On the isoH$_v$-numbers

Thomas Vougiouklis
Democritus University of Thrace, School of Education,
68 100 Alexandroupolis, Greece
tvougiou@eled.duth.gr

Abstract

A basic question on Santilli's isotheory is "what are the numbers?" This question in hyperstructures is "what are the hypernumbers?" We present special classes of the largest class of hyperstructures, called H_v-structures, which give to this theory a variety of mathematical models.

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1 Introduction

We deal with hyperstructures called H_v-structures introduced in 1990 by Vougiouklis [14], which satisfy the weak axioms where the non-empty intersection replaces the equality.

Some basic definitions are the following [15]:

In a set H equipped with a hyperoperation (abbreviation: *hyperoperation = hope)*

$$\cdot : H \times H \to P(H) - \{\varnothing\},$$

we abbreviate by
WASS the *weak associativity*: $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ and by
COW the *weak commutativity*: $xy \cap yx \neq \varnothing, \forall x, y \in H$.

The hyperstructure $(H, \cdot)$ is called an *H_v-semigroup* if it is *WASS*, it is called *H_v-group* if it is reproductive H_v-semigroup, i.e., $xH = Hx = H, \forall x \in H$. The hyperstructure $(R, +, \cdot)$ is called an *H_v-ring* if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \varnothing,\ (x+y)z \cap (xz+yz) \neq \varnothing,\ \forall x, y, z \in R.$$

An H_v-ring is called additive if its addition is a hope and the multiplication is an ordinary operation and is called multiplicative if its product is a hope and addition is an operation.

Motivation for H_v-structures: We know that the quotient of a group with respect to an invariant subgroup is a group. F. Marty from 1934, states that, the quotient of a group with respect to any subgroup is a hypergroup. Finally, the quotient of a group with respect to any partition (or equivalently to any equivalence relation) is an H_v-group.

In an H_v-semigroup the powers of an element $h \in H$ are defined as follows: $h^1 = \{h\}, h^2 = h \cdot h, ..., h^n = h \circ h \circ ... \circ h$, where $(\circ)$ denotes the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An H_v-semigroup $(H, \cdot)$ is called *cyclic of period s*, if there exists an element h, called *generator*, and a natural number s, the minimum one, such that $H = h^1 \cup h^2 ... \cup h^s$. Analogously the cyclicity for the infinite period is defined. If there is an element h and a natural number s, the minimum one, such that $H = h^s$, then $(H, \cdot)$ is called *single-power cyclic of period s.*

Let $(H, \cdot), (H, *)$ be H_v-semigroups defined on the same set H. $(\cdot)$ is called *smaller* than $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an $f \in Aut(H, *)$ such that $xy \subset f(x * y),\ \forall x, y \in H$. Then we write $\cdot \leq *$ and we say that $(H, *)$ *contains* $(H, \cdot)$. If $(H, \cdot)$ is a structure then it is called *basic structure* and $(H, *)$ is called $H_b - structure$.

Theorem 1.1. *(The Little Theorem). Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.*

So we have posets on H_v-structures. In the problem of enumeration of classes of H_v-structures we have results by using computers [7]. The partial

order in H_v-structures restrict the problem in finding the minimal, up to isomorphisms, H_v-structures.

During last decades hyperstructures seem to have a variety of applications not only in other branches of mathematics but also in many other sciences including social studies. These applications range from biomathematics -conchology, inheritance- and hadronic physics, to mention but a few. The hyperstructure theory is closely related to fuzzy theory; thus, hyperstructures can now be widely applicable in industry and production, too.

In several books and papers one can find numerous applications [3],[15]. An new application, which combines hyperstructure theory and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis [23].

Definition 1.1. *[12],[15] Let $(G,\cdot)$ be a groupoid then for every $P\subset G$, $P\neq\varnothing$, we define the following hopes called P-hopes: for all $x,y\in G$*

$$\underline{P}: x\underline{P}y=(xP)y\cup x(Py),$$

$$\underline{P}_r: x\underline{P}_r y=(xy)P\cup x(yP),$$

$$\underline{P}_l: x\underline{P}_l y=(Px)y\cup P(xy).$$

The $(G,\underline{P})$,$(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called P-hyperstructures. The most usual case is if $(G,\cdot)$ is semigroup, then $x\underline{P}y=(xP)y\cup x(Py)=xPy$ and $(G,\underline{P})$ is a semihypergroup but we do not know about $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$. In some cases, depending on the choice of P, the $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ can be associative or WASS. If more operations are defined in G, then for each operation several P-hopes can be defined.

Definition 1.2. *[18] Let H a set with n operations (or hopes) $\otimes_1,...,\otimes_n$ and a map (or multivalued map) $f:H\to H(or f:H\to P(H)-\{\varnothing\}$, resp.), then n hopes $\partial_1,\partial_2,...,\partial_n$ on H can be defined, called theta-hyperoperations (theta-hopes and write ∂-hope) by putting $x\partial_i y=\{f(x)\otimes_i y, x\otimes_i f(y)\},\forall x,y\in H, i\in\{1,2,...,n\}$, in case where $\otimes_i$ are hopes or f is multivalued we have $x\partial_i y=(f(x)\otimes_i y)\cup(x\otimes_i f(y)),\forall x,y\in H, i\in\{1,2,...,n\}$. If $\otimes_i$ is associative then ∂_i is WASS. A special case for a map f, is to take the union of this with the identity id. Thus, we consider the map $\underline{f}\equiv f\cup(id)$, so*

$\underline{f}(x) = \{x, f(x)\}$, $\forall x \in G$, *which is called* $b-\partial-$*hope, we denote it by* $(\underline{\partial})$, *so we have*

$$x\underline{\partial}y = \{xy, f(x)\cdot y, x\cdot f(y)\}, x, yG.$$

Remark that $\underline{\partial}$ *contains the operation* $(\cdot)$, *so it is b-operation. Moreover, if* $f : G \to P(G)$ *is multivalued then the* $b-\partial$*-hopes is defined by using the* $f\underline{f}(x) = \{x\} \cup f(x), \forall x \in G$.

Motivation for the definition of the theta-hope is the map derivative where only the multiplication of functions can be used. Thus, for two functions $s(x), t(x)$, we have $s\partial t = \{s't, st'\}$, $(')$ denotes the derivative.

Examples 1.1. *(a) For first degree polynomials* $g_i(x) = a_ix + b_i$ *we have* $g_1\partial g_2 = \{a_1a_2x + a_1b_2, a_1a_2x + b_1a_2\}$ *so it is a hope. All polynomials x+c, where c be a constant, are units.*

(b) The constant map. Let $(G, \cdot)$ *group and* $f(x) = a$, *thus* $x\partial y = \{ay, xa\}, \forall x, y \in G$. *If f(x)=e, then* $x\partial y = \{x, y\}$, *the smallest incidence hope.*

Properties 1.1. *If* $(G, \cdot)$ *semigroup:* $\forall f$, *the* ∂*-hope is WASS.* $\forall f$, *the b-*∂*-hope* $(\underline{\partial})$ *is WASS. If f is projection and homomorphism, then* (∂) *is associative. If* $(\cdot)$ *is reproductive then* (∂) *is also reproductive:*

$$x\partial G = \bigcup_{g\in G}\{f(x)\cdot g, x\cdot f(g)\} = G.$$

If $(\cdot)$ *is commutative then* (∂) *is commutative. If f is into the centre of G, then* (∂) *is commutative. If* $(\cdot)$ *is COW then,* (∂) *is COW. u is right unit element if f(u)=e, where e a unit in* $(G, \cdot)$. *The elements of the kernel of f, are the units of* (G, ∂). *Let* $(G, \cdot)$ *a monoid with unit e and u a unit in* (G, ∂), *then f(u)=e. The* $x' = (f(x))^{-1}u$ *and* $x' = u(f(x))^{-1}$, *are the right and left inverses of x, resp. We have two-sided inverses iff* $f(x)u = uf(x)$.

Proposition 1.1. *Let* $(G, \cdot)$ *be a group then, for all maps* $f : G \to G$, *the hyperstructure* (G, ∂) *is an* H_v*-group.*

Hopes on any type of matrices can be defined [20],[21],[22].
There are methods to enlarge or to reduce hopes [16]:

Definition 1.3. *Let $(H, \cdot)$ be hypergroupoid. We remove $h \in H$, if we take the restriction of $(\cdot)$ in the set $H - \{h\}$. $\underline{h} \in H$ absorbs $h \in H$ if we replace h by $\underline{h}$ and h does not appear. $\underline{h} \in H$ merges with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class.*

2 H_v-rings, H_v-fields, Representations

The main tool to study hyperstructures are the fundamental relations β^*, γ^* and ϵ^*, which are defined, in H_v-groups, H_v-rings and H_v-vector spaces, resp., as the smallest equivalences so that the quotient would be group, ring and vector space, resp. The relation β^* was introduced by Koskas in 1970, the γ^*, ϵ^*, by Vougiouklis and he named them Fundamental. A way to find the fundamental classes is given by theorems as the:

Theorem 2.1. *Let $(H, \cdot)$ be an H_v-group and denote by $\boldsymbol{U}$ the set of all finite products of elements of H. We define the relation β in H by setting $x\beta y$ iff $\{x, y\} \subset \boldsymbol{u}$ where $\boldsymbol{u} \in \boldsymbol{U}$. Then β^* is the transitive closure of β.*

An element is called *single* if its fundamental class is singleton.

Analogous theorems for γ^* in H_v-rings, ϵ^* in H_v-modules and H_v-vector spaces, are also proved. he analogous theorem in the case of an H_v-ring:

Theorem 2.2. *Let $(R, +, \cdot)$ be an H_v-ring. Denote by $\boldsymbol{U}$ the set of all finite polynomials of elements of R. We define the relation γ in R as follows $x\gamma y$ iff $\{x, y\} \subset \boldsymbol{u}$ where $\boldsymbol{u} \in \boldsymbol{U}$. Then γ^* is the transitive closure of the relation γ.*

Definition 2.1. *Let $(R, +, \cdot)$ be ring and $f : R \to R$, $g : R \to R$ be two maps. We define two hopes (∂_+) and $(\partial.)$, called both ∂-hopes, on R as follows*

$$x\partial_+ y = \{f(x) + y, x + f(y)\} \text{ and } x\partial. y = \{g(x) \cdot y, x \cdot g(y)\}, \forall x, y \in G.$$

A hyperstructure $(R, +, \cdot)$, where $(+)$, $(\cdot)$ are hopes which satisfy all H_v-ring axioms, except the weak distributivity, will be called *H_v-near-ring.*

Proposition 2.1. *(a) Let $(R,+,\cdot)$ ring and $f: R \to R$, $g: R \to R$ maps. The $(R, \partial_+, \partial_.)$, is an H_v-near-ring. (∂_+) is commutative.*

(b) Let $(R,+,\cdot)$ a ring and $f: R \to R, g: R \to R$ maps, then $(R, \partial_+, \partial_.)$, is an H_v-ring.

(c) In the group of integers $(\mathbf{Z},+)$ take $n \neq 0$ a natural. Take $f: f(0) = n, f(x) = x, \forall x \in \mathbf{Z} - \{0\}$. Then $(\mathbf{Z}, \partial)/\beta^* \cong (\mathbf{Z}_n, +)$.

(d) In the ring of integers $(\mathbf{Z},+,\cdot)$ fix a natural $n \neq 0$. Take f with $f: f(0) = n, f(x) = x, \forall x \in \mathbf{Z} - \{0\}$. Then $(\mathbf{Z}, \partial_+, \partial_.)$ is an H_v-near-ring, with $(\mathbf{Z}, \partial_+, \partial_.)/\gamma^* \cong \mathbf{Z}_n$.

(e) In $(\mathbf{Z},+,\cdot)$ and $n \neq 0$ a natural. Take f with $f: f(0) = n, f(x) = x, \forall x \in \mathbf{Z} - \{n\}$. Then $(\mathbf{Z}, \partial_+, \partial_.)$ is an H_v-ring, moreover $(\mathbf{Z}, \partial_+, \partial_.)/\gamma^* \cong \mathbf{Z}_n$.

Fundamental relations are used for general definitions. Thus we have [13],[14],[15]:

Definition 2.2. *An H_v-ring $(R,+,\cdot)$ is called H_v-field if R/γ^* is a field. The elements of a hyperfield are called hypernumbers. In the special case when strong axioms are valid then $(R,+,\cdot)$ is called hyperfield.*

In the Proposition 2.1 (e) remark that in the case for $n = p$, prime, then $(\mathbf{Z}, \partial_+, \partial_.)$ is an H_v-field.

Using again the fundamental relations we may obtain more general hyperstructures [16],[17].

Classifying the several classes of hyperfields similar to hyperrings we have the following:

Definition 2.3. *An H_v-field is additive if the addition is hope and the multiplication is ordinary operation. An H_v-field is multiplicative if its multiplication is hope and the addition is ordinary operation.*

Several weak properties can take stronger forms as for example $x(y+z) \subset xy + xz, \forall x, y, z \in R$, instead of the weak distributivity then we have the *inclusion distributivity.*

One can see the enormous number of hyperfields, even in the finite case, we may obtain by enlarging the ordinary fields by putting in the results of special couples of special elements more, extra, elements. Then we obtain H_v-fields where all the weak properties are valid.

H_v-structures are used in Representation Theory. Representations (abbreviated by rep) of H_v-groups can be considered by generalized permutations or by H_v-matrices [13],[15],[17]. Reps by generalized permutations can be achieved using translations. In this theory the single elements are playing a crucial role. We present here the hypermatrix rep in H_v-structures:

Definition 2.4. *H_v-matrix is called a matrix with entries elements of an H_v-ring or H_v-field. The hyperproduct of two H_v-matrices (a_{ij}) and (b_{ij}), of type $m \times n$ and $n \times r$ respectively, is defined, in the usual manner, and it is a set of $m \times r$ H_v-matrices. The sum of products of elements of the H_v-ring is the union of the sets obtained with all possible parentheses put on them, i.e. the n-ary circle hope on the hyperaddition. The hyperproduct of H_v-matrices does not necessarily satisfy WASS.*

The problem of the H_v-matrix representations is the following:

Let $(H, \cdot)$ be H_v-group. Find an H_v-ring R, a set $M_R = \{(a_{ij}) | a_{ij} \in R\}$ and a map

$$T : H \to M_R : h \mapsto T(h) \text{ such that } T(h_1h_2) \cap T(h_1)T(h_2) \neq \varnothing, \forall h_1, h_2 \in H.$$

The map T is called *H_v-matrix rep.* If the $T(h_1h_2) \subset T(h_1)(h_2), \forall h_1, h_2 \in H$ is valid, then T is called *inclusion rep.* If $T(h_1h_2) = T(h_1)(h_2) = \{T(h) | h \in h_1h_2\}$, $\forall h_1, h_2 \in H$, then T is called *good rep* and then an induced rep T^* for the hypergroup algebra is obtained. If T is one to one and good then it is a *faithful rep.*

The problem of reps is complicated because the cardinality of the product of H_v-matrices is very big. Bu it can be simplified in special cases such as the following:

(a) The H_v-matrices are over H_v-rings with 0 and 1 and if these are scalars.

(b) The H_v-matrices are over *very thin* H_v-rings.

(c) The case of 2×2 H_v-matrices, since the circle hope coincides with the hyperaddition.

(d) The case of H_v-rings in which the strong associativity in hyperaddition is valid.

(e) The case of H_v-rings which contains singles, then these act as absorbings.

The main theorem of reps is the following:

Theorem 2.3. *A necessary condition in order to have an inclusion rep T of an H_v-group $(H, \cdot)$ by $n \times n$ H_v-matrices over the H_v-ring $(R, +, \cdot)$ is the following: For all classes $\beta^*(x), x \in H$ there must exist elements $a_{ij} \in H, i, j \in \{1, ..., n\}$ such that*

$$T(\beta^*(a)) \subset \{A = (a'_{ij}) | a'_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, ..., n\}\}$$

So every inclusion rep $T : H \to M_R$: $a \mapsto T(a) = (a_{ij})$ induces a homomorphic rep T^* of the group H/β^* over the ring R/γ^* by setting $T^*(\beta^*(a)) = [\gamma^*(a_{ij})], \forall \beta^*(a) \in H/\beta^*$, where the $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. Then T^* is called *fundamental induced rep* of T.

For more results on rep theory one can see [13],[15],[17],[21].

3 The e-hyperstructures

The Lie-Santilli theory on isotopies was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a "lifting" of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1996 [10] and they are called *e-hyperfields*. The H_v-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics [8],[9] or biology. We present in

the following the main definitions and results restricted in the H_v-structures. This construction is based on the partial ordering of the H_v-structures and the Little Theorem [4], [5], [10], [11], [19], [20].

Definition 3.1. *A hyperstructure $(H, \cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure we normally assume that for every element x, there exists an inverse x^{-1}, i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$. The inverses are not necessarily unique.*

Definition 3.2. *A hyperstructure $(F, +, \cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called e-hyperfield if the following axioms are valid:*

(a) $(F, +)$ *is an abelian group with the additive unit 0,*

(b) $(\cdot)$ *is WASS,*

(c) $(\cdot)$ *is weak distributive with respect to* $(+)$*,*

(d) 0 is absorbing element: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$*,*

(e) there exists a multiplicative scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x, \forall x \in F$*, and*

(f) for every $x \in F$ *there exists a unique inverse* x^{-1}*, such that* $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$*.*

The elements of an e-hyperfield are called e-hypernumbers. In the case that the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$*, is valid, then we say that we have a strong e-hyperfield.*

Definition 3.3. *The Main e-Construction. Given a group $(G, \cdot)$, where e is the unit, then we define in G, a large number of hopes $(\otimes)$ as follows:*

$$x \otimes y = \{xy, g_1, g_2, ...\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, ... \in G - \{e\}$$

g_1, g_2,... are not necessarily the same for each pair (x,y). Then $(G, \otimes)$ becomes an H_v-group, in fact is H_b-group which contains the $(G, \cdot)$. The H_v-group $(G, \otimes)$ is an e-hypergroup. Moreover, if for each x,y such that $xy = e$, so we have $x \otimes y = xy$, then $(G, \otimes)$ becomes a strong e-hypergroup

The proof is immediate. Moreover one can see that the unit e is a unique scalar and for each x in G, there exists a unique inverse x^{-1}, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$ and if this condition is valid then we have $1 = x \cdot x^{-1} = x^{-1} \cdot x$. So the hyperstructure $(G, \otimes)$ is a strong e-hypergroup.

Remark. The above main e-construction gives an extremely large class of e-hopes. These e-hopes can be used in the several more complicate hyperstructures to obtain appropriate e-hyperstructures. However, notice that the most useful are the ones where only few products are enlarged.

Examples 3.1. *Consider the finite-non-commutative quaternion group* $\mathbf{Q} = \{1, -1, i, -i, j, -j, k, -k\}$. *Using this operation one can obtain several hopes which define very interesting e-groups. For example, denoting* $\underline{i} = \{i, -i\}, \underline{j} = \{j, -j\}, \underline{k} = \{k, -k\}$ *we may define the* $(*)$ *hope by the table:*

$*$	*1*	*-1*	i	*-i*	j	*-j*	k	*-k*
1	*1*	*-1*	i	*-i*	j	*-j*	k	*-k*
-1	*-1*	*1*	*-i*	i	*-j*	j	$\underline{k}$	k
i	i	*-i*	*-1*	*1*	k	*-k*	*-j*	j
-i	*-i*	$\underline{i}$	*1*	*-1*	*-k*	k	j	*-j*
j	j	*-j*	*-k*	k	*-1*	*1*	i	*-i*
-j	*-j*	j	k	*-k*	*1*	*-1*	*-i*	i
k	k	$\underline{k}$	$\underline{j}$	*-j*	*-i*	i	*-1*	*1*
-k	*-k*	k	*-j*	j	i	*-i*	*1*	*-1*

The hyperstructure $(Q, *)$ is strong e-hypergroup because 1 is scalar unit and the elements -1, i, -i, j, -j, k and -k have unique inverses the elements -1, -i, i, -j, j, -k and k, resp., which are the inverses in the basic group. Thus, from this example one can have more strict hopes.

In [4], [5] a P-hope was introduces which is appropriate to obtain e-hyperstructures:

Construction 3.1. *Let* $(G, \cdot)$ *be an abelian group and* P *any subset of* G *with more than one elements. We define the hyperoperation* $\times_P$ *as follows:*

$$x \times_p y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \text{if } x \neq e \text{ and } c \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G, \times_p)$ is an abelian H_v-group.

4 H_v-Lie algebras

Definition 4.1. *[15],[21],[22] Let $(F, +, \cdot)$ be an H_v-field, $(\mathbf{V}, +)$ be a COW H_v-group and there exists an external hope*

$$\cdot : F \times \mathbf{V} \to P(\mathbf{V}) - \{\emptyset\} : (a, x) \to ax$$

such that, $\forall a, b \in F$ and $x, y \in \mathbf{V}$ we have $a(x+y) \cap (ax+ay) \neq \emptyset, (a+b)x \cap (ax+bx) \neq \emptyset, (ab)x \cap a(bx) \neq \emptyset$, then $\boldsymbol{V}$ is called an H_v-vector space over F. In the case of an H_v-ring instead of an H_v-field then the H_v-modulo is defined. In these cases the fundamental relation ϵ^* is the smallest equivalence relation such that the quotient $\boldsymbol{V}/\epsilon^*$ is a vector space over the fundamental field F/γ^*.

The general definition of an H_v-Lie algebra over F is given [11], [21], [22], as follows:

Definition 4.2. *Let $(\mathbf{L}, +)$ be an H_v-vector space over the H_v-field $(F, +, \cdot)$, take the canonical map $\phi : F \to F/\gamma^*$ with $\omega_F = \{x \in F : \phi(x) = 0\}$, 0 is the zero of F/γ^*. Similarly, ω_L the core of $\phi' : L \to L/\epsilon^*$ and denote again 0 the zero of L/ϵ^*. Consider the bracket (commutator) hope: $[,] : \mathbf{L} \times \mathbf{L} \to P(\mathbf{L}) : (x, y) \to [x, y]$ then $\boldsymbol{L}$ is an H_v-Lie algebra over F if the following axioms are satisfied:*

(L1) The bracket hope is bilinear, i.e. $\forall x, x_1, x_2, y, y_1, y_2 \in \mathbf{L}, \lambda_1, \lambda_2 \in F$
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \varnothing$
$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \varnothing,$

(L2) $[x, x] \cap \omega_L \neq \varnothing, \quad \forall x \in \mathbf{L}$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \varnothing, \quad \forall x, y \in \mathbf{L}$

Now we can see theta hopes in H_v-vector spaces and H_v-Lie algebras:

Theorem 4.1. *Let $(\mathbf{V},+,\cdot)$ be an algebra over the field $(F,+,\cdot)$ and $f:\mathbf{V}\to\mathbf{V}$ be a map. Consider the ∂-hope defined only on the multiplication of the vectors $(\cdot)$, then $(\mathbf{V},+,\partial)$ is an H_v-algebra over F, where the related properties are weak. If, moreover f is linear then we have more strong properties.*

Definition 4.3. *Let $(\mathbf{A},+,\cdot)$ be an algebra over the field F. Take any map $f:\mathbf{A}\to\mathbf{A}$, then the ∂-hope on the Lie bracket $[x,y]=xy-yx$, is defined as follows*

$$x\partial y=\{f(x)y-f(y)x,f(x)y-yf(x),xf(y)-f(y)x,xf(y)-yf(x)\}.$$

Remark that if we take the identity map $f(x)=x,\forall x\in A$, then $x\partial y=\{xy-yx\}$, thus we have not a hope and remains the same operation.

Proposition 4.1. *Let $(\mathbf{A},+,\cdot)$ be an algebra over the field F and $f:\mathbf{A}\to\mathbf{A}$ be a linear map. Consider the ∂-hope defined only on the multiplication of the vectors $(\cdot)$, then $(\mathbf{A},+,\partial)$ is an H_v-algebra over F, with respect to the ∂-hopes on Lie bracket, where the weak anti-commutativity and the inclusion linearity is valid.*

Let $(\mathbf{A},+,\cdot)$ be an algebra and $f:\mathbf{A}\to\mathbf{A}:f(x)=a$ be a constant map. Consider the ∂-hope defined only on the multiplication of the vectors $(\cdot)$, then $(\mathbf{A},+,\partial)$ is an H_v-Lie algebra over F. If we take a=e, the unit of the multiplication, then the properties become more strong.

The H_v-structures can be used as models mainly as an organized devise in other branches of mathematics and for several applied sciences as well. One application of this type is the realization of the graded classical Lie algebras. These realizations are not only for finite dimensional Lie-algebras but for infinite dimensional, for example the Kac-Moody Lie algebras, as well [7], [11], [22]. The main point of the realizations is the hyperproduct of any two elements to be homogeneous, i.e. to have the same degree. Therefore the fundamental classes are smaller than the homogeneous subspaces.

5 Isohypernumbers, Genohypernumbers

According to Santillis iso-theory and geno-theory, we have the following basic definitions [8], [9]:

Definition 5.1. *On a field $F = (F, +, \times)$, a general isofield $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ is defined to be a field with elements $\hat{a} = a \times \hat{1}$, called isonumbers, where $a \in F$, and $\hat{1}$ is a positive-defined element generally outside F, equipped with two operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with the conventional additive unit 0, and $\hat{\times}$ is a new multiplication*

$$\hat{a}\hat{\times}\hat{b} := \hat{a} \times \hat{T} \times \hat{b}, \text{ with } \hat{1} = \hat{T}^{-1}, \forall \hat{a}, \hat{b} \in \hat{F} \ (i)$$

called iso-multiplication, for which $\hat{1}$ is the left and right unit of $\hat{F}$,

$$\hat{1}\hat{\times}\hat{a} = \hat{a} \times \hat{1} = \hat{a}, \forall \hat{a} \in \hat{F} \ (ii)$$

called iso-unit. The rest properties of a field, are reformulated analogously and there are valid.

Definition 5.2. *Genotopies were introduced by Santillis from the Greek meaning of "inducing topologies and they contain isotopies as special case. In isotopies there is no ordering but in genotopies there is. The multiplication of two quantities is ordered to the right and denoted by $>$, when the first quantity multiplies the second to the right, while it is ordered to the left, and denoted by $<$, when the second quantity multiplies the first to the left. On a field $F = (F, +, \times)$, a genofield to the right $\hat{F}^{>} = \hat{F}^{>}(\hat{a}, +, \hat{>})$ is defined to be a field with elements $\hat{a}^{>} = \hat{a}^{>} \times \hat{1}^{>}$, called genonumbers to the right, where $a \in F$, and $\hat{1}^{>}$ is a quantity generally outside F and $\hat{F}(\hat{a}, +, \hat{\times})$, equipped with two operations $+$ and $\hat{>}$ where $+$ is the sum with the conventional additive unit 0, and $\hat{>}$ is a new multiplication*

$$\hat{a}\hat{>}\hat{b} := \hat{a} \times \hat{Q} \times \hat{b}, with \hat{1}^{>} = \hat{Q}^{-1}, \forall \hat{a}, \hat{b} \in \hat{F} \ (iii)$$

called genomultiplication to the right, for which $\hat{1}^{>} = \hat{Q}^{-1}$ is the left and right unit of $\hat{F}^{>}$,

$$\hat{1}^{>}\hat{>}\hat{a}^{>} \equiv \hat{a}^{>}\hat{>}\hat{1}^{>} \equiv \hat{a}^{>}, \forall \hat{a}^{>} \in \hat{F} > \ (iv)$$

called genounit to the right. The rest properties of a field, are reformulated analogously and there are valid.

We transfer this theory to hypernhmbers only on the case of isonumbers and analogously we can transfer this to genonumbers. To transfer this theory into the hyperstructure case we generalize only the new multiplication $\hat{\times}$ from (i), by replacing with a hope including the old one. We introduce two general constructions on this direction as follows:

Construction 5.1. *The general enlargement. On a field $F = (F, +, \cdot)$ and on the isofield $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ we replace in the results of the iso-product*

$$\hat{a}\hat{\times}\hat{b} = \hat{a} \times \hat{T} \times \hat{b}, \text{ with } \hat{1} = \hat{T}^{-1}$$

of the element $\hat{T}$ by a set of elements $\hat{H}_{ab} = \{\hat{T}, \hat{x}_1, \hat{x}_2, ...\}$ where $\hat{x}_1, \hat{x}_2, ... \in \hat{F}$, containing $\hat{T}$, for all hyperproducts $\hat{a}\hat{\times}\hat{b}$ for which

$$\hat{a}, \hat{b} \notin \{\hat{0}, \hat{1}\} \text{ and } \hat{x}_1, \hat{x}_2, ... \in \hat{F} - \{\hat{0}, \hat{1}\}$$

If one of $\hat{a}, \hat{b}$, or both, is equal to $\hat{0}$ or $\hat{1}$, then $\hat{H}_{ab} = \hat{T}$. Therefore the new iso-hope is

$$\hat{a}\hat{\times}\hat{b} = \hat{a} \times \hat{H}_{ab} \times \hat{b} = \hat{a} \times \{\hat{T}, \hat{x}_1, \hat{x}_2, ...\} \times \hat{b}, \forall \hat{a}, \hat{b} \in \hat{F} \ (iii)$$

$\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ becomes isoH$_v$-field, and the elements of $\hat{F}$ are called ***isoH$_v$-numbers*** *or* ***isonumbers****.*

Remark 5.1. *(a) More important hopes, of the above construction, are the ones where only for few ordered pairs $(\hat{a}, \hat{b})$ the result is enlarged, even more, the extra elements $\hat{x}_i$, are only few, preferable exactly one. Thus, this special case is if there exists only one pair $(\hat{a}, \hat{b})$ for which*

$$\hat{a}\hat{\times}\hat{b} = \hat{a} \times \{\hat{T}, \hat{x}\} \times \hat{b}, \forall \hat{a}, \hat{b} \in \hat{F}$$

and the rest are ordinary results, then we have a hyperstructure called very thin isoH$_v$-field [1], [10], [11], [19].

(b) The assumption that $\hat{H}_{ab} = \{\hat{T}\}$, if one of $\hat{a}, \hat{b}$, is equal to $\hat{0}$ or $\hat{1}$, together with the assumption that $\hat{x}_i$, are not $\hat{0}$ or $\hat{1}$, guarantee that the isoH$_v$-field has exactly one scalar absorbing element $\hat{0}$, one exactly scalar $\hat{1}$, and every element $\hat{a} \in \hat{F}$, has exactly one inverse element.

Construction 5.2. *The P-hope. Consider any isofield* $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ *with* $\hat{a} = a \times \hat{1}$*, the isonumbers, where* $a \in F$*, and* $\hat{1}$ *is a positive-defined element generally outside F, with two operations* $\hat{+}$ *and* $\hat{\times}$*, where* $\hat{+}$ *is the sum with the conventional additive unit 0, and* $\hat{\times}$ *is the iso-multiplication*

$$\hat{a}\hat{\times}\hat{b} := \hat{a} \times \hat{T} \times \hat{b}, with \hat{1} = \hat{T}^{-1}, \forall \hat{a}, \hat{b} \in \hat{F}$$

Take any set $\hat{p} = \{\hat{T}, \hat{p}_1, ..., \hat{p}_s\}$*, with* $\hat{p}_1, ..., \hat{p}_s \in \hat{F} - \{\hat{0}\}$*, we define the isoP-*$H_v$*-field,* $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times}_P)$ *where the hope* $\hat{\times}_P$ *is defined as follows:*

$$\hat{a}\hat{\times}_P\hat{b} := \begin{cases} \hat{a} \times \hat{P} \times \hat{b} = \{\hat{a} \times \hat{h} \times \hat{b} | \hat{h} \in \hat{P}\} & if\ \hat{a} \neq \hat{1}\ and\ \hat{b} \neq \hat{1} \\ \hat{a} \times \hat{T} \times \hat{b} & if\ \hat{a} = \hat{1}\ or\ \hat{b} = \hat{1} \end{cases} (iv)$$

The elements of $\hat{F}$ *are called isoP-*H_v*-numbers.*

Remark 5.2. *The most important of this construction is when* $\hat{P} = \{\hat{T}, \hat{p}\}$*, that is that* $\hat{P}$ *contains only one* $\hat{p}$ *except* $\hat{T}$*. The inverses in isoP-*H_v*-fields, are not necessarily unique.*

Examples 5.1. *Non degenerate examples on definition 3.3 and on construction 3.1 on the small finite field* $(Z_5, +, \cdot)$*:*

1. *In the* $\hat{Z}_5 = \hat{Z}_5(\hat{a}, \hat{+}, \hat{\times})$*, where we denote* $\hat{Z}_5 = \{\hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{4}\}$*, the weak associative multiplicative hope is described by the Table 1:*

$\hat{\times}$	$\underline{\hat{0}}$	$\underline{\hat{1}}$	$\underline{\hat{2}}$	$\underline{\hat{3}}$	$\underline{\hat{4}}$
$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$
$\underline{\hat{1}}$	$\underline{\hat{0}}$	$\underline{\hat{1}}$	$\underline{\hat{2}}$	$\underline{\hat{3}}$	$\underline{\hat{4}}$
$\underline{\hat{2}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{4}}$	$\underline{\hat{1}}$	$\underline{\hat{3}}$,$\underline{\hat{2}}$
$\underline{\hat{3}}$	$\underline{\hat{0}}$	$\underline{\hat{3}}$	$\underline{\hat{1}}$	$\underline{\hat{4}}$	$\underline{\hat{2}}$
$\underline{\hat{4}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{3}}$	$\underline{\hat{2}}$	$\underline{\hat{1}}$

Table1

$\hat{\times}$	$\underline{\hat{0}}$	$\underline{\hat{1}}$	$\underline{\hat{2}}$	$\underline{\hat{3}}$	$\underline{\hat{4}}$
$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$
$\underline{\hat{1}}$	$\underline{\hat{0}}$	$\underline{\hat{1}}$	$\underline{\hat{2}}$	$\underline{\hat{3}}$	$\underline{\hat{4}}$
$\underline{\hat{2}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{1}}, \underline{\hat{4}}$	$\underline{\hat{1}}, \underline{\hat{4}}$	$\underline{\hat{2}}$,$\underline{\hat{3}}$
$\underline{\hat{3}}$	$\underline{\hat{0}}$	$\underline{\hat{3}}$	$\underline{\hat{1}}, \underline{\hat{4}}$	$\underline{\hat{1}}, \underline{\hat{4}}$	$\underline{\hat{2}}$,$\underline{\hat{3}}$
$\underline{\hat{4}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{2}}$,$\underline{\hat{3}}$	$\underline{\hat{2}}$,$\underline{\hat{3}}$	$\underline{\hat{1}}$,$\underline{\hat{4}}$

Table2

2. *In order to define a generalized P-hope on* $\hat{Z}_5 = \hat{Z}_5(\underline{\hat{a}}, \hat{+}, \hat{\times})$*, where we take* $\hat{P} = \{\underline{\hat{1}}, \underline{\hat{4}}\}$*, the weak associative multiplicative hope is described by the Table 2. The hyperstructure* $\hat{Z}_5 = \hat{Z}_5(\underline{\hat{a}}, \hat{+}, \hat{\times})$*, is commutative and associative on the multiplication hope.*

6 Santilli's hyper-admissibility

The Lie-Santilli admissibility on square matrices is not faced in this presentation. However we can present this problem on the non-square case. This problem can be faced in two ways:

1. using ordinary numbers, as real or complex numbers, so using ordinary matrices and hopes, instead of operations on non-square matrices,

2. using hypernumbers (e-hypernumbers) as entries and the ordinary operations on non-square hypermatrices.

The general definition, is the following [11], [21]:

Construction 6.1. *Let* $(\mathbf{L} = \mathbf{M}_{m\times n}, +)$ *be a* H_v*-vector space of* $m \times n$ *hyper-matrices over the* H_v*-field* $(\mathbf{F}, +, \times), \phi : \mathbf{F} \to \mathbf{F}/\gamma^*$*, the canonical map and* $\omega_F = \{x \in F : \phi(x) = 0\}$*, where 0 is the zero of the fundamental field* $\mathbf{F}/\gamma^*$*. Similarly, let* ω_L *be the core of the canonical map* $\phi' : \mathbf{L} \to \mathbf{L}/\epsilon^*$ *and denote by the same symbol 0 the zero of* $\mathbf{L}/\epsilon^*$*. Take any two subsets* $\mathbf{R}, \mathbf{S} \subseteq \mathbf{L}$ *then a Santillis Lie-admissible hyperalgebra is obtained by taking the Lie bracket, which is a hope:*

$$[,]_{RS} : \mathbf{L} \times \mathbf{L} \to \mathbf{P}(\mathbf{L}) : [x, y]_{RS} = x\mathbf{R}^t y - y\mathbf{S}^t x.$$

Notice that $[x, y]_{RS} = x\mathbf{R}^t y - y\mathbf{S}^t x = \{xr^t y - ys^t x/r \in \mathbf{R}$ *and* $s \in \mathbf{S}\}$
Special cases, but not degenerate, are the "small" and "strict" ones:

- $\mathbf{R} = e$*. Then,* $[x, y]_{RS} = xy - y\mathbf{S}^t x = \{xy - ys^t x/s \in \mathbf{S}\}$
- $\mathbf{S} = e$*. Then,* $[x, y]_{RS} = x\mathbf{R}^t y - yx = \{xr^t y - yx/r \in \mathbf{R}\}$
- $\mathbf{R} = \{r_1, r2\}$*, and* $\mathbf{S} = \{s_1, s_2\}$ *then*

$$[x, y]_{RS} = x\mathbf{R}^t y - y\mathbf{S}^t x =$$

$$\{xr_1^t y - ys_1^t x, xr_1^t y - ys_2^t x, xr_2^t y - ys_1^t x, xr_2^t y - ys_2^t x\}$$

Remark 6.1. *In the above constructions whenever a "shift" of elements is needed, as in Santilli's isotheory [8], [9], then the elements for the subsets* $\boldsymbol{S}$ *and* $\boldsymbol{R}$ *must belong to a set of hypermatrices where a reversibility could be applied.*

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