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ISOMINKOWSKIAN FORMULATION OF GRAVITATION AND COSMOLOGY

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Abstract

We submit the viewpoint that, perhaps, some of the controversies in gravitation occurred during this century are not due to insufficiencies of Einstein's field equations, but rather to insufficiencies in the mathematics used for their treatment. For this purpose we treat the same equations with the novel, broader isomathematics and related isominkowskian geometry for the representation of the gravity of matter and their anti-isomorphic versions called isodual for the representation of the gravity of antimatter. We then show an apparent resolution in favor of existing relativities of controversies such as: the lack of invariance of the basic units of space and time; lack of compatibility between gravitational and relativistic conservation laws; lack of meaningful relativistic limit of gravitation; lack of classical characterization of antimatter via negative energies; and others. An apparent necessary condition for the resolution of these controversies is the abandonment of the notion of curvature used in this century in favor of a conceptual and mathematical broader notion. Available experimental verifications of the isominkowskian geometry are briefly outlined. The note ends with the identification of the main elements of the ensuing cosmology, and its intriguing implications.

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1. Introduction.

One of the most majestic achievements of this century for mathematical beauty, axiomatic consistency and experimental verifications has been the *special theory of relativity* (STR)[1]. By comparison, despite equally outstanding achievements, the *general theory of relativity* (GTR)[2] has remained afflicted by numerous problematic aspects at both classical and quantum levels, some of which are considered in these proceedings[3a](see also Refs.[3b,3c]).

The view submitted in this note is that, perhaps, some of the controversies in gravitation (as well as in cosmology) are not due to insufficiencies in current gravitational theories, but rather to insufficiencies in their mathematical treatment.

More specifically, we argue that contemporary mathematics (consisting of conventional numbers and fields, vector and metric spaces, differential calculus and functional analysis, etc.) has produced an outstanding physical consistency when applied to *relativistic* theories. Yet the same mathematics has produced unsettled controversies when applied to *gravitation.*

For an illustration, let us consider the following concrete example. As it is well known, the unit I = diag. ([1, 1, 1], 1) of the Minkowskian geometry representing in a dimensionless form the basic units of space and time, e.g., $I = Diag.([1cm, 1cm, 1cm], 1sec)$.The above basic unit is indeed invariant under the Poincaré symmetry, as well known. By comparison, we have the following:

Theorem 1[7r]. *All classical and quantum theories based on geometries with non-null curvature (thus including the Riemannian geometry) do not possess invariant units of space and time, by therefore lacking un-ambiguous applications to real measurements.*

In fact, the transition from the Minkowskian metric $\eta = Diag(1, 1, 1, -1)$ to a (3+1)-dimensional Riemannian metric g(x) is characterized by a *noncanonical* transformation $x \rightarrow x' = U \times x, U \times U^t \neq I$, for which (by ignoring the dash) $g(x) = U \times \eta \times U^t$. Corresponding theories of quantum gravity are then generally nonunitary when formulated on conventional Hilbert spaces over conventional complex fields. The lack of the invariance of the basic units then follows at both the classical and operator levels from the very definition of noncanonical and nonunitary transforms for all gravitational theories with curvature.

Theorem 1 implies rather serious ambiguities in the application of gravitational theories to actual measurements, evidently because we cannot possibly have a physically valid measure, say, of length, via a stationary meter varying in time. The hope that the problem is resolved by the joint change of the entire environment does not resolve the shortcoming because, e.g., the impasse remain for measures related to far away objects which, as such, are independent from our local environment.

We here argue that Theorem 1 is a specific manifestation of the insufficiency of the mathematics currently used for gravitation, because no corresponding shortcoming exists for flat relativistic theories.

We also argue that the shortcoming of Theorem 1 is at the foundation in a rather subtle way with a number of controversies in gravitation existing in the literature. For instance, as we shall see in this note, the achievement of a formulation of gravity with invariant basic units will automatically provide a novel unambiguous operator formulation of gravity as axiomatically consistent as relativistic quantum mechanics. After all, no axiomatically consistent operator theory of gravitation should be expected without the invariance of the basic units.

This paper is devoted to a summary presentation of alternative theories of gravitation and cosmology based on new mathematics under the condition of preserving Einstein's field equations and related experimental verifications, while possessing invariant basic units of space and time. A comprehensive study is presented elsewhere[5g].

The preceding literature can be summarized as follows: Refs.[4] deal with the class of mathematical methods used in this study, which are known under the name of *isotopies* interpreted in the Greek sense of being *axiom-preserving*; Refs.[5,6] present the particular isotopies needed for our analysis, those for numbers and fields, vector and metric spaces, algebras and geometries, etc.; Refs.[7] deal with the image of quantum mechanics under isotopies; papers[8] study the isotopies of the rotational, Lorentz and Poincaré symmetries; papers[9] study the isotopic formulation of gravity; papers[10] extend the results to antimatter; papers[11] present available applications and experimental verifications; papers[12] present other applications and developments; monographs[13] present comprehensive and independent reviews; papers[14] deal with the rather serious problems of physical consistency of q-, k- and other quantum deformations with nonunitary time evolutions on conventional Hilbert spaces over conventional fields, such as the lack of preservation of the original Hermiticity-observability, violation of causality and probability laws, besides the evident lack of invariant basic units, which prevent their use for real physical applications; papers[15] deal with the celebrated E-P-R argument which, as we shall see, is directly relevant for our analysis; papers[16] deals with the local variation of the speed of light within physical media, another topic of primary relevance for this analysis; papers[17] treat the forgotten Freud identity of the Riemannian geometry; papers[18] by P. A. M. Dirac present the first isotopies ever done for the Minkowskian geometry with intriguing connections to this study; and Refs.[19] deal with alternative approaches with significant connections with this study.

The apparently most important result of this analysis is that the isotopies of the Minkowskian geometry, first introduced by Santilli[8a] under the name of the *isominkowskian geometry,* characterize the only known geometry which, on one side, preserves the majestic invariance and other properties of the conventional Minkowskian geometry while, on the other side, admits (well behaved, symmetric and signature preserving) metrics with arbitrary functional dependence, thus including all infinitely possible Riemannian, Finslerian, non-Desarguesian and other symmetric (3+1)-dimensional metrics as particular cases.

The above universality of the isominkowskian geometry, when joined with its unique invariance properties, then voids the use of any other geometry for physical applications, according to our best knowledge at this writing.

2. Basic Assumptions of the Isominkowskian Gravity for Matter. As indicated in Sect. 1, the origin of the majestic axiomatic consistency of the special relativity is the invariance of the basic unit of the Minkowski geometry, the quantity $I = Diag[(1,1,1),1] = Diag.(I_s, I_t)$, which is the unit of the fundamemntal spacetime symmetry, the celebrated Poincaré symmetry P(3.1).

The main assumptions of this study, first presented by the author at mg7[9a], are given by:

1) factorization of all possible (3+1)-dimensional Riemannian metrics $g(x)$ into the Minkowskian metric

$$g(x) = \hat{T}(x) \times \eta, \tag{1}$$

where $\hat{T}(x)$ is a 4×4-dimensional matrix which is always positive definite (because of the local Minkowskian character of Riemann);

2) construction of a new mathematics, called for certain technical reasons *isomathematics*[5], which is based on the following positive-definite, left and right generalized unit

$$\hat{I}(x) = 1/\hat{T}(x), \tag{2}$$

3) formulate the geometry via the above isomathematics, that is, with respect to the generalized unit $\hat{I}$.

The above conditions characterize a new geometry, first introduced by Santilli[8a] in 1983 under the name of *isominkowskian geometry*, where the prefix "iso" is used to indicate the "axiom-preserving character", namely, the preservation of all original abstract axioms of the Minkowskian geometry. In fact, it was shown in the original proposal[8a] that the joint liftings

$$\eta \rightarrow \hat{\eta}(x, ...) = T(x, ...) \times \eta, I \rightarrow \hat{I}(x, ...) = 1/\hat{T}(x, ...), \tag{3}$$

preserves all original, abstract, Minkowskian axioms for all positive-definite matrices $\hat{T}(x, ...)$ irrespective of their functional dependence (a basic characteristic hereon assumed) and, as such, they characterize an isotopy.

The *isominkowskian characterization of exterior gravitation in vacuum for matter*, or *isominkowskian gravity* for short, proposed in Ref.[9a] is then the formulation based on the above assumptions 1), 2), 3).

Its main characteristic is that of *eliminating curvature* evidently in favor of a covering notion to be identified in this paper. This is due to the basic mechanism of the above assumptions, that is, *the formulation of Riemannian metrics $g(x) = \hat{T}(x) \times \eta$ with respect to a generalized unit $\hat{I} = 1/\hat{T}(x)$ which is the inverse of the term $\hat{T}(x)$ truly representing gravitation, the remaining term η being constant and flat.* Equivalently, the elimination of the conventional notion of curvature is established by te validity of the *Minkowskian* axioms for all possible *Riemannian* metrics $g(x)$.

Rather than being a drawback, the elimination of curvature has permitted rather intriguing and novel advances, such as:

A) The formulation of conventional gravitational theories in a form possessing invariant units of space and time[5e];

B) The achievement of a *universal symmetry* (rather than covariance) for all possible Riemannian line elements, called by the author *isopoincaré* *symmetry*[8] $\hat{P}(3.1)$, which is locally isomorphic to the conventional symmetry P(3.1);

C) A novel *geometric unification of the special and general relativities*[8d] in which the relativities are differentiated by the *unit*, rather than by the metric;

D) A novel *operator formulation of gravity*[7r,9] which verifies the axioms of *relativistic* quantum mechanics, thus avoiding the known problems of axiomatic consistency of conventional quantum gravity;

E) The apparently first, axiomatically consistent inclusion of gravity in unified gauge theories of electroweak interactions presented at mg8[9e,9f];

F) The consequential existence of a basically novel cosmology with rather intriguing characteristics;

G) The resolution of at least some of the controversies in gravitation that have afflicted the scientific literature of this century;

and other advances studied later on.

The main objective of this paper is a study of the isominkowskian geometry underlying the above advances. Such a geometry has been studied until now solely from the viewpoint of *Minkowskian* axioms[7,8,9]. In this paper we shall show, apparently for the first time, that the geometry also admits the machinery of the *Riemannian* axioms, such as connections, covariant derivatives, etc., thus resulting to be a symbiotic unification of both, the Minkowskian and Riemannian geometries. In turn, this will permit us to identify the covering of the conventional notion of curvature in a space which is flat at the abstract level. A detailed presentation is available in Ref.[5g].

It appears that the experimental validity of advances A)-G) above is beyond scientific or otherwise credible doubts because of the preservation unchanged of Einstein's field equations and related experimental verifications. Moreover, the isominkowskian geometry nowadays possesses a number of additional applications and experimental verifications in fields other than gravitation, including particle physics, nuclear physics, astrophysics, superconductivity, chemistry, biology and other fields (see Sect. 12 for an outline).

It should be noted that this paper is a continuation of paper[8d] on the isopoincaré symmetry and provides the foundation of papers[9e,9f] on the Iso-Grand-Unification. This study was possible thanks to the achievement of sufficient maturity of mathematical methods in memoir[5e], physical formulations in memoir[7r] and generalized symmetry principles in memoir[6c].

3. Isominkowskian Geometry for Matter.

The novel axiom-preserving mathematical methods underlying this study, the so-called *isotopies*, are rather old, dating back to the origin of the *latin squares*[4a], and have been applied to a variety of mathematical structures, such as the Jordan algebras[4b] (see Tomber's bibliography[4c]).

The particular isotopies needed for the isominkowskian geometry are given by *maps, called 'liftings', of any given linear, local, and canonical/unitary structure into all possible nonlinear, nonlocal and noncanonical/nonunitary generalizations which are however capable of reconstructing linearity, locality and canonicity/unitarity on certain generalized spaces over generalized fields.*

The latter isotopies imply the lifting of all aspects of mathematics used in physics, such as numbers and fields, metric and Hilbert spaces, algebras and symmetries, geometries, etc. They were first submitted by Santilli[5a] back in 1978, and their mathematical study has been continued by a number of authors[6,13].

The emerging new mathematics, submitted by Santilliu under the name of *isomathematics*, has reached operational maturity only recently in memoir[5e] of 1996 due to the lack of form-invariance of the equations of motion of preceding formulations. The origin of the non-invariance escaped identification for over a decade because occurring where one would expect it the least, in the *ordinary differential calculus.*

In essence, treateses in the differential calculus have no consideration for the basic unit because it is (tacitly) assumed to be the trivial number $I = +1$, thus having a trivially null differential, $dI = 0$. The actual dependence of the differential calculus from the assumed unit was first identified by Santilli[5e] on grounds that, whenever the unit has a functional dependence on the local variables, as it is the case for the generalized unit $\hat{I}(x)$ of the isominkowskian gravity, its differential is no longer null, $d\hat{I}(x) \neq 0$. Memoir[5e] therefore constructed the isotopies of the differential calculus, or *isodifferential calculus* for short. The achievement of form-invariance and sufficient maturity for applications was then consequential.

An axiomatically consistent and invariant formulation of the isominkowskian geometry is therefore submitted in this paper for the first time, thanks precisely to the recent identification of the isodifferential calculus.

The fundamental isotopies of the isominkowskian geometry, first introduced in Ref.[5a] of 1978, are:

1) The lifting of the unit of conventional theories, I = diag. (1, 1, 1, 1), into a well behaved, nowhere singular, Hermitean and positive–definite 4×4-dimensional matrix $\hat{I}$ whose elements have an arbitrary dependence on local quantities and, therefore, can depend on the spacetime coordinates x, their derivatives $\dot{x}$ and any other needed variable,

$$I = Diag.(1,1,1,1) \rightarrow \hat{I} = \hat{I}(x, \dot{x}, ...) = Diag.(\hat{I}_1^1, \hat{I}_2^2, \hat{I}_3^3, \hat{I}_4^4) > 0. \qquad (4)$$

where the diagonal form is always possible (because of the positive-definiteness of $\hat{I}$) and it is hereon assumed.

2) Jointly, all conventional associative products $A \times B$ among generic quantities A, B (numbers, vector-fields, operators, etc.) must be lifted by the *inverse* amount,

$$A \times B \rightarrow A \hat{\times} B = A \times \hat{T} \times B, \hat{I} = \hat{T}^{-1}. \qquad (5)$$

Under these assumptions $\hat{I}$ is the correct (left and right) generalized unit of the new theory,

$$\hat{I} \hat{\times} A = A \hat{\times} \hat{I} \equiv A, \forall A, \qquad (6)$$

in which case (only) $\hat{I}$ is called the *isounit* and $\hat{T}$ is called the *isotopic element.*

For consistency, the *totality* of the original theory must be reconstructed to admit $\hat{I}$ as the correct (left and right) unit. Any exception generally implies inconsistencies which often remain undetected by nonexpert in the field.

The new product $A \hat{\times} B$ is called *isoassociative* because it preserves the original associative character, i.e., $A \hat{\times} (B \hat{\times} C) = (A \hat{\times} B) \hat{\times} C$.

Note that we are studying the isotopies with an unrestricted functional dependence, including nonlinear and/or nonlocal-integral dependence, which is evidently broader than the dependence needed for the *exterior* gravitation in vacuum. As we shall see, the emerging broader formulation is naturally set for a more realistic description of *interior* gravitational problems (see Sect. 5), while preserving the abstract axioms of the exterior problem and admitting the latter as a simple particular case.

The explicit construction of the isominkowskian geometry first requires the lifting of real numbers n and field $R = R(n, +, \times)$ into the *isofield*[5d] $\hat{R} = \hat{R}(\hat{n}, \hat{+}.\hat{\times})$ which is a ring of elements $\hat{n} = n \times \hat{I}$ called *isonumbers*, equipped with the *isosum* $\hat{n}\hat{+}\hat{m} = (n + m) \times \hat{I}$ and *isoproduct* $\hat{n}\hat{\times}\hat{m} = \hat{n} \times \hat{T} \times \hat{m} = (n \times m) \times \hat{I}$ with consequential simple generalization of conventional operations (see Ref.[5d] for brevity).

Note that the *additive unit* is conventional, $\hat{0} = 0, \hat{n}\hat{+}\hat{0} = \hat{0}\hat{+}\hat{n} = \hat{n}$ and only the *multiplicative unit* is generalized. As a result, we shall continue to use the conventional symbol + for the sum and use the new symbol $\hat{\times}$ only for the multiplication.

It is easy to see that $\hat{R}$ verifies all axioms of a field and, therefore, the lifting $R \to \hat{R}$ is an isotoy. In actuality, since we have assumed that $\hat{I} > 0$, R and $\hat{R}$ *coincide* at the abstract level, *as desired.*

Next, we need the lifting of the Minkowski space space $M = M(x, \eta, R)$ with coordinates $x = (x^\mu)$, $\mu = 1, 2, 3, 4, x^4 = c_o t$, c_o being the speed of light in vacuum, unit I = diag. (1, 1, 1, 1) and metric $\eta = Diag.(1, 1, 1, -1)$ over R, into the *isominkowskian space* $\hat{M} = \hat{M}(\hat{x}, \hat{\eta}, \hat{R})$, first introduced by Santilli[8a] in 1983 and then studied in various works[5,7,8,9,10], which is characterized by the *isocoordinates* $\hat{x}^\mu = x^\mu \times \hat{I}$, *isounit*$\hat{I}$ and *isometric* $\hat{N}_{\mu\nu} = \hat{T}_\mu^\beta \times \eta_{\beta\nu} \times \hat{I}$ with *isoinvariant* over $\hat{R}$

$$(\hat{x} - \hat{y})^{\hat{2}} = (\hat{x} - \hat{y})^\mu \hat{\times} \hat{N}_{\mu\nu} \hat{\times} (\hat{x} - \hat{y})^\nu = [(x - y)^\mu \times \hat{\eta}_{\mu\nu} \times (x - y)^\nu] \times \hat{I} =$$

$$= \quad [(x^1 - y^1) \times \hat{T}_{11} \times (x^1 - y^1) + (x^2 - y^2) \times T_{22} \times (x^2 - y^2) +$$

$$+(x^3 - y^3) \times T_{33} \times (x^3 - y^3) - (x^4 - y^4) \times T_{44} \times (x^4 - y^4) \times \hat{I}. \quad (7)$$

It easy to see that $\hat{M}$ is locally isomorphic to M and the lifting $M \to \hat{M}$ is also an isotopy, as desired. Thus, M and $\hat{M}$ coincide at the abstract level by conception and construction. As a consequence, the isospace $\hat{M}$ is *isoflat*, i.e., it verifies the axiom of flatness *in isospace over the isofields.*

Alternatively, we can say that the conventional notion of curvature is no longer applicable when $\hat{\eta}$ is referred to the generalized unit $\hat{I}$, otherwise $\hat{M}$ is evidently curved when referred to the conventional unit I owing to the dependence $\hat{\eta} = \hat{\eta}(x, ...)$. However, the above occurrence is only the result of a first inspection of the novel isominkowskian spaces and a deeper, more appropriate characterization of the applicable notion of curvature will soon emerge.

Note that $\hat{M}$ and $\hat{R}$ have the *same isounit* $\hat{I}$. The conventional Minkowskian setting admitted for $\hat{I} = I$ is therefore that in which both, the Minkowski space and the base field have the same unit I = diag. (1, 1, 1,1). This implies a trivial redefinition of conventional fields hereon tacitly assumed.

Note also that: the basic invariant of the theory has the structure $[Length]^2 \times [Unit]^2$; the isocoordinates can be used in the form x^μ (rather than $\hat{x}^\mu$) and the isometric can be the simplified expression $\hat{\eta}_{\mu\nu}$ (rather than $\hat{N}_{\mu\nu} = \hat{\eta}_{\mu\nu} \times \hat{I}$) in view of the simplifications in Eqs. (7); repeated indices in isospace imply contractions via the isometric, e.g., $\hat{x}^{\hat{2}} = \hat{x}^\mu \hat{\times} \hat{x}_\mu = [x^\mu \times \hat{\eta}_{\mu\nu} \times x^\nu] \times \hat{I}$, while repeated indices between the isotopic element/isounits and other variables, e.g., $T_\mu^\nu \times \eta_{\nu\beta}$ imply an ordinary sum.

Note finally that conventional spaces have only *one* formulation, as well known. On the contrary, the isominkowski space has *two* different formulations, that as an isospace over an isofield and its *projection* on a conventional space over a conventional field. This dual formulation will soon be fundamental to understand how Einstein's field equations, which are traditionally formulated on a curved space, can be identically written in a flat isospace.

The *isominkowskian geometry*[8,7s,7t] is the geometry of isospaces $\hat{M}$. Its study can be initiated with the notions of: *isocontinuity*, introduced by Kadeisvili[6a,13b]; *isomanifold*, introduced by Tsagas and Sourlas[6b]; *isotopology*, introduced by Tsagas-Sourlas[6b] and Santilli[5e]; *isospecial functions and transforms*, introduced by Santilli[5f], Aringazin, Kirukhin and Santilli[6f], Kadeisvili[6a]; and other notions (see also the topological studies by Vacaru[6e], Aslander and Keles[6f] and others).

First, the novel isominkowskian geometry preserves all geometric properties of the conventional *Minkowskian* geometry, including the light cone[7s,8]. The axiom-preserving character of the geometry is so strong that the maximal causal speed on $\hat{M}$ remains the speed of light *in vacuum*, c_o. These Minkowskian aspects are now known and they will be merely indicated for brevity (one may consult monographs[7s,7t] for their detailed study).

A primary objective of this paper is to identify the *Riemannian* properties of the isominkowski geometry. The central tool for this task is the isodifferential calculus[5e] on $\hat{M}(\hat{x}, \hat{\mu}, \hat{R})$, which is characterized by the *isodifferentials, isoderivatives* and related properties

$$\hat{d}x^\mu = \hat{I}^\mu_\nu \times dx^\nu, \hat{d}x_\mu = \hat{T}^\nu_\mu \times dx_\nu, \hat{\partial}_\mu = \hat{\partial}/\hat{\partial}x^\mu = \hat{T}^\nu_\mu \times \partial/\partial x^\nu,$$

$$\hat{\partial}^\mu = \hat{\partial}/\hat{\partial}x_\mu = \hat{I}^\mu_\nu \times \partial/\partial x_\nu, \hat{\partial}x^\mu/\partial x^\nu = \delta^\mu_\nu, \hat{\partial}x_\mu/\hat{\partial}x^\nu = \hat{\eta}_{\mu\alpha} \times \hat{\partial}x^\alpha/\hat{\partial}x^\nu = \hat{\eta}_{\mu\nu},$$

$$\hat{\partial}x^\mu/\hat{\partial}x_\nu = \hat{\eta}^{\mu\alpha} \times \hat{\partial}x_\alpha/\hat{\partial}x^\nu = \hat{\eta}^{\mu\nu}. \tag{8}$$

Note that the original axioms must be preserved for an isotopy. Thus, the isodifferential calculus is isocommutative, i.e., commutative on $\hat{M}$ over $\hat{R}$, $\hat{\partial}_\alpha\hat{\partial}_\beta = \hat{\partial}_\beta\hat{\partial}_\alpha$. However, the same isocalculus *is not*, in general, commutative in its projection on M over R, evidently because in the latter case $\hat{T}^\alpha_\mu \times \partial_\alpha \times \hat{T}^\beta_\nu \times \partial_\beta \neq \hat{T}^\beta_\nu \times \partial_\beta \times \hat{T}^\alpha_\mu \times \partial_\alpha$ in view of the functional dependence of $\hat{T}$.

Note also the hidden *isoquotient*[5d] $A\hat{/}B = (A/B) \times \hat{I}$ and isoproduct $\hat{\partial}\hat{\times}\hat{\partial}$. Thus, by including the isoquotient, the quantity $\hat{\partial}\hat{\partial}$ should be more rigorously written $\hat{\partial}\hat{\times}\hat{\partial}$. This results in an inessential final multiplication of the expression considered by $\hat{I}$ and, as such, it will be ignored hereon for simplicity.

The entire formalism of the *Riemannian* geometry[2] can then be introduced in the *isominkowskian* space via the isodifferential calculus. This aspect is studied in details elsewhere[5g]. We here mention the: *isochristoffel symbols*

$$\hat{\Gamma}_{\alpha\beta\gamma} = \frac{\hat{1}}{2}\hat{\times}(\hat{\partial}_\alpha\hat{\eta}_{\beta\gamma} + \hat{\partial}_\gamma\hat{\eta}_{\alpha\beta} - \hat{\partial}_\beta\hat{\eta}_{\alpha\gamma}) \times \hat{I}; \tag{9}$$

isocovariant differential

$$\hat{D}\hat{X}^\beta = \hat{d}\hat{X}^\beta + \hat{\Gamma}^\beta_{\alpha\gamma}\hat{\times}\hat{X}^\alpha\hat{\times}\hat{d}\hat{x}^\gamma; \tag{10}$$

isocovariant derivative

$$\hat{X}^\beta_{\hat{|}\mu} = \hat{\partial}_\mu\hat{X}^\beta + \hat{\Gamma}^\beta_{\alpha\mu}\hat{\times}\hat{X}^\alpha; \tag{11}$$

isocurvature tensor,

$$\hat{R}^\beta_{\alpha\gamma\delta} = \hat{\partial}_\beta\hat{\Gamma}^\beta_{\alpha\gamma} - \hat{\partial}_\gamma\hat{\Gamma}^\beta_{\alpha\delta} + \hat{\Gamma}^\beta_{p\delta}\hat{\times}\hat{\Gamma}^p_{\alpha\gamma} - \hat{\Gamma}^\beta_{p\gamma}\hat{\times}\hat{\Gamma}^p_{\alpha\delta}; \tag{12}$$

and similarly for all other properties.

The preservation on $\hat{M}$, this time, of the *Riemannian* properties is illustrated by the following:

Lemma 1[7s]: *The isocovariant derivatives of all isometrics on $\hat{M}$ over $\hat{R}$ are identically null,*

$$\hat{\eta}_{\alpha\beta\hat{|}\gamma} \equiv 0, \alpha, \beta, \gamma = 1, 2, 3, 4. \tag{13}$$

This illustrates that the Ricci Lemma also holds under the *Minkowskian* axioms as well as for an *arbitrary* functional dependence of the metric, evidently when treated with the novel isomathematics. The understanding is that the same results are simply impossible with conventional mathematics.

A similar occurrence holds for all other properties, including the five identities of the Riemannian geometry (where the fifth is the forgotten *Freud identity*[17], as studied in details elsewhere[5g].

In summary, the isominkowskian geometry characterizes a new notion of curvature, that of Eq.s (12), here called *pseudocurvature*, to illustrate the fact that conventional curvature is in reality absent because of the structure of the basic invariant $[Length]^2 \times [Unit]^2$ with metric $\hat{T} \times \eta$ referred to the unit $\hat{I} = 1/\hat{T}$. The term "pseudocurvature" is also introduced to achieve epistemological compatibility with a space whose abstract structure is flat.

4. Classical Exterior Isominkowskian Gravity for Matter.

We are now equipped to present, apparently for the first time, the classical equations of our isominkowskian formulation of gravity for matter, here called *isoeinstein equations* on $\hat{M}$ over $\hat{R}$, which can be written

$$\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{\hat{1}}{2}\hat{\times}\hat{N}_{\mu\nu} \times \hat{R} = \hat{k}\hat{\times}\hat{\tau}_{\mu\nu}, \tag{14}$$

where $\hat{\tau}_{\mu\nu}$ is the source *isotensor* on $\hat{M}, \frac{\hat{1}}{2} = \frac{1}{2} \times \hat{I}, \hat{N}_{\mu\nu} = \hat{\eta}_{\mu\nu} \times \hat{I} = g_{\mu\nu} \times \hat{I}, \hat{k} = k \times \hat{I}$ and k is the usual constant.

The compatibility of the above equations with available experimental evidence on gravitation is discussed later on in Sect. 12.

The novel notion of pseudocurvature characterizing the isoeinstein equations can be best illustrated via the fact that *isominkowsline elements co-*

incide with the Minkowskian ones. For instance, the isominkowskian representation of Schwarzschild's gravitation[2d] is given by

$$\hat{ds}^{\hat{2}} = \hat{dr}^2 + r^2 \times (\hat{d\theta}^2 + sin^2\theta \times \hat{d\phi}^2) - \hat{dt}^2 \times c_0^2,$$
$$\hat{dr} = \hat{I}_s \times dr, \hat{dt} = \hat{I}_t \times dt, \hat{I}_s = (1 - 2M/r)^{-1}, \hat{I}_t = 1 - 2M/r, \qquad (15)$$

with an inessential extension to a fully isotopic formalism, inclusive of isoproducts, etc., which is here ignored (because requiring the isotrigonometry[5f] we cannot possibly review for brevity).

The lack of the conventional curvature in formulation (15) is evident and this illustrates the notion of pseudocurvature introduced above.

Almost needless to say, isorepresentation (15) is the simplest possible one introduced here merely for illustrative purposes. In fact, a more adequate isorepresentation would be that of the Schwarzschild gravity in *nondiagonal* form, as available, e.g., in Ref.[2f], whose existence is assured by the universality of the isominkowskian representation of all possible (3+1)-dimensional Riemannian metrics.

We here assume the reader is aware from Ref.[9e,9f] of the *necessity* of the elimination of the conventional curvature to achieve an axiomatically consistent inclusion of gravitation in unified gauge theories, besides the need to resolve the impasse of Theorem 1.

To be explicit in this fundamental point, the conventional formulation of Schwarzschild's gravitation *cannot* be consistently included in unified gauge theories of electroweak interactions, again, because the former is based on a curved space, while the latter are based on a flat space. The unification of Ref.[9e,9f] is based on the reduction of the former to the axiomatic structure of the latter, as manifestly expressed by isorepresentation (15).

Note that the other approach, that of generalizing gauge theories of electroweak interactions into a form based on a curved spacetime, first of all, has escaped throughout this century attempts initiated by Einstein and, second, it is faced with the rather severe problematic aspects of physical character of all quantum theories of gravity on a curved manifold[3a,3a,7r,7t,14], including the physical shortcomings of Theorem 1.

Needless to say, no claim of uniqueness is here implied, as standard in truly scientific inquiries; tother approaches are indeed possible; they have indeed been proposed elsewhere[19]; their study is indeed encouraged; and their connections with the isominkowskian approach are intriguing, although they cannot regrettably be studied here for brevity.

5. General Isominkowskian Geometry for Matter.

As stressed earlier, the isotopies leave unrestricted the functional dependence of the isometric. Its sole dependence on the coordinates is therefore a *restriction* which has been used so far for a representation of *exterior gravitation in vacuum.* The above ocurrence has a number of implications which can be summarized as follows (see monograph[7t] for details).

5.A. Classical Isominkowskian Geometrization of Interior Gravitation. In exterior problems, astrophysical bodies can be well approximated as *massive points* as a necessary condition for the very applicability of a local and (first-order) Lagrangian geometry.

Interior problems such as gravitational collapse are not composed by isolated points, and are more realistically composed by extended and hyperdense hadrons in conditions of complete mutual penetration and compression in large numbers into very small regions of smale. The latter conditions imply the emergence of the most general conceivable equations which are arbitrarily nonlinear in all variables, arbitrarily nonlocal and clearly non-Lagrangian (i.e., violating the integrability conditions for the existence of a Lagrangian[5c]).

Independently from the problems caused by the lack of invariance of the basic units of space and time, the Riemannian geometry cannot possibly be *exactly* valid for the latter general conditions, evidently in view of its strictly local and Lagrangian character. As a consequence, all results on interior gravitational problems based on the Riemannian geometry (e.g., theorems on singularities, black holes, etc.) should be considered only as an *approximation* of physiucal reality.

The isominkowskian geometry in its most general possible realization permits a representation of interior gravitational problems which, when compared to that permitted by the Riemannian geometry, is not only invariant, but also more realistic because admitting precisely the desired arbitrary dependence of the isometric.

In fact, in the general case we have isometrics $\hat{\eta} = \hat{T}(x, \dot{x}, ...) \times \eta$ which can represent *interior gravitation problems* with a well behaved but otherwise *unrestricted nonlinearity in the velocities* and other variables, as well as *nonlocality* (e.g., characterized by surface or volume integrals), as expected in realistic interior models, e.g., of collapsing stars.

Moreover, the isominkowskian geometry reconstructs linarity, locality and the (first-order) Lagrangian characters on isospaces over isofields, as a necessary condition to be an axiom-preserving image of the conventional Minkowskian geometry, because all possible nonlinear, nonlocal and non-Lagrangians terms are embedded in the isounit (see monograph[7t] for details).

Note that the addition of interior gravitational problems occurs without altering the axioms of the exterior problem in vacuum. This evidently permits a geometric unification of exterior and interior gravitational problems which are solely differentiated by the functional dependence of the isounit.

5.B. Direct Geometrization of Arbitrary Speeds of Light. As it is well known, the "universal constancy of the speed of light" is a philosophical abstractions, because in the physical reality light has a local speed $c = c_o/n$, where n is the familiar index of refraction. As an example, light has a speed in our atmosphere locally varying with the density, and then different speeds in water, glasses, oil, etc.

One of the first studies on locally varying speeds of light $c < c_o$ was conducted Lorentz[16a] who, as typical for founders of new insights, also identified the limitation of its celebrated symmetry (see also the quotation by Pauli[16b] of this little known historical work by Lorentz).

More recently, apparent experimental evidence has been identified on the complementary case $c > c_o$, such as the measures of photons traveling on certain guides at speeds higher than that in vacuum[16c,16d]; the expulsion of matter in astrophysical explosions at speeds apparently higher than that of light in vacuum[16e,16f,16g]; solutions of conventional wave equations with arbitrary speeds[16h]; and other cases (see[16i,16j] and references quoted therein).

The above occurrences establish the need for systematic geometric studies of *arbitrary* speeds $c = c_o/n$ of electromagnetic waves, with the speed c_o in vacuum as a particular case, which cannot evidently be conducted via the conventional formulation of the special and general relativities because they were conceived and developed solely for conditions in vacuum.

The conventional approach of reducing the propagation of light within physical media to photons moving in vacuum and scattering among molecules is no longer credible for various reasons, such as: 1) a *classical* electromagnetic wave, say, with one meter in wavelength propagating in our atmosphere cannot be ctredible reduced to photons in *second quantization* without a prior classica-geometricl representation; 2) the reduction evidently prevents any quantitative study of superluminal speeds as experiemntally detected; 3) the reduction eliminates altogether rather crucial physical characteristics, such as the *inhomogeneity and anisotropy* of physical media which, as we shall see, have experimentally measurable predictions; and other insufficiencies.

Our isominkowskian geometry has been conceived precisely for the *direct geometrization* of interior conditions[8a], that is, their representation directly via the isometric. In fact, the direct geometrization of arbitrary speeds of electromagnetic waves is simply permitted by the diagonal isotopic element[7t]

$$\hat{T}_{\mu\mu} = \hat{T}^{Int.}_{\mu\mu} \times \hat{T}^{Ext}_{\mu\mu}, \hat{T}^{Int.}_{\mu\mu} = 1/n^2_\mu$$

$$\hat{x}^2 = (x_1^2/n_1^2 + x_2^2/n_2^2 + x_3^2/n_3^2 - t^2 \times c_o^2/n_4^2) \times \hat{I}, \tag{16}$$

where there is no summation; $\hat{T}^{Ext}_{\mu\mu}$ is the conventional, exterior, gravitational isotopic element of the preceding section; $n_4 = n$ is the ordinary index of refraction; and the $n_k's$ emerge from the spacetime symmetrization of n, e.g., via the use of the conventional Lorentz transforms (or, more appropriately, their isotopic formulation studied in Sect. 8).

As an illustration, under the assumption of a space isotropy for which $n_1 = n_2 = n_3 = n_3 = n_s$, the interior isominkowskian formulation of Schwarzschild's metric, Eq.s (15), is lifted into a form whose projection in ordinary spacetime is given (in the (1+1)-case) by

$$\hat{\eta}^{Int} = Diag.[1/(1 - 2M/r) \times n_s^2, (1 - 2M/r) \times c_o^2/n_4^2] \tag{17}$$

with evident extension to three space dimensions, thus providing a "direct geometrization" of local speeds of electromagnetic waves, i.e., its geometrization directly via the metric.

Moreover, the isominkowskian geometry permits the direct geometrization of the *inhomogeneity* of physical media (e.g., due to a local variation of the density easily represented via a radial dependence of the isotopic element $\hat{T}^{Int}$), as well as their possible *anisotropy* (e.g., due to an intrinsic angular momentum which creates a preferred direction in the physical medium, and *not* in the underlying vacuum, easily representable via a factorization in $\hat{T}^{Int}$ of the preferred direction, or via different values of n_s and n_4).

The advantage of the above treatment of interior gravitation over conventional lines is then evident, e.g., for more realistic studies of the region outside gravitational horizons where the speed of electromagnetic waves *is not* c_o, but rather $c = c_o/n_4 < c_o$. In fact, the region considered is not empty, but it is instead filled up with huge and hyperdense chromospheres.

The above direct geometrization of local speeds of electromagnetic waves is also preferable over the conventional treatment via photons scattering among molecules for various reasons, such as:

1) the former is purely classical, while the latter is valid only in second quantization, thus implying the yet unsolved problematic aspects of quantum gravity;

2) the former implies a geometrization of the inhomogeneity and anisotropy of physical media with predicted novel contributions for the redshift (see later on Sect. 12), while the latter reduces the event to photons scattering in empty space, thus being manifestly unable to represent the inhomogeneity and anisotropy of physical media;

3) The former implies no restriction on the local value of the speed, thus permitting one of the most important predictions of the isominkowskian geometry (see Sect. 12), while the latter restricts aprioristically the maximal causal speed to that in vacuum.

5.C. Isominkowskian Classification of Physical Media. Studies on interior problems[5e,7t] appear to suggest the general rule according to which *physical media alter the geometry of empty space.* In fact, no alteration of the speed of light in vacuum is possible without an alteration of space-time itself, and the same occurs under the inhomogeneity and anisotropy of physical media.

The isominkowskian geometrization of physical media was constructed because: 1) it permits the direct geometrization indicated above; 2) it allows the preservation of Einstein's axioms under locally varying speeds of light; and 3) it is "direct universal"[11k], that is, it applies for all infinitely possible, signature-preserving deviations from the Minkowskian settings (universality) in the fixed frame of the experimenter (direct universality).

In particular, the isominkowskian geometry has permitted the classification of physical media into *three different Classes each having three different*

Types (see monograph[7t], Sect. 8.5). In fact, for the case of space isotropy with $n_1 = n_2 = n_3 = n_3 = n_4$, we have the following classification:

Class I: $n_s = n_4$; Type 1 $n_4 = 1$, Type 2 $n_4 > 1$, Type 3 $n_4 < 1$;

Class II: $n_s < n_4$; Type 4 $n_4 = 1$, Type 6 $n_4 > 1$, Type 6 $n_4 < 1$;

Class III: $n_s > n_4$; Type 7 $n_4 = 1$, Type 8 $n_4 < 1$, Type 9 $n_4 < 1$.

The above classification is significant because the knowledge of the isominkowskian Class of a given medium permits the identification of some main characteristics, such as the behaviour of the frequency shift. In fact, for Clas I we have no change in the conventional Doppler's shift, for Class II we have an increase in redshift called ***isoredshift***, and for Class III we have an increase of the blueshift called ***isoblueshift*** (see Sects. 8, 12 and 14 for more details and monograph[7t] for a comprehensive treatment).

With the understanding that the studies are at their initiation and so much remains to be finalized, we should also mention that examples of all nine isominkowskian media have alrady been identified. Type 1 is evidently the ordinary vacuum; Type 2 characterizes water and other homogeneous and isotropic media of low density; Type 3 charactrizes the recently identified superluminal solutions of wave equations[16e]; Type 4 charactrizes a first type of anisotropic propagation of light in the universe; Type 5 characterizes planetary atmosphere or astrophysical chromospheres of low density; Type 6 characterizes a form of superconductivity; Type 7 characterizes a second form of anisotropioc propagation of light in the universe (complementary to Type 4); Type 8 characterizes astrophysical chromosphere of high density; and Type 9 appear to hold universally for all media with a density beginning with that of Kaons[11e].

6. Explicit Construction of Classical Isominkowskian Gravity and its Form-Invariance.

It is significant to indicate that the above exterior and interior isominkowskian gravity admits a rather simple construction in all its aspects, including those of the underlying isomathematics.

Recall from Sect. 1 that Riemannian metrics $g(x)$ are noncanonical images of the Minkowskian one η. Recall also that the component of the Riemannian metric $g(x) = \hat{T}(x) \times \eta$ truly representing gravitation is precisely the deviation $\hat{T}(x)$ from η. The methods here proposed for the explicit construction of the isominkowskian gravity is therefore given by *the systematic application to all aspects of the Minkowskian geometry of a noncanonical* *transformation selected in such a way that*

$$U \times U^t = \hat{I}(x) = 1/\hat{T}(x) \neq I, \tag{18}$$

where t represents transpose.

As a specific example, the construction of the isominkowskian representation of the Schwarzschild gravity requires the selection of a noncanonical transformation according to rule (18) where the isounit acquires values (15).

Note that the above rule is applicable for both *exterior and interior* gravitational models, and it is readily extendable to *nondiagonal* realizations of the metric. As such, the methods herein considered is quite general.

It is easy to see that, when defined on conventional spaces over conventional fields, the noncanonical image of Minkowski geometry yields precisely the Riemann geometry. However, such a setting activates Theorem 1, thus lacking invariant units of space and time.

A *necessary* condition for the resolution of the above impasse is that noncanonical transform (18) is applied to the *totality* of the formalism of the Minkowskian geometry, beginning with the underlying *numbers,* and then passing to spaces, metrics, etc. Then, the emerging structure is precisely the isominkowskian geometry which, as we shall see shortly, does indeed resolve the problematic aspects of Theorem 1.

In fact, under transform (18) we have the following liftings: the trivial multiplicative unit I of $R = R(n, +, \times)$ is lifted into the isounit $I \rightarrow \hat{I} = U \times I \times U^t$; the additive unit of R remains unchanged, $0 \rightarrow \hat{0} = 0 \times U \times U^t = 0$; numbers are lifted precisely into isonumbers $n \rightarrow \hat{n} = U \times n \times U^t = n \times \hat{I}$; the product is precisely lifted into the isoproduct $n \times m \rightarrow \hat{n} \hat{\times} \hat{m} = \hat{n} \times \hat{T} \times \hat{m}$, where $\hat{T}$ has the correct form $\hat{T} = \hat{I}^{-1}$ and the correct symmetry $\hat{T} = \hat{T}^t$; and, therefore, the original field is lifted into the needed isofield, $R(n, +, \times) \rightarrow \hat{R}(\hat{n}, \hat{+}, \hat{\times})$.

Similarly, the Minkowski invariant is lifted precisely into the isominkowskian form (7),

$$x^2 \rightarrow x'^2 = U \times x^2 \times U^t = [(x^t \times U^t) \times (U^{t-1} \times U^{-1}) \times \eta \times (U \times x)] \times (U \times U^t) = \hat{x}^t \times \hat{N} \hat{\times} \hat{x}, \tag{19}$$

which clarifies that the coordinates x on M are lifted into the form $U \times x$ on $\hat{M}$ and then in the isoform, i.e., multiplied by $\hat{I}$.

It is an instructive exercise for the reader interested in learning the isotopic techniques to prove that the above noncanonical liftings yield all remaining aspects of the isominkowskian geometry.

As recalled in Sect. 1, the majectic consistency of the special relativity originates from its form-invariance, while one of the problematic aspects of the Riemannian formulation of gravity is its lack of form-invariance, as emphasized by Theorem 1. As an example, it is well known that the Minkowski metric is invariant under the applicable transformation theory, that via canonical transforms, while the Riemannian metrics are not invariant under the applicable transformation theory, this time given by noncanonical transforms.

Another main reason for the construction of the novel isomathematics and the isominkowskian formulation of gravity is that of resolving the above impasse and achieving an *invariant classical representation of gravity.*

In fact, once constructed via the above noncanonical lifting, isominkowskian gravity is indeed invariant, *provided that the transformation theory is formulated via the isomathematics.* This essentially requires that any additional noncanonical transform be written in the *isocanonical form*

$$W \times W^t = \hat{I}(x), W = \hat{W} \times \hat{T}^{1/2},$$
$$W \hat{\times} W^t = \hat{W} \hat{\times} \hat{W}^t = \hat{W}^t \hat{\times} \hat{W} = \hat{I}. \tag{20}$$

Isominkowskian gravity is then form-invariant,

$$[\hat{I} \rightarrow \hat{I}' = \hat{W} \hat{\times} \hat{I} \hat{\times} \hat{W}^t = \hat{I},$$
$$[\hat{A} \hat{\times} \hat{B} \rightarrow \hat{W} \hat{\times} (\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{B} = \hat{A}' \hat{\times} \hat{B}',$$
$$\hat{W} \hat{\times} \hat{N}_{\mu\nu} \hat{W}^t = \hat{\eta}_{\mu\nu} \times \hat{W} \hat{\times} \hat{I} \hat{\times} \hat{W}^t = \hat{N}_{\mu\nu}, etc. \tag{21}$$

The invariant of all other structures then follows, including the invariance of the isoconnections, covariant isoderivatives, etc. Note that the above invariance implies not only the preservation of the *form*, but also the preservation of the *numerical value* of the isounit.

The reader should meditate a moment here and note the complete lack of any similar form-invariance in the Riemannian formulation of gravity on curved space-time.

The reader should also note that the above study on form-invariance is only preliminary. The true form-invariance of the isominkowskian gravity follows from its universal symmetry, the isopoincaré symmetry studied below.

Note that the use of noncanonical transforms $W \times W^t = \hat{I} \neq \hat{I}$ implies the transition to different physical systems and, as such, they should not be considered for the form invariance of a gravity with a given $\hat{I}$ (otherwise it would be like attempting to study the form-invariance of Minkowskian settings with a transformation $W \times W^t = \hat{I} \neq I$).

7. Operator Isominkowskian Gravity for Matter.

As indicated earlier, the isominkowskian formulation of gravity permits a *geometric unification of the special and general relativities into one single relativity, the isospecial relativity*[8], where for $\hat{I} = I = diag.(1,1,1,1)$ we have the special and for $\hat{I} = \hat{I}(x)$ we have the general.

One of the implications of such a *classical* unification is that it permits the identification of a unique *operator* counterpart, called *operator isogravity* (OIG), as first submitted by Santilli at mg7[9a].

It should be indicated from the outset that OIG is structurally different than the conventional *quantum gravity* (QG)[20] on numerous grounds, e.g., because OIG and QG have *different units, Hilbert spaces, and fields.* In particular, the word "operator" in OIG is suggested to keep in mind the differences with "quantum" mechanics (as it should also be the case for QG).

To identify the explicit form of OIG, we note that the operator image of the *noncanonical* transform (18) is a *nonunitary* transform on a conventional Hilbert space $\mathcal{H}$ over the complex field $C = C(c, +, \times)$. The isounit of the operator theory is therefore assumed to be

$$\hat{I}(x, p, \Psi, \partial\Psi, ...) = U \times U^\dagger = \hat{I}^\dagger, \hat{T} = (U \times U^\dagger)^{-1} = T^\dagger = \hat{I}^{-1}, \tag{22}$$

where we have indicated the most general possible functional dependence for the operator case, thus including nonlinearity in the wavefunctions and its derivatives. The representation of exterior gravity occurs by *restricting* the above general functional dependence as per Eqs. (1) and (2). Then, OIG requires the isotopies of the *totality* of *relativistic quantum mechanics*

(RQM) resulting in a formulation known as *relativistic hadronic mechanics* (RHM)[7r].

Besides the preceding isotopies $R \rightarrow \hat{R}$ and $\hat{M} \rightarrow \hat{M}$, RHM is based on the lifting of the Hilbert space $\mathcal{H}$ with states $|\Psi>, |\Phi>, \ldots$ and inner product $<\Phi|\Psi> \in C(c, +, \times)$ into the *isohilbert space*[7d,7r] $\hat{\mathcal{H}}$ with *isostates, isoproduct* and *isonormalization*

$$[|\hat{\Psi}>= U \times |\Psi>, |\hat{\Phi}>= U \times |\Phi>, \ldots,$$

$$<\hat{\Phi}\hat{|}\hat{\Psi}> = U \times <\Phi|\Psi> \times U^{\dagger} =<\hat{\Phi}| \times \hat{T} \times |\hat{\Psi}> \times \hat{I},$$

$$<\hat{\Psi}| \times \hat{\hat{T}} \times |\hat{\Psi}> = I, \tag{23}$$

defined on the isofield $\hat{C} = \hat{C}(\hat{c}, \hat{+}, \hat{\times})$.

We then have the *iso-four-momentum operator*

$$\hat{p}_{\mu}\hat{\times}|\hat{\Psi}>= -\hat{i}\hat{\times}\hat{\partial}_{\mu}|\hat{\Psi}>= -i \times \hat{T}_{\mu}^{\nu} \times \partial_{\nu}|\hat{\Psi}>; \tag{24}$$

and *fundamental isocommutation rules*

$$[\hat{x}_{\mu}, \hat{p}_{\nu}] = U \times [x_{\mu}, p_{\mu}] \times U^{\dagger} = \hat{x}_{\mu} \times \hat{T} \times \hat{p}_{\nu} - \hat{p}_{\nu} \times \hat{T} \times \hat{x}_{\mu} = \hat{i}\hat{\times}\hat{N}_{\mu\nu} \tag{25}$$

The (nonrelativistic) *isoheisenberg equation*, first submitted in Ref.[7a], and the *isoschroedinger equation*, first submitted in Refs.[7c,7d], can be written in terms of the isodifferential calculus of Ref.[5e]

$$\hat{i}\hat{\times}\hat{d}A/\hat{d}t = i \times \hat{I}_t \times dA/dt = [A \hat{,} H] = A \times \hat{T}_s \times H - H \times \hat{T}_s \times A, \ \hat{I} = \hat{I}_s \times \tilde{I}_t,$$

$$\hat{i}\hat{\times}\hat{\partial}_t|\hat{\Psi}>= i \times \hat{I}_t \times \partial_t|\hat{\Psi}>= H\hat{\times}|\hat{\Psi}>=$$

$$= H \times \hat{T}_s \times |\hat{\Psi}>= \hat{E}\hat{\times}_s|\hat{\Psi}>= (E \times \hat{I}_s) \times \hat{T}_s \times |\hat{\Psi}> \equiv E \times |\hat{\Psi}>. \tag{26}$$

Note that the final numbers of the theory are conventional. We also have the lifting of expectation values into the form

$$\hat{<}A\hat{>} =<\hat{\Psi}| \times \hat{T} \times A \times \hat{T} \times |\hat{\Psi}> / <\hat{\Psi}| \times \hat{T} \times |\hat{\Psi}> \tag{27}$$

and the compatible liftings of the remaining aspects of RQM here omitted for brevity (the interested reader may consult Refs.[7r,7t]).

It has been proved that *RHM preserves all conventional properties and axioms of RQM*[(7r)]. In particular, *isohermiticity coincides with conventional Hermiticity,* $H^{\hat{\dagger}} \equiv H^{\dagger}$, as one can verify. As a result, all quantities which are originally observables for RQM remain so for RHM. Similarly, *the isoeigenvalues of isohermitean operators are isoreal,* thus conventional (because of the identity $\hat{E}\hat{\times}|\hat{\Psi}> \equiv E \times |\hat{\Psi}>$).

The reader is here suggested to meditate a moment on the above results of OIG and compare them with the corresponding *difficulties* of QG.

It has also been proved that *RHM preserves all physical laws of RQM*[7r]. This additional important property can be verified by showing that, via the use of isocommutators (25) and isoexpectation values (27), *the isouncertainties coincide with the conventional uncertainties.* It is also easy to prove the *exact preservation of Pauli's exclusion principle* evidently due to the preservation of the spin eigenvalue $1/2$[7r].

The reader should here compare the *validity* of conventional quantum laws for OIG with the *departures* from the same implied by the nonunitary structure of QG.

In a way fully similar to the corresponding classical case,*RHM is form invariant under non-unitary transforms* $U \times U^{\dagger} = \hat{I} \neq I$, provided that they are written in the *isounitary form* $U = \hat{U}\times\hat{T}^{1/2}$, $U\times U^{\dagger} = \hat{U}\hat{\times}\hat{U}^{\dagger} = \hat{U}^{\dagger}\hat{\times}\hat{U} = \hat{I}$. In fact, we have the invariance of: the isounit, $\hat{I} \rightarrow \hat{I}' = \hat{U}\hat{\times}\hat{I}\hat{\times}\hat{U}^{\dagger} \equiv \hat{I}$, the isoassociative product , $\hat{U}\hat{\times}(A\hat{\times}B)\hat{\times}\hat{U}^{\dagger} = A'\hat{\times}B'$; etc; and the same occurs for all other properties (including causality).

It should be stressed that RHM *is not a new theory, but merely a new realization of the abstract axioms of RQM.* In fact, RHM and RQM coincide at the abstract, realization–free level where all distinctions are lost between I and $\hat{I}, R$ and $\hat{R}$, M and $\hat{M}, \mathcal{H}$ and $\hat{\mathcal{H}}$, etc. Yet, RHM is inequivalent to RQM evidently because the two theories are related by a *nonunitary* transform. Also, RHM is broader than RQM; it recovers the latter identically for $\hat{I} = I$; and can approximate the latter as close as desired for $\hat{I} \approx I$.

Note that RHM is highly *nonlinear* (in the wavefunctions and their derivatives) because of the unrestricted functional dependence of the isounit for which the explicit form of the isoeigenvalue equation reads $H\hat{\times}| >= H(t,r,p) \times \hat{T}(t,r,p,\Psi,...)| >= E| >$. RHM is also *nonlocal* in the sense of admitting integral effects, e.g., representing deep wave-overlappings. Finally, RHM is *nonhamiltonian* (and, therefore, nonunitary), in the sense of admitting novel nonhamiltonian interactions represented by $\hat{T}$, besides all conventional interactions represented by H.

However, RHM is *isolinear, isolocal and isohamiltonian*[7r], namely, it reconstructs linearity, locality and unitarity in isospaces over isofields[7r], as an evident *necessary* condition to preserve quantum axioms and physical laws, as well as to achieve abstract unity between RHM and RQM.

The representation of systems via RQM requires the knowledge of only *one* quantity, the Hamiltonian H, under the generally tacit assumption $h = 1$. The representation of systems via RHM requires instead the knowledge of *two* quantities, the conventional Hamiltonian H(t, r, p) and the isounit $\hat{I}(x, p, \Psi, ...)$, where the former represents all action-at-a-distance, potential interactions, while the latter represent all interactions and effects which are beyond the representational capability of the Hamiltonian.

Note that, in view of the unrestricted functional dependence of $\hat{T}$, *RHM is directly universal*, i.e., capable of representing all infinitely possible (well behaved) linear and nonlinear, local and nonlocal, and Hamiltonian as well as nonhamiltonian operator systems directly in the fixed frame of the experimenter.

Yet another property important for this introductory study is the *axiomatic and physical consistency of RHM*, as guaranteed beyond scientific or otherwise credible doubts by *the verification of the abstract axioms of RQM.*

The above consistency should be compared with the litany of problematic aspects of other deformations of quantum mechanics, including quantum gravity, as studied in details in Refs.[14]. As an example, conventional nonlinear theories $H(t, r, p, \Psi, ...) \times | >= E \times | >$ are known to violate the superposition principle, thus lacking a physically meaningful application to composite systems.

By comparison, RHM fully verifies the superposition principle in isospace over isofields, trivially, because of the reconstruction at that level of linearity, thus permitting, apparently for the first time, physically consistent applications to composite systems under unrestricted nonlinearities. Moreover, all conventional nonlinear models can be *identically* rewritten in the axiomatically consistent isolinear form via the simple re-definition $H(t, r, p, \Psi, ...) \times | >= H_o(t, r, p) \times \hat{T}(\Psi, ...) \times | >= E \times | >$, i.e., via the embedding of all nonlinear terms in the isounit. A form-invariant description of the same is then consequential.

It is easy to see that the entire formulation of RHM of memoir[7r] can be specialized into the desired OIG via the mere selection of the desired gravitational isounit $\hat{I}$ according to rules (1) and (2). Since the functional dependence of the latter is unrestricted in RHM, OIG can represent *exterior and interior operator gravity.* The study of explicit examples is here left to the interested reader for brevity.

Note that OIG resolves the historical impasse that has prohibited the achievement until now of a consistent quantum theory of gravity, the fact that, on one side, RQM requires a well defined Hamiltonian for the consistent description of physical systems, while gravitation admits an *identically null* Hamiltonian[2].

By conception *OIG does not require the representation of gravity via a Hamiltonian,* and that is the very reason why it was submitted[9a]. In OIG gravitation is represented via $\hat{I}$, while H represents all conventional (e.g., electromagnetic) interactions.

An aspect of OIG which is important for this introductory study is the "hidden" character of OIG. To see it, let us first note that gravity is represented again with the unit of the theory which, as such, verifies all axioms of a unit, $\hat{I}^{\hat{n}} = \hat{I}\hat{\times}\hat{I}\hat{\times}...\hat{\times}\hat{I} = \hat{I}$, $\hat{I}^{1/2} = \hat{I}$, $\hat{I}\hat{/}\hat{I} = \hat{I}$, $\hat{I}\hat{\times}A = A\hat{\times}\hat{I} = A$, etc. Moreover, $\hat{I}$ is the fundamental invariant of the theory, $i\hat{d}\hat{I}/\hat{d}t = \hat{I}\hat{\times}H - H\hat{\times}\hat{I} \equiv 0$.

The "hidden character of OIG then follows from the property that *the isoexpectation value of the isounit recovers the conventional quantum unit,*

$$\hat{<}\hat{I}\hat{>} =< \hat{\Psi}| \times \hat{T} \times \hat{T}^{-1} \times \hat{T} \times |\hat{\Psi} > / < \hat{\Psi}| \times \hat{T} \times |\hat{\Psi} >= I. \tag{28}$$

It then follows that the proposed OIG is a "completion" of RQM much along the celebrated E-P-R argument[15a] for which van Neumann's theorem[15b] and Bell's inequalities[15c] do not apply in view of the nonunitary character of the theory. Alternatively, we can say that *OIG is an explicit and concrete realization of the theory of "hidden variables"* λ which are actually realized via the *operator* $\hat{T}$. For a detailed study of these aspects one may consult Ref.[15e].

Equivalently, the "hidden" character of OIG can be seen from the fact that it originates from the following *hitherto unknown degrees of freedom of the Minkowskian and Hilbert spaces* (where n is a scalar),

$$[x^2 \times I = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I = [x^\mu \times (n^{-2} \times \eta_{\mu\nu}) \times x^\nu] \times (n^2 \times I) = (x^\mu \times \hat{\eta}_{\mu\nu} \times x^\nu) \times \hat{I},$$

$$< \Phi | \times | \Psi > \times I = < \Phi | \times n^{-2} \times | \Psi > \times (n^2 \times I) = < \Phi | \times \hat{T} \times | \Psi > \times \hat{I}, \quad (29)$$

It should not be surprising that the above new symmetries of well known spaces escaped detection throughout this century, because they required the prior discovery of *new numbers*, those with *arbitrary units.*

In summary, the viewpoint submitted by Santilli at mg7[9a] is that, *perhaps, an axiomatically consistent operator version of gravity "always" existed. It did creep in un-noticed because embedded where nobody looked for, in the "unit" of quantum mechanics.*

8. The Poincaré-Santilli Isosymmetry for Matter.

All the preceding results at the exterior and interior, as well as classical and operator levels can be uniquely and unambiguously derived from the *isotopies of the Poincaré symmetry* $\hat{P}(3.1)$, first identified by Santilli[8] under the name of *isopoincaré symmetry*, and today called the *Poincaré-Santilli isosymmetry*[6,11,12,13].

The primary significance of the isosymmetry $\hat{P}(3.1)$ for this study is that of establishing that the fundamental symmetry of the *special* relativity also holds for all possible *gravitations*, when merely subjected to the broader isolinear, isolocal and isocanonical *realization*, thus confirming the achievement of a mathematically consistent and physically meaningful unification of the special and general relativities into the isospecial one.

The isosymmetry $\hat{P}(3.1)$ is the invariance of isointerval (7) where now the isotopic element has an unrestricted functional dependence, $\hat{T} = \hat{T}(x, p, \Psi, \partial\Psi, ..)$, and can be constructed via the *isotopies of Lie's theory* first proposed by Santilli[5] via the lifting of universal enveloping algebras, Lie algebras, Lie group, transformation and representation theories, etc., and today called *Lie-Santilli isotheory*[6,11,12,13]. The latter theory essentially consists in the reconstruction of all branches of Lie's theory for the generalized unit $\hat{I} = \hat{T}^{-1}$. Since $\hat{I} > 0$, one can see from the inception that the Poincaré-Santilli isosymmetry is isomorphic to the conventional one, $\hat{P}(3.1) \approx P(3.1)$ (see the recent study by Kadeisvili[6c]).

Note that all simple Lie algebras are known from Cartan's classification. Therefore, the Lie-Santilli isotheory cannot produce new Lie algebras, but only *new realizations* of known Lie algebras of nonlinear, nonlocal and nonhamiltonian type.

Moreover, a primary function of the Lie-Santilli isotheory is that of reconstructing as exact conventional spacetime and internal symmetries when believed to be broken. In particular, one of the primary functions of the Poincaré-Santilli isosymmetry is to establish that the abstract axioms of the conventional Poincaré symmetry remain exact under nonlinear, nonlocal and nonhamiltonian interactions, evidently when properly treated.

In this section we shall show in particular that, contrary to a rather popular belief, the rotational, Lorentz and Poincaré symmetry are indeed *exact* for all possible *gravitational* models. As indicated in Sect. 1, the most important contribution on the isosymmetry $\hat{P}(3.1)$ is that of Ref[8d], with Refs.[8e,8f] presenting the covering isospinorial symmetry. The reason for its re-inspection in this section is the discovery of the novel isosymmetries (29) which imply an evident increase of the dimensions with respect to studies of Refs.[8]¿

The operator version of the isosymmetry $\hat{P}(3.1)$ is characterized by the conventional generators and parameters only reformulated on isospaces over isofields

$X = \{X_k\} = \{M_{\mu\nu} = x_\mu p_\nu - x_\nu p, p_\alpha\} \rightarrow \hat{X} = \{\hat{M}_{\mu\nu} = \hat{x}_\mu \times \hat{p}_\nu - \hat{x}_\nu \times \hat{p}_\mu, \hat{p}_\alpha\}$,
$\mathrm{w} = \{\mathrm{w}_k\} = \{(\theta, v), a\} \in R \rightarrow \hat{w} = w \times \hat{I} \in \hat{R}(\hat{n}, +, \hat{\times}), k = 1, 2, ..., 10, \mu, \nu = 1, 2, 3, 4.$(30)

Since the generators of spacetime symmetries represent conventional total conservation laws, the preservation under isotopies of conventional generators ensures *ab initio* the preservation for the isominkowskian formulation of gravity of conventional total conservation laws.

Isotopic liftings preserve the connectivity properties of the original symmetries[7t]. The connected component of $\hat{P}(3.1)$ is then given by $\hat{P}_o(3.1) = S\hat{O}(3.1)\hat{\times}\hat{T}(3.1)$, where $S\hat{O}(3.1)$ is the *connected Lorentz-Santilli isosymmetry*[(8a)] and $\hat{T}(3.1)$ is the group of *isotranslations*[8d]. $\hat{P}_o(3.1)$ can be written via the *isoexponentiation* $\hat{e}^A = \hat{I} + A/1! + A\hat{\times}A/2! + ... = (e^{A\times\hat{T}}) \times \hat{I}$ characterized by the *isotopic Poincaré-Birkhoff-Witt theorem*[5a,5c,13c] of the underlying enveloping isoassociative algebra

$$\hat{P}_0(3.1) : \hat{A}(\hat{w}) = \Pi_k \hat{e}^{i \times X \times w} = (\Pi_k e^{i \times X \times \hat{T} \times w}) \times \hat{I} = \tilde{A}(w) \times \hat{I}. \qquad (31)$$

Note the appearance of the gravitational isotopic element $\hat{T}(x, ...)$ in the *exponent* of the group structure. This illustrates the nontriviality of the lifting and its *nonlinear* character, as evidently necessary for any symmetry of gravitation. One should however keep in mind that $\hat{P}(3.1)$ is *isolinear* on $\hat{M}$ over $\hat{C}$ (i.e., when referred to the isounit $\hat{I}$), and that the nonlinearity emerges only in its *projection* on M over C (when referred to the conventional unit I).

Conventional linear transforms on M *violate* isolinearity on $\hat{M}$ and must then be lifted into the *isotransforms*

$$\hat{x}' = \hat{A}(\hat{w})\hat{\times}\hat{X} = \hat{A}(\hat{w}) \times \hat{T}(x) \times \hat{x}, \tag{32}$$

which can be written from (31) for computational purposes (only) $\hat{x}' = \tilde{A}(w) \times \hat{x}$.

The preservation of the original dimension is ensured by the *isotopic Baker–Campbell–Hausdorff Theorem*[5a,5c,13c]. Structure (31) then forms a connected *Lie–Santilli isogroup*[5,6,13] with laws $\hat{A}(\hat{w})\hat{\times}\hat{A}(\hat{w}') = \hat{A}(\hat{w}')\hat{\times}\hat{A}(\hat{w}) = \hat{A}(\hat{w}+\hat{w}'), \hat{A}(\hat{w})\hat{\times}\hat{A}(-\hat{w}) = \hat{A}(0) = \hat{I}(x) = T^{-1}$.

As one can see, $\hat{P}_o(3.1)$ is *noncanonical* on M over R at the classical level and *nonunitary* at the operator level. As such, it *does not* preserve the conventional unit I. However, $\hat{P}_o(3.1)$ is *isocanonical* at the classical level and *isounitary* at]the operator level.

This proved Theorem 1 and guarantees the achievement of the primary objective of this study, a classical and operator formulation of gravitation with invariant basic units.

The use of the isodifferential calculus on $\hat{M}$ then yields the Poincaré-Santilli isoalgebra $\hat{p}_o(3.1)$[5–13]

$$[\hat{M}_{\mu\nu}\hat{,}\hat{M}_{\alpha\beta}] = i \times (\hat{\eta}_{\nu\alpha} \times \hat{M}_{\mu\beta} - \hat{\eta}_{\mu\alpha} \times \hat{M}_{\nu\beta} - \hat{\eta}_{\nu\beta} \times \hat{M}_{\mu\alpha} + \hat{\eta}_{\mu\beta} \times \hat{M}_{\alpha\nu}),$$

$$[\hat{M}_{\mu\nu}\hat{,}\ \hat{p}_\alpha] = i \times (\hat{\eta}_{\mu\alpha} \times \hat{p}_\nu - \hat{\eta}_{\nu\alpha} \times \hat{p}_\mu), [\hat{p}_\alpha\hat{,}\hat{p}_\beta] = 0,\ \hat{\eta}_{\mu\nu} = g_{\mu\nu}(x), \tag{33}$$

where $[A\hat{,}B] = A \times \hat{T}(x) \times B - B \times \hat{T}(x) \times A$ is the *isoproduct* (originally proposed in [5a,7a]), which does indeed satisfy the Lie axioms in isospace, as one can verify. Note the appearance of the Riemannian metric $\hat{\eta}_{\mu\nu} = g_{\mu\nu}(x)$, this time, as the "structure functions" $\hat{\eta}_{\mu\nu}$ of the isoalgebra[5a]. Note also that the *momentum components isocommute* (while they are notoriously noncommutative for QG). This confirms the isoflat character of the isominkowskian gravity, as *necessary* for a consistent grand unification[9e,9f].

The local isomorphism $\hat{p}_o(3.1) \approx p_o(3.1)$ is ensured by the positive–definiteness of $\hat{T}$. In fact, the use of the generators in the form $\hat{M}^\mu_\nu = \hat{x}^\mu \hat{\times} \hat{p}_\nu - \hat{x}^\nu \hat{\times} \hat{p}_\mu$ would yield *conventional* structure constants under a *generalized* Lie product, as one can verify. The above local isomorphism is sufficient, per se', to guarantee the axiomatic consistency of RHM in general, and of OIG in particular.

The *isocasimir invariants* of $\hat{p}_0(3.1)$ are simple isotopic images of the conventional ones, and can be written

$$C^0 = \hat{I}(x, p, \Psi, \partial\Psi, ...) = [\hat{T}(x)]^{-1},$$

$$C^{(2)} = \hat{p}^{\hat{2}} = \hat{p}_\mu \hat{\times} \hat{p}^\mu = \hat{\eta}^{\mu\nu} \times \hat{p}_\mu \hat{\times} \hat{p}_\nu,$$

$$C^{(4)} = \hat{W}_\mu \hat{\times} \hat{W}^\mu, \hat{W}_\mu = \epsilon_{\mu\alpha\beta\pi} \hat{M}^{\alpha\beta} \hat{\times} \hat{p}^\pi. \tag{34}$$

for exterior gravity[9a].

From them, one can construct any needed *gravitational relativistic equation*, such as the *isodirac equation*

$$(\hat{\gamma}^\mu \hat{\times} \hat{p}_\mu + \hat{i} \hat{\times} \hat{m}) \hat{\times} | >= [\hat{\eta}_{\mu\nu}(x) \times \hat{\gamma}^\mu(x) \times \hat{T}(x) \times \hat{p}^\nu - i \times m \times \hat{I}(x)] \times \hat{T}(x) \times | >= 0,$$

$$\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = \hat{\gamma}^\mu \times \hat{T} \times \hat{\gamma}^\nu + \hat{\gamma}^\nu \times \hat{T} \times \hat{\gamma}^\mu = 2 \times \hat{\eta}^{\mu\nu} \equiv 2 \times g^{\mu\nu}, \hat{\gamma}^\mu = \hat{T}^{1/2}_{\mu\mu} \times \gamma^\mu \times \hat{I}\ , \tag{35}$$

where γ^μ are the conventional gammas and $\hat{\gamma}^\mu$ are the *isogamma matrices.*

Note that *the anti-isocommutators of the isogamma matrices yield (twice) the Riemannian metric $g(x)$,* thus confirming the representation of Einstein's (or other) gravitation in the *structure* of Dirac's equation.

As an illustration, we have the *Dirac–Schwarzschild equation* given by Eqs. (37) with

$$\hat{\gamma}_k = (1 - 2M/r)^{-1/2} \times \gamma_k \times \hat{I}, \hat{\gamma}_4 = (1 - 2M/r)^{1/2} \times \gamma^4 \times \hat{I} \tag{36}$$

although, as indicated in earlier, the use of the nondiagonal representation in of Ref.[2f] is preferable. Similarly one can construct the isogravitational version of all other equations of RQM.

The *Poincaré-Santilli isotransforms* are given by:

1) Isorotations. The space components $S\hat{O}(3)$, called *isorotations*[8b], can be computed from isoexponentiations (31) with the explicit form in the (x,y)–plane (were we ignore again the factorization of $\hat{I}$ for simplicity)

$$x' = x \times \cos(\hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \theta_3) - y \times \hat{T}_{11}^{-\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \sin(\hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \theta_3),$$

$$y' = x \times \hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{-\frac{1}{2}} \times \sin(\hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \theta_3) + y \times \cos(\hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \theta_3), \quad (37)$$

(see[7t] for general isorotations in all three Euler angles). Isotransforms (37) leave invariant all ellipsoidical deformations $x \times \hat{T}_{11} \times x + y \times \hat{T}_{22} \times y + z \times \hat{T}_{33} \times z = R$ of the sphere $x \times x + y \times y + z \times z = r$. Such ellipsoid become perfect spheres $\hat{r}^{\hat{2}} = (\hat{r}^t \times \hat{\delta} \times \hat{r}) \times \hat{I}_s$ in *isoeuclidean spaces*[(3h,3r)] $\hat{E}(\hat{r}, \hat{\delta}, \hat{R}), \hat{r} = \{\hat{r}^k\} = \{r^k\} \times \hat{I}_s, \hat{\delta} = \hat{T}_s \times \delta, \delta = diag.(1,1,1), \hat{T}_s = diag.(\hat{T}_{11}, \hat{T}_{22}, \hat{T}_{33}), \hat{I}_s = \hat{T}_s^{-1}$, called *isospheres.*

In fact, the deformation of the semi-axes $1_k \rightarrow \hat{T}_{kk}$ while the related units are deformed of the *inverse* amounts $1_k \rightarrow \hat{T}_{kk}^{-1}$ preserves the perfect spheridicity (because, as noted in Sect. **3**, the invariant in isospace is $[Length]^2 \times [Unit]^2$). Note that this perfect sphericity in $\hat{E}$ is the geometric origin of the isomorphism $\hat{O}(3) \equiv O(3)$, with consequential preservation of the exact *rotational* symmetry for the space–components $g(r)$ of all possible *Riemannian* metrics (becomes the isogeodesics are perfect circles).

2) Isoboosts. The connected Lorentz-Santilli isosymmetry $S\hat{O}(3.1)$ is characterized by the isorotations and the *isoboosts*[8a] which can be written in the $(3,4)$–plane

$$\begin{aligned}
x^{3\prime} &= x^3 \times \sinh(\hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44}^{\frac{1}{2}} \times v) - x^4 \times \hat{T}_{33}^{-\frac{1}{2}} \times \hat{T}_{44}^{\frac{1}{2}} \times \cosh(\hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44} \times v) = \\
&= \tilde{\gamma} \times (x^3 - \hat{T}_{33}^{-\frac{1}{2}} \times \hat{T}_{44}^{\frac{1}{2}} \times \hat{\beta} \times x^4) \\
x^{4\prime} &= -x^3 \times \hat{T}_{33}^{\frac{1}{2}} \times c_0^{-1} \times \hat{T}_{44}^{-\frac{1}{2}} \times \sinh(\hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44} \times v) + x^4 \times \cosh(\hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44}^{\frac{1}{2}} \times v) = \\
&= \tilde{\gamma} \times (x^4 - \hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44}^{-\frac{1}{2}} \times \tilde{\beta} \times x^3) \\
\tilde{\beta}^2 &= v_k \times \hat{T}_{kk} \times v_k / c_0 \times \hat{T}_{44} \times c_0, \quad \tilde{\gamma} = (1 - \tilde{\beta}^2)^{-\frac{1}{2}}. \qquad (38)
\end{aligned}$$

Note that the above isotransforms are formally similar to the Lorentz transforms, as expected from their isotopic character. Isotransforms (38) characterize the *light isocone*[7t,8], i.e., the perfect cone in isospace $\hat{M}$. In a way similar to the isosphere, we have the deformation of the light cone axes $1_\mu \rightarrow \hat{T}_{\mu\mu}$ while the corresponding units are deformed of the *inverse* amount $1_\mu \rightarrow \hat{T}^{-1}_{\mu\mu}$, thus preserving the perfect cone in isospace.

In particular, the isolight cone also has the conventional characteristic angle, as a necessary condition for an isotopy (the proof of the latter property requires the use of isotrigonometric and isohyperbolic functions[5f]). Thus, *the maximal causal speed in isominkowski space is the conventional speed in vacuum* c_0. The identity of the light cone and isocones is the geometric origin of the isomorphism $S\hat{O}(3.1) \approx SO(3.1)$ and, thus, of the exact validity of the *Lorentz* symmetry for all possible *Riemannian* metrics $g(x)$ in (3+1)-dimensions.

3) Isotranslations. The *isotopies of translations* can be written

$$x' = (\hat{e}^{i \times \hat{p} \times a}) \hat{\times} \hat{x} = [x + a \times A(x)] \times \hat{I}, \hat{p}' = (\hat{e}^{i \times \hat{p} \times a}) \hat{\times} \hat{p} = \hat{p},$$

$$A_\mu = \hat{T}^{1/2}_{\mu\mu} + a^\alpha \times [\hat{T}^{1/2}_{\mu\mu}, \hat{p}_\alpha]/1! + ... \tag{39}$$

and they are also nonlinear, as expected.

4) Isoselftransforms. Intriguingly, the isotopies identify a new symmetry, that of Eqs. (29), which is absent in the conventional case. It is here called *isoselfscalar invariance* and it is given by

$$\hat{I} \rightarrow \ \hat{I}' = n^2 \times \hat{I}, \eta \rightarrow \ \hat{\eta} = n^{-2} \times \eta, \tag{40}$$

where n is a novel 11-th parameter absent in the presentations of Refs[8].

Note that, even though n^2 is factorizable, the corresponding isosymmetry is not trivial, e.g., because n^2 enters into the *argument* of the isolorentz transforms (38). Note also that the isominkowskian representation of gravity is permitted precisely by the latter isoinvariance. In fact, isoinvariance (29) holds also for the conventional Poincaré symmetry, by introducing in this way the generalized unit at the foundation of the isominkowskian gravity.

5) Isoinversions. The *isodiscrete transforms*[8d] are

$$\pi \times x = (-r, x^4), \hat{\tau} \hat{\times} x = \tau \times x = (r, -x^4), \hat{\pi} = \pi \times \hat{I}, \hat{\tau} = \tau \times \hat{I}, \tag{41}$$

where π, τ are the conventional inversion operators. Despite their simplicity, the physical implications of isoinversions are nontrivial because of *the* *possibility of reconstructing as exact discrete symmetries when believed to be broken.* This is studied by embedding all symmetry breaking terms in the isounit, and it has ben already successfully done for tjhe isospin symmetry[8c], the Lorentz symnmetry[8a], and parity[7t].

Note that the *isorepresentations* of $\hat{P}(3.1)^{7t}$ can be easily constructed from the conventional representations of $P(3.1)$ via the methods of nonunitary transforms indicated in Sect. 8.

The *general Poincaré-Santilli isosymmetry* is defined as the 11-dimensional set of *isorotations, isoboosts, isotranslations, isoselftransforms and isoinversions.* The *restricted Poincaré-Santilli isosymmetry* is defined as the general isosymmetry in which the isounit is averaged into constants.

It is easy to se that the isolorentz transforms (38) confirm the anomalous behaviour of the frequency shift for isominkowskian media of Classes I, II, III, as outlined in Sect. 5.C. In fact, the general expression for the conventional Doppler shift for 90-degrees aberration is given by $\omega = \omega_o/\gamma$ and can be written in first approximation $\omega = \omega_o[1 - (v/c_o) + ...]$.

For the isolorentz transforms (38) with $\hat{T}_{\mu\mu} = 1/n_\mu^2$ and $n_1 = n_2 = n_3 = n_s$, the corresponding isodoppler shift is given by $\hat{\omega} = \omega_o/\hat{\gamma}$ and can be written in first approximation $\hat{\omega} = \omega_o[1 - (v/c_o) \times (n_4/n_s) + ...]$. Therefore, for Class I ($n_s = n_4$) we have $\hat{\omega} = \omega$, for Class II ($n_s < n_4$) we have $\hat{\omega} > \omega$; and for Class III ($n_s > n_4$) we have $\hat{\omega} < \omega$.

The reader interested in learning the Poincaré-Santilli isosymmetry is suggested to specialize isolorentz transforms (38) for all nine different types of isominkowskian media as outlined in Sect. 5.C. in fact, the above cases are important for the experimental verification of the isominkoskian geometry (Sect. 12), besides having rather intriguing implications in cosmology (Sect. 14).

9. Direct universality of the Poincaré-Santilli Isosymmetry for Exterior and Interior Gravitations of matter.

The results of this paper imply the following:

Theorem 2. *The 11-dimensional, general, Poincaré-Santilli isosymmetry on isominkowski spaces over isoreal fields with well behaved, positive-definite isounits is the largest possible isolinear, isolocal and isocanonical invariance of isoseparation (7) (universality) in the fixed x-frame of the experimenter (direct universality).*

The verification of the above invariance is instructive. Note that for any arbitrarily given (diagonal) Riemannian metric $g(x)$ (such as Schwarzschild, Krasner, etc.[2]) *there is nothing to compute* because one merely *plots* the $\hat{T}_{\mu\mu}$ terms of the decomposition $g_{\mu\mu} = \hat{T}_{\mu\mu} \times \eta_{\mu\mu}$ (no sum) in the above given isotransforms. The invariance of the separation $x^t \times g \times x$ is then ensured. The maximal character of the isosymmetry can be proved as in the conventional case.

The $(2+2)$–de Sitter or other cases can be derived from Theorem 2 via mere changes of signature or dimension of the isounits. The extension to positive-definite yet nondiagonal isounit is assured by the method of nonunitary lifting of Sect. 8 and it will be implied hereon.

Note finally that isosymmetry $\hat{P}(3.1)$ cannot be even defined, let alone constructed in conventional Riemannian spaces (as well as in all their possible isotopies), thus rendering the isominkowskian formulation of gravity rather unique for our purposes.

10. The Isospecial Relativity for Matter.

It may be of interest to indicate in more details that all preceding studies belong to the so-called *isospecial relativity*, first submitted by Santilli[8a] in 1983 and then studied in a variety of works[5,7,9,10], which can be defined as *the conventional special relativity only formulated on the isominkowskian space $\hat{M}$ over the isoreal field $\hat{R}$.*

In particular, this implies, by conception and construction, the validity on isospace over isofield of *all* axioms and physical laws of the special relativity, including the constancy of the speed of light c_o, light cone, time dilation, space contraction, Doppler shift, etc. (see monograph[7t] for brevity).

A primary reason for the submission of the isospecial relativity is to disprove a rather popular belief these days that any deviation from conventional settings, such as a speed of light $c = c_o/n$ different than c_o, implies a "violation of Einstein's special relativity". In fact, the isospecial relativity reconstructs as exact the special relativity for all possible speeds c different than c_o.

As a matter of fact, the isospecial relativity renders Einstein's axioms truly "universal" in the sense of holding irrespective of whether we have exterior problems in vacuum or interior problems within physical media. By comparison, as well known, the conventional formulation of the special relativity was solely conceived in vacuum, and it remains exact only within such a setting.

Yet another objective for the submission of the isospecial relativity is the geometric unification of the special and general relativities indicated earlier.

The latter objective was necessary, to our best knowledge at this writing, to achieve an axiomatically consistent inclusion of gravity in the unified gauge theories of electroweak interactions[9e,9f], as well as to achieve an axiomatically consistent operator gravity[9a].

The primary motivation for the submission of the isospecial relativity is however an axiomatically consistent transition from the current linear, local and potential formulations of particle physics to broader nonlinear, nonlocal and nonpotential conditions, with specific reference to novel studies in the structure of hadrons. These latter aspects are not considered in this paper[7s,7t].

11. Apparent Resolution of Some of the Controversies in Gravitation.

The preceding results permit the apparent resolution of some of the controversies that have been lingering in the gravitational literature throughout this century, thus providing support to the study of gravitation via the novel isomathematics.

First, we note that Theorem 2 provides a rigorous proof of Theorem 1. In fact, the universal symmetry of gravitation, $\hat{P}(3.1)$ does not leave invariant the basic spacetime unit of the Riemannian formulation of gravity. Theorem 2 also allows to resolve the shortcoming. In fact, the spacetime isounit is indeed invariant under the isosymmetry $\hat{P}(3.1)$ by conception and construction.

Theorem 2 also permits the resolution of the controversy whether the total conservation laws of general relativity are compatible with those of the special relativity via a mere *visual examination.*

Recall that the generators of all space-time symmetries characterize total conserved quantities. The compatibility of the total conservation laws of the general and special relativities is therefore established by the visual observation that *the generators of the conventional and isotopic Poincaré symmetries coincide.* In fact, only the operations on them are changed in the transition from the relativistic to the gravitational case.

Yet another controversy which appears to be resolved by our isominkowskian treatment of gravity is the apparent lack of a meaningful *relativist limit* in conventional gravitational theories, all limits of existing formulations being primarily of *Euclidean* character[2]. In fact, such a limit is now clearly and unequivocally established by $\hat{I} \rightarrow I$ under which the special relativity is recovered identically in all its aspects.

The isominkowskian treatment of gravity also permits a resolution of some of the limitations of conventional gravitational models, such as their insufficiency to provide an effective representation of *interior* gravitational problems. In fact, conventional formulations of gravity admit only a limited dependence on the velocities, while being strictly local-differential and derivable from a first-order Lagrangian (variationally self-adjoint[5c]). These characteristics are evidently exact for exterior problems in vacuum.

By comparison, interior gravitational problems, such as all forms of gravitational collapse, are constituted by extended and hyperdense hadrons in conditions of total mutual penetration in large numbers into small regions of space. It is well known that these conditions imply effects which are *arbitrarily nonlinear in the velocities as well as in the wavefunctions and possibly their derivatives, nonlocal-integral on various quantities and variationally nonselfadjoint*[5c,7t], (i.e. not representable via first-order Lagrangians). It is evident that the latter conditions are beyond any scientific expectation of quantitative treatment via conventional gravitational theories.

The isominkowskian formulation of gravity resolve this limitation too and shows that it is equally due to insufficiencies in the underlying mathematics. In fact, isogravitation extends the applicability of Einstein's axioms to a form which is "directly universal" for exterior and interior gravitations.

As indicated earlier, this extension is due to the fact that the functional dependence of the metric in Riemannian treatments is restricted to the sole dependence on the local coordinates, g = g(x), while under isotopies the same dependence becomes unrestricted, $g = g(x, p, \Psi, \partial\Psi....)$ *without altering the original geometric axioms.* This results in *geometric unification of exterior and interior problems,* despite their sizable structural differences of topological, analytic and other characters. The latter unification was studied in details in ref.[7t] under the *isoriemannian* geometry and it is studied with the *isominkowskian* geometry in this paper for the first time.

Yet another controversy which appears to be resolved by the isominkowskian formulation of gravity is the achievement of an axiomatically consistent operator version of gravity, that with: invariance of the basic units;

preservation of the original Hermiticity at all times; uniqueness and invariance of the numerical predictions; preservation of causality and probability laws; consistent PCT and other theorems; etc.

Even though far from being a complete theory, our OIG does indeed offer realistic hopes of achieving such an axiomatically consistent operator form of gravity, as expected from the validity of the *conventional* axioms of RQM.

Yet another resolution of an existing controversy is that related to gravitational singularities. This controversy is due to the fact that all existing studies in the field are based on the conventional Riemannian geometry, while the same geometry is not expected to be exactly valid under the extreme conditions of gravitational collapse, as indicated earlier.

In fact, the isominkowskian formulation of gravity implies the following property of self-evident proof.

Theorem 3[7t,9]**.** *Gravitational singularities (horizons) are the zeros of the space (time) component of the isounit.*

The above properties are trivially equivalent to the conventional ones for Riemannian metrics. The novelty is that the same properties also apply for an unrestricted functional dependence of the metric, thus including the missing internal, nonlocal and nonlagrangian effects.

The isominkowskian reformulation of existing theorems and properties on gravitational singularities will be done elsewhere.

12. Experimental Verifications and Novel Predictions

It is important to see that the proposed isominkowskian representation of gravity verifies all available experimental data at both classical and operator level.

Despite apparent differences, isoeinstein equations (14) *numerically coincide* with Einstein's equations both in isospace as well as in their projection in ordinary spaces.

The preservation in isospace of the *numerical value* of the conventional field equations stems from a general property of the isotopies of preserving all original numerical values[7r,7t], as it was illustrated earlier with the preservation of the maximal causal speed c_o in isospace over isofield. In fact, the isoderivative $\hat{\partial}_\mu = \hat{T}^\alpha_\mu \times \partial_\alpha$ deviates from the conventional derivative ∂_μ by the isotopic factor $\hat{T}$. But its numerical value must be referred to $\hat{I} = \hat{T}^{-1}$, rather than I. This implies the preservation in isospace of the original value of ∂_μ and, consequently, of the original field equations.

For the case of the projection of Eqs. (14) into ordinary spaces, the isoequations are reducible to the conventional equations multiplied by common isotopic factors which, as such, are inessential and can be eliminated. In fact, the isochristoffel symbols (9) deviate from the conventional symbols by the same factor $\hat{T}$ (again, because $\hat{\eta} \equiv g$), and the same happens with other terms, except for possible re-definition of the source when needed, thus preserving again the conventional field equations and related experimental verifications also in our space-time.

The verification of all experimental data at the operator level by the isominkowskian gravity is equally incontrovertible. In fact, isogravitational field equations, such as Eqs. (35), establish the compatibility of OIG with experimental data in particle physics in view of the much smaller contribution of gravitational over electromagnetic, weak and strong contributions.

Our unification of the special and general relativities, therefore, appears to be compatible with experimental evidence at both classical and operator levels.

The reader should be aware that the isominkowskian geometry also has a number of applications and experimental verifications in other fields. By using the isominkowskian 'classification of physical media of Sect. 5.C, we indicate::

1) An exact isominkowskian fit[11a] of the experimental data on the behavior of meanlife of the K^o particle (isominkowskian medium of Type 9) with energies from 30 to 400 GeV, where the Minkowskian anomaly is predicted from expected internal nonlocal effects under a conventional behavior of the center-of-mass;

2) An exact isominkowskian fit[11b] of the experimental data on the Bose-Einstein correlation for the two-point-isocorrelation function[11c] deriving the correlation from the nonlocality of the p–$\bar{p}$ fireball (isominkowskian medium also of Type 9) from, first axiomatic principles without ad hoc "semiphenomenological approximations" with unknown parameters, and by reconstructing the exact Poincaré symmetry in isospace under nonlocal interactions;

3) An exact confinement of quarks on isominkowskian spaces of Type 9[11d] (i.e., a confinement with an identically null probability of tunnel effects)

even in the absence of a potential barrier, which is quite simply permitted by the isotopies due to the incoherence of the internal and external Hilbert spaces, under conventional unitary symmetries, conventional quantum numbers and conventional experimental data on mass spectra;

4) An exact representation on isominkowskian space of Type 9 of the synthesis of the neutron as occurring in stars at their formation, from protons and electrons only[8e,8f], which has been able to represent the totality of the characteristic of the neutron;

5) The apparently first exact representation of total nuclear magnetic moments[11e] via isominkowskian media of Type 4, verifying conventional quantum axioms and physical laws, and representing the 1this century despite all possible relativistic corrections;

6) An exact reconstruction of the SU(2) isospin symmetry[8e] with equal masses for protons and neutrons in isominkowskian space of Type 9 and physical masses in conventional spaces;

7) An exact representation of the large difference in cosmological redshifts between quasars and their associated galaxies when physically connected according to spectroscopic evidence via the isominkowskian geometrization of Type 5 of quasars chromospheres11f;

8) An exact isominkowskian representation of Type 5 of the internal quasars redshift and blueshift[11g];

9) The achievement of the apparently first attractive force between the two identical electrons of the Cooper pair in superconductivity in excellent agreement with experimental data via the isominkowskian geometrization of the electron pair of Type 5 [11h];

10) The apparently first achievement of explicitly attractive forces between the neutral atoms of molecular bonds in chemistry[11i] capable of representing the 2by quantum chemistry through this century;

11) The apparently first capability of representing main characteristics of biological structures, such as their irreversibility, time-rate-of-variations of sizes and shapes, etc.[11j]; and others.

To achieve a technical understanding of the novel isominkowskian geometry, the reader is suggested to verify the extreme difficulties, if not the impossibility of achieving the above results with a theory based on conventional mathematics.

Some of the novel predictions of the isotopic grand unification and underlying isominkowskian geometry are the following:

A) The prediction that the speed of electromagnetic waves is a local quantity which can be arbitrarily smaller or bigger than the speed in vacuum depending on local conditions[16];

B) The prediction that the inhomogeneity and anisotropy of the media in which light propagates has a new measurable contribution to the Doppler's red- or blue-shift[11f,11g];

C) The prediction of a new isocosmology which is characterized the first time by a universal symmetry, the Poincaré-Santilli isosymmetry, without the need for the "missing mass", a direct geometrization of the anisotropy in the propagation of light in the universe, and other features[7t];

D) The prediction of a new geometric propulsion called *isolocomotion*[7s], in which motion occurs via the reduction of distances due to very large local amounts of energy without any Newtonian propulsion;

E) A new notion of spacetime[8] in which the novelty rests in its basic units, thus implying local notions of space and time different than those of conventional relativities.

Needless to say, the above novel predictions can be solely resolved via experiments.

We here limit ourselves to indicate the differences between the conventional spacetime and isospacetime can be illustrated via the *isobox*[7s] which is an ordinary cube with two observers, a conventional Minkowskian observer in the outside and an isominkowskian observer in the inside. The novelty of isospacetime emerges from the fact that *the same object has dramatically different shapes and dimensions for the two observers*, evidently in view of the arbitrariness of the units of the interior observer (which imply different dimensions), as well as their differences for different space axes (which implies different shapes).

The novelty of isospacetime is further illustrated by the fact that *the same object can belong to dramatically different times, both in the future or in the past.* In fact, the isospacetime permits the *mathematical* formulation of a fully causal *space-time machine*[7s,7t].

The difference between the conventional and isotopic spacetime then reaches its climax with the fact that *exactly the same object can have different space dimensions for the two observers.* This prediction is implicit in some of the last papers written by P. A. M. Dirac[18] who, in his notorious intuitive brilliance, submitted without his awareness one of the firsts, yet most general possible isotopies of the Minkowskian geometry. Dirac's papers[18] of 1971-1972 escaped technical understanding until the advent of the novel isomathematics, and were studied in details in Ch. 10 of Ref.[7t] of 1994.

In essence, Dirac proposed a generalization of his celebrated equation based precisely on the isogamma matrices of Eqs. (35), in which the isotopic element $\hat{T}$ (denoted β by Dirac) is positive-definite, thus invertible, yet it is *nondiagonal* and such to lift the *three* space dimensional Minkowski space into an isominkowskian space in only *one* space dimension according to the isotopy $x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I \rightarrow \hat{x}^{\hat{2}} = (x^\mu \times \hat{\eta}_{\mu\nu} \times x^\nu) \times \hat{I} = (-2 \times x^2 \times x^4) \times \hat{I}$.

As we hope to illustrate in future works, rather than being a mathematical curiosity, Dirac's papers[18] have an apparently fundamental character in understanding anomalous conditions of his electron, such as the *coupling of electrons* in the Cooper pair in superconductivity or in molecular bondings, and in other anomalous cases transparently outside any realistic hopes of quantitative treatment via the conventionak Minkowskian geometry.

In closing we should also indicated that the isotopies with basic lifting $I \rightarrow \hat{I}(x, \Psi, ...) = \hat{I}^\dagger$ constitute only the first step of a chain of generalized methods[(5e)]. The second class is given by the *genotopies*[5a,5e] in which the isounit is no longer Hermitean. This broader class geometrizes in a natural way the irreversibility of interior gravitational problems and it has been used, e.g., for the black hole model of Refs.[12i,12j].

A third class of methods is given by the (multi–valued) *hyperstructures*[5e], in which the generalized unit is constituted by a *set* of non-Hermitean quantities. The latter class appears to be significant for quantitative studies of biological structures with their typical irreversibility and variation of physical characteristics[11j]. in the latter biological conditions the conventional RQM is manifestly inapplicable due to its reversibility as well as intrinsically conservative character.

13. Elements of isodual representations of antimatter.

Another structural incompatibility between gravitation and unified gauge theories, besides that due to curvature studied earlier, is that the latter are *bona fide* relativistic theories, thus characterizing antimatter via *negative-energy* solutions, while the former characterize antimatter via *positive-definite* energy-momentum tensors.

The above structural incompatibility is only the symptom of deeper problems in the contemporary treatment of antimatter. To begin, matter is treated nowadays at *all* levels, from Newtonian to electroweak interactions, while antimatter is treated only at the level of *second quantization.* Since there are serious indications that half of the universe could well be made up of antimatter, the need for a more effective theory of antimatter holding at *all* levels of study becomes compelling.

At any rate, recall that charge conjugation in quantum mechanics is an *anti-automorphic map.* As a result, no classical theory of antimatter can be axiomatically consistent via the mere change of the sign of the charge, because it must be an anti-automorphic (or, more generally, anti-isomorphic) image of that of matter.

The current dramatic disparity in the treatment of matter and antimatter also has its predictable problematic aspects. Since we currently use only *one* type of quantization (whether naive of symplectic), it is easy to see that *the operator image of the contemporary treatment of antimatter is not the correct charge conjugate state, but merely a conventional state of particles with a reversed sign of the charge.*

The view here submitted is that, as it is the case for curvature, the resolution of the above general shortcomings for antimatter requires a yet *novel mathematics.*

Santilli[8b,10] therefore entered into a further laborious search for another novel mathematics under the uncompromisable condition of being an anti-isomorphic image of the preceding isomathematics. After inspecting a number of alternatives, this author[8b] submitted in 1985 the following map of an arbitrary quantity Q (i.e., a number, or a vector field or an operator) under the name of *isoduality*

$$Q = Q(x, p, \psi, ...) \rightarrow Q^d = -Q^\dagger(-x^\dagger, -p^\dagger, \psi^\dagger, ...). \tag{42}$$

When applied to the *totality* of quantities and their operations of a given theory of matter, map (42) yields an anti-isomorphic image, as axiomatically needed for antimatter. Moreover, while charge conjugation is solely applicable within operator settings, isoduality (42) is applicable at *all* levels of study, beginning at the *Newtonian* level.

It is evident that map (42) implies a new mathematics, that with *negative units* called *isodual mathematics*[10], which includes new numbers, new spaces, new calculus, new geometries, etc. In reality we have two different isodual mathematics, the first is the anti-isomorphic image of the *conventional* mathematics used for *exterior* problems of antimatter, and the second is the anti-isomorphic image of the preceding *isomathematics* used for *interior* problems of antimatter.

Isodualities imply the transition from the conventional space-time units of matter $I = diag(1,1,1,1) > 0$ to their negative images $I^d = -I < 0$. As a result, *under isoduality all characteristics of matter change sign in the transition to antimatter*, thus yielding the correct conjugation of charge, as well as negative energy, negative energy-momentum tensor, and, inevitably, negative time.

The historical objections against these negative values are inapplicable, because they are tacitly referred to the conventional positive units. In fact, negative energy and time referred to negative units are fully equivalent, although antiautomorphic, to the conventional positive energy and time referred to positive units.

The above characteristics have permitted the construction of the novel *isodual theory of antimatter* which holds at *all* levels of study[10]. Thus, the theory begins with the *isodual Newton and iso-Newton equations*, continues with the *isodual Lagrangian-Hamiltonian and iso-Lagrangian and iso-Hamiltonian mechanics,* and evidently includes the *isodual quantum and hadronic mechanics*, at which latter level it results to be equivalent to charge conjugation for massive particles (see later on for photons)[10c].

Most importantly, the isodual theory of antimatter has resulted in agreement with all available classical and quantum experimental data on antimatter, those based on the various interactions except gravitation (for which there are no experimental data pertaining to antimatter).

It is an instructive exercise for the reader interested in learning the new techniques to work out the isodualities of the *conventional* relativistic field theory (rather than of their isotopies), and show that they essentially provide a mere reinterpretation of the usually discarded, advanced solutions as characterizing antiparticles referred to negative units.

Therefore, in our theory, *retarded* solutions are associated with *particles, advanced* isodual solutions are associated with *antiparticles,* and no numerical difference is expected in the above reformulation.

It is also recommendable for the interested reader to verify that the isodualities are indeed equivalent to charge conjugation for all massive particles, with the exception of the photon.

Isodual theories predict that the antihydrogen atom emits a new photon, tentatively called by this author the *isodual photon*, which coincides with the conventional photon for all possible interactions, thus including electroweak interactions, *except gravitation* [loc. cit.]. This indicates that the isodual map is inclusive of charge conjugation for massive particles, but it is broader than the latter.

Isodual theories in general, thus including the grand unification of Ref.[9e] and the new cosmology of the next section, predict that all *stable* isodual particles, such as the isodual photon, the isodual electron (positron), the isodual proton (antiproton) and their bound states (such as the antihydrogen atom), experience *antigravity* in the field of the Earth (defined as the reversal of the sign of the curvature tensor). A number of experiments are under consideration for the resolution of these basic issues.

The reader should also be aware that the isodual theory of antimatter was born from properties of the *conventional* Dirac equation

$$[\gamma^{\mu} \times (p_{\mu} - e \times A_{\mu}/c) + i \times m] \times \Psi(x) = 0, \tag{43a}$$

$$\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \tag{43b}$$

$$\gamma^4 = i \times \begin{pmatrix} I_s & 0. \\ 0 & -I_s \end{pmatrix}. \tag{43c}$$

As one can see, *the negative unit* $I_s^d = Diag.(-1, -1)$ *appears in the very structure of* γ_4. The isodual theory was then constructed precisely around Dirac's unit I_s^d[10]

In essence, Dirac assumed that the negative-energy solutions of his historical equation behaved in an unphysical way because tacitly referred to the conventional mathematics of his time, that with *positive units* $I_s > 0$. Ref.s[10] showed that, when the same negative-energy solutions are referred to the *negative units* $I_s^d < 0$, they behaved in a fully physical way. This eliminates the need of second quantization for the treatment of antiparticles (as expected in a theory of antimatter beginning at the Newtonian level), and permits the reformulation of the equation in the form

$$[\tilde{\gamma}^{\mu} \times (p_{\mu} - e \times A/c) + i \times m] \times \tilde{\Psi}(x) = 0, \tag{44a}$$

$$\tilde{\gamma}_k = \begin{pmatrix} 0 & \sigma_k^d \\ \sigma_k & 0 \end{pmatrix}, \quad \tilde{\gamma}^4 = i \begin{pmatrix} I_s & 0, \\ 0 & I_s^d \end{pmatrix}, \tag{44b}$$

$$\{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2\eta_{\mu\nu}, \quad \tilde{\Psi} = -\tilde{\gamma}_4 \times \Psi = i \times \begin{pmatrix} \Phi \\ \Phi^d \end{pmatrix}, \tag{44c}$$

where $\Phi(x)$ is now two-dimensional, which is fully asymmtrized between particles and antiparticles.

As it was the case for the preceding isotopies, the isodual theory of antimatter also sees its solid roots in two additional novel symmetries, also unknown until recently, and first presented in memoir[7r], the first holding for the conventional Minkowski interval

$$x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I = [x^\mu \times (-n^{-2} \times \eta_{\mu\nu}) \times x^\nu] \times (-n^2 \times I)$$

$$= (x^\mu \times \hat{\eta}^d_{\mu\nu} \times x^\nu) \times \hat{I}^d = x^{d2d} \tag{45}$$

and the second holding for the Hilbert space

$$< \phi| \times |\psi > \times I =< \phi| \times (-n^{-2}) \times |\psi > \times(-n^2 \times I) =< \phi| \times \hat{T}^d \times |\psi > \times \hat{I}^d, \tag{46}$$

which ensure that all physical laws for matter also hold for antiparticles under our isodual representation, with corresponding symmetries for the isodual expressions.

The axiom-preserving lifting of the parameter n to an explicit x-dependence then yields the *isodual isominkowskian treatment of gravity for antimatter* with basic structures

$$g(x) = \hat{T}(x) \times \eta \rightarrow g^d(x) = -g(x) = \hat{T}^d(x) \times^d \eta^d, \eta \rightarrow \eta^d = -\eta, \tag{47a}$$

$$\hat{I}(x) = [\hat{T}(x)]^{-1} \rightarrow \hat{I}^d(x) = [\hat{T}^d(x)]^{-1}. \tag{47b}$$

It should also be recalled that isodualities imply yet another new symmetry. A quantity Q is said to be *isoselfdual*[10] when it is invariant under the isodual map (42)

$$Q(x, p, \psi, ...) \rightarrow Q^d = -Q^\dagger(-x^\dagger, -p^\dagger, \psi^\dagger, ...) = Q(x, p, \psi, ...). \tag{48}$$

A property which is fundamental for the isodual theory of antimatter is that *Dirac's gamma matrices are isoselfdual* (from which property the symmetry itself was derived in the first place). In fact, $\gamma_\mu \rightarrow \gamma_\mu^d = -\gamma_\mu^\dagger = \gamma_\mu$.

This property disproves a popular belief held throughout this century, according to which the Poincaré symmetry is the total symmetry of Dirac's equations. This belief is disproved because a non-isoselfdual structure such as the Poincaré symmetry cannot possibly be the correct symmetry of a structure which is invariant under isoduality.

In fact, the correct total symmetry of the conventional Dirac equation was identified in Ref.[10c], and it is given by the following 22-dimensional isoselfdual symmetry with underlying isoselfdual geometry and unit

$$S_{Tot} = \{SL(2.C) \times T(3.1)\} x \{SL^d(2.C^d) \times^d T^d(3.1)\}, \qquad (49a)$$

$$M_{Tot} = \{M(x, \eta, R) \times S_{spin}\} x \{M^d(x^d, \eta^d, R^d) \times^d S^d_{spin}\}, \qquad (49b)$$

$$I_{Tot} = \{I_{orb} \times I_{spin}\} x \{I^d_{orb} \times^d I^d_{spin}\}, \qquad (49c)$$

To understand the dimensionality of symmetry one must first recall that isodual spaces are independent from conventional spaces and so are the related parameters. The doubling of the conventional dimensionality then yields *twenty* dimensions. The additional two dimensions are given by the novel isoselfscalarity (40) and their isoduals.

The reader should not be surprised that the four invariances (isoself-scalarity and isoselfduality) remained undetected throughout this century. In fact, their identification required the prior discovery of *new numbers*, first the numbers with arbitrary positive units for isoselfscalarity, and then the additional new numbers with arbitrary negative units for isoselfduality.

14. Iso-, Geno- and Hyper-Selfdual Cosmologies. As recalled above,in Ref.[9e] we submitted the Iso-Grand- Unification with the inclusions of gravity for matter and antimatter which is based on the abstract axioms of the true symmetry of the conventional Dirac equations, Eqs. (48), in its most general possible isotopic realization, the Kronecker product of the Poincaré-Santilli isosymmetry and its isodual with underlying, generalized, isoselfdual spaces and units.

$$\hat{S}_{Tot} = \{\hat{SL}(2.C)\hat{\times}\hat{T}(3.1)\}x\{\hat{SL}^d(2.C^d)\hat{\times}^d\hat{T}^d(3.1) = \hat{S}^d_{Tot}, \qquad (50a)$$

$$\hat{M}_{Tot} = \{\hat{M}(\hat{x},\hat{\eta},\hat{R})\hat{\times}\hat{S}_{spin}\}x\{\hat{M}^d(\hat{x}^d,\hat{\eta}^d,\hat{R}^d)\hat{\times}^d\hat{S}^d_{spin} = \hat{M}^d_{Tot}\},(50b)$$

$$\hat{I}_{Tot} = \{\hat{I}_{orb}\hat{\times}\hat{I}_{spin}\}x\{\hat{I}^d_{orb}\hat{\times}^d\hat{I}^d_{spin}\} = \hat{I}^d_{Tot}. \qquad (50c)$$

In this note we would like to indicate that, in turn, the Iso-Grand-Unification implies a new cosmology here submitted under the name of *Isoselfdual Cosmology*, which can be entirely defined by the above universal isosymmetry, and exhibits the following main properties:

1) The proposed cosmology is the only one known to this author which is characterized by a universal *symmetry* valid for relativistic and gravitational, exterior and interior, classical and operator, as well as matter and antimatter systems. In turn, the latter symmetry characterizes an (isoselfdual) *universal metric* $\hat{(}_{Tot} = \hat{\eta} \times \hat{\eta}^d = ^d_{Tot}$ with unrestricted functional dependence on local spacetime, velocity, density, temperature, and any other needed variable.

2) The proposed cosmology is *isoselfdual* (i.e., invariant under isoduality), thus implying as limit conditions equal amounts of matter and antimatter in the universe. A novelty here is the treatment of antimatter with new mathematics possessing negative-definite units. As a result, *all* characteristics of antimatter (and not just the charge as in other cosmological models) are opposite to those of matter. Another novelty is that antimatter belongs to a spacetime physically different than that of matter yet coexistent with the same, hereon referred as *matter and antimatter "warps"* (we use here the term "warp" rather than "dimension" because the latter would be technically inappropriate in view of the identity of the dimension of the two spaces).

3) The proposed cosmology predicts the possibility for future experimental identification whether a far away galaxy or quasar is made up of matter of of antimatter. This possibility is permitted by the predictions that antimatter emits a light different than that of matter (the isodual light of Ref.[10c]) which is *repelled* by the gravitational field of matter. Astrophysical studies along these lines are under way.

4) The proposed cosmology predicts *a universe with null total physical characteristics*, that is, null total mass, null total energy, null total time, etc., as measured by one observer made up of matter or, equivalently, of antimatter. This conception renders meaningless the question of the "age of the universe"; it removes the huge singularity of the big bang theory at creation (because total physical characteristics are null before and after creation); and it permits one of the firsts mathematically consistent representations of the creation of the universe.

5) The proposed cosmology admits a *novel notion of local spacetime*, the isospacetime described earlier, according to which different regions of the universe generally have different space and time characteristics due to *differences in the related units*, resulting in nine possible different isominkowskian Types. This renders meaningless the question of "the age of the matter (or, separately, antimatter) warp of the universe", the only physically meaningful questions being "the average age of the matter (and, independently, antimatter) warp of the universe" whose infinite value cannot be *a priori* excluded.

6) The proposed cosmology predicts speeds $c = c_o/n_4$ of electromagnetic waves which depend on the local physical conditions, which are *smaller* than the speed in vacuum c_o for propagation within physical media of low density such as planetary atmospheres or astrophysical chromospheres (isominkowskian media of Types 2, 6, 5, 8, and which are *bigger* than c_o for propagation within hyperdense media such as the interior of hadrons or in the core of stars (isominkowskian media of Types 3, 5, 9)[11l], thus admitting by conception and construction the superluminal speeds of Refs.[16].

7) The proposed cosmology can provide a direct geometrization (i.e., a representation via the universal metric itself) of any anisotropic and/or inhomogeneous distribution of light in the universe as established by experimental observations via isominkowskian media of Type 4 or 8.

8) The proposed cosmology predicts a size of the detected universe which is significantly *smaller* than that of current views[11f,11g]. This is due to the *isodoppler redshift* according to which light exits a galaxy already redshifted. The argument is that the medium in the interior of galaxies is not "perfectly empty" but it is in reality an ordinary physical medium of low density isominkowskian media of Type 5) because of the presence of dust, particles, etc. The speed of light within such a medium is then *decreased* resulting in the first contribution to redshift $\hat{\omega} = \omega_o[1 - v/(c_o/n_4) + ...]$ with full

isotopic contribution due to a (necessary) space-time symmetrization $\hat{\omega}$ =-$\omega_o[1-(v/c_o)\times(n_4/n_s)+...]$. Since $n_s.n_4$ for media of low density, the isotopy then increases the redshift originating in the interior of the galaxy. As a result, the inclusion of the physical medium inside a galaxy decreases the distance of the same from us on a comparative basis with current estimates. An additional reduction of the size of the universe is possible due the fact that the intergalactic space itself is a physical medium which, even though of extremely low density, it is nevertheless expected to yield a measurable contribution because of the extremely large intergalactic distances, thus yielding an additional isoredshift. Since no astrophysical information is currently available on the above media, the isominkowskian geometry can provide either a small correction to the current estimates of expansion of the universe or, as a limit case, its elimination altogether. The reader should also be aware that the isominkowskian geometry predicts for each cosmological isoredshift the existence of an internal isored- or isoblue-shift because the effects here considered depends on the frequency[11g], that is, the n's depend on ω_o.

9) The proposed cosmology eliminates the need for the "missing mass". In fact, the latter emerges because the total energy of the universe is computed under the *tacit assumption* of the universality of the speed of light in vacuum with the familiar expression $E_{Tot} = M_{Tot}c_o^2$, while the expression predicted by the proposed cosmology is $\hat{E}_{Tot} = M_{Tot} \times c^2$. The "missing mass" is then characterized by $M_{Miss} = M_{Tot}(1 - c_o^2/c^2) = M_{Tot}(1 - n_4^2)$. As a result, *the "missing mass" may imply that the average speed of light in the universe is bigger than that in vacuum.* Such a result should not be surprising because it essentially confirms all available evidence of superluminal speeds within hyperdense hadronic matter[11l], as well as the importance of considering interior problem in the computation of the average speed c. Equivalently, we can say that the isominkowskian geometry predicts that the total energy of each star is bigger than that currently estimated because of the internal superluminal character of the speed of light (isominkowskian media of Type 9)[11l] with consequential value $E = m \times c^2$ bigger than the value $E_o = m \times c_o^2$ currently estimated. In turn, this prediction evidently implies realistic possibility of eliminating the vexing problem of the "missing mass".

We should also indicate that the isoselfdual cosmology is a particular case of the broader *genoselfdual cosmology*, i.e., the cosmology constructed with the broader genomathematics and its isodual[5e]. The main difference is that the universe in the former is *closed-conservative*, while in the latter it is *open-nonconservative*, namely, the genoselfdual cosmology predicts a continuous creation throughout the universe.

In turn, the genoselfdual cosmology is a particular case of the *hyperselfdual cosmology*, namely, a cosmology based on the hypermathematics and its isodual[5e]. The main difference is that the universe in the former has only *one matter and one antimatter warp* while in the latter it admits *infinitely many countable and different, yet co-existing matter and antimatter warps.*

The reader with a technical knwovledbe of the iso-, geno- and hypermathematics will note that *all the above models of the universe are compatible with our sensory perception.* In fact, the abstract geometric axioms of space remain precisely the Euclidean axioms of our sensory perception, and they are only realized in their most general possible form. To be specific in this important point, the axiom-preserving character implies that our visual observation of a far away galaxy or quasars, not only does not imply that it is necessarily made of matter, but also it does not imply that it necessarily belongs to our own matter warp, because it could exist in a warp different than our own (where, again, we use the term "warp" because the "dimensions" remain the same, while the spaces themselves are different).

By remembering that the iso-, geno- and hyper-mathematics have been also constructed to provide quantitative studies of biological structures and that isodualities appears to be necessary for quantitative representation of bifurcations and other aspects in biology[11j], we can say that *the proposed iso-, geno- and hyper-selfdual cosmologies have been conceived to represent a "universe" inclusive of biological structures.* As a matter of fact, the inverse viewpoint appears to be more appropriate, namely, modern conceptions of the universe should be based on mathematical models primarily applicable to *biological* rather than *physical* systems.

15. Concluding Remarks. Perhaps the most significant result of this analysis is the identification that, to the author's best knowledge, *the isominkowskian geometry is the "only" geometry which is as invariant as the conventional Minkowskian geometry, yet admits all possible (3+1)-dimensional Riemannian metrics and all their (symmetric, signature preserving) generalizations to an arbitrary functional dependence*, the latter being requested for more realistic studies of interior gravitational problems.

Specifically, the "majestic beauty" of the Minkowskian geometry recalled Sect. 1 implies the property that, if the geometry yields a given measure, say, the length 2.5 cm for an object at rest in the observer frame at the time t = 3.7 sec, the same value 2.5 cm persists at all subsequent times in the absence of motion (invariance under the time evolution as well as the symmetries of the line element).

By comparison, Theorem 1 implies that, if the Riemannian, Finslerian, nondesarguesian or other geometries whose metrics have an explicit x-depemnence predict a given measure, say the length, 2.5 cm for an object at rest with respect to the observer at the time t = 3.7 sec, =the same measure is *altered* at subsequent times without motion, e.g., it may become 25.4 cm at t = 1,231 sec (lack of invariance under the time evolution as well as the symmetries of the line element).

It is easy to see that the same fate also holds for all isotopies of geometries with non-null curvature, such as the isoriemannian and isofinslerian geometries.

On the contrary, if the isominkowskian geometry predicts a certain measure, say, 2.5 cm at t = 3.7 sec, the same measure persists at all subsequent times exactly as it is the case for the conventional Minkowskian geometry.

The understanding of the above occurrence is the best proof of having understood the "axiom-preserving" character of the isotopies. In fact, if given measures are not preserved in time, this is evidence of the incorrect construction of the isotopies, rather than insufficiencies of the isominkowskian geometry.

For the case of the isoriemannian, isofinslerian, isonondesarguesian and other geometries the lack of invariance exists in the original geometries and merely carries over at the isotopic level.

The "secret" for the achievement of invariance is quite simple, and merely consists in the *absorption of all functional dependence in the isounit*, that is, invariance is guaranteed only when a given metric with nontrivial functional dependence $g = g(x, v, ...)$ is factorized into the conventional Minkowski metric, $g(x, v, ...) = \hat{T}(x, v, ...) \times \eta$, and the unit is assumed to be the *inverse* of the entire deviation from Minkowski, $\hat{I}(x, v, ...) = 1/\hat{T}(x, v, ...)$. This yields again the fundamental relations of this analysis, Eqs. (1) and (2) for the isominkowskian representation of gravity.

The isoriemannian, isofinslerian, isonondesarguesian and other geometries also have an arbitrary functional dependence, $\hat{g} = \hat{g}(x, v, ...)$. However, the factorization is done with respect to the original metric g(x, ...) (whether Riemann, Finsler, etc.), $\hat{g}(x, v, ...) = \hat{T}(x, v, ...) \times g(x, v, ...)$. The isounit is then constructed as in the preceding case, $\hat{I} = 1/\hat{T}$. The loss of invariance then follows because inherent in the original metric.

Alternatively and equivalently, we can say that *invariance is ensured when the geometry admits a bona-fide "symmetry" (and not "covariance") of the line element.* This condition too uniquely selects the isominkowskian geometry as the only invariant one. In fact, as stressed earlier in the text (as well as in the literature), the universal Poincaré-Santilli isosymmetry can be *only* constructed for the isominkowski space, while no such construction is possible for the Riemannian, Finslerian and other spaces and all their infinitely possible isotopies.

In the final analysis, one should keep in mind that *the isominkowskian geometry is "directly universal"*, in the sense that it admits as particular cases all conceivable, well behaved (3+1)-dimensional metrics, thus including all possible Riemannian, Finslerian, nondesarguesian and other metrics and all their isotopies (universality), directly in the fixed frame of the experimenter without any need of the transformation theory (direct universality)[11k]. The above direct universality, when joined with its unique invariance properties, then voids the need for any other spacetime geometry, to our best understanding at this writing.

We should also indicate that the analysis of this paper has a number of connections with various other lines of inquiries presented in these proceedings [3a], which we regret to be unable to point out at this time for brevity, but which we hope to point out at some future time.

All in all, it is hoped that this study is another illustration of the fact that *physics is a discipline which will never admit "final theories"*. By following the teaching of the Founders of contemporary knowledge, such as Schwarzschild's *two* articles[2d], the well known one on the *exterior* problem and the little known additional article on the *interior* problem, we can additional say that *the maturity in the formulation of a new theory is also given by the joint identification of its limits of applicability.*

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References

[1] H.A.Lorentz, Amst. Proc.6, 809 (1904) [1a]; H.Poincaré, Compte Rendues, Paris 140. 1504 (1905) [1b]; A.Einstein, Ann. Phys. 17, 891 (1905) [1c]; H.Minkowski, Nachr. Ges. Wiss. Gottingen 43 (1908) [1d].

[2] B.Riemann, Gott. Nachr. **13,** 133 (1868) and *Collected Works*, H.Weber, Editor, Dover, New York (1953) [2a]; D.Hilbert, Nachr. Kgl. Ges. Wissench. Gottingen, 1915, p.395 [2b]; A.Einstein, Sitz. Ber. Preuss. Akad. Wissench. Berlin, 1915, p.844 [2c]; K.Schwartzschild, Sitzber. Deut. Akad. Wiss. Berlin. Kl. Math.–Phys. Tech., 189 and 424 (1916) [2d]; H.Weyl, *Raum–Zeit–Materie*, Springer, Berlin (1916) [2e]; S. Adler, A. Bazin and L. Schieffer, *Introduction to General Relativity* (Mc.Graw-Hill, New York, 1984) [2f]; C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, Freeman, San Francisco (1970) [2g]; D.Lovelock and H.Rund, *Tensors, Differential Forms and Variational Principles*, J. Wiley, New York (1975) [2h].

[3] E. I. Guendelman, Editor, *Proceedings of the Workshop on Modern Modified Theories of Gravitation and Cosmology*, Ben Gurion Univ, Israel, June 1997, Hadronic J **21**, issues 1, 2, 3 (1998) [3a]. M. Barone and F. Selleri, Editors, *Frontiers of Fundamental Physics*, Plenum, New York (1994) [3b]; H. E. Wilhelm and K. Liu, Editords, *Space, Time and Motion: Theory and Experiments*, Chinese J. Syst Eng. Electr., **6**, issue 4 (1995) [3c]; .

[4] R. H. Bruck, A Survey of Binary Systems, Springer-Verlag, Berlin (1958) [4a]; K. McCrimmon, Isotopies of Jordan Algebras, Pacific J. Math. 15, 925-962 (1965) [4b]; D. M. Norris *et al, Tomber's Bibliography and Index in Nonassociative Algebras*, Hadronic Press, Palm Harbor, FL (1984) [4c].

[5] R. M. Santilli, Hadronic J. 1, 224, and 1267 (1978) [5a]; R. M. Santilli, Phys. Rev. D20, 555 (1979) [5b]; *Foundations of Theoretical Mechanics,* Vol. I (1978) and II (1983), Springer-Verlag, Heidelberg-New York [5c]; Algebras, Groups and Geometries 10, 273 (1993) [5d]; Rendiconti Circolo Matematico Palermo, Suppl. 42, 7 (1996) [5e]; "Isotopies of trigonometric and hyperbolic functions", Ann. Univ. of Costanta, Romania, **5**,113 (1996) [5f]; "Isominkowskian geometry for the gravitational treatment of matter and its isodual for antimatter", Intern. J. Modern Phys. D, in press [5g].

[6] J. V. Kadeisvili, Algebras, Groups and Geometries **9**, 283 and 319 (1992) [6a]; Gr. T. Tsagas and D. S. Sourlas, Algebras, Groups and Geometries 12, 1 and 67 (1995) [6b]; J. V. Kadeisvili, Math. Methods in Applied Sciences, 19, 1349 (1996) [6c]; A. K. Aringazin, D. A. Kirukhin and R. M. Santilli, Algebras, Groups and Geometries **12**, 255 and 359 (1995) [6d]; P. Vacaru, Algebras, Groups and Geometries **14**, 211 (1997) [6e]; R. Aslander and S. Keles, Algebras, Groups and Geometries **14**, 283 (1997) [6f].

[7] R. M. Santilli, Hadronic J. **1**, 574 (1978) [7a]; C. N. Ktorides, H. C. Myung and R. M. Santilli, Phys. Rev. D **22**, 892 (1980) [7b]; G.Eder,

Hadronic J. **4,** (1981) and **5,** 750 (1982)[7c]; H. C. Myung and R. M. Santilli, Hadronic J. **5**, 1277 (1982) [7d]; G.Eder, Hadronic J. **4,** (1981) and **5,** 750 (1982)[7e]; S. Okubo, Hadronic J. **5**, 1667 (1982) [7f]; A. J. Kalnay, Hadronic J. **6**, 1 (1983) [7d]; A. J. Kalnay and R. M. Santilli, Hadronic J. **6**, 1798 (1983) [7g]; R. M. Santilli, Lett. Nuovo Cimento **3,** 509, (1983) [7h]; R. Mignani, H. C. Myung and R. M. Santilli, Hadronic J. **6**, 1873 (1983) [7i]; R. Mignani, Lettere Nuovo Cimento **39**, 406 (1984) [7j]; M.Nishioka, Nuovo Cimento **82A,** 351 (1984) [7k]; M.Gasperini, Hadronic J. 7, 971 (1984) [7l]; D. Constantoupoulos and C. N. Ktorides, J. Phys. A bf 17, L29 (1984) [7m]; E.B,Lin, Hadronic J. 11, 81 (1988) [7o]; A. Jannussis and R. Mignani, Physica A **187,** 575 (1992) [7p]; A.O.E.Animalu, Adronic J. **17,** 349 (1995) [7q]; R. M. Santilli, Found. Phys **27**, 625 (1997) [7r]; R. M. Santilli, *Elements of Hadronic Mechanics*, Vol. I [7s] and II [7t] Second edition (1995), Ukraine Academy of Sciences, Kiev.

[8] R.M.Santilli, Nuovo Cimento Lett. **37,** 545 (1983) [8a]; Hadronic J. **8,** 25 and 36 (1985) [8b]; JINR Rapid Comm. **6,** 24 (1993) [8c]; J. Moscow Phys. Soc. **3,** 255 (1993) [8d]; JINR Comm. E4-93-352 (1993) [8e]; Chinese J. Syst. Eng. & Electr. **6,** 177 (1996) [8f].

[9] R.M.Santilli, in *Proceedings of the VII M. Grossmann Meeting on General Relativity*, R. T. Jantzen, G. Mac Kaiser and R. Ruffinni, Editors, World Scientific, Singapore (1996), p. 500 [9a]; in *Gravity, Particles and Space–Time*, P. Pronin and G. Sardanashvily, Editors, World Scientific, Singapore (1995), p. 369 [9b]; Comm. Theor. Phys. **3,** 47 (1994) [9c]; Comm. Theor. Phys. **4,** 1 (1995) [9d]; Found. Phys. Letters **10,** 305 (1997) [9e]. in *Proceedings of the VIII Marcel Grossmann meeting on General Relativity and Gravitation*, World Scientific, Singapore, in press [9f].

[10] R.M.Santilli, Comm. Theor. Phys. **3,** 153 (1994) [10a]; Hadronic J. **17,** 257 (1994) [10b]; Hyperfine Interaction, in press. [10c]; in *New Frontiers of Hadronic Mechanics*, T.L.Gill, Editor, Hadronic Press, Palm

Harbor, FL (1996) [10d]; "Classical isodual theory of antimatter", submitted for publication [10e].

[11] F. Cardone, R. Mignani and R. M. Santilli, J. Phys. G **18**, L61 and L141 (1992) [11a]; F. Cardone and R. Mignani.. Univ. of Rome preprint 894 [11b], R. M. Santilli, Hadronic J. **15**, 1 (1992) [11c]; R. M. Santilli, Comm. Theor. Phys **4**, 1, (1995) [11d]; "Exact representation of total nuclear magnetic moments via relativistic hadronic mechanics", in press [11e]; R. M. Santilli, Hadronic J. Suppl. **4A**, issue no.s 1 and 2 (1988) and R. Mignani, Physics Essays **5**, 531 (1992) [11f]; R. M. Santilli, in *Frontiers in Fundamental Physics,* M. Barone and F. Selleri, eds., Plenum, new York (1994) , p. 41 [11g]; A. O. E. Animalu, Hadronic J. **17**, 349 (1995), A. O. E. Animalu and R. M. Santilli, Int. J. Quantum Chemistry **29**, 175 (1995) [11h]; R. M. Santilli and D. Shillady, "Ab initio isochemistry, I and II", Hadronic J. 20 (1997), in press [11i]; R. M. Santilli, *Isotopic, Genotopic and Hyperstructural Methods in Theoretical Biology,* Ukraine Academy of Sciences, Kiev (1996) [11j]; A. K. Aringazin, Hadronic J. **12**, 71 (1989), A. K. Aringazin and K. M. Aringazin, in *Frontiers of Fundamental Physics,* M. Barone and F. Selleri, Editors, Plenum, New York (1995), p. 153 [11k]; Yu Arestov, R. M. Santilli and V. Solovianov, "Apparent experimental evidence on suprluminal deviations from the Minkowskian geometry within the hyperdense hadrons", to appear (1998) [11e].

[12] S.L. Adler, Phys.Rev. **17,** 3212 (1978) [12a]; S.Okubo, Hadronic J. **5**, 1667 (1982) [12b]; Cl.George, F. Henin, F.Mayene and I.Prigogine, Hadronic J. **1,** 520 (1978) [12c]; A.Jannussis, R.Mignani and D.Skaltsas Physics A **187,** 575 (1992) [12d]; D.Rapoport–Campodonico, Algebras, Groups and Geometries **8,** 1 (1991) [12e]; R.Mignani, Physics Essays **5,** 531 (1992) [12f]; T.Gill, J.Lindesay and W.W.Zachary, Hadronic J. **17,** 449 (1994) [12g]; A. Jannussis, M. Miatovic and B. Veljanowski, Physics Essays 4, 202 [1991] [12h]; J. Ellis, in *Proceedings of the Seventh Marcel Grossmann meeting on general Relativity,* R. T. Jantsen, G. Mac Kaiser and R. Ruffini, Editors, World Scientific (1966), p., 755 [12i]; J. Ellis, N. E. Mavromatos and D. V. Nanopoulos in *Proceedings of the Erice Summer School, 31st Course: From Superstrings to the Ori-*

gin of Space-Time, World Scientific [1996] [12j]; M. Battler, M. McBee and S. Smith http://home1.gte.net/ibr [12k].

[13] A.K. Aringazin, A.Jannussis, D.F.Lopez, M.Nishioka and B.Veljanosky, *Santilli's Lie–Isotopic Generalization of Galilei's Relativities* (1990), Kostarakis Publisher, Athens, Greece [13a]; J.V.Kadeisvili, *Santilli's Isotopies of contemporary Algebras, Geometries and Relativities*, Hadronic Press, **FL** (1991), Second Edition, Ukraine Academy of Sciences, Kiev, in press [13b]; D.S.Sourlas and G.T.Tsagas, *Mathematical Foundations of the Lie–Santilli Theory*, Ukraine Academy of Sciences, Kiev (1993) [13c]; J.Lohmus, E.Paal and L.Sorgsepp, *Nonassociative Algebras in Physics*, Hadronic Press, Palm Harbor, **FL**, USA (1994) [13d]; S. Vacaru *Interactions, Strings and Isotopies in Higher Order Anisotropic Spaces*, hadronic Press, in press (1998) [13e].

[14] D.F. Lopez, Hadronic J. **16**, 429 (1993) [14a]; A. Jannussis, R. Mignani and R.M. Santilli, Ann. Fond. L. de Broglie **18**, 371 (1993) [14b]; A. Jannussis and D. Skaltsas, Ann. Fond. L. de Broglie **18**, 275 (1993) [14c]; D. Schuch, Phys. Rev. A **55** (1997), in press [14d].

[15] A.Einstein, B.Podolsky and N.Rosen, Phys. Rev. **47,** 777 (1935) [15a]; J.Von Neumann, *The Mathematical Foundations of Quantum Mechanics*, Princeton Univ. Press, Princeton, N. J. (1955) [15b]; J.S. Bell, Physics **1,** 195 (1965) [15c]; D.Bohm, *Quantum Theory*, Dover Publications, New York (1979) [15d]; R. M. Santilli, "Isotopies of the SU(2) symmetry and its application to local realism", Acta Appl. Math, in press [15e].

[16] H.A.Lorentz, *Versuch einer Theorie der Elektrishen und Magnetishen Erscheinungen in bewenglen Körpern*, Leyda (1895) [16a]; W.Pauli, *Theory of Relativity*, Pergamon Press, (1958) [16b]; A.Enders and G.Nimitz, J. Phys. France **2,** 1693 (1992) [16c]; A.M.Steuimnberg, P.C.Kwait and R.Y.Chiaoo, Phys. Rev. **Lett.** 68, 2421 (1992) [16d]; F.Mirabel and F.Rodriguez, Nature 371, 464 (1994) [16e]; J.Tingay

et al., Nature 374, 141 (1995) [16f]; D.Baylin *et al.*,IAU Comm. 6173 (1995) [16g]; E. Recami, Hadronic J. **21** (1998), in press [16h]; H. Nimtz, Progress Quantum Electr. 1, 81 (1997) [16i]; P. Saari and K. Reivelt, Phys. Rev. Letters **79**, no. 21 (1997) [16j]; E. Recami, Physica A,, in press (1998) [16k].

[17] 17. P. Freud, Ann. Math. 40 [2], 417 (1939) [17a]; H. Rund, Algebras, Groups and Geometries 8, 267 (1991) [17b].

[18] 18. P. A. M. Dirac, Proc. Roy. Soc. A 322, 435 (1971) and 328, 1 (1972).

[19] 1. J. Julve et al., Gen. Rel. Grav. **28**, 759 (1996) [19a]; A. Lopez-Pinto et al, "Ordinary matters in nonlinear affine gauge theories", Gen. Rel. Grav. **28** (1997), in press [19b]; E. Schruefer et al., Gen. Rel. Grav. **19**, 197 (1987) [19c]; M. Kaku, *Quantum Field Theory,* Oxford Univ. Press, New York (1993) [19d]; D. Hestenes and G. Sobczyk, *Clifford Algebras to Geometric Calculus,* Reidel, Dorfdrecht (1984) [19e].

[20] M.J.G.Veltman, in *Methods in Field Theory*, R. Ballan and J. Zinn–Justin, Editors, North–Holland, Amsterdam (1976) [20a]; C.J.Isham, R.Penrose and D.W.Sciama, Editors, *Quantum Gravity 2*, Oxford University Press, Oxford (1981) [20b]; *Proceedings of the VII M. Grossmann Meeting on General Relativity*, R. T. Jantzen, G. Mac Kaiser and R. Ruffinni, Editors, World Scientific, Singapore (1996) [20c]; W.A.Rodrigues, jr, and J.–Yu Lu, "On the existence of undistorted progressive waves of arbitrary speed", Found. Phys., to apper [20f].