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FOUNDATIONS OF THE HADRONIC GENERALIZATION OF THE ATOMIC MECHANICS, II: MODULAR–ISOTOPIC HILBERT SPACE FORMULATION OF THE EXTERIOR STRONG PROBLEM

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Abstract

This paper is devoted to a first formulation of the axiomatic Hilbert space foundations of the branch of the Hadronic Mechanics dealing with the exterior treatment of strong non–Hamiltonian systems, and which admits a Lie–isotopic algebraic character. In particular, we are interested in generalizing the conventional eigenvalue equations of the Atomic Mechanics into the broadest possible equations which are permitted by an associative algebra of operators on a one–sided, modular, Hilbert space. The objective is made possible by the isotopic generalization of the associative algebra $\mathcal{A}$ of operators A, B, ... on a Hilbert space, such as the isotope $\mathcal{A}^{(T)}$ characterized by the product $A*B = ATB$, where T is a fixed, bounded, and nonsingular operator of $\mathcal{A}$. By using the T–isotopic product, we first introduce a generalization of the Hermitean conjugate, transpose, Hermitean, skew–Hermitean, unitary, skew–unitary, projection, and exponential operators. We then pass to the isotopic generalization of the notion of determinant, trace, and eigenvalue of a linear operator. Some essential properties of this modular–isotopic formulation of the Hilbert space theory are identified. The results are then applied to Hadronic Mechanics. In particular, we present the hadronic generalization of a number of postulates of Atomic Mechanics, ranging from observables, states, and their time evolution, to total probability, its conservation in time, and the expectation values. We also prove that the isotopic generalization of Schrödinger's equations presented in Paper I are equivalent to the isotopic generalization of Heisenberg's equations. We then pass to a central aspect of the exterior hadronic problem, the achievement of total conservation laws under non–Hamiltonian internal forces. For this purpose, we review the classical Birkhoffian solution of the problem, and point out its possible hadronic counterpart. As an application, the structure model of the π^0 particle proposed by one of us in 1978 is re–examined and shown to possess a modular–isotopic structure. The existence of rather intriguing and virtually endless possibilities of further developments of mathematical, theoretical, and experimental character, are self–evident.

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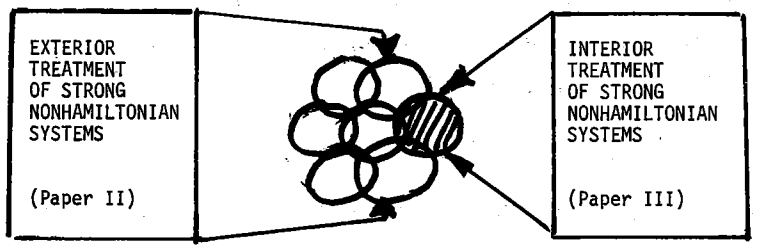
I. INTRODUCTION

Recent physical studies, outlined in the adjoining Paper I by one of us, have indicated a realistic possibility of generalizing quantum mechanics for strong interactions via suitable generalization of the conventional associative algebra of operators on a Hilbert space.

Two primary lines of study have been identified. The first is based on an associativity preserving generalization of the associative operator algebra, and it is called *isotopic* generalization. The second is based on the use of more general, nonassociative, Lie-admissible algebras, and it is sometimes referred to as the *genotopic* generalization.

The two generalizations are predictably inter-related, mathematically and physically. In fact, the Lie algebra which is attached to the Lie-admissible generalization is of isotopic, rather than conventional type. Thus, the study of the isotopic context appears to be important for those of Lie-admissible type. Also, on physical grounds, the isotopic generalization is often preferred for the exterior description of a system with strong internal forces (where total conservation laws suggest the use of an anti-commutative algebra in the time evolution). On the contrary, the genotopic generalization is often preferred for the study of each constituent of a system under strong internal forces (where nonconservation laws suggest the use of a non-anticommutative algebra in the time evolution). The physical compatibility of these exterior and interior descriptions then results, mathematically, in the notion of isotopic Lie algebra attached to a non-associative Lie-admissible algebra of operators.

The objective of this Paper II is to attempt the identification of the foundations of the isotopic branch of the hadronic mechanics, that is, of the exterior treatment of systems with strong internal interactions which are of generally non-Hamiltonian type, while Paper III of this series will deal with the interior treatment of the same problem according to the figure



On physical grounds, one of our primary problems is the achievement of a theory permitting the actual realization of experimental measures for systems which are non-Hamiltonian by assumption, that is, under the condition that the conventional Atomic Mechanics is inapplicable.

On mathematical grounds, one of our primary problems is the formulation of a non-Hamiltonian generalization of the Atomic Mechanics which preserves the underlying Hilbert space although in a predictaly generalized way.

The solution to both problems has been inspired by the Birkhoffian generalization of the Hamiltonian mechanics which, on one side, is universal for local non-Hamiltonian systems, while, on the other side, leaves unchanged the underlying carrier space (the cotangent bundle T*M of the symplectic

geometry or its contact extension R x T*M). We simply have the replacement of the fundamental (canonical) structure with the most general possible exact symplectic (or contact) structure.

Inspired by these results, we search in this paper for the most general possible formulation of the eigenvalue problem which is permitted by a modular, one-sided, Hilbert space theory, while preserving the associative character of the enveloping algebra of operators.

Predictably, the solution results to be characterized by an isotopy. The emerging isotopic generalization of the Atomic Mechanics then results to be the expected hadronic image of the Birkhoffian Mechanics.

The analysis is conducted as follows. In Section II we present a mathematical study of the isotopic generalization of the conventional operator theory on Hilbert spaces, with particular reference to the isotopic generalization of: Hermitian conjugate, transpose, Hermitean, skew-Hermitean, unitary, antiunitary, projective, and exponential operators. We then pass to the identification of the isotopic generalization of determinants, traces, and eigenvalues. The analysis is completed with the identification of a number of mathematical properties, with the understanding that much remains to be done, particularly for aspects related to topology, spectral decompositions, boundedness, domains, etc.

The results of Section II are sufficient to permit in Section III the initiation of the identification of the axiomatic foundations of the Hadronic-isotopic Mechanics which appear most plausible at this moment. Even though tentative and partial, the results are sufficient to confirm the expectation that each and every aspect of the Atomic Mechanics admits a consistent hadronic generalization.

The possibilities for further mathematical, theoretical, and experimental developments are therefore truly intriguing and virtually endless.

II. ISOTOPES OF LINEAR OPERATIONS ON A HILBERT SPACE

2.1. Conventional Definitions. (See, for instance, Ref.s 7). Throughout this section, by a linear space V we mean a vector space over the complex number field C with addition $x + y$ and scalar multiplication λx, $x, y \in V$ and $\lambda \in C$. For a complex number λ, $\bar{\lambda}$ denotes the conjugate of λ . In the essence, we shall consider only finite-dimensional spaces.

Important linear spaces are the inner product spaces. Recall that if V is a linear space then a mapping $(\ , \ ) : V \times V \to C$ is called an *inner (scalar) product* on V if it satisfies the axioms

$$(x,x) > 0 \quad \text{for all} \quad x \neq 0 \ \text{in} \ V \ , \tag{2.1}$$

$$(x,y) = \overline{(y,x)} \ , \tag{2.2}$$

$$(x,y+z) = (x,y) + (x,z) \ , \tag{2.3}$$

$$(x,\lambda y) = \lambda(x,y), \ \lambda \in C \tag{2.4}$$

for all $x, y, z \in V$. Note that the inner product (x,y) is also denoted by the Dirac's notation $\langle x|y\rangle$ in other literature. A linear space V equipped with inner product is called an *inner product (Euclidean) space.*

In an inner product space V , we define the *norm* $||x||$ of each vector $x \in V$ by

$$||x|| = \sqrt{(x,x)} \ . \tag{2.5}$$

Thus every inner product space V turns out to be a metric space under the metric $d(x,y) = ||x - y||$. A complete inner product space in this metric is called a (complex) *Hilbert space.* An inner product space and so Hilbert spaces can be defined for the real number field. Since complex Hilbert spaces are more exclusively used in quantum physics, in this section we only treat complex Hilbert spaces.

Due to the technical problem of developing the concept of isotope, we will essentially restrict ourself to finite-dimensional Hilbert spaces. If H is a finite-dimensional Hilbert space, there exists a special type of basis $e_1, \cdots, e_n$ of H such that

$$(e_i,e_j) = \delta_{ij}, \ i, j = 1, 2, \cdots, n \ . \tag{2.6}$$

Such basis $e_1,\cdots, e_n$ is called an *orthonormal basis* of H . Henceforth, we fix an orthonormal basis $e_1, \cdots, e_n$ of H . Thus every vector $x \in H$ is uniquely expressed as

$$x = \alpha_1 e_1 + \cdots + \alpha_n e_n, \ \alpha_i \in C \ . \tag{2.7}$$

where $\alpha_i = (e_i, x)$. Given $y = \beta_1 e_1 + \cdots + \beta_n e_n$, we have

$$(x,y) = \sum_{i=1}^{n} \bar{\alpha}_i \beta_i \ . \tag{2.8}$$

Definition 2.1. Let x be given by (2.7). Then the *conjugate vector* $\bar{x}$ of x is defined by $\bar{x} = \bar{\alpha}_1 e_1 + \cdots + \bar{\alpha}_n e_n$. #

Definition 2.2. Let V be a linear space. A *linear operator* A on V is a mapping $A : x \to Ax$ of V into itself satisfying:

$$A(\lambda x + \mu y) = \lambda(Ax) + \mu(Ay)$$

for all $x, y \in V$ and $\lambda, \mu \in C$. If $A : V \to V$ satisfies the relation

$$A(\lambda x + \mu y) = \bar{\lambda}(Ax) + \bar{\mu}(Ay)$$

for all $x, y \in V$ and $\lambda, \mu \in C$ then A is called an *antilinear operator* on V. #

In view of Definition 2.2., *all linear operators on a Hilbert space H are assumed in this section to be defined on the entire set H*, whereas a more general definition requires only that a linear operator in H is defined on a subspace of H. Antilinear operators in a Hilbert space play a very minor role in quantum mechanics by comparison with linear operators. Thus, for most of the cases, we focus on linear operators. For the sake of generality, we introduce the following.

Definition 2.3. A *complex (associative) algebra* $\mathcal{A}$ is a linear space with addition $A+B$ and scalar multiplication λA such that there is defined an operation : $(A,B) \to AB$ of $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$ satisfying

(i) $(AB)C = A(BC)$, associativity,

(ii) $(A+B)C = AC + BC$, $A(B+C) = AB + AC$, distributivity,

(iii) $\lambda(AB) = (\lambda A)B = A(\lambda B)$, assoc. and commut. of scalar multiplic.

for all $A, B, C \in \mathcal{A}$ and complex numbers λ. The algebra $\mathcal{A}$ is called *commutative* if $AB = BA$ for all $A, B \in \mathcal{A}$. An element $1 \in \mathcal{A}$ is called an *identity element* of $\mathcal{A}$ if $1A = A1 = A$ for all $A \in \mathcal{A}$. #

An algebra $\mathcal{A}$ is in general neither commutative nor has an identity element. We are particularly interested in algebras of linear operators in a Hilbert space H. Denote by $L(H)$ the set of all linear operators on H. For $A, B \in L(H)$, as usual we define

$$(A + B)x = Ax + Bx, \tag{2.9}$$

$$(AB)x = A(Bx), \tag{2.10}$$

$$(\lambda A)x = \lambda(Ax), \; x \in H, \; \lambda \in C. \tag{2.11}$$

It is readily seen that these operations convert $L(H)$ into a complex associative algebra in the sense of Definition 2.3. Note that $L(H)$ is not commutative in general but has an identity element I which is the identity operator on H; i.e., $Ix = x$ for all $x \in H$.

Definition 2.4. Let H be a Hilbert space. By a (linear) *operator algebra* $\mathcal{A}$ of H, we mean a subalgebra $\mathcal{A}$ of $L(H)$ which contains the identity element I. Namely, $\mathcal{A}$ is a subspace of $L(H)$ such that $I \in \mathcal{A}$ and $AB \in \mathcal{A}$ for all $A, B \in \mathcal{A}$. #

Definition 2.5. Let $\mathcal{A}$ be an (associative) algebra with multiplication AB and with identity element I. An element A of is said to be *invertible* in $\mathcal{A}$ if there exists an element $A^{-1} \in \mathcal{A}$ such that $AA^{-1} = A^{-1}A = I$. In this case, call A^{-1} the *inverse* of A. An invertible linear operator A in a Hilbert space is customarily called nonsingular with inverse operator A^{-1}. #

A linear operator A on a Hilbert space H is called *bounded* if

$$||Ax|| \leq c\,||x||, \quad x \in H$$

for some fixed positive real number c. Note that if A is a bounded nonsingular linear operator then the inverse operator A^{-1} is bounded also. We assume the reader is familiar with the boundedness of operators on finite-dimensional Hilbert spaces.

2.2. Isotope of an operator algebra. The concept of isotope of an operator algebra $\mathcal{A}$ is to convert $\mathcal{A}$ into another algebra defined on the same underlying linear space as $\mathcal{A}$ where the algebra $\mathcal{A}$ can be recovered as a special case of the isotope. Therefore, it is very important in our investigation to distinguish the notions of product, inverses and identity element in $\mathcal{A}$ from those in the isotope of $\mathcal{A}$. This distinction will be the central scheme of our attempt to introduce a more generalized notion of unitary, symmetric, hermitian and exponential operators in a Hilbert space. For a general approach, we define an isotope in an arbitrary complex associative algebra.

Definition 2.6. Let $\mathcal{A}$ be a complex associative algebra with product AB and with identity element I. Let T be a fixed element of $\mathcal{A}$. The algebra with multiplication

$$A * B = ATB, \quad A, B \in \mathcal{A} \tag{2.12}$$

defined on the same underlying linear space as $\mathcal{A}$ is called the T-*homotope* of $\mathcal{A}$ and is denoted by $\mathcal{A}^{(T)}$. If T is invertible in $\mathcal{A}$ then the T-homotope $\mathcal{A}^{(T)}$ is called the T-*isotope* of $\mathcal{A}$. #

Notice that $\mathcal{A}^{(T)}$ is an associative algebra; $(A * B) * C = A * (B * C)$ for all $A, B, C \in \mathcal{A}$. The notion of homotope and isotope has been a useful tool for the structure of associative and Jordan algebras. The interested reader may be referred to Jacobson[1], Myung[2], Osborn[3], and Myung and Santilli[4]. An isotopic generalization of Lie's theorems can be found in Santilli[14].

Remark 2.1. Homotopes and isotopes are closely related to some classes of Lie-admissible algebras. Let $\mathcal{A}$ be an associative algebra with product AB. Let R, S be fixed elements of $\mathcal{A}$. Denote by $\mathcal{A}(R,S)$ the algebra with multiplication

$$A \circ B = ARB - BSA, \quad A, B \in \mathcal{A} \tag{2.13}$$

defined on the same linear space as $\mathcal{A}$. The algebra $\mathcal{A}(R,S)$ has been called the (R,S)-*mutation* of $\mathcal{A}$ and arises from a generalization of the Heisenberg equation by Santilli. Denote by $\mathcal{A}(R,S)^-$ the algebra with the commutator product

$$[A,B]^\circ = A \circ B - B \circ A, \quad A, B \in \mathcal{A}$$

defined on the linear space $\mathcal{A}(R,S)$. Then $\mathcal{A}(R,S)$ is a Lie-admissible algebra in the sense that $\mathcal{A}(R,S)^-$ is a Lie algebra; that is, $[\ ,\ ]^\circ$ satisfies the anticommutative law, $[A,B]^\circ = -[B,A]^\circ$ and the Jacobi identity, $[[A,B]^\circ, C]^\circ + [[B,C]^\circ, A]^\circ + [[C,A]^\circ, B]^\circ = 0$. However, $\mathcal{A}(R,S)$ is not in general associative; i.e., $(A \circ B) \circ C \neq A \circ (B \circ C)$ for some $A, B, C \in \mathcal{A}(R,S)$. By direct computation, one sees

$$[A, B]^\circ = A(R + S)B - B(R + S)A,$$
$$= ATB - BTA = [A, B]^*$$

which implies that the commutator $[A, B]^\circ$ in $\mathcal{A}(R,S)$ equals the commutator $[A, B]^* = A * B - B * A$ in the $T = (R + S)$-homotope $\mathcal{A}^{(T)}$, being associative. The structure of $\mathcal{A}(R,S)$ has been studied in considerable detail in relation with $\mathcal{A}^{(T)}$ by a number of authors (see Ref. 2, 3, 4 and Oehmke[5]). #

Given a homotope $\mathcal{A}^{(T)}$ of $\mathcal{A}$, define a mapping g of $\mathcal{A}^{(T)}$ into $\mathcal{A}$ by

$$g(A) = AT, \quad A \in \mathcal{A} . \tag{2.14}$$

Then g is clearly linear and satisfies

$$g(A * B) = g(A)g(B), \quad A, B \in \mathcal{A} . \tag{2.15}$$

Thus g is an algebra homomorphism of $\mathcal{A}^{(T)}$ into $\mathcal{A}$. Assume that T is invertible in $\mathcal{A}$ with inverse T^{-1}, so that $\mathcal{A}^{(T)}$ is an isotope of $\mathcal{A}$. Then g is an isomorphism of $\mathcal{A}^{(T)}$ onto $\mathcal{A}$ with inverse $g^{-1} = f : \mathcal{A} \to \mathcal{A}^{(T)}$ given by

$$f(A) = AT^{-1}, \quad A \in \mathcal{A} . \tag{2.16}$$

Though $\mathcal{A}$ has an identity element, in a homotope $\mathcal{A}^{(T)}$ there is no guarantee for the existence of identity element, unless T is invertible in $\mathcal{A}$. This leads to

Definition 2.7. Let $\mathcal{A}$ be an associative algebra with product AB and with identity element 1. Let $\mathcal{A}^{(T)}$ be the T-homotope of $\mathcal{A}$.

If $\mathcal{A}^{(T)}$ has an identity element, it is denoted by I^*; i.e., $I^* * A = A * I^* = A$ for all $A \in \mathcal{A}$. Let $\mathcal{A}^{(T)}$ have an identity element I^*. If an element A in $\mathcal{A}^{(T)}$ is invertible in $\mathcal{A}^{(T)}$, the inverse of A in $\mathcal{A}^{(T)}$ is denoted by A^{-1^*}; i.e., $A * A^{-1^*} = A^{-1^*} * A = I^*$. #

The following result is very important for our discussion and may be found in Myung[2] (or easily verified).

Theorem 2.1. Let $\mathcal{A}$ be an associative algebra with product AB and with identity element I. Let $\mathcal{A}^{(T)}$ be the T-homotope of $\mathcal{A}$ for a fixed element $T \in \mathcal{A}$. Then we have

(1) $\mathcal{A}^{(T)}$ has an identity element I^* if and only if T is invertible in $\mathcal{A}$. In this case, $I^* = T^{-1}$.

(2) If T is invertible in $\mathcal{A}$ then an element $A \in \mathcal{A}$ is invertible in $\mathcal{A}$ if and only if A is invertible in $\mathcal{A}^{(T)}$. In this case, $A^{-1^*} = T^{-1} A^{-1} T^{-1}$.

(3) If T is invertible in $\mathcal{A}$ then the set of invertible elements in $\mathcal{A}$ coincides with the set of invertible elements in $\mathcal{A}^{(T)}$. #

Another important concept for our investigation is the notion of one-sided modules for an associative algebra and its isotope.

Definition 2.8. Let $\mathcal{A}$ be an associative algebra with product AB and with identity element I. A linear space V is called a *left unital $\mathcal{A}$-module* if there is defined a mapping : $(A,x) \to Ax$ of $\mathcal{A} \times V$ into V satisfying :

(i) $A(\lambda x + \mu y) = \lambda(Ax) + \mu(Ay)$,

(ii) $(\lambda A + \mu B)x = \lambda(Ax) + \mu(Bx)$,

(iii) $(AB)x = A(Bx)$,

(iv) $Ix = x$

for all $x, y \in V$, $A, B \in \mathcal{A}$, and $\lambda, \mu \in C$. #

Let $\mathcal{A}$ be the same as in Definition 2.8 and V be a left $\mathcal{A}$ -module. Let $\mathcal{A}^{(T)}$ be the T-isotope of $\mathcal{A}$. Define a mapping $(A,x) \to A * x$ of $\mathcal{A}^{(T)} \times V$ into V by

$$A * x = (AT)x = A(Tx), \quad x \in V, \ A \in \mathcal{A} , \tag{2.17}$$

where Ax is the module action in the $\mathcal{A}$ -module V . It is easily checked that V becomes a unital left $\mathcal{A}^{(T)}$-module under the composition $A * x$, since $I^* * x = T^{-1}(Tx) = (T^{-1}T)x = Ix = x$ for all $x \in V$. Thus we can put

Definition 2.9. Let $\mathcal{A}$ and V be the same as in Definition 2.8. Then the unital left $\mathcal{A}^{(T)}$-module defined on V under the composition $A * x$ given by (2.17) is called the T-*isotope* of the $\mathcal{A}$-module V and is denoted by $V^{(T)}$. #

The definitions and results above are in particular applied to any operator algebra $\mathcal{A}$ in a Hilbert space H where Ax is simply the linear operator action of A on H . Note that if $T = I$ is the identity element, then $\mathcal{A}^{(I)} = \mathcal{A}$ and $V^{(I)} = V$. #

Our aim is to introduce a *generalized notion of unitary, hermitian,*

symmetric and exponential operators, and of eigenvalues by means of isotope. It is important to note that the hermitian conjugate and transpose of linear operators in a Hilbert space is an involutional mapping in the sense of

Definition 2.10. Let $\mathcal{A}$ be an algebra. A linear (or antilinear) mapping $i : \mathcal{A} \to \mathcal{A}$ is called an *involution* (or *anti-involution*) of $\mathcal{A}$ if $i(i(A)) = A$ and $i(AB) = i(B)\, i(A)$ for all $A, B \in \mathcal{A}$. #

The following result is useful for later discussion.

Theorem 2.2. Let $\mathcal{A}$ be an associative algebra with product AB and with identity element I . Let T be an invertible element in $\mathcal{A}$. Let i be an involution (or anti-involution) of $\mathcal{A}$. Then the mapping $i_T : \mathcal{A}^{(T)} \to \mathcal{A}^{(T)}$ defined by

$$i_T(A) = i(T)\, i(A)\, T^{-1}, \quad A \in \mathcal{A} .$$

is an involution (or anti-involution) of the T-isotope $\mathcal{A}^{(T)}$ of $\mathcal{A}$.

Proof. The mapping i_T is the composition $f \circ i \circ g$ of the mappings

$$\mathcal{A}^{(T)} \xrightarrow{\ g\ } \mathcal{A} \xrightarrow{\ i\ } \mathcal{A} \xrightarrow{\ f\ } \mathcal{A}^{(T)}$$

where f and g are algebra isomorphisms defined by (2.16) and (2.14), respectively. Hence i_T is an involution (or anti-involution) of $\mathcal{A}^{(T)}$. #

2.3. Isotope of unitary and hermitian operators. Let H be a Hilbert space with inner product $(\, , \,)$ and let $\mathcal{A} = L(H)$ be the operator algebra of all linear operators on H, with addition $A+B$, multiplication AB and scalar multiplication λA defined by (2.9) - (2.11). We also fix T as a (bounded) nonsingular linear operator in H (so T^{-1} is bounded also). Thus the T-isotope $\mathcal{A}^{(T)}$ and $H^{(T)}$ of $\mathcal{A}$ and H are defined by Definitions 2.6 and 2.9, respectively. In other words, H is regarded as a unital left $\mathcal{A}^{(T)}$-module by the operation (2.17). The following definition is of extreme importance for our programs.

Definition 2.11. Given a linear operator A, define the linear operator A^{hT} by the relation

$$(A * x, y) = (x, A^{hT} * y), \; x, y \in H . \tag{2.18}$$

Call A^{hT} the T-*hermitian conjugate* of A. Similarly, we define the T-*transpose* A^{tT} of A as a linear operator satisfying

$$(x, A * y) = (\bar{y}, A^{tT} * \bar{x}), \; x, y \in H . \quad \# \tag{2.19}$$

When $T = I$ is the identity operator, the I-hermitian conjugate and I-transpose of A are simply the usual hermitian conjugate A^h and transpose A^t. In this case, (2.18) and (2.19) are given by

$$(Ax, y) = (x, A^h y) , \tag{2.20}$$

$$(x, Ay) = (\bar{y}, A^t \bar{x}) . \tag{2.21}$$

Theorem 2.3. Let H be a Hilbert space, and let $\mathcal{A} = L(H)$. Then, for each $A \in \mathcal{A}$, the T-hermitian conjugate A^{hT} and the T-transpose A^{tT} exist and satisfy

$$A^{hT} = T^h A^h T^{-1} , \tag{2.22}$$

$$A^{tT} = T^t A^t T^{-1} \tag{2.23}$$

where A^h amd A^t are the usual hermitian conjugate and transpose. Furthermore,

$$(A^{hT})^{hT} = (A^{tT})^{tT} = A , \tag{2.24}$$

$$(\lambda A + \mu B)^{hT} = \bar{\lambda} A^{hT} + \bar{\mu} B^{hT} , \tag{2.25}$$

$$(A * B)^{hT} = B^{hT} * A^{hT} , \tag{2.26}$$

$$(A * B)^{tT} = B^{tT} * A^{tT} \tag{2.27}$$

for all $A, B \in \mathcal{A}$ and $\lambda, \mu \in C$.

Proof. First, note that (2.24) - (2.27) hold for A^h and A^t. In view of (2.18) and (2.20), we have

$$\begin{aligned} (A * x, y) &= (x, A^{hT} * y) = (x, A^{hT} T y) \\ &= (ATx, y) = (x, (AT)^h y) = (x, T^h A^h y) \end{aligned}$$

and this implies $A^{hT} T = T^h A^h$ and so (2.22). Similarly, $(x, A * y) = (x, AT y) = (\bar{y}, (AT)^t \bar{x}) = (\bar{y}, T^t A^t \bar{x}) = (\bar{y}, A^{tT} * \bar{x})$ and this gives (2.23). Since the mapping $i : \mathcal{A} \to \mathcal{A}$ defined by $i(A) = A^h$ or A^t is an anti-involution of $\mathcal{A}$, it follows from Theorem 2.2. that

the mapping $i_T : \mathcal{A}^{(T)} \to \mathcal{A}^{(T)}$ defined by $i_T(A) = A^{hT}$ or A^{tT} is an anti-involution of the T-isotope $\mathcal{A}^{(T)}$. This implies the relations (2.24) - (2.27). #

Definition 2.12. A linear operator S is said to be T-*symmetric* if

$$S^{tT} = S \tag{2.28}$$

and S is called T-*skew-symmetric* if

$$S^{tT} = -S \tag{2.29}$$

Similarly, a linear operator B is called T-*hermitian* (or T-*self-adjoint*) or T-*skew-hermitian* if

$$B^{hT} = B \quad \text{or} \quad B^{hT} = -B . \quad \#$$

The following result is immediate from Theorem 2.3.

Theorem 2.4. A linear operator A is T-symmetric or T-skew-symmetric if and only if

$$A = T^t A^t T^{-1} \quad \text{or} \quad A = -T^t A^t T^{-1} . \tag{2.30}$$

Similarly, B is T-hermitian or T-skew-hermitian if and only if

$$B = T^h B^h T^{-1} \quad \text{or} \quad B = -T^h B^h T^{-1} . \quad \# \tag{2.31}$$

Note that (2.31) is equivalent to:

$$(x, B * y) = (B * x, y) \quad \text{or} \quad (x, B * y) = -(B * x, y) \tag{2.32}$$

for all $x, y \in H$. The usual definition of symmetry, hermitian, etc. is recovered when $T = I$ is the identity operator.

Definition 2.13. A linear operator A is called T-*isometric* if

$$||A * x|| = ||x||, \; x \in H . \quad \# \tag{2.33}$$

If $T = I$, Definition 2.13 leads to the usual definition of isometry. By the standard linearization, one sees that (2.33) is equivalent to the relation

$$(A * x, A * y) = (x, y), \; x, y \in H . \tag{2.34}$$

Theorem 2.5. A linear operator A is T-isometric if and only if

$$A^{hT} * A = I^* \tag{2.35}$$

where $I^* = T^{-1}$ is the identity element of $\mathcal{A}^{(T)}$. Any T-isometric linear operator is bounded.

Proof. By (2.18) and (2.34), we have $(x,y) = (A * x, A * y) = (x, A^{hT} * (A * y)) = (x, (A^{hT} * A) * y) = (x, (A^{hT} * A)Ty)$ for all $x, y \in H$, since $H^{(T)}$ is a unital left $\mathcal{A}^{(T)}$-module. This gives $(A^{hT} * A)T = I$ and $A^{hT} * A = T^{-1} = I^*$. Clearly, AT is bounded and since T is bounded by the assumption, so is A. #

Definition 2.14. A linear operator U is called T-*unitary* if

$$(U * x, U * y) = (x, y) \tag{2.36}$$

for all $x, y \in H$ and U is surjective; i.e., $UH = H$. # Clearly, T-unitarity and T-isometry are equivelent for finite-dimensional Hilbert spaces.

Note that a T-unitary linear operator is T-isometric.

Theorem 2.6. A linear operator U is T-unitary if and only if

$$U^{hT} * U = U * U^{hT} = I^* \text{ or } U^{hT} = U^{-1^*} . \tag{2.37}$$

The set of all T-unitary linear operators of H forms a group with identity element I^* under the T-isotopic product " $*$ " defined by (2.12).

Proof. Assume U is T-unitary. Since U is T-isometric, by (2.35) $U^{hT} * U = I^*$. On the other hand, $(U * x,y) = (x, U^{hT} * y) = (U * x,(U * U^{hT}) * y)$. Since UT is also surjective, we have $U * U^{hT} = I^*$. Conversely, if U satisfies (2.37), U is clearly T-unitary. It remains to show that if U_1, U_2 are T-unitary then $U_1 * U_2$ is T-unitary. From (2.26), it follows that $(U_1 * U_2)^{hT} * (U_1 * U_2) = U_2^{hT} * U_1^{hT} * U_1 * U_2 = I^*$ and likewise $(U_1 * U_2) * (U_1 * U_2)^{hT} = I^*$. #

The group of all T-unitary linear operators on H is called the T-*unitary group*.

An antilinear operator U satisfying (2.36) is called T-*antiunitary*. Again, notice that if $T = I$ then the definition of T-unitary agrees with the usual definition of unitary operators U satisfying

$$U^h U = UU^h = I . \tag{2.38}$$

In fact, the T-unitary group is isomorphic to the unitary group.

Theorem 2.8. Let $U_T(H)$ and $U(H)$ be the T-unitary and the unitary group of H , respectively. Then the mapping f given by

$$f(U) = UT^{-1}, \; U \in U(H) \tag{2.39}$$

is a group isomorphism of $U(H)$ to $U_T(H)$.

Proof. Let U be unitary. Then $(UT^{-1})^{hT} * (UT^{-1}) = T^h(UT^{-1})^h T^{-1}TUT^{-1} = T^h(T^{-1})^h U^h UT^{-1} = T^{-1} = I^*$ and likewise $(UT^{-1}) * (UT^{-1})^{hT} = I^*$. Thus UT^{-1} is T-unitary. Assume U is T-unitary. Then $(UT)^h UT = T^h U^h UT = (T^h U^h T^{-1}) * (UT) = U^{hT} * (U * I) = (U^{hT} * U) * I = I^* * I = I$ and similarly $(UT)(UT)^h = I$. Hence UT is unitary. Clearly, $f(U_1U_2) = f(U_1) * f(U_2)$ and f is an isomorphism of $U(H)$ to $U_T(H)$. #

Theorem 2.9. Suppose that W is a linear operator on H and satisfies

$$|(x,y)| = |(W * x, W * y)| \text{ for all } x, y \in H . \tag{2.40}$$

Then we have

$$W * x = e^{i\varphi(x)} U * x, \; x \in H \tag{2.41}$$

for some T-unitary or T-antiunitary operator U where $\phi(x)$ is a real valued function of x .

Proof. The proof is immediate from Theorem 2.8. and the known result that if $|(x,y)| = |(Ax,Ay)|$ for a linear operator A and $x, y \in H$ then $Ax = e^{i\phi(x)} U_1 x$ for some unitary or antiunitary operator U_1 . Since WT is a linear operator and $U = U_1 T^{-1}$ is T-unitary or T-antiunitary by Theorem 2.8, we can express as $W * x = (WT)x = e^{i\phi(x)} U_1 x = e^{i\phi(x)} (U_1 T^{-1}) * x = e^{i\phi(x)} U * x$. #

When $T = I$, Theorem 2.9. is well known for the study of symmetry properties of physical systems. Theorem 2.9 gives the interpretation that if (2.40) or (2.41) holds then the phases of all vectors can be adjusted in such a way that the T-isotopic mapping $x \to W * x$ by W is effected by either a T-unitary or T-antiunitary operator.

Theorem 2.10. If U is T-unitary and A is T-hermitian then $U * A * U^{-1^*}$ is T-hermitian.

Proof. Note $U^{-1^*} = U^{hT}$ since U is T-unitary. Thus by Theorem 2.3. $(U * A * U^{-1^*})^{hT} = (U^{-1^*})^{hT} * A^{hT} * U^{hT} = U * A * U^{-1^*}$ and hence $U * A * U^{-1^*}$ is T-hermitian. #

2.4. Isotope of projection operators. We discuss the isotopic generalization of a special type of linear operators, called projection operators.

Definition 2.15. A linear operator P on H is called a T-*idempotent operator* if

$$P * P = P \quad \text{or} \quad PTP = P . \quad \# \tag{2.42}$$

Consider now a subspace H_0 of H. Letting

$$H_0^{\perp} = \{x \in H \mid (x,y) = 0,\ y \in H_0\} ,$$

we have a linear space direct sum

$$H = H_0 \oplus H_0^{\perp}. \tag{2.43}$$

and any vector $x \in H$ is uniquely expressed as $x = y + z$, $y \in H_0$, $z \in H_0^{\perp}$. [Note that if H is of infinite dimension then (2.43) is still valid for a closed subspace H_0 of H.]

Definition 2.16. Given a subspace H_0 of H. Let any vector $x \in H$ be expressed as $x = y + z$ for $y \in H_0$, $z \in H_0^{\perp}$. Then the mapping $P_{H_0} : H \to H$ defined by

$$P_{H_0} * x = (P_{H_0} T)x = y \tag{2.44}$$

is linear and is called the T-*projection onto* H_0. A mapping $P : H \to H$ is called a T-*projection* if P is the T-projection onto some subspace H_0 of H; i.e., $P = P_{H_0}$. #

Theorem 2.11. A linear operator P on H is a T-projection if and only if P is T-hermitian and T-idempotent, that is, $P^{hT} = P$ and $P * P = P$.

Proof. Assume P is the T-projection onto a subspace H_0 of H. Let $x_i = y_i + z_i$, $y_i \in H_0$, $z_i \in H_0^{\perp}$, $i = 1,2$. Then $(x_1, P * x_2) = (y_1 + z_1, y_2)$ $= (y_1, y_2) = (y_1, y_2 + z_2) = (P * x_1, x_2)$ and by (2.32) $P^{hT} = P$. Clearly, $P * P = P$. Suppose that a linear operator P is T-hermitian and T-idempotent. Let $H_0 = \{y \in H \mid P * y = y\}$. Then H_0 is a subspace of H. For each vector $x \in H$ and $y \in H_0$, $x - P * x \in H_0^{\perp}$ since $(x - P * x, y)$ $= (x - P * x, P * y) = (P * (x - P * x), y) = (P * x - P * P * x, y) = 0$. But since $x = P * x + (x - P * x)$ and $P * x \in H_0$ for $x \in H$, $P = P_{H_0}$, the T-projection onto H_0. #

Note that if $T = I$, a T-projection is the usual projection. In fact, we have

Corollary 2.12. A linear operator U is a projection if and only if UT^{-1} is a T-projection.

Proof. If P is a projection then $(PT^{-1}) * (PT^{-1}) = PPT^{-1} = PT^{-1}$ and by (2.22) $(PT^{-1})^{hT} = T^h(PT^{-1})^hT^{-1} = T^h(T^{-1})^hPT^{-1} = PT^{-1}$, so PT^{-1} is T-hermitian. Conversely, assume PT^{-1} is a T-projection. Let $P_1 = PT^{-1}$, so $P = P_1T$. Then $PP = P_1TP_1T = (P_1 * P_1)T = P_1T = P$ and $P^h = (P_1T)^h = T^hP_1^h = (T^hP_1^hT^{-1})T = P_1^{hT}T = P_1T = P$, since P_1 is T-idempotent and T-hermitian. By Theorem 2.11, this completes the proof. #

Definition 2.16. T-projections $P_1, \cdots, P_n$ are said to be *mutually T-orthogonal* if

$$P_i * P_j = 0 \quad \text{for} \quad i \neq j = 1,2, \cdots, n . \quad \# \tag{2.45}$$

Since the mapping $f : A \to AT^{-1}$ is an algebra isomorphism of the operator algebra $\mathcal{A} = L(H)$ onto $\mathcal{A}^{(T)}$, by Corollary 2.12 and the known result for projections, the following is immediate.

Theorem 2.13. T-projections $P_1, \cdots, P_n$ are mutually T-orthogonal if and only if the sum $P = \oplus_{i=1}^{n} P_i$ is a T-projection. In this case, we have $P = P_{H_0}$ and $P_i = P_{H_{0i}} (i = 1, \cdots, n)$ for some subspaces H_0, H_{0i} of H, and $H_0 = H_{01} \oplus \cdots \oplus H_{0n}$ as a linear space direct sum where $H_{01}, \cdots, H_{0n}$ are mutually orthogonal, i.e., $(H_{0i}, H_{0j}) = 0$ for $i \neq j$. In particular, $H_0 = H$ if and only if

$$\sum_{i=1}^{n} P_i = I^* = T^{-1} . \quad \#$$

2.5. Isotope of exponential operators. If $\mathcal{A}$ is a finite-dimensional real or complex algebra with product AB and with identity element 1 then it is a well known fact that, for each $A \in \mathcal{A}$, the exponential $e^A = \exp A$ defined by

$$e^A = \exp A = 1 + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \cdots \tag{2.46}$$

exists and is a uniformly continuous function defined on $\mathcal{A}$. Besides the central role of the exponential function in the analytic and continuous group theory, it also plays an important role for the structure of some classes of nonassociative algebras (see Remark 2.3 below). If $\mathcal{A}$ is infinite dimensional, $\exp A$ does not in general exist. But the convergence of $\exp A$ can be guaranteed for some element $A \in \mathcal{A}$. For example, if A is a bounded operator in a Hilbert space then $\exp A$ is defined in $\mathcal{A} = L(H)$. For the sake of simplicity, we assume that H is a finite-dimensional Hilbert space. Let $\mathcal{A} = L(H)$.

Our aim is to develop the isotopic generalization of the following two important problems.

(1) If a linear operator $A \in \mathcal{A}$ is also a function of a real parameter t, then, for a fixed $H \in \mathcal{A}$, the solution of the equation

$$\frac{dA}{dt} = i[H,A] = i(HA-AH) \tag{2.47}$$

is expressed by

$$A(t) = A = e^{itH}A(0)e^{-itH} . \tag{2.48}$$

The equation (2.47) is recognized as the Heisenberg equation.

(2) If H is a hermitian operator then

$$U(t) = e^{itH} \tag{2.49}$$

is a unitary operator for all real numbers t.

We consider these problems in the T-isotope $\mathcal{A}^{(T)}$. We first generalize Heisenberg's equations (2.47) in the T-isotopic form according to Santilli (see Ref. 6):

$$\frac{dA}{dt} = \frac{1}{i}[A,B]^* = \frac{1}{i}(A * B - B * A) = \frac{1}{i}(ATB - BTA) \tag{2.50}$$

Let us begin by noticing $[A,B]^* = 0$ is not equivalent to the equation $[A,B] = 0$, unless T commutes in $\mathcal{A}$ with A and B. Denote by A^{*n} the n th power of A in $\mathcal{A}^{(T)}$. It is readily seen that

$$A^{*n} = (AT)^{n-1}A = A(TA)^{n-1}, \quad n = 1, 2, \cdots \tag{2.51}$$

where A^n indicates the n th power of A in $\mathcal{A}$.

Definition 2.17. We define the T-*exponential* $e_T^A = \exp_T A$ of A in the T-isotope $\mathcal{A}^{(T)}$ by

$$e_T^A = \exp_T A = I^* + A + \frac{1}{2!}A^{*2} + \cdots + \frac{1}{n!}A^{*n} + \cdots . \quad \# \tag{2.52}$$

Since the mappings $f, g \in \mathcal{A} \to \mathcal{A}^{(T)}$ defined by

$$f(A) = AT^{-1}, \quad g(A) = T^{-1}A, \quad A \in \mathcal{A} \tag{2.53}$$

are algebra isomorphisms, they map the exponential e^A in $\mathcal{A}$ to the exponential $e_T^{f(A)}$ and $e_T^{g(A)}$ of $f(A)$ and $g(A)$ in $\mathcal{A}^{(T)}$. Thus

$$f(e^{AT}) = e_T^{f(AT)} = e_T^A = e^{ATT^{-1}},$$

$$g(e^{TA}) = T^{-1}e^{TA} = e_T^{g(TA)} = e_T^A .$$

Hence we have the following result (also see Myung[2]).

Theorem 2.14. For every linear operator $A \in \mathcal{A}$, we have

$$e_T^A = T^{-1}e^{TA} = e^{AT}T^{-1} . \tag{2.54}$$

If $[A,B]^* = A * B - B * A = 0$ then

$$e_T^A * e_T^B = e_T^{A+B}, \tag{2.55}$$

$$e_T^{-A} = (e_T^A)^{-1^*} . \quad \# \tag{2.56}$$

By direct computation, one sees that the solution of the generalized equation (2.50) is given by

$$A = e^{itHT}A(0)e^{-itTH} \tag{2.57}$$

which can also be written as

$$A = e_T^{itH} * A(0) * e_T^{-itH} . \tag{2.58}$$

It is interesting to compare (2.57) and (2.58) with the conventional form (2.48). The solution (2.58) has an infinite series expansion by the formula

$$e_T^A * B * e_T^{-A} = B + [A,B]^* + \frac{1}{2!}[A,[A,B]^*]^* + \frac{1}{3!}[A,[A,[A,B]^*]^*]^* + \cdots . \tag{2.59}$$

Noting the fact that the T-isotopic product $A * B$ is associative, (2.59) can be proved by the same argument as in Ref. 8 (see p. 159).

For the isotopic generalization of Problem (2) above, we need

Theorem 2.15. Let H be a Hilbert space (of arbitrary dimension). The mappings f, g defined by (2.53) are bijections from the set of hermitian or skew-hermitian (symmetric or skew-symmetric) linear operators in $\mathcal{A}$ to the set of T-hermitian or T-skew-hermitian (T-symmetric or T-skew-symmetric) linear operators.

Proof. Let A be a hermitian linear operator in $\mathcal{A}$. Then $f(A)^{hT} = (AT^{-1})^{hT} = T^h(AT^{-1})^h T^{-1} = AT^{-1}$, so $f(A)$ is T-hermitian. If B is T-hermitian then, letting $A = BT$, we have $A^h = (BT)^h = (T^hB^hT^{-1})T = B^{hT}T = BT = A$ and hence A is hermitian. The same argument applies to g and to the symmetric or skew-hermitian operators. #

Since it is known that every hermitian linear operator in any Hilbert space is bounded and T is bounded by the assumption, in view of Theorem 2.15, we have

Corollary 2.16. Any T-hermitian linear operator is bounded. #

Remark 2.2. The hermitian and T-hermitian operators are not closed under the product AB and the T-isotope product $A * B$. Define the anticommutator $\{A,B\}$ and $\{A,B\}^*$ of A,B in $\mathcal{A}$ and in $\mathcal{A}^{(T)}$ by

$$\{A,B\} = AB + BA,\ \{A,B\}^* = A * B + B * A .$$

If A and B are hermitian or T-hermitian then so is $\{A,B\}$ or $\{A,B\}^*$. This follows from Theorem 2.3. By a similar reason, if A and B are skew-hermitian or T-skew-hermitian then so is $[A,B]$ or $[A,B]^*$. #

In terms of T-isotope, we can state Problem (2) above as follows.

Theorem 2.17. Let H be an arbitrary Hilbert space. If B is a T-hermitian linear operator on H then

$$U_T(w) = e_T^{iwB} \tag{2.60}$$

are defined for all real numbers w and are T-unitary linear operators satisfying

$$U_T(0) = T^{-1} , \tag{2.61}$$

$$U_T(w_1 + w_2) = U_T(w_1) * U_T(w_2), \tag{2.62}$$

$$\text{The infinitesimal} = \frac{d}{dw}U_T(w)\Big|_{w=0} = iB . \tag{2.63}$$

Proof. Note that if A is hermitian, e^{iwA} is unitary. We use this and Theorems 2.8 and 2.15. Let B be T-hermitian. By Theorem 2.15, BT is hermitian and hence e^{iwBT} is unitary. Thus, by Theorem 2.8,

$$f(e^{iwBT}) = e_T^{iwf(BT)} = e_T^{iwB}$$

is T-unitary. The remaining relations follow from Theorem 2.14. #

Remark 2.3. The exponential e_T^A has a close relation with the exponential defined on the class of Lie-admissible algebras of the (R,S)-mutation type mentioned in Remark 2.1. Let $\mathcal{A}$ be an associative algebra with product AB and with identity element I. Let $\mathcal{A}(R,S)$ be the

(R,S)-mutation of $\mathcal{U}$ with product $A \circ B = ARB - BSA$ as in Remark 2.1. In general, the n th power of each element A in $\mathcal{U}^{(T)}$ is not definable. Given $A \in \mathcal{U}$, define $A^{\circ n}$ inductively as $A^{\circ 1} = A$, $A^{\circ n} = A^{\circ(n-1)} \circ A$. Then $\mathcal{U}(R,S)$ is called *power-associative* if $A^{\circ n} \circ A^{\circ m} = A^{\circ(m+n)}$ for all positive integers m,n and all $A \in \mathcal{U}$. If there exists an element 1° in $\mathcal{U}(R,S)$ such that $1^\circ \circ A = A \circ 1^\circ = A$ for all $A \in \mathcal{U}$ then 1° is called an *identity element* of $\mathcal{U}(R,S)$. In this case, we can define the exponential $e_\circ^A$ of A in $\mathcal{U}(R,S)$ by

$$e_\circ^A = 1^\circ + A + \frac{1}{2!} A^{\circ 2} + \cdots + \frac{1}{n!} A^{\circ n} + \cdots .$$

The important fact is that if $\mathcal{U}(R,S)$ is power-associative with identity element 1° then $1^\circ = (R - S)^{-1}$ and

$$e_\circ^A = e_{R-S}^A$$

for all $A \in \mathcal{U}$ where e_{R-S}^A is the exponential of A in the $(R - S)$-isotope $\mathcal{U}^{(R-S)}$ defined by (2.52). In other words, the exponential in some (nonassociative) Lie-admissible algebra can be realized as that in an isotope (so associative) of an associative algebra. This problem has been discussed in detail by Myung[2] and Myung and Santilli[4]. #

An important special case of isometric operators is the *Cayley transform* $c(A)$ of a linear operator A:

$$c(A) = (A - iI)(A + iI)^{-1}, \tag{2.64}$$

which plays a key role in the hermitian operator theory. The T-isotope generalization of (2.64) can be stated as

Definition 2.18. Given a linear operator $A \in \mathcal{U}$. If the linear operator $c_T(A)$ given by

$$c_T(A) = (A - iI^*) * (A + iI^*)^{-1^*} \tag{2.65}$$

is definable then $c_T(A)$ is called the T-*Cayley transform* of A. #

Since any unitary operator U on H is expressed by (2.64) for some hermitian operator A on H, in view of Theorems 2.8 and 2.15, the following is immediate.

Theorem 2.18. Any T-unitary operator on H is given by the T-Cayley transform $c_T(B)$ for some T-hermitian operator B. #

2.6. Isotope of eigenvalues, determinants, and traces.

Until now we have essentially considered the isotopic generalization of some elementary definitions and results in the linear operator theory of Hilbert spaces. On the other hand, some notion may not be preserved under the isotope in its original form. Consider, again, a finite-dimensional Hilbert space. Then, for a linear operator A on H, we define $\mathrm{Tr}\, A$ as the usual trace of A. As is well known, $\mathrm{Tr}[A,B] = 0$ in $\mathcal{U}$. But $\mathrm{Tr}[A,B]^* = \mathrm{Tr}(ATB - BTA) \neq 0$ in $\mathcal{U}^{(T)}$, in general. However, if T commutes in $\mathcal{U}$ with A and B, then $\mathrm{Tr}[A,B]^* = 0$.

For another example, given a linear operator A on H, consider the equation

$$A * x = \lambda * x = (\lambda T)x \tag{2.66}$$

for some scalar (complex number) λ and a nonzero vector $x \in H$. If one attempts to make the isotopic generalization of eigenvalues and

eigenvectors by means of (2.66), one encounters an inconsistency, namely, H *does not become a linear space over* C *under the scalar multiplication* $\lambda * x$ *given by* (2.66), since $(\lambda\mu) * x = (\lambda\mu T)\, x \neq \lambda * (\mu * x) = (\lambda\mu T^2)x$, in general.

This situation suggests to introduce *an isotopic generalization of the notion of scalars*. For this, one can proceed as follows.

Definition 2.19. We define $C^{(T)}$ as the set of all linear operators in $\mathcal{A}$ given by

$$C^{(T)} = \{\lambda I^* = \lambda T^{-1} | \lambda \in C\} . \tag{2.67}$$

We denote $\lambda_T = \lambda T^{-1}$ and call each element in $C^{(T)}$ a T-*scalar*. $C^{(T)}$ may be called the T-*isotope* of C . We also call λ_T T-*complex* or T-*real* if λ is complex or real. #

Define the addition $\lambda_T + \mu_T$ in $C^{(T)}$ as the same as in $\mathcal{A}$ or in $\mathcal{A}^{(T)}$ but the multiplication $\lambda_T * \mu_T$ in $C^{(T)}$ by

$$\lambda_T * \mu_T = \lambda_T T \mu_T = (\lambda\mu)T^{-1} . \tag{2.68}$$

Then it can be easily seen that $C^{(T)}$ is a field isomorphic to C . In fact, the mapping $\lambda \to \lambda_T = \lambda T^{-1}$ is an isomorphism of C onto $C^{(T)}$ and the inverse $\lambda_T^{-1^*}$ of λ_T and the identity element 1_T in $C^{(T)}$ is given by

$$\lambda_T^{-1^*} = \lambda_T^{-1} = \lambda^{-1} T^{-1}, \quad 1_T = I^* = T^{-1} . \tag{2.69}$$

We can now convert H into a linear space over the field $C^{(T)}$ of T-scalars by defining the T-scalar multiplication $\lambda_T * x$:

$$\lambda_T * x = (\lambda_T T)x = \lambda x, \; x \in H , \tag{2.70}$$

namely, by defining $\lambda_T * x$ as the same as the scalar multiplication λx in the linear space H over C . It can be directly checked that this definition makes H a linear space over $C^{(T)}$, for example, $(\lambda_T * \mu_T) * x = (\lambda\mu)x = \lambda_T * (\mu_T * x)$. It is important to note that each element λ_T of $C^{(T)}$ is not in the center of $\mathcal{A} = L(H)$ but is in the center of the T-isotope $\mathcal{A}^{(T)}$; for, if $A \in \mathcal{A}$ then $A * \lambda_T = AT(\lambda T^{-1}) = \lambda A = \lambda_T * A$. Thus a T-scalar is not a scalar linear operator on H but is on the T-isotope $H^{(T)}$ (see Definition 2.9). This leads to

Definition 2.20. Given a linear operator A , a T-scalar λ_T is called a T-*eigenvalue* of A if there is a nonzero vector $x \in H$ such that

$$A * x = \lambda_T * x . \tag{2.71}$$

Then x is called a T-*eigenvector* of A corresponding to λ_T . #

From the definition, it is clear that a T-scalar λ_T is a T-eigenvalue of A with T-eigenvector x if and only if λ is an eigenvalue of AT with eigenvector x . Thus the T-eigenvalues λ_T of A are obtained by the eigenvalues λ of AT , namely, by the solutions of the equation

$$\det (AT - \lambda I) = 0 . \tag{2.72}$$

We rewrite (2.72) as $0 = \det (AT - \lambda I) = \det [(A - \lambda T^{-1})T]$ $= [\det (A - \lambda T^{-1})] \det T$. Since $\det T \neq 0$ (T nonsingular), we have $\det (A - \lambda T^{-1}) = 0$. Therefore, we have

Theorem 2.19. Let A be a linear operator on H . Then a T-scalar λ_T is a T-eigenvalue of A if and only if λ is an eigenvalue of AT . The T-eigenvalues of A are the solutions of the equation, called the T-*characteristic polynomial* of A ,

$$\det (A - \lambda_T * I^*) = 0 . \quad \# \tag{2.73}$$

The following is the T-isotope generalization of the known result for hermitian operators.

Corollary 2.20. Let B be a T-hermitian operator on H . Then the T-eigenvalues of B are all T-real.

Proof. If B is T-hermitian then, by Theorem 2.15, BT is hermitian. Hence, by the known result, the eigenvalues of BT are all real and by Theorem 2.19 the T-eigenvalues of B are all T-real. #

The discussion above and Theorem 2.19 also suggest the following T-isotope version of trace and determinant.

Definition 2.21. Let $\dim H = n$. Given a linear operator A on H , let $\lambda_T^1, \cdots, \lambda_T^n$ be the T-eigenvalues of A . The T-*trace*, $\mathrm{Tr}_T A$, of A is defined to be the T-scalar:

$$\mathrm{Tr}_T A = \lambda_T^1 + \cdots + \lambda_T^n = (\lambda^1 + \cdots + \lambda^n)T^{-1} .$$

Define the T-*determinant*, $\det_T A$, of A as the T-scalar given by

$$\det_T A = (\det AT)T^{-1} . \quad \#$$

The proof of the following result is straightforward.

Theorem 2.20. Let H be a finite-dimensional Hilbert space. For linear operators A, B, P, we have

(1) $\mathrm{Tr}_T A = (\mathrm{Tr}\ A)T^{-1}$,

(2) $\mathrm{Tr}_T (P * A * P^{-1^*}) = \mathrm{Tr}_T A$,

(3) $\det_T (A * B) = (\det_T A) * (\det_T B)$,

(4) $\det_T A^{-1^*} = (\det_T A)^{-1^*}$,

(5) $\mathrm{Tr}_T (A * B) = \mathrm{Tr}_T (B * A)$,

(6) $\det_T (\exp_T A) = \exp_T (\mathrm{Tr}_T A)$. #

Note that if $T = I$ then the T-trace and T-determinant coincide with the usual ones (multiplied by T^{-1}).

2.7. Isotopes of Hilbert spaces and C*-algebras

Our next step is to generalize the inner product $(\ ,\ )$ of H in such a way as to permit the study of the compatibility with the generalized envelope $\mathcal{O}\mathcal{L}^{(T)}$ and field $C^{(T)}$.

Definition 2.22. Let (x,y) be the inner product of a Hilbert space H and let T be an operator on H which is nonsingular and satisfies all needed topological conditions (regarding domain, range, etc.). Then the T-isotopic extension of (x,y) is given by

$$(x,y)^* = (xT^h,y)I^* = I^*(x,Ty) \tag{2.74}$$

where T^h is the usual Hermitean conjugate of T.

In particular, if T, besides being invertible, is positive on H (that is, $(x,Tx) > 0$ for all $x \neq 0$ in H), then the following result can be easily proved.

Theorem 2.21. Let T be an invetible and positive operator as per Definition 2.22 under which the product $(x,y)^*$ is an inner product mapping $H \times H \to C^{(T)}$, given by (2.74). Then it satifies

$$\begin{aligned} &(x,x)^* > 0 \quad \text{for all } x \neq 0 \text{ on } H \\ &(x,y)^* = \overline{(y,x)^*} \\ &(x,y+x)^* = (x,y)^* + (x,z)^* \\ &(x,\lambda y)^* = \lambda (x,y)^* = \lambda_T^* (x,y)^*, \quad \lambda \in C \end{aligned} \tag{2.75}$$

where we have used the notion of T-scalar in the last expression.

The realization of the product $(x,y)^*$ we shall use in the next section can be expressed in terms of L^2-functions x and y and a (suitably selected) measure dm (we ignore the multiplication by I* here)

$$(x,Ty) = \langle |*| \rangle = \int \bar{x} * y \; dm = \int \bar{x} \, T \, y \, dm \tag{2.76}$$

and can be interpreted as the "expectation value" of the isotopy operator T.

We should insist, however, on the fact that, since the ordinary product has no algebraic meaning in $\mathcal{A}^{(T)}$, the inner product is now given by (2.74) and not by (x,y). This implies a number of predictable restrictions on the admissible operators T which we hope to study in a separate paper.

We are finally equipped to identify the generalized form of Hilbert spaces that will be used in the physical applications.

Definition 2.23. Let H be a Hilbert space over the field C of complex numbers, with enveloping associative algebra of operators $\mathcal{A}$ and inner product (x,y). We shall denote with $H^{(T)}$ the linear vector space over the

field $C^{(T)}$ of T-scalars with isotopic enveloping associative algebra of operators $\mathcal{A}^{(T)}$ and isotopic inner product $(x,y)^*$.

Remark 2.4. Since all the mappings

$$\begin{aligned} C &\longrightarrow C^{(T)} \\ \mathcal{A} &\longrightarrow \mathcal{A}^{(T)} \\ (x,y) &\longrightarrow (x,y)^* \end{aligned} \tag{2.77}$$

are isomorphisms we shall continue to use the name "Hilbert space" for the space $H^{(T)}$ However, we would like to stress that a considerable amount of additional studies (not conducted in this paper) are needed for the full, technical qualific ation of the space $H^{(T)}$ as a Hilbert space.

Note the appearance of the generalized unit I^* in the definition of inner product (2.74). It is essential to ensure that the values $(x,y)^*$ belong to $C^{(T)}$ rather than C. In turn, this is essential for $H^{(T)}$ to have the pre-requisites of a Hilbert space. Finally, note that $(x,y)^* = (x,y)$ whenever $T \simeq I$.

Thus, $H^{(T)}$ is a meaningful generalization of H only for non-trivial (i.e., operator) isotopies.

We pass now to a few considerations for the extension of the T-trace to arbitrary dimensions. Let λ_T be a T-eigenvalue of A defined by (2.71). Thus, $(A - \lambda_T)*x = 0$ for some x in $H^{(T)}$. But $(A - \lambda_T)*x = (AT - \lambda)x = 0$. Thus, $(A - \lambda_T)$ is not invertible in $\mathcal{A}^{(T)}$ if and only if $(A - \lambda)$ is not invertible in $\mathcal{A}$. In light of theorem 2.19, this leads to the following definition of spectrum in $H^{(T)}$.

Definition 2.24. Let A be a linear operator on $H^{(T)}$. The T-spectrum, $Sp_T A$, is defined as the set of T-scalars λ_T such that $A - \lambda_T$ is not invertible in $\mathcal{A}^{(T)}$.

The following result is immediate from this and Theorem 2.19.

Theorem 2.22. For any linear operator in $H^{(T)}$,

$$Sp_T A = (SpAT)I^* \tag{2.78}$$

where Sp is the conventional spectrum of operators on H.

The following additional results can also be readily verified.

Theorem 2.23. Let A be a linear operator on $H^{(T)}$.

(1) If A is T-Hermitean, then each element of $Sp_T A$ is T-real; and

(2) If A is T-unitary, then, for each $\lambda_T \in Sp_T A$,

$$\| \lambda_T \|_T = \| \lambda \| I^* = I^*$$

Theorem 2.24. Let A be T-Hermitean in $\mathcal{OL}^{(T)}$. Then, the followings are equivalent.

(a) $Sp_T A > 0$ (i.e., $\lambda_T > 0$ for all $\lambda_T \in Sp_T A$)

(b) A is of the form $B*B^{hT}$ for some $B \in \mathcal{OL}^{(T)}$.

(c) A is of the form $B^{hT}*B^{hT}$ for some T-Hermitean operator B.

It can be seen that each of the conditions in Theorem 2.24 is equivalent to the condition

$$(x, A*x) > 0 \quad \text{for all} \quad x \in H^{(T)} \tag{2.79}$$

when, when T is positive, can be equivalently written

$$(x, A*y)^* = (x, ATy)I^* > 0 \tag{2.80}$$

This permits to call a T-hermitean operator A on $H^{(T)}$ T-positive when (2.79) is verified. In particular, a T-hermitean operator A is T-positive if and only if AT is positive in the conventional sense.

A linear operator A on $H^{(T)}$ is said to be of the T-trace class if, for any orthonormal basis $\{e_\alpha\}$ of $H^{(T)}$ the series $S = \sum_\alpha (e_\alpha, A*e_\alpha) I^*$ converges to a unique T-scalar (that is, in a way independent of the choice of the basis). In this case, the value of the series S is defined as the T-trace of A in $H^{(T)}$,

$$Tr_T A = \sum_\alpha (e_\alpha, A*e_\alpha) I^* \tag{2.81}$$

If $H^{(T)}$ is finite-dimensional, the definition above coincides with that given before in Definition 2.21. Note that not every linear operator on $H^{(T)}$ possesses a T-trace. However, for a T-positive operator (essentially those needed for Equations (2.75)), the existence of a T-trace can be proved under certain conditions that will be presented elsewere.

Let H be a (conventional) Hilbert space of arbitrary dimensions, and $\mathcal{OL}$ the (conventional) associative algebra of linear operators on H. We regard $\mathcal{OL}$ as a complex C*-algebra under the usual norm $\|A\|$, $A \in \mathcal{OL}$. Consider now the following generalization of the norm

$$\| A \|_T = \| AT \| I^* \tag{2.82}$$

Assume T to be a (conventionally) positive operator for simplicity. Then the ordering of T-scalars follows as for the ordinary case.

To show that norm (2.82) also defined a form of C*-algebras, it is sufficient to verify the following properties.

$$\| A*B \|_T \leq \| A \|_T * \| B \|_T \tag{2.83}$$

$$\| A^{hT} \|_T = \| A \|_T \tag{2.84}$$

$$\| A \|_T^{2*} \; (= \| A \|_T * \| A \|_T \;) = \| A^{hT} * A \|_T \tag{2.85}$$

This is left here as an (instructive) exercise to the interested reader.

We then call (2.82) the T-norm of A. The resulting algebra is here called $C^*_{(T)}$- algebra. The conventional C*-algebra is then admitted as the particular case when I* = I. Additional studies are in progress and will be reported elsewhere.

III. APPLICATIONS TO THE LIE-ISOTOPIC BRANCH OF HADRONIC MECHANICS

The applications of the theory of the preceding section are virtually endless. In fact, as we shall indicate in this section, the theory permits the isotopic generalization of each and every aspect of Atomic Mechanics (the ordinary quantum mechanics) into a non-Hamiltonian form which appears to be valuable for strong non-Hamiltonian interactions as well as other physical phenomena where contact/nonpotential interactions are possible (e.g., statistical ensembles of extended molecules with inelastic collisions; electronic systems; neural systems; etc.).

In Paper I we have attempted the presentation of the Lie-isotopic generalization of Heisenberg's mechanics via only one postulate. The theory of the preceding section permits a more refined study of the results, as well as, and perhaps more importantly, the formulation of postulates encompassing other representations, such as those of Schrödinger-type.

In this section we shall therefore attempt the identification of the physical foundations of the Lie-isotopic branch of the Hadronic Mechanics, in the hope that our work results to be valuable for applications to specific systems (e.g., the deuteron, the neutron, the π° , the nonpotential scattering theory, etc.).

A first understanding is that this section is, by far, incomplete and a considerable amount of additional work is needed.

A second understanding is that the contents of this section should be considered as of tentative nature until the theory is completed, tested in a variety of cases, and confirmed by experiments.

A third understanding is related to the primary objective of the Lie-isotopic branch of the Hadronic Mechanics, the *exterior treatment* of composite systems with non-Hamiltonian internal forces. Classically, the compatibility of total conservation laws with non-Hamiltonian forces is achieved via subsidiary constraints (Paper I). At the hadronic level, the difficulties are then two-fold. We must first reach a non-Hamiltonian generalization of Atomic Mechanics, and then we must implement it in such a way to permit total conservation laws. This section is primarily devoted to the first task, and only incidental comments will be presented for the second objective. Thus, the reader should keep in mind that the systems treated in this section are, in general, of *open non-Hamiltonian* character.

A sound knowledge of related theoretical studies is recommendable, with particular references to the research by Eder[10], Mignani[11], Okubo[12], Kapuscik[13] and others.

A sound knowledge of the Birkhoffian generalization of the classical Hamiltonian mechanics[14] appears to be also recommendable for an in-depth understanding of this section as well as for possible further developments.

In regard to numerical applications of the theory of this section, we refer specifically to the studies by Eder[10] (e.g., to represent conventional values of the spin while the magnetic moment is anomalous).In regard to the problem of structure, we refer specifically to the studies by Santilli[15] for which the Hadronic Mechanics was suggested in the first place. The reader should however be aware that the model of structure of hadrons of ref.[15] (as closed variationally nonselfadjoint systems) is considerably different than those of current trends. For instance, by recalling that the number of constituents increases in the transition from the hydrogen atom to the helium, and from the deuteron to the tritium, a fundamental condition of the model of ref.[15] is that the number of constituents increases from the π^o to the $\pi^\pm$ and to heavier hadrons [for a theoretical description of what is meant by "constituent" see Paper III].

3.1. Existing applications. A number of applications of the theory of the preceding section have already been identified, as indicated in Section III of Paper I. We are referring here to the isotopic generalization of

(1) Enveloping associative algebra of Heisenberg's mechanics;

(2) Lie algebras;

(3) Heisenberg's equations;

(4) Canonical commutation rules;

(5) Cononical quantization;

(6) Heisenberg's uncertainty principle;

(7) Planck's constant;

(8) Galilei's relativity;

(9) Representation theory.

It is an instructive exercise for the interested reader to reformulate each and every topic (1) through (9) within the context of the theory of the preceding section.

3.2. Hadronic-isotopic observables. We begin our study with the identification of the mathematical characteristics of observable quantities under strong non-Hamiltonian interactions. The conventional Atomic Mechanics, since it is strictly Hamiltonian in character, is inapplicable. As a result, the old characterization of observability under electromagnetic interactions via Hermitian operators is inapplicable. Once the non-Hamiltonian character of the theory of the preceding section is seen, the following generalized postulate appears plausible.

POSTULATE I. *Measurable hadronic quantities of the exterior strong problems (such as the total energy H, the total linear momentum* $\vec{P}$*, the total angular mementum* $\vec{J}$*, etc.) are characterized by T-Hermitian operators (in the sense of Definition 2.12), i.e.,*

$$H^{hT} = T^h H^h T^{-1} = H, \quad \text{etc.} \tag{3.1}$$

The generalized observability submitted here is clearly a covering of the conventional atomic one, in the sense that the latter is admitted as a particular case of a broader mathematical and physical context.

Note that the conventional observability is recovered not only when the hadronic unit $I^* = T^{-1}$ is the conventional one I, but also when I^* is a real scalar ($\neq I$). In fact, in this latter case we have

$$H^{hT} = H^h T^h T^{-1} = H^h \tag{3.2}$$

under which T-Hermiticity and conventional Hermiticity are equivalent.

As it will be evident later on, the transition from conventional Hermiticity to T-Hermiticity implies that physical quantities are generally *nonconserved*. The "germs" of the new mechanics are therefore already contained in Postulate I. In fact, the Atomic Mechanics was conceived to achieve a representation of the *stability of the orbit* of the electrons in the atomic structure. An objective of the Hadronic Mechanics is instead that of achieving a representation of *orbits as unstable as dynamically possible*, because we have in mind the treatment of extended hadrons moving within a medium of other hadrons. When total conservation laws are regained, the internal dynamics is left as unrestricted as possible, in the hope of achieving "global stability" via irreversible, nonconservative, and non-Hamiltonian internal dynamics, along the idea of "closed variationally nonselfadjoint systems" (Paper I).

The transition from atomic to hadronic observability therefore implies the *loss of quantum levels*, that is, the loss of Planck's constant $\hbar$, and its replacement with the operator T^{-1}. As recalled in Paper I, this generalization is submitted in the hope of achieving a more adequate representation of the processes of emission and absorption of energy for particles under conditions of mutual penetration within other particles.

But there are other reasons to suggest a departure from the original idea of Atomic Mechanics. They are given by the *clear experimental evidence according to which, in the transition from the two-body problem under electromagnetic interactions to that under strong interactions, there is the disappearance of exited states*. In fact, while the hydrogen atom and the positronium admit an infinite variety of exited states, no exited state has been experimentally established until now for the deuteron.# The same situation may occur also for other composite particles supposed to be of two-body nature, such as the π°.[15]

This drastic change in physical behaviour is, perhaps, the most forceful experimental evidence suggesting a revision of the Atomic Mechanics into a form specifically conceived for the strong interactions (an application of Hadronic Mechanics for the suppression of the atomic spectrum will be reviewed later on).

3.3. Hadronic-isotopic states. The state of an atomic system is characterized by a vector $|>$ on a Hilbert space on which Hermitian observables A act according to the familiar eigenvalue equation

$$A|> = a|> \quad . \tag{3.3}$$

It takes a minimum of five nucleons to reach a nucleus with exited states vaguely reminiscent of the quantum levels of the Atomic Mechanics, and this is a reason why the efforts here are to abandon the notion of quantum in favor of more general notions (Paper I).

This equation is intrinsically Hamiltonian in character, and, thus, inapplicable to the system considered. To lift the restriction, we introduce the following second postulate where hereon "Hilbert spaces" are referred to Def. 2.23.

POSTULATE II. *A hadronic state of the exterior strong problem is characterized by a vector* $|>$ *of a Hilbert space on which the hadronic observables* A *act according to the T-isotopic eigenvalue equations (in the sense of Definition 2.20).*

$$A * |> = a_T * |> \quad , \tag{3.4a}$$

$$A * |> = AT|> \quad , \quad a_T * |> = a_T T|> = a|> \quad . \tag{3.4b}$$

As it is the case in the Atomic Mechanics, not all observables can be measured simultaneously. Consider a second observable B with eigenvalue equation

$$B * |> = b_T * |> \quad . \tag{3.5}$$

Then we can write

$$B * A * |> = B * a_T * |> = a_T * B * |> = a_T * b_T * |> \quad , \tag{3.6}$$

$$A * B * |> = A * b_T * |> = b_T * A * |> = b_T * a_T * |> \quad ,$$

i.e.

$$A * B * |> - B * A * |> \equiv 0 \quad , \tag{3.7}$$

which can hold if and only if

$$[A,B]^* = A * B - B * A \equiv 0 \quad . \tag{3.8}$$

Lemma 3.1. A necessary and sufficient condition for two or more hadronic observables to be measurable simultaneously is that they T-commute in the sense of Equation (3.8). #

The reader should keep in mind the remark after Equation (2.50) in regard to the lack of equivalence of the commutators $[A,B]^*$ and $[A,B]$. In fact, ordinary commutativity *is not* in general sufficient to permit the simultaneous hadronic observability. It then follows that quantities which are measurable simultaneously in Atomic Mechanics are not necessarily measurable simultaneously in the Hadronic Mechanics.

Needless to say, the ordinary commutativity is a particular case of the T-commutativity. In particular, if the isotopy operator T is a scalar, ordinary commutativity and T-commutativity are equivalent.

The study of the "maximal and complete set of T-commuting observables" of a hadronic system will not be conducted here. We merely assume that this set exists, and write

$$\begin{aligned} A_1 * |> &= a_1 * |> \ , \\ A_2 * |> &= a_2 * |> \ , \\ &\vdots \\ A_n * |> &= a_n * |> \ . \end{aligned} \tag{3.9}$$

The state vector can then be labeled via the collection, say, α, of all T-eigenvalues

$$|> = \sum_{\alpha} \Psi(\alpha) * |\alpha\rangle \qquad \alpha = (a_1, a_2, \cdots, a_n) \tag{3.10}$$

where

$$\Psi(\alpha) = <\alpha| * |> \tag{3.11}$$

The functions $\Psi(\alpha)$ will be called *hadronic state functions*. Note that, if one preserves the operator in the transition from the atomic to the hadronic setting, the state functions are generally different because of the isotopy in the eigenvalue equations.

Equation (3.10) identifies the existence of *hadronic representations* in much of the way as occurring in Atomic Mechanics. Our immediate objective is that of identifying at least some of these representations; identify the methods for passing from one to the other; and study some of their properties (e.g., normalization).

It is recommendable to identify the alteration (called *mutation* in Ref.[15], p. 690) of eigenvalues under isotopy. This mutation is at the basis of the physical applications of the Hadronic Mechanics, such as the anomalous magnetic moments in nuclear physics[10], the theoretical representation of nuclear irreversibility[11], the models of the structure of the pions and of the neutron[15], etc.

Suppose that the atomic state function is given by

$$\Psi = Ne^{-irs} \tag{3.12}$$

where r and s represent given physical quantities. Suppose that an operator A verifies the atomic eigenvalue law

$$A\Psi = i\frac{\partial}{\partial s}\Psi = r\Psi \ . \tag{3.13}$$

Suppose now that this law is implemented into the isotopic form with a scalar T quantity. Then we have

$$A * \Psi = i\frac{\partial}{\partial s} T\Psi = r'\Psi \quad . \tag{3.14}$$

Thus, the *mutated eigenvalue* r' is given by

$$r' = Tr \quad . \tag{3.15}$$

If $T = T(r,s)$, the mutated eigenvalue is given by the more general expression

$$r' = i\frac{\partial T}{\partial s} + Tr \tag{3.16}$$

where we have assumed that atomic and hadronic Hermiticity coincide. If this is not the case, a further, more general mutation occurs which can be computed via the rules of Section 2.6 and the explicit form of the T-quantity.

Note that, if the T-eigenvalues are used, then Eq. (3.14) can be written

$$A * \Psi = r'_T * \Psi \tag{3.17}$$

where

$$r'_T = r \ , \ r'_T = T^{-1}(i\frac{\partial T}{\partial s} + Tr) \ , \text{ etc.} \tag{3.18}$$

3.4. Hadronic-isotopic generalization of Schrödinger's representation. This representation was achieved in Paper I via the techniques of the Birkhoffian Mechanics[14]. Our objective here is that of identifying the underlying Hilbert space structure, that is, to achieve a formulation of the representation within the context of the Hilbert-isotopic theory of the preceding section.

It may be recommendable to review the most important steps. Consider an open nonselfadjoint system in first order form

$$(\dot{a}^\mu) - (\hat{\Xi}^\mu_{(t,a)}) = (\dot{a}^\mu) - (\Xi^\mu_{SA}) - (r^\mu_{NSA})$$

$$= \begin{pmatrix} \dot{\vec{r}}_k \\ \dot{\vec{p}}_k \end{pmatrix} - \begin{pmatrix} \vec{p}_k/m_k \\ \vec{F}^{SA}_k(\vec{r}) - \vec{F}^{NSA}_k(t,\vec{r},\vec{p}) \end{pmatrix}, \ a = (\vec{r},\vec{p}) \tag{3.19}$$

$$\mu = 1, 2, \cdots, 6N, \ k = 1, 2, \cdots, N$$

where the symbol "p" represents the physical linear momentum $(m\dot{\vec{r}})$. The physical angular momentum is then given by $\vec{J} = \vec{r} \times \vec{p}$. The energy H is that of the maximal selfadjoint subsystem (recall that the notion of potential energy has no physical basis for contact nonselfadjoint interactions).

By using the techniques of Ref.[14], the next step is that of constructing a representation of system (3.19) in terms of the semiautonomous Birkhoff's equations

$$\left\{ [\frac{\partial R_\nu(a)}{\partial a^\mu} - \frac{\partial R_\mu(a)}{\partial a^\nu}]\dot{a}^\nu - \frac{\partial B(t,a)}{\partial a^\mu} \right\}^{SA} = 0 \ , \tag{3.20}$$

$$\det (\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}) \neq 0$$

with underlying generalized variational principle of Pfaff's type

$$\delta A^g = \delta \int_{t_1}^{t_2} [R_\mu(a)da^\mu - B(t,a)dt] \tag{3.21}$$

$$\overset{\text{def.}}{=} \delta \int_{t_1}^{t_2} [P_k(\vec{r},\vec{p})dr^k + Q^k(\vec{r},\vec{p})dp_k - B(t,\vec{r},\vec{p})^{dt}] = 0 \ .$$

The representation is called of "semiautonomous" type to indicate that the geometric (symplectic) tensor of the theory

$$\Omega_{\mu\nu}(a) = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \tag{3.22}$$

does not depend explicitly on time in order to be consistent with the conventional geometric realizations in local coordinates of the most general possible (exact) symplectic two-forms

$$\Omega_2 = \frac{1}{2}\Omega_{\mu\nu}(a)da^\mu \wedge da^\nu \ . \tag{3.23}$$

The dependence of the vector field on time is then represented via the explicit time dependence of the Birkhoffian B .

Under topological restrictions inessential here, all systems (3.19) admit a representation in terms of Equations (3.20) in the local variables of the observer. This is the so-called *direct universality of Birkhoff's equations*.

Note that the Hamiltonian representations occur as a particular case of the Birkhoffians when

$$(R_\mu) \overset{\text{def.}}{=} (R^0_\mu) = (\vec{p},\vec{0}) \tag{3.24}$$

in which case the Birkhoffian and the Hamiltonian coincide.

By using the degrees of freedom characterized by the Birkhoffian guages

$$R_\mu \to R^+_\mu = R_\mu + \frac{\partial G}{\partial a^\mu} \ ; \ B \to B^+ = B - \frac{\partial G}{\partial t} \ . \tag{3.25}$$

Equations (3.20) can be turned into an equivalent form verifying the additional condition

$$\det \left(\frac{\partial R_\mu}{\partial a^\nu}\right) \neq 0 \tag{3.26}$$

under which the following Birkhoffian generalization of the Hamilton-Jacobi theory exists (Paper I).

$$\frac{\partial A^g}{\partial t} + \mathcal{B}(t,R(a)) = 0 \ , \tag{3.27a}$$

$$R_\mu(a) = \frac{\partial A^g}{\partial a^\mu} \ , \quad \mathcal{B}(t,R) = B(t,a(R)) \ . \tag{3.27b}$$

The Hamiltonian particularization is now given by the guaged case

$$R^{0+}_\mu = -\frac{1}{2}\omega_{\mu\nu}a^\nu \ , \quad (\omega_{\mu\nu}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ . \tag{3.28}$$

The use of the naive quantization rules

$$-\mathcal{B} = \frac{\partial A^g}{\partial t} \to i\frac{\partial}{\partial t} \ , \tag{3.29a}$$

$$R_\mu = \frac{\partial A^g}{\partial a^\mu} \to \frac{1}{i}\frac{\partial}{\partial a^\mu} \overset{\text{def.}}{=} \mathcal{R}_\mu \tag{3.29b}$$

finally permitted in Paper I the identification of the following generalization of Schrödinger's equation

$$i\frac{\partial}{\partial t}\Psi(t,a) = \mathcal{B}(t,R)\Psi(t,a) \ , \tag{3.30a}$$

$$\mathcal{R}_\mu\Psi(t,a) = \frac{1}{i}\frac{\partial}{\partial a^\mu}\Psi(t,a) = R_\mu\Psi(t,a) \ . \tag{3.30b}$$

The reader should keep in mind the necessity that the hadronic wave function Ψ depends on both, the space coordinates $\vec{r}^k$ *and* linear momentum $\vec{p}_k$ (as well as on time), as pointed out in Paper I. The reformulation of the conventional Schrödinger's equations via Birkhoff's guages (3.28) to achieve an equivalent theory in terms of the atomic wave functions $\Psi(t,\vec{r},\vec{p})$ should be kept also in mind.

Our task is now that of interpreting Equations (3.30) in terms of the isotopic Hilbert space theory of the preceding section. For this purpose we note that the classical equations (3.20) are characterized by

(1) The 2n functions $R_\mu(a)$ which essentially characterize the underlying symplectic isotopy or Lie-isotopy, that is, the replacement of the exact canonical, symplectic structure with the most general possible exact symplectic structure (*symplectic isotopy*[14])

$$\omega_2 = \frac{1}{2}\omega_{\mu\nu}da^\mu\wedge da^\nu = dp_k\wedge dr^k \to \Omega_2 = \frac{1}{2}\Omega_{\mu\nu}(a)da^\mu\wedge da^\nu \tag{3.31}$$

or, equivalently, the replacement of the fundamental Poisson brackets with the most general possible Lie realization of fundamental brackets in mechanics (*Lie isotopy*)[14]

$$([a^\mu,a^\nu]) = \begin{pmatrix} [r^i,r^j] & [r^i,p_j] \\ [p_i,r^j] & [p_i,p_j] \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\omega^{\mu\nu})$$

$$= \left(\left| \left| \frac{\partial R^0_\alpha}{\partial a^\beta} - \frac{\partial R^0_\beta}{\partial a^\alpha} \right| \right|^{-1} \right)^{\mu\nu} \to \quad ([a^\mu,a^\nu]^*) = (\Omega^{\mu\nu}(a)) \tag{3.32}$$

$$= \left(\left| \left| \frac{\partial R_\alpha}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\alpha} \right| \right|^{-1} \right)^{\mu\nu} .$$

(2) The Birkhoffian function $B(t,a)$ which is the generator of the noncanonical but generalized canonical time evolution[14]

$$a(t) = e^{t\Omega^{\mu\nu}(a)\frac{\partial B(t,a)}{\partial a^\nu}\frac{\partial}{\partial a^\mu}\Big|_{a(t_0)}} \tag{3.33}$$

(where $\mathcal{A}$ is the conventional associative algebra);

(3) The preservation of the underlying carrier space of the Hamiltonian Mechanics (the cotangent bundle T^*M although without the conventional meaning of phase space ; it is called instead the *dynamical space* [14]).

In the transition to the Hadronic Mechnaics, the following possibilities to characterize the generalized Schrödinger's representation appear plausible.

(1*) Identification of the generalized unit operator $I^* = T^{-1}$ needed replace the <u>conventional</u> Heisenberg's commutation rules

$$([a^\mu,a^\nu]) = (a^\mu Ta^\nu - a^\nu Ta^\mu) = i(\omega^{\mu\nu})\,I \tag{3.34}$$

$$= i\left(\left| \left| \frac{\partial R^0_\alpha}{\partial a^\beta} - \frac{\partial R^0_\beta}{\partial a^\alpha} \right| \right|^{-1} \right)^{\mu\nu} = i\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \begin{matrix} T = \hbar^{-1} I \\ a = (r,p) \end{matrix}$$

(where now a, R^0 , etc. are operators) with their isotopic form

$$([a^\mu,a^\nu]^*) = (a^\mu T a^\nu - a^\nu T a^\mu) = i\Omega^{\mu\nu}(a)$$

$$= i\left(\left|\left|\frac{\partial R_\alpha}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\alpha}\right|\right|^{-1}\right)^{\mu\nu}, \quad T = T(a) \quad . \tag{3.35}$$

(2*) Identification of the T-Hermitian Birkhoffian operator B which characterizes a nonunitary, T-unitary time evolution according to structure (2.60), i.e.,

$$U_T(t) = e_T^{itB} = I^* e^{iBTt} = I^* e^{iB*t} \quad . \tag{3.36}$$

(3*) Preservation of the Hilbert space of the Atomic Mechanics, although interpreted in the generalized form of Definition 2.23. It should be stressed that remarks (1*)-(3*) are introduced here on pure grounds of *similarities* with the classical cases (1)-(3). We assume the reader is familiar with the noncanonical character of the Birkhoffian time evolution (3.33), and the consequently necessary nonunitary character of the hadronic time evolution.

These occurrences render plausible the following third postulate.

POSTULATE III. *The time evolution of a hadronic state in the isotopic generalization of Schrödinger's representation is characterized by the generalized unit operator* $I^* = T^{-1}$ *and by the T-unitary time evolution induced by a T-Hermitian Birkhoffian operator* B *within the context of the isotopic formulation of the Hilbert space.*

Let $|t_0>$ be the hadronic state vector at time t_0. According to Postulate III, the state vector at a later time t is given by

$$|t> = U(t,t_0) * |t_0> = U(t,t_0)T|t_0> \tag{3.37}$$

where $U(t,t_0)$ admits representation (3.36) and, in particular, it is such that

$$U(t,t-\delta t) \cong I^* + \frac{i\delta t B}{1!} \quad . \tag{3.38}$$

Since T-unitary transformations form a Lie-isotopic group, they verify property (2.62), i.e.,

$$\begin{aligned} &U(t,t-\delta t) * U(t-\delta t,t_0) = U(t,t_0) \\ &= (I^* - i\delta t B) * U(t-\delta t,t_0) \quad . \end{aligned} \tag{3.39}$$

We can then write

$$\begin{aligned} &\lim_{\delta t\to 0} \frac{U(t,t_0) - U(t-\delta t, t_0)}{\delta t} = \frac{\partial}{\partial t} U(t,t_0) \\ &= -iB * U(t,t_0) \end{aligned} \tag{3.40}$$

and, by recalling (3.37) we finally have the desired isotopic interpretation of Equation (3.30a), i.e.,

$$i\frac{\partial}{\partial t}|t> = B * |t> = BT|t> \quad . \tag{3.41}$$

This essentially implies the following interpretation of the Birkhoffian operator $\mathcal{B}$ as emerging from the naive quantization rules (3.29a)

$$\mathcal{B} = BT \quad . \tag{3.42}$$

The Hamiltonian particularization is intriguing, inasmuch it can be written

$$i\frac{\partial}{\partial t}|t\rangle = H * |t\rangle = HT|t\rangle \quad , \quad T = \hbar^{-1} \tag{3.43}$$

by therefore confirming that the conventional Schrödinger's representation has indeed an isotopic structure, although of trivial character, as it was the case for Equations (3.34).

This latter point can be made more precise by passing to the isotopic interpretation of the remaining equations (3.30b). Consider the atomic wave structure

$$\Psi(r) = Ne^{\frac{i}{\hbar}pr} = Ne^{iKr}, \quad K = \frac{p}{\hbar} \quad . \tag{3.44}$$

It can be written in the trivial isotopic form

$$\Psi(r) = Ne^{ip*r} = Ne^{ipTr} = Ne^{ik_T*r}, \quad T = \hbar^{-1} \quad . \tag{3.45}$$

Similarly, the atomic eigenvalue equation for the linear momentum

$$p\Psi(r) = -i\hbar\frac{\partial}{\partial r}\Psi(r) = K\Psi(r) \tag{3.46}$$

can be turned into the trivial isotopic form

$$p * \Psi(r) = -i\frac{\partial}{\partial r}\Psi(r) = K_T * \Psi(r) \quad . \tag{3.47}$$

By using these guidelines, we therefore postulate that the hadronic generalization of structure (3.45) is given by

$$\Psi(a) = Ne^{iR_\mu * a^\mu} = Ne^{i\alpha_{T\mu} * a^\mu} \quad . \tag{3.48}$$

The naive quantization rules (3.29b) can then be reformulated in the isotopic form

$$R_\mu * \Psi(a) = -i\frac{\partial}{\partial a^\mu}\Psi(a) = \alpha_{T\mu} * \Psi(a) \quad . \tag{3.49}$$

Under the assumption that the new unit $I^* = T^{-1}$ is known, the operator $\mathcal{R}_\mu$ originating from the naive quantization (3.29b) can be decomposed according to the rule

$$\mathcal{R}_\mu = R_\mu T \quad . \tag{3.50}$$

To see the joint equations (3.30), recall that the complete atomic wave packet for a free particle is given by

$$\Psi(t,a) = Ne^{i(p*r - H*t)} = Ne^{i(K_T * r - E_T * t)} \tag{3.51}$$

The conventional Schrödinger's equation is then justified on grounds of the identities

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi &= E_T * \Psi \quad , \\ p * \Psi &= K_T * \Psi \quad , \\ H = \frac{p^2}{2m} \rightarrow i\frac{\partial}{\partial t}\Psi &= H * \Psi \quad . \end{aligned} \tag{3.52}$$

The Hadronic generalization is straightforward. First, we shall write the *hadronic wave packet* in the form

$$\Psi(t,\vec{r},\vec{p}) = \Psi(t,a) = Ne^{i(R * a - B * t)}$$

$$= Ne^{i(\vec{P} * \vec{r} + \vec{Q} * \vec{p} - B * t)} \tag{3.53}$$

$$= Ne^{i(\vec{\alpha}_T * \vec{r} + \vec{\beta}_T * \vec{p} - \gamma_T * t)} = Ne^{i(\rho_T * a - \gamma_T * t)}$$

by therefore recovering the form achieved in Paper I. Equations (3.30a) or (3.41) are then readily obtained via a straightforward generalization of (3.52), i.e.,

$$i\frac{\partial}{\partial t}\Psi = \gamma_T * \Psi ,$$

$$-i\frac{\partial}{\partial a^\mu}\Psi = \rho_T^\mu * \Psi , \tag{3.54}$$

$$B = B(t,R) \to i\frac{\partial}{\partial t}\Psi = B * \Psi .$$

The advancement in the transition from the atomic to the hadronic wave packet is selfevident. In fact, *the atomic wave packet (3.51) represents a free particle, while the hadronic wave packet (3.53) represents a particle under the most general possible collection of local, potential, and nonpotential interactions.*

Recall that wave packet (3.51) is, in actuality, an approximation of the form

$$\Psi(t,r) = Ne^{iT\int_{t_o}^{t}(pdr - Hdt)} , \quad T = \hbar^{-1} . \tag{3.55}$$

Equivalently, we can say that the phase of the atomic wave packet is characterized by the canonical action functional.

On similar grounds, wave packet (3.53) should be considered an approximation of the more adequate form

$$\Psi(t,a) = Ne^{iT\int_{t_o}^{t}(R_\mu da^\mu - Bdt)} \tag{3.56}$$

in which the phase is characterized by the action of Birkhoffian Mechanics, (with the understanding that the nature of T needs specific study).

To avoid possible misrepresentations, we should stress that the quantities "r" and "p" are no longer canonically conjugate in the Hadronic Mechanics. In fact, the fundamental brackets (3.34) are mutated into the isotopic form (3.35).

Intriguingly, there are new quantities which may be formally interpreted as "canonically conjugated". In fact, a comparison of the Hamiltonian and Birkhoffian actions

$$A_{Ham.} = \int_{t_o}^{t}(pdr - Hdt) , \quad A_{Birk.} = \int_{t_o}^{t}(Rda - Bdt) \tag{3.57}$$

reveals the possibility that the quantities "a" and "R" may be interpreted as "canonically conjugated", again, under regularity condition (3.26) to ensure their independence. In fact, we may formally introduce the "conventional" commutation rules.

$$[a^\mu,R^\nu] = a^\mu R^\nu - R^\nu a^\mu = i\delta^{\mu\nu}, \quad \mu,\nu = 1,2, \ldots,2n \tag{3.58}$$

with the understanding that equations (3.58) now imply

$$[r^i, p_j] = r^i p_j - p_j r^i \equiv 0 \tag{3.59}$$

Note that, if the atomic operators r and p are $2n$, the corresponding hadronic operators a and R are $4n$.

These remarks may assist the interested researcher for further studies in the construction of the Hadronic Mechanics, as well as for specific applications.

Until now we have been interested in studying the formal structure of the isotopic generalization of Schrödinger's representation. Its explicit construction demands the selection of the identity and the Birkhoffian operators. A formal solution was given in Section IV of Paper I, and it can be reviewed according to the steps summarized below with their isotopic formulation.

STEP 1. *Identify the desired Newtonian system (3.19) and compute its Birkhoffian representation in the variables* $a = (\vec{r}, \vec{p})$ *with* $\vec{p}_k = m\dot{\vec{r}}_k$
This implies, in particular, the identification of the $(2n + 1)$ functions

$$(R_\mu) = R_\mu(a)) = (\vec{P}_k(\vec{r},\vec{p}), \vec{Q}_k(\vec{r},\vec{p})) \; ; \; B = B(t,a) = B(t,\vec{r},\vec{p}) \; ,$$

$$\mu = 1,2, \cdots, 2n = 6N, \; k = 1,2, \cdots, N \; , \tag{3.60}$$

as well as of the energy E, angular momentum J, and all needed physical quantities

$$E = \sum_k \frac{\vec{p}_k^2}{2m_k} + V(\vec{r}) \; , \; \vec{J} = \sum_k \vec{r}_k \times \vec{p}_k \; , \text{ etc.} \tag{3.61}$$

STEP 2. *Upon verification of regularity condition (3.26) via guage (3.25), compute the dependence of the a-variables in the R-ones, i.e., identify* $a = a(R)$, *and reduce all functional dependences to the R-variables,*

$$(a^\mu) = (a^\mu(R)) = (\vec{r}_k^{\,k}(\vec{P},\vec{Q}) \; , \; \vec{p}_k(\vec{P},\vec{Q})) \; ,$$

$$B(t,a(R)) = \mathcal{B}(t,R) = \mathcal{B}(t,\vec{P},\vec{Q}) \; , \tag{3.62}$$

$$E = \sum_k \frac{\vec{p}_k^2}{2m_k} + V(\vec{r}) = \sum_k \frac{\vec{p}_k(\vec{P},\vec{Q}) \cdot \vec{p}_k(\vec{P},\vec{Q})}{2m_k} + V'(\vec{P},\vec{Q}) \; , \text{ etc.}$$

STEP 3. *Perform the naive (nonisotopic) quantization (3.29), and identify the operational form of* $a(R)$, *i.e.,*

$$\mathcal{R}_\mu = \frac{1}{i}\frac{\partial}{\partial a^\mu} \; , \; \mathcal{B} = i\frac{\partial}{\partial t} \; . \tag{3.63}$$

STEP 4. *Compute the operator image of the tensor* $\Omega^{\mu\nu}$ *and search for the operator* T *such that*

$$[a^\mu, a^\nu]^* = a^\mu T a^\nu - a^\nu T a^\mu = i\Omega^{\mu\nu} \tag{3.64}$$

where

$$a^\mu = (a^\mu)^{hT} \; . \tag{3.65}$$

STEP 5. *Perform the isotopic reformulation of the eigenvalue equations obtained via the naive quantization*

$$i\frac{\partial}{\partial t}\Psi(t,a) = B * \Psi(t,a) \tag{3.66}$$

where

$$B = \mathcal{B}\, T^{-1} = B^{hT} \tag{3.67}$$

The procedure above solves an important point left open in Paper I. Recall that in Example (3.26) we had assumed $p = i^{-1} \partial / \partial r$ also for the Hadronic Mechanics, lacking specific rules for the identification of a more adequate operator form of p. These rules are now identified, although on formal grounds, in a way admitting a limit into Birkhoff's equations. In fact, the rules are now provided by the functional dependence $a = a(R)$, and then the operator form $R = i^{-1} \partial / \partial a$, $a = (r,p)$. In this way, the existence of a correspondence into the Birkhoffian generalization of the Hamilton-Jacobi equations is ensured.

Note that, <u>if one assumes $p = i^{-1} \partial / \partial r$, then this correspondence into the Birkhoffian mechanics is lost</u>. The point is important to stress again the departure of the Hadronic from the Atomic Mechanics.

The status of the current knowledge for the explicit computation of isotopic rules (3.64) is therefore the following.

(a) Identify the operator form of a(R) via the Birkhoffian method above.

(b) Identify the operator form of $\hat{\Omega}^{\mu\nu}(a)$ via suitable symmetrization of the classical expression under quantization rule $a \to a(R)$; and

(c) when the operator form of a and $\hat{\Omega}$ are so determined, solve Equations (3.64) in the unknown isotopic operator T.

Note that, in the Atomic Mechanics, $a = (r,p)$ has the realization via noncommuting operators r and $p = i^{-1} \partial / \partial r$. In the transition to the Hadronic Mechanics the r and p operators are more directly replaced by the R_μ operators Their realization $R_\mu = i^{-1} \partial / \partial a$ may give the impression that such operators commutes. This is true conventionally, but not in the hadronic case where the isotopic product must be used. It is then trivial to see that, in general, $[R_\mu, R_\nu]^* \neq 0$. Thus, not only the hadronic operator are noncommuting, but actually the hadronic generalization of the components of $\underset{\sim}{r}$ (or $\underset{\sim}{p}$)are noncommuting, as expectable from gravitational arguments.

We should indicate that the assignment of classical quantities $R_\mu(a)$ and $B(t,a)$ where $(R_\mu) \neq (\vec{p},\vec{0})$ is generally sufficient to incorporate non-potential forces. Nevertheless, in this case the mathematical symbol "p" does not, in general, represent the physical linear momentum.

The interested reader is therefore urged to compute explicitly the *Newtonian* equations of motion in their first-order form (3.19) for any given Birkhoffian quantities R_μ and B. The first set of equations in $\dot{r}^k$ will then provide the actual meaning of "p".

This is an often forgotten peculiarity also of conventional Hamiltonian Mechanics. In fact, the expression $p = m\dot{r}$ is lost already for Hamiltonians of the type quadratic in p

$$H = \alpha(t)p^2 + \beta(t,r)p + \gamma(t,r) \tag{3.68}$$

and the departure of p from the linear momentum becomes even greater for more complex functional dependences (e.g. quartic polynomials cp^4).

The identification of the physical meaning of the algorithms at hand has been stressed in paper[15], as well as in much of the literature of Lie-admissible algebras, in order to prevent mathematical statements (e.g., $\Delta r \ \Delta p \geq \frac{1}{2}\hbar$) which are physically vacuous (e.g., $p = c_1 \log c_2 \, r \cdot \dot{r}$).

To close this subsection, we can now complement Postulate III with the following operational postulate essentially based on Steps I through 5.

<u>POSTULATE IV</u>. *All classical Birkhoffian quantities can be characterized via functions depending on time and 6N independent variables $a = (\vec{r},\vec{p})$ which are dynamically conjugate in the sense of the Lie-isotopy (3.32).*

The hadronic operators which can be interpreted as corresponding to these classical quantities are obtained by replacing the dynamical variables via T-Hermitian hadronic operators verifying the Lie-isotopic commutation rules (3.35). It is understood that the resulting Hadronic Mechanics can admit quantities, properties, and physical laws which are not shared by the Birkhoffian (or the Atomic) Mechanics.

3.5. Hadronic-isotopic generalization of Heisenberg's representation. As is well known, the transition from Schrödinger's to Heisenberg's representation, within the context of the Atomic Mechanics, implies the transition from a representation in which the states are time dependent to another, in which the states become time independent. We shall assume that this feature persists in the transition to the covering Hadronic Mechanics.

The Lie-isotopic generalization of Heisenberg's equations have been proposed by Santilli (see the review in Paper I). The following postulate can then be formulated by combining these generalized equations with the constancy of the states.

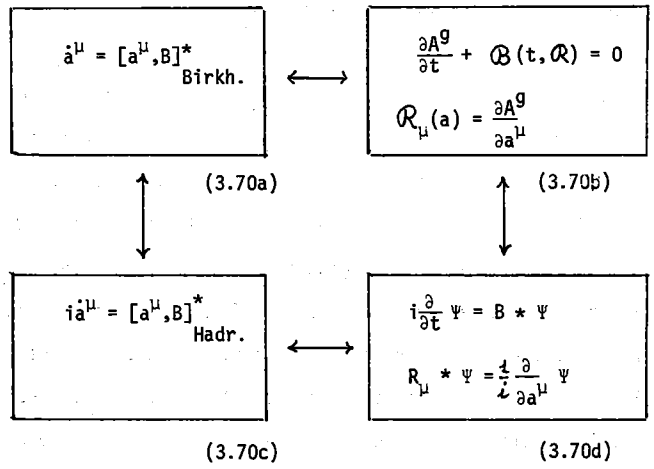
POSTULATE V. *The time evolution of an operator in the hadronic generalization of Heisenberg's representation for the exterior strong problem is characterized by the isotopic operator T and by the T-Hermitian Birkhoffian operator B within the context of the isotopic formulation of the Hilbert space.*

Explicitly, we can write for an arbitrary hadronic operator A

$$i\frac{d}{dt} A *| > = [A,B]^{*}_{*}| > , \quad \frac{d}{dt}| > \equiv 0 . \tag{3.69}$$

Our task is now that of proving that *the isotopic generalizations of Schrödinger's and Heisenberg's representations are equivalent.* This task clearly calls for the prior generalization of the transformation theory. With the understanding that this objective cannot be achieved in one single paper, the following subsection may be useful to identify some basic aspects.

3.6. Hadronic-isotopic generalization of the unitary transformation theory. Let us recall that Birkhoff's equations are the classical images of the generalized Heisenberg's equations, while the classical image of the generalized Schrödinger's equations is given by the Hamilton-Jacobi form according to the scheme (see Paper I, Section IV, for comments)



In order to avoid inconsistencies under the correspondence principle, the hadronic transformation theory must therefore be built as a suitable image of the Birkhoffian transformation theory (studied in detail in monograph[24]).

What is important for this introductory study is to recall that the transition from Equations (3.70a) to (3.70b) is performed via a noncanonical, generalized canonical transformation. These are regular, class C^∞ transformations $a \to a'(a)$ which *do not preserve the fundamental Poisson brackets*

$$\omega^{\mu\nu} \to \omega'^{\mu\nu} = \frac{\partial a'^{\mu}}{\partial a^{\alpha}} \omega^{\alpha\beta} \frac{\partial a'^{\nu}}{\partial a^{\beta}} \neq \omega^{\mu\nu} . \tag{3.71}$$

Nevertheless, the transformations preserve the generalized fundamental brackets (3.32), i.e., verify the rules

$$\overset{\mu\nu}{\Omega}(a) \to \Omega'^{\mu\nu}(a') = \frac{\partial a'^{\mu}}{\partial a^{\alpha}} \Omega^{\alpha\beta}(a(a')) \frac{\partial a'^{\nu}}{\partial a^{\beta}} \equiv \Omega^{\mu\nu}(a') . \tag{3.72}$$

A first fundamental requirement for the hadronic transformation theory is therefore that it must not be of unitary type. By keeping in mind that the physical laws and relativities of the Atomic Mechanics are based on unitary transformations, the remark is sufficient to provide additional elements indicating the nontriviality of the generalization under consideration.

The Birkhoffian Mechanics is a covering of the Hamiltonian Mechanics, in the sense that the latter can be recovered as a particular case of the former. For the case of generalized canonical transformations, one can see that, when all nonselfadjoint forces are identically null, the generalized Lie tensor $\Omega^{\mu\nu}$ reduces to the fundamental ones $\omega^{\mu\nu}$

$$\Omega^{\mu\nu}(a) \Big|_{F^{NSA} = 0} = \omega^{\mu\nu} \tag{3.73}$$

and the generalized canonical transformations become identical with the conventional ones.

A second fundamental requirement of the transformation theory of the Hadronic Mechanics is therefore that of admitting the conventional unitary transformation theory as a particular case.

These (and other) requirements can be met via the use of the T-unitary transformations of Section 2, i.e.,

$$\begin{aligned} A' &= U * A * U^{hT} = UTATU^{hT} , \\ U * U^{hT} &= U^{hT} * U = I^* = T^{-1} \end{aligned} \tag{3.74}$$

which recover the conventional unitary transformations at the value $I^* = I$ (the transformations were proposed by Santilli under the name of *unitary-admissible* transformations in Ref. 15, p. 743, precisely to stress the latter feature).

In the following we shall identify some of the essential properties of the T-unitary transformation theory.

First, we must verify that the generalized unit I^* is invariant under T-unitary transformations. This property is clearly important for physical aspects related to the preservation of the hadronic probability, as we shall see. It is a matter of elementary calculations to show that

$$I^{*\prime} = U * I^* * U^{hT} = UTT^{-1}TU^{hT} = UTU^{hT} = U * U^{hT} \equiv I^* . \tag{3.75}$$

Lemma 3.2. T-unitary transformations leave invariant the operator unit $I^* = T^{-1}$ of the Hadronic Mechanics. #

Next, by using the trivial identities

$$I = TT^{-1} = TI^* = TU^{hT}TU = TUTU^{hT}$$
$$= T^{-1}T = I^*T = U^{hT}TUT = UTU^{hT}T \ , \tag{3.76}$$

one can see that the isotopic product $A * B$ of operators preserves its structure, i.e.,

$$U * A * B * U^{hT} = UTATBTU^{hT}$$
$$= \underbrace{UTATU^{hT}}T\underbrace{UTBTU^{hT}} = A'TB' = A'* B' \ . \tag{3.77}$$

Theorem 3.1. The Lie-isotopic product of operators is left invariant by T-unitary transformation in the sense of the rules

$$U * [A,B]^*_* U^{hT} = UT(ATB - BTA)TU^{hT}$$
$$= [A',B']^* = A'TB' - B'TA' \ . \quad \# \tag{3.78}$$

It should be recalled from Paper I that, unlike the simplest possible Lie product $AB - BA$, the isotopic product $A * B - B * A$ preserves its structure under (nonsingular) nonunitary and non- T-unitary transformations. Under these latter transformations, however, the T-operator is changed, i.e., we have transformations of the type

$$ATB - BTA \to A'T'B' - B'T'A' \ . \tag{3.79}$$

On the contrary, in transformations (3.78), the operators A and B are transformed but the *structure of the product* (that is, the isotopy operator T) is left invariant. This confirms the special role of the T-unitary transformations for the Hadronic Mechanics.

Next, we recall that the usual differential rules of Atomic Mechanics

$$[AB,C] = [A,C]B + A[B,C] \ ; \ \ [A,BC] = [A,B]C + B[A,C] \tag{3.80}$$

generalize directly into the hadronic form

$$[A * B,C]^* = [A,C]^*_* B + A * [B,C]^* \ ,$$
$$[A,B * C]^* = [A,B]^*_* C + B * [A,C]^* \tag{3.81}$$

with the understanding that, in general,

$$[AB,C]^* \neq [A,C]^* B + A[B,C]^* \ ,$$
$$[A,BC]^* \neq [A,B]^* C + B[A,C]^* \ . \tag{3.82}$$

It is then easy to see that, if A and C are operators verifying time evolution (3.69), their isotopic product $A * B$ verifies the law

$$\frac{d}{dt} A * C = \frac{dA}{dt} * C + A * \frac{dC}{dt} = [A * C,B]^* \ . \tag{3.83}$$

We are now equipped to identify the transformation rules under T-unitary transformation of the isotopic generalization of Heisenberg's equation. In the case of Birkhoff's equations, the transformation rules under a generalized canonical transformation $a \to a'$ are given by

$$R_\mu(a) \to R'_\mu(a') = R_\alpha \frac{\partial a^\alpha}{\partial a'^\mu} = R_\mu(a') + \frac{\partial F(t,a)}{\partial a^\mu} \ , \tag{3.84a}$$

$$B(t,a) \to B'(t,a') = B(t,a(a')) + \frac{\partial F}{\partial t} \tag{3.84b}$$

where rule (3.84a) expresses the form-invariance of the generalized Lie product, Equations (3.72), while rule (3.84b) gives the new Birkhoffian.

Consider now the hadronic time evolution of the transformed operator A'

$$i\frac{d}{dt}A' = i\frac{d}{dt}(U * A * U^{hT})$$

$$= i\frac{dU}{dt} * A * U^{hT} + iU * \frac{dA}{dt} * U^{hT} + iU * A * \frac{dU^{hT}}{dt}$$

$$= i\frac{dU}{dt} * U^{hT} * U * A * U^{hT} + iU * A * U^{hT} * U * \frac{dU^{hT}}{dt}$$

$$+ U * (A * B - B * A) * U^{hT} \qquad (3.85)$$

$$= i\frac{dU}{dt} * U^{hT} * A' + iA' * U * \frac{dU^{hT}}{dt} + (A' * B' - B' * A') .$$

Introduce a T-Hermitian operator D' such that

$$D' = i\frac{dU^{hT}}{dt} * U = -iU * \frac{dU^{hT}}{dt} . \qquad (3.86)$$

Then Equations (3.85) become

$$i\frac{dA'}{dt} = (D' * A' - A' * D') + (A' * B' - B' * A') \qquad (3.87)$$

and we have proved the following important property.

Theorem 3.2. The Lie-isotopic generalization of Heisenberg's equations transform under a T-unitary transformation U according to the rule

$$i\frac{dA'}{dt} = [A', K']^* = A'TK' - K'TA' \qquad (3.88)$$

where

$$A' = U * A * U^{hT}, \; U * U^{hT} = U^{hT} * U = I^* = T^{-1} , \qquad (3.89a)$$

$$K' = U * B * U^{hT} - i\frac{dU^{hT}}{dt} * U . \quad \# \qquad (3.89b)$$

It should be indicated that the calculations above are based on right-modular actions. They can be repeated for left modular actions by therefore gaining information important for the identification of the intrinsic irreversibility of the Hadronic Mechanics. This aspect will not be investigated here (see Paper III).

Next it is important for our program to identify the behavior of T-eigenvalues under T-unitary transformations. Recall that, in conventional Atomic Mechanics, the eigenvalues of an operator are left unchanged by unitary transformations. It is easy to see that the property persists in the covering hadronic context.

In fact, suppose that the following eigenvalue equation holds

$$A * |\ > = AT|\ > = \alpha_T * |\ > = \alpha|\ > . \qquad (3.90)$$

Then we can write

$$U * A * |\ > = U * A * U^{hT} * U * |\ > = A' * (U * |\ >) = A' * |\ >'$$

$$= U * \alpha_T * |\ > = U * \alpha|\ > = \alpha U * |\ > = \alpha_T * (U * |\ >) = \alpha_T * |\ >' .$$

Theorem 3.3. The T-eigenvalues of an operator are left unchanged by T-unitary transformations. #

3.7. Equivalence of the hadronic-isotopic generalizations of Schrödinger's and Heisenberg's representations. Birkhoff's equations (3.70a) and their Hamilton-Jacobi form (3.70b) are equivalent in the sense that the former reduce to the latter under a generalized canonical transformation for which the new Birkhoffian is identically null (Paper I).

$$B' = B(t,a(a')) + \frac{\partial F}{\partial t} \equiv 0 \quad . \tag{3.92}$$

It is possible to prove that this classical property admits a consistent hadronic image. In fact, the T-unitary operator U can be selected, for a given Birkhoffian B and identity I^*, in such a way that the transformed Birkhoffian (3.89b) is identically null, i.e.,

$$K' = U * B * U^{hT} - i\frac{dU}{dt} * U^{hT} \equiv 0 \quad . \tag{3.93}$$

As a matter of fact, this expression can be assumed as a condition on the transformation U. Its existence (under sufficient topological conditions) can then be proved in a way similar to the atomic case.

By using identities (3.76), Equation (3.93) can then be written in the generalized Schrödinger's form

$$i\frac{dU}{dt} = B' * U \tag{3.94}$$

We now indicate, for notational convenience, quantities belonging to the representation of Postulate III with a subscript (1) and those belonging to the representation of Postulate V with the subscript (2). Then Equation (3.94) can be written

$$i\frac{dU}{dt} = B_{(1)} * U \ , \ B_{(1)} = B' = U * B_{(2)} * U^{hT} \ . \tag{3.95}$$

By recalling the expressions

$$|t>_{(1)} = U|0>_{(2)} \ , \ \frac{\partial}{\partial t}|0>_{(2)} \equiv 0 \ , \tag{3.96}$$

we finally have the following important result.

Theorem 3.4. The hadronic-isotopic generalization of Schrödinger's representation

$$\begin{cases} i\frac{\partial}{\partial t}|t>_{(1)} = B_{(1)} * |t>_{(1)} & (3.97a) \\ \frac{dA_{(1)}}{dt} \equiv 0 & (3.97b) \end{cases}$$

is equivalent ot the hadronic-isotopic generalization of Heisenberg's representation

$$\begin{cases} i\frac{d}{dt}(A_{(2)} * |>_{(2)}) = [A_{(2)}, B_{(2)}]^* * |>_{(2)} & (3.99a) \\ \frac{d}{dt}|>_{(2)} \equiv 0 & (3.99b) \end{cases}$$

in the sense that, under sufficient topological conditions, there exists a T-unitary transformation under which

$$|>_{(1)} = |t,a>_{(1)} = U * |>_{(2)} = U * |a>_{(2)} \tag{3.100}$$

and such a transformation is given by

$$U = e^{-iB} * t = Te_T^{-iBt} \quad . \quad \# \tag{3.101}$$

Note that the theorem confirms the time-dependence of the hadronic wave function (3.53).

Theorem 3.4 clearly generalizes the corresponding Hamiltonian property of the Atomic Mechanics. In addition, the theorem identifies in technical terms an insufficiency of the contemporary formulation pointed out in Paper I.

It is given by the fact that the states of the Heisenberg's representation are dependent on both $\vec{r}$ and $\vec{p}$

$$| >_{(2)} = | a>_{(2)} = | \vec{r},\vec{p}>_{(2)} \quad . \tag{3.102}$$

In the reduction of this representation to the Schrödinger's one, one gains the time dependence, *but the $\vec{p}$-dependence cannot be, in general, suppressed.* As a result, the wave function of contemporary quantum mechanics is expected to be dependent on both $\vec{r}$ and $\vec{p}$

$$\Psi_{Schr.} = \Psi(t,\vec{r},\vec{p}) \quad . \tag{3.103}$$

In turn, this implies the existence of an *equivalent* formulation of Schrödinger's equations in terms of wave function (3.104). This problem has been identified in Paper I, and a solution has been proposed via the use of the Birkhoffian gauge (3.28). Since it deals with an equivalent formulation of the Atomic Mechanics (which has apparently escaped the attention of the researchers in the field) we shall not consider it any longer. We have mentioned it here because the atomic particularization of Theorem 3.4 , while involving the conventional Heisenberg's equations, implies the indicated reformulation of Schrödinger's equations.

The study presented until now is sufficient to indicate the existence of the hadronic generalization of other atomic representations, e.g., the interaction representation, that via path integrals, etc. The study of these additional aspects of the Hadronic Mechanics is left here to the interested researcher.

3.8. Hadronic-isotopic generalization of Kronecker's and Dirac's deltas. To make further progress we need the generalization of the *Kronecker's delta*

$$\delta_{ij} = \begin{cases} I \ , & i = j \\ 0 \ , & i \neq j \end{cases} \tag{3.104}$$

and of *Dirac's delta*

$$\delta(x) = \frac{I}{2\pi} \int_{-\infty}^{+\infty} e^{izx} \ dz \tag{3.105}$$

which is applicable to Hadronic Mechanics. The need for the generalization is selfevident. In fact, the unit of the Hadronic Mechanics is no longer I, and the Kronecker's δ_{ij} must be suitably generalized. The generalization of the quantity $\delta(x)$ is equally needed inasmuch exponentials of the type appearing in (3.105) are generally meaningless in Hadronic Mechanics.

Definition 3.1. The hadronic generalization δ^*_{ij} of Kronecker's δ_{ij} is given by

$$\delta^*_{ij} = I^*\delta_{ij} = \begin{cases} I^* \ , & i = j \\ 0 \ , & i \neq j \end{cases} \ . \tag{3.106}$$

The hadronic generalization of Dirac's $\delta(x)$ is given by

$$\delta^*(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iz * x} \ dz = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{izTx} \ dz$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e_T^{izx} \ T dz \quad . \quad \# \tag{3.107}$$

As in the conventional case, there exist numerous reformulations of the "δ-function" which we cannot possibly study at this time. We limit ourselves to the indication of the properties

$$\delta^*(x) = \delta(\hat{T}x) = \frac{1}{T}\,\delta(x) \ , \qquad \int f(x)\,\delta^*(x)dx = I^*\,f(0) = f_T(0) \ . \tag{3.108}$$

The generalization of the symmetry properties [e.g., $\delta(x) = \delta(-x)$] will be studied at a later time after having identified the basic structure of the symmetries of hadronic systems.

3.9. Hadronic-isotopic generalization of the atomic orthonormality, normalization. Two atomic states $|\alpha_i>$ and $|\alpha_j>$ are said to be orthonormal when they verify the rule

$$<\alpha_i|\alpha_j> = \delta_{ij} \ . \tag{3.109}$$

Definition 3.2. Two hadronic states $|\alpha_i>$ and $|\alpha_j>$ are said to be $\hat{I}^*$-*orthonormal*, when they verify the generalized rule

$$<\alpha_i| * |\alpha_j> = <\alpha_i|T|\alpha_j> = \delta^*_{ij} = T^{-1}\delta_{ij} \ . \quad \# \tag{3.110}$$

Note that when T is a scalar, the atomic and hadronic orthonormality are equivalent.

As is well known, one of the fundamental points of the Atomic Mechanics is the normalization of states $|\alpha>$ to the unit element I of the enveloping associative algebra of operators

$$<\alpha|\alpha> = \int \overline{\Psi}\,(\alpha)\Psi(\alpha)d\alpha = I \ . \tag{3.111}$$

In turn, this normalization is at the basis of the probabilistic interpretation of the Atomic Mechanics.

Let us recall that one of the primary objectives of the Lie-isotopic branch of the Hadronic Mechanics is the exterior treatment of strong non-Hamiltonian interactions, that is, the *experimental observation* of strong interactions. For such a task, it is clearly important to preserve a form of probabilistic interpretation, in order to reach a theory which is consistent with available experimental data. In turn, this can be apparently achieved via the requirement of the *existence* of the unit operator $I^* = T^{-1}$ (plus additional topological conditions, besides regularity, which are inessential here).

Definition 3.3. A hadronic state $|\alpha>$ is said to be $\hat{I}^*$-*normalized* when it verifies the rule

$$<\alpha| * |\alpha> = <\alpha|T|\alpha> = \int \overline{\Psi}\,(\alpha) * \Psi(\alpha)d\alpha = I^* = T^{-1} \tag{3.112}$$

Along similar lines, two hadronic wave-functions $\Psi_\beta\,(\alpha)$ and $\Psi_{\beta'}(\alpha)$ are said to be $\hat{\delta}^*$-normalized when they verify the rule

$$\int \overline{\Psi}_{\beta'}(\alpha) * \Psi_\beta(\alpha)d\alpha = \delta^*(\beta-\beta') \ . \tag{3.113}$$

Similar normalization occur for other hadronic generalizations of atomic normalizations. #

It is recommendable to give a simple example. Consider two hadronic wave packets

$$\Psi_{R'}(a) = N^* e^{iR'a} \ , \quad \Psi_R(a) = N^* e^{iRa} \tag{3.114}$$

Normalization (3.113) now reads (assuming, for simplicity, T = const.)

$$\int \overline{\Psi_{R'}}(a) * \Psi_R(a)da = N^{*2}T \int e^{i(R - R')a}da$$

$$= N^{*2}T2\pi\delta(R - R') = N^{*2}T^2 2\pi\delta^*(R - R') , \qquad (3.115)$$

thus yielding the *hadronic normalization coefficient*

$$N^* = \frac{1}{T} \frac{I}{\sqrt{2\pi}} = \frac{I^*}{\sqrt{2\pi}} \qquad (3.116)$$

which is a direct generalization of the atomic ones

$$N = \frac{I}{\sqrt{2\pi}} \qquad (3.117)$$

3.10. Hadronic-isotopic probability and expectation values. The physical implications of Definition 3.3. are intriguing. In fact, it implies a nontrivial generalization of the atomic probabilistic interpretation. The total probability of an atomic state is given by the familiar form

$$P = \langle\alpha|\alpha\rangle = \int \overline{\Psi}(\alpha)\Psi(\alpha)d\alpha = I \qquad (3.118)$$

and its unit value represents the certainty. We now define as *hadronic total probability* the corresponding quantity

$$P^* = \langle\alpha| * |\alpha\rangle = \int \overline{\Psi}(\alpha) * \Psi(\alpha)d\alpha = I^* \qquad (3.119)$$

of which (3.118) is a particular case. The nontriviality of the generalization is due to the fact that the total probability is no longer the ordinary unit, and it is instead a more general quantity depending on the coordinates and momenta, $I^* = I^*(r,p)$. However, this quantity provides a perfectly acceptable

characterization of certainty because it is the (right and left) *unit* of the algebra.

The consistency of the hadronic probability can be further illustrated by showing that it is conserved in time. This is a consequence of Lemma 3.1. Nevertheless, owing to its importance, it is appropriate to prove the property within the context of each hadronic-isotopic representation. Consider first the representation of Postulate III. Then we have

$$i\frac{\partial P^*}{\partial t} = i \int \overline{\dot{\Psi}}(\alpha) * \Psi(\alpha)d\alpha$$

$$= i \int [(\frac{\partial}{\partial t}\overline{\Psi}) * \Psi + \overline{\Psi} * \frac{\partial \Psi}{\partial t}]d\alpha \qquad (3.120)$$

$$= \int (\overline{\Psi} * B * \Psi - \overline{\Psi} * B * \Psi)d\alpha \equiv 0$$

where we have used Equations (3.30a) and the condition of T-Hermiticity of the Birkhoffian.

The proof of the conservation of I^* within the context of the representation of Postulate V is even simpler. In fact, we trivially have

$$i\frac{dI^*}{dt} = [I^*,B]^* = [T^{-1},B]^*$$

$$= T^{-1} * B - B * T^{-1} = T^{-1}TB - BTT^{-1} = B - B \equiv 0 . \qquad (3.121)$$

POSTULATE VI. *The total hadronic probability for the exterior strong problem is characterized by the generalized unit* $I^* = T^{-1}$ *which is conserved in the hadronic-isotopic generalizations of Heisenberg's and Schrödinger's representations, as well as under all possible T-unitary transformations.*

For readers not yet familiar with the Birkhoffian Mechanics, it should be indicated here that the lack of explicit time dependence of I^* originates

in the semiautonomous character of Birkhoff's equations (3.20). In fact, the general nonautonomous equations

$$\{[\frac{\partial R_\nu(t,a)}{\partial a^\nu} - \frac{\partial R_\mu(t,a)}{\partial a^\nu}]\dot{a}^\nu - [\frac{\partial B(t,a)}{\partial a^\mu} - \frac{\partial R_\mu(t,a)}{\partial t}]\}^{SA} = 0 \quad (3.122)$$

imply an explicit time dependence of the tensor $\Omega_{\mu\nu}(t,a)$ as well as of the tensor $\Omega^{\mu\nu}(t,a) = (||\ \Omega_{\alpha\beta}\ ||^{-1})^{\mu\nu}$.

This implies a hadronic T-operator with an explicit time dependence. However, the brackets of the time evolution of equations (3.122) violate the necessary condition for constituting an algebra (that is, not only we lose the Lie algebra, but we actually lose the condition for admitting any algebra)[14].

This is sufficient to clarify the reason for the restriction of our analysis to the semiautonomous equations. At any rate, these latter equations are already directly universal in mechanics. The transition to Equations (3.1_{22}) is therefore not needed.

The remarks, however, serve the purpose of indicating that the theory has the necessary mathematical structure to permit a time rate of variation of the "probability", in case needed in future developments.

The atomic expectation values of a (Hermitian) operator A are given by the familiar form

$$\langle A\rangle = \frac{\langle\alpha|A|\alpha\rangle}{\langle\alpha|\alpha\rangle} \quad (3.123)$$

which, in view of normalization (3.111), is generally written in the reduced form

$$\langle A\rangle = \langle\alpha|\ A\ |\alpha\rangle = \int \overline{\Psi}(\alpha) A \Psi(\alpha) d\alpha \quad . \quad (3.124)$$

We consider now the problem of the generalization of the atomic expectation values for the case of the Hadronic Mechanics.

Recall that the conventional field C cannot be used for the isotopic Hilbert space $H^{(T)}$ of Definition 2.23 (because $H^{(T)}$ would not act linearly on C). As a result of this occurrence, the conventional "numbers" of the Atomic Mechanics had to be generalized to the "T-numbers" of the field $C^{(T)}$.

This occurrence leads to the following definition of hadronic expectation values

$$\langle A\rangle_T = I^* \frac{\langle |*A*| \rangle}{\langle |*| \rangle} \quad (3.125)$$

which trivially coincides with the conventional ones when T is itself a scalar,

$$\langle A\rangle_T \equiv \langle A\rangle \quad (3.126)$$

with the understanding that, when T is an operator, the hadronic and the atomic expectation values are structurally different.

Note that values (3.125) are composed of two parts, an operator part expressed by the unit I*, and an (ordinary) scalar part expressed by the rest of the right hand side. Actual physical measurements are, of course, associated to the latter part, and this is the reason for the selection of normalization (3.112) rather than a corresponding normalization via the use of product (2.74), i.e. $\langle |*| \rangle I^*$.

POSTULATE VII. *The values expected in the measurements of hadronic observables in the exterior strong problem are given by rule (3.125) and, when representing total quantities, can be conserved (see Section 3.11).*

Owing to the open/non-Hamiltonian character of the mechanics under study, we expect the last part of Postulate VII to be selfevident.

By noting the following trivial properties

$$P = \int \overline{\Psi'} \ (\alpha) * \Psi(\alpha) d\alpha \equiv \int \overline{\Psi} \ (\alpha) * U^{hT} * U * \Psi(\alpha) d\alpha$$

$$= \int \overline{\Psi'} \ (\alpha) * \Psi'(\alpha) d\alpha = I^* \qquad (3.127a)$$

$$\int \overline{\Psi'} \ (\alpha) * A * \Psi(\alpha) d\alpha \equiv \int \overline{\Psi} \ (\alpha) * U^{hT} * U * A * U^{hT} * U * \Psi(\alpha) d\alpha$$

$$= \int \overline{\Psi'} \ (\alpha) * A' * \Psi'(\alpha) d\alpha, \qquad (3.127b)$$

we have the following result.

Lemma 3.3. Hadronic-isotopic expectation values are left invariant by T-unitary and by T-antiunitary transformations. #

By recalling the invariance of the Lie-isotopic product and the preservation of the T-eigenvalues, we then reach the following important theorem.

Theorem 3.5. All hadronic systems which can be connected by a T-unitary or a T-antiunitary transformation are equivalent. #

It is evident that the theorem above generalizes the corresponding property of the Atomic Mechanics for unitary and antiunitary transformations.

Note that, if the hadronic systems can be connected by a unitary or an antiunitary transformation, they *are not* necessarily equivalent (e.g., because the eigenvalues can be now different).

3.11. Implementation of the theory with subsidiary operator constraints for total conservation laws. Consider a closed selfadjoint Hamiltonian system

$$\begin{cases} \dot{a}^\mu = \Xi^\mu(a) = \omega^{\mu\nu} \dfrac{\partial H(a)}{\partial a^\nu} \ , \ (\Xi^\mu) = \begin{pmatrix} \vec{p}\ /m \\ \vec{f}^{SA}(\vec{r}) \end{pmatrix} & (3.128a) \\ \dfrac{dX_k}{dt} = \dfrac{\partial X_k}{\partial t} + \dfrac{\partial X_k}{\partial a^\mu} \dot{a}^\mu = \dfrac{\partial X_k}{\partial t} + [X_k, H] \equiv 0 & (3.128b) \\ \mu = 1, 2, \cdots, 6N \ , \ k = 1, 2, \cdots, 10 \ , \end{cases}$$

$$X = \{H, \vec{P}_{tot}, \vec{J}_{tot}, \vec{G}_{tot}\} \ . \qquad (3.128c)$$

As well known, the total Galilean conservation laws are first integrals of the equations of motion.

Generalize now the system via additive nonpotential (non-Hamiltonian) foraces. This yields the closed nonselfadjoint Birkhoffian systems

$$\begin{cases} \dot{a}^\mu = \hat{\Xi}^\mu(t,a) = \Omega^{\mu\nu}(a) \dfrac{\partial B(t,a)}{\partial a^\nu} \ , \\ (\hat{\Xi}^\mu) = (\Xi^\mu) + (I^\mu) \ ; \ (I^\mu) = \begin{pmatrix} 0 \\ \vec{f}^{NSA}(\vec{r},\vec{r},\vec{p}) \end{pmatrix} & (3.129a) \\ \dfrac{dX_k}{dt} = \dfrac{\partial X_k}{\partial t} + \dfrac{\partial X_k}{\partial a^\mu} \dot{a}^\mu = 0 \ , \quad \mu = 1, 2, \ldots, 6N, \ k = 1,2,\cdots,10, & (3.129b) \end{cases}$$

$$X = \{H, \vec{P}_{tot}, \vec{J}_{tot}, \vec{G}_{tot}\} \qquad (3.129c)$$

in which the total conservation laws are no longer first integrals of the equation of motion, but "bona fide" subsidiary constraints.

As reviewed in Paper I, one of the simplest realizations of systems (3.129) is to restrict the nonpotential vector Γ^μ to be the null eigenvector of the matrix $\partial X_k/\partial a^\mu$, i.e.,

$$\frac{\partial X_k}{\partial a^\mu}\,\Gamma^\mu \equiv 0 \quad , \quad k = 1, 2, \cdots, 10 \tag{3.130}$$

which results into the conditions

$$\begin{aligned} &\sum_k \vec{F}_k^{NSA} = 0 \quad , \\ &\sum_k \vec{p}_k \cdot \vec{F}_k^{NSA} = 0 \quad , \\ &\sum_k \vec{r}_k \times \vec{F}_k^{NSA} = 0 \quad . \end{aligned} \tag{3.131}$$

The existence and consistency of classical models (3.129) is then selfevident.

Until now we have considered the formulation of the Hadronic-isotopic Mechanics for *open* strong systems. The implementation to the closed non-Hamiltonian form is also possible, in full analogy with the classical-Birkhoffian case, as it was the case for the preceding aspects.

Consider a closed Hamiltonian atomic system, say, in Heisenberg's representation

$$i\frac{d}{dt}\,a^\mu|\,> = [a^\mu,H]|\,> \quad , \quad [a^\mu,a^\nu]|\,> = i\omega^{\mu\nu}|\,> \quad , \tag{3.132a}$$

$$i\frac{d}{dt}\,X_k|\,> = (i\frac{\partial X_k}{\partial t} + i\frac{\partial X_k}{\partial a^\mu}\,\dot{a}^\mu)|\,> = (i\frac{\partial X_k}{\partial t} + [X_k,H])|\,> \equiv 0 \quad , \tag{3.132b}$$

$$X = \{H, \vec{P}_{tot}, \vec{J}_{tot}, \vec{G}_{tot}\} = X^h \quad . \tag{3.132c}$$

The ten conserved operators are the generators of unitary symmetries of the Hamiltonian.

But, as stressed in Paper I, the symmetries of the total energy are not necessarily symmetries of the equations of motion, owing to the fact that internal nonpotential interactions have no representatives in the total energy.

This leads in a natural way to the implementation of hadronic-isotopic representations via subsidiary constraints. e.g., of the type

$$i\frac{d}{dt}\,a^\mu * |\,> = [a^\mu,B]^*_* |\,> \; ; \; [a^\mu,a^\nu]^*_* |\,> = i\Omega^{\mu\nu} * |\,> \quad , \tag{3.133a}$$

$$i\frac{d}{dt}\,X_k * |\,> = (i\frac{\partial X_k}{\partial t} + [X_k,B]^*)\,*\,|\,> \stackrel{\circ}{=} 0 \quad , \tag{3.133b}$$

$$X = \{H, \vec{P}_{tot}, \vec{J}_{tot}, \vec{G}_{tot}\} = X^{hT} \quad , \tag{3.133c}$$

which were originally proposed in Schrödinger's form in Ref.[15], p. 707.

Note that, in the transition from Eqs. (3.128) to (3.129) the physical quantities have been left unchanged as a fundamental condition for consistency. The situation is somewhat altered at the hadronic level, inasmuch conventional Hermiticity of quantities (3.132c) should be replaced by T-Hermiticity in the hadronic extension, and this generally implies different symmetrizations. Also, the explicit realization of the operators a^μ changes in the transition from the atomic to the hadronic setting, as by now familiar.

With an open mind on these and other aspects, systems (3.133) are expected to be consistent as it is the case for systems (3.129). In fact, the consistency reduces to the search of a generalized identity I^* and a Birkhoffian B under which Equations (3.133) are consistent for given pre-

assigned total observables X_k .

The detailed study of this problem will not be considered here, and we shall meanly limit ourselves to the presentation of a simple but physically nontrivial application.

3.12. Application to the structure of the π° particle. As indicated earlier, the applications of Hadronic Mechanics are already considerable, even though they are scattered in the literature with a terminology different than that of this paper. Also a rather comprehensive effort toward specific applications appears to be under way, beginning with the attempts to eliminate the rather unphysical spreading of the atomic wave packets (which, in nuclear cases may reach the size of the experimental apparatus); and passing to a systematic study of the departures from atomic settings of the nuclear magnetic moments; and the problem of strong nonpotential scattering, etc.

To conclude this paper, we would like to review a structure model of the π° proposed by Santilli in Ref.[15], Section 5, and reformulate it in terms of the isotopic theory of this paper. We expect the basic in ideas and methods to be extendable to other systems (such as the deuteron or the neutron) upon suitable implementations.

The starting point of the model was the inability of the Atomic Mechanics tp provide a nonrelativistic structure model of the π° as a bound state of a particle, say ε^-, and its antiparticle ε^+, under the condition that their mass m_ε is much smaller than that of the π° (~135 MeV) . This feature was explicitly worked out by showing that the atomic equations

$$[-\frac{\hbar^2}{m_\varepsilon}\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr}) - (\frac{e^2}{r} + V_0\frac{e^{-br}}{1-e^{-br}})]\Psi = E^{Bind.}\Psi \ ,$$

$$E^{Rest}_{\pi^\circ} = 2E^{Kin}_\varepsilon + 2E^{rest}_\varepsilon - E^{Bind} = 135 \text{ MeV},$$

$$\tau^{-1}_{\pi^\circ} = 4\pi\hbar^2 |\Psi(0)|^2 \frac{\partial^2 E^{Kin}_\varepsilon}{\hbar} = 10^{16} \text{ Sec}^{-1} \ , \qquad (3.134)$$

$$b^{-1}_{\pi^\circ} = 10^{-13} \text{ cm}, \ E^{Rest}_\varepsilon = .5 \text{ MeV} << E^{Rest}_{\pi^\circ} \ ,$$

are *inconsistent*, in the sense that there exists no real solution for the total energy. In fact, under the condition that the masses of the constituents are much smaller than the total mass of the bound state, the indicial equation admits only complex solution, thus preventing the achievement of a real total energy.

To resolve the problem, Santilli therefore proposed a "hadronic mutation" of Equations (3.134) which we now rewrite in the isotopic form

$$[-\frac{\hbar^2}{m_\varepsilon}\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr}) - (\frac{e}{r} + V_0\frac{e^{-br}}{1-e^{-br}})] * \Psi = E^{Bind.}\Psi \ ,$$

$$E^{Rest}_{\pi^\circ} = 135 \text{ MeV} \ , \quad E^{Rest}_\varepsilon = .5 \text{ MeV} \quad , \qquad (3.135)$$

$$\tau^{-1}_{\pi^\circ} = 10^{16} \text{ Sec}^{-1}, \ b^{-1}_{\pi^\circ} = 10^{-13} \text{cm} \ .$$

Under the condition that the isotopic operator is averaged to a constant

$$T \to \frac{1}{V}\int_V T dV = T_0 \ , \qquad (3.136)$$

equations (3.135) coincide with equations (5.1.14) of Ref.[15].

Intriguingly, the isotopy permitted the achievement of a consistent structure equations of the π° , that is, equations capable of reproducing

the *real* values of the total energy, the mean life, the charge radius, as well as all other additional total data of the particle (such as the charge, the space and charge parity, etc.), via the following value

$$T_0 = 1.73 \times 10^{-3} \ . \qquad (3.137)$$

In particular, the isotopy suppressed energy spectrum of the atomic equations, by therefore resulting into one single total value of the energy. The π° particle resulted to be a unique structure, fundamentally different than the $\pi^\pm$ (which were interpreted in Ref.[15] as a hadronic *three-body* structure).

The model permitted a quantitative study of the hypothesis that the constituents of hadrons are the ordinary *electrons* and *positrons*, although in mutated states (called *eletons* and *antieletons*).

In the final analysis, the construction of the Hadronic Mechanics was proposed in paper[15] to study the possibility of the rather suggestive reduction of the entire physical universe to photons, neutrinos, and electrons.

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