Iso-calculus of variations

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February 22, 2014

Abstract

In this article we make lift of the basic problem of the calculus of variations in the isolanguage and deduct the main iso-Euler-Lagrange equations..

1 Introduction

Genious ideas is the Santilli's generalization of the basic unit of quantum mechanics into an integro - differential operator $\hat{I}$ which is as positive - definite as $+1$ and it depends of local variables and it is assumed to be the inverse of the isotopic element $\hat{T}$

$$+1 > 0 \longrightarrow \hat{I}(t, r, p, a, E, \dots) = \frac{1}{\hat{T}} > 0$$

and it is called Santilli isounit. Santilli introduced a generalization called lifting of the conventional associative product ab into the form

$$ab \longrightarrow a\hat{\times}b = a\hat{T}b$$

called isoproduct for which:

$$\hat{I}\hat{\times}a = \frac{1}{\hat{T}}\hat{T}a = a\hat{\times}\hat{I} = a\hat{T}\frac{1}{\hat{T}} = a.$$

for every element a of the field of real numbers, complex numbers and quaternions.

The Santilli isonumbers are defined as follows: for given real number or complex number or quaternion a,

$$\hat{a} = a\hat{I},$$

with isoproduct

$$\hat{a}\hat{\times}\hat{b} = \hat{a}\hat{T}\hat{b} = a\frac{1}{\hat{T}}\hat{T}b\frac{1}{\hat{T}} = ab\frac{1}{\hat{T}} = \hat{ab}.$$

If $a \neq 0$ the corresponding isoelement of $\frac{1}{a}$ will be denoted with $\hat{a}^{-1}$ or $\hat{I} \,\hat{/}\, \hat{a}$.

With $\hat{F}_{\mathbb{R}}$ we will denote the field of the isonumbers $\hat{a}$ for which $a \in \mathbb{R}$ and basic unit $\hat{I}_1$.

In [1] are defined isocontinuous isofunctions and isoderivative of isofunction and in [1] are proved some of their properties.

The aim of this article is to be made lift of the basic problem of the calculus of variations in the isolanguage. In the next section we deduct the main iso-Euler-Lagrange equations.

2 Main results

Here we suppose that $L : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is given smooth function, hereafter called *Lagrangian*. We introduce the following notations

$$L = L(q, x) = L(q_1, q_2, \ldots, q_n, x_1, x_2, \ldots, x_n), \quad q, x \in \mathbb{R}^n,$$

$$D_q L = (L_{q_1}, L_{q_2}, \ldots, L_{q_n}),$$

$$D_x L = (L_{x_1}, L_{x_2}, \ldots, L_{x_n}).$$

Let $x, y \in \mathbb{R}^n$ and $t > 0$. Let $\hat{T} \in \mathcal{C}^2([0, t])$, $\hat{T} > 0$ on $[0, t]$. We define the set

$$\mathcal{A} = \{w \in \mathcal{C}^2([0, t] : w(0) = y, w(t) = x\}$$

and the functional

$$I(w, \hat{T}) = \hat{\int}_0^t \hat{L}(\hat{w}^{\wedge\circledast}(\hat{s}), \hat{w}^{\wedge}(\hat{s})) \hat{\times} \hat{d}\hat{s}.$$

Using the introduced isofunctions, iso-differential and iso-integral in [1] we can rewrite the above functional in the following form

$$I(w, \hat{T}) = \int_0^t L\Big(\frac{w'(s)\hat{T}(s) - w(s)\hat{T}'(s)}{\hat{T}^2(s) - s\hat{T}(s)\hat{T}'(s)}, \frac{w(s)}{\hat{T}(s)}\Big) \frac{\hat{T}(s) - s\hat{T}'(s)}{\hat{T}^2(s)} ds.$$

Definition 2.1. *The problem*

$$\min_{w \in \mathcal{A}} I(w, \hat{T})$$

will be called basic problem of the iso-calculus of variations of first kind.

Theorem 2.2. *Let $x \in \mathcal{A}$ be a solution of the basic problem of iso-calculus of variations of first kind. Then $x(\cdot)$ satisfies the iso-Euler-Lagrange equation*

$$-\hat{T}(s)\tfrac{d}{ds} D_q L(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s})) + D_x L(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))(\hat{T}(s) - s\hat{T}'(s))$$

$$+ D_q L(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))\hat{T}'(s) = 0, \quad 0 \le s \le t.$$

Proof. Let v be a smooth function defined on $[0,t]$ and $v(0) = v(t) = 0$. We define the function

$$w(\cdot) = x(\cdot) + \tau v(\cdot)$$

for $\tau \in [0,t]$. Since $x \in \mathcal{C}^2([0,t])$ and v is a smooth function then $w \in \mathcal{C}^2([0,t])$. Also,

$$w(0) = x(0) + \tau v(0) = y + 0 = y,$$

$$w(t) = x(t) + \tau v(t) = x + 0 = x.$$

Consequently $w \in \mathcal{A}$.

From the definition of $x(\cdot)$ we have

$$I(x(\cdot), \hat{T}) \leq I(w(\cdot), \hat{T}).$$

Now we define the function

$$i(\tau) :=$$

$$\int_0^t L\left(\tfrac{(x'(s)+\tau v'(s)))\hat{T}(s)-(x(s)+\tau v(s)))\hat{T}'(s)}{\hat{T}^2(s)-s\hat{T}(s)\hat{T}'(s)}, \tfrac{x(s)+\tau v(s)}{\hat{T}(s)}\right)\tfrac{\hat{T}(s)-s\hat{T}'(s)}{\hat{T}^2(s)} ds.$$

Because

$$i(0) = I(x(\cdot), \hat{T})$$

then

$$i(0) \leq i(\tau) \quad \text{for} \quad \forall \tau \in [0,t].$$

Therefore the function i has minimum at the point $\tau = 0$. From here it follows that there exists $i'(0)$ and $i'(0) = 0$. For $i'(0)$ we have

$$i'(0) = \int_0^t \sum_{i=1}^n L_{q_i}(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))\tfrac{(v'(s)\hat{T}(s)-v(s)\hat{T}'(s))}{\hat{T}^2(s)-s\hat{T}(s)\hat{T}'(s)}\tfrac{\hat{T}(s)-s\hat{T}'(s)}{\hat{T}^2(s)} ds$$

$$+ \int_0^t \sum_{i=1}^n L_{x_i}(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))\tfrac{v(s)}{\hat{T}(s)}\tfrac{\hat{T}(s)-s\hat{T}'(s)}{\hat{T}^2(s)} ds$$

$$= \int_0^t \sum_{i=1}^n L_{q_i}(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))(v'(s)\hat{T}(s) - v(s)\hat{T}'(s))\tfrac{1}{\hat{T}^3(s)} ds$$

$$+ \int_0^t \sum_{i=1}^n L_{x_i}(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))v(s)\tfrac{\hat{T}(s)-s\hat{T}'(s)}{\hat{T}^3(s)} ds$$

$$= \int_0^t \sum_{i=1}^n L_{q_i}(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))v'(s)\tfrac{1}{\hat{T}^2(s)} ds$$

$$+ \int_0^t \sum_{i=1}^n (L_{x_i}(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))(\hat{T}(s) - s\hat{T}'(s)) - L_{q_i}(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))\hat{T}'(s))\tfrac{v(s)}{\hat{T}^3(s)} ds.$$

After integration by parts in the first term in the last expression and using that $v(0) = v(t) = 0$ we get

$$i'(0) = -\int_0^t \sum_{i=1}^n \frac{d}{ds}\Big(L_{q_i}(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))\frac{1}{\hat{T}^2(s)}\Big)v(s)ds$$

$$+\int_0^t \sum_{i=1}^n (L_{x_i}(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))(\hat{T}(s) - s\hat{T}'(s)) - L_{q_i}(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))\hat{T}'(s))\frac{v(s)}{\hat{T}^3(s)}ds$$

$$= \sum_{i=1}^n \int_0^t \Big[-\frac{d}{ds}\Big(L_{q_i}\frac{1}{\hat{T}^2(s)}\Big) + L_{x_i}\frac{\hat{T}(s)-s\hat{T}'(s)}{\hat{T}^3(s)} - L_{q_i}\frac{\hat{T}'(s)}{\hat{T}^3(s)}\Big]v(s)ds = 0.$$

Since v was arbitrary chosen smooth function from the last equality we obtain that

$$-\frac{d}{ds}\Big(L_{q_i}(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))\frac{1}{\hat{T}^2(s)}\Big) + L_{x_i}(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))\frac{\hat{T}(s)-s\hat{T}'(s)}{\hat{T}^3(s)}$$

$$-L_{q_i}(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))\frac{\hat{T}'(s)}{\hat{T}^3(s)} = 0 \quad \text{for} \quad i = 1, 2, \ldots, n.$$

□

Now we define the functional

$$J(w, \hat{T}) = \hat{\int}_0^t \hat{L}^{\wedge}(\hat{w}^{\wedge\circledast}(\hat{s}), \hat{w}^{\wedge}(\hat{s}))\hat{\times}\hat{d}\hat{s}.$$

Using the introduced isofunctions, iso-differential and iso-integral in [1] we can rewrite the above functional in the following form

$$J(w, \hat{T}) = \int_0^t L\Big(\frac{w'(s)\hat{T}(s) - w(s)\hat{T}'(s)}{\hat{T}(s) - s\hat{T}'(s)}, w(s)\Big)\frac{\hat{T}(s) - s\hat{T}'(s)}{\hat{T}^2(s)}ds.$$

Definition 2.3. *The problem*

$$\min_{w\in\mathcal{A}} J(w, \hat{T})$$

will be called basic problem of the iso-calculus of variations of second kind.

Theorem 2.4. *Let $x \in \mathcal{A}$ be a solution of the basic problem of iso-calculus of variations of second kind. Then $x(\cdot)$ satisfies the iso-Euler-Lagrange equation*

$$-\frac{d}{ds}\Big(D_qL(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))\frac{1}{\hat{T}(s)}\Big) + D_xL(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))\frac{\hat{T}(s)-s\hat{T}'(s)}{\hat{T}^2(s)}$$

$$-D_qL(\hat{x}^{\wedge\circledast}(\hat{s}), \hat{x}^{\wedge}(\hat{s}))\frac{1}{\hat{T}(s)} = 0, \quad 0 \le s \le t.$$

Let us define the functional

$$K(w, \hat{T}) = \hat{\int}_0^t L^{\wedge}(\hat{w}^{\wedge\circledast}(\hat{s}), \hat{w}^{\wedge}(\hat{s}))\hat{\times}\hat{d}\hat{s}.$$

Using the introduced isofunctions, iso-differential and iso-integral in [1] we can rewrite the above functional in the following form

$$K(w,\hat{T})=\int_0^t L\Big(\frac{w'(s)\hat{T}(s)-w(s)\hat{T}'(s)}{\hat{T}(s)-s\hat{T}'(s)},w(s)\Big)\frac{\hat{T}(s)-s\hat{T}'(s)}{\hat{T}(s)}ds.$$

Definition 2.5. *The problem*

$$\min_{w\in\mathcal{A}} K(w,\hat{T})$$

will be called basic problem of the iso-calculus of variations of third kind.

Theorem 2.6. *Let $x\in\mathcal{A}$ be a solution of the basic problem of iso-calculus of variations of third kind. Then $x(\cdot)$ satisfies the iso-Euler-Lagrange equation*

$$-\hat{T}(s)\tfrac{d}{ds}D_qL(\hat{x}^{\wedge\circledast}(\hat{s}),\hat{x}^{\wedge}(\hat{s}))+D_xL(\hat{x}^{\wedge\circledast}(\hat{s}),\hat{x}^{\wedge}(\hat{s}))(\hat{T}(s)-s\hat{T}'(s))$$

$$+D_qL(\hat{x}^{\wedge\circledast}(\hat{s}),\hat{x}^{\wedge}(\hat{s}))\hat{T}'(s)=0,\quad 0\leq s\leq t.$$

We define the functional

$$M(w,\hat{T})=\hat{\int}_0^t L(\hat{w}^{\wedge\circledast}(\hat{s}),\hat{w}^{\wedge}(\hat{s}))\hat{\times}\hat{d}\hat{s}.$$

Using the introduced isofunctions, iso-differential and iso-integral in [1] we can rewrite the above functional in the following form

$$M(w,\hat{T})=\int_0^t L\Big(\frac{w'(s)\hat{T}(s)-w(s)\hat{T}'(s)}{\hat{T}^2(s)-s\hat{T}(s)\hat{T}'(s)},\frac{w(s)}{\hat{T}(s)}\Big)\frac{\hat{T}(s)-s\hat{T}'(s)}{\hat{T}(s)}ds.$$

Definition 2.7. *The problem*

$$\min_{w\in\mathcal{A}} M(w,\hat{T})$$

will be called basic problem of the iso-calculus of variations of fourth kind.

Theorem 2.8. *Let $x\in\mathcal{A}$ be a solution of the basic problem of iso-calculus of variations of fourth kind. Then $x(\cdot)$ satisfies the iso-Euler-Lagrange equation*

$$-\hat{T}(s)\tfrac{d}{ds}D_qL(\hat{x}^{\wedge\circledast}(\hat{s}),\hat{x}^{\wedge}(\hat{s}))+D_xL(\hat{x}^{\wedge\circledast}(\hat{s}),\hat{x}^{\wedge}(\hat{s}))(\hat{T}(s)-s\hat{T}'(s))=0,\quad 0\leq s\leq t.$$

3 Acknowledgements

This research has been supported in part by a grant from the R. M. Santilli Foundation.

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