Foundations of Isomathematics; A mathematician's curiosity

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Historical Background

- Modern mathematics has a strong foundation laid down by Dr.Bertrand Russel and Dr.Whitehead through 'Principia Mathematica'.
- Algebraic structures like Group, Ring, Field, Algebra, Vector space and related structures had already found vast applications in the modern mathematical and scientific developments.

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- More general structures like Groupoids, Semigroups, Monoids, Quasigroups and Loops were also being studied which were to find vast applications in future.
- The detailed consolidated account of these generalized structures is found in 'Survey of Binary Systems' by R.H.Bruck [1] in 1958.

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Need for New Mathematics

- While the scientific discoveries and mathematical knowledge were moving hand in hand, towards the end of 20th century there were few mathematically unexplained physical phenomena in Quantum mechanics and Quantum Chemistry.
- These situations called for a more generalized mathematical structure, latter called as 'Isofield'.

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Exterior Dynamical Systems

System
Exterior Dynamical Systems. Point-like particles are moving in a homogeneous and isotropic vacuum with local-differential and potential-canonical equations of motion.

$\longmapsto$

Nature
Linear,Local, Newtonian Lagrangian and Hamiltonian

$\longmapsto$

Mathematics
Conventional Mathematical Structures such as Algebras, Geometries, Analytical Mechanics, Lie Theory.

Interior Dynamical Systems

System
Interior Dynamical Systems. Extended non-spherical deformable particles moving within non-homogeneous anisotropic physical medium

$\longmapsto$

Nature
non-linear,non-local,non-newtonian,non-lagrangian and non-hamiltonian

$\longmapsto$

Mathematics
Non-conventional most general possible mathematical Structures which are axiom preserving non-linear non-local formulations of current mathematical structures.

Santilli's Achievement

Problem
To represent the non-local, non-linear, non-lagrangian, non-hamiltonian, and non-newtonian system characterizing the motion of extended particle within physical media.

$\longmapsto$

Solution
Isotopic Generalization of Contemporary mathematical structures, like Field, Vectorspace, Transformations, Lie Algebra and conventional geometries.

$\longmapsto$

Span
This Isotopic generalization by Santill leads to the maximum generalization of Gallili's, Einstein's relativity and contemporary mathematical structures.

Emergence of New Mathematics

It was Santilli in early 1992 who first discovered the concept of 'Isofield' which further led to a plethora of new concepts and a whole new 'Isomathematics' which is a step further in Modern Mathematics.

Emergence of New Mathematics

- This work aims at exploring the very basics of Isomathematics as formulated by Santilli [8] and [9].
- The concept of 'Isotopy' plays a vital role in the development of this new age mathematics.

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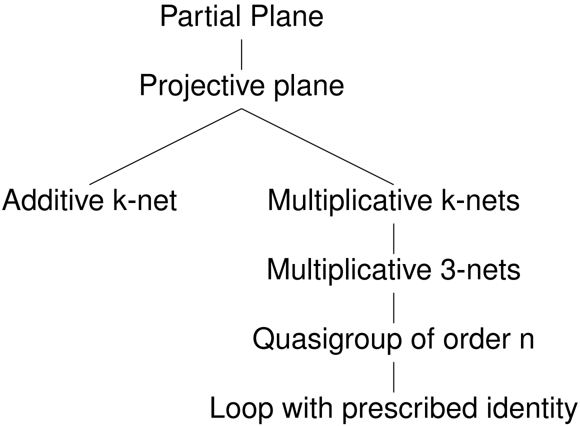
Emergence of New Mathematics

- Starting with Isotopy of groupoids we develop the study of Isotopy of quasi groups and loops via Partial Planes, Projective planes, 3-nets and multiplicative 3-nets.

From partial Plane to Loop



Partial Plane

Definition

*A **partial plane** is a system consisting of a non-empty set G partitioned into two disjoint subsets(one of which may be empty), namely the point-set and the line-set together with a binary relation, called incidence, such that (i) (Disjuncture) If x is incident with y in G then one of x, y is a line of G and the other is a point, (ii) (Symmetry) If x is incident with y in G then y is incident with x in G, and (iii) If x, y are distinct elements of G there is at most one z in G such that x and y are both incident with z in G.*

Projective Plane

Definition

*A **Projective plane** is a special kind of a partial plane G such that; (iv) if x and y are distinct points or distinct lines of G, there exists a z in G such that x and y are both incident with z in G ; (v) there exists at least one set of four distinct points of G no three of which are incident in G with the same element.*

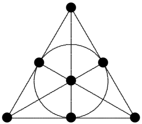
It is easy to show that in the presence of (i)- (iv), postulate (v) is equivalent to ; (vi) there exists at least one set of four distinct lines of G no three of which are incident in G with the same element.

Example

A *projective plane* of order n has

$n^2 + n + 1$ points;
$n^2 + n + 1$ lines;
$n + 1$ points on each line;
$n + 1$ lines through each point.

Example: The projective plane of order $n = 2$



7 points
7 lines
3 points on each line
3 lines through each point

k-net

Definition

*A **k-net** is a partial plane N whose line-set has been partitioned into k disjoint classes such that (a) N has at least one point, (b) Each point of N is incident in N with exactly one line of each class, and (c) Every two lines of distinct classes in N are both incident in N with exactly one point.*

If some line of a k-net N is incident with exactly n distinct points in N, so is every line of N. The cardinal number n is called the **order of N.**

Since a net is a partial plane, every net may be embedded in at least one projective plane. Every projective plane contains nets and, of these, two types additive 3-net and multiplicative 3-net have special significance.

Example; From Geometry to Algebra

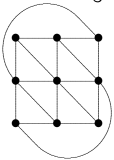
A quasigroup $(X, *)$ of order n determines a *3-net,* eg.

Quasigroup $(X, *)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \longleftrightarrow$$

$n \times n$

3-Net $\mathcal{N}_3$



n^2 points;
$3n$ lines;
3 parallel classes of n lines each;
n points on each line

From 3-net to Loop

We can have an **additive 3-net** and **multiplicative 3-net** of a projective plane.

- *Every 3-net N of order n gives rise to a class of quasigroups* $(Q, \circ)$ *of order n* by defining one-to-one mappings $\theta(i)$ with $i = 1, 2, 3$ of Q upon the class of i-lines of N.
- Two quasigroups obtainable from the same 3-net by different choices of the set Q or of the mappings $\theta(i)$ are said to be **isotopic**.
- *For any Q, the* $\theta(i)$ *can be so chosen that* $(Q, \circ)$ *is a loop with a prescribed element e of Q is as identity element.*

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Quasigroup

Definition

*A **Quasigroup** is a groupoid G such that, for each ordered pair $a, b \in G$, there is one and only one x such that $ax = b$ in G and one and only one y such that $ya = b$ in G.*

In other words a quasigroup is groupoid whose composition table is a Latin square.

Definition

*A **loop** is a quasigroup with an identity.*

An associative loop is a group.

Isotopy of Groupoids

Definition

Let$(G,\cdot)$ and $(H,\circ)$ be two groupoids. An ordered triple (α,β,γ) of one-to-one mappings α,β,γ of G upon H is called an ***isotopism*** *of $(G,\cdot)$ and $(H,\circ)$. provided $(x\alpha)\circ(y\beta)=(x.y)\gamma$. $(G,\cdot)$ is said to be isotopic with $(H,\circ)$ or $(G,\cdot)$ is said to be an* ***isotope*** *of $(H,\circ)$.*

Isotopy of Groupoids

- Isotopy of groupoids is an equivalence relation.
- Every isotope of a quasigroup is a quasigroup.

Origin of Isotopy

- "The concept of isotopy seems very old. In the study of Latin squares (which were known to BACHET and certainly predate Euler's problem of the 36 0fficers) the concept is so natural to creep in unnoticed; and latin squares are simply the multiplication tables of finite quasigroups."
- "It was consciously applied by SCHÖNHART, BAER and independently by ALBERT. ALBERT earlier had borrowed the cocept from topology for application to linear algebras; in the latter theory it has virtually been forgotten except for applications to the theory of projective planes."

Example1: Isotopy of Groupoids

Consider the two groupoids $G = \{1,2,3\}$ and $G' = \{a,b,c\}$ defined by the following composition tables.

.	1	2	3
1	1	3	2
2	3	1	3
3	2	3	2

and

*	a	b	c
a	a	c	b
b	b	b	c
c	a	a	b

Then the ordered triple (α, β, γ) defined by the permutations $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 \\ b & c & a \end{pmatrix}$ and $\gamma = \begin{pmatrix} 1 & 2 & 3 \\ c & a & b \end{pmatrix}$ is an isotopy.

Thus $(G, .)$ and $(G, *)$ are **isotopic.**

Note that an isomorphism is just a particular case of isotopy wherein $\alpha = \beta = \gamma$. If I is the identity mapping then (α, β, I) is called a ***principal isotopy*** between the two groupoids.

Example 2: Isotopy of Quasigroups

Consider the groupoids $(L, .)$ and $(L', *)$ with multiplication tables as;

.	0	1	2	3	4
0	0	1	3	4	2
1	1	0	2	3	4
2	3	4	1	2	0
3	4	2	0	1	3
4	2	3	4	0	1

and

*	0	1	2	3	4
0	1	0	4	2	3
1	3	1	2	0	4
2	4	2	1	3	0
3	0	4	3	1	2
4	2	3	0	4	1

Here the ordered triple (α, β, γ) defined as

$$\alpha = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 0 & 3 \end{pmatrix} \text{ and}$$

$$\gamma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 0 & 4 & 3 \end{pmatrix}$$

is an Isotopism. Note that L and L' are quasi groups.

Example 3: Principal Isotopy of Groupoids

Consider the two groupoids G and G' defined by the following composition tables.

.	1	2	3
1	1	3	2
2	3	1	3
3	2	3	2

and

*	1	2	3
1	1	2	2
2	3	2	1
3	1	3	3

Then the ordered triple (α, β, γ) defined by the permutations
$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$,
$\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ is a principal isotopy.

Consider the groupoids and their isotopy as defined in Example 1, we can define $\delta = \alpha\gamma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$,
$\eta = \beta\gamma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.
Note that (δ, η, I) is the principal isotopy corresponding to (α, β, γ).
In this manner every isotopy gives rise to a principal isotopy such that the isotope and the principal isotope are isomorphic.
In general, it is sufficient to consider only the principal isotopies in view of the following Theorem.

Theorem

If G and H are isotopic groupoids then H is isomorphic to a principal isotope of G.

Proof

Let (α, β, γ) *be an isotopy of G on to H. Let* $\delta = \alpha\gamma^{-1}$ *and* $\eta = \beta\gamma^{-1}$. *We have* $(\alpha, \beta, \gamma) = (\delta\gamma, \eta\gamma, \gamma)$. *Hence there exists a groupoid K such that* (δ, η, i) *is a principal isotopy of G on to K and* γ *is an isomorphism of K on to H.*

'Necessary and sufficient conditions that a groupoid possess an isotope with identity element are that the groupoid have a right nonsingular element and a left nonsingular element' Ref.[1]p.57.

All the elements of a quasigroup are left nonsingular and right nonsingular (as every element occurs only once in every row and column). Therefore **every quasigroup is isotopic to a loop**.

- **This lifting of the multiplicative quasigroup to a loop with the prescribed identity gives rise to a multiplicative group with the ’prescribed identity’ which Santilli termed as ’Isounit’. The resulting field with the multiplicative isounit is called as an Isofield [4].**
- Without loss of generality we can say that the words ’Isotopy’ and ’Axioms preserving’ are synonymous.

Santilli's Isofield

Definition

Given a field F with elements $\alpha, \beta, \gamma, \ldots$, sum $\alpha + \beta$, multiplication $\alpha\beta$, and respective units 0 and 1, "Santilli's isofields" are rings of elements $\hat{\alpha} = \alpha\hat{1}$ where α are elements of F and $\hat{1} = T^{-1}$ is a positive- definite $n \times n$ matrix generally outside F equipped with the same sum $\hat{\alpha} + \hat{\beta}$ of F with related additive unit $\hat{0} = 0$ and a new multiplication $\hat{\alpha} * \hat{\beta} = \hat{\alpha} T \hat{\beta}$, under which $\hat{1} = T^{-1}$ is the new left and right unit of F in which case $\hat{F}$ satisfies all axioms of a field.

Santilli's Isofield

The 'isofields' $\hat{F} = \hat{F}(\hat{\alpha}, +, \hat{\times})$ are given by elements $\hat{\alpha}, \hat{\beta}, \hat{\gamma} \ldots$ characterized by one-to-one and invertible maps $\alpha \to \hat{\alpha}$ of the original element $\alpha \in F$ equipped with two operations $(+, \hat{\times})$, the conventional addition $+$ of F and a new multiplication $\hat{\times}$ called "isomultiplication" with corresponding conventional additive unit 0 and a generalized multiplicative unit $\hat{1}$, called "multiplicative isounit" under which all the axioms of the original field F are preserved.

- If the conventional field is chosen to be alternative under the operation of conventional multiplication then the resulting isofield is also isoalternative under isomultiplication.
- If the given algebraic structure is a noncommutative division ring (e.g. ring of quaternions) then the resulting isoalgebraic structure is also noncommutative under the isomultiplication.

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- **This new algebraic structure has revolutionized contemporary mathematics and found its applications in so far unexplored (unexplained) and unknown territories of quantum mechanics and quantum chemistry.**
- The resulting new theory of numbers has become the basis of recent studies of **nonlinear-nonlocal, nonhamiltonian systems** in nuclear particle and statistical physics.

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Isofields and Isonumbers

- Isofields are of two types, **isofield of first kind**; wherein the isounit does not belong to the original field, and **isofield of second kind**;
- wherein the isounit belongs to the original field. The elements of the isofield are called as **isonumbers.**
- This leads to a a plethora of new terms and parallel development of conventional mathematics.

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Generalized Isostructures

- Isonumbers is the generalization of conventional numbers formed by lifting conventional unit 1 to $\hat{1}$.
- In fact this lifting leads to a variety of algebraic structures which are often used in physics.
- The following flowchart is self explanatory.
  Isonumbers $\longrightarrow$ Isofields $\longrightarrow$ Isospaces $\longrightarrow$ Isotransformations $\longrightarrow$ Isoalgebras $\longrightarrow$ Isogroups $\longrightarrow$ Isosymmetries $\longrightarrow$ Isorepresentations $\longrightarrow$ Isogeometries etc.

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The New structure

- **In view of the definition of an isofield [9], we can say that an isofield is an additive abelian group equipped with a new unit (called isounit) and isomultiplication defined appropriately so that the resulting structure becomes a field.**
  If the original field is alternative then the isofield also satisfies weaker *isoalternative laws* as follows.
  $\hat{a}\hat{\times}(\hat{b}\hat{\times}\hat{b}) = (\hat{a}\hat{\times}\hat{b})\hat{\times}\hat{b}$ **and** $\hat{a}\hat{\times}(\hat{a}\hat{\times}\hat{b}) = (\hat{a}\hat{\times}\hat{a})\hat{\times}\hat{b}$.
  We mention two important proposition by Santilli.

Propositions

Proposition

The necessary and sufficient condition for the lifting (where the multiplication is lifted but elements are not)
$F(a, +, \times) \longrightarrow (\hat{F}, +, \hat{\times}), \hat{\times} = \times T \times, \hat{1} = T^{-1}$ *to be an isotopy (that is for* $\hat{F}$ *to verify all axioms of the original field F) is that T is a non-null element of the original field* F.

Proposition

The lifting (where both the multiplication and the elements are lifted)
$F(a, +, \times) \longrightarrow (\hat{F}, +, \hat{\times}), \hat{a} = a \times \hat{1}, \hat{\times} = \times T \times, \hat{1} = T^{-1}$
constitutes an isotopy even when the multiplicative isounit $\hat{1}$ *is not an element of the original field.*

We propose three propositions which directly follow from the definition of isofield.

Proposition

If $(F, +, \times)$ *is a field and* $(\hat{F}, \hat{+}, \hat{\times})$ *is the corresponding isofield such that the isounit* $\hat{I} \in F$ *then* $(F, +, \times) \cong (\hat{F}, \hat{+}, \hat{\times})$.

Propositions

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If $(F, +, \times)$ is a field and $(\hat{F}, \hat{+}, \hat{\times})$ is the corresponding isofield such that the isounit $\hat{I} \notin F$ then $(F, +, \times)$ is isotopic to $(\hat{F}, \hat{+}, \hat{\times})$.

Proposition

Isofield corresponding to a non-commutative field is isotopic to the original field.
The noncommutative ring of Quaternions is an example of this type.

Some Open Problems

- Can we construct finite isofields of first kind ?
- Can we construct finite isofields of second kind ?
- What is the structure of Quaternionic isofields ?
- Can a more generalized isofield be defined with a prescribed additive identity ?

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THANK YOU

FOR YOUR ATTENTION