations.

(B) Algebraic significance of the conditions of variational selfadjointness. They guarantee that the brackets of the time evolution law of the canonical aequations are lie Lie brackets , that is, they are the generalized Poisson brackets for Birkhoff's equations, or the conventional Poisson brackets for Hamilton's equations.

(C) Geometrical significance of the conditions of variational selfadjointness. They guarantee that the vector fields are locally Hamiltonian (Birkhoffian) for in general, or Hamiltonian for $\Omega_{\mu\nu} \equiv \omega_{\mu\nu}$.

This illustrates the symbiotic characterization of the variational approach to selfadjointness of certain elemental analytic, algebraic and geometrical aspects of mechanics.

The geometrical meaning of the conditions of variational selfadjointness brings into focus in a natural way the possible existence of a geometrical image of the notion of algebraic isotopy (Sections 1.2 and 1.5). We here define as symplectic isotopic mapping any C^{∞}, invertible mapping of a symplectic structure which preserves its symplectic character.* This is clearly the geometric counterpart of the Lie isotopic mapping of a Lie product, e.g., the transition from the conventional to the generalized Poisson brackets and viceversa. Thus, the transition from the fundamental to a general symplectic structure, or viceversa, is a symplectic isotopy.

Clearly, the notion of symplectic isotopy is inclusive of that of Pauli's transformations.

The emphasis in these latter notions, however, is in the coordinate transformations, while the isotopy occurs irrespective of whether the coordinates are transformed or not.

A typical example is given by

$$\theta_1 = p_{k a}\, dz^{k a} \longrightarrow \omega'_1 = -p_{k a}\, dz^{k a} + z^{k a}\, dp_{k a} \quad (4.4.69)$$

This mapping occurs within the same local variables by construction. Neverthless, it is such to preserve the fundamental symplectic form (up to a numerical multiplicative constant). As such, mapping (4.4.69) is a simple example of symplectic isotopy.

Within a fixed system of local coordinates, we shall therefore write the symplectic isotopy in the form

* The condition that the mapping does not change the cohomology class should be here included. Regrettably, this more adequate approach goes beyond the rudimentary treatment of this volume.

$$\begin{pmatrix} \Omega_2 = \\ \Omega_{\mu\nu}(a)\, da^\mu \wedge da^\nu \end{pmatrix} \longrightarrow \begin{pmatrix} \Omega'_2 \\ = \Omega'_{\mu\nu}(a)\, da^\mu \wedge da^\nu \end{pmatrix} \quad (4.4.70)$$

If the original form is exact, the isotopy is assumed to preserve this character, and we shall write

$$\Omega_2 = d\left(R_\mu(a)\, da^\mu\right) \longrightarrow \Omega'_2 = d\left(R'_\mu(a)\, da^\mu\right) \quad (4.4.71)$$

In the next section, we shall consider a possible generalization of the above notion of isotopy into that of <u>geometric genotopy</u>. We are referring here to a mapping that, this time, does not preserve the symplectic character by construction, i.e., to a mapping of the type

$$\begin{pmatrix} \omega_2 \\ = \omega_{\mu\nu}\, da^\mu \wedge da^\nu \\ = dp_k \wedge dz^k \end{pmatrix} \longrightarrow \begin{pmatrix} s_2 \\ = \omega_{\mu\nu}\, da^\mu \wedge da^\nu + t_{\mu\nu}\, da^\mu \times da^\nu, \\ t_{\mu\nu} = t_{\nu\mu} \end{pmatrix} \quad (4.4.72)$$

which, as we shall see, induces a fundamental symplectic-admissible structure.

It is evident that the geometric notions of isotopy and genotopy are sumbitted here in such a way to follows as closely as possible the corresponding algebraic notions worked out in Chapters 1 and 2.

To complete our analysis, we introduce now the Lie derivative. Let $G = G_a(t)$ be a one-parameter Lie group of transformations on $M(a, \Omega_2)$ and let X be its generator. Let also F(a) be a differentiable function on $M(a, \Omega_2)$. The <u>Lie derivative</u> of the

function F(a) can be defined as follows[178]

$$\mathcal{L}_X F = \lim_{t \to 0} \frac{F \circ G_a(t) - F \circ G_a(0)}{t} = XF \quad (4.4.73)$$

On similar grounds, the <u>Lie derivative of a covariant p-form</u> on $M(a, \Omega_2)$ can be defined by

$$\mathcal{L}_X A_p = \lim_{t \to 0} \frac{A_p \circ G_a(t) - A_p \circ G_a(0)}{t}, \quad G_a(0) = 1 \quad (4.4.74)$$

where

$$A_p \circ G_a(t) = \left(A_{\mu_1 \ldots \mu_p} \circ \psi\right)\left(\psi^* da^{\mu_1}\right) \wedge \ldots \wedge \left(\psi^* da^{\mu_p}\right) \quad (4.4.75)$$

It is possible to prove that

$$\mathcal{L}_X A_p = X \lrcorner\, dA_p + d\left(X \lrcorner\, A_p\right) \quad (4.4.76)$$

Thus, if the A_p form is closed,

$$\mathcal{L}_X A_p = d\left(X \lrcorner\, A_p\right) \quad (4.4.77)$$

Let $\Xi^\mu(a)$ be a vector field in the notation of Section 4.3. Suppose that such vector field is complete i.e., it characterizes the generator of a one-parameter connected Lie group according to

$$e^{\theta X} = e^{\theta \Omega^{\mu\nu} \Xi_\nu \frac{\partial}{\partial a^\mu}} = e^{\theta \Xi^\mu \frac{\partial}{\partial a^\mu}} \quad (4.4.78)$$

Then the Lie derivative can be redefined in terms of Ξ and we shall write

$$\mathcal{L}_{\Xi}\,\Omega_2 = d\,(\Xi \rfloor \Omega_2) \qquad (4.4.79)$$

where Ω_2 is a symplectic structure. The vector field Ξ^μ is then called locally Hamiltonian when Ω_2 is an invariant two-form of Ξ^μ, i.e.,

$$\mathcal{L}_{\Xi}\,\Omega_2 = 0 \qquad (4.4.80)$$

The <u>time evolution law</u> can then be written

$$\mathcal{L}_{\Xi} = XF = \Xi^\mu \frac{\partial}{\partial a^\mu} F = \frac{\partial F}{\partial a^\mu}\,\Omega^{\mu\nu}\,\frac{\partial B}{\partial a^\nu} = [F, B]^* \qquad (4.4.81)$$

yielding, as expected, the time evolution law for Birkhoff's equations.

However, if Ξ^μ is Hamiltonian, we have

$$\mathcal{L}_{\Xi} F = XF = \Xi^\mu \frac{\partial}{\partial a^\mu} F = \frac{\partial F}{\partial a^\mu}\,\omega^{\mu\nu}\,\frac{\partial H}{\partial a^\nu} = [F, H] \qquad (4.4.82)$$

by recovering the time evolution law of Hamilton's equations.

For completeness we now outline the case of <u>nonautonomous systems</u>. The underlying differential equations in this case are given by

$$C_{\mu\nu}(t,a)\,\dot{a}^\nu + D_\mu(t,a) = 0 \qquad (4.4.83)$$

The tensor $C_{\mu\nu}$ also induces a geometric structure but now in the (2n+1)-dimensional space $T^*M \times \mathbb{R}$ where the real line $\mathbb{R}$ is representative of the time variable (Section 4.3). By introducing the notation

$$\{\hat{a}^i\} = \{t, a^\mu\}, \quad i = 0, 1, 2, \ldots, 2n$$

$$\hat{\Omega}_{i0} = -\hat{\Omega}_{i0} = D_i, \quad \hat{\Omega}_{ij} = \Omega_{ij}, \quad i,j = 1, 2, \ldots, 2n \qquad (4.4.84)$$

we can write the extension of form (4.4.3) on $T^*M \times \mathbb{R}$ as follows

$$\hat{\Omega}_2 = \hat{\Omega}_{ij}\, d\hat{a}^i \wedge d\hat{a}^j \qquad (4.4.85)$$

with underlying closure conditions

$$\hat{\Omega}_{ij} + \hat{\Omega}_{ji} = 0, \qquad (4.4.86a)$$

$$\frac{\partial \hat{\Omega}_{ij}}{\partial \hat{a}^k} + \frac{\partial \hat{\Omega}_{jk}}{\partial \hat{a}^i} + \frac{\partial \hat{\Omega}_{ki}}{\partial \hat{a}^j} = 0. \qquad (4.4.86b)$$

The form $\hat{\Omega}_2$ is said to be of <u>maximal rank</u> (2n) when its restriction Ω_2 to TM is nowhere degenerate.

When the space $T^*M \times \mathbb{R}$ is equipped with an ∞-differentiable contravariant two-form $\hat{\Omega}_2$ of maximal rank, it is turned into a (Hausdorff, ∞-differentiable, second countable, (2n+1)-dimensional) manifold $M(\hat{a}, \hat{\Omega}_2)$. This manifold is called a <u>contact manifold</u> when the two form $\hat{\Omega}_2$ is closed in which case $\hat{\Omega}_2$ is called a <u>contact form</u> or <u>structure</u>. This is the extension of the notion of symplectic manifold to the nonautonomous case. We leave it as an exercise for the interested reader the proof that brackets (4.4.18) can still be introduced and, for a contact manifold, they are Lie.

A contact manifold can also be "constructed" starting from a symplectic manifold. Let $M(a, \Omega_2)$ be a manifold of this type.

Consider the product manifold $M(a, \Omega_2) \times \mathbb{R}$ and introduce the mapping

$$\Pi : M(a,\Omega_2)\times R \rightarrow M(a,\Omega_2) \qquad (4.4.87)$$

Then the manifold $(M \times \mathbb{R})(\hat{a}, \hat{\Omega}_2)$ with $\hat{\Omega}_2 = \Pi_* \Omega_2$ is a contact manifold and $\hat{\Omega}_2$ a contact structure. If Ω_2 is canonical

$$\Omega_2 = \omega_2 = d\theta, \quad \theta_1 = p_{ka} dz^{ka} \rightarrow \hat{\theta}_1 = dt + \Pi_* \theta \qquad (4.4.88)$$

and the underlying analytic equations are Hamilton's equations. The Hamiltonians now have an explicit dependence on time. If Ω_2 is symplectic, then (locally) the analytic equations are given by the complete form of the nonautonomous Birkhoff's equations

$$\left[\left(\frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial R_\nu}{\partial a^\mu}\right)\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right]_{SA} = 0 \qquad (4.4.89)$$

In this case the system is determined by 2n+1 quantities, the 2n functions R_μ and B, with the "gauge degrees of freedom"

$$R_\mu \rightarrow R_\mu - \frac{\partial G}{\partial a^\mu}, \quad B \rightarrow B + \frac{\partial G}{\partial t} \qquad (4.4.90)$$

The following property has been proved by W. Sarlet and F. Cantrijn.[119]

THEOREM 4.4.3: Eqs. (4.4.67) are locally Hamiltonian in

the sense that there always exists a local diffeomorphism under which they become Hamilton's equations for nonautonomous systems.

The above property implies the following generalization of Pauli's theorem for the contact (nonautonomous) case.

THEOREM 4.4.4: Given a contact form $\hat{\Omega}_2$ on a (2n+1)-dimensional manifold $M(\hat{a}, \hat{\Omega}_2)$ with local coordinates $\hat{a}^i$, $i = 0, 1, 2, \ldots 2n$, $a^0 = t$, there always exists a diffeomorphism $\hat{\varphi} : M(\hat{a}, \hat{\Omega}) \rightarrow M(\hat{a}', \hat{\omega}_2)$ realizable through class C^∞, everywhere invertible transformations

$$\hat{a}^i \rightarrow \hat{a}'^i = \hat{a}'^i(\hat{a}) = \hat{a}'^i(t, a) \qquad (4.4.91)$$

under which the form $\hat{\omega}_2$, when restricted to T^*M coincides with the fundamental symplectic form ω_2.

$$\hat{\varphi} : \hat{\Omega}_2 \rightarrow \hat{\Omega}'_2\Big|_{T^*M} = \omega_2 \qquad (4.4.92)$$

The geometric counterpart of this theorem can be formulated as follows[171]

THEOREM 4.4.5: In any contact manifold $M(\hat{a}, \hat{\Omega}_2)$ there always exists a contact chart $(\hat{\theta}, \hat{\varphi})$ at each $\hat{m} \in M$ with

$$\hat{\varphi}(\hat{m}) = \{t(m), \tau(m), p(m)\} \qquad (4.4.93)$$

such that

$$\hat{\Omega}_2 \Big| \vartheta = dp_{k\hat{a}} \wedge dz^{ka} \qquad (4.4.94)$$

If the form $\hat{\Omega}_2$ is exact, then there always exists a contact chart such that

$$\hat{\Omega}_2 = d\hat{\theta}_1 \ , \quad \hat{\theta}_1 = dt + p_{ka}\, dz^{ka} \qquad (4.4.95)$$

We are now in a position to clarify the geometrical significance of the conditions of variational self-adjointness for the nonautonomous case, i.e., the equations (Section I.2.7)

$$C_{\mu\nu} + C_{\nu\mu} = 0 \ , \qquad (4.4.96a)$$

$$\frac{\partial C_{\mu\nu}}{\partial a^\rho} + \frac{\partial C_{\nu\rho}}{\partial a^\mu} + \frac{\partial C_{\rho\mu}}{\partial a^\nu} = 0 , \qquad (4.4.96b)$$

$$\frac{\partial C_{\mu\nu}}{\partial t} = \frac{\partial D_\mu}{\partial a^\nu} - \frac{\partial D}{\partial a} \ , \qquad (4.4.96c)$$

In essence, for the autonomous case, Eqs. (4.4.96a) and (4.4.96b) constitute necessary and sufficient conditions for the underlying two-form C_2 to be closed, Eqs. (4.4.6). Thus, in this case they (locally) guarantee a symplectic structure.

For the nonautonomous case the full set of conditions (4.4.96) is needed from a geometric profile. But, under identifications (4.4.84) conditions (4.6.96) coincide with the necessary and sufficient conditions for the extended two-form (4.4.85) to be closed, Eqs. (4.4.86). Thus, Eqs. (4.4.96) (locally) guarantee a contact structure.

This illustrates the viewpoint of Section I.2.7, to the effect that the conditions of variational self-adjointness represent (locally) a symbiotic, characterization of certain elemental aspects of analytic mechanics, Lie algebras, and symplectic (or more general, contact) geometry. As such, they constitute a valuable arena for the study of the deep interrelationships among these disciplines.

4.5: THE MAIN IDEAS OF THE SYMPLECTIC-ADMISSIBLE COVERING OF THE SYMPLECTIC GEOMETRY

As by now familiar, the objective of this monograph is the study of open (nonconservative), nonself-adjoint, Newtonian systems

$$\left\{\left[m_k \ddot{r}_{ka} - f_{ka}(t, r, \dot{r})\right]_{SA} - F_{ka}(t, r, \dot{r})\right\}_{NSA} = 0 \qquad (4.5.1)$$

$$k = 1, 2, \ldots, N, \quad a = x, y, z$$

in terms of Birkhoff-admissible equations (Section 2.5), either in their contravariant form(*)

$$\dot{b}^\mu = S^{\mu\nu}(t,b)\frac{\partial B(t,b)}{\partial b^\nu}, \quad \mu = 1, 2, \ldots, 6N, \qquad (4.5.2a)$$

$$S^{\mu\nu} = \left(\left\|\frac{\partial R_\alpha}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\alpha}\right\|^{-1}\right)^{\mu\nu} + T^{\mu\nu}, \qquad (4.5.2b)$$

$$(b^\mu) = \begin{pmatrix} r^{ka} \\ p_{ka} \end{pmatrix}, \quad T^{\mu\nu} = T^{\nu\mu}, \qquad (4.5.2c)$$

or in their equivalent covariant form

$$S_{\mu\nu}(t,b)\,\dot{b}^\nu - \frac{\partial B(t,a)}{\partial b^\mu} = 0, \qquad (4.5.3a)$$

$$S_{\mu\nu} = \frac{\partial R_\nu}{\partial b^\mu} - \frac{\partial R_\mu}{\partial b^\nu} + T_{\mu\nu}, \quad T_{\mu\nu} = T_{\nu\mu} \qquad (4.5.3b)$$

subject to the general conditions (2.5.9), i.e.,

$$\det(S^{\mu\nu})(R) \neq 0, \quad \det(S^{\mu\nu} - S^{\nu\mu})(R) \neq 0 \qquad (4.5.4a)$$

$$S^{\mu\nu} = \Omega^{\mu\nu} + T^{\mu\nu}, \quad S_{\mu\nu} = \Omega_{\mu\nu} + T_{\mu\nu} \qquad (4.5.4b)$$

(*) Throughout this section we shall use the symbol "b" for local coordinates to stress the difference with the symplectic context of the preceding section.

$$S_{\mu\nu} = \left(\|S^{\alpha\beta}\|^{-1}\right)_{\mu\nu}, \qquad (4.5.4c)$$

$$\Omega_{\mu\nu} = \left(\|\Omega^{\alpha\beta}\|^{-1}\right)_{\mu\nu}, \quad T_{\mu\nu} \neq \left(\|T^{\alpha\beta}\|^{-1}\right)_{\mu\nu}. \qquad (4.5.4d)$$

The objective of this section is to identify the rudiments of the geometry which characterizes Eqs. (4.5.2) or (4.5.3).

For such objective, the following difficulty soon emerges. It is related to the identification of the applicable calculus. Let us recall in this respect the role of the calculus of exterior forms for the symplectic geometry. This calculus is, in essence, based on the antisymmetric nature of the exterior product,

$$da^\mu \wedge da^\nu = -\, da^\nu \wedge da^\mu \qquad (4.5.5)$$

In turn, the antisymmetric property is not only consistent within the context of the symplectic geometry, but necessary on a number of counts. For instance, property (4.5.5) can, in the final analysis, be interpreted as the geometric image of the antisymmetric nature of the Lie product

$$[A, B]^* = -[B, A]^*, \quad [A, B]^* = \frac{\partial A}{\partial b^\mu}\,\Omega^{\mu\nu}(a)\,\frac{\partial B}{\partial b^\nu} \qquad (4.5.6)$$

Equivalently, property (4.5.5) is the geometric image of the antisymmetric nature of Hamilton's tensor $\omega^{\mu\nu}$ or Birkhoff's tensor $\Omega^{\mu\nu}$. We can therefore conclude by saying that the calculus of exterior forms is the "natural calculus" for the study of the symplectic formulations of Analytic Mechanics.

In the transition to the Lie-admissible formulations, the situation is different. One of the fundamental properties of the Lie-admissible product is that of being neither totally symmetric nor totally antisymmetric. As such, it admits a nontrivial decomposition into these parts

$$(A,B)^* = \frac{\partial A}{\partial b^\mu} S^{\mu\nu} \frac{\partial B}{\partial b^\nu} = [A,B]^* + \{A,B\}^* = \frac{\partial A}{\partial b^\mu} \Omega^{\mu\nu} \frac{\partial B}{\partial b^\nu} + \frac{\partial A}{\partial b^\mu} T^{\mu\nu} \frac{\partial B}{\partial b^\nu} \quad (4.5.7)$$

The algebraic notion of Lie-admissibility can then be expressed with the rule

$$(A,B) - (B,A) \equiv 2[A,B]^* \quad (4.5.8)$$

These elemental aspects of the Lie-admissible formulations are clearly nonrepresentable with the calculus of exterior forms. For instance, if one uses such calculus to construct a two-form with the Lie-admissible tensor $S_{\mu\nu}$, from antisymmetry property (4.5. 5), we have

$$S_2 = S_{\mu\nu} da^\mu \wedge da^\nu \equiv \frac{1}{2}(S_{\mu\nu} - S_{\nu\mu}) da^\mu \wedge da^\nu = \frac{1}{2}\Omega_{\mu\nu} da^\mu \wedge da^\nu \equiv \frac{1}{2}\Omega_2 \quad (4.5.9)$$

that is, the calculus under consideration is such to always eliminate the totally symmetric part of $S_{\mu\nu}$ and restrict the two-forms to its antisymmetric part.

The inapplicability of the calculus of exterior forms to the characterization of Lie-admissible formulations then follows.

In order to overcame this difficulty, we here introduce the

rudiments of a more general calculus which, for reasons to be clarified below, we call <u>exterior-admissible calculus</u>. It is essentially based on the generalization of the exterior product $da^\mu \wedge da^\nu$ to a form called <u>exterior-admissible product</u>, denoted with $db^\mu \circ db^\nu$, and which is essentially given by the conventional tensorial product, i.e.,

$$db^\mu \circ db^\nu = \frac{1}{2} db^\mu \times db^\nu + \frac{1}{2} db^\mu \wedge db^\nu \quad (4.5.10)$$

$$db^\mu \times db^\nu = db^\nu \times db^\mu, \quad db^\mu \wedge db^\nu = -db^\nu \wedge db^\mu$$

Thus, by central property, the exterior-admissible product $db^\mu \circ db^\nu$ is neither totally symmetric nor totally antisymmetric, but such to admit a nontrivial decomposition into these parts according to Eqs. (4.5.10) where $\times$ is the ordinary symmetric product. This is clearly in line with the notion of Lie-admissible product recalled earlier.

Product (4.5.10) is exterior-admissible in the sense that, by construction, its attached antisymmetric product

$$db^\mu \circ db^\nu - db^\mu \circ db^\nu \equiv db^\mu \wedge db^\nu, \quad (4.5.11)$$

is the exterior one.

Next, we define <u>exterior-admissible p-form</u> the following forms

$$\hat{A}_0 = \rho(b) \equiv A_0, \quad (4.5.12a)$$

$$\hat{A}_1 = A_\mu db^\mu \equiv A_1, \quad (4.5.12b)$$

$$\hat{A}_2 = A_{\mu\nu} db^\mu \circ db^\nu = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) db^\mu \times db^\nu + \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu}) db^\mu \wedge db^\nu \quad (4.5.12c)$$

etc.

These forms are exterior-admissible in a two-fold way.

(A) Their attached forms

$$\hat{A}_2 - \hat{A}_2^{Alt} \equiv A_2 \tag{4.5.13a}$$

$$A_{\mu\nu}\, db^\mu \circ db^\nu - A_{\mu\nu}\, db^\nu \circ db^\mu = A_{\mu\nu}\, db^\mu \wedge db^\nu \tag{4.5.13b}$$

are conventional (exterior) p-forms.

(B) At the limit when the A-tensors become totally antisymmetric, the exterior-admissible forms become exterior forms, i.e.,

$$\lim_{A_{\mu\nu} = -A_{\nu\mu}} \hat{A}_2 \equiv A_2 \tag{4.5.14}$$

In conclusion, the exterior-admissible forms can be decomposed into two generally non-null parts, one totally symmetric and one totally antisymmetric. As such, they are adequate for the characterization of the Lie-admissible tensors.

Next, we need the operations on exterior-admissible forms. The exterior-admissible sum is the ordinary sum, i.e.,

$$\hat{A}_1 + B_1 = A_\mu db^\mu + B_\mu db^\mu = (A_\mu + B_\mu) db^\mu, \tag{4.5.15a}$$

$$\hat{A}_2 + \hat{B}_2 = A_{\mu\nu}\, db^\mu \circ db^\nu + B_{\mu\nu}\, db^\mu \circ db^\nu = (A_{\mu\nu} + B_{\mu\nu})\, db^\mu \circ db^\nu, \tag{4.5.15b}$$

etc.

The exterior-admissible product of two one-forms is precisely a form of type (4.5.12c), i.e.,

$$\hat{A}_1 \circ \hat{B}_1 = (A_\mu db^\mu) \circ (B_\nu db^\nu)$$

$$= A_\mu B_\nu\, db^\mu \circ db^\nu = \frac{1}{2}(A_\mu B_\nu + A_\nu B_\mu)\, db^\mu \otimes db^\nu + \frac{1}{2}(A_\mu B_\nu - A_\nu B_\mu)\, db^\mu \wedge db^\nu = \hat{A}_2, \tag{4.5.16}$$

etc

and similarly for the case of higher forms.

Finally, we introduce the left* exterior-admissible derivative in the natural way

$$\hat{d}\hat{A}_0 = \frac{\partial \rho}{\partial b^\mu} db^\mu, \tag{4.5.17a}$$

$$\hat{d}\hat{A}_1 = \frac{\partial A_\mu}{\partial b^\nu} db^\nu \circ db^\mu, \tag{4.5.17b}$$

$$\hat{d}\hat{A}_2 = \frac{\partial A_{\mu\nu}}{\partial b^\tau} db^\tau \circ db^\mu \circ db^\nu, \text{ etc.} \tag{4.5.17c}$$

with corresponding right versions hereinafter ignored. A left exact exterior-admissible form then occurs when there exists a primitive form such that

$$\hat{A}_1 = dA_0, \quad A_\mu = \frac{\partial \rho}{\partial b^\mu}, \tag{4.5.18a}$$

$$\hat{A}_2 = d\hat{A}_1, \quad A_{\mu\nu} = \frac{\partial A_\mu}{\partial b^\nu}, \tag{4.5.18b}$$

etc.

Thus, the notion of exact exterior form carries over to the exterior-admissible forms. However, the notion of closure is lost. And indeed, the Poincaré lemma does not extend to the calculus of exterior-admissible forms because, trivially,

* Notice that this operation is not a derivative as commonly understood because, for in stance, $\hat{d}\,(df \circ dg) \neq d(df) \circ dg + df \circ d(dg)$.

$$\hat{d}(\hat{d}\hat{A}_p) \neq 0 \qquad (4.5.19)$$

Both these properties will turn out to be crucial for the geometrical treatement of the Birkhoff-admissible equations. Specifically, the notion of exactness is crucial for the representation of vector fields on manifolds in terms of the Birkhoff-admissible equations, while the lack of closure is crucial to distinguish the symplectic geometry from its intended generalization.

The notions introduced until now are sufficient for the rudimentary treatement of this section. In the following we shall indicate the extension of conventional, additional, notions of the symplectic geometry to the exterior-admissible calculus as they occur.

By following the pattern of Section 4.4, we are again interested in the study of general first-order systems, but now under the condition thet they are variationally nonselfadjoint, i.e.,

$$\left[C_{\mu\nu}(b)\,\dot{b}^{\nu} + D_{\mu}(b)\right]_{NSA} = 0 \qquad (4.5.20)$$

The exterior-admissible form

$$\hat{C}_2 = C_{\mu\nu}\, db^{\mu} \circ db^{\nu} = \frac{1}{2}(C_{\mu\nu}+C_{\nu\mu})\, db^{\mu} \times db^{\nu} + \frac{1}{2}(C_{\mu\nu}-C_{\nu\mu})\, db^{\mu} \wedge db^{\nu} \qquad (4.5.21)$$

satisfies the requirements of Section 4.3 to qualify as an ∞-differentiable structure on the manifold M with local coordinates b^{μ}. As

a result, systems (4.5.20) can be interpreted as vector fields on a (Hausdorff, second countable, ∞-differentiable, 2n-dimensional) manifold $M(b,\hat{C}_2)$. This manifold, however, by construction, is not a symplectic manifold because, for instance, the two-form $\hat{C}_2$ is not closed. Neverthless, a number of properties of the symplectic manifolds admit consistent extensions in $M(b,\hat{C}_2)$. For instance, realizations (4.4.1), (4.4.2) and (4.4.3) admit the following images for the characterization of the form C_2.

(a) Tensorial realization

$$\hat{C}_2 : C_{\mu\nu} \qquad (4.5.22)$$

(b) Matrix realization

$$\hat{C}_2 : \hat{C}_2 = \{\hat{C}_2(e^{\mu}, e^{\nu})\} \qquad (4.5.23)$$

where now $\{e^{\mu}\}$ is a basis of $M(b,\hat{C}_2)$, and

(c) Differential realization, Eq. (4.5.21).

The notion of non- degeneracy also extends, because independent from the antisymmetric nature of the tensor, in which case the exterior-admissible, covariant, two-form $\hat{C}_2$ uniquely characterizes a contravariant two-form $\hat{C}^2$ expressible in terms of

(a') Tensorial characterization

$$\hat{C}^2 : C^{\mu\nu} ; \quad (C^{\mu\nu}) = (C_{\mu\nu})^{-1} \qquad (4.5.24)$$

(b') Matrix realization

$$\hat{C}^2 : \hat{C}^2 = \{\hat{C}_2(e^{\mu}, e^{\nu})\}^{-1} \qquad (4.5.25)$$

(c') Differential characterization

$$\hat{C}^2 : \hat{C}^2 = C^{\mu\nu} \frac{\partial}{\partial b^\mu} \circ \frac{\partial}{\partial b^\nu}, \qquad (4.5.26)$$

where $\partial / \partial b^\mu$ is the dual of $d b^\mu$ as in Eq. (4.4.12).

The forms $\hat{C}_2$ and $\hat{C}^2$ also preserve the basic functions of the corresponding symplectic and cosymplectic forms, Ω_2 and Ω^2, respectively. For instance, they can be used for the mapping from TM to T*M, i.e.,

$$C_b : TM \longrightarrow T^*M, \qquad (4.5.27)$$

as well as, more importantly for our analysis, they can be used for the raising and lowering of the indeces, i.e.,

$$b_\mu = C_{\mu\nu} b^\nu, \quad b^\nu = C^{\nu\mu} b_\mu, \qquad (4.5.28a)$$

$$\Xi_\mu = C_{\mu\nu} \Xi^\nu, \quad \Xi^\nu = C^{\nu\mu} \Xi_\mu . \qquad (4.5.28b)$$

To summarize, our stating point is a system of first-order, ordinary differential equations, Eqs. (4.5.20) which, by assumption, is nonselfadjoint. This implies that the tensor $C_{\mu\nu}$ is neither totally antisymmetric nor totally symmetric. To properly preserve this character in the geometric treatement, we have introduced an exterior-admissible two-form $\hat{C}_2$, Eq. (4.5.21). Despite these departures from the conventional symplectic treatement, system (4.5.20) is fully characterizable as a vector field on manifolds, here intended in the generalized meaning of Section 4.3, and the tensor $C_{\mu\nu}$ preserves its geometrical meaning of lowering indeces as in the symplectic case.

We are now equipped to introduce a crucial concept of our analysis. Again, we shall consider a dual approach, a first analytic approach and a second more geometrical approach.

I. Analytic Approach. When the cotangent bundle T*M is equipped with the ∞-differentiable, non-degenerate, two-form $\hat{C}^2$, it is turned into a (Hausdorff, second countable, ∞-differentiable, 2n-dimensional) manifold $M(b, \hat{C}^2)$. When restricted to exact differentials of functions on $M(b, \hat{C}^2)$, the two-form $\hat{C}^2$ induces the bilinear composition law in the local coordinates b^μ

$$\hat{C}^2(df, dg) = f \Box g = \frac{\partial f}{\partial b^\mu} C^{\mu\nu} \frac{\partial f}{\partial b^\nu} \qquad (4.5.29)$$

which also verifies properties (4.4.19). The corresponding covariant form can be characterized also as in Eqs. (4.4.21a), i.e.,

$$\hat{C}_2 = \hat{C}_2(\lambda_f, \lambda_g) = \hat{C}^2(f, g), \qquad (4.5.30)$$

yielding the dual composition law

$$\hat{C}_2(df, dg) = f \times g = \frac{\partial b^\mu}{\partial f} C_{\mu\nu} \frac{\partial b^\nu}{\partial g}, \qquad (4.5.31)$$

and the mapping $M(b, \hat{C}^2) \to M(b, \hat{C}_2)$.

We shall call $M(b, \hat{C}_2)$ a symplectic-admissible manifold[25] when structure (4.5.29) satisfies the properties

$$\hat{J}(f, g) = [f, g] - [g, f] = 0, \qquad (4.5.32a)$$

$$\hat{J}(f, g, h) = [[f, g], h] + [[g, h], f] + [[h, f], g] = 0, \qquad (4.5.32b)$$

where

$$[f,g] = f\square g - g\square f \qquad (4.5.33)$$

The form $\hat{C}_2$ will then be called a symplectic-admissible form or structure. The associated contravariant form $\hat{C}^2$ will be called cosymplectic-admissible structure.

Conditions (4.5.32) essentially ensure that brackets (4.5.29) satisfy the laws of Lie-admissibility, Eqs. (1.4.2), i.e., they are the general Lie-admissible brackets (1.5.21), and we shall write

$$f\square g \equiv (f,g)^* \qquad (4.5.34)$$

Equivalently, $M(b,\hat{C}_2)$ is a symplectic-admissible manifold when the elements of the matrix $(C^{\mu\nu}) = (C_{\mu\nu})^{-1}$ satisfy all Eqs. (1.5.8). It then follows that brackets (4.5.34) are the general, inverse, Lie-admissible brackets, and we shall write

$$f\times g = \widetilde{f,g}^{\,*} \qquad (4.5.35)$$

The dual relationship between $\hat{C}^2$ and $\hat{C}_2$ is then reflected in the preservation of rule (4.4.26), i.e.,

$$\sum_{k=1}^{6N} (f_i, f_k)^* \, \widetilde{f_k, f_i}^{\,*} = \delta_{ij} \qquad (4.5.36)$$

II. Geometric Approach. Let $M(b, \hat{C}_2)$ be a (Hausdorff, second countable, ∞-differentiable, 2n-dimensional) manifold equipped with a non-degenerate covariant two form (4.5.21). The manifold $M(b, \hat{C}_2)$ is called a symplectic-admissible manifold when the attached two form

$$\Omega_2 = \hat{C}_2 - \hat{C}_2^{Alt} = \frac{1}{2}(C_{\mu\nu} - C_{\nu\mu})\, db^\mu \wedge db^\nu \qquad (4.5.37)$$

is symplectic.[25] Thus, by central assumption, the two-form $\hat{C}_2$ is not., in general, closed, and we shall write

$$\hat{d}\,\hat{C}_2 \neq 0 \qquad (4.5.38)$$

However, also by central assumption, the form $\hat{C}_2$ is such that the attached form (4.5.37) is closed, i.e.,

$$\hat{d}\,(\hat{C}_2 - \hat{C}_2^{Alt}) = d\Omega_2 \equiv 0 \qquad (4.5.39)$$

In essence, this notion is here introduced to attempt a geometric counterpart of the notion of Lie-admissible algebra as presented in Chapyer 1. In full analogy with Lemma 1.4.1 we have the following

LEMMA 4.5.1: Any closed symplectic-admissible form is symplectic.

On similar grounds we have

LEMMA 4.5.2: Any symplectic manifold is symplectic-admissible.

And indeed, when $\hat{C}_2 = \Omega_2$ is symplectic, the attached two-form (4.5.37) is trivially symplectic

$$\Omega_2 - \Omega_2^{Alt} = 2\,\Omega_2 \qquad (4.5.40)$$

Alternatively we can say that symplectic-admissible manifolds admit the conventional symplectic manifolds as a particular case. This is clearly a crucial requirement to attempt the construction of a covering of the symplectic geometry.

The use of the exterior-admissible forms trivially yields the following

LEMMA 4.5.3: A symplectic-admissible manifold is not necessarily symplectic.

that is, symplectic-admissible manifolds are a nontrivial generalization of the symplectic manifolds. Their classification can be also conducted by closely following the classification of Lie-admissible algebras of Section 1.4. Therefore, a manifold $M(b, \hat{C}_2)$ will be called

(a) - a general symplectic-admissible manifold when the associated contravariant tensor $C^{\mu\nu}$ satisfies the general conditions of Lie-admissibility

$$\begin{aligned}&(C^{\mu\rho} - C^{\rho\nu})\frac{\partial}{\partial b^\rho}(C^{\nu\tau} - C^{\tau\nu})\\ &+ (C^{\nu\rho} - C^{\rho\nu})\frac{\partial}{\partial b^\rho}(C^{\tau\mu} - C^{\mu\tau})\\ &+ (C^{\tau\rho} - C^{\rho\tau})\frac{\partial}{\partial b^\rho}(C^{\mu\nu} - C^{\nu\mu}) = 0\end{aligned} \qquad (4.5.41)$$

(b) - a flexible symplectic-admissible manifold when the associated tensor $C^{\mu\nu}$ satisfies the flexible conditions of Lie-admissibility

$$C^{\mu\rho}\frac{\partial C^{\tau\nu}}{\partial b^\rho} + C^{\tau\rho}\frac{\partial C^{\nu\mu}}{\partial b^\rho} - \frac{\partial C^{\mu\nu}}{\partial b^\rho}\rho\tau - \frac{\partial C^{\tau\nu}}{\partial b^\rho}C^{\rho\mu} = 0, \qquad (4.5.42a)$$

$$\begin{aligned}&(C^{\tau\rho} - C^{\rho\tau})\frac{\partial C^{\mu\nu}}{\partial b^\rho} + (C^{\mu\rho} - C^{\rho\mu})\frac{\partial C^{\nu\tau}}{\partial b^\rho}\\ &+ (C^{\nu\rho} - C^{\rho\nu})\frac{\partial C^{\tau\mu}}{\partial b^\rho} = 0,\end{aligned} \qquad (4.5.42b)$$

(c) - a symplectic manifold when the associated tensor satisfies the Lie conditions,

$$C^{\mu\nu} + C^{\nu\rho} = 0, \qquad (4.5.43a)$$

$$C^{\mu\rho}\frac{\partial C^{\nu\tau}}{\partial b^\rho} + C^{\nu\rho}\frac{\partial C^{\tau\mu}}{\partial b^\rho} + C^{\tau\rho}\frac{\partial C^{\mu\nu}}{\partial b^\rho} = 0. \qquad (4.5.43b)$$

A nontrivial symplectic-admissible manifold is either a general or a flexible symplectic-admissible manifold. When no specification is introduced, a symplectic-admissible manifold will be referred to any of possibilities (a), (b) and (c).

Until now, we have generally considered in this section the case of Birkhoff-admissible equations. However, the reader should recall from Chapter 2 that our fundamental equations are the historical equations originally conceived by Hamilton. These ARE NOT the equations customarily called "Hamilton's equations" in the contemporary literature of the symplectic and contact geometries. Instead, they are the equations with external terms

$$\begin{cases}\dot{r} = \dfrac{\partial H}{\partial p}\\[2mm] \dot{p} = -\dfrac{\partial H}{\partial r} + F\end{cases} \qquad (4.5.44)$$

which we write in the contravariant Hamilton-admissible form

$$\dot b^\mu = S^{\mu\nu}(t,b)\frac{\partial H}{\partial b^\nu} = (\omega^{\mu\nu} + t^{\mu\nu})\frac{\partial H}{\partial a^\nu}$$

$$\omega^{\mu\nu} = \left(\left\|\frac{\partial R^\circ_\alpha}{\partial b^\beta} - \frac{\partial R^\circ_\beta}{\partial b^\alpha}\right\|^{-1}\right)^{\mu\nu}, \quad R^\circ = (\underline{p}_\mu, \underline{0}) \qquad (4.5.45)$$

$$(t^{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & F/(p/m) \end{pmatrix} = (t^{\nu\mu})$$

or in the equivalent covariant form

$$S_{\mu\nu}(t,b)\,\dot b^\nu - \frac{\partial H}{\partial b^\mu} = (\omega_{\mu\nu} + t_{\mu\nu})\,\dot b^\nu - \frac{\partial H}{\partial b^\mu} = 0$$

$$\omega_{\mu\nu} = \frac{\partial R^\circ_\nu}{\partial b^\mu} - \frac{\partial R^\circ_\mu}{\partial b^\nu} = \left(\|\omega^{\alpha\beta}\|^{-1}\right)_{\mu\nu} \qquad (4.5.46)$$

$$(t_{\mu\nu}) = \begin{pmatrix} -F/\dot r & 0 \\ 0 & 0 \end{pmatrix} \neq \left(\|t^{\alpha\beta}\|^{-1}\right)_{\mu\nu}$$

in order to bypass the lack of consistent algebra in the brackets of time evolution of Eqs. (4.5.44) (Section 2.2).

The geometer reading these lines should recall the remarks of Chapter 1, as well as of Volume I of this series, and of monograph[65], stressing the apparent Hamilton's awareness that the "truncated equations" (those without external term) may imply the accaptance of the perpetual motion in our environment, owing to contact-nonpotential forces which are well known since Hamilton's time.

Eqs. (4.5.45) characterize the simplified version of brackets (4.5. 7)

$$(A,B) = \frac{\partial A}{\partial b^\mu} S^{\mu\nu}\frac{\partial B}{\partial b^\nu} = [A,B] + \{A,B\} \qquad (4.5.47)$$

$$= \frac{\partial A}{\partial b^\mu}\omega^{\mu\nu}\frac{\partial B}{\partial b^\nu} + \frac{\partial A}{\partial b^\mu} t^{\mu\nu}\frac{\partial B}{\partial b^\nu}$$

we have called <u>fundamental Lie-admissible brackets</u>. It is then

evident that the exterior-admissible two-form

$$\hat S_2 = S_{\mu\nu}\, db^\mu \circ db^\nu \qquad (4.5.48)$$

$$= \tfrac{1}{2}\omega_{\mu\nu}\, db^\mu \wedge db^\nu + \tfrac{1}{2} t_{\mu\nu}\, db^\mu \wedge db^\nu$$

shall be called <u>fundamental symplectic-admissible two-form</u>.

The problem of the possible generalization of Darboux's theorem to the covering geometry under consideration, will be studied in Appendix 4.A.

We pass now to the generalization of the notion of Hamiltonian and Birkhoffian vector-fields of Section 4.4. Consider a general, symplectic-admissible two-form

$$\hat{\mathcal S}_2 = \mathcal S_{\mu\nu}\, db^\mu \circ db^\nu \qquad (4.5.49)$$

$$= \tfrac{1}{2}\Omega_{\mu\nu}\, db^\mu \wedge db^\nu + \tfrac{1}{2} T_{\mu\nu}\, db^\mu \times db^\nu$$

To contract $\hat S_2$ with a contravariant vector field Ξ^μ, we introduce the prescriptions

$$\Xi_1^{\hat S_2} = \Xi \boxtimes \hat S_2 \overset{def}{=} S_{\mu\nu}\,\Xi^\nu\, db^\mu \qquad (4.5.50)$$

which we call <u>inner-admissible product</u>. Even though, again, this is not a product in the conventional sense, operation (4.5.50) admits the conventional inner product in a dual way. First, by following the rule of symplectic-admissibility, we recover the inner product of Eqs. (4.4.47) according to

$$\Xi_1^{\mathcal{R}} = \frac{1}{2}\left(\Xi \boxtimes \hat{S}_2 - \Xi \boxtimes \hat{S}_2^{Alt}\right) = \frac{1}{2}(S_{\mu\nu} - S_{\nu\mu})\Xi^\nu db^\mu = \frac{1}{2}\mathcal{R}_{\mu\nu}\Xi^\nu db^\mu \qquad (4.5.51)$$

Second, the inner product can be recovered under the value of all null nonself-adjoint forces, in which case the <u>general</u> symplectic-admissible tensor reduces to the <u>fundamental</u> symplectic tensor, i.e.,

$$\hat{S}_2 \Big|_{F^{NSA}\equiv 0} \equiv \omega_2 = \frac{1}{2}\omega_{\mu\nu}\, db^\mu \wedge db^\nu \qquad (4.5.52)$$

Note that for zero- and one-forms we have

$$\Xi \boxtimes \hat{A}_{(0)} \equiv \Xi \lrcorner A_{(0)} \equiv 0$$
$$\Xi \boxtimes \hat{A}_{(1)} \equiv \Xi \lrcorner A_{(1)} = \Xi^\mu A_\mu \qquad (4.5.53)$$

The reader can then identify the generalized, inner-admissible product of a vector field with a p-forms ($p > 2$) on a symplectic-admissible manifold.

A vector field Ξ is called <u>Hamilton-admissible</u> when the one-form characterized by the inner-admissible product with the <u>fundamental</u> symplectic-admissible two-form is exact, and we shall write

$$\Xi_1^{\hat{S}_2} = \Xi \boxtimes \hat{s}_2 = s_{\mu\nu}\Xi^\nu db^\mu = dH = \frac{\partial H}{\partial b^\mu} db^\mu \qquad (4.5.54)$$

The function H(b) is then called the <u>Hamiltonian</u> to stress the preservation of the conventional meaning of this function (as representing the total energy).

A vector field Ξ is called <u>Birkhoff-admissible</u> when the one form characterized by the inner-product with a <u>general</u> symplectic-admissible two-form is exact, and we shall write

$$\Xi_1^{\hat{S}_2} = \Xi \boxtimes \hat{S}_2 = S_{\mu\nu}(b)\,\Xi^\nu(b)\, db^\mu = dB(b) = \frac{\partial B}{\partial b^\mu} db^\mu \qquad (4.5.55)$$

In this case the function B(b) will be called the <u>Birkhoffian</u>, to stress the differences of its physical meaning with the Hamiltonian (Chapter 2).

The achievement of a covering notion with respect to that of Hamiltonian and Birkhoffian vector fields can be expressed via the following property.

<u>LEMMA 4.5.4. Every Hamiltonian (Birkhoffian) vector field is Hamilton-admissible (Birkhoff-admissible). However, the inverse property is not generally true.</u>

Theorem 2.4.1 also implies the following property.

<u>LEMMA 4.5.5. Every vector-field which is not Hamiltonian in the local variables considered, is always Hamilton-admissible under sufficient topological conditions.</u>

We reach in this way the following "direct universality" of the symplectic-admissible geometry for the characterization of local, non-conservative, autonomous Newtonian systems.

THEOREM 4.5. All local, autonomous, class $\mathcal{C}^{\infty}$, regular, essentially nonself-adjoint Newtonian systems in the time and Cartesian coordinates of its experimental detection

$$\left\{\left[m_k \ddot{r}_{ka} - f_{ka}(r, \dot{r})\right]_{SA} - F_{ka}(r, \dot{r})\right\}_{NSA} = 0 \qquad (4.5.56)$$
$$k = 1, 2, \ldots, N, \quad a = x, y, z$$

can be characterized by the symplectic-admissible geometry via the equivalent first-order form

$$\begin{pmatrix} \dot{b}^{\mu} \end{pmatrix} = \begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} (p + \alpha)/m \\ f^{SA} + F^{NSA} \end{pmatrix} \qquad (4.5.57)$$
$$f = -\frac{\partial U}{\partial r} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}}, \quad U = \alpha_{ka}(t, r)\dot{r}^{ka} + \beta_{ka}(t, r)$$
$$p = \frac{\partial L}{\partial \dot{r}} = , p - \alpha, \quad L = \frac{1}{2} m_k p_k \cdot p^k - U$$

which is always Hamilton-admissible, that is, it is such to admit always characterization (4.5.54).

Simple examples are provided in Appendix 2.C.

For the nonautonomous case, we consider the most general possible class of first-order systems on $\mathbb{R} \times T^*M$, where $\mathbb{R}$ represents time as in Section 4.4,

$$\left[C_{\mu\nu}(t, b)\dot{b}^{\nu} + D_{\mu}(t, b)\right]_{NSA} = 0 \qquad (4.5.58)$$
$$\mu = 1, 2, \ldots, 6N$$

The two-form in (6N+1)-dimensional spaces in notation (2.8.15)

$$\hat{C}_2 = \hat{C}_{\mu\nu}(\hat{b})\, d\hat{b}^{\mu} \circ d\hat{b}^{\nu}$$
$$= \frac{1}{2}(C_{\mu\nu} - C_{\nu\mu})\, d\hat{b}^{\mu} \wedge d\hat{b}^{\nu} + \frac{1}{2}(\hat{C}_{\mu\nu} + \hat{C}_{\nu\mu})\, d\hat{b}^{\mu} \times d\hat{b}^{\nu} \qquad (4.5.59)$$
$$\hat{C}_{00} = \frac{\partial B}{\partial b^{\mu}} T^{\mu\nu} \frac{\partial B}{\partial b^{\nu}}, \quad \hat{C}_{0\mu} = -\frac{\partial B}{\partial b^{\mu}} = -\hat{C}_{\mu 0}$$
$$\hat{C}_{\mu\nu} = C_{\mu\nu}, \quad \mu, \nu = 1, 2, \ldots, 6N, \quad \hat{b} = (t, b)$$

will said to possess maximal rank when both its restriction to T^*M and its antisymmetric (exterior) component have maximal rank 6N

$$\text{Rank}\, \hat{C}_2\Big|_{T^*M} = \text{Rank}\, C_2 = 6N \qquad (4.5.60)$$
$$\text{Rank}\left(\hat{C}_2 - \hat{C}_2^{Aet}\right)\Big|_{T^*M} = \text{Rank}\, \Omega_2 = 6N$$

A two-form (4.5.59) of maximal rank is called contact-admissible when the attached antisymmetric two-form is closed and therefore contact

$$\hat{d}\hat{C}_2 \neq 0 \qquad (4.5.61)$$
$$\hat{d}\left(\hat{C}_2 - \hat{C}_2^{Aet}\right) = d\hat{\Omega}_2 = 0$$

A contact-admissible manifold is the space $\mathbb{R} \times T^*M$ equipped with a contact-admissible two-form, which in our notion can be written $M = M(\hat{b}, \hat{C}_2)$. The direct universality of the contact-admissible geometry for all class $\mathcal{C}^{\infty}$, regular, nonautonomous systems (4.5.58) is then a trivial consequence of Theorems 4.5.1 and 2.4.1.

A few concluding remarks are now in order. The capability of the symplectic-admissible geometry to geometrize Lie-admissible algebras is self-evident. Particularly intriguing is the fact that this characterization occurs while preserving the conventional symplectic-Lie cases as particular cases. Needless to say, and as anticipated in Section 4.1, we have merely indicated the existence of the symplectic-admissible (and contact-admissible) geometry. Their actual construction in all the diversified aspects, including comohomological aspects, will predictably take time.

The first aspect which should be brought to the attention of the reader is that the symplectic-admissible geometry appears to be at the foundations of the structure of the conventional Poisson brackets. Recall from Chapter 3 that the notion of Lie algebra can be ultimately conceived, not as a primitive structure, but as a structure attached to the truly primitive algebra, the enveloping algebra U, via Lie's fundamental rule of Lie-admissibility

PRIMITIVE ALGEBRA:
the universal enveloping algebra U with abstract product AB
DERIVED ALGEBRA:
The attached Lie algebra with product [A,B] = AB-BA (4.5.61)

Now, for the case of the Poisson brackets

$$[A,B] = \frac{\partial A}{\partial z^{ka}}\frac{\partial B}{\partial p_{ka}} - \frac{\partial B}{\partial z^{ka}}\frac{\partial A}{\partial p_{ka}} \qquad (4.5.63)$$

the underlying primitive algebra is precisely a general, nonassociative, Lie-admissible algebra characterized by the brackets (Section 1.5)

$$A \times B \overset{\text{def}}{=} \frac{\partial A}{\partial z^{ka}}\frac{\partial B}{\partial p_{ka}} \qquad (4.5.64)$$

$$(A\times B)\times C \neq A\times(B\times C)$$

It is then rather natural to argue along Lie's idea (4.5.62), that the primitive geometric notion underlying the structure of Poisson's brackets is not the fundamental symplectic structure

$$\omega_2 = d p_{ka} \wedge d z^{ka} \qquad (4.5.65)$$

but instead the symplectic-admissible structure

$$s_2 = d p_{ka} \circ d z^{ka} = d p_{ka} \wedge d z^{ka} + d p_{ka} \times d z^{ka} \qquad (4.5.66)$$

according to the geometric counterpart of algebraic rules (4.5.62)

PRIMITIVE GEOMETRY:
the symplectic-admissible geometry with fundamental forms s_2
DERIVED GEOMETRY:
the attached symplectic geometry with fundamental forms $\omega_2 = s_2 - s_2^{Alt}$. (4.5.67)

The best way to illustrate the importance of approach (4.5.67) is via an inspection of contemporary geometric studies on quantization, which, as well know, consist in the search of a map (and other quantities) suitable for the transition from the Poisson brackets of functions A, B, ... on T*M to quantum mechanical (Heisenberg's) product of operators $\tilde{A}$, $\tilde{B}$,... acting on a Hilbert space $\mathcal{H}$,

$$\begin{pmatrix} [A,B] \\ = \frac{\partial A}{\partial r^{ka}}\frac{\partial B}{\partial p_{ka}} - \frac{\partial B}{\partial r^{ka}}\frac{\partial A}{\partial p_{ka}} \end{pmatrix} \xrightarrow[\text{quantization}]{\text{contemporary}} \begin{pmatrix} [\hat{A},\hat{B}] \\ = \hat{A}\hat{B} - \hat{B}\hat{A} \end{pmatrix} \qquad (4.5.68)$$

The nature of the envelope is ignored in these geometric studies, to my best knowledge. In fact, the studies are centered in the symplectic geometry alone, by ignoring the existence of a possible primitive notion. The results in the field are well known. In fact, a consistent (full) quantization has not yet been achieved at this moment.

The use of the more general view (4.5.68), even if it may eventually result to be insufficient to resolve this difficult problem, at least it sets the mind in the identification of unnecessary inconsistencies in the formulation of the problem itself. In fact, once the notion of symplectic-admissible geometry as the primitive notion underlying Poisson brackets is understood, one immediately searches for the corresponding notion in Heisenberg's product. It is then rather natural to see that the algebra underlying Heisenberg's algebra is associative in its current form. As a result, no mapping between a CLASSICAL NONASSOCIATIVE and a QUANTUM MECHANICAL ASSOCIATIVE algebra is expected to be possible

$$\left\{ \begin{array}{l} \text{POISSON BRACKETS:} \\ [A,B] = (A,B) - (B,A); \quad (A,B) = \text{nonassociative product} \\ \text{HEISENBERG'S BRACKETS:} \\ [\tilde{A},\tilde{B}] = \tilde{A}\tilde{B} - \tilde{B}\tilde{A}; \quad \tilde{A}\tilde{B} = \text{associative product} \end{array} \right. \qquad (4.5.69)$$

To put it differently, it is true that both Poisson and Heisenberg's brackets characterize a Lie algebra. However, their primitive envelopes are structurally different and algebraically nonequivalent. No consistent quantization is expected to exist under these conditions.

Once the problem has been identified via the use of the symplectic-admissible geometry, one can search for possible solution, such as the possibility of redefining Heisenberg's product identically via a nonassociative, Lie-admissible, enveloping algebra of the type

$$\begin{array}{l} [\tilde{A},\tilde{B}] = \tilde{A}\tilde{B} - \tilde{B}\tilde{A} \equiv \tilde{A}\times\tilde{B} - \tilde{B}\times\tilde{A} \\ \tilde{A}\tilde{B} = \text{ASSOCIATIVE LIE-ADMISSIBLE} \\ \tilde{A}\times\tilde{B} = \tilde{A}\tilde{B} + \frac{1}{2}(\tilde{A}\tilde{B} + \tilde{B}\tilde{A}) = \text{NONASSOCIATIVE LIE-ADMISSIBLE} \end{array} \qquad (4.5.70)$$

The importance of these issues for our analysis is self-evident. In fact, in the next volume we have to confront the problem of quantization for the considerably more general (and difficult) case of nonpotential/non-Hamiltonian systems. It is evident that the identification of the ultimate algebraic-geometric roots for the lack of achievement of a consistent (full) quantization for the simpler Hamiltonian case must be identified, in order to make genuine progress.

Another aspect that should be brought to the attention of the interested (and open minded) geometer is the insufficiency of the local-differential geometry for genuine advances in the problem of interactions. This geometry is fully acceptable for the old notion of interactions, that based on point-like particles. However, particles can be effectively approximated as massive points only in certain conditions, such as our planetary system or the atomic system. Whenever short-range interactions and collisions occur, the point-like characterization and the underlying local-differential geometry should be abandoned for more suitable treatments.

The symplectic-admissible geometry has been prosposed in this volume, by no means as a terminal geometry, and instead only as an intermediary step prior to a full nonlocal-integral treatment.

The physical motivations underlying this step are the following. It is known that the electromagnetic interactions can be described in a fully satisfactory way via local-differential, symplectic approaches, and that a nonlocal-integral treatment has been suggested only for the strong interactions. However, it is unlike that the physics community will readily abandon the local-differential approach to electromagnetism. This suggest the intermediary possibility of an "integro-differential geometry", here intended as a geometry which is based on the conventional local-differential settings, but which nevertheless admits some representative of nonlocal-integral settings.

Although our arguments have been strictly presented for local differential equations, the hope for the proposed symplectic-admissible geometry is that it will indeed result to be valuable as an intermediary step toward the search of more adequate geometries. This point is important because it is at the foundation of the unified treatment of strong, weak, and electromagnetic interactions to be proposed in Volume III. Its understanding at the primitive Newtonian level is therefore valuable.

Consider an extended particle moving in a resistive medium, such as a satellite during re-entry in atmopshere. The trajectory of the center-of-mass can be described via a local-differential geometry, trivially, because the center-of-mass is a point. Nonlocal effects due to the extended character of the satellite occur in the forces. These forces can be classified as being of two cathegory. The first is of

action-at-a-distance/potential/self-adjoint type (e.g., gravitational). The second is of nonlocaltype due to the motion of the extended object in the resistive medium. These latter forces have been approximated in our analysis, as customarily done in mechanics, via a power-series in the velocities, thus resulting in variational nonself-adjoint forces. Their nonpotential character is evident because the notion of potential energy has no physical foundation or meanining for contact interactions.

We reach in this way the structure of the systems considered

$$m\ddot{\underset{\sim}{r}} + \underset{\sim}{f}^{SA}(t,\underset{\sim}{r},\dot{\underset{\sim}{r}}) + \underset{\sim}{F}^{NSA}(t,\underset{\sim}{r},\dot{\underset{\sim}{r}}) = 0 \qquad (4.5.71)$$

MASS·ACCELERATION	+	LOCAL/POTENTIAL FORCES	+	NONLOCAL-NONPOTENTIAL FORCES IN LOCAL SERIES APPROXIMATION

The geometric characterization we have proposed is via our simplectic-admissible two-forms according to the structure

$$\hat{S}_2 = \left(\Omega_{\mu\nu}\, db^\mu \wedge db^\nu\right) + \left(T_{\mu\nu}\, db^\mu \times db^\nu\right) \qquad (4.5.72)$$

SYMPLECTIC TWO-FORM	SYMMETRIC TWO-FORM
LOCAL-POTENTIAL FORCES	LOCAL-NONPOTENTIAL FORCES
POINT-LIKE CHARACTERIZATION OF PARTICLES	CORRECTIONS FOR EXTENDED CHARACTER

The main idea of our unified treatment of strong, weak and electromagnetic interactions is as follows. Recall that the unified gauge theory of weak and electromagnetic interactions are effectively treated via the symplectic geometry, and underlying point-like approximation of particles. Hadrons, however, are extended charge distributions in conditions of mutual penetration, as necessary to activate the strong interactions. This calls for a suitable generalization of conventional, unified, gauge theory, in order to achieve a more adequate representations of physical reality. The geometric approach we shall work out in the next volume is essentially based on the proposal to accept the entirety of the current treatments of gauge theories of strong, weak, and electromagnetic interactions via symplectic geometry, and implement it with corrective, symmetric, two-forms for the representation of the extended character of hadrons and of their constituents, according to the proposal

SYMPLECTIC-ADMISSIBLE GEOMETRY	=	SYMPLECTIC GEOMETRY	+	SYMMETRIC TWO-FORMS
Unification of strong, weak, and electromagnetic interactions		Conventional treatment of unified theories of action-at-a-distance/potential type among point-like approximations of particles		Representation of extended character of hadrons and related contact nonpotential effects

(4.5.73)

Consider now the case of true nonlocal-integral treatments of the nonself-adjoint forces via the symmetric two-forms along Eqs. (2.4.25). Then, if predictable problems of topology, cohomology, etc. are resolved, the symplectic-admissible geometry could become a bona fide "integrodifferential geometry", in the sense that the symplectic part is the conventional local-differential one, and nonlocal-integral effects are represented only by the departure from the symplectic geometry characterized by the symmetric two-forms.

It is therefore tempting to conclude with the following remarks.

[1] The conventional, Hamiltonian-Lie-symplectic formulations for point-like approximation of particles and only action-at-a-distance/potential/self-adjoint forces, appear to admit consistent coverings of Hamilton-admissible/Lie-admissible/symplectic-admissible character for the treatment of the additional presence of contact/nonpotential/nonself-adjoint forces.

[2] The deep inter-relation and mutual compatibility betwen the conventional analytic, algebraic, and geometric formulations appear to carry over in the covering formulations; and

[3] The covering formulations reproduce the conventional formulations identically when all nonpotential forces are null, that is, when only point-like approximations are admitted.

By recalling that, according to incontrovertible experimental evidence, a necessary condition to activate the strong interactions is that hadrons enter into a conditions of mutual penetration of their charge volume, the foundational character of the methods we have identified for possible advances in strong interactions becomes self-evident.

APPENDIX 4.A: A TRIVIAL FORMULATION OF DARBOUX'S THEOREM FOR THE SYMPLECTIC-ADMISSIBLE GEOMETRY.

Recall that our fundamental Hamilton-admissible brackets are jointly Lie-admissible and symplectic-admissible in the sense

$$\begin{aligned} &S^{\mu\nu} = \omega^{\mu\nu} + t^{\mu\nu}, \quad t^{\mu\nu} = t^{\nu\mu} \\ &\det(S^{\mu\nu})(R) \neq 0, \quad \det(S^{\mu\nu} - S^{\nu\mu}) \\ &S_{\mu\nu} = (\|S^{\alpha\beta}\|^{-1})_{\mu\nu} = \omega_{\mu\nu} + t_{\mu\nu} \\ &\omega_{\mu\nu} = (\|\omega^{\alpha\beta}\|^{-1})_{\mu\nu} = \frac{\partial R^o_\nu}{\partial b^\mu} - \frac{\partial R^o_\mu}{\partial b^\nu} \\ &t_{\mu\nu} \neq (\|t^{\alpha\beta}\|^{-1})_{\mu\nu}, \quad R^o = (p, 0) \end{aligned} \quad (4.A.1)$$

Recall also that all the above characteristics are preserved by our Birkhoff-admissible brackets

$$\begin{aligned} &S^{\mu\nu} = \Omega^{\mu\nu} + T^{\mu\nu}, \quad T^{\mu\nu} = T^{\nu\mu} \\ &\det(S^{\mu\nu})(R) \neq 0, \quad \det(\Omega^{\mu\nu})(R) \neq 0 \\ &S_{\mu\nu} = (\|S^{\alpha\beta}\|^{-1})_{\mu\nu} = \frac{\partial R_\nu}{\partial b^\mu} - \frac{\partial R_\mu}{\partial b^\nu} \\ &T_{\mu\nu} \neq (\|T^{\alpha\beta}\|^{-1})_{\mu\nu} \\ &R = R(b) = \{P(r, p), Q(r, p)\} \neq R^o \end{aligned} \quad (4.A.2)$$

to the point that brackets (4.A.1) are a particular case of brackets (4.A.2).

Finally, recall that arbitrary, generally noncanonical transformations transform the fundamental into the general brackets (Section 2.8), as it can be seen more readily via the symplectic-admissible tensors

$$\begin{aligned} S'_{\mu\nu}(b') &= \frac{\partial b^\alpha}{\partial b'^\mu}\left[\left(\frac{\partial R_\beta}{\partial b^\alpha} - \frac{\partial R_\alpha}{\partial b^\beta}\right) + T_{\alpha\beta}\right]\frac{\partial b^\beta}{\partial b'^\nu} \\ &= \left(\frac{\partial R'_\nu}{\partial b'^\mu} - \frac{\partial R'_\mu}{\partial b'^\nu}\right) + T'_{\mu\nu} \end{aligned} \quad (4.A.3)$$

Upon achievement of a general, joint Lie-admissible and symplectic-admissible structure of the type indicated above, the theory preserves its algebraic/geometric character under all possible (regular) transformations, not only on T*M, but also more generally on R x T*M. (*)

For the case of the conventional symplectic geometry, Darboux's theorem deals with the inverse reduction of brackets or, equivalently, two-forms from their general to their fundamental version.

The problem of the possible extension of Darboux's theorem to the case of the symplectic-admissible geometry consists of the study of the inverse reduction from the general Lie-admissible/symplectic-admissible structures to the fundamental ones. In turn, this problem demands a prior knowledge of the degrees of freedom of the forms considered, which are considerable broader than those for the symplectic

(*) As studied by Sarlet (ref.198), there exist two-forms which are Lie-admissible but not jointly symplectic-admissible, and viceversa. These forms are not needed for our analysis owing to the achievement of universality with our Hamilton- and Birkhoff-admissible equations with underlying brackets (4.A.1) and (4.A.2) and related forms.

cted. This is done in purpose, to stress that, each given Hamiltonian system admits an infinite number of generalizations via nonpotential/ non-Hamiltonian forces.

It is evident that the inverse of transformation (4.A. 3) always exists, under the topology of symplectic-admissible manifolds, and under the terminology assumed. In fact, the reduction simply implies that of the antisymmetric part, which is ensured by the conventional Darboux's theorem. However, during the reduction, the symmetric part remains unrestricted, and we shall write

$$S_{\mu\nu}(b) = \left(\frac{\partial R^o_\nu}{\partial b^\mu} - \frac{\partial R^o_\mu}{\partial b^\nu}\right) + \left(t_{\mu\nu}(b)\right) = \frac{\partial b'^\alpha}{\partial b^\mu}\left[\left(\frac{\partial R'_\beta}{\partial b'^\alpha} - \frac{\partial R'_\alpha}{\partial b'^\beta}\right) + T_{\alpha\beta}(b')\right]\frac{\partial b'^\beta}{\partial b'^\nu} \qquad (4.A.4)$$

To put it differently, it is possible to prove the lack of existence of one single transformation which reduces the antisymmetric part to the fundamental one, and reduces at the same time the symmetric part to one predetermined form.

We shall call the transition from one fundamental symplectic-admissible two-form to another one, only with a different symmetric part

$$S_2 = \omega_2 + t_2 \quad \Longrightarrow \quad S^*_2 = \omega_2 + t^*_2 \qquad (4.A.5)$$

a fundamental, symplectic-admissible isotopy in the sense that, not only the isotopy preserves the symplectic-admissible character (as necessary for the mapping to qualify as isotopy), and it occurs in the

geometry.

The primary difference between the fundamental symplectic and symplectic-admissible two-forms is given by the fact that, in the former case, the form can be reduced to the antisymmetric matrix with constant elements

$$\omega_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (4.A.4)$$

while this is no longer possible for the latter case. In fact, we have the matrix decomposition into an antisymmetric and a symmetric form

$$S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \left(t_{\mu\nu}(b)\right) \qquad (4.A.5)$$

Thus, while the antisymmetric part is the conventional one, the symmetric component preserves a dependence on the local coordinates.

Finally, under the assumption that the Hamiltonian is the total energy, one should recall (Section 2.4) that the symmetric part represents the nonpotential/nonself-adjoint forces, while no representation of forces occurs via the fundamental symplectic part. Under these conditions, different symmetric parts characterize different systems, that is, Newtonian systems having the same potentila forces, but different nonpotential ones.

We learn in this way that there does not exist, in our terminology, one, single, fundamental, symplectic-admissible form. Instead, we have an infinite number of these forms in which the attached antisymmetric part is the fundamental one, but the symmetric part is unrestriwithin a fixed system of local variables, but the isotopy actually preserves the antisymmetric-fundamental part.

The following extension of Pauli's theorem 4.4.1 is then trivial

THEOREM 4.A.1: Given a general, symplectic-admissible two-form S'_2 on a 2n-dimensional manifold $M(b';S'_2)$ with local coordinates b'^{μ}, $\mu = 1,2, \quad ,2n$, there always exists a diffeomorphism

$$\varphi : M(b', S'_2) \Longrightarrow M(b, s_2) \qquad (4.A.6)$$

realizable via a class C^{∞} and invertible transformation

$$b' \Longrightarrow b = b(b') \qquad (4.A.7)$$

under which the two-form S'_2 reduces to a fundamental symplectic-admissible two-form s_2 up to local, fundamental, symplectic-admissible isotopy.

The geometric reformolation of Darboux's theorem for the symplectic-admissible geometry is more involved, inasmuch it calls for topological aspects which have not yet been investigated. It is therefore proposed according to the following

CONJECTURE 4.A.1. Suppose that the two-form on a manifold M

$$S_2 = S_{\mu\nu}\, db^{\mu} \otimes db^{\nu} \qquad (4.A.8)$$

is non-degenerate jointly with the attached exterior two-form. Then S_2 is a symplectic-admissible two-form, i.e.

$$d\left(S_2 - S_2^{Att}\right) = 0 \qquad (4.A.9)$$

if and only if there exist a chart (U, φ) at each point $m \in M$ such that $\varphi(m) = 0$ and with

$$\varphi(u) = \{x^1, x^2, \dots, x^n, p_1, p_2, \dots, p_n\}$$

the form S_2 can be reduced to a fundamental symplectic-admissible two-form up to local isotopy.

The ideas presented in this appendix were prepared for paper[25] (1977) which subsequently was printed in 1978[191,192] (note, in particular, the footnote 30 at the end of paper 192).

APPENDIX 4.B: A GENERALIZATION OF LIE'S DERIVATIVE FOR THE SYMPLECTIC-ADMISSIBLE GEOMETRY.

In Section 4.4 we recalled that the notion of Lie derivative, Equations (4.4.74),

$$\mathcal{L}_X F = \lim_{t\to 0} \frac{F \circ G_b(t) - F \circ G_b(0)}{t} = XF \qquad (4.B.1)$$

can be first subjected to the conventional, standard realization of the generator X in terms of the vector field

$$X \longrightarrow \omega^{\mu\nu} \Xi_\nu \frac{\partial}{\partial b^\mu} = \omega^{\mu\nu} \frac{\partial H}{\partial b^\nu} \frac{\partial}{\partial b^\mu} \qquad (4.B.2)$$

This yields a canonical realization of the group $G_b(t)$ and of the time evolution law for Hamiltonian vector fields

$$\mathcal{L}_{\Xi} F = [F, H] = \frac{\partial F}{\partial b^\mu} \omega^{\mu\nu} \frac{\partial H}{\partial b^\nu} \qquad (4.B.3)$$

Secondly, Eq. (4.B.1) can be subjected to a Lie covering realization of the generator X (in the terminology of Section 3.3), that in terms of a general symplectic structure Ω_2 according to which Eq. (4.B.2) is replaced by

$$X \longrightarrow \Omega^{\mu\nu} \Xi^*_\nu \frac{\partial}{\partial b^\mu} = \Omega^{\mu\nu} \frac{\partial B}{\partial b^\nu} \frac{\partial}{\partial b^\mu} \qquad (4.B.4)$$

This yields an isotopic image $G^*_b(t)$ of the group $G_b(t)$ and the generalized time evolution law for Birkhoffian vector fields

$$\mathcal{L}_{\Xi^*} F = [F, B]^* = \frac{\partial F}{\partial b^\mu} \Omega^{\mu\nu}(b) \frac{\partial B}{\partial b^\nu} \qquad (4.B.5)$$

The existence of a third, symplectic-admissible realization is then expected. We here define as the Lie-admissible derivative of a function $F(b)$ on $M(b, \hat{S}_2)$ the limit

$$\mathcal{L}_{\hat{X}} F \equiv \hat{\mathcal{L}}_X F = \lim_{t\to 0} \frac{F \circ \hat{G}_b(t) - F \circ \hat{G}_b(0)}{t} \qquad (4.B.6)$$

where now $\hat{G}_b(t)$ is a Lie-admissible genotopic image of the group $G_b(t)$ (Definition 3.3.3) with generator

$$\hat{X} \longrightarrow S^{\mu\nu}(b) \Xi_\nu \frac{\partial}{\partial b^\mu} = S^{\mu\nu} \frac{\partial H}{\partial b^\nu} \frac{\partial}{\partial b^\mu} \qquad (4.B.7)$$

Notice that the existence of limit (4.B.6) is ensured by the fact that the genotope $\hat{G}_b(t)$ is a Lie group. Thus, it can be realized in terms of the standard realization under which limit (4.B.6) exists. The notion of Lie-admissible derivative then follows through redefinition (4.B.7) of the basis.

Quantity (4.B.6) is here called Lie-admissible because, besides being realized in terms of a Lie-admissible group, it admits the conventional canonical realization of a different group, the original group $G_b(t)$, at the limit of null nonconservative and/or symmetry breaking forces, i.e.,

$$\lim_{S^{\mu\nu} \to \omega^{\mu\nu}} \hat{\mathcal{L}}_X F \equiv \mathcal{L}_X F \qquad (4.B.8)$$

Further studies on the Lie-admissible derivative are left to the interested reader. Here we limit ourself to recalling the effectiveness of Lie's derivative for the representation of conservation laws, as evident from its antisymmetric character

$$\mathcal{L}_{\Xi} H = [H, H] \equiv 0 \qquad (4.B.9)$$

These same characteristics become the insufficiencies of Lie's derivative for nonconservative mechanics because now, as familiar from the analysis of Chapters 2 and 3, the objective is to represent time rates of variations. Intriguingly, our Lie-admissible derivatives, at least formally, appears to permit the representation of nonconservation laws In fact, for the case of the total energy, we have the rule

$$\hat{\mathcal{L}}_{\Xi} H = (H, H) = \dot{r} \cdot F^{NSA} \neq 0 \qquad (4.B.10)$$

which provides the time rate of variation of the Hamiltonian exactly as established by experiments.

The most intriguing aspect is that the capability of the Lie-admissible "derivative" to achieve result (4.B.10) is due exactly to its loss of true derivative character, as evident from the loss of totally antisymmetric character.(*)

(*) The preservation of the derivative character is conceivable for flexible symplectic — admissible manifolds, although the point demands specific studies.

APPENDIX 4.C: REPRESENTATION OF NONCONSERVATIVE NEWTONIAN SYSTEMS VIA THE SYMPLECTIC-ADMISSIBLE GEOMETRY

It may be advantageous to illustrate the "mechanics" of the representation offered by the symplectic-admissible geometry. Consider first the conservative vector-field

$$(\dot{b}^{\nu}) = \begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p/m \\ f^{SA}(r) \end{pmatrix} = \left(\Gamma^{\nu}(b) \right) \qquad (4.C.1)$$

$$\nu = 1, 2, \ldots, 2n, \quad r \in \mathbb{R}_n, \quad f^{SA} = -\frac{\partial V}{\partial r}$$

which is manifestly Hamiltonian, that is, it verifies local rule (4.4.53),

$$\Gamma_1^{\omega_2} = \frac{1}{2} \Gamma^1 \rfloor \omega_2 = \omega_{\mu\nu} \Gamma^{\nu} db^{\mu} = \frac{\partial H}{\partial b^{\mu}} db^{\mu}, \qquad (4.C.2)$$

$$H = T + V,$$

which can be explicitly written

$$(\omega_{\mu\nu})(\Gamma^{\nu}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p/m \\ f^{SA} \end{pmatrix} = \begin{pmatrix} -f^{SA} \\ p/m \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial r} \\ \frac{\partial H}{\partial p} \end{pmatrix} \qquad (4.C.3)$$

Suppose now that the "perpetual motion system" (4.C.1) is implemented into a more realistic form inclusive of nonself-adjoint forces

$$(\Gamma^1) \rightarrow (\Xi) = \begin{pmatrix} p/m \\ f^{SA}(r) + F^{NSA}(r, \dot{r}) \end{pmatrix} \qquad (4.C.4)$$

Then, the extended vector-field is no longer Hamiltonian in the physical coordinates r, and $p = m\dot{r}$. Nevertheless, the vector-field is Hamilton-admissible, that is, it verifies the covering (local) rule (4.5.54),

$$\Xi^{\hat{s}_2}_{1} = \Xi \boxtimes \hat{s}_2 = s_{\mu\nu}\,\Xi^{\nu}\,db^{\mu} = (\omega_{\mu\nu} + t_{\mu\nu})\,\Xi^{\nu}\,db^{\mu} = dH = \frac{\partial H}{\partial b^{\mu}}\,db^{\mu} \qquad (4.C.5)$$

which can be explicitly written

$$(\omega_{\mu\nu} + t_{\mu\nu})(\Xi^{\nu}) = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{F^{NSA}}{(p/m)} & 0 \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} p/m \\ f^{SA} + F^{NSA} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{F^{NSA}}{(p/m)} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p/m \\ f^{SA} + F^{NSA} \end{pmatrix} = \begin{pmatrix} -f^{SA} \\ p/m \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial r} \\ \frac{\partial H}{\partial p} \end{pmatrix} \qquad (4.C.6)$$

Eqs. (4.C.5) constitute the geometric counterpart of our Hamilton-admissible equations

$$s_{\mu\nu}\,\dot{b}^{\nu} - \frac{\partial H}{\partial b^{\mu}} = 0 \qquad (4.C.7)$$

Thus, the tensor $s_{\mu\nu}$ is indeed the correct tensor for the lowering of the indeces of the vector field Ξ^{ν}, that is, $s_{\mu\nu}$ is the tensor characterizing the underlying symplectic-admissible geometry. Needless to say, for given equations (4.C.4) the tensor $t_{\mu\nu}$ <u>is not</u> unique. In fact, there exist a variety of different symmetric tensors $t_{\mu\nu}$ capable of representing the considered nonself-adjoint forces in the fixed local coordinates (r,p). Furtermore, additional degrees of freedom are permitted by the Birkhoff-admissible representation of the same system. In fact, besides (4.C.5), we have

also representation (4.5.55), i.e.,

$$[\Omega_{\mu\nu}(b) + T_{\mu\nu}(b)]\,\Xi^{\nu} = \frac{\partial B}{\partial b^{\mu}} \qquad (4.C.8)$$

in which case, in addition to the degrees of freedom of the symmetric tensor $T_{\mu\nu}$, we have additional ones in the structure of the symplectic component $\Omega_{\mu\nu}$, as well as the gauge and isotopic equivalence transformations studied in Chapter 2.

Note that all these degrees of freedom occur within a fixed system of local variables, and they are all such to preserve the joint Lie-admissible/symplectic-admissible character of the representation. Upon implementation of the transformation theory, this dual algebraic/geometric character is preserved, thus giving hopes for a possible coordinate-free, globalization of the symplectic-admissible geometry.

The reader should keep in mind that the Hamiltonian has remained unchanged in the transition from Eqs. (4.C.3) to (4.C.6). Also, the right-hand-sides of these two rules has remained unchanged, i.e.,

$$\omega_{\mu\nu}\,\Gamma^{\nu} \equiv (\omega_{\mu\nu} + t_{\mu\nu})\,\Xi^{\nu} \equiv \frac{\partial H}{\partial b^{\mu}} \qquad (4.C.9)$$

To put it explicitly, the joint implementation of

(a) the vector field, from a conservative to a nonconservative form in the same variables; and

(b) the underlying geometry , ; from the fundamental/symplectic form to a fundamental/symplectic-admissible form characterized by a suitable, additive, symmetric, two-form,

leaves the original one form dH unchanged.

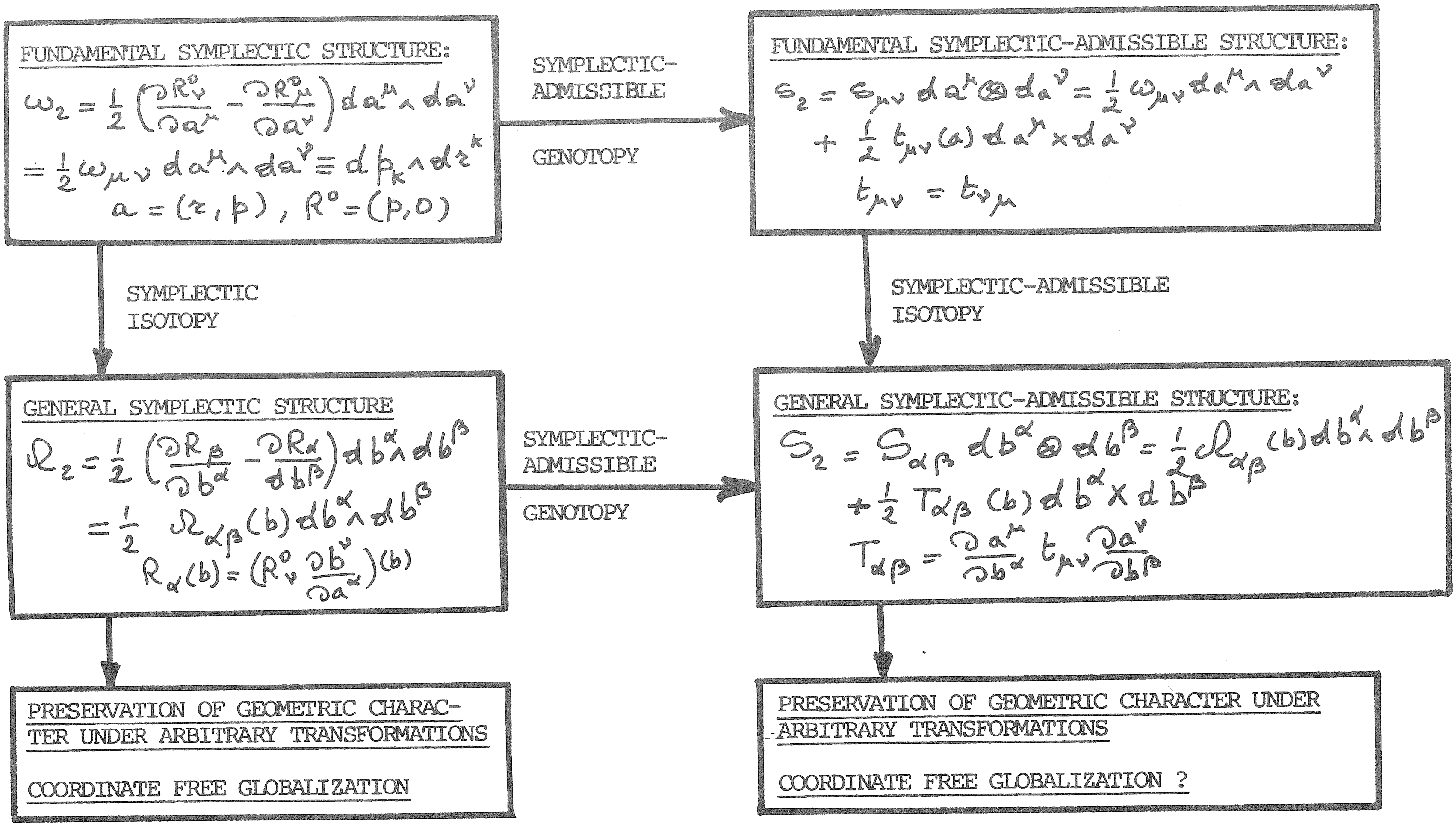


TABLE 4.1: A schematic view of the proposed symplectic-admissible generalization of conventional symplectic two-forms. The fundamental one characterizes Hamilton-admissible equations, while the general one characterizes the Birkhoff-admissible equations in a way fully parallel to the Hamilton/ Birkhoff case. The transition from the fundamental to a general two-form can be done via an arbitrary noncanonical (regular) transformation of the local variables in both cases. The preservation of the symplectic-admissible character under all possible transformations has been proved in Chapter 2. These results give hopes for the possible achievement of a global, coordinate free formulation of the symplectic-admissible geometry in a way parallel, although generalized, of the corresponding symplectic case.

CHAPTER 5

GENERALIZATION OF GALILEI AND EINSTEIN SPECIAL RELATIVITY FOR CLOSED SYSTEMS WITH NON–HAMILTONIAN INTERNAL FORCES ?

5.1: STATEMENT OF THE PROBLEM.

We shall now combine the analysis conducted in this volume and in the preceding one of this series, as well as that of monographs[65,189], and submit a conjecture for a conceivable generalization of the Galilei and Einstein relativity ideas for open nonself-adjoint systems.

Before entering into this task, a few understandings should be stated clearly. Our task is manifestly nontrivial. The reader should, therefore, not expect the construction of the generalized relativities in all necessary details. Instead, as stressed in the Preface, we shall content ourselves with the identification of the structural foundations which are mathematically and physically promising for the future construction of the generalized relativities in all necessary details.

Also, the reader should be aware that we have been forced to eliminate virtually all references to the preceding analysis to prevent an excessive cluttering of the language of this chapter. The reader without at least a superficial knowledge of the Birkhoffian and of the Birkhoffian-admissible generalizations of Hamiltonian mechanics, should therefore not expect a full understanding in a first reading of this chapter.

Finally, it should be understood that the Galilei and Poincaré transformations are, and will remain, the largest possible linear transformations from inertial systems to inertial systems in their respective (non-relativistic and relativistic) carrier spaces. Our analysis is not devoted to linear transformations of inertial systems for numerous reasons. For example, we have identified generalizations of Lie's theory which, by conception, admit only nonlinear representations. Similarly, we are interested in the relativity laws of open systems

which, also by conception, are not in inertial conditions, as we shall see soon.

The central physical notion at the foundation of our generalized relativities is that of closed systems of extended particles with Hamiltonian and non-Hamiltonian internal forces, also called closed, variationally nonself-adjoint . For the nonrelativistic case, the existence of these systems is majectically established by our Earth when seen by an outside observer. For the "relativistic" case (that is, when the speeds of the constituents is of the order of magnitude of that of light in vacuum), we expect that stars are closed nonself-adjoint systems, evidently, because of the conditions of mutual penetration of the charge distributions of hadrons at least in its core, as well as for the existence of a considerable phenomenology essentially dependent on contact interactions. Similarly, and this will be the central topic of Volume III of this series, we expect that hadrons are also closed nonself-adjoint systems. As a matter of fact, the reader familiar with our work will know that all our classical studies, including the entirety of this chapter, are conceived as a primitive Newtonian model of the structure of a hadron.

The arena of applicability of conventional relativities has been subjected to a significant study throughout our analysis, by resulting to be that of closed systems of point-like particles with only action-at-a-distance, potential forces, also called closed, variationally self-adjoint systems.

The primary physical difference between the closed self-adjoint and nonself-adjoint systems is, therefore, given by the transition from point-like to extended particles. In fact, points, being dimensionless, can only interact at a distance. The internal forces can then only be of potential type, and the system is trivially Hamiltonian. On the contrary, when the constituents are extended, and the interactions are of sufficiently short range, the systems possess additional, internal, <u>contact interactions</u> for which the notion of potential energy has no physical meaning. The emerging systems are, therefore, non-Hamiltonian by conception. The prior identification of suitable methods for their treatment is then self-evident.

During our scientific journey, we have identified two different, complementary formulations. They are intended for the dual treatment of the exterior and of the interior case. The advisability to work out this duality of formulations was a consequence of the non-Hamiltonian character of the systems. In fact, only one class of formulations is sufficient for the exterior treatment of a closed Hamiltonian system as well as for the characterization of one of its constituent, as well known.

PART I: EXTERIOR TREATMENT OF CLOSED NONHAMILTONIAN SYSTEMS

The basic nonrelativistic equations are given by the familiar form

$$m_k \ddot{r}_{ka} - f^{SA}_{ka}(\underset{\sim}{r}) - F^{NSA}_{ka}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) = 0 \qquad (5.1.1)$$

$$k = 1, 2, \ldots, N, \quad a = x, y, z$$

subject to the conventional, Galilean, ten conservation laws of total quantities

$$\dot{X}_z(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) = 0, \quad \{X_z\} = \{E_{tot}, \underset{\sim}{P}_{tot}, \underset{\sim}{J}_{tot}, \underset{\sim}{G}_{tot}\}$$

$$z = 1, 2, \ldots, 10 \qquad (5.1.2)$$

which are now interpreted as subsidiary constraints to Eqs. (5.1.1).

We shall avoid unnecessary complexities here, and consider the case when the nonself-adjoint forces verify the conditions

$$\sum_{k=1}^{N} \dot{\underset{\sim}{r}}_k \cdot \underset{\sim}{F}_k^{NSA} = 0, \qquad (5.1.3a)$$

$$\sum_{k=1}^{N} \underset{\sim}{F}_k^{NSA} = 0, \qquad (5.1.3b)$$

$$\sum_{k=1}^{N} \underset{\sim}{r}_k \times \underset{\sim}{F}_k^{NSA} = 0, \qquad (5.1.3c)$$

under which conservation laws (5.1.2) are automatically verified. Eqs. (5.1.3) constitute seven algebraic equations in 3N unknowns, the components of $\underset{\sim}{F}_k^{NSA}$. Trivial solutions therefore exist for $N \geq 3$. The case N = 2 is somewhat special inasmuch as it calls for non-Newtonian forces (e.g., forces dependent on the acceleration).

To identify the analytic, algebraic, and geometric formulations for the characterization of the systems, we construct the equivalent first-order form

$$(\dot{a}^\mu) = \begin{pmatrix} \dot{r}_{ka} \\ \dot{p}_{ka} \end{pmatrix} = (\Xi^\mu) = \begin{pmatrix} p_{ka}/m_k \\ f^{SA} + F^{NSA} \end{pmatrix} \qquad (5.1.4a)$$

$$\mu = 1, 2, \ldots, 6N$$

$$H = T + V; \quad \underset{\sim}{f}_k^{SA} = -\frac{\partial V}{\partial \underset{\sim}{r}^k} \qquad (5.1.4b)$$

$$\underset{\sim}{p}_k = m_k \dot{\underset{\sim}{r}}_k \qquad (5.1.4c)$$

in which one can see that the Hamiltonian H is that of the maximal self-adjoint subsystem.

By conception, vector field (5.1.4) is not Hamiltonian. Thus, <u>all</u>

canonical realizations of analytic, algebraic, and geometric formulations are inapplicable. Nevertheless, under our topological conditions, the vector field is always Birkhoffian. Thus, the exterior treatment of closed non-Hamiltonian systems can be characterized via the following set of mutually compatible formulations.

(I): Analytic profile: Birkhoff's generalization of Hamilton's equations

(II): Algebraic profile: Lie-isotopic generalization of Lie's theory

(III): Geometric profile: Symplectic geometry with arbitrary, exact, (noncanonical) symplectic structures in local realization.

For relativity aspects, it is further assumed that the maximal self-adjoint subsystem is Galilei form-invariant. The nonself-adjoint forces are, therefore, the Galilei symmetry-breaking forces.

Our analysis has made clear that, by no means, space-time symmetry must necessarily be expressed in their simplest possible Lie form. In fact, they can acquire also the structurally more general Lie-isotopic form. This lead to the formulation of the Lie-isotopic generalization of Galilei relativity for the exterior treatment of closed non-Hamiltonian systems proposed in monograph[189].

The relativistic extension of the class of systems considered has been proposed by Santilli in ref.[202]. The main idea is to ensure by construction that the system, when observed from the outside, complies in full with Einstein's special relativity, and that possible deviations-generalizations are admitted only in the interior. We are referring

here, for instance, to a proton in a particle accelerator. The compliance of the center-of-mass behaviour of this system with the special relativity is incontrovertible. Nevertheless, by no means this is evidence that exactly the same relativity and physical laws must hold for the constituents of the proton, as we shall see in detail in Volume III.

Let M(3.1) be a Minkowski space with metric tensor $(g^{\mu\nu}) = (-,+,+,+)$. Let X^μ be the center-of-mass four-vector. The first necessary conditions to ensure that the system is conform with Einstein's special relativity is that the center-of-mass four-vector verifies the familiar Lorentz-invariant separation

$$dX^\mu dX_\mu = d\underset{\sim}{X}\cdot d\underset{\sim}{X} - dX_0^2 = -c^2 d\tau^2 \qquad (5.1.5)$$

where τ is the proper time of the system [or, more exactly, of the center-of-mass trajectory]. Next, we introduce the total four-momentum

$$P^\mu = M_0 c V^\mu \;, \quad V^\mu = dX^\mu/d\tau \qquad (5.1.6)$$

where M_0 is the total rest mass. The total angular momentum tensor is then given by the familiar form

$$J^{\mu\nu} = (X^\mu P^\nu - P^\nu X^\mu) - I^{\mu\nu} \qquad (5.1.7)$$

where $I^{\mu\nu}$ represents the intrinsic component. The relativistic generalization of the ten, Galilean conservation laws (5.1.2) is then given by the familiar equations

$$\frac{d}{d\tau} P^\mu = 0 \;, \quad \frac{d}{d\tau} J^{\mu\nu} = 0 \qquad (5.1.8)$$

$$\mu, \nu = 0, 1, 2, 3$$

which imply the conservation of Casimir invariant of the Poincaré algebra

$$\frac{d}{d\tau}(P^\mu P_\mu) = 0 \quad , \quad \frac{d}{d\tau}(W^\mu W_\mu) = 0 \qquad (5.1.9)$$

where W^μ is the Pauli-Lubanski four-vector

$$W^\mu = \frac{1}{2} \varepsilon^{\mu\alpha\beta\gamma} J_{\alpha\beta} P_\gamma \qquad (5.1.10)$$

Eqs. (5.1.8) ensure that the system considered is closed in the most conventionally possible way. The ensurance of full compatibility with the special relativity calls for the handling of a number of additional aspects which are known in the literature (see, for instance, the presentation by van Dam and Ruijgrok[203]). These aspects will be ignored here for brevity, and kept in mind for possible subsequent refinements.

PART II: INTERIOR TREATMENT OF CLOSED NONSELF-ADJOINT SYSTEMS

We consider now ONE constutuent of a closed nonself-adjoint system, called eleton in Volume I, and interpret the rest of the system as external. Thus, the eleton is in nonconservative conditions as a necessary condition to be in interaction [recall that when all its physical characteristics are conserved, a particle is free]. We shall then write the equation of motion of the Newtonian form of the eleton

$$m\ddot{\underset{\sim}{r}} - \underset{\sim}{f}^{SA}(\underset{\sim}{r}) - \underset{\sim}{F}^{NSA}(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) = 0 \qquad (5.1.11)$$

now without any subsidiary condition.

For the case of a Galilean system , the particle considered would essentially experience the nonconservation of the energy and of the linear momentum. However, the symmetry under rotation is preserved. In fact, Galilean systems achieve stability via a collection of orbits each one of which is stable.

At any rate, Galilean particles are massive points which, being dimensionaless, do not experience collisions, contact interactions, and other effects causing orbit instabilities.

In the broader class of closed non-Galilean systems, eletons are in the highest conceivable conditions of nonconservation, clearly, as a necessary condition to maximize the interactions. We reach in this way the ten nonconservation laws of a constituent of a closed non-Hamiltonian system, as established by clear, Newtonian, experimental evidence

$$\dot{X}_r = \frac{\partial X_r}{\partial \dot{\underset{\sim}{r}}} \cdot \underset{\sim}{F}^{NSA} \neq 0 \qquad (5.1.12a)$$

$$\{X_r\} = \{E_{tot}, \underset{\sim}{p}, \underset{\sim}{J}, \underset{\sim}{G}\}_{particle} \qquad (5.1.12b)$$

The insistence in the nonconservation of the angular momentum should be emphasized, to prevent the illusion of novelty. The physical cases we are referring here are those occurring in Earth's environment (interior problem), such as the trajectory of a satellite in Earth's atmosphere. The decaying of the orbit then calls, as a necessary condition of consistency, for the nonconservation of the angular momentum. Unless this crucial nonconservation is realized, one risks the preservation of the Galilean treatment for the satellite, that is, its approximation as a massive point, with consequential ignorance of contact interactions, and of the instability of the orbit.

We recover in this way an important property of closed nonself-adjoint systems, that the global stability of the system is achieved via a collection of orbits each of which is unstable. This is clearly illustrated by the case of the Earth's satellite. In fact, the nonconservation of its angular momentum, by no means, affects the conservation of the total angular momentum of Earth.

Thus, the nonconservation laws of the physical characteristics of the individual constituents should always be compatible with the conservation laws of total quantities.

As familiar to the readers who have followed our journey, the Lie-isotopic product is totally antisymmetric and, therefore, naturally set for conservation laws. In order to represent more effectively the non-conservative character of the system considered, we advocate the treatment of the interior problem via the following additional, generalized, formulations.

(I'): <u>Analytic profile</u>: Birkhoff-admissible generalization of Birkhoff's equations with Hamilton-admissible particularization.

(II'): <u>Algebraic profile</u>: Lie-admissible generalization of Lie's theory.

(III'): <u>Geometric profile</u>: Symplectic-admissible generalization of the Birkhoffian realization of the symplectic geometry, characterized by general, symplectic-admissible structures on T^*M.

These are the essentially tools we shall use to submit the conjecture of a second generalization of Galilei relativity, this time for the constituents of a closed non-Hamiltonian system.

The relativistic extension on $M(3.1)$ is straightforward[202]. Decompose the total linear and angular momentum into the familiar component form

$$P^\mu = \sum_{k=1}^{N} p_k^\mu \ , \quad J^{\mu\nu} = \sum_{k=1}^{N} j_k^{\mu\nu} \qquad (5.1.13)$$

An open non-Hamiltonian system is truly reached when each component verifies the ten <u>relativistic nonconservation laws</u>

$$\frac{d}{d\tau} p_k^\mu \neq 0 \ , \quad \frac{d}{d\tau} j_k^{\mu\nu} \neq 0 \ , \quad \mu,\nu = 0,1,2,3, \ k = 1,2,\ldots,N \qquad (5.1.14)$$

where, we keep in mind a multiple time theory, again, for possible future refinements.

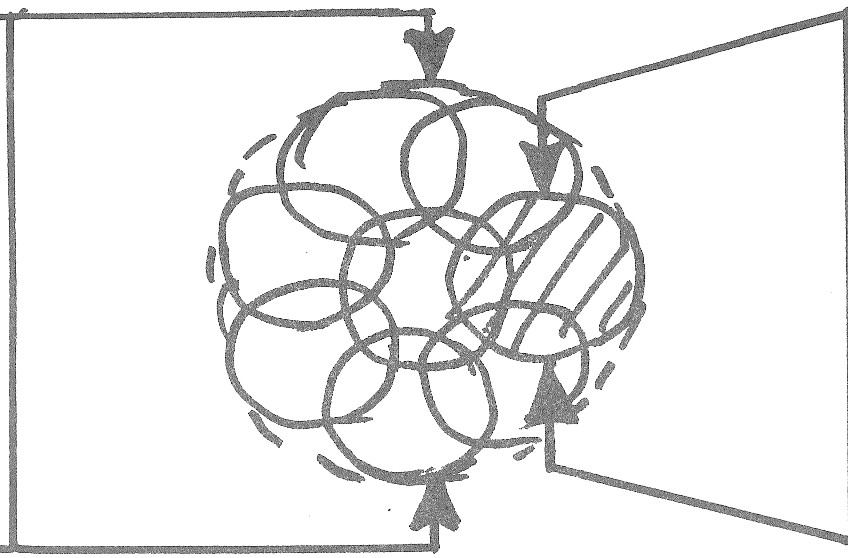
The insistence in the nonconservation law of the angular momentum should be emphasized again. In fact, in case the orbits of the constituents are individually stable, the operator image of the system on a Hilbert space essentially recovers the atomic structure offered by quantum mechanics. One, therefore, has only the illusion of novelty. When the interactions of the systems are realized instead in their maximal conceivable form, that is, in the form permitting the <u>maximal possible nonconservations of the characteristics of the constituents</u>, then the foundations for possible novel advances are set.

The compatibility of the systems of equations (5.1.8) and (5.1.13) is evident. In fact, there always exist 10N functions p_k^μ and $j_k^{\mu\nu}$ verifying the ten equality conditions (5.1.8) and any desired number of

CLOSED VARIATIONALLY NONSELF-ADJOINT SYSTEMS

[Isolated systems of extended particles with Hamiltonian and non-Hamiltonian internal forces]

EXTERIOR PROBLEM: Representation of total conservation laws under non-Hamiltonian internal forces via:
1-Birkhoff's generalization of Hamilton's equations;
2-Lie-isotopic generalization of Lie algebras; and
3-Symplectic geometry in noncanonical local forms.



INTERIOR PROBLEM: Representation of the maximal possible nonconservation laws under non-Hamiltonian forces via:
1-Birkhoff-admissible generalization of Birkhoff's equations;
2-Lie-admissible generalization of Lie-isotopic algebras; and
3-Symplectic-admissible generalization of symplectic geometry.

FIGURE 5.1

inequality conditions. Equivalently, the existence of a consistent relativistic extension of closed nonself-adjoint systems can be imposed to relativistic treatments as a necessary condition of compliance with physical reality.

We now pass to the identification of the reference frames to be used in the analysis. It is evident that we have three frames.

(A) THE OBSERVER FRAME. This is evidently an external fram at rest with the measuring apparatus. This frame is not inertial, strictly speaking. Nevertheless, it is often approximated as being inertial.

(B) THE CENTER-OF-MASS FRAME. This is the frame at rest with the particle and with origin at its center-of-mass. This frame should be generally assumed to be noninertial, to avoid excessive approximations. As an example, the center-of-mass of a proton in a particle accelerator is noninertial.

(C) THE CONSTITUENT FRAME. This is the frame at rest with the constituent considered. It is evident that, as a necessary condition for the consistency of the model, this frame must be noninertial.

Our dual generalizations of the Galilei relativity for the exterior and for the interior problem are constructed in the center-of-mass frame. Once the analytic-algebraic-geometric structures of the relativities have been identified in this fram, then the transition to different frames is permitted.

Thus, one can first conceive the closed nonself-adjoint system as being at rest in the observer frame. Subsequently, one can allow motions

of the center-of-mass.

This aspect brings into light an important feature of our methods, the preservation of:

- the derivability from a variational principle;
- the algebraic character, and
- the geometric structure,

under all possible, generally nonlinear and noncanonical transformations. Note that this is the case individually for the Birkhoff/Lie-isotopic/symplectic formulations and for the Birkhoff-admissible/Lie-admissible/symplectic-admissible ones.

We hope that the reader can begin to see the reasons for our rather long journey. In fact, the achievement of analytic/algebraic/geometric methods which preserve their structure under arbitrary (noncanonical) transformations, was an evident pre-requisite for attempting the formulation of covering relativities.

We pass now to the identification of the notion of form-invariance used in the analysis.

The Galilei and Einstein relativities essentially represent systems which are form invariant under Galilei's and Poincaré's transformations

$$\begin{aligned} G &: \underset{\sim}{r} \rightarrow \underset{\sim}{r}' = R\,\underset{\sim}{r} + \underset{\sim}{v}_0 t + \underset{\sim}{r}_0 , \quad R \in O(3) \\ P &: x \rightarrow x' = \Lambda x + x_0 , \quad \Lambda \in O(3.1) \end{aligned} \qquad (5.1.15)$$

As stressed throughout our analysis, these relativities can only treat closed systems of point-like particles with action-at-a-distance/potential internal forces.

PRIMARY REFERENCE FRAMES OF A CLOSED NON-HAMILTONIAN SYSTEM

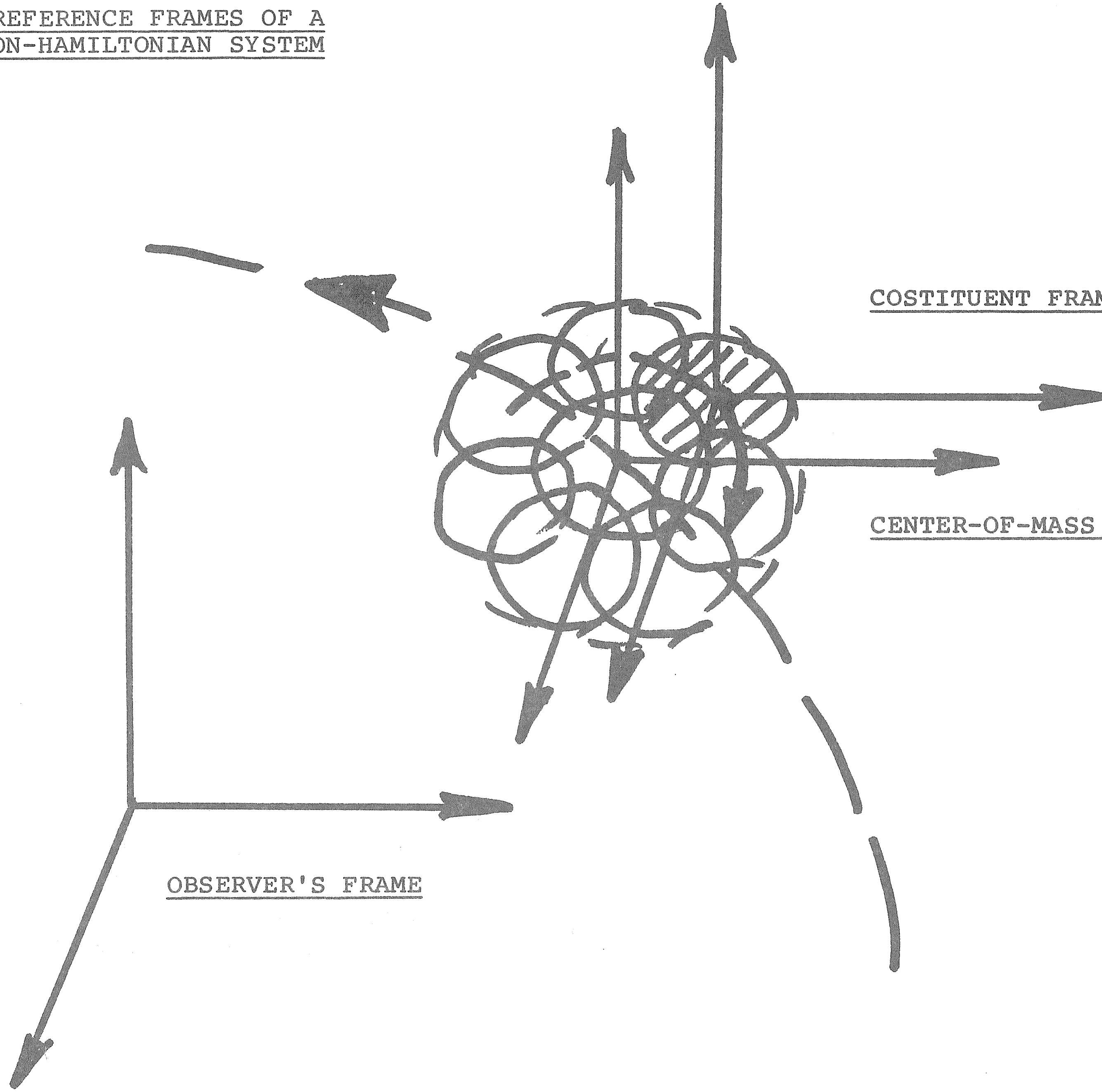


FIGURE 5.2

In order to characterize the more general class of closed non-Hamiltonian systems, we search for covering symmetry trnasformations, that is, transformations which:

(1) constitute Lie-isotopic (Lie-admissible) groups for the exterior (interior) case;

(2) constitute symmetries of the vector fields in the conventional sense reviewed earlier; and, last but not least,

(3) constitute coverings of transformations (5.1.15) in the sense that the latter are recovered identically when all nonselfadjoint/symmetry-breaking forces are null.

To put is differently, rather then abandoning fundamental transformations (5.1.15), we assume them at the foundation of our studies. Nevertheless, we do not assume them of terminal physical value, and search instead for generalization which avoids excessively simplistic approximations of particles and their interactions.

5.2: LIE-ISOTOPIC AND LIE-ADMISSIBLE GENERALIZATION OF THE GROUP OF ROTATIONS

There is little doubt that a generalization of the group of rotations is a necessary prerequisite for any attempt to generalize the Galilei and Einstein relativities. In fact, the group of rotations is of such fundamental character, mathematically and physically, to the point that the preservation of its conventional, Lie, exact character may, imply, whether directly or indirectly, the preservation of the entire Galilei and Poincaré symmetries.

The contemporary formulation of the group of rotations fulfills in its entirety the physical objectives for which it was conceived, with particular reference to the characterization of the orbital and intrinsic angular momentum of particles assumed as point-like.

Our particles (eletons), however, are extended. In turn, this is per se sufficient to imply a necessary generalization of the group of rotations for the very simple reason that rigid objects do not exist in nature. Thus, the main idea of the desired covering of the group of rotations is that of jointly representing rotations and deformations, under the condition of time rate of variation of the angular momentum, whether orbital or intrinsic.

Needless to say, we may wish to preserve the value of the intrinsic angular momentum in specific conditions in a way compatible with deformations of the spherical symmetry of the eletons (i.e., of its charge distribution). However, the desired tools should be conceived first to provide, in general, time rate of variations of physical quantities for the interior case, along the lines of Section 5.1.

It is evident that the conventional group of rotations is unable to achieve the objectives considered. In fact, the group has been historically associated with the rotations of rigid bodies.

Let us elaborate better the novel physical situation we are referring to.

Consider a neutron as depicted by conventional quantum mechanics, that is, as a point-like particle. Under these assumptions, the rotational symmetry of the neutron CANNOT be broken, no matter what interaction is permitted.

Consider now the neutron as it is in the physical reality: an extended charge distribution with a charge radius of about one Fermi (10^{-13}cm). Then, the breaking of the rotational symmetry becomes trivially possible. In fact, it may occur under collisions/impact/interactions with other particles which are sufficient to produce a deformation of its charge distribution [as expected, for instance, when the neutron collides and penetrates within a nucleus].

Once one recognizes that perfectly rigid objects exists only in the imagination of a physicist, but not in the physical reality, the BREAKING of the rotational symmetry becomes unavoidable for extended particles. The only debatable aspect is the AMOUNT OF BREAKING, which must be resolved via experiments (currently well under way, as we shall review in Volume III).

Another understanding which should be stressed again at the risk of being repetitive, is that the time rate of variations of angular momenta are conceivable for the INTERIOR PROBLEMS ONLY [e.g., for the neutron considered earlier while the collision with a nucleus is considered as

external]. It is evident that the conservation of the angular momentum for the closed extension of the neutron, that inclusive also of the nucleus, is out of the question.

A final understanding to reach the level of studies presented here is that, while a broken symmetry is left mathematically and physically undefined in the contemporary literature, we are interested here in reaching a COVERING SYMMETRY which replaces the old, insufficient, broken, one.

To put it differently, the terms "broken rotational symmetry" may ultimately be reduced to semantic. In fact, these terms have sense in our study only if referred to the SIMPLEST POSSIBLE realization of the group of rotations, that of the contemporary literature. If, instead, the terms "rotational symmetry" are referred to the most general possible form , then the symmetry is exact for our analysis, as we shall soon see.

The contemporary (physical) definition of the group of rotations, denoted with O(3), is the set of orthogonal transformations in E(3), i.e., the transformations

$$\underset{\sim}{r} \rightarrow \underset{\sim}{r}' = R\,\underset{\sim}{r} \tag{5.2.1}$$

where the R-matrices are orthogonal

$$R R^t = R^t R = I \tag{5.2.2}$$

This implies that the transformations leave invariant the bilinear form in E(3)

$$\underset{\sim}{r}^2 = r^t r \longrightarrow \underset{\sim}{r}'^2 = r'^t r' = r^t r, \quad r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{5.2.3}$$

and have determinant ±1, i.e.,

$$\det(RR^t) = (\det R)^2 = 1 \qquad (5.2.4)$$

If one calls $J_i = (\underline{r} \times \underline{p})_i$, $i=1,2,3$, the infinitesimal (Hermitian) generators of rotations, Lie's theory of O(3) can be formulated beginning with the universal enveloping associative algebra $A(\underline{O}(3))$ with infinite dimensional basis characterized by Theorem 3.4.1, i.e.,

$$A(\underline{O}(3)):\ 1,\ J_i,\ J_iJ_j,\ J_iJ_jJ_k,\ \ldots \qquad i \leq j,\quad i \leq j \leq k \qquad (5.2.5)$$

where J_iJ_j is the conventional associative product. The Lie algebra $\underline{O}(3)$ of the group O(3) is then (homomorphic to) the attached $[\underline{A}(O(3))]^-$, and verifies the familiar commutation rules

$$\underline{O}(3):\ [J_i, J_j]_A = J_iJ_j - J_jJ_i = i\varepsilon_{ijk} J_k \qquad (5.2.6)$$

Under sufficient integrability conditions, basis (5.2.5) permits the exponentiation to the finite group

$$O(3):\ R(\theta) = R(\theta_1,\theta_2,\theta_3) = e^{i\theta_1 J_1}\, e^{i\theta_2 J_2}\, e^{i\theta_3 J_3} \qquad (5.2.7)$$

where the three parameters are Euler's angles.

These rudimentary notions are sufficient for our objectives. The extension of the treatment to the conventional SU(2) covering of O(3) will be considered as a refinement [for a recent, comprehensive study of the conventional rotational symmetry, we refer the interested reader to the two monographs by L. C. Biedenharn and J. D. Louck[204]].

To construct the desired generalizations, we simply put to work the Lie-isotopic and the Lie-admissible generalizations of Lie's theory worked out in Chapter 3. As a matter of fact, this is the best way to test their capabilities.

PART I: LIE ISOTOPIC COVERING OF THE ROTATION GROUP

The basic algebra of the rotation group is the associative algebra A of infinitesimal generators J_i in its simplest most conceivable product J_iJ_j. The isotopic covering of the rotation group is essentially produced by the generalization of the algebra A into in the following simple, yet less trivial isotopic form

$$\hat{A}(\underline{O}(3)):\quad J_i * J_j \stackrel{\text{def}}{=} J_i\, T\, J_j\,, \qquad (5.2.8)$$

where T is a fixed nonsingular element for all possible products, whose nature, dependence, etc. will be specified below.

The non-triviality of generalization (5.2.8) can be seen from the fact that the conventional unit of the old associative algebra A

$$I:\quad IA = AI = A \qquad (5.2.9)$$

is now changed into the isotopic unit

$$\hat{I}:\quad \hat{I} * A = A * \hat{I} = A\,,\quad \hat{I} = T^{-1} \qquad (5.2.10)$$

This is sufficient to imply a generalization of the entire Lie theory of the group of rotations (as well as of the entire Galilei and Poincaré groups, as we shall see). In fact, isotopic product (5.2.8) generalizes the enveloping algebra (5.2.5). The generalization of the attached Lie algebra (5.2.6) and of the Lie group (5.2.7) is then a mere consequence.

We cannot possible reconstruct in this section the entire theory of the group of rotations, as presented, say, in monographs[204]. We shall, therefore, content ourselves with the identification of the foundations of the isotopic covering. More details are provided in a separate, forthcoming paper.

The basic invariant (5.2.3) of the old algebra becomes under isotopy

$$\underset{\sim}{r}^{\hat{2}} = r^{t} * r = r^{t} T r \qquad (5.2.11)$$

Consider now the isotopic transformations

$$r' = \hat{R} * r = \hat{R} T r \; ; \; r'^{\hat{t}} = r^{\hat{t}} * \hat{R}^{\hat{t}} = r^{\hat{t}} T \hat{R}^{\hat{t}} \qquad (5.2.12)$$

where $\hat{t}$ denotes the operation of transpose in the new algebra $\hat{A}$ to be identified below. We shall call (5.2.12) the <u>isotopic covering of rotations</u>, or <u>isotopic rotations</u> for short, when they leave invariant the correct length of the vector $\underset{\sim}{r}$, i.e., when

$$\begin{aligned} r'^{\hat{t}} * r' &= r^{\hat{t}} * \hat{R}^{\hat{t}} * \hat{R} * r = r^{\hat{t}} T \hat{R}^{\hat{t}} T \hat{R} T r \\ &= r^{t} * r = r^{t} T r \end{aligned} \qquad (5.2.13)$$

It is easy to see that this is the case, if and only if

$$\begin{aligned} \hat{R}^{\hat{t}} * \hat{R} &= \hat{R} * \hat{R}^{\hat{t}} = \hat{I} \\ (\det \hat{R})^{2} \det(T) &= \det(T^{-1}) \end{aligned} \qquad (5.2.14)$$

or, equivalently, in terms of matrix elements, if and only if

$$R_{rs} T_{rp} R_{pq} = \hat{\delta}_{sq} = \hat{I} \delta_{sq} \qquad (5.2.15)$$

where $\hat{\delta}_{sq}$ is the isotopic generalization of the Kronecker product

introduced by Myung and Santilli in ref.[205].

The covering nature of conditions (5.2.14) over the old ones (5.2.2) is self-evident.

We must now identify the relation between the "transpose" and the "inverse". For this purpose, we introduce the inverse transformations

$$r = \hat{R}^{-\hat{1}} * r' = \hat{R}^{-\hat{1}} T r' \qquad (5.2.16)$$

The identities

$$\begin{aligned} r &= \hat{R}^{-\hat{1}} * r' = \hat{R}^{-\hat{1}} * \hat{R} * r \\ r' &= \hat{R} * r = \hat{R} * \hat{R}^{-\hat{1}} * r' \end{aligned} \qquad (5.2.17)$$

can then hold if and only if

$$\hat{R}^{-\hat{1}} * \hat{R} = \hat{R} * \hat{R}^{-\hat{1}} = \hat{I} \qquad (5.2.18)$$

which is the correct definition of inverse under isotopy.[205] Consid now the operation

$$\mathcal{G} = \hat{R}^{t} * \hat{R} * \hat{R}^{-\hat{1}} \qquad (5.2.19)$$

and compute it according to the following different associations

$$\begin{aligned} \mathcal{G} &= (\hat{R}^{\hat{t}} * \hat{R}) * \hat{R}^{-\hat{1}} = (\hat{R}^{\hat{t}} T \hat{R}) T \hat{R}^{-\hat{1}} \\ &= \hat{R}^{\hat{t}} * (\hat{R} * \hat{R}^{-\hat{1}}) = \hat{R}^{\hat{t}} T (\hat{R} T \hat{R}^{-\hat{1}}) \end{aligned} \qquad (5.2.20)$$

By using first (5.2.14) and then (5.2.18) we trivially reach the rule

$$\hat{R}^{\hat{t}} = \hat{R}^{-\hat{1}} \qquad (5.2.21)$$

which is the desired isotopic covering of the old relation.

The nontriviality of the generalization should be stressed. In fact, by using (5.2.14) the full expression of the transpose in $\hat{A}$ in terms of the inverse in A is given by

$$\hat{R}^{\hat{t}} = \hat{I}\, R^{-1}\, \hat{I} = T^{-1} R^{-1} T^{-1} \qquad (5.2.22)$$

Next, in order to reach a covering symmetry, we must verify that the isotopic rotations constitute a group. Evidently, expansion (5.2.7) is meaningless in $\hat{A}$. By following Myung and Santilli[205], we, therefore, introduce the <u>isotopic exponentiation</u> for one generator J and one parameter θ

$$\hat{O}(1):\ \hat{R}(\theta) = \hat{e}^{i\theta * J} = \hat{I}\left(1 + \frac{i\theta * J}{1!} + \frac{(i\theta * J)*(i\theta * J)}{2!} + \dots\right) \qquad (5.2.23)$$

which can be expressed in terms of the exponentiation in A according to the rules

$$\hat{e}^{i\theta * J} = \hat{I}\, e^{i\theta T J} \equiv e^{iJT\theta}\, \hat{I} \qquad (5.2.24)$$

It is easy to see that the set of transformations (5.2.23) for all possible values of the parameter θ form a one-dimensional group of transformations continuously connected to the identity transformation. In fact, the continuity is trivially ensured by the property

$$\hat{R}(\theta)\Big|_{\theta=0} = \hat{e}^{i\theta * J}\Big|_{\theta=0} = \hat{I} \qquad (5.2.25a)$$

$$r'\Big|_{\theta=0} = \hat{R} * r\Big|_{\theta=0} \equiv r \qquad (5.2.25b)$$

while the group composition law reads

$$\hat{R}(\theta) * \hat{R}(\theta') = \hat{R}(\theta + \theta') \qquad (5.2.26)$$

and similarly one can verify other conditions, such as the existence of the inverse

$$\hat{R}(\theta) * \hat{R}(-\theta) = \hat{R}(0) = \hat{I} \qquad (5.2.27)$$

The generalization to more than one dimension is possible provided that the reader understands that the conventional commutators (5.2.6) are mathematically <u>inconsistent</u> for the framework under consideration, and must be replaced by the <u>isotopic commutation rules</u>

$$[A, B]_{\hat{A}} \stackrel{def}{=} [A, B]^* = A * B - B * A \qquad (5.2.28)$$

As a result, the isotopic generalization of the Baker-Campbell-Hausdhorff formula reads

$$\hat{e}^{\theta * A} * \hat{e}^{\theta * B} = \hat{e}^{\theta * C} \qquad (5.2.29)$$

$$C = A + B + \frac{1}{2}[A, B]^* + \frac{1}{12}\Big[(A - B), [A, B]^*\Big]^* \qquad (5.2.29b)$$

These results permit us to define the <u>isotopic generalization of the group of rotations</u>, or <u>isotopic rotation group</u> for short, via the structure

$$\hat{O}(3):\ \hat{R}(\theta_1, \theta_2, \theta_3) = \hat{e}^{i\theta_1 * J_1} * \hat{e}^{i\theta_2 * J_2} * e^{i\theta_3 * J_3} \qquad (5.2.30)$$

To reach a preliminary understanding of this structure, we must identify its dimension. In turn, this calls for the identification of the Lie-isotopic algebra underlying structure (5.2.30).

For this purpose, we shall assume that the generators of O(3) and of $\hat{O}(3)$ coincide, as it is typically the case for all isotopies. This implies the preservation of the realization of the generators J_i and of their commutation rules (5.2.6) also for the isotopic setting [as we shall see in Volume III, this is not necessarily the case in all (isotopic) Hilbert space because, even under the preservation of the conventional realization $\underset{\sim}{J} = \underset{\sim}{r} \times \underset{\sim}{p}$, the operator $\underset{\sim}{p}$ does not necessarily acquire the conventional realization $\underset{\sim}{p} = i\,\partial/\partial \underset{\sim}{r}$]. Under these assumptions, we consider the empirical rule

$$[J_i, J_j]^* = \hat{I}\,[TJ_i, TJ_j] \qquad (5.2.31)$$

which permits the computation of the new, isotopic, commutation rules vis the old ones. Trivial calculations, then yield the following <u>isotopic generalization of the Lie algebra of the rotation group</u>, or <u>isotopic rotation algebra</u> for short

$$\hat{\underline{O}}(3):\ [J_i, J_j]_{\hat{A}} = \varepsilon_{ijk}\, T J_k + [J_i, T]_A J_j + [T, J_i]_A J_j \qquad (5.2.32)$$

At this point, additional information on the isotopic element T is needed for further progress. As we shall see below in the examples, the isotopic element T does depend in general on the base manifold, that is, for the case considered, on the Carthesian coordinates $\underset{\sim}{r}$, on the linear momentum $\underset{\sim}{p}$, as well as possible additional parameters outside the carrier space of $\hat{O}(3)$, such as time, angles, densities, etc.,

$$T = T(t, \underset{\sim}{r}, \underset{\sim}{p}, \theta_1, \theta_2, \theta_3, \ldots) \qquad (5.2.33)$$

Under these conditions, it is easy to see that, while the original algebra $\underline{O}(3)$ is three-dimensional, as well known, the isotopic image (5.2.32) is in general of <u>infinite-dimension</u>. This is clearly due to the fact that the commutators

$$Z_i = [J_i, T]_A \qquad (5.2.34)$$

can produce elements outside A, in such a way to prevent the achievement of closure rules

$$[J_i, J_j]_{\hat{A}} = \hat{C}_{ij}^{\ k}\, J_k \qquad (5.2.35)$$

for finite dimension.

However, in most of the cases in which we are interested, the isotopic element T is in the center of the original algebra A, i.e.,

$$[J_i, T]_A \equiv 0\,, \quad i = 1,2,3 \qquad (5.2.36)$$

Under these conditions, that is,

(a) under the assumption of the preservation of the generators $\underset{\sim}{J} = \underset{\sim}{r} \times \underset{\sim}{p}$ of O(3) also for the isotopic generalization;

(b) under the assumption that the realization of the quantity "$\underset{\sim}{p}$" is the same under isotopy and that, therefore, the commutation rules of $\underline{O}(3)$ remain unchanged; and

(c) under the assumption that the isotopic element T commutes with all elements of $A(\underline{O}(3))$,

the isotopic rotational algebra is of dimension three and verifies the commutation rules

$$\hat{\underline{O}}(3):\quad [J_i, J_j]_{\hat{A}} = [J_i, J_j]^* = \varepsilon_{ijk}{}^*\, J_k \qquad (5.2.37)$$

The reader should be warned that the similarity with commutation rules (5.2.3) is deceptive, as we shall illustrate below.

The isotopic generalization of the $\underset{\sim}{O}(3)$ Casimir invariant, is evidently given by

$$\hat{C}_2 = J^{\hat{2}} = J^t * J = J^t T J, \qquad J = \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} \quad (5.2.38)$$

The use of the isotopic differential rule

$$[A * B, C]^* = A * [B, C]^* = [A, C]^* * B \quad (5.2.39)$$

then permits the calculation of the expression for the general case

$$[J^{\hat{2}}, J_k]_{\hat{A}} = J_i * [J_i, J_k]_{\hat{A}} + [J_i, J_k]_{\hat{A}} * J_i$$
$$= J_i * \varepsilon_{ijk} * J_e + \varepsilon_{ike} * J_e * J_i \quad (5.2.40)$$

which is generally nonnull. However, for the case of rules (5.2.37), we have

$$[J^{\hat{2}}, J_k]_{\hat{A}} = \varepsilon_{i\ e} * \{J_i, J_e\}_{\hat{A}} \equiv 0 \quad (5.2.41)$$

that is, $J^{\hat{2}}$ is an <u>isotopic Casimir invariant</u>, i.e., it is an element of the center of the isotopic enveloping algebra $\hat{A}$.

Almost needless to say, the old Casimir is no longer invariant for

$\hat{\underset{\sim}{O}}(3)$, i.e.,

$$[J^2, J_k]_{\hat{A}} = J^t J T J_k - J_k T J^t J \neq 0 \quad (5.2.42)$$

as it can be seen, e.g., from the breakdown of the differential rule

$$[AB, C]^*_{\hat{A}} \neq A[B, C]_{\hat{A}} + [A, C]_{\hat{A}} B \quad (5.2.43)$$

We would like now to identify the isotopic generalization of the transformation laws under rotations. Consider a unit vector $\underset{\sim}{u}$ characterizing the direction of rotation in E(3), and let $\underset{\sim}{u} \cdot \underset{\sim}{J}$ be the component of $\underset{\sim}{J}$ along $\underset{\sim}{u}$. Then the isotopic rotation around $\underset{\sim}{u}$ can be written

$$\hat{R}_u(\theta) = e^{\hat{}\, i\, \theta * (\underset{\sim}{u} \cdot \underset{\sim}{J})} \quad (5.2.44)$$

and the components of $\underset{\sim}{J}$ transform according to the law

$$J'_k = \hat{R}_u(\theta) * J_k * \hat{R}_u(\theta)^{-\hat{1}} \quad (5.2.45)$$

which can be explicitly written

$$J'_k = \hat{I}\, e^{i\theta T(\underset{\sim}{u} \cdot \underset{\sim}{J})}\, T J_k T\, e^{-i(\underset{\sim}{u} \cdot \underset{\sim}{J}) T\theta}\, \hat{I}$$
$$= J_k + [i\theta T(\underset{\sim}{u} \cdot \underset{\sim}{J}), J_k]_{\hat{A}} \quad (5.2.46)$$
$$+ [i\theta T(\underset{\sim}{u} \cdot \underset{\sim}{J}), [i\theta T(\underset{\sim}{u} \cdot \underset{\sim}{J}), J_k]_{\hat{A}}]_{\hat{A}} + \cdots$$

When T is a (dimensionless) scalar we, therefore, have the isotopic transformation rules

$$J'_k = J_k \cos(\theta T) + u_k(\underline{u}\cdot\underline{J})[1-\cos(\theta T)] + \underline{u}\times\underline{J}\,\sin(\theta T) \qquad (5.2.47)$$

with the understanding that, when T is not a scalar, the transformation laws are nontrivial.

We are now sufficiently equipped to identify the significance of the Lie-isotopic group $\hat{O}(3)$ for qeneralized relativities.

The reader will recall that the fundamental role of the conventional rotation group within the context of the relativities of point-like particles is the characterization of the ISOTROPY OF THE MEDIUM in which the particle move. As well known, for the case of Galilean-Einstenian particles, this medium is the vacuum.

The generalized relativities we are interested in are conceived, instead, for particles moving in a material medium constituted by other particles (called hadronic medium in Volume I for the case of strong interactions). The primary purpose of the proposed generalization of the rotation group is to represent the ANISOTROPY OF THE MEDIUM in which the particle move. In fact, once the transition from vacuum to ordinary matter is performed, all conceptual, theoretical, and mathematical prerequisities for isotropy fail to apply in favor of the anisotropy of our material world.

The realization of anisotropy permitted by our generalized rotation group is so direct and simple, to appear trivial. Recall that the isotropy of space is rooted in the foundation of the theory, the basic

invariant

$$r^2 = r^1 r^1 + r^2 r^2 + r^3 r^3 \qquad (5.2.48)$$

Thus, in order to represent the anisotropy of the medium considered, we must alter invariant (5.2.48) into the anisotropic form suggested by experimental data (e.g., variation of the density in time, or in space, etc.). We reach in this way the following generalization of invariant (5.2.48)

$$\begin{aligned} r^{\hat{2}} &= r^1 t_{11}(t,r,p,\dots) r^1 + r^1 t_{12}(t,r,p,\dots) r^2 + \\ &\quad \dots + r^3 t_{33}(t,r,p,\dots) r^3 \\ &= \sum_{i,j=1}^{3} r^i t_{ij}(t,r,p,\dots) r^j \end{aligned} \qquad (5.2.49)$$

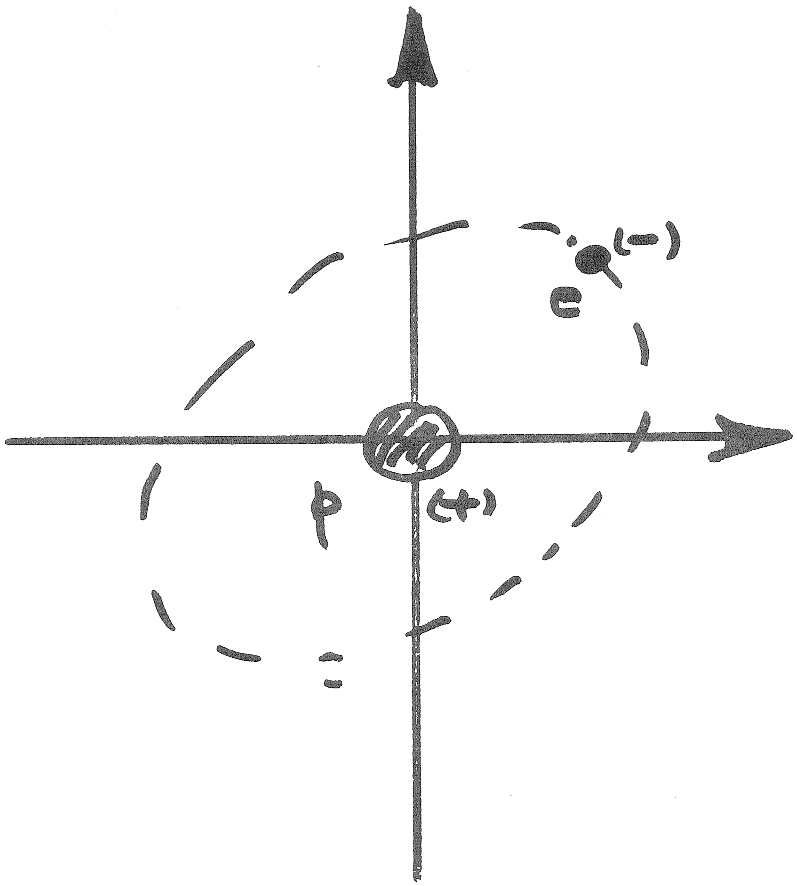
which is exactly that expressed by our fundamental isotopic form (5.2.1),

$$r^{\hat{2}} = \widetilde{(r^1\ r^2\ r^3)} \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{3} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \begin{pmatrix} r^1 \\ r^2 \\ r^3 \end{pmatrix} \qquad (5.2.50)$$

In turn, this is exactly the form left invariant by $\hat{O}(3)$. This confirms the complete achievement of our objective.

ATOMIC STRUCTURE:

Isotropy of the medium (vacuum) in which the electron moves, thus demanding an exact rotational symmetry.



HADRONIC STRUCTURE:

Unisotropy of the medium in which a hadronic constituent moves, thus demanding a broken rotational symmetry for extended charge distributions.

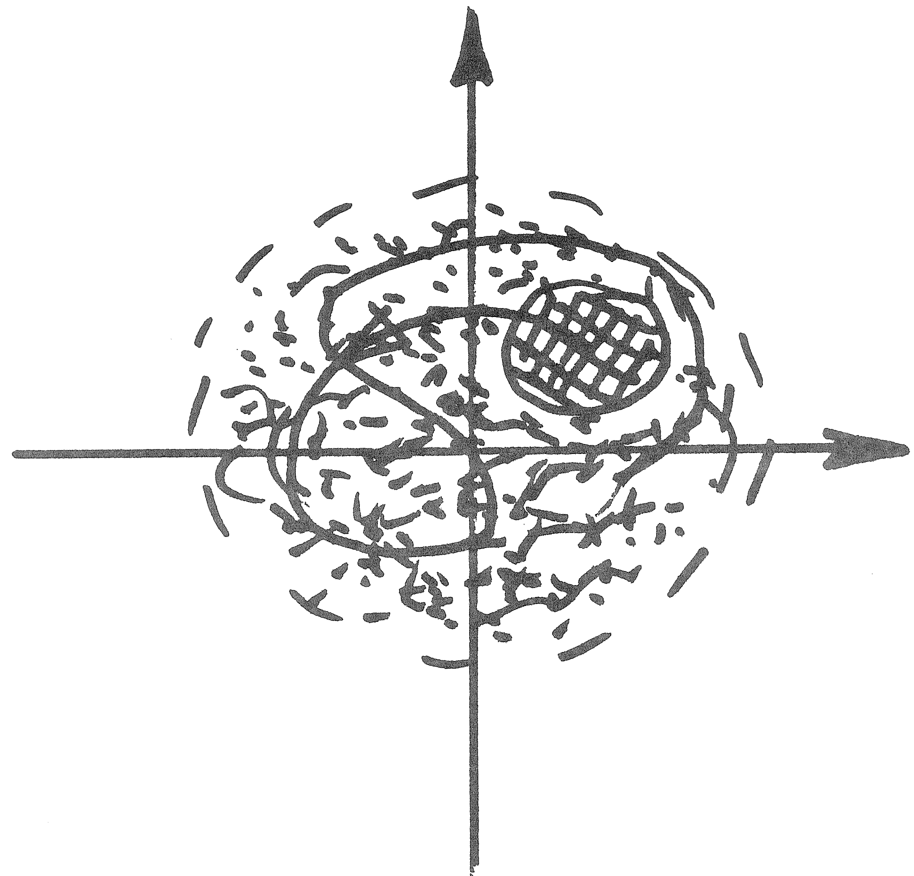


FIGURE 5.3

A simple illustration is in order at this point. Consider the particular case of invariant (5.2.49)

$$z^{\hat{2}} = z^1 t_1 z^1 + z^2 t_2 z^2 + z^3 t_3 z^3 \qquad (5.2.51)$$

for which the isotopic element is given by the matrix with constant elements

$$T = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \qquad (5.2.52)$$

The knowledge of T then determines the entirety of the generalized theory. Consider, for instance, a rotation around the z-axis. Then the conventional infinitesimal generator can be written

$$J_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (5.2.53)$$

The use of isotopic exponentiation (5.2.23), then produces the explicit form of the isotopic rotation which we shall write in the form

$$\hat{R}(\theta) = T^{-1} \begin{pmatrix} \cos\theta\sqrt{t_1 t_2} & \sqrt{\frac{t_2}{t_2}}\,\sin\theta\sqrt{t_1 t_2} \\ -\sqrt{\frac{t_2}{t_1}}\,\sin\theta\sqrt{t_1 t_2} & \cos\theta\sqrt{t_1 t_2} \end{pmatrix} \qquad (5.2.54)$$

The transformed vector is then given by

$$z' = \hat{R} * z = \hat{R} T z = \begin{pmatrix} z^1\cos\theta\sqrt{t_1 t_2} + z^2\sqrt{\frac{t_2}{t_1}}\,\sin\theta\sqrt{t_1 t_2} \\ -z^1\sqrt{\frac{t_1}{t_2}}\,\sin\theta\sqrt{t_1 t_2} + z^2\cos\theta\sqrt{t_1 t_2} \\ z^3 \end{pmatrix} \qquad (5.2.55)$$

under which we reach the form-invariance of anisotropic separation (5.2.51),

$$\begin{aligned} z'^t * z' &= z'^1 t_1 z'^1 + z'^2 t_2 z'^2 + z'^3 t_3 z'^3 \\ &= t_1\Big[(z^1)^2\cos^2\theta\sqrt{t_1 t_2} + (z^2)^2\frac{t_2}{t_1}\sin^2\theta\sqrt{t_1 t_2} \\ &\qquad + 2z^1 z^2(\cos\theta\sqrt{t_1 t_2})\sqrt{\frac{t_2}{t_1}}(\sin\theta\sqrt{t_1 t_2})\Big] \\ &\quad + t_2\Big[(z^1)^2\frac{t_1}{t_2}\sin^2\theta\sqrt{t_1 t_2} + (z^2)^2\cos^2\theta\sqrt{t_1 t_2} \\ &\qquad - 2z^1 z^2\sqrt{\frac{t_1}{t_2}}\sin\theta\sqrt{t_1 t_2}\cos\theta\sqrt{t_1 t_2}\Big] \\ &\quad + t_3 z^3 z^3 \\ &= z^1 t_1 z^1 + z^2 t_2 z^2 + z^3 t_3 z^3 \equiv z^t * z \end{aligned} \qquad (5.2.56)$$

As concluding remarks, the reader may note the achievement of symmetries and conservation laws under non-Hamiltonian forces, which is the primary objective of the Lie-isotopic generalization of Galilei's relativity[189].

In fact, the notion of symmetry is preserved, e.g., in the form-invariance of the Casimir

$$J'^{\hat{2}} \equiv J^{\hat{2}} \tag{5.2.57}$$

which is, therefore, conserved

$$J^{\hat{2}} = \hat{C}_2 = Const. \tag{5.2.58}$$

while the theory is manifestly non-Hamiltonian, e.g., because it is not of conventional Lie character.

Finally, even though we shall not consider the representation theory, we should prevent the erroneous impression that the theory is (conventionally) linear. This is manifestly illustrated by the possible space-dependence of the isotopic element under which the generalized rotations produce the highly nonlinear transformations

$$r' = \hat{R} * r = \hat{R}(\theta, r, \ldots) T(\theta, r, \ldots)\, r \tag{5.2.59}$$

It is hoped the reader will see the possibilities of our isotopic liftings of Lie structures for the representation theory, e.g., because of the possibility of turning a nonlinear theory in the old carrier space into a formally linear one in the new space.

As stressed in Volume I, this is a fundamental point of our analysis. In fact, we have conducted our long scientific journey because of our convinction that Einstein's special relativity is insufficient for the strong interactions BECAUSE OF THE LINEARITY OF THE POINCARÉ TRANSFORMATIONS, and the rather general expectation that hadrons call for a nonlinear transformation theory.

PART II: LIE-ADMISSIBLE COVERING OF THE ROTATION GROUP

We shall now indicate that the results of the preceding Lie-isotopic analysis are still preliminary and that our methods permit the identification of much deeper insights.

Consider again the isotropic invariant of the group of rotations

$$r^2 = r^t r = \underline{\overline{r^1 r^2 r^3}} \begin{pmatrix} r^1 \\ r^2 \\ r^3 \end{pmatrix} \tag{5.2.60}$$

We shall show here that the characterization of the isotropy of space via this invariant is due to the selection of the simplest possible realization of Lie's theory. On the contrary, if one assumes the Lie-admissible generalization of Lie's theory, a more profound and rich situation emerges.

The basic idea of the Lie-isotopy is that of generalizing the enveloping associative algebra $A(\underline{O}(3))$ into isotope (5.2.8), but without any differentiation of the actions to the right and to the left, that is, by assuming that the isotopy element is the same for both actions.

The basic idea of the Lie-admissible generalization of the group of rotations is that of generalizing the enveloping associative algebra

A(O(3)) into two different isotopes, one for the action to the right and one for the action to the left.

We therefore introduce the following two products. First, the isotopic associative product to the right

$$A^{\triangleright}: \quad A \triangleright B \stackrel{def}{=} A T^{\triangleright} B \qquad (5.2.61)$$

and, second, the isotopic product to the left

$${}^{\triangleleft}A: \quad A \triangleleft B \stackrel{def}{=} A\, {}^{\triangleleft}T B \qquad (5.2.62)$$

each one possessing a generalized identity

$$I^{\triangleright} = (T^{\triangleright})^{-1} \in A^{\triangleright}; \quad {}^{\triangleleft}I = ({}^{\triangleleft}T)^{-1} \in {}^{\triangleleft}A \qquad (5.2.63)$$

The understanding is that the right and left isotopic elements are (nonsingular and) generally different

$${}^{\triangleleft}T \neq T^{\triangleright} \qquad (5.2.64)$$

In fact, it is this difference that produces Lie-admissible algebras, as we shall see. Evidently, the right and left isotopies can be the same, as a particular case, thus recovering the Lie-isotopy. Finally, the right and left isotopies can be the conventional identities, in which case we recover the contemporary simplest possible formulation of Lie's theory.

This will provide an illustration of the hierarchy of generalizations we have studied in detail throughout our scientific journey

(LIE ALGEBRAS) ⊂ (LIE-ISOTOPIC ALGEBRAS) ⊂ (LIE-ADMISSIBLE ALGEBRAS) (5.2.65)

The construction of the desired Lie-admissible generalization of Ô(3) is fully defined, as far as its structure is concerned, by the basic assumptions of the two, generalized, left and right units (5.2.63), and it can be reduced to a mere duplication of the Lie-isotopic analysis, one per each side.

For the reader's convenience, we present here the main points. The transformation laws of the coordinates are now given by

$$\begin{aligned} r' &= R^{\triangleright} \triangleright r = R^{\triangleright} T^{\triangleright} r \\ r'^{t} &= r^{\hat{t}} \triangleleft {}^{\triangleleft}R^{\hat{t}} = r^{t}\, {}^{\triangleleft}T\, {}^{\triangleleft}R^{\hat{t}} \end{aligned} \qquad (5.2.66)$$

where $R^{\triangleright}$ and ${}^{\triangleleft}R$ and the right and left isotopic rotations.

Introduce the basic invariant

$${}^{\triangleleft}r^{2\triangleright} = r^{\hat{t}} \,\Box\, r \qquad (5.2.67)$$

where □ represents a generic isotopy, generally different than the left and right ones. Then, the transformation law reads

$$\begin{aligned} r'^{t} \,\Box\, r' &= r^{\hat{t}} \triangleleft {}^{\triangleleft}R^{\hat{t}} \,\Box\, R^{\triangleright} \triangleright r \\ &\equiv r^{\hat{t}} \,\Box\, r \end{aligned} \qquad (5.2.68)$$

and holds if and only if

$${}^{\triangleleft}R^{\hat{t}} \,\Box\, R^{\triangleright} = {}^{\triangleleft}I \,\Box\, I^{\triangleright} = {}^{\triangleleft}R \,\Box\, R^{\triangleright\hat{t}} \qquad (5.2.69)$$

The equations above are the desired generalization of the Lie-isotopic rules (5.2.14). Note that isotopy $\square$ can be the ordinary identity. In this way, the basic invariant remains isotropic. This is the objective we had in mind, that is, to represent the anisotropy of space via the <u>generalization of Lie theory</u>, more than via the generalization of the basic invariant.

The subsequent steps are trivial. To avoid degrees of freedom which are unnecessary (at this moment), we restrict each action to be isotopic rotational, i.e.,

$$R^{\triangleright \hat{t}} \triangleright R^{\triangleright} = R^{\triangleright} \triangleright R^{\triangleright \hat{t}} = I^{\triangleright}$$
$${}^{\triangleleft}I = {}^{\triangleleft}R \triangleleft {}^{\triangleleft}R^{\hat{t}} = {}^{\triangleleft}R^{\hat{t}} \triangleleft {}^{\triangleleft}R \qquad (5.2.70)$$

Generalization of Eqs. (5.2.18) then yields

$$(R^{\triangleright})^{-\hat{1}} \triangleright R^{\triangleright} = R^{\triangleright} \triangleright (R^{\triangleright})^{-\hat{1}} = I^{\triangleright}$$
$${}^{\triangleleft}I = {}^{\triangleleft}R \triangleleft ({}^{\triangleleft}R)^{-\hat{1}} = ({}^{\triangleleft}R)^{-\hat{1}} \triangleleft {}^{\triangleleft}R \qquad (5.2.71)$$

that is,

$$(R^{\triangleright})^{\hat{t}} = (R^{\triangleright})^{-\hat{1}}; \quad ({}^{\triangleleft}R)^{\hat{t}} = ({}^{\triangleleft}R)^{-\hat{1}} \qquad (5.2.72)$$

a number of other properties can be derived accordingly.

We introduce now the important <u>left and right exponentiations</u>

$$O(1)\triangleright: \quad R^{\triangleright}(\theta) = I^{\triangleright} e^{i\theta \triangleright J} = e^{iJ \triangleright \theta} I^{\triangleright}$$
$$\triangleleft O(1): \quad {}^{\triangleleft}R(\theta) = {}^{\triangleleft}I\, e^{i\theta \triangleleft J} = e^{iJ \triangleleft \theta}\, {}^{\triangleleft}I \qquad (5.2.73)$$

This yields the following generalization of the Lie-isotopic group (5.2.45)

$$\triangleleft O(1)\triangleright: \quad A'(\theta) = R^{\triangleright}(\theta) \triangleright A(0) \triangleleft {}^{\triangleleft}R(\theta)^{-\hat{1}}$$
$$= I^{\triangleright} e^{i\theta \triangleright J} \triangleright A(0) \triangleleft e^{iJ \triangleleft \theta}\, {}^{\triangleleft}I \qquad (5.2.74)$$

It is important to understand that the structure above does indeed constitute a one-parameter group of transformations which is continuously connected to the identity transformation. The latter property is manifestly verified by the rule

$$A(\theta)\big|_{\theta=0} = A(0) \qquad (5.2.75)$$

The group composition law follows from the properties

$$R^{\triangleright}(\theta) \triangleright R^{\triangleright}(\theta') = R^{\triangleright}(\theta + \theta')$$
$${}^{\triangleleft}R(\theta) \triangleleft R^{\triangleleft}(\theta') = {}^{\triangleleft}R(\theta + \theta') \qquad (5.2.76)$$

Similarly, for the inverse we have

$$R^{\triangleright}(\theta) \triangleright R^{\triangleright}(-\theta) = I^{\triangleright}$$
$${}^{\triangleleft}I = {}^{\triangleleft}R(-\theta) \triangleleft R(\theta) \qquad (5.2.77)$$

Despite the preservation of the group character, we have the <u>LOSS OF THE LIE AND OF THE LIE-ISOTOPIC ALGEBRAS IN THE NEIGHBORHOOD OF THE</u>

IDENTITY TRANSFORMATION IN FAVOR OF THE BROADER LIE-ADMISSIBLE ALGEBRAS, as manifest in the rules

$$i\frac{d\theta}{d\theta} \hat{=} i\frac{A(\theta)-A(0)}{d\theta}$$

$$= i\left[(I + id\theta J) \triangleright A(0) \triangleleft (I - id\theta J) - A(0)\right]/d\theta \qquad (5.2.78)$$

$$= A \triangleleft J - J \triangleright A$$

$$= A^{\triangleleft} T J - J T^{\triangleright} A \stackrel{def}{=} (A, J)^*$$

We therefore call structure (5.2.78) a Lie-admissible generalization of the group of rotations, or Lie-admissible rotation group for short. The generalization to more than one dimension is evidently given by

$$\triangleleft O(3) \triangleright : \qquad A(\theta) = \qquad (5.2.79)$$

$$= I^{\triangleright} e^{i\theta_1 \triangleright J_1} \triangleright e^{i\theta_2 \triangleright J_2} \triangleright e^{i\theta_3 \triangleright J_3} \triangleright A(0) \triangleleft$$

$$\times\ e^{-iJ_3 \triangleleft \theta_3} \triangleleft e^{-iJ_2 \triangleleft \theta_2} \triangleleft e^{-iJ_1 \triangleleft \theta_1} {}^{\triangleleft} I$$

It results to be a structure of considerable complexity which demands extensive and separate research. We therefore merely content ourselves here with the Lie-admissible generalization of each individual one-dimensional subgroup of the old Lie group.

The capability to represent the time rate of variation of the angular momentum is self-evident, and can be realized in more than one way. The most direct one is the following. The conventional O(3)-theory leaves invariant the magnitude of the angular momentum by construction:

$$J^2 = J^t I J \equiv J'^t I J' = C_2 = \text{Const.} \qquad (5.2.80)$$

thus providing an example of the conventional symbiotic characterization of symmetries and conservation laws (for Hamiltonian systems).

In the passage to the more general group $\triangleleft O(3) \triangleright$ the situation is profoundly altered. Suppose that isotopy $\square$ is given by a function of time, say, $g^{-1}(t)$. The quantity which is left invariant by the group $\triangleleft O(3) \triangleright$ is now given by

$$\triangleleft J^2 \triangleright = J^t \square J = J^t J g^{-1}(t) = \hat{C}_2 = \text{const} \qquad (5.2.81)$$

By recalling that the physical magnitude of the angular momentum is still the old one J^2, we then trivially reach the time rate of variation

$$J^2 = g(t)\hat{C}_2 \qquad (5.2.82)$$

GLOBAL STABILITY OF A CLOSED NON-HAMILTONIAN SYSTEM VIA THE MAXIMAL POSSIBLE INSTABILITY OF THE ORBIT OF EACH CONSTITUENT.

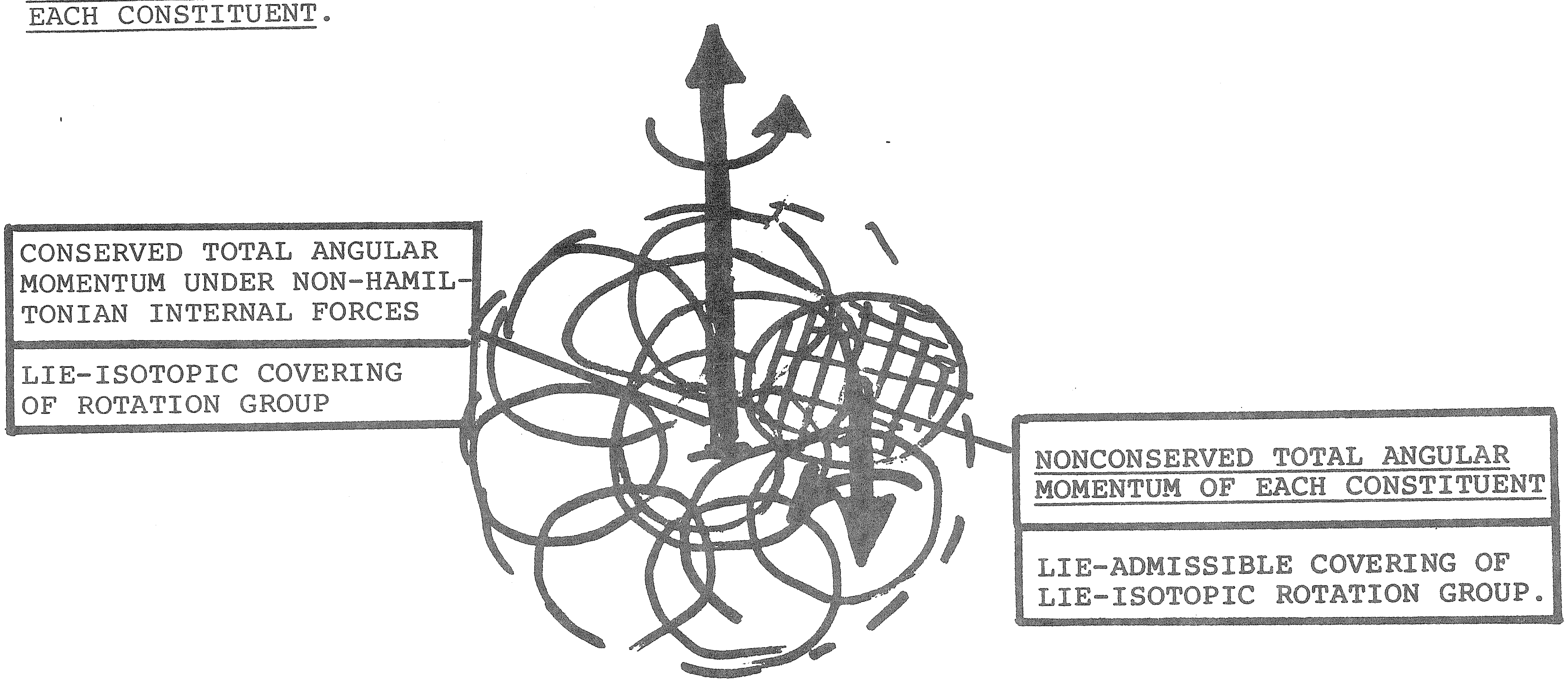


FIGURE 5.4

We discover also as an incidental note that the isotopy $\hat{\mathfrak{a}}$ can be representative of the Lie-admissible time evolution in the sense

$$\frac{dJ^2}{dt} = (J^2, H) = f(t) = \frac{dg(t)}{dt}. \qquad (5.2.83)$$

It is then evident that the Lie-admissible generalization of the rotation group does indeed achieve our main objective, that is, the symbiotic <u>characterization of symmetries and nonconservation laws</u>. In fact, we do have the form-invariance of $\langle J^2 \rangle$. Yet, the angular momentum is not conserved. Most importantly, we reach this generalized formulation while preserving the conventional characterization of symmetries and conservation laws as a trivial particular case.

The reader familiar with our preceding analysis will have no difficulty in understanding the mathematical structure and physical motivations underlying the transition from the Lie-isotopy to the Lie-admissible genotopy.

On mathematical grounds, structure (5.2.45) is, in essence, a Lie-bymodule, where, we should recall, the Lie character originates from the identity of the left and right actions.

Structure (5.2.74) is a considerably more general one. In fact, it is a Lie-admissible bimodule (according to the concept introduced by Santilli in ref.[207]), where, again, the loss of Lie algebra character in favor of the Lie-admissible one is generated by the differentiation of the left and right actions.

The hope is to achieve the notion of a particle (our eleton) as a representation of the structurally most general possible groups of

space-time transformations.

On physical grounds, the objective at hand are also known. The preservation of the conventional, isotropic invariant (5.2.60) under a Lie-admissible bimodular theory is intended to illustrate the basic idea of our approach to strong interactions, that effects and dynamical behaviours in the interior problem are not expected to be visible in the outside (exterior problem). We are referring here to the proton in the particle accelerator recalled in Section 5.1. Even though its center-of-mass complies strictly with Einstein special relativity, by no means, this implies that the interior dynamics must necessarily obey Einstein's laws. To indicate the plausibility of this occurrence, it is sufficient to recall that the center-of-mass of the proton moves in an isotropic medium, the vacuum of the accelerator. On the contrary, the hadronic constituents, once assumed as extended charge distributions to avoid excessive approximations, must move within the hadronic medium, that is, the medium constituted by the other particles. The loss of isotropy in the interior problem only is then consequential. The Lie-admissible generalization of the group of rotations provides a technical context for the quantitative treatment of the problem, that is, for the loss of the rotational symmetry in the interior problem only, while no such loss is detectable from the outside.

We close this section with a few incidental remarks. The reader has noted that we have used in this section the language of the abstract theory of algebras, as presented in Chapter 1 [rather than realizations more commonly used in Newtonian mechanics].

This is due to the fact that our ultimate objective is the <u>structure of hadrons</u>. The use of the abstract approach is then of self-evident relevance. In fact, in this section we have already put the basis for the generalized, Lie-admissible notion of spin (<u>hadronic spin</u>) which will be developed and applied in Volume III. The main ingredient missing in this section, but worked out by Myung and Santilli in ref.s[205], is the modular and bimodular-isotopic liftings of the Hilbert space.

The reader should however keep in mind that our abstract algebraic formulations admit a direct realization in the Newtonian language of Chapters 2. In fact, the Lie-isotopic product (5.2.28) admits the Birkhoffian realization

$$\begin{pmatrix} [A,B]^* \\ = ATB - BTA \end{pmatrix} \Longrightarrow \begin{pmatrix} [A,B]^* \\ = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu}(a) \frac{\partial B}{\partial a^\nu} \end{pmatrix} \qquad (5.2.84)$$

and the Lie-admissible product (5.2.78) admits the Birkhoffian-admissible realization

$$\begin{pmatrix} (A,B)^* \\ = A^{\triangleleft}TB - BT^{\triangleright}A \end{pmatrix} \Longrightarrow \begin{pmatrix} (A,B)^* \\ = \frac{\partial A}{\partial a^\mu} \left(\Omega^{\mu\nu} + T^{\mu\nu}\right) \frac{\partial B}{\partial a^\nu} \end{pmatrix} \qquad (5.2.85)$$

The reader can now see the reason for the selection of the Birkhoff-admissible equations among a considerable variety of other alternatives, and it is related to the existence of an exponentiated, bimodular, group structure which is of difficult [if not impossible] identification for non-Lie theories that are not Lie-admissible.

Next, we would like to clarify the reasons for our use in this section of the word "generalized" rather than "covering". The best way to illustrate the point is via an example. In particular, to show the existence of the Newtonian realizations of the abstract theory, we shall use the latter.

As worked out in Volume I, the groups O(3) and O(2.1) are isotopically related as symmetries of the equivalent Lagrangians

$$L = \frac{1}{2}\left(\dot{z}_1^2 + \dot{z}_2^2 + \dot{z}_3^2\right) - \frac{1}{2}\left(z_1^2 + z_2^2 + z_3^2\right) \Rightarrow L^* = \frac{1}{2}\left(\dot{z}_1^2 - \dot{z}_2^2 + \dot{z}_3^2\right) - \frac{1}{2}\left(z_1^2 - z_2^2 + z_3^2\right) \qquad (5.2.86)$$

which both lead to the conservation laws of the angular momentum. We also stressed the importance of <u>realizing the Lorentz group via the angular momentum components</u>, trivially, because the conserved quantities for the case considered are exactly the components of the angular momentum. Finally, in Chapter 3 we worked out a realization of the Lorentz algebra as an isotope of the rotation algebra.

It is easy to show that these results are all contained in the theory worked out in this section.

In fact, the transition from the O(3)-to the O(2.1)-invariant is given by the isotopy

$$z_1^2 + z_2^2 + z_3^2 \Longrightarrow z_1^2 - z_2^2 + z_3^2 = z^t T z \qquad (5.2.87)$$

and it is a trivial subcase of isotopy (5.2.49) realized via the matrix with constant elements

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \stackrel{def}{=} (g^i{}_j) \qquad (5.2.88)$$

The conventional Poisson bracket realization of O(3)

$$[J_i, J_j] = \frac{\partial J_i}{\partial r^\ell} \delta^\ell_m \frac{\partial J_j}{\partial p_m} - \frac{\partial J_j}{\partial r^\ell} \delta^\ell_m \frac{\partial J_i}{\partial p_m} \qquad (5.2.89)$$

is then turned into its isotope

$$[J_i, J_j]^* = \frac{\partial J_i}{\partial r^\ell} g^\ell_m \frac{\partial J_j}{\partial p_m} - \frac{\partial J_j}{\partial r^\ell} g^\ell_m \frac{\partial J_i}{\partial p_m} \qquad (5.2.90)$$

which, as the reader will recall, verifies the correct commutation rules of O(2.1) even though the generators are, again, the components of the angular momentum.

The point we wanted to illustrate is that isotopy (5.2.87) DOES NOT constitute a covering of the group of rotations, in the sense that O(3) is not obtained as a particular case of $O^*(3) \approx O(2.1)$.

This aspect is important to understand that the Lie-isotopic and Lie-admissible generalizations of O(3) do not provide, automatically, covering symmetries as defined (at the end of) Section 5.1. This is due to the richness of their structure, and the consequential variety of possibilities of realizations.

In order to achieve the covering rotational symmetry needed for the construction of covering relativities, we must impose that the generalized groups contain the conventional rotation group O(3) as particular case. This occurs, for instance, when the isotopic admits the decomposition

$$T = I + G(F^{NIA}) \qquad (5.2.91)$$

where the additional term becomes identically null when the system considered reduces to a self-adjoint form (i.e., when all nonpotential forces are null).

In closing this section, it is a pleasant duty for me to thank Professor Eder whose pioneering studies on the Lie-admissible generalization of spin[208] have been simply invaluable for the writing of this section.

Finally, it is a pleasure to thank Professor Myung, as well as a number of other mathematicians. Their contributions in Lie-admissible algebras have been invaluable for the inception of the generalized relativities, and shall remain to be invaluable for further developments, as indicated in Section 5.5.

5.3: THE HYPOTHESIS OF A LIE-ADMISSIBLE GENERALIZATION OF GALILEI RELATIVITY.

We shall now formulate our generalized relativity according to the following lines.

PART I:

Step 1: Assumption of a conservative, Galilean, single, particle;

Step 2: Review of the Hamilton/Lie/symplectic formulation of the Galilei relativity;

Step 3: Identification of Galilei relativity as a form-invariant description of the system considered.

PART II:

Step 4: Assumption that the particle experiences contact, non-potential, Galilei-symmetry-breaking forces;

Step 5: Identification of the Hamilton-admissible/Lie-admissible/symplectic-admissible methods underlying the proposed generalized relativity;

Step 6: Formulation of the generalized relativity;

Step 7: Implications, comments, and examples.

Steps 1 through 6 can be expressed via the following

HYPOTHESIS 5.3.1[25,191]: Consider a local, class C^∞, unconstrained Newtonian particle moving in vacuum under long range, self-adjoint (SA), conservative forces without collisions, represented in the local variables

$$\{t, b^\mu\} = \{t, \underset{\sim}{r}, \underset{\sim}{p}\}, \quad \mu = 1,2,\ldots,6 \qquad (5.3.1)$$

constituting the time and the Cartesian coordinates $\underset{\sim}{r} \in E(3)$ of its experimental detection, and physical linear momentum $\underset{\sim}{p} = m\dot{\underset{\sim}{r}}$, and consider the equations of motion in the vector-field form

$$\Gamma = \Gamma^\mu(b)\partial/\partial b^\mu \qquad (5.3.2a)$$

$$(\Gamma^\mu) = \begin{pmatrix} \underset{\sim}{p}/m \\ \underset{\sim}{f}^{SA}(\underset{\sim}{r}) \end{pmatrix}, \quad \underset{\sim}{f}^{SA} = -\frac{\partial V}{\partial \underset{\sim}{r}} \qquad (5.3.2b)$$

Then the applicable relativity, the GALILEI RELATIVITY, can be characterized in terms of the following formulations.

(A) ANALYTIC FORMULATIONS, essentially consisting of the representation of the equations of motion via the conventional Hamilton's equations without external terms

$$\Gamma^\mu = \omega^{\mu\nu}\frac{\partial H}{\partial b^\nu}, \quad H = T(\underset{\sim}{p}) + V(\underset{\sim}{r}) \qquad (5.3.3a)$$

$$\omega^{\mu\nu} = \left(\left\|\frac{\partial R^\circ_\alpha}{\partial b^\beta} - \frac{\partial R^\circ_\beta}{\partial b^\alpha}\right\|^{-1}\right)^{\mu\nu}, \quad R^\circ = (\underset{\sim}{p}, \underset{\sim}{0}) \qquad (5.3.3b)$$

the Hamiltonian characterization of conserved quantities

$$\begin{aligned} \dot{A}(b) &= [A, H] = \frac{\partial A}{\partial b^\mu}\omega^{\mu\nu}\frac{\partial H}{\partial b^\nu} \\ &= \frac{\partial A}{\partial \underset{\sim}{r}}\cdot\frac{\partial H}{\partial \underset{\sim}{p}} - \frac{\partial H}{\partial \underset{\sim}{r}}\cdot\frac{\partial A}{\partial \underset{\sim}{p}} \equiv 0 \end{aligned} \qquad (5.3.4)$$

and related theory;

(B) ALGEBRAIC FORMULATIONS, essentially consisting of the universal enveloping associative algebra $A(\underset{\sim}{G})$ of the Galilei algebra G

$$A(\underline{G}) := \mathcal{T}/R \qquad (5.3.5a)$$

$$\mathcal{T} = 1 \oplus \underline{G} \oplus \underline{G} \otimes \underline{G} \oplus \cdots \qquad (5.3.5b)$$

$$R = [X_i, X_j] - (X_i \otimes X_j - X_j \otimes X_i) \qquad (5.3.5c)$$

$$\{X_i\} = \{H ; \underset{\sim}{p} ; \underset{\sim}{r} \times \underset{\sim}{p} ; m\underset{\sim}{r} ; I\} \qquad (5.3.5d)$$

the attached Galilei algebra G

$$\underline{G} \approx [A(\underline{G})]^- : \qquad (5.3.6a)$$

$$[X_i, X_j] = C_{ij}^{\ k} X_k \quad ; i,j = 1,2,\ldots,11 \qquad (5.3.6b)$$

the Lie group G of Galilei transformations

$$G : b' = \prod_{k=1}^{11} e^{\theta_k \omega^{\alpha\beta} \frac{\partial X_k}{\partial b^\beta} \frac{\partial}{\partial b^\alpha}} b \qquad (5.3.7a)$$

$$\{\theta_k\} = \{\tilde{t} ; \tilde{\underset{\sim}{r}} ; \theta_1, \theta_2, \theta_3 ; \underset{\sim}{v} , m\} \qquad (5.3.7b)$$

and related theory;

(C) GEOMETRIC FORMULATIONS, essentially consisting of the characterization of the vector field Γ as Hamiltonian

$$\Gamma \rfloor \omega_2 = \omega_{\mu\nu} \Gamma^\nu db^\mu \equiv dH \qquad (5.3.8)$$

with respect to the fundamental, exact, symplectic structure on $E_{\underset{\sim}{r}}(3) \times E_{\underset{\sim}{p}}(3)$

$$\omega_2 = dR_1^\circ = d(R_\mu^\circ db^\mu) \qquad (5.3.9)$$

$$= \frac{1}{2}\left(\frac{\partial R_\nu^\circ}{\partial b^\mu} - \frac{\partial R_\mu^\circ}{\partial b^\nu}\right) db^\mu \wedge db^\nu \equiv d\underset{\sim}{p} \wedge d\underset{\sim}{r}$$

the characterization of conservation laws via Lie's derivative (for the autonomous case)

$$L_\Gamma A = [A, H] = 0 \qquad (5.3.10)$$

and related theory.

We suppose now that the particle enters within a dissipative medium by therefore experiencing contact/nonpotential interactions due to its extended si e, which characterize one of the possible breakings of Galilei relativity (semicanonical, canonical, and essentially nonself-adjoint) according to the new vector field

$$\Xi = \Xi^\mu(t, b)\,\partial/\partial b^\mu \qquad (5.3.11a)$$

$$(\Xi^\mu) = \begin{pmatrix} \underset{\sim}{p}/m \\ \underset{\sim}{f}^{SA}(\underset{\sim}{r}) + \underset{\sim}{F}^{NSA}(t, \underset{\sim}{r}, \underset{\sim}{p}) \end{pmatrix} \qquad (5.3.11b)$$

where the original system is the maximal self-adjoint subsystem. Then the applicable relativity, called GALILEI-ADMISSIBLE RELATIVITY, can be characterized via the following covering formulations.

(A') COVERING ANALYTIC FORMULATIONS, essentially consisting of the representation of the equations of motion via the original equations conceived by Hamilton (those with external terms), written in our Hamilton-admissible form

$$\Xi^\mu = S^{\mu\nu}(t,b)\frac{\partial H(b)}{\partial b^\nu}, \quad S^{\mu\nu} = \omega^{\mu\nu} + t^{\mu\nu} \qquad (5.3.12a)$$

$$t^{\mu\nu} = \begin{pmatrix} -\frac{F^{NSA}}{(p/m)} & 0 \\ 0 & 0 \end{pmatrix} = t^{\nu\mu} \qquad (5.3.12b)$$

the Hamilton-admissible characterization of the time rate of variations of physical quantities

$$\dot A = (A,H) = \frac{\partial A}{\partial b^\mu} S^{\mu\nu}\frac{\partial H}{\partial b^\nu} \qquad (5.3.13)$$

$$= [A,H] + \frac{\partial A}{\partial b^\mu} t^{\mu\nu}\frac{\partial H}{\partial b^\nu} = \frac{\partial A}{\partial \underline{p}}\cdot \underline{F}^{NSA} \neq 0$$

and related theory.

(B') COVERING ALGEBRAIC FORMULATIONS, essentially consisting of the general, nonassociative, Lie-admissible, genotope of the associative enveloping algebra A(G)

$$\tilde A = \tilde{\mathcal{T}}/\tilde{\mathcal{R}} \qquad (5.3.14a)$$

$$\tilde{\mathcal{T}} = 1 \oplus \underline{G} \oplus \underline{G} \odot \underline{G} \oplus \cdots \qquad (5.3.14b)$$

$$\tilde{\mathcal{R}} = [(X_i,X_j) - (X_j,X_i)] - (X_i \odot X_j - X_j \odot X_i) \qquad (5.3.14c)$$

$$\{X_i\} = \{H, \underline{p}, \underline{r}\times\underline{p}, \underline{G}, I\}, \quad \dot X_i \neq 0 \qquad (5.3.14d)$$

the Lie-admissible covering of the Galilei Lie algebra

$$\tilde{\underline{G}} : (X_i,X_j) = \tilde U_{ij}^{\ k}(t,b,X)\, X_k \qquad (5.3.15a)$$

$$\tilde U_{ij}^{\ k} = C_{ij}^{\ k} + D_{ij}^{\ ke} X_e, \quad C_{ij}^{\ k} = -C_{ji}^{\ k}, \quad D_{ij}^{\ ke} = D_{ji}^{\ ke} = D_{ij}^{\ ek} \qquad (5.3.15b)$$

the Lie-admissible covering group $\tilde G$ of the Galilei group G

$$\tilde G : \tilde b = \prod_{k=1}^{11} e^{\theta_k(\omega^{\alpha\beta} + t^{\alpha\beta})\frac{\partial X_k}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}}\, b \qquad (5.3.16a)$$

$$\{\theta_k\} = \{\tilde t\,;\ \tilde{\underline{r}}\,;\ \theta_1,\theta_2,\theta_3\,;\ \tilde{\underline{v}}\,;\ m) \qquad (5.3.16b)$$

and related theory;

(C') COVERING GEOMETRIC FORMULATIONS, essentially consisting of the characterization of the vector field Ξ as Hamilton-admissible

$$\Xi \boxtimes s_2 = s_{\mu\nu}\,\Xi^\nu db^\mu \equiv dH \qquad (5.3.17)$$

with respect to the fundamental symplectic-admissible structure on $E_r(3) \times E_p(3)$

$$s_2 = s_{\mu\nu}\, db^\mu \otimes db^\nu = \tfrac{1}{2}\omega_{\mu\nu}\, db^\mu \wedge db^\nu + \tfrac{1}{2} t_{\mu\nu}\, db^\mu \times db^\nu ; \quad s_{\mu\nu} = \left(\| s^{\alpha\beta} \|^{-1}\right)_{\mu\nu} = \omega_{\mu\nu} + t_{\mu\nu} \tag{5.3.18}$$

the characterization of the time rate of variation of physical quantities via the Lie-admissible generalization of Lie's derivative,

$$\hat{\mathcal{L}}_{\Xi} A = (A, H) = \frac{\partial A}{\partial p} \cdot F^{NSA} \tag{5.3.19}$$

and related theory.

The Galilei-admissible relativity is further characterized by the condition of providing a covering form-invariant description of the nonconservative conditions of the system, that is, the form-invariance under nonself-adjoint forces

$$\Xi(t,b) = \Xi^\mu \partial/\partial b^\mu = \Xi^\mu(t, b(\tilde{b})) \frac{\partial \tilde{b}^\alpha}{\partial b^\mu} \frac{\partial}{\partial \tilde{b}^\alpha} = \tilde{\Xi}^\alpha(t,\tilde{b})\, \partial/\partial \tilde{b}^\alpha \equiv \Xi^\alpha(t,\tilde{b})\, \partial/\partial \tilde{b}^\alpha \tag{5.3.20}$$

for all transformations (5.3.16) while its time component

$$\hat{b} = \tilde{b}(\tilde{t}) = e^{\tilde{t}(\omega^{\alpha\beta} + t^{\alpha\beta}) \partial H/\partial b^\beta\, \partial/\partial b^\alpha}\, b(0), \tag{5.3.21}$$

characterizes the time rate of variation of physical quantities as

experimentally established in the Newtonian reality

$$\frac{dX_k}{dt} = \frac{X_k(t) - X_k(0)}{dt} \cong \left[e^{dt(\omega^{\alpha\beta} + t^{\alpha\beta}) \frac{\partial H}{\partial b^\beta} \frac{\partial}{\partial b^\alpha}} - 1 \right] A/dt = (X_k, H) = \frac{\partial X_k}{\partial p} \cdot F^{NSA} \tag{5.3.22}$$

under the condition that, when all nonself-adjoint forces are identically null, the generalized relativity and underlying formulations, Equations (5.3.11) through (5.3.22), recover identically and in their entirety the conventional relativity and underlying formulations, Equations (5.3.2) through (5.3.10).

It is important to understand that Hypothesis 5.3.1 is submitted mainly as a working ground for the future construction of a complete, diversified, and comprehensive relativity which is applicable to nonconservative and Galilei-noninvariant systems.

In this section, we would like to identify the part of the relativity which is sufficiently understood and provide a partial identification of the rather numerous problems to be yet resolved.

The part which can be considered as sufficiently (though incompletely) identified is the "time component". This includes all various aspects dealing with the time evolution, such as:

(I) Lie-admissible group (5.3.16) constitutes a tool for the actual construction of the transformation in time for all given self-adjoint and nonself-adjoint forces. This is merely given by computing the infinite series

$$\begin{cases}\tilde{r} = r + \frac{\tilde{t}}{1!}(r,H) + \frac{\tilde{t}^2}{2!}((r,H),H) + \cdots \\ \tilde{p} = p + \frac{\tilde{t}}{1!}(p,H)+\frac{\tilde{t}^2}{2!}((p,H),H)+\cdots\end{cases} \qquad (5.3.23)$$

whose convergenge is guaranteed by the topological conditions assumed in our analysis. The understanding is that, even though the existence of the finite Lie-admissible group (5.3.23) is guaranteed, this does not imply that all possible infinite series (5.3.23) can be summed in the needed finite form.

(II) Lie-admissible group (5.3.16) leaves form-invariant all vector-fields (5.3.11). This is due to a well-known property of differential geometry that, locally, all vector fields are form-invariant under the time evolution generated by themselves, i.e.,

$$\tilde{b} = e^{t\Xi}\, b \quad \Longrightarrow \quad \Xi(t,b) = \tilde{\Xi}(t,\tilde{b}) \equiv \Xi(t,\tilde{b}) \qquad (5.3.24)$$

(for more detail on this aspect, the reader may consult ref.[192], p. 1308). As a result, there is no need to verify form-invariance (5.3.20) under time-evolution (5.3.23), because the generalized relativity has been proposed in such a way to provide this invariance authomatically. The only requirement is the proper identification of the symmetric component $t^{\mu\nu}$ of the Lie-admissible tensor $s^{\mu\nu}$ in representation (5.3.12) (see Chapter 2 for details).

(III) Lie-admissible group (5.3.16) characterizes, also by construction, time rates of variations (5.3.22). This is manifestly expressed by Eqs. (5.3.22) themselves.

To summarize, the time component of the proposed Galilei-admissible relativity can be proved as providing a covering description of the conventional, Galilean, time-component, for ALL nonconservative systems of the class admited in the frame of the observer (direct universality).

As an simple example, consider the nonconservative particle (2.9.21)

$$(\Xi^\mu) = \begin{pmatrix} p \\ -\gamma p^2 \end{pmatrix}; \mu = 1; p \in \mathbb{R}_1 \qquad (5.3.25)$$

with transformations (2.9.23), i.e.,

$$\begin{cases}\tilde{r} = r + \frac{1}{\gamma}\ln(1+\gamma\tilde{t}p) \\ \tilde{p} = p/(1+\gamma\tilde{t}p)\end{cases} \qquad (5.3.26)$$

and related form-invariance (2.9.24), i.e.,

$$\begin{cases}\frac{d\tilde{r}}{d\tilde{t}} = \tilde{p} \\ \frac{d\tilde{p}}{d\tilde{t}} = -\gamma\tilde{p}^2\end{cases} \qquad (5.3.27)$$

It is an instructive exercise for the interested reader to prove that transformations (5.3.26) have been constructed by summing up infinite series (5.3.23) and, therefore, they constitute a Lie-admissible group. The verification of the time-rate of variation (5.3.22) is trivial and it is ommitted. Equally trivial is the verification that the covering setting, that is,

– the Lie-admissible symmetry and underlying time rates of variation;

recover identically the conventional setting, i.e.,

– the Lie symmetry and underlying conservation laws,

when the nonself-adjoint force is identically null (i.e., when $\gamma = 0$).

For several additional examples, the reader may consult ref.[192]. An example in which the Lie-admissible symmetry is characterized by a transcendental function was worked out in the original proposal of Hypothesis 5.3.1, ref.[25] (see ref.[191], p. 399). We assume the reader is aware that the equations of motion we are interested in are <u>nonlinear</u>, and that their solution is not, therefore, a trivial task.

The situation above <u>does not</u> extend to the remaining components of the proposed covering relativity, trivially, because of the lack of knowledge of the fundamental Lie-admissible tensors $s^{\mu\nu}_{(k)}$ per each generator X_k. To put it differently, in the case of the time component, we knew both the generator H and the Lie-admissible tensor $s^{\mu\nu}_{(1)}$. In the case of the remaining components, we assumed as known the generators X_k. However, we need additional methods for the identification of the Lie-admissible tensors.

A number of <u>formal</u> methods are suitable for this task. They are indicated here in case of possible usefulness to the interested researcher.

<u>FORMAL METHOD I</u>: <u>USE OF DARBOUX THEOREM</u>. By central assumption, the vector field Ξ is not Hamiltonian in the local physical variables considered (t, r, p). However, under the assumed topological conditions, ALL systems considered admit transformations

$$t \to t' \equiv t, \quad b^\mu \to b'^\mu(t, b), \quad \mu = 1, 2, \dots, 6 \qquad (5.3.28)$$

under which they assume a Hamiltonian form

$$\Xi(t, b(t,b')) \rfloor \omega_2(b) = \Xi'(t,b') \rfloor \omega_2(b') = dH'(t,b') \qquad (5.3.29)$$

This is, in essence, the Theorem of Indirect Universality of Hamiltonian Formulations studies in detail in monograph[189]. In particular, one can prove (e.g., via the additional use of canonical transformations), that all vector fields (5.3.11) admit a new system of coordinates in which they assume the "free" form

$$\left(\Xi'^\mu(b')\right) = \begin{pmatrix} p'/m \\ 0 \end{pmatrix} \qquad (5.3.30)$$

for which the conventional Galilei symmetry

$$b'' = \prod_{k=1}^{11} e^{\theta_k \omega^{\alpha\beta} \frac{\partial X'_k}{\partial b'^\beta} \frac{\partial}{\partial b'^\alpha}} b' \qquad (5.3.31)$$

$$\{X'_k\} = \{H'; p'; r' \times p'; m r'; I\}$$

holds.

To put it differently, all systems considered admit an hypothetical system of coordinates in which they verify Galilei symmetry. As a necessary condition of consistency, however, this system of coordinates is "unphysical" in the sense that it <u>cannot</u> be the frame of of the observer.

But one of the fundamental conditions of physical consistency of any mathematical algorithm is that of admitting a realization in the frame actually used in the experimental observation. To fulfill this fundamental requirement, we, therefore, subject symmetry (5.3.31) to the inverse transformation to the physical coordinates $b^{\mu}(t, b')$. This implies in particular that the fundamental cosymplectic tensor transform into the general cosymplectic (Birkhoffian) tensor according to the rules

$$\hat{\prod_{k=1}^{11}} e^{\theta_k \omega^{\alpha\beta} \frac{\partial X'_k}{\partial b'^{\beta}} \frac{\partial}{\partial b'^{\alpha}}} \equiv \hat{\prod} e^{\theta_k \Omega^{\rho\sigma} \frac{\partial Z_k}{\partial b^{\sigma}} \frac{\partial}{\partial b^{\rho}}} \quad (5.3.32a)$$

$$\Omega^{\rho\sigma} = \frac{\partial b'^{\rho}}{\partial b^{\alpha}} \omega^{\alpha\beta} \frac{\partial b'^{\sigma}}{\partial b^{\beta}} \quad (5.3.32b)$$

$$Z_k(t, b) = X'_k(t, b'(t, b)) \quad (5.3.32c)$$

The reformulation of transformations (5.3.32) into the form of the Galilei-admissible group (5.3.16) is trivial. In fact, one has to solve the algebraic equations

$$\Omega^{\rho\sigma} \frac{\partial Z_k}{\partial b^{\sigma}} = S^{\rho\sigma}_{(k)} \frac{\partial X_k}{\partial b^{\sigma}}, \quad k = 1, 2, .., 11 \quad (5.3.33)$$

where the generators X_k are now fixed, and the only unknowns are the Lie-admissible tensors. The case of the time component has been worked out in Chapter 2. The case of the other components is left to the interested reader.

The disadvantage of this approach is that, even though the existence of transformations (5.3.29) for each given system (5.3.11) is guaranteed by the existence theory of the Inverse Problem, its explicit construction is by far nontrivial (often involving the solution of nonlinear, hyperbolic, partial differential equations), as illustrated in the examples of monograph[189].

The distinct advantage of the method is that of confirming the expectation that Galilei-admissible group (5.3.16) does indeed exist and it is finite-dimensional. In fact, the groups (5.3.31) and (5.3.32) ALWAYS exist, while the Galilei-admissible form is provided by mere reformulation (5.3.33).

A number of comments are in order. Eqs. (5.3.7) constitute a simbolic representation of the scalar extension of the Galilei group, as needed because the mass of the particle is assumed to be nonnull. For a presentation of this scalar extension, which also includes a treatment of the one-particle Galilean case (the only considered here), we refer the interested reader to Sudarshan and Mukunda.[131]

The submitted covering symmetry, Eq. (5.3.16) is proposed by keeping in mind the original scalar extension. This is the reason for eleven parameters and generators. It should be stressed, however, that structure (5.3.16) is submitted at this time as a collection of disjoint, one-parameter, Lie-admissible groups. This is due to a considerable number of technical aspects which need specific investigations for the understanding and treatment of the full 11-parameter structure (5.3.16). To put it differently, we are submitting Hypothesis 5.3.1 as one-by-one generalization of each component of the conventional Galilei relativity.

FORMAL METHOD II: USE OF THE ORIGINAL LIE METHOD. It is not sufficiently known in the physical literature that a method for the explicit construction of symmetries from given equations of motion has been proposed by Sophus Lie. The method has been revived by Eliezer and his associates (see ref.[206] and quoted papers). A review is also provided in the Appendix A of monograph[189].

Ref.[209] initiated the study for constructing the Galilei-admissible symmetry according to the following steps.

(a) construction of a sufficient number of symmetries from given equations or motion;

(b) representation of these symmetries as Lie-admissible groups; and

(c) selection of the symmetries which are covering of the conventional Galilean ones in the sense of (the end of) Hypothesis 5.3.1.

For further work along these lines, we refer the interested reader to a recent study by Kobussen.[210]

It is hoped that this volume has established the need to abandon the Lie and the Lie-isotopic formulation of symmetries for open systems in favor of their Lie-admissible form. It is understood that the conventional Lie formulation is fully sufficient for systems that are Hamiltonian in the frame of the observer. In monograph[189], we have established the sufficiency of the Lie-isotopic formulation of symmetries for closed non-Hamiltonian systems. Thus, the class of systems for which the Lie-admissible formulations of symmetries is advocated, is that of open and non-Hamiltonian systems. The need of the Lie-admissible structure then follows from the requirement that, not only the local coordinates have a direct physical meaning (which is permitted by the Birkhoffian mechanics), but also that all the generators of the symmetries are conventional quantities with a direct physical significance, e.g., as in case (5.3.5d). It is also hoped the reader will keep in mind the lack of Lie algebra character of the full, inhomogeneous Birkhoff's equations and the consequential, inevitable difficulties for exponentiation into a group structure.

At the risk of being repetitive, it should be stressed again that the clear, unequivocal, and direct physical meaning of the algorithms at hand is a necessary prerequisite for the achievement of a physically meaningful quantization, that is, to avoid the achievement of operator formulations in a Hilbert space that are mathematically consistent but physically vacuous, as we shall see in detail in Volume III.

FORMAL METHODS III: VIA DIFFERENTIAL GEOMETRY AND OTHER DISCIPLINES. A number of additional methods for the construction of symmetries from given systems of differential equations are scattered throughout the literature of differential geometry and other disciplines. All these methods are potentially applicable to the construction of the Galilei-admissible relativity. In fact, they generally hold in the local variables of the experimenter. Thus, the implementation of guidelines (a), (b), (c) above may yield the desired Galilei-admissible covering of the Galilei symmetry (or at least part of it).

We assume the reader is aware of the inapplicability of (the direct and inverse) Noether theorem for the construction and the application of the Galilei-admissible relativity. This is the case for numerous reasons.

First, Noether theorem demands the prior knowledge of a Hamiltonian (or a Lagrangian), while the systems considered are non-Hamiltonian (and non-Lagrangian), by conception;

Second, Noether theorem demands the prior knowledge of a generally manifest symmetry, while the symmetries we are referring here are highly nonmanifest (to the point of possibly demanding transcendental functions).

Third, and perhaps most importantly, Noether's theorem is only applicable to Lie symmetries and conservation laws, while our setting is that of Lie-admissible symmetries and time rates of variation.

Needless to say, we do not exclude the possibility that the Galilei-admissible symmetry, being a conventional symmetry in the sense of Eqs. (5.3.20), may be associated to a set of first-integrals. The aspect we are stressing here is that these first integrals CANNOT be the generators (5.3.14d), trivially, because the particle considered is in the highest possible nonconservative conditions.

To put it differently, the conventional setting of Galilei's relativity is based on the identification of conserved quantities (first integrals) which are assumed as the generators of the Galilei symmetry. The generator of the Galilei-admissible symmetry, instead, are nonconserved by central physical condition. The first integrals which may be associated to the symmetry, therefore, do not possess a direct physical meaning. Lacking the technical characterization of the NONCONSERVATION of the generators, one risks the illusion of novelty in the study of open non-Hamiltonian systems.

We close the section with a number of consequences that appear to be implied by Hypothesis 5.3.1.

HYPOTHESIS 5.3.1A: The Galilei-admissible relativity provides a NONGEODESIC characterization of the motion of the particle considered, in the sense that its trajectory does not characterize a geodesic in the topological manifold of group (5.3.16).

This aspect was implicit in Appendix 3.G. It is significant to recall that Galilei (as well as Einstein special) relativity provides a GEODESIC characterization of the trajectory of particles. The intrinsically nongeodesic nature of the proposed Galilei-admissible relativity is, therefore, another technical characterization of the presence of nonself-adjoint forces, as illustrated, say, by the trajectory of a satellite in Earth's atmosphere.

HYPOTHESIS 5.3.1B: The Galilei-admissible relativity provides a NONINERTIAL characterization of systems.

This is also evident from the fact that the particle considered is in strictly noninertial conditions. Thus, the frame considered is initially noninertial. The (generally nonlinear and noncononical) Galilei-admissible transformations, therefore, map noninertial frames into other noninertial frames.

Again, as stressed since the beginning of Section 5.1, Galilei transformations remain the largest possible group of linear (nonrelativistic) transformations for mapping inertial systems into inertial systems. The point is that the systems considered are noninertial and Galilei-noninvariant by assumption. At any rate, the imposition of the Galilei relativity in the physical frame of the experimenter implies an often excessive restriction on the class of forces admitted. Different views literally imply the validity of the perpetual motion in our environment [this is a necessary consequence of the imposition of Galilei symmetry under the condition of direct physical meaning of the generators and of the local coordinates].

We hope this volume has established the need of reversing the contemporary attitude regarding relativity. We are referring here to the conventional attitude of first assuming a given relativity, and then restricting the admissible dynamics to that permitted by the relativity. The inverse attitude we have advocated is instead that of, first assuming systems and equations of motion as suggested by the physical reality, and then identifying a relativity capable of their description.

HYPOTHESIS 5.3.1C: The Galilei-admissible relativity provides a NONLINEAR characterization of the eleton.

We are referring here to the characterization of the nonconservative particle of Hypothesis 5.3.1, which we call "eleton" from Volume I, via nonlinear transformations, nonlinear representations, etc. This is evident from Appendix 3.B.

Again, the nongeodesic, noninertial, and nonlinear nature of the eleton, rather than being drawbacks, should all be seen as different manifestations of the achievement of a dynamics which is nontrivially more general than the conventional one for planetary motions (and for the atomic structure).

HYPOTHESIS 5.3.1D: The Galilei-admissible relativity generally characterizes a PRIVILEDGED reference frame, that at rest with the medium in which the motion occurs.

This additional aspect, even though more phylosophical (and potentially open to a variety of views), is also a consequence of our assumptions of Section 5.1. We are referring here to the assumption that the covering relativity is formulated in the frame which is at rest with the center-of-mass of the CLOSED non-Hamiltonian system of which the eleton

is a constituent.

This condition is automatically verified for our experiments, because the Earth can be considered as a closed non-Hamiltonian system and our measuring apparata are at rest with respect to Earth.

The fact that the assumed frame is priviledged follows from the fact that the essential physical characteristics of the particle considered are generally lost for other frames as illustrated by the fact that the nonconservative system (5.3.11) becomes "free" in frame (5.3.30).

To put it differently, the NONSELF-ADJOINT CHARACTER of the forces considered exists in the actual frame of the experimenter. The forces preserve their form under Galilei-admissible transformations. However, these transformations are highly nonlinear. As a result, the transformed frames are (generally) nonrealizable via experiments. The original frame at rest with the center-of-mass of the closed non-Hamiltonian system, therefore acquires a priviledged character for the experimenter, in the sense that it is the frame actually realizable for the measure and characterization of the nonconservative character of the particle. Other frames are mathematically possible, but either they are not physically realizable, or they imply an alteration of the explicit form of the forces.

HYPOTHESIS 5.3.1E: The Galilei-admissible relativity describes a class of systems nontrivially more general than the (conservative) systems of the conventional relativity, and implies nontrivially more general relativity laws (such as nongeodesic, noninertial, and nonlinear characterizations of particles). Nevertheless, it is a covering relativity by construction, in the sense that the conven-

AN EXAMPLE OF APPLICABILITY OF THE PROPOSED GALILEI-ADMISSIBLE GENERALIZATION OF GALILEI RELATIVITY:

SATELLITE DURING RE-ENTRY IN EARTH'S ATMOSPHERE.

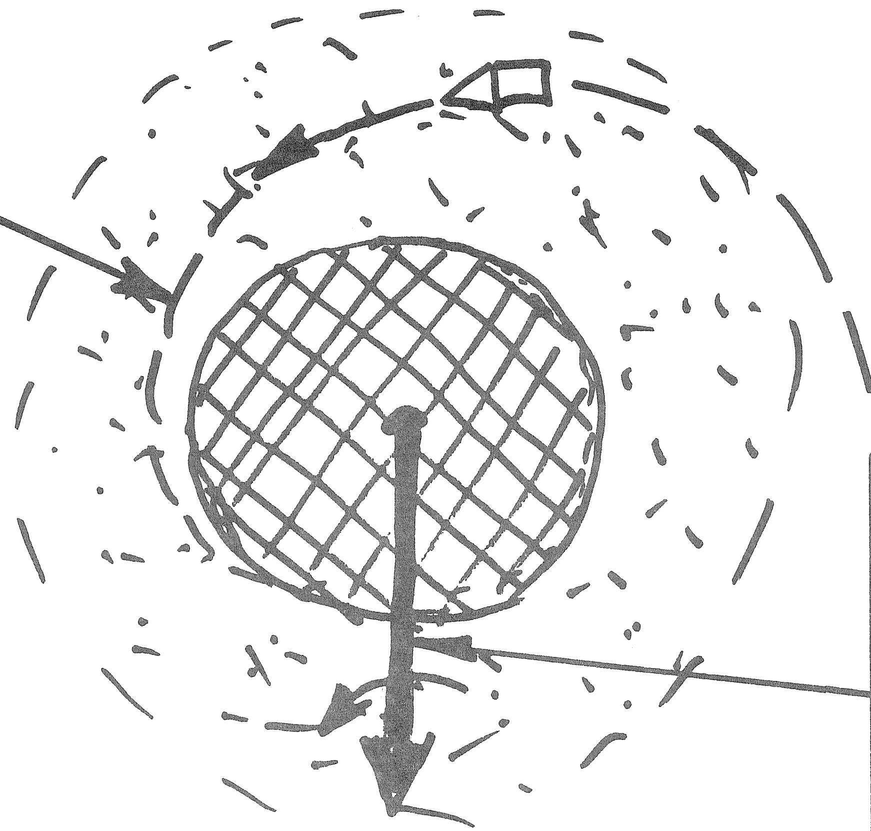


FIGURE 5.5

tional relativity is recovered identically when all nonself-adjoint/Galilei-relativity-breaking forces are null.

This last comment has been introduced to stress that novel advances in physics never "destroy" existing results of clear physical value. They merely generalize their conceptual, physical, and mathematical structure. Thus, under no circumstance the proposed Galilei-admissible relativity should be interpreted as intended to void Galilei relativity. Instead, we have merely identified the arena of unequivocal applicability and physical relevance of Galilei relativity, and proposed a plausible generalization ONLY for broader physical conditions.

5.3: THE HYPOTHESIS OF A LIE-ADMISSIBLE GENERALIZATION OF EINSTEIN SPECIAL RELATIVITY.

We are now equipped to formulate the most important hypothesis of this volume.

HYPOTHESIS 5.4.1: Einstein special relativity for the characterization of one point-like particle under action-at-a-distance/potential/self-adjoint forces, admits a covering relativity, called EINSTEIN-ADMISSIBLE RELATIVITY, for the representation of the particle as extended under the additional presence of contact/nonpotential/nonself-adjoint forces. The covering relativity is characterized by a Lie-admissible bimodular generalization of the Poincaré group verifying the conditions:

(a) of recovering the conventional Einstein relativity identically under the point-like approximation of particles;

(b) of recovering the conventinal Galilei relativity identically whenever, jointly with the point-like approximation of particles, the speeds are nonrelativistic; and

(c) of admitting the implementation into a closed non-Hamiltonian system, inclusive of the particle considered and all external terms, whose center-of-mass motion complies with Einstein relativity.

A most important implication of the above hypothesis is that THE SPEED OF LIGHT IS NOT EXPECTED TO BE NECESSARILY THE MAXIMAL POSSIBLE SPEED FOR ORDINARY, MASSIVE PARTICLES.

To illustrate the point, consider again the proton in the particle accelerator of Section 5.1. Its maximal possible speed, as well known, is that of light in vacuum

$$\begin{pmatrix}\text{MAXIMAL SPEED OF PROTON}\\ \text{IN PARTICLE ACCELERATOR}\end{pmatrix} = c \qquad (5.4.1)$$

The implication we are referring to here is that, under the full validity of the above limit for the center-of-mass of the proton, and depending on the interior physical characteristics, the maximal speed of massive constituents of the proton can be higher, equal, or smaller than c[202]

$$\begin{pmatrix}\text{MAXIMAL SPEED OF THE MASSIVE}\\ \text{CONSTITUENTS OF PROTON}\end{pmatrix} \gtrless c \qquad (5.4.2)$$

The quantitative treatment of the above occurrence which is permitted by our Lie-isotopic and Lie-admissible generalizations of Lie's theory is rather intriguing. For brevity, we shall limit ourselves to a mere reformulation of the results of Section 5.2.

Consider the component of the conventional Lorentz transformations which is responsible for the maximal speed (5.4.1), here expressed in only one space-dimension for simplicity but without loss of generality,

$$\begin{cases} x^{0\prime} = \gamma(x^0 - \beta x^1), & \gamma = (1-\beta^2) \\ x^{1\prime} = \gamma(-\beta x^0 + x^1), & \beta = v/c \end{cases} \qquad (5.4.3)$$

As well known, it is characterized by the Minkowski invariant

$$x'^t x' = x'^2 - x'^2 \equiv x - x^2 = x^t x \qquad (5.4.4)$$

Transformations (5.4.3) can be written in the well-known matrix

rotation

$$x' = \Lambda x, \quad x = \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \qquad (5.4.5a)$$

$$\Lambda = e^{uM}, \quad M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad (5.4.5b)$$

$$\Lambda = \begin{pmatrix} \cosh u & -\sinh u \\ -\sinh u & \cosh u \end{pmatrix}, \quad \tanh u = \beta \qquad (5.4.5c)$$

We subject now invariant (5.4.4) to our isotopy

$$x^t x \Rightarrow x^{\hat{t}} T x = x^\mu T_{\mu\nu}(t, x, \frac{dx}{d\tau}, \dots) x^\nu \stackrel{\text{def}}{=} x^{\hat{t}} * x \qquad (5.4.6)$$

and consider the simplest possible case of interest here, the diagonal isotopy

$$T = \begin{pmatrix} t_0^2 & 0 \\ 0 & t_1^2 \end{pmatrix} \qquad (5.4.7)$$

which characterizes the following isotopic covering of Lorentz transformations (5.4.5)

$$\hat{\Lambda} = \hat{I} e^{u*M} = e^{M*u} \hat{I}, \quad \hat{I} = T^{-1} \qquad (5.4.8)$$

which, after simple calculations, can be written in the explicit form

PROTON IN A PARTICLE ACCELERATOR

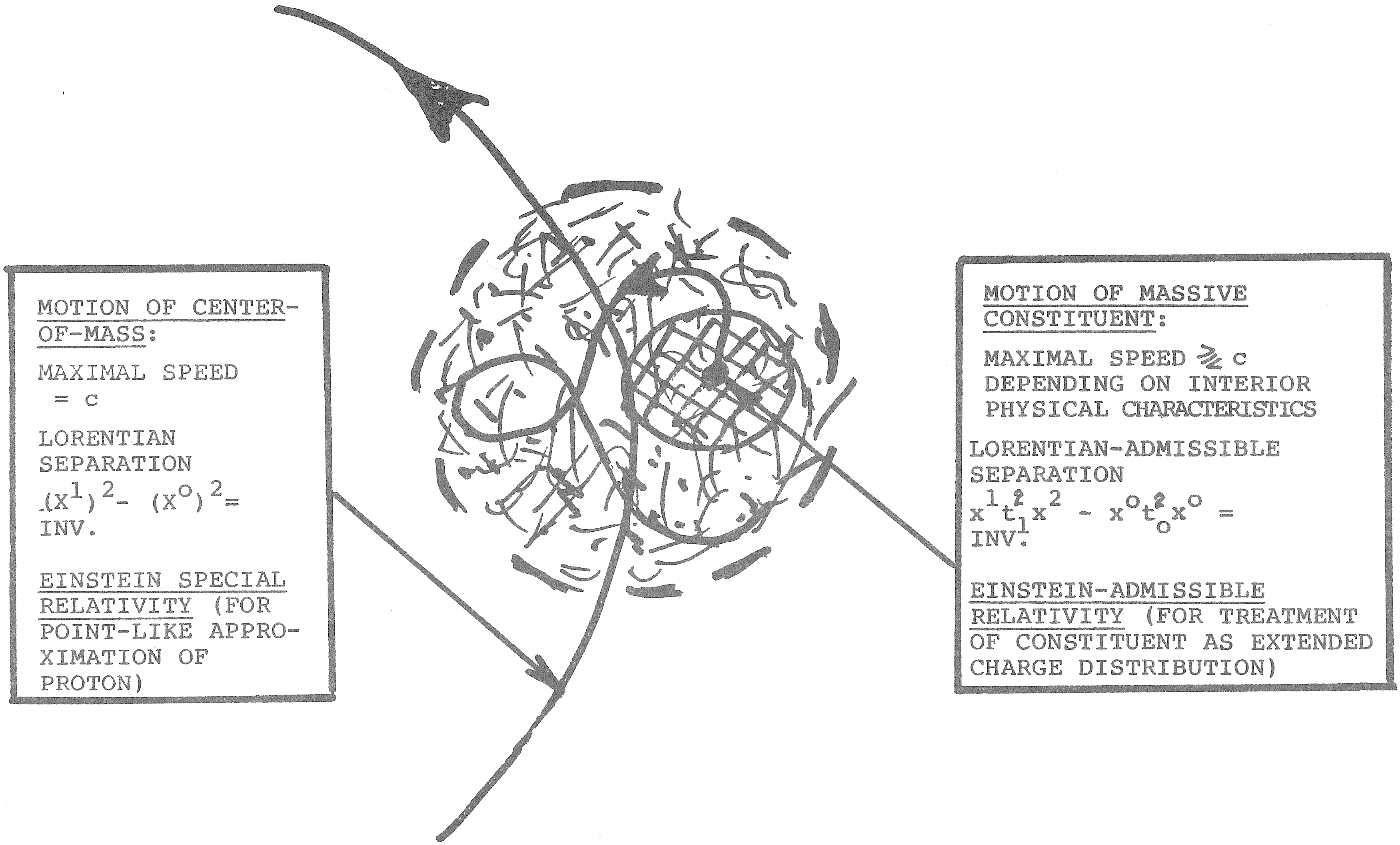


FIGURE 5.6

$$\hat{\Lambda} = T^{-1}\begin{pmatrix} \cosh(ut_0t_1) & -\frac{t_1}{t_0}\sinh(ut_0t_1) \\ -\frac{t_0}{t_1}\sinh(ut_0t_1) & \cosh(ut_0t_2) \end{pmatrix} \tag{5.4.9}$$

The verification of the form-invariance of the new separation is trivial,

$$x'^{\hat{t}} * x' = x^{\hat{t}} * \hat{\Lambda}^{\hat{t}} * \hat{\Lambda} * x \equiv x^{\hat{t}} * x \tag{5.4.10}$$

as one can see. The bimodular Lie-admissible generalization is trivial and will be ignored at this time.

We now assume the particular value $t_1 = 1$. This is sufficient to provide an alteration of the speed of light into the new value

$$c' \quad t_0 c \gtrless c \tag{5.4.11}$$

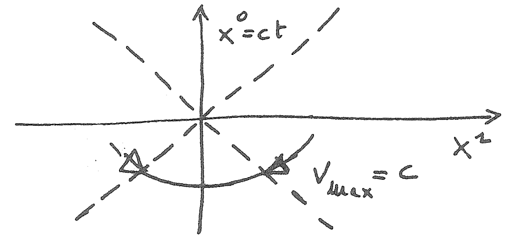
which can be bigger, equal, or smaller than c depending on the physical characteristics of the medium in which the particle considered moves. This confirms possibility (5.3.2). In particular, it provides a group theoretical explanation to the hypothesis

$$\left(\begin{matrix}\text{MAXIMAL SPEED OF LIGHT INSIDE} \\ \text{A PROTON}\end{matrix}\right) \cong 75\ c \tag{5.4.12}$$

submitted by De Sabbata and Gasperini[211] following the proposal by Santilli of the relativistic extension of closed non-Hamiltonian systems[202].

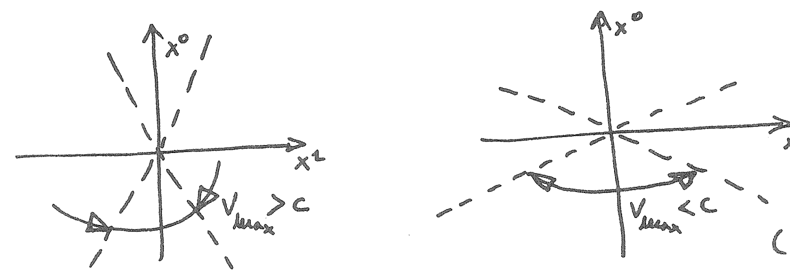
What we have here is, in essence, a deformation of the light cone

of Einstein's special relativity



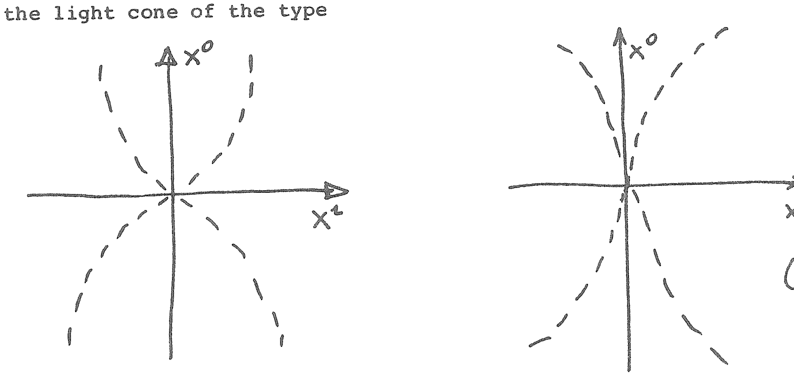
(5.4.13)

into one of the following modified forms



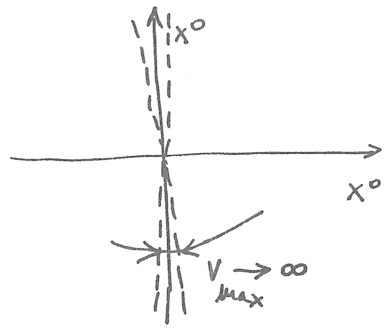
(5.4.14)

with the understanding that, when the space unisotropy of the hadronic medium is taken into account, we may have more general deformations of the light cone of the type



(5.4.15)

with the limiting "Galilean case" within a full relativistic setting



(5.4.16)

which evidently implies possible infinite speeds of particles within hadronic media, of course, under limiting cases such as the core of stars undergoing gravitational collapse.

We assume the reader is aware of the fact that the possible achievement of speeds higher than c within hadronic matter is referred here to CONVENTIONAL MASSIVE PARTICLES AND NOT TO TACHYONS.

The reader familiar with our scientific journey will find hypothesis (5.4.2) quite natural and plausible. In fact, the isotopic deformations of the Minkowski invariant are admissible, on physical grounds, only for non-Hamiltonian systems of extended particles admitting contact interactions for which the notion of potential energy has no physical foundation.

These interactions can accelerate massive particles without any need of potential energy and instead via contact effects. Besides, contact interactions must be instantaneous by conception, and, thus, they are outside the technical capabilities of Einstein special relativity. But, then, the achievement by ordinary particles of speed higher than

c , and even infinite, under suitable limiting conditions, is a mere consequence.

This concludes the preparatory phase of our study of the problem of the hadronic structure, and, in particular, of the hadronic constituents, which we contemplate to present in the forthcoming Volume III.

A more detailed treatment of Hypothesis 5.4.1, including an initial construction of a Lie-admissible bimodular generalization of the Lorentz group, will be presented in the forthcoming paper.[212]

A self-evident gravitational implication is presented in Figure 5.7.

THE POSSIBLE LOCALLY NONLORENTIAN CHARACTER
OF THE INTERIOR PROBLEM OF GRAVITATION
FOR EXTENDED CONSTITUENTS OF
MASSIVE BODIES

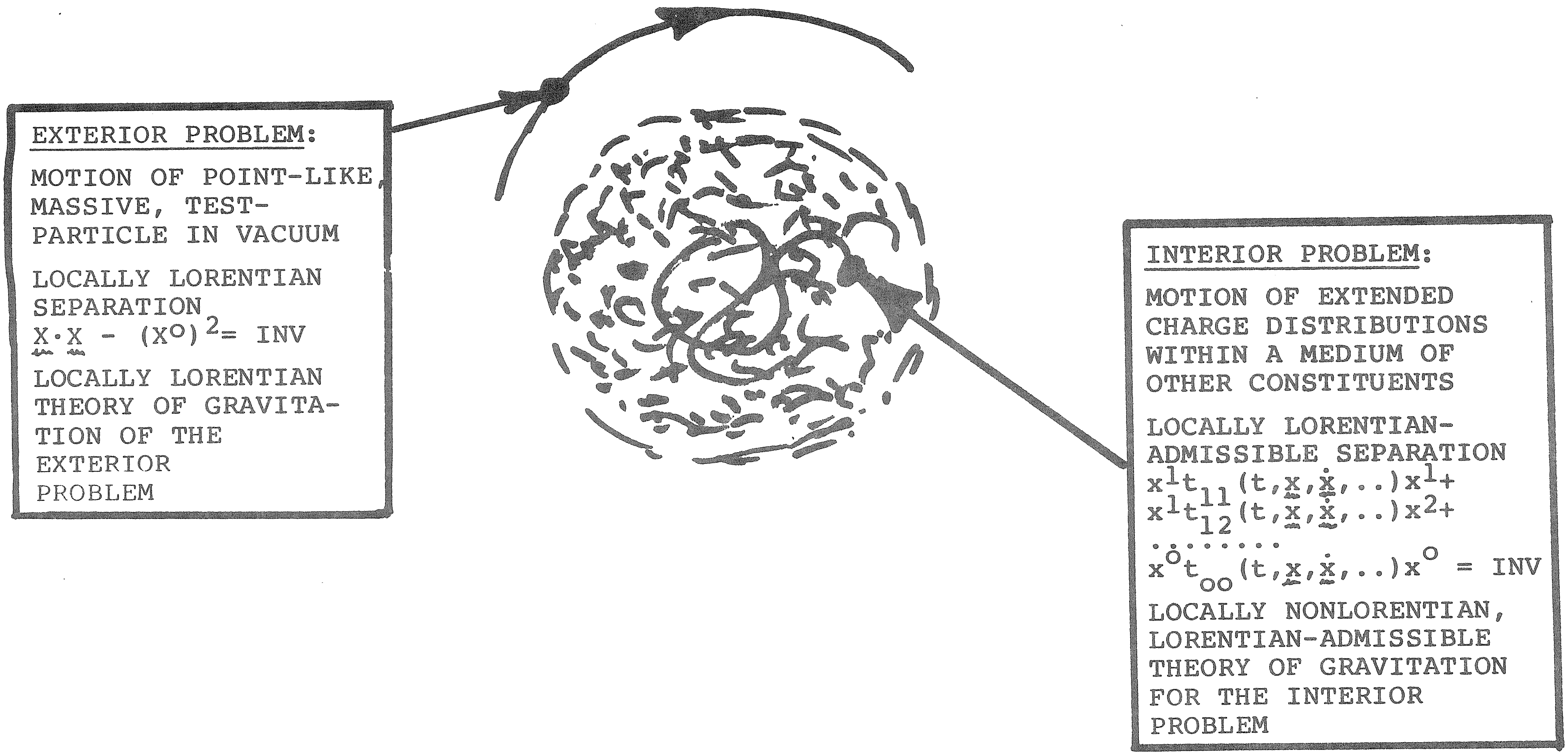


FIGURE 5.7

5.5: RELEVANCE OF THE MATHEMATICAL STUDIES BY BENKART, BRITTEN, ILAMED, KÔIV, LÔHMUS, MYUNG, OKUBO, OSBORN, SAGLE, SORGSEPP, TOMBER, WENE, ET AL. FOR THE POSSIBLE GENERALIZATION OF GALILEI AND EINSTEIN RELATIVITIES FOR EXTENDED PARTICLES.

As indicated in the Preface and in the note added at the end of Section 1.5, this volume was conceived and written in 1977, and that a rather considerable amount of mathematical studies on Lie-admissible algebras have appeared since that time.

Regrettably, I have been unable to apply these mathematical studies to the topic of this volume, because this would have called for its complete rewriting, a task outside my possibilities at this time.

Nevertheless, this volume would be grossly deficient without at least an indication that the mathematical studies on Lie-admissible algebras in general, and on Lie-isotopic algebras as a particular case, conducted by

G. M. BENKART, D. J. BRITTEN, Y. ILAMED, M. KÔIV, J. LÔHMUS,
H. C. MYUNG, R. H. OEHMKE, S. OKUBO, A. A. SAGLE, L. SORGSEPP,
M. L. TOMBER, G. P. WENE, et al.,

are particularly relevant for the achievement of technical maturity in the study of the generalization of Galilei and Einstein relativities for extended particles under Hamiltonian and non-Hamiltonian forces, with particular reference to hadrons, their interactions, and their structure.

It is virtually impossible to identify here all the possibilities for advancements, and quote all the pertinent literature. I therefore refer the interested reader to the papers by the above authors listed

in Tomber's Bibliography[183-185] and in Proceedings[186-188]. As an indication of possibilities, we quote here the following

[1] The theorems of classification of flexible Lie-admissible algebras admitting a given attached simple Lie algebra (Tomber, Myung, Okubo, Osborn, Benkart, et al) are clearly important for the construction of a possible Lie-admissible covering of the rotation and of the Lorentz group, under the condition of preservation of the old algebra in the attached form, as well as of flexibility.

[2] The studies of exponentiation on associative isotopic and non-associative Lie-admissible algebras (Myung, et al) are clearly at the foundation of the very structure of the generalized relativities submitted in this volume;

[3] The geometric characterization of Lie-admissible algebras (Oehmke) are clearly important for the possible globalization of generalized Lie-admissible relativities. In turn, globalization is important both classically and quantum mechanically, as well as in regard to quantization.

[4] The geometric characterization of Lie-admissible algebras via suitable changes of the local coordinates (Sagle) is clearly important for the practical construction of Lie-admissible generalizations of the Galilei and Poincaré transformations.

[5] The studies of the mutation of the Lie product (ab - bc) into the Lie-admissible form (acb - bda), c,d=fixed (Osborn, et al), are clearly important for any physical model dependent on the local behaviour of a bimodular Lie-admissible structure.

[6] The studies of the deformation of Lie into Lie-admissible algebras (Kòiv, Lôhmus, Sorgsepp, Myung, et al) are manifestly relevant for the construction of the generalized relativity via (a suitable generalization of the conventional) deformation theory.

[7] General studies regarding the structure, realizations, and properties of the Lie-admissible algebras (Benkart, Ilamed, Wene, et al) are clearly important for corresponding specific, technical aspects of the generalized relativities.

It is hoped that these possibilities are considered by other independent researchers interested in the magnificent open problem of the generalization of contemporary relativities for physical conditions of particles (e.g., strong interactions) which were unknown at the time of their inception.

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