Foundations of Iso-Differential Calculas
Volume I

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FOREWORD

As it is well known, Isaak Newton had to develop the *differential calculus,* (jointly with Gottfried Leibniz), with particular reference to the historical definition of velocities as the time derivative of the coordinates, $v = dr/dt$, in order to b to write his celebrated equation $mdv/dt = F(t, r, v)$, where $a = dv/dt$ is the acceleration and $F(t, r, v)$ is the Newtonian force acting on the mass m. Being local, the differential calculus solely admitted the characterization of *massive points.* The differential calculus and the notion of massive points were adopted by Galileo Galilei and Albert Einstein for the formulation of their their relativities, thus acquiring a fundamental role in 20th century sciences.

In his 1966 Ph. D. thesis at the University of Turin, Italy, the Italian-American scientist Ruggero Maria Santilli[1] pointed out that Newtonian forces are the most general known in dynamics, including action-at-a-distance forces derivable derivable from a potential, thus representable with a Hamiltonian, and other forces that are not derivable from a potential or a Hamiltonian, since they are contact dissipative and non-conservative forces caused by the motion of the mass m within a physical medium. Santilli pointed out that, due to their lack of dimensions, massive points can solely experience action-at-a-distance Hamiltonian forces.

On this ground, Santilli initiated a long scientific journey for the generalization of Newton's equation into a form permitting the representation of the actual extended character of massive bodies whenever moving within physical media, as a condition to admit non-Hamiltonian forces. Being a theoretical physicist, Santilli had a number of severe physical conditions for the needed representation. One of them was the need for a representation of extended bodies and their non-Hamiltonian forces to be invariant over time as a condition to predict the same numerical values under the same conditions but at different times.

When he was a member the Department of Mathematics at Harvard University in the early 1980s under support by the U.S. Department of Energy, Santilli achieved a step-by-step isotopic (that is, axiom preserving) lifting of the various branches of Lie's theory based on the generalizion $J_i \hat{\times} J_j = J_i \hat{T} J_j$ of the associative product $J_i J_j$ of the universal enveloping associative algebra between Hermitean generators J_i, J_j, with consequential generalized Lie's second theorem $[J_i \hat{,} J_j] = J_i \hat{\times} J_j - J_j \hat{\times} J_i = J_i \hat{T} J_j - J_j \hat{T} J_i = C_{ij}^k J_k$,

[1]Prof. Santilli's curriculum is available in from the link http://www.world-lecture-series.org/santilli-cv

where $\hat{T}$ is a fixed positive-definite operator and the C's can be constants or, unlike the Lie's case, can be functions.

Since the generalized product $J_i \hat{\times} J_j$ remains associative, and the generalized brackets $[J_i \hat{,} J_j]$ verify the Lie axioms, Santilli called the operator $\hat{T}$ the *isotopic element,* the product $J_i \hat{\times} J_j$ the *isoproduct,* and the brackets $[J_i \hat{,} J_i]$ the *Lie-isotopic product.* Santilli then proved that the emerging isotopically lifted Lie theory, today called the *Lie-Santilli isotheory,* permits the representation of the actual extended shapes of bodies as well as all possible (sufficiently smooth and regular) Newtonian forces, e.g., via realizations of the isotopic element of the type $\hat{T} = Diag.(1/n_1^2, n_2^2, n_3^2)\Gamma(t, r, v, ...)$, where the $n_k^2,\ k = 1, 2, 3$ represent the shape of the body considered and $\Gamma(t, r, v, ...)$ represents all non-Hamiltonian forces, the remaining forces derivable from a potential being representable with a conventional Hamiltonian (for these early studies, see the two 1978 volumes of *Foundations of Theoretical Mechanics* [16] written at Harvard University and published by Springer-Verlag).

Subsequently, Santilli showed that, when the quantities C_{ij}^k are all constants, the isotopies of Lie's theory can be achieved via a non-unitary transformation of Lie's theory such as $U(J_i J_j)U^\dagger = J_i' \hat{T} J_j',\ J' = UJU^\dagger,\ \hat{T} = (UU^\dagger)^{-1}, UU^\dagger \neq I$, in which case we have the so-called *regular isotopies.* However, Santilli showed that, under isotopies (only), the C's can also be functions/In this case, the isotopies of Lie's theory cannot be obtained via non-unitary or other transforms of the conventional Lie's theory, are called *irregular isotopies* and characterize a *bona fine* (still mostly unexplored) new theory (for more recent studies, see the two 1995 volumes of *Elements of Hadronic Mechanics* [] published by the Ukraine Academy of Sciences).

Despite these notable advances, Santilli remained inquisitive and self-critical. In this way, he discovered that his original isotopic formulation of Lie's theory was not invariant over time because the formulation was non-canonical at the classical level and non-unitary at the operator level, thus causing insufficiencies that Santilli called "catastrophic," such as: the lack of preservation over time of the basic unit +1 and related numeric field; the loss over time of Hermiticity and, consequently, of observability; the violation of causality laws, and other insufficiencies.

To attest at Santilli's commitment to serious science, we should report the statement in his works that the prestigious Volume II of *Foundations of Theoretical Mechanics* had "no physical value." In this second volume, Santilli achieved a covering of classical Hamiltonian mechanics with a Lie-isotopic structure that he called *Birkhoffian mechanics* for certain historical reasons, and proved its remarkable "direct universality," that is, the capability of representing all possible Hamiltonian and non-Hamiltonian Newtonian systems ("universality") directly in the reference frame of the observer ("direct universality"). However, the emerging mechanics was non-canonical because elaborated with the mathematics of Hamiltonian mechanics. Consequently, in Santilli's own words: *Birkhoffian mechanics elaborated with the mathematics of Hamiltonian mechanics has no physical value because it is unable to predict the same numerical values under the same condition at different times.*

These basic, self-identified insufficiencies forced Santilli to re-examine the mathematics originating them, namely, the conventional, 20th century applied mathematics defined over a numeric field of characteristic zero, thus including the need for a re-inspection of numbers, functions, metric spaces, geometries, topologies, etc. The inspiration was the teaching of the history of science establishing that the protracted lack of solution of physical problems is generally due to insufficiencies of the used mathematics.

In 1993, when he was visiting the *Joint Institute for Nuclear Research* in Dubna, Russia, Santilli had the courage to re-inspect the historical classification of numbers into real, complex and quaternionic numbers and discovered that the axioms of a numeric field do not require the basic multiplicative unit to be the trivial number $+1$, since the axioms of a field also admit generalized multiplicative units $\hat{I}$ provided that: the generalized units are positive-definite, thus invertible $\hat{I} = 1/\hat{T} > 0$; conventional numbers n are lifted into the form $\hat{n} = n\hat{I}$; and the conventional multiplication of numbers nm is lifted into that at the foundation of the Lie-Santilli isotheory, $\hat{n}\hat{\times}\hat{m} = (nm)\hat{I}$ so that the generalized units $\hat{I}$ verify the basic axiom $\hat{I}\hat{\times}n = n\hat{\times}\hat{I} = n$ for all elements of the set considered. These foundations led Santilli to the discovery of new numbers, today known as *Santilli isoreal, isocomplex and isoquaternionic isonumbers* (see memoir [43] of 1993 that may represent in due time one of the most significant mathematical discoveries of the 20th century due to its implications for all sciences, including new industrial applications).

Santilli's driving motivation was, again, physical. As indicated above, Santilli was looking for an invariant representation of extended masses under non-Hamiltonian forces. After a number of trials and errors, he selected the needed representation via a generalization of the basic unit since the unit is the fundamental invariant of any theory, a solution that we believe will resist the test of time. In turn, the isotopic lifting of the basic unit forced Santilli to construct compatible isotopies of conventional numeric fields. Still in turn, the isotopies of fields stimulated a flurry of studies for the construction of compatible isotopies of all of 20th century applied mathematics, including the reformulate over isofields of functional analysis, Lie-Santilli isotheory,

metric spaces, geometries, etc.
These isotopies were achieved in the early 1990s thanks also to contributions by a number of pure and applied mathematicians, including Gr. Tsagas, D. S. Sourlas, H. C. Myung, C-X. Jiang, J. V. Kadeisvili, A. Aringazin, A. Kirukin, and others. All these efforts set the foundation of what is today called *Santilli isomathematics,* which is referred to the isotopies of the entirety of 20th century applied mathematics with no exclusion to prevent insidious inconsistencies that generally remain undetected by non-experts in the field.
Despite these additional, equally notable advances, Santilli continued to remain dissatisfied because, in his strong self-criticism, he proved that the physically important invariance of the isotopies over time was still missing. Therefore, Santilli spent years in re-examining the achieved systematic isotopies of 20th century applied mathematics in the hope that some of them was missing, with no avail.
Finally, in 1995, when he was at the *Institute for Basic Research,* Castle Prince Pignatelli, Molise, Italy, Santilli had the courage, "out of desperation" in his words, to re-inspect the Newton-Leibniz differential calculus and discovered that, contrary to a popular belief in mathematics for centuries, *the differential calculus generally depends on the basic numeric field* because, whenever the multiplicative unit of the base field depends on the differentiation variables, the conventional calculus is inapplicable.
In the 1966 memoir [52] published by the *Rendiconti Circolo Matematico Palermo,* Santilli introduced the generalized differential $\hat{d}\hat{r} = \hat{T}d[r\hat{I}(t,r,v,...)]$, which he called *isodifferential,* and the corresponding generalized derivative $\hat{\partial}\hat{f}(\hat{r})/\hat{\partial}\hat{r} = \hat{I}\ (\partial\hat{f}(\hat{r})/\partial\hat{r}$, which he called *isoderivative,* where the lifting of coordinates r into $\hat{r} = r\hat{I}$ is necessary for consistency becase their values must be isonumbers, and the same holds for functions. As one can see, for $\hat{I}$ independent from the differentiation variables or a constant, the generalized differential and derivatives coincide with the conventional form, $\hat{d}\hat{r} \equiv dr$, $\hat{\partial}\hat{f}(\hat{r})/\hat{\partial}\hat{r} \equiv \partial f(r)/\partial r$ (where, in the latter expression, one should not forget the insidious isotopy of the fraction), and this may explain the reason that the isotopies of the differential calculus remained undetected for centuries until the 1966 memoir [52] (that may also represent in due time another major mathematical discovery of the 20th century due to its impact on all sciences).
In this way, thirty years following the identification of the problem in 1966, Santilli finally achieved the desired structural generalization of Newton's equation $\hat{m}\hat{\times}\hat{d}\hat{v}/\hat{d}\hat{t} = -\hat{\partial}\hat{V}(\hat{t},\hat{r},\hat{v})\hat{\partial}\hat{r}$ which achieves direct universality for the invariant representation of extended bodies moving within physical me-

dia under the most general known Hamiltonian and non-Hamiltonian forces, today called the *Newton-Santilli isoequation.* One should note the representation of Hamiltonian forces via the potential V and the embedding of all non-Hamiltonian forces in the isodifferential calculus (see memoir [52], Section 2.2, pages 30 to 39 the Theorem of Direct Universality and other features).

The resulting new calculus, today known as *Santilli IsoDifferential Calculus,* or IDC for short, stimulated a further layer of studies that finally signaled the achievement of mathematical and physical maturity. In particular, we note: the isotopies of Euclidean, Minkowskian, Riemannian and symplectic geometries; the isotopies of classical Hamiltonian mechanics, today known as the *Hamilton-Santilli isomechanics;* and the isotopies of quantum mechanics, today known as the *isotopic branch of Hadronic mechanics.*

The latter structurally important isotopies were identified in memoir [52] with additional studies provided by a number of pure and applied mathematicians, including (in addition to those mentioned above) R. M. Falcon Ganfornina, J. Nunez Valdes, T. Vougiouklis, C. Corda, A. Bhalekar, S. Georgiev, J. V. Kadeisvili, and others. It should be noted that, nowadays, thanks to its consistency, isomathematics and related classical and operator isomechanics have applications and experimental verifications in classical mechanics, particle physics, superconductivity, chemistry, biology, statistical mechanics, astrophysics and cosmology (see monographs [34,35] for extended presentations).

Independently from the above studies, and also when he was at the Department of Mathematics of Harvard University in the early 1980s, Santilli realized that 20th century mathematics had another major insufficiency for physical studies, the inability to provide a representation of antimatter at the *classical* level beginning with Newton's equations, because their only classical conjugation from matter to antimatter was the sign of the charge, while predicted antimatter asteroids, stars and galaxies have to be assumed as being neutral. Particularly insufficient for antimatter was the conventional differential calculus due to the absence of any differentiations between matter and antimatter, due to its independence from the charge.

In this way, while working at the above outlined isotopies of 20th century applied mathematics specifically intended for the representation of *matter,* Santilli conducted parallel studies for the construction of a new mathematics capable of representing *antimatter* from Newtonian mechanics to second quantization. In particular, the needed new mathematics had to be an anti-isomorphic image of 20th century mathematics as a condition to achieve compatibility with charge conjugation at the operator level. Again, Santilli

main objective was physical, namely, the achievement of another generalization of Newton's equations for neutral or charged antimatter masses. Again, the biggest difficulties for the achievement of such a generalization was the conventional differential calculus.
After a number of trials and errors, Santilli was forced to achieve the needed anti-isomorphic character by embedding it in the mathematical foundations, namely, in the notions of numbers, their multiplication and their unit, after which the anti-isomorphism of the new mathematics with respect to conventional mathematics was assured.
In 1993, while he was also visiting the *Joint Institute for Nuclear Research* in Dubna, Russia, and while re-inspecting the axioms of a numeric field, Santilli discovered that, besides the isofields with *positive-definite* multiplicative units, the axioms of a field admit additional new fields with *negative-definite* multiplicative units (hereon indicated with an upper index d), $I^d < 0$, provided that the numbers n are lifted to the anti-isomorphic form $n^d = nI^d$, where n is a conventional number, and the multiplication of numbers nm is lifted into the equally anti-isomorphic form $n^d \times^d m^d = n^d \times (1/I^d) \times m^d = (nm)I^d$. This allowed the discovery of yet new solutions of the axioms of a numeric field, that Santilli called *isodual fields* in the sense of being conjugated in an axiom-preserving way (see paper [43]). Note that isodual fields are the isoduals of conventional fields and not of isofields. Hence, this first layer of isodualities can be solely used for the characterization of *point-like antimatter masses.*
Despite these additional also notable advances for antimatter, Santilli remained unable to formulate the desired generalization of Newton's equation for neural or charged "antimatter massive points," because of the insufficiency of the conventional differential calculus. Following the discovery in paper [43] of the isodual fields, and the discovery in memoir [52] of 1966 of the dependence of the differential calculus on the assumed basic fields, it was easy for santilli to introduce yet a new calculus that he called *isodual differential calculus* with basic expressions $d^d r^d$ and $\partial^d f^d(r^d)/\partial^s r^d$ after which he was finally able to write the desired generalized newton equation for antimatter $m^d \times^d d^d v^d / d^d t^d = F^d(t^d, r^d, v^d)$, today known as the *Newton-Santilli isodual equation* (see monographs [34.35]).
Santilli]s isodual mathematics is nowadays referred to the anti-isomorphic image of the entire 20th century applied mathematics characterized by the *isodual map* of all possible quantities $Q(t, r, v, ...) \rightarrow Q^d(t^d, r^d, v^d, ...) = -Q^\dagger(-t^\dagger, -r^\dagger, -v^\dagger, ...)$ and all their possible operations, thus including the isoduality of numbers, functional analysis, metric spaces, Lie's theory, geometries, topologies, etc. It should be noted that the fundamental unit of

isodual mathematics is the negative unit $I^d = -1$ at all possible levels. The emerging new *isodual theory of antimatter* verifies all known experimental data since the Newton-Santilli equation verifies all known classical experimental data on antimatter and the representation of all data available at the particle level is assured by the equivalence of isoduality and charge conjugation. Note that the *isodual time* and *isodual energy* are negative definite, $t^d = -t, \ E^d = -E$. Their historical inconsistencies are resolved by the isodual mathematics because said quantities are now referred to *isodual unit,* namely, negative-definite unit. In fact, negative-definite time and energy referred to negative-definite units are as causal as our positive-definite time and energy referred to positive-definite units.(see Refs. [34.35] for details, including an identical re-formulation of Dirac's equation representing an electron-positron pair without any need for the hole theory).

20th Cent. Mathematics Point-like particles in vacuum	**IsoMathematics Extended particles within media**	**GenoMathematics Extended particles in irreversible conditions**	**HyperMathematics Biological structures**
Isodual mathematics Point-like antiparticles in vacuum	**Isodual isomathematics Extended antiparticles within media**	**Isodual genomathematics Extended antiparticles in irreversible conditions**	**Isodual hypermathematics Antimatter biological structure**

Figure 1: *An overview of Santilli's fifty year long scientific journey. The top indicates conventional and generalized mathematics with increasing complexity characterized by multiplicative units of increasing complexity for the characterization of corresponding complex systems of matter. The bottom indicates anti-isomorphic mathematics with corresponding progressive increase of complexities characterized by negative-definite multiplicative units for the representation of systems of antimatter with increasing complexity, which mathematics are constructed via the isodual map of the preceding mathematics for matter. All these mathematics together referred to as 'hadronic mathematics' because they characterize the various branches of hadronic mechanics and chemistry [35].*

Santilli isodual isomathematics is the image of isomathematics under isoduality. It should be indicated that Santilli also identified the foundation of two additional yet more general mathematics we cannot possibly review in this volume called *geno- and hyper-mathematics* and their *isodual geno- and isodual-hyper-mathematics* for the description of more complex systems of

matter and antimatter, that includes corresponding yet broader generalizations of Newtonian, Hamiltonian and quantum mechanics (see Figure 1 for an overview).
In Chapter 1 of this volume we have elected to review Santilli's scientific journey, and identify its most important references, in the hope that interested colleagues may be inspired to identify possible alternative routes and/or additional advances in a large number of still open mathematical problems.
In chapter 2 we introduce isoreals, basic operations with them and we give their properties.
In chapter 3 we define sequences of isoreals and deduct their properties.
In chapter 4 we give definitions for four kinds isofunctions and outline their properties.
In chapter 5 we introduce limit of isofunctions and continous isofunctions.
In chapter 6, we present the first comprehensive study of the IsoDifferential calculus for the specific intent of showing its non-triviality, as well as the generation of a series of new properties and methods.
In chapter 7 we reflect the integral calculas in the language of isomathematics.
In Chapter 8, as appendix, we outline the isodual isomathematics and present the first comprehensive study of the isodual isodifferential calculus.
The authors would appreciate any comments by interested colleagues. Prof. Santilli is also available at his email basicresearch@i-b-r.org for additional technical aspects.

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Chapter 1

PHYSICAL ORIGIN OF THE ISODIFFERENTIAL CALCULUS:

Santilli's Generalization of Newton's Equations for Extended Bodies Moving within Physical Media

1.0.1 The Birth of the Differential Calculus

As it is well known, Isaac Newton [1] had to construct first the modern version of the differential calculus (jointly with Gottfried Leibniz) in order to formulate his equations. In fact, Newton had to achieve first the notion of velocity for a system of n bodies $v_k, k = 1, 2, ..., n$, as the derivative of the coordinates r_k, k with respect to time t

$$v_k = \frac{dr_k}{dt},$$

where each coordinate r_k is defined in a three-dimensional Euclidean space, after which notion Newton was able to formulate his celebrated equations

$$m_k \frac{dv_k}{dt} = F_k(t, r, v), \quad k = 1, 2, ..., n. \tag{1.1}$$

where: dv_k/dt represents the acceleration of the body with mass m_k and $F_k(t, r, v)$ represents the force experienced by the mass m_k during its motion. Newton's equations were adopted by the founders of analytic mechanics, resulting in their representation via*Lagrange equations* [2],

$$\frac{d}{dt}\frac{\partial L(t, r, v)}{\partial v_k} - \frac{\partial L(t, r, v)}{\partial r_k} = F_k(t, r, v), \tag{1.2a}$$

where L is the *Lagrangian*

$$L = \frac{1}{2} m_k v_k^2 - V(t, r, v), \tag{1.2b}$$

and *Hamilton's equations* [3]

$$\frac{dr_k}{dt} = \frac{\partial H(t, r, p)}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H(t, r, p)}{\partial r_k} + F_k(t, r, p), \tag{1.3a}$$

where H is the *hamiltonian*

$$H = \frac{p_k^2}{2m_k} + V(t, r, p), \tag{1.3b}$$

As one can see, Newton's forces are partially represented with the potential V, and the residual forces are represented with external terms.
Newton's equations (1.1) and their analytic representations via Lagrange's equations (1.2) or Hamilton's equations (1.3) with external terms remained at the foundation of science until the early 20th century, when the external terms were removed in order to restrict the systems to a form derivable from an action principle due to its need for quantization.

1.0.2 The Notion of Point-Like Mass

A feature with historical implications is that *Newton's equations can solely characterize point-like (dimensionless) masses.* This feature is requested by the mathematical structure of the equations, namely, the differential calculus which. being local (i.e., defined at the points r_k) is solely able to characterize a finite number of dimensionless points. This characteristics of Newton's equations has been more recently identified on rigorous mathematical grounds, such as via the well known *Euclidean topology.* .
The historical implications of the point-like character of all masses is that it was assumed by *Galileo Galilei* at the foundation of his celebrated 1963 *Dialogus de Systemate Mundi* [4]. Therefore, Galileo's relativity too can solely characterize systems with a finite number of dimensionless masses.
Albert Einstein had no other choice than that of adopting Newton's and Galileo's notion of point-like masses because the differential calculus was the only available mathematical method for quantitative representations of our physical reality at the dawn of the 20th century. Consequently, Einstein's special relativity [5] and general relativity [6] too can solely characterize a finite number of point-like masses.

1.0.3 Interior and Exterior Dynamical Problems

The Italian-American scientist Ruggero maria Santilli conducted his graduate studies at the University of Torino, Italy, in the 1960s and, since that time, he has dedicated his research life to the study of *interior dynamical problems,* referred to extended, generally non-spherical and deformable bodies and electromagnetic waves propagating within a physical medium. By contrast, the problems represented by Newton's mechanics, Galileo's relativity and Einstein's relativities are known as *exterior dynamical problems,* referred to point-like masses and electromagnetic waves moving in vacuum conceived as empty space.

In his Ph. D. thesis, Santilli argued that point-like masses can only experience acting at a distance forces derivable from a potential because, when moving within a physical medium, a point-like mass cannot experience any resistance. Therefore, Santilli initiated a long scientific journey aimed at the achievement of a quantitative representation of the extended character of bodies as a pre-requisite for the consistent admission of contact non-potential forces.

Figure 1.1: *An illustration of Santilli's Theorem* **??** *according to which an interior dynamical system (such as a spaceship during re-entry in our atmosphere of this picture), which is irreversible over time and non-conservative, cannot be consistently decomposed into a finite number of elementary constituents all in reversible and conservative conditions as requested by quantum mechanics. Therefore, Santilli's Theorem 2.0.1 establishes the need for a covering of quantum mechanics capable of representing irreversible and no-conservative conditions at the most elementary level of nature.*

Since the advent of special relativity, it was generally believe that interior dynamical problems are "inessential" because they can be reduced to a collection of particles in exterior dynamical conditions. As part of his 1966 Ph. D. thesis, Santilli proved the following property (see later on monographs [40] and Fig. 1):

Theorem 1.0.1. *Interior dynamical systems within physical media cannot be consistently reduced to a finite number of elementary particles all in exterior conditions in vacuum and, vice-versa, a finite systems of elementary particles all in exterior conditions cannot consistently characterize exterior systems under statistical, thermodynamical or other principles.*

The proof was based on the use of Newton's equations (1.1) with acting forces not derivable from a potential for an extended body moving within a physical medium, while all forces acting on elementary particles in vacuum are derivable from a potential. It is then evident that a collection of the latter forces cannot reproduce the former.
A parallel proof of Theorem 2.0.1 was based on thermodynamical arguments. It is well know that Galileo's and Einstein's relativities are incompatible with thermodynamical laws for several technical reasons, such as the fact that they lack an "arrow ofd time." Santilli then proved that the reduction of interior to exterior systems implies that thermodynamical laws are "illusory" since they can be made to disappear by reducing systems to their elementary constituents, and equally illustory would be the entropy tdespite its ever increasing character.
In short, Santilli proved that the forces experiences by extended bodies moving within a physical medium, rather than "disappearing" evidently to achieve compatibility with Galileo's and Einstein's relativities, originate instead at the most elementary level of nature.
As an example, Santilli quoted the contact interactions of a spaceship during re-entry in our atmosphere (Fig. 1) generated by the most general known, non-linear, non-local and non-potential interactions between the electron orbitals of peripheral atoms of the spaceship with corresponding electron orbitals of atmospheric atoms.

1.0.4 Santilli's Lie-admissible Treatment of Open Irreversible Systems

While exterior dynamical problems in vacuum are reversible over time (because Lagrangians and Hamiltonians are time reversal invariant for all p[physically meaningful potentials), interior dynamical problems are *irreversible over*

time, in the sense that their time reversal image violates causality, energy conservation and other physical laws. Additionally, the most general irreversible systems are *open* in the sense that their total energy is not conserved due to interaction with other systems that are assumed as being external.
Santilli decided to dedicate his research life to the identification of methods for the quantitative treatment of irreversible processes in their most general open formulation because all energy-releasing processes are open and irreversible thus not being quantitatively treatable with Galileo's and Einstein's relativities. besides, systems that are reversible over time are an evident particular case of irreversible systems.
During his Ph. d. studies, Santilli was deeply influenced by Lagrange's original papers, some of which were written in Italian in Torino and were available in the library of the University of Torino. In these works, Lagrange stresses that some of the forces of nature are representable with the quantity we call nowadays the Lagrangian, and other are not, thus being representable with external terms in the historical analytic equations (1.2).
Therefore, Santilli conducted extensive studies on the *integrability conditions for the existence of a Lagrangian or a Hamiltonian,* also called the *conditions of variational selfadjointness,* which he released for publication by Springer-Verlag later on 9see monograph [16a] when he was at Harvard University and following the delivering of a seminar course in the field. Thanks to these studies, santilli could identify Newton's forces more technically and write the equations in the form

$$m_k \frac{dv_k}{dt} = F_k^{SA}(t, r, v) + F_k^{NSA}(t, r, v, ...).$$

where SA (NSA) stands for verification (violation) of the integrability conditions to admit a potential. Hence, all SA forces are hereon represented with a Lagrangian or a Hamiltonikan and all NSA forces were represented with external terms.
During his intent to study the most general possible open and irreversible systems, Santilli soon discovered that their representation via the analytic equations with external terms has serious limitations because the brackets of the time evolution of a generic observable A characterized by Hamilton's equations with external terms

$$\frac{dA}{dt} = (A, H) = \frac{\partial H}{\partial r_k}\frac{\partial H}{\partial p_k} - \frac{\partial H}{p_k}\frac{\partial H}{\partial r_k} + \frac{\partial A}{\partial p_k} F_k^{NSA}(t, r, v, ...) \qquad (1.4)$$

violate the right scalar and distributivity axioms to characterize an algebra. Being an applied mathematician by instinct, Santilli could not use

Lagrange's and Hamilton's equations with external terms as the foundation of an irreversible covering of 20th century reversible theories and, therefore, he had to search for basically new methods.
By noting that *Lie's theory* (see the English translation of Lie's original thesis [7]) is at the foundation of 20th century theories, as part of his Ph. D/ thesis Santilli noted that their reversibility over time is due to the *invariance of the Lie product under anti-Hermiticity*

$$[A, B] = AB - BA = -[A, B]^{\dagger}. \tag{1.5}$$

where A, B are Hermitean and AB is the conventional associative product. Consequently, Santilli proposed the *embedding of Lie algebras into a covering algebra whose product is neither antisymmetric nor symmetric* of the type

$$(A, B) = \lambda AB - \mu BA = w[A, B] + z\{A, B\}, \tag{1.6}$$

where $\lambda = w + z,\ \mu = -w + z$ are non-null scalars.
Santilli then spent several months of search in European mathematical libraries to identify the type of algebras characterized by his product (A, B) and finally discovered that it characterized a jointly *Lie-admissible and Jordan-admissible algebra* according to the American mathematician A. A. Albert [8], in the sense that the attached antisymmetric and symmetric products verify the axioms of a Lie and Jordan algebras, respectively.
It was only following this identification of paternity that Santilli released the 1967 paper [9] for publication. It should be noted that paper [9] is the second paper following Albert's paper [8] in Lie-admissible algebras, and paper [9[is the origination paper of the simpler q-deformations of Lie algebras with product AB - qBA that appeared decades later.
In 1967 Santilli was invited by the *Center for Theoretical Physics* of the *University of Miami,* Coral Gables, Florida, under NASA support in view of his Lie-admissible studies on irreversibility since spaceship during reentry constitute irreversible systems (Fig. 1). Therefore, Santilli moved in Summer 1967 to the University of Miami where he wrote papers [10.11] proposing the following *parametric Lie-admissible generalization of Heisenberg equations* in its infinitesimal form

$$i\frac{dA}{dt} = (A, H) = \lambda AH - \mu HA = w[A, H] + z\{A, H\}, \tag{1.7a}$$

and integrated form

$$A(t) = e^{H\mu ti} A(0) e^{-it\lambda H}. \tag{1.7b}$$

Following his stay at the University of Miami, Santilli accepted the position of Associate Professor of Physics at the it Department of Physics of *Boston University* where he remained until 1974 to teach physics and mathematics from prep courses to advanced seminar course and to write "Phys. Rev" papers on various open problems of the time. The only paper in Lie-admissibility written during this period is that of Ref. [12] with P. Roman on the reformulation of the time evolution of the density matrix of an dissipative plasma with a Lie-admissible structure. This work remains to this day the sole characterization of dissipative plasma with a consistent algebra in the time evolution.

Following a stay at the *Institute for Theoretical Physics* of MIT from 1974 to 1977, Santilli moved to the *Lyman Laboratory of Physics* of *Harvard University* to be transferred in 1978 at the *Department of Mathematics* of the same university.[1] On arrival at Harvard on September 7, 1977, Santilli was invited by the DOE (then ERDA) to apply for a grant for the Lie-admissible treatment of irreversible processes because, as indicated earlier, all energy releasing processes are irreversible, while Galileo's relativity, Einstein's relativities and relativistic quantum mechanics are strictly reversible over time.

Under the backing of the DOE, Santilli resumed full time research on Lie-admissible formulations and wrote in 1978 two seminal memoirs [13,14], the first memoir on the status of our knowledge at that time in the Lie-admissible covering of Lie's theory (we cannot possibly review here for brevity), and the second memoir on physical applications of lie-admissible theories. he also wrote in 1978 monographs [14,16] and other papers.

In memoirs [13], Santilli introduced the the most general known, jointly Lie-admissible and Jordan-admissible product on a conventional *Hilbert space* $\mathcal{H}$ over a conventional field of complex numbers $\mathcal{C}$ (Sect. 3.7, p. 349, Ref. [13] and Eq. (94.14.11). p. 719, Ref. [14])

$$(A\hat{,}B) = ARB - BSA =$$

$$= (ATB - BTA) + (AWB + BWA) = [A\hat{,}B] + \{A\hat{,}B\} \qquad (1.8)$$

where $R = T + W, S = -T + W$ are this time operators that, besides being non-singular, have otherwise an unrestricted functional dependence on all needed local variables, $R = R(t, r, p, ...,\ S = A(t, r, v, ...)$.

In the same memoir [13] Santilli proved the *direct universality* of the algebras with [product $(A\hat{,}B)$ in the sense of admitting all possible algebras defined

[1]A recollection of his stay at MIT and Harvard is available in Santilli's lectures IIA and IIB of the World Lecture series, http://www.world-lecture-series.org

over a field of characteristic zero ("universality") without the use of the transformation theory ("direct universality").
In memoir [13] Santilli proposed the construction of a new mathematics for the quantitative treatment of Lie-admissible formulations under the name of *genomathematics,* where the prefix "geno" was suggested in the Greek sense of "inducing new axioms."
The covering character of lie-admissible algebras was soon noted by S. Adler who wrote paper [17] immediately following the appearance of memoirs [13.14] pointing out that supersymmetric and other algebras are indeed a particular case of Santilli's Lie-admissible algebras, thus identifying their irreversibility induced by the symmetric component of the product. Adler's analysis remains valid to this day, particularly in view of the physical insufficiencies of supersymmetric theories, thus suggesting their replacement with Lie-admissible algebras as for a quantitative study of the essential physical content, the irreversibility of the systems, recently suggested also by Santilli[2]
Thanks to the mathematical foundations set forth in the preceding memoir [13], Santilli proposed in the second memoir [14] the following *Operator Lie-admissible generalization of of Heisenberg equation,* also called *Heisenberg-Santilli genoequations* in the infinitesimal form (Eq. (4.15.34), p. 746, Re, [14])

$$\frac{dA}{dt} = (A\hat{,}H) =$$
$$= ARB - BSA = (ATB - BTA) + (AWB + BWA) =$$
$$= [A\hat{,}B] + \{A\hat{,}B\} \tag{1.9a}$$

and in the exponentiated form

$$A(t) = U((t)A(0)W^{\dagger}(t) = e^{HSti}A(0)e^{-itRH} \tag{1.9b}$$

where the Hamiltonian H represents all potential. therefore reversible forces, S represents all non-potential forces for motion forward in time, R represents the non-potential forces for motion backward in tine, and irreversibility is ensure by $R \neq S$.
Santilli then proposed in memoir [14] the construction of a covering of quantum mechanics based on Lie-admissible equations (1.9) with the name of *hadronic mechanics,* under the conditions of recovering quantum mechanics identically for the particular values $R = S = I$ (see, later on, Santilli's monographs [34,35] and independent general review [36]).

[2]See the website announcement http://www.santilli-foundation.org/Announcement-Super.php

In Section 5 of the same memoir [14], Santilli illustrated the validity of hadronic mechanics with the representation of all physical characteristics of the π^0 meson in its synthesis from a positronium (the bound state of an electron and a positron)

$$(e^-_\uparrow + e^+_\downarrow)_{J=0} \quad \rightarrow \quad \pi^0.$$

This synthesis is impossible for quantum mechanics because the rest energy of the π^0 meson is about 134-times the rest energy of the positronium, thus requiring a ""positive binding energy" under which quantum mechanical equations become inconsistent.[3]
Following the systematic liftings of all main spacetime symmetries,[4] Santilli solved for the first time the problem of the synthesis of the neutron from the Hydrogen atom inside a star (see review [18])

$$p^+_\uparrow + e^-_\downarrow \quad \rightarrow \quad n + \nu.$$

This synthesis constituted the main motivation by Santilli to propose the construction of hadronic mechanics in memoir [14] since it is the first synthesis inside stars prior to any possible synthesis of natural elements, thus illustrating again the intent of applying Lie-admissible formulation for irreversible energy releasing processes. The synthesis of the neutron is also outside any the capabilities of quantum mechanics because the rest energy of the neutron is 0.782 MeV bigger than the sum of the rest energy of the protons and of the electron, with ensuing lack ogf p-hysically m,eaningful solutions of the Schrödinger and Heisenberg equations.
Santilli's proposal originating from Harvard University to build the novel hadronic mechanics as a covering of quantum mechanics was an immediate success that stimulated world wide interest resulting in an estimated number of over $1,000$ papers, 30 post Ph. D. monographs and about 50 volumes of conferences proceedings. It is evident we cannot possibly review this volume of scientific publications and have to restrict ourself to indicate a few representative publications (see the extended bibliography in Vol. I of ref.s [34] for partial listing up to 2008).
We should mention the organization and conduction by the mathematician H. C. Myung and R. M. Santilli of five *International Workshops on*

[3]The only bound states admitted by quantum mechanics are those with a "negative binding energy" resulting in a "mass defect" of the bound state for which the rest energy of the bound state is "smaller" than the sum of the rest energies of the constituents, as it is the case for nuclear fusions.

[4]see website http://www.santilli-foundation.org/LPS-references.php for references all available in free pdf download

Lie-admissible Formulations held at Harvard University from 1078 to 197 to 1982 (se representative proceedings [19,20]). In 1981, the physicist J. Fronteau of the *Université d'Orleans,* France, and R. M. Santilli organized the *First International Conference on the Lie-admissible treatment of Irreversible systems* (see the four volumes of proceedings [21]). Then, over twenty *Workshops on Hadronic mechanics* were organized by Santilli and various other scientists held in the U.S.A, Europe and China (see the representative proceedings of the first workshop of 1982 [22]. An excellent collection of reprinted articles edited by A. Schoeber on irreversibility at the mechanical, statistical and thermodynamical levels is available in Ref. [23]. Important representative papers of this initial period in the construction of hadronic mechanics are: the identification by H. C. Myung and R. M. Santilli of the generalized Hilbert space requested by Lie-admissible dynamical equations [24]; the first known treatment of the irreversibility of nuclear fusions by R. M. santilli for which scope hadronic mechanics was built for [25]; the first known Lie-admissible treatment of open statistical systems by J. Fronteau, r. M. Santilli and A. Tellez-Arenas [26]; and numerous other important contributions listed in the proceedings.
Following this initial period in the construction of hadronic mechanics, there were either *Workshops on Hadronic Mechanics* whose proceedings were published by various scientific houses; the *Second International Conference on the lie-0admissible treatment of Irreversible Systems* held in 1995 at the Castle Prince Pignatelli, Molise, Italy, with fifteen volumes of proceedings published by Hadronic Press; and the *Third International, Conference on the Lie-admissible Treatment of Irreveresioble Systems* held in 2011 at the *Kathmandu University,* Nepal (see proceedings [27]).
Along a considerable list of additional important contributions during this second period for the construction of hadronic mechanics, we should quote: the achievement by R. M. Santilli of the invariance over time of Lie-admissible formulations in general and q-deformations in particular [28] as well as the achievement of mathematical and physical maturity for the Lie-admissible formulation of irreversible systems [29]; the first known Lie-admissible characterization of interior astrophysical systems by [30] J. Ellis, N. E. Mavromatos and D. V. Nanopoulos [30]; the first known Lie-admissible formulation of thermodynamical laws by J. Dunning-Dabvies [31]; the first known studies of compatibility of lie-admissible mechanics with thermodynamics by A. Bhalekar (see the recent paper [32[and references quoted therein); and the multi-valued hyperstructural formulation of the lie-admissible branch of hadronic mechanics by R. M. Santilli and the mathematician T. Vougiouklis which is the most general mathematics that can be conceived by the human

mind nowadays opening a new scientific era in quantitative treatment of biological structures, that are notoriously irreversible over time, with particular reference to the initiation of studies on the DNA code.[5]

1.0.5 Santilli Lie-Isotopic Representation of Closed Irreverible Systems

While writing the seminal memoir [13], Santilli realized that the genomathematics needed for the elaboration of Lie-admissible formulations was too complex for full comprehension by the general physics audience since Lie-admissible algebras were known at that time only by a few mathematicians and they had remained essentially unknown in the physics community, despite the need to study irreversible energy releasing processes as presented in papers [9-12] .

Consequently, Santilli introduce for the first time, also in Ref. [13], a simpler particular case of genomathematics under the name of *isomathematics,* where the prefix "iso" was introduced in the Greek meaning of being "axiom-preserving." Santilli recommended its study and development prior to addressing full Lie-admissible formulations, a suggestion that remains valid today and which has been adopted in this monograph for the presentation of *Santilli IsoDiifferential Calculus* (IDC) as a preparatory ground for broader studies.

The central physical notion of of isomathematics is that of *Santilli's closed-isolated non-Hamiltonian/NSA systems*, namely, systems that verify the conventional ten conservation laws (for the conservation of the total energy, total linear momentum, total angular momentum and the uniform motion of the center of mass), yet the internal forces are partially SA and partially NSA. As a consequence, the systems are not representable with a Hamiltonian or a Lagrangian. hence, closed non-Hamiltonian systems are irreversible over time.

These new systems were introduced first in memoir [13,14] and then treated via the conditions of variational selfadjointness in monograph [16b), Section 6.3, Eqs. (6.3.36 and Fig. 2), resulting in the following conditions for the

[5]Fully in line with the original motivation of the studies recalled earlier, and besides the scientific advances outlined above, Santilli Lie-admissible treatments of energy releasing, thus irreversible processes has already permitted the development of a number of new, patented, clean fuels and energies, such as the fuel*magnegas* with the new chemical structure of *magnecules* developed by the U. S. publicly traded company Magnegas Corporation (www.magnegas.com), the new nuclear fusions without radiations developed by the U.S. publicly traded company Thunder Fusion Corporation (www.thunder-fusion.com), and other new technologies.

internal non0-conservative forces to allow total conservation laws

$$\Sigma_{k=2,..,n} F_k^{NSA} = 0. \tag{1.10a}$$

$$\Sigma_{k=1...,n} p_k \star F_k^{NSA} = 0, \tag{1.10b}$$

$$\Sigma_{k=1,...,n} p_k \wedge F_k^{NSA} = 0, \tag{1.10c}$$

where one should note that $k \geq 2$ since one single particle, when isolated in vacuum, cannot experience non-Hamiltonian forces.

Figure 1.2: *Santilli illustrates closed non-Hamiltonian systems with Jupiter because it verifies the ten conservation laws of total physical quantities when considered as isolated from the rest of the universe, yet its interior dynamics is highly non-conservative as shown by atmospheric vortices with variable angular momenta, entropy, etc.*

The physical problem addressed by santilli for the treatment of closed irreversible systems was the identification of an algebra in the brackets of the time evolution which needs to be antisymmetric as an evident condition to represent the indicated ten conservation law, yet the algebra could not be Lie otherwise the represented systems would have no internal NSA forces. The solution found by Santilli signals the birth of isomathematics, and consists in the assumption in Ref. [13] of the following axiom-preserving, thus isotopic, generalization, also called *isotopic lifting* of the conventional associative product into the form today known as *Santilli isoproduct* between two generic quantities A, B (numbers, functions, matrices, operators, etc.)

$$A \times B \ \rightarrow \ A \hat{\times} B = A \times \hat{T} \times B, \quad T > 0, \tag{1.11}$$

where $\hat{T}$, called the *isotopic element,* is a positive-definite quantity (number, function, matrix, operator, etc.) which is fixed for the problem considered, but otherwise admits an unrestricted functional dependence on all needed local quantity, $\hat{T} = T(t, r, p, ...) > 0$
Via the use of the isoproduct, Santilli then presented, also for the first time in memoir [13] (see monograph [16a] for extended studies), the isotopic lifting of all main branches of Lie's theory, including the isotopies of the universal enveloping associative algebra, Lie algebras, Lie transformation groups, and the representation theory, today called the *Lie-Santilli isotheory* (see the independent studies [37,38]).
By recalling that Lie's theory can solely characterize linear, local and Hamiltonian systems, Santilli proposed the covering theory for the treatment of non-linear, non-local and non-Hamiltonian systems, with particular reference to closed irreversible systems.
We cannot possibly review here the Lie-Santilli isotheory, and can merely mention that the isotopies $\hat{\xi}(\hat{L})$ of the universal enveloping algebra $\xi(L)$ of a Lie algebra L, characterize the Lie-Santilli isoalgebra $\hat{L} \approx [\hat{\xi}(\hat{L})]^{-}$ with isoproduct

$$[A \hat{,} B] = A \hat{\times} B - B \hat{\times} A = A \times \hat{T} \times B - B \times \hat{T} \times A, \tag{1.12}$$

that evidently verifies the Lie axioms thus confirming its isotopic character.
Thanks to the isotopies of Lie's theory of memoir [13], Santilli then introduced in memoir [14] the isotopic lifting of Heisenberg's equations, today known as the *Heisenberg-Santilli isoequations* in their infinitesimal form 9see Eqs. (4.15.59), p. 752, Ref. [14])

$$i \times \frac{dA}{dt} = [A \hat{\times} H - H \hat{\times} A = A \times \hat{T} \times H - H \times \hat{T} \times A, \tag{1.13a}$$

and in the finite form

$$A(t) = U(t) \times A(0) \times U^{\dagger}(t) = e^{H\hat{T}ti} \times A(t) \times e^{-it\hat{T}H}, \tag{1.13b}$$

which were introduced to characterize the *isotopic branch of hadronic mechanics,* also known as *isomechanics.*

As one can see, dynamical equations. (1.13) do achieve the intended primary aims, namely: 1) Representing all potential interactions with the conventional Hamiltonian H; 2) Representing non-potential/NSA interactions with the isotopic element T; and 3) verifying all needed conservation law due to the antisymmetric character of isoproduct (1.12) as a necessary c condition to represent closed irreversible systems, as it is the case of conservation of the total energy

$$i\frac{dH}{dt} = [H\hat{;}H] = H\hat{\times}H = H\hat{\times}H \equiv 0,$$

The above representation implies that the systems are assumed as being isolated from the rest of the universe. Yet. the systems are generally irreversible because the isotopic element is generally non-invariant under time-reversal

$$T(t, ...) \neq T(-t, ...). \tag{1.14}$$

The above characteristics confirm the possible achievement of compatibility between isomechanics originally aimed at [23] and thermodynamics recently under study by A. Bhalekar [32] and others.

1.0.6 Lorentz-Poincaré-Santilli isosymmetry

Thanks to the prior construction in Refs. [13,14,16b] of the isotopies of Lie's theory, Santilli solved in paper [39] of 1983 the historical *Lorentz problem,* namely, the universal invariance of the locally varying speeds of light within physical media

$$C = c/n(t, r, d, \tau, ...), \tag{1.15}$$

where c is the speed of light in vacuum, and n is the familiar index of refraction with a rather complex dependence on the characteristic of the medium and light considered and other physical quantities.

As it is well known to historians, Lorentz did achieve the universal invariance of the *constant* speed of light c, but failed to achieve the invariance of locally varying speeds (1.15) because of the insufficiency of Lie's theory to treat locally varying speeds of light that constitute non-linear, non-local and non-Hamiltonian systems.

In essence, Santilli first constructed the isotopies of the Minkowski space $M(x, \eta, I)$ with local coordinates $x = (x^\mu)$, $\mu = 1, 2, 3, 4$, metric $\eta = Diag.(1, 1, 1, -1)$,and unit $I = Diag.(1, 1, 1, 1)$, today known as *Minkowski-Santilli isospaces,* with line element

$$\hat{x}^{\hat{2}} = x^\mu(\hat{T}^\rho_\mu \eta_{\rho\nu} x^\nu = x^\mu \hat{\eta}_{\mu\nu} x^\nu = \frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} - t^2 \frac{c^2}{n_4^2}, \qquad (1.16a)$$

$$\hat{T} = Diag.(1/n_1^2, 1/n_2^2, 1/n_3^2, 1/n_4^2) > 0, \qquad (1.16b)$$

where

$$n_\mu = n_\mu(t, r, v, e, \rho, \omega, \tau, ...) > 0, \quad \mu = 1, 2, 3, 4, \qquad (q.17)$$

where: the n's are called the *characteristic quantities* of the medium considered; n_4 is the conventional index of refraction providing a geometrization of the density of the medium normalized to the value $n_4 = 1$ for the vacuum; n_1, n_2, n_3 provide a geometrization of the shape of the medium considered normalized to the values $n_1 = n_2, n_3 = 1$ for the sphere; the general *inhomogeneity* of the medium is represented by the dependence of the characteristic quantity on the local variables (e.g., the elevation for the case of our atmosphere); and the general *anisotropy* of the medium (e.g., the anisotropy of our atmosphere caused by Earth's rotation) is represented by different values of the type $n_4 \neq n_s$.[6]
The identification of the isotopic element (1.16b) and its application of the Lie-isotopic theory) then permitted Santilli to solve the historical Lorentz problem in one page of ref. [39], resulting in the generalized transformations (Eqs. (15) of Ref. [39]), today known as the *Lorentz-Santilli (LS) isotransforms* [35-42] which we write in the currently used symmetrized form

$$x^{‘1} = x^1, \quad x^{‘2} = x^2, \qquad (1.19a)$$

$$x^{‘3} = \hat{\gamma}(x^3 - \hat{\beta}\ \frac{n_3}{n_4} x^4), \qquad (1.19b)$$

$$x^{‘4} = \hat{\gamma}(x^4 - \hat{\beta}\ \frac{n_4}{n_3} x^3), \qquad (1.19c)$$

where

$$\hat{\beta} = \frac{v_3/n_3}{c_o/n_4}, \quad \hat{\gamma} = \frac{1}{\sqrt{1 - \hat{\beta}^2}}, \qquad (1.19d)$$

which leave leaving invariant the isoline element (1.16a), thus providing the invariance of the varying speeds of light (1.12) (see Ref. [34b] for the general treatment).

[6] Scientific caution is suggested before dubbing the n-characteristic quantities as "free parameters" because that would imply that, e.g., the index of refraction n_4 is a free parameter when in reality it is measured for a given medium, or that the $g_{\mu\mu}$ elements of the Schwartzchild metric are free parameters, etc.

A main feature of the LS isosymmetry is that it is isomorphic to the conventional Lorentz symmetry, thus confirming its isotopic character. In reality, this property is a consequence of the fact that, despite its dependence on all needed local variables, *the Minkowski-isotopic space is locally isomorphic to the conventional Minkowski space.*
Subsequently, Santilli constructed a systematic, step by step isotopic lifting of every aspect of the conventional as well as spinorial covering of the Poincaré symmetry, resulting in an axiom-preserving covering symmetry today known as the *Lorentz-Poincaré-Santilli (LPS) isosymmetry* (see Ref. [41] for complete literature available in free pdf download and monographs [34] for a comprehensive treatment,) which isosymmetry is at the foundation of the *relativistic isomechanics* and its various scientific and industrial applications.
To understand the implications, the reader is suggested to note that the LPS isosymmetry has achieved, for the first time, the universal invariance of *all* possible spacetime elements in (3+1)-dimensions, including Riemannian, Fynslerian, and other spacetimes with important applications from particle physics to cosmology indicated later on.

1.0.7 Inconsistencies of Earlier Non-Unitary Theories

Following the above advances, Santilli conducted their in depth critical analysis by discovering that they had rather serious mathematical and physical insufficiencies today known under the name of the *Theorem of catastrophic Inconsistencies of Non0canonical and Non-unitary theories Elaborated with the mathematics of canonical and unitary theories, respectively* [42], that we can summarize as follows

INCONSISTENCY THEOREM 1.2: Non-canonical and non-unitary theories formulated with the mathematics of canonical and unitary theories, respectively, are mathematically and physically inconsistent.

In essence, time evolution (1.12b) is non-unitary on a conventional Hilbert space $\mathcal{H}$ over the field of complex number $\mathcal{C}$, i.e.

$$U\,U^{\dagger}\ \neq I.$$

Consequently, said time evolution does not preserve over time the basic unit and, therefore, does not preserve over time the basic numeric field, with consequently loss over time of the entire mathematics defined over a field.
Equally serious are the physical insufficiencies under the indicated conditions because the unit, e.g., of the Euclidean space $I = Diag(1, 1, 1)$ physically

provides a dimensionless representation of the units of measurements, such as $I = Diag.(1cm, 1cm, 1cm)$. Consequently, the lack of preservation over time of the unit physically implies the *inability of the theory to predict the same numerical values under the same conditions at different time,* the lack of preservation over time of Hermiticity with consequential loss of observables, the violation of causality laws, and other insufficiencies Santilli calls "catastrophic."

1.0.8 Isonumbers

Inconsistency Theorem 1.2 established the necessity of constructing the novel *isomathematics* via the step-by-step isotopic lifting of *all* aspect of mathematics formulated over a field of characteristic zero.

A central problem addressed by Santilli was the identification of representation of NSA interactions which is invariant over time. After numerous attempts, Santilli decided to *represent NSA forces via an isotopic generalization of the multiplicative unit of the theory,* because the unit is the basic invariant of any theory.

This lead to the introduction a generalized multiplicative unit as the inverse of the isotopic element of Refs. [13,14], today known as *Santilli isounit* [43], with explicit realizations in (3+1)-dimensions of the time

$$\hat{I} = Diag.(n_1^2, n_2^2, n_3^2, n_4^2)\ e^{\Gamma(t,r,p,\psi,\partial\psi,...)} = 1/\hat{T} > 0. \tag{1.20}$$

that clearly allows the representation of extended, non spherical and deformable bodies with shape represented by the characteristic quantities $n_k^2, k = 1, 2, 3$, the geometrization of the medium in their interior with the characteristic quantity n_4^2 and the representation of NSA interactions via the exponent Γ, as illustrated below.

Santilli's hesitation in using the isotopies of the unit was motivated by the fact that all physically consistent theories must be formulated over a numeric field as a prerequisite for their experimental verifications, while the isotopies of of the multiplicative units clearly imply the loss of all numeric fields known at the time 9early 1900) with the consequential inability to conduct experimental verification.

Due to his great respect for Gauss, Cayley, Hamilton and the other founders of the modern number theory, Santilli accepted as final their classification of numbers into real, complex and quaternionic numbers Octonions being excluded from the classification due to their violation of the axioms of associativity of the product.

As indicated in his writing, Santilli reinspected the historic classification of numbers "out of desperation" due to the impending loss of decades of research. In summer 1993 while visiting the *Joint Institute for Nuclear Research* in Dubna, Russia, Santilli to discovered that *the axioms of a numeric field do not necessary require that the multiplicative unit is the trivial number 1, since said unit can be an arbitrary non-singular quantity* $\hat{I}$.

This observation led to one of the most important mathematical discoveries of the 20th century with far reaching implications in all sciences: the isotopies of numeric fields presented for the first time in Santilli's paper [43] of 1993 and subsequently studies by numerous independent authors (see the monograph by the Chinese mathematician C-X. Jiang [44], the review by the Italian physicist C. Corda [45] and references quoted therein).

Regrettably, we cannot review *Santilli isonumber theory.* We merely mention that, given a numeric field $F(n, \times, 1)$ with real, complex or quaternionic numbers $n, m, ...$, conventional associative product $n \times m$ and basic unit 1, $1 \times n \equiv n \times I \equiv n \ \forall \ n \ \in \ F$, the ring $\hat{F}(\hat{n}, \hat{\times}, \hat{I})$, with elements $\hat{n} = n \times \hat{I}$ equipped with the isoproduct $\hat{n}\hat{\times}\hat{m} = n \times m \times \hat{I} \in \hat{F}$ and multiplicative isounit $\hat{I} = 1/\hat{T}$, $\hat{I}\hat{\times}\hat{n} \equiv \hat{n}\hat{\times}\hat{I} \equiv \hat{n} \ \forall \hat{n} \ \in \hat{F}$, verifies all axioms of a numeric field and are called *Santilli isofields* and the elements $\hat{n}$ are called *isonumbers.*.

Santilli additionally noted in Ref. [43] that *the axioms of a field do not necessarily require that the new multiplicative unit has to be an element of the original field.* This lead to the classification isofields into *isofields of the first kind* when the isounit is not an element of the original field, and *isofields of the second kind* when the isounit is an element of the original field.

The significance of Santilli's isonumber theory can be illustrated with the fact that, contrary to a popular belief throughout 20th century mathematics, *prime numbers do not have an absolute meaning because their value depends on the assumed multiplicative unit.* Consider the *real isofield* $\hat{\mathcal{R}}(n, \hat{\times}, \hat{I})$, $\hat{I} \in \mathcal{R}$ for which isonumbers coincide with ordinary numbers because $\hat{n} = n \times \hat{I} \in \mathcal{R}$. Then, for $\hat{I} = 3$, we have $2\hat{\times}3 = 2$ and 4 is a prime numbers.

We should also mention that Santilli's isonumbers have stimulated new *isocryptograms,* namely, cryptograms based on an infinite number of periodically changing multiplicative units, thus not being solvable in a finite period of time [34a].

1.0.9 Isofunctions

The discovery of isofields $\hat{F}(\hat{n}, \hat{\times}, \hat{I})$ required the reformulation of all studies conducted on isotopies prior to 1993 beginning with the lifting of functions into *isofunctions* of the form for a generic variable

$$\hat{f}(\hat{x}) = [f(x \times \hat{I})] \times \hat{I}, \tag{1.21}$$

because consistency requires that the dependence must be on *variables* and the value of isofunctions must be *isoscalars,* namely, elements of $\hat{F}$.
The birth of isofunctions stimulated a second group of mathematical studies in addition to those for isofields that we, regrettably, cannot review (see monograph [34a] for a general presentation with large literature up to 1995). We merely mention the notion of *isoexponential* first identified by H. C. Myung and R. M. Santilli in 1982 [46] via the use of ordinary numbers, and then finalized by Santilli [34a] in terms of isofunction,

$$\hat{e}^{\hat{x}} = \hat{I} + \hat{x}/1! + \hat{x}\hat{\times}\hat{x}/2! + ... = [e^{\hat{x}\hat{\times}\hat{T}}] \times \hat{I} = \hat{I} \times [e^{\hat{T}\hat{\times}\hat{x}}]. \tag{1.22}$$

which allowed Santilli to achieve maturity of formulation of the *isogroups*, e.g., for the one-dimensional case of time evolution (1.12b) that acquires the mathematically consistent form over isofields

$$\hat{A}(\hat{t}) = \hat{e}^{\hat{H}\hat{\times}\hat{t}\hat{i}}\hat{\times}\hat{A}(\hat{0})\hat{\times}\hat{e}^{-\hat{i}\hat{\times}\hat{t}timesH} \tag{1.23}$$

with similar expressions for n-dimensional isotransforms.
The isoexponent allowed the lifting of the conventional Dirac's delta function $\delta(r - r_o)$ into the expression introduced in Ref. [46]

$$\hat{\delta}(\hat{r} - \hat{r}_0) = \frac{\hat{I}}{2\pi}\hat{\times}\hat{\int}_{-\infty}^{+\infty} \hat{e}^{\hat{i}\hat{\times}\hat{k}\times(\hat{r}-\hat{r}_0)}\hat{\times}\hat{d}\hat{k}, = \frac{1}{2\pi} \times \int_{-\infty}^{+\infty} e^{i\times k\times T\times(r-r_o)} \times dk, \tag{1.24a}$$

$$\hat{T} = \Sigma_{k=1}^{n}\hat{c}_k \times (\hat{r} - \hat{r}_0)^{-k}, \quad \hat{c}_k \in \hat{\mathcal{C}}. \tag{1.24b}$$

$$\hat{\delta}(r - r_0) = \delta[T \times (r - r_0)]. \tag{1.24c}$$

which lifting was called by M. Nishioka [47] the *Dirac-Myung-Santilli (DMS) isofunction.*
The significance of the DMS isofunction is quite remarkable because, as illustrated in Fig. 3, under the suitable selection of the isotopic element, the DMS isofunction implies the elimination of the singularity of the conventional Dirac delta at $t = r_o$ which singularity is the origin for the divergences in quantum mechanics and quantum, field theory.

As a matter of fact, Santilli proposed the isotopies of 20th century applied mathematics by having particularly in mind the absence of divergences whose eliminations is somewhat arbitrary, thus implying unsettle numerical. results.

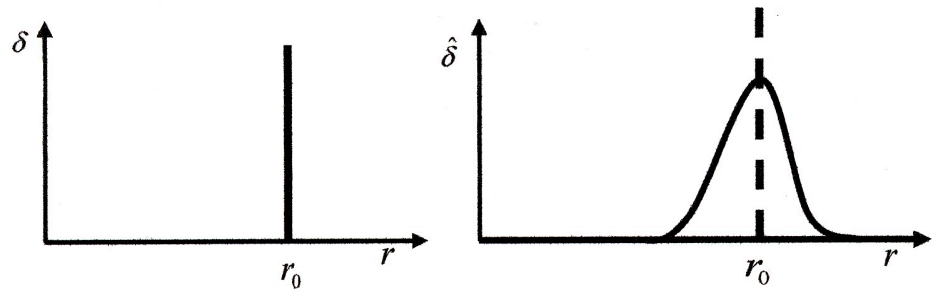

Figure 1.3: *A schematic view in the left of the conventional Dirac delta function $\delta(r-r_0)$ illustrating its divergence at r_0, and a schematic view in the right of the Dirac-Myung-Santilli isodelta isofunction of hadronic mechanics $\hat{\delta}(r-r_0)$, illustrating the absence of the above divergence at r_0, a feature allowing the removal of divergencies of quantum mechanics and quantum field theory.*

Among a number of additional initial contributions in isofunctions, we mention the studies by J. V. Kadeisvili [48,49] of 1992 on the isoanalysis with particular reference to the *Fourier-Santilli isotransforms*, and the studies by A. Aringazin et al [50] on various special isofunctions.
The authors have no words to stress that Santilli's isotopic theories must be elaborated with isofunctions due to the emergence of insidious inconsistencies in the use of conventional function that often remain undetected by non-experts in the field.

1.0.10 Isospaces

Following the isotopies of numeric fields and of functional analysis, the next isotopies necessary for physical applications were those of isospaces that were first lifted by Santilli in paper [29] of 1983 and then reformulated in various subsequent works, yielding the current notion of *isospaces* characterized by isotopic lifting of conventional (metric or pseudo-metric, compact or non-compact) spaces when defined over isofields, that Santilli studied in great details, including isospaces today called *Euclid-Santilli, Minkowski-Santilli, Riemannian-Santilli and other isospaces* (see monographs [34] and independent review [51).

We merely indicate for completeness that the correctly formulated Minkowski-Santilli isospace is given by $\hat{M}(\hat{x}, \hat{\eta}., \hat{I})$ with *isocoordinates* $\hat{x} = x \times \hat{I}$, *isometric* $\hat{\eta} = \hat{T} \times \eta$, where η is the conventional Minkowski metric, isounit $\hat{I} = 1/\hat{T}$ and infinite family of *isospacetimes*

$$\hat{x}^{\hat{2}} = \hat{x}^{\mu} \hat{\times} \hat{\eta}_{\mu\nu} \hat{\times} \hat{x}^{\nu} = (x^{\mu} \times \hat{\eta}_{\mu\nu} \times x^{\nu}) \times \hat{I} \tag{1.25}$$

where one should note the final multiplication by the isounit for the projection of the isospacetime in the conventional Minkowski coordinates for the value of the isoline element to be an isoscalar.

This seemingly trivial multiplication by the isounit has the deep implications that *the isotopies of the Minkowski spacetime are a kind of a hidden symmetry of the conventional spacetime* due to the identity

$$x^2 = (x^{\mu} \times \eta_{\mu\nu} \times x^{\nu}) \times I \equiv (\hat{x}^{\mu} \hat{\times} \hat{\eta}_{\mu\nu} \hat{\times} \hat{x}^{\nu}) \times \hat{\equiv} \hat{x}^{\hat{2}}, \tag{1.26a}$$

$$\hat{I} = 1/\hat{T} = K \in F, K \neq 0. \tag{1.26b}$$

A technical knowledge of Santilli isospaces is recommended prior to venturing any mathematical or physical interpretation. For instance, to understand the axiom-preserving character of the isotopies of the Minkowski space, it is necessary to know that, despite its deformed appearance, the *light isocone* in the (3, 4)-isoplane

$$\hat{x}^{\hat{2}} = \frac{\hat{x}_3^2}{n_3^2} - \frac{\hat{t}^{\hat{2}} \hat{\times} \hat{c}^{\hat{2}}}{n_4^2} = (\frac{x_3^2}{n_3^2} - \frac{t^2 c^2}{n_4^2}) \times \hat{I} = 0, \tag{1.27a}$$

$$\hat{I} = Diag.(n_3^2, n_4^2), \tag{1.27a}$$

is a *perfect cone* when properly formulated on $\hat{M}(\hat{x}, \hat{\times}, \hat{I})$,m to such an extent that the angle of the isocone is the same as that of the conventional cone, namely, the maximal causal speed on $\hat{M}$ remains $\hat{c}$.

Alternatively, one can see from Eqs. (1.27) that the conventional light cone id deformed with the characteristic quantities $1/n_3^2, 1/n_4^2$ to achieve the desired local variation of the speed of light $C = c/n_4$ in a properly symmetrized way. However, the selected isounit provide the *inverse* deformations, thus preserving the exact light cone in isospaces over isofields.

1.0.11 IsoDifferential Calculus

By the early 1994, all main aspects of mathematics defined over a field of characteristic zero had been isotopically lifted. Yet, Santilli remained dissatisfied because the fundamental dynamical equations, such as Eqs. (1.13) remained non-invariant over time. Consequently, hadronic mechanic was unable to predict the same numerical values under the same conditions at different times, thus being "without physical value" in Santilli's own words. Since the discovery of the isonumbers [43], Santilli spent a great effort to identify the origin of the lack of invariance, without any result. Finally, during the *Second International Conference on the Lie-admissible Treatment of Irreversible Processes* at the Castle Prince Pignatelli, Molise, Italy, Santilli had the courage to reinspect the differential calculus that had remained basically unchanged for the past four centuries.
As a result of a critical analysis, Santilli soon discovered that, contrary to a popular belief in mathematics and physics for centuries, *the differential calculus depends on the unit of the assumed basic field* because, in the event said unit is dependent on the differentiation variables, the conventional nations of differentials and derivatives are inapplicable.
This lead Santilli to another fundamental mathematical discovery of the 20th century, today known as *Santilli IsoDifferential Calculus* (IDC), which he first presented in Section 1.5, pages 19-23 of the mathematical memoir [52] published in 1996. with *isodifferential* of an isocoordinate $\hat{r}$[7]

$$\hat{d}\hat{r} = \hat{T} \times d(r \times \hat{I}) \tag{1.28}$$

and *isoderivative* of an isofunction $\hat{f}(\hat{r})$

$$\frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{r}} = \hat{I} \times \frac{\partial \hat{f}}{\partial \hat{r}} \tag{1.29}$$

In this way, following the isotopies of Heisenberg equations of 1978 [14], Santilli finally achieved their invariant formulation only in 1995 thanks to the use of the isodifferential calculus [52][8]

$$\hat{i}\hat{\times}\frac{\hat{d}\hat{A}}{\hat{d}\hat{t}} = \hat{A}\hat{;}\hat{H} = \hat{A}\hat{\times}\hat{H} - \hat{H}\hat{\times}\hat{A}. \tag{1.30}$$

[7]It should be noted that Santilli first wrote memoir [52] in 1995 and immediately thereafter the second edition of monographs [34] reformulated with the IDC, even though memoir [52] appeared in print one year later than the publication of monographs [34].

[8]It should be noted that Ref. [52] used the isounit and isotopic element in an interchanged form as compared to that of this book which is the notation nowadays widely used.

The achievement by hadronic mechanics of invariance over time stimulated the appearance of a large number of papers in mathematics, physics, chemistry, biology, astrophysics and other fields we can only briefly indicate later on.
Note that, for $\hat{I}$ constant or independent from the variable of differentiation, we have the trivial identities [52]

$$\hat{d}\hat{r} \equiv dr, \quad \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{r}} \equiv \frac{\partial f}{\partial r} \tag{1.31}$$

thus illustrating the reason for which the limitations of the differential calculus remained undetected for centuries.

1.0.12 Newton-Santilli Isoequations

Finally , after the above laborious scientific journey, Santilli was in a position to formulate in the mathematical memoir [52] of 1996 (as well as in monographs [34]) the desired structural generalization of Newton's equations for the representation of extended, non-spherical and deformable bodies as a condition to admit non-conservative/NSA forces when moving within a physical medium.
The generalized equations are defined on the Kronecker product of Euclid-Santilli isospaces for time, coordinates and velocities

$$\hat{S}(\hat{t}, \hat{r}, \hat{v}) = \hat{E}(\hat{t}, \hat{\delta}_t \hat{I}_t) \times \hat{E}(\hat{r}, \hat{\delta}_r, \hat{I}_r) \times \hat{E}(\hat{v}, \hat{\delta}_v, \hat{I}_v), \tag{1.32}$$

where: $\hat{I}_t = 1/\hat{T}_t,\ \hat{I}_r = 1/\hat{T}_r,\ \hat{I}_v = 1/\hat{T}_v$ are the isounits for time, coordinates and velocities, respectively all generally different among themselves, e.g., due to different dimensionalities; $\hat{t} = t \times \hat{I}_t$ is the isotime, $\hat{r} = (r_k \times \hat{I}_r)$ are the isocoordinates, $\hat{v} = (v_k \times \hat{I}_r)$ are the isovelocities; $\hat{\delta}_t = \hat{T}_t,\ \hat{\delta}_r = \hat{T}_r \times \delta,\ \hat{\delta}_v = \hat{T}_v \times \delta$ are the isometrics for time, coordinates and velocities, respectively; $\delta = Diag.(1, 1, 1)$ is the conventional metric of the Euclidean space; and $\hat{\delta}_t$ is evidently one-dimensional to comply with our current notion of time.[9]
The resulting generalized Newton's equations are then written in the form (see Eqs. (2.5), page 31, Ref. [52] and Chapter 1 of Ref. [34]), today called the *Newton-Santilli isoequations*

$$\hat{m}_k \hat{\times} \frac{\hat{d}\hat{v}_k}{\hat{d}\hat{t}} = \hat{F}_k^{SA}(\hat{t}, \hat{r}, \hat{v}) + \hat{F}_k^{NSA}(\hat{t}, \hat{r}, \hat{v}), \tag{1.33}$$

[9] It should be noted that the restriction for time being one-dimensional is lifted under hyperstructural formulations [35].

By assuming for simplicity $\hat{I}_t = \hat{I}_r = 1$, when projected in our time and space, and after eliminating redundant terms, the above equations assume to the simple form (where we ignore the conventional multiplication $\times$ to be in line with the formalism of Newtonian mechanics)

$$m_k \frac{d(v_k I_k)}{dt} = m_k \frac{dv_k}{dt} \hat{I}_v + m_k v_k \frac{d\hat{I}_v}{dt} = [F_k^{SA}(t,r,\hat{v}) + F_k^{NSA}(t,r,\hat{v})\hat{I}_v,$$

By assuming the realization of the isounit $\hat{I}_v$ in the exponential form, which is necessary for the generalized equations to be a covering of the conventional; equations.

$$\hat{I}_v = \Sigma_{k=1,2,3} Diag.(n_k^2 e^{\Gamma_k(t,r,v)}, \tag{1.34}$$

and assuming for simplicity that the $n_k 62$ are constants, thus tthey cancel out from the r.h.s and the l.h.s., the Newton-Santilli isoequations (1.33) represent identically the conventional Newton's equations (1.1) via the solutions of the equations

$$\frac{d\Gamma_k}{dt} = \frac{1}{m_k v_k} F_k^{NSA}, \tag{1.35}$$

that always exist under sufficient continuity and regularity conditions (see Theorem **??** of "direct universality" of the Newton-Santilli isoequations in page 31, Ref. [52]).[10]
Note the spirit of Santilli's isotopies of Newton equations, namely, the use of their lifting solely for the characterization of non-conservative/NSA forces via the isodifferential calculus. This is a truly crucial feature since it allows, as indicated below, the achievement of a "directly universal" isoaction principle for all (sufficiently smooth and regular) non-conservative/NSA forces that, in turn allows a unique and unambiguous map into the isotopic branch of hadronic mechanics, with a resulting new vistas in various scientific fields.
We should mention that Santilli formulated his isotopies of Newton's equations via the use of *Kadeisvili isocontinuity* [48] (see Ref. [52], Section 1.7, page 23 and ff.). The ultimate maturity of the isoequations is expressed by the isotopies of the conventional Euclidean topology, known as *isotopology*, that was first studied by Gr. Tsagas and D. S. Sourlas in Refs. [53,54] (see Ref. [52] page 24). Tsagas-Sourlas studies were re-reformulated by Santilli in Ref. [52] over isofields (because originally formulated over a conventional field); and the isotopology was finally studied in great mathematical details by R. M. Falcon Ganfornina and J. Nunez Valdes, first in monograph [55] and

[10]It should be indicated again that the use in Ref. [52] of the symbols $\hat{I}$ and $\hat{T}$ in interchanged with that of this chapter that later became of wide adoption.

then in memoir [56] specifically devoted to the isotopology, which is therefore referred to as the *Tsagas-Sourlas-Santilli-Ganfornina-Valdez (TSSGV) isotopology.*

We should also indicate that, far from being trivial, the isotopies of time, coordinates and velocities have rater deep geometric meanings due to the general property studied in details in Ref. [34b] that *the numerical value of a physical quantity in our time and space is* $q_{t,r}$ *is preserved under isotopies*

$$q_{t,r} \equiv \hat{q}_{\hat{t},\hat{r}} = q_{\hat{t},\hat{r}} \hat{I}_q, \tag{1.36}$$

as illustrated in the light isocone, Eq., (1.27) for which the locally varying speed of light within physical media $C = c/n$ is turned into the speed of light $\hat{c}$ on isospace over isofields due to the identity $c/n = \hat{c} = c \times I_v,\ I_c = 1/n$ (see Fig. 4 for comments)

It should be finally noted that Eqs. (1.zzz) are only one out a total of *seven* generalizations of Newton's equations introduced by Santilli following the prior constructions of the underlying mathematics,, and are known today as *Newton-Santilli iso,, geno, hyper-,* for the study of *matter* in conditions of increasing complexity,and the *Newton-Santilli isodual, isodual isotopic, isodual genotopic and isodual hyperstructural equations* for the study of *antimatter* in conditions also of increasing complexity 9see memoir [zzz] and monographs [zzz] for details) 9see memoir [57] and general treatment in monographs [35]).

1.0.13 Universal Isoaction Principle

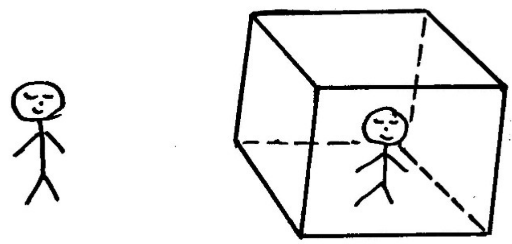

Figure 1.4: *The topological implications of the notions of isotime and isocoordinates were illustrated by Santilli [34a] via the drawing of this figure depicting an observer* O_{ext} *in our time* t_{ext} *and coordinates* r_{ext} *and an observer* $\hat{O}_{int}$ *in isotime* $\hat{t} = t_{int}\hat{I}_t$ *and isocoordinates* $\hat{r} = r_{int}\hat{I}_r$*, that observe the same box from the outside and the inside, respectively. Since the numerical values of time and coordinates are preserved under isotopies, Eq. (1.36), the internal observer still evolves forward in time, but can exist in the future or in the past with respect to the external observer, depending on the value of the time isounit. Under the same condition (1.36), the external observer sees a cube, while the internal observer can see a much different structure, again, depending on the coordinate isounit. At the limit, the internal observer can even see a cathedral under the multivalued hyperstructural extension of the isotopies [57]. The full understanding of Santilli's isotopies requires a knowledge of these topological anomalies including, perhaps more importantly, the fact that the internal anomalous features are not perceived by the sensory perception of the external observer due to law (1.36).*

Chapter 2

Isoreals

Let $\hat{F}_{\mathbb{R}}$ is a set of isoreals with basic isounit $\hat{I}_1 = \frac{1}{\hat{T}_1}$, where $\hat{T}_1$ is positive real constant.

Definition 2.0.2. *We will say that two elements* $\hat{a}$, $\hat{b} \in \hat{F}_{\mathbb{R}}$ *are equal and will write*

$$\hat{a} = \hat{b} \tag{2.0.1}$$

if

$$a = b \tag{2.0.2}$$

The relation equality is well defined. Really, if $a, b \in \mathbb{R}$ and (2.0.1) holds then

$$\frac{a}{\hat{T}_1} = \frac{b}{\hat{T}_1}, \tag{2.0.3}$$

from where (2.0.2) holds, and the inverse, if (2.0.2) holds then the equality (2.0.3) is valid, from where (2.0.1) holds.
The equality of the isoreals has the following properties

1. $\hat{g} = \hat{g}$ for every $\hat{g} \in \hat{F}_{\mathbb{R}}$,
2. if $\hat{g} = \hat{h}$ then $\hat{h} = \hat{g}$ for every $\hat{g}, \hat{h} \in \hat{F}_{\mathbb{R}}$,
3. if $\hat{g} = \hat{h}$ and $\hat{h} = \hat{l}$ then $\hat{g} = \hat{l}$ for every $\hat{g}, \hat{h}, \hat{l} \in \hat{F}_{\mathbb{R}}$.

These properties we can consider as direct corrolaries of the properties of the equality of real numbers therefore their proof we left to the reader.

Definition 2.0.3. *A nonempty set $\hat{F}_{\mathbb{R}}$ of isonumbers in which are defined two operations* addition $+$ *and* isomultiplication $\hat{\times}$ *so that*

1. $\hat{g} + \hat{h} = \hat{h} + \hat{g}$, $\hat{g}\hat{\times}\hat{h} = \hat{h}\hat{\times}\hat{g}$ *for every* $\hat{g}, \hat{h} \in \hat{F}_{\mathbb{R}}$,
2. $(\hat{g} + \hat{h}) + \hat{l} = \hat{g} + (\hat{h} + \hat{l})$, $(\hat{g}\hat{\times}\hat{h})\hat{\times}\hat{l} = \hat{g}\hat{\times}(\hat{h}\hat{\times}\hat{l})$ *for every* $\hat{g}, \hat{h}, \hat{l} \in \hat{F}_{\mathbb{R}}$,
3. $\hat{g}\hat{\times}(\hat{h} + \hat{l}) = \hat{g}\hat{\times}\hat{h} + \hat{g}\hat{\times}\hat{l}$ *for every* $\hat{g}, \hat{h}, \hat{l} \in \hat{F}_{\mathbb{R}}$,
4. *if* $\hat{g} = \hat{h}$ *and* $\hat{h} = \hat{l}$ *then* $\hat{g} = \hat{l}$ *for every* $\hat{g}, \hat{h}, \hat{l} \in \hat{F}_{\mathbb{R}}$,
5. $\hat{g} + \hat{h} + \hat{l} := (\hat{g} + \hat{h}) + \hat{l}$ *for every* $\hat{g}, \hat{h}, \hat{l} \in \hat{F}_{\mathbb{R}}$,
6. *the equation* $\hat{g} + \hat{r} = \hat{h}$, $\hat{g}, \hat{h} \in \hat{F}_{\mathbb{R}}$ *are given,* $\hat{r} \in \hat{F}_{\mathbb{R}}$ *is unknown, has a solution in* $\hat{F}_{\mathbb{R}}$,
7. *for every* $\hat{g} \in \hat{F}_{\mathbb{R}}$ *is valid only one of the relations* $\hat{g} = \hat{0}$ *or* $\hat{g} \neq \hat{0}$,
8. *if* $\hat{g} \neq \hat{0}$ *the equation* $\hat{g}\hat{\times}\hat{r} = \hat{h}$, $\hat{g}, \hat{h} \in \hat{F}_{\mathbb{R}}$ *has a solution in* $\hat{F}_{\mathbb{R}}$

will be called isoreal isofield.

Below we will consider $\hat{F}_{\mathbb{R}}$ as isoreal isofield.

Corollary 2.0.4. *Every solution of the equation* $\hat{g} + \hat{r} = \hat{g}$, $\hat{g} \in \hat{F}_{\mathbb{R}}$, *is a solution of the equation* $\hat{h} + \hat{r} = \hat{h}$, $\hat{h} \in \hat{F}_{\mathbb{R}}$.

Proof. Let $\hat{r} \in \hat{F}_{\mathbb{R}}$ is a solution of the equation $\hat{g} + \hat{r} = \hat{g}$ and $\hat{y}$ is a solution to the equation $\hat{g} + \hat{y} = \hat{h}$. Then

$$\hat{h} + \hat{r} = (\hat{g} + \hat{y}) + \hat{r} = \hat{g} + (\hat{y} + \hat{r})$$

$$= \hat{g} + (\hat{r} + \hat{y}) = (\hat{g} + \hat{r}) + \hat{y} = \hat{g} + \hat{y} = \hat{h}.$$

□

Corollary 2.0.5. *The equation $\hat{g} + \hat{r} = \hat{g}$, $\hat{g} \in \hat{F}_{\mathbb{R}}$, has unique solution.*

Proof. From the definition of isoreal isofield follows that the considered equation has a solution $\hat{r} \in \hat{F}_{\mathbb{R}}$. Let us suppose that it has and other one solution $\hat{y} \in \hat{F}_{\mathbb{R}}$, i.e.

$$\hat{g} + \hat{r} = \hat{g}, \quad \hat{g} + \hat{y} = \hat{g}.$$

From Corollary 2.0.3 follows that we have

$$\hat{y} + \hat{r} = \hat{y}, \quad \hat{r} + \hat{y} = \hat{r},$$

from here and the definition of isoreal isofield follows that $\hat{r} = \hat{y}$. □

Corollary 2.0.6. *Let $\hat{g}, \hat{h} \in \hat{F}_{\mathbb{R}}$. Then the equation $\hat{g} + \hat{r} = \hat{h}$ has unique solution.*

Proof. Let us suppose that the considered equation has two solutions $\hat{r}$, $\hat{y}$, for them we have

$$\hat{g} + \hat{r} = \hat{h}, \hat{g} + \hat{y} = \hat{h},$$

from where follows that

$$\hat{g} + \hat{r} = \hat{g} + \hat{y}.$$

From the definition for isoreal isofield follows that the equation $\hat{g} + \hat{z} = \hat{0}$ has a solution. Then

$$\hat{y} = \hat{y} + \hat{0} = \hat{y} + (\hat{g} + \hat{z})$$

$$= (\hat{y} + \hat{g}) + \hat{z} = (\hat{g} + \hat{y}) + \hat{z} = (\hat{g} + \hat{r}) + \hat{z}$$

$$= (\hat{r} + \hat{g}) + \hat{z} = \hat{r} + (\hat{g} + \hat{z}) = \hat{r} + \hat{0} = \hat{r}.$$

□

Consequently for every two isoreals $\hat{g}$, $\hat{h}$ the equation $\hat{g} + \hat{r} = \hat{h}$ has unique solution. This unique solution will be denoted with $\hat{h} - \hat{g}$ and we have

$$\hat{g} + (\hat{h} - \hat{g}) = \hat{h}.$$

From Corollary 2.0.5 follows that the equation $\hat{g} + \hat{r} = \hat{0}$ has unique solution which will be denoted with $-\hat{g}$ and we have

$$\hat{g} + (-\hat{g}) = \hat{0}.$$

The isoreal $-\hat{g}$ is called isoopposite of the isoreal $\hat{g}$.

Corollary 2.0.7. *Let $\hat{g} \in \hat{F}_{\mathbb{R}}$. Then*

$$\hat{g} - \hat{g} = \hat{g} + (-\hat{g}) \quad \text{and} \quad -(-\hat{g}) = \hat{g}.$$

Proof.

$$\hat{g} + (\hat{h} + (-\hat{g})) = \hat{g} + ((-\hat{g}) + \hat{h})$$

$$= (\hat{g} + (-\hat{g})) + \hat{h} = \hat{0} + \hat{h} = \hat{h},$$

in other words $\hat{h} + (-\hat{g})$ is a solution of the equation $\hat{g} + \hat{r} = \hat{h}$ which has, in accordance with Corollary 2.0.5, unique solution which is denoted with $\hat{h} - \hat{g}$. Therefore $\hat{h} + (-\hat{g}) = \hat{h} - \hat{g}$.
also,

$$-(-\hat{g}) = -(-\hat{g}) + \hat{0} = -(-\hat{g}) + (\hat{g} + (-\hat{g}))$$

$$= (\hat{g} + (-\hat{g})) - (-\hat{g}) = (\hat{g} + (-\hat{g})) + (-(-\hat{g}))$$

$$= \hat{g} + ((-\hat{g}) + (-(-\hat{g}))) = \hat{g} + \hat{0} = \hat{g}.$$

□

Corollary 2.0.8. *Let $\hat{g}, \hat{h} \in \hat{F}_{\mathbb{R}}$, $\hat{g} \neq \hat{0}$. Then the equation*

$$\hat{g} \hat{\times} \hat{r} = \hat{h}$$

has unique solution.

Proof. Since $\hat{g} \neq \hat{0}$, from the definition for isoreal isofield follows that the considered equation has a solution $\hat{r} \in \hat{F}_{\mathbb{R}}$. Let us suppose that it has other solution $\hat{y}$. Let $\hat{z}$ is a solution of the equation $\hat{g}\hat{\times}\hat{r} = \hat{I}_1$. Then

$$\hat{y} = \hat{y}\hat{\times}\hat{I} = \hat{y}\hat{\times}(\hat{g}\hat{\times}\hat{z}) = (\hat{y}\hat{\times}\hat{g})\hat{\times}\hat{z}$$

$$= (\hat{g}\hat{\times}\hat{y})\hat{\times}\hat{z} = \hat{h}\hat{\times}\hat{z} = (\hat{g}\hat{\times}\hat{r})\hat{\times}\hat{z}$$

$$= (\hat{r}\hat{\times}\hat{g})\hat{\times}\hat{z} = \hat{r}\hat{\times}(\hat{g}\hat{\times}\hat{z}) = \hat{r}\hat{\times}\hat{I}_1 = \hat{r}.$$

□

Consequently for every $\hat{g}, \hat{h} \in\in \hat{F}_{\mathbb{R}}$, $\hat{g} \neq \hat{0}$ the equation $\hat{g}\hat{\times}\hat{r} = \hat{h}$ has unique solution which will be denoted with $\hat{h} \nwarrow \hat{g}$ and will be called quotient of $\hat{h}$ and $\hat{q}$ and we have

$$\hat{g}\hat{\times}(\hat{h} \nwarrow \hat{g}) = \hat{h}.$$

Corollary 2.0.9. *Let $\hat{g}, \hat{h} \in \hat{F}_{\mathbb{R}}$. Then*

1. $\hat{0} = -\hat{0}$,
2. $\hat{g} - \hat{0} = \hat{g}$,
3. $\hat{g}\hat{\times}\hat{0} = \hat{0}$,
4. $\hat{I}_1 \neq \hat{0}$,
5. $(-\hat{I}_1)\hat{\times}\hat{a} = -\hat{a}$,
6. $(-\hat{I}_1)\hat{\times}(-\hat{I}_1) = \hat{I}_1$,
7. $-(\hat{g} - \hat{h}) = \hat{h} - \hat{g}$.

Proof. 1. The isoreals $\hat{0}$ and $-\hat{0}$ are solutions of the equation $\hat{0} + \hat{r} = \hat{0}$, which has unique solution. Therefore $\hat{0} = -\hat{0}$.

2. Since $\hat{0} + (\hat{g} - \hat{0}) = \hat{g}$ then $\hat{g} - \hat{0}$ is a solution of the equation of the equation $\hat{0} + \hat{r} = \hat{g}$, but its solution is $\hat{g}$ because $\hat{0} + \hat{g} = \hat{g} + \hat{0} = \hat{g}$. Then from Corollary 2.0.5 it has unique solution, therefore $\hat{g} - \hat{0} = \hat{g}$.

3.
$$\hat{g}\hat{\times}\hat{0} = \hat{g}\hat{\times}\hat{0} + \hat{0} = \hat{g}\hat{\times}\hat{0} + (\hat{g} + (-\hat{g}))$$
$$= (\hat{g}\hat{\times}\hat{0} + \hat{g}) + (-\hat{g}) = (\hat{g}\hat{\times}\hat{0} + \hat{g}\hat{\times}\hat{I}_1) + (-\hat{g})$$
$$= \hat{g}\hat{\times}(\hat{0} + \hat{I}_1) + (-\hat{g}) = \hat{g}\hat{\times}(\hat{I}_1 + \hat{0}) + (-\hat{g})$$
$$= \hat{g}\hat{\times}\hat{I}_1 + (-\hat{g}) = \hat{g} + (-\hat{g}) = \hat{0}.$$

4. Since there exists $\hat{a} \in \hat{F}_{\mathbb{R}}$ so that $\hat{a} \neq \hat{0}$ then if we suppose that $\hat{I}_1 = \hat{0}$ we will have
$$\hat{a} = \hat{a}\hat{\times}\hat{I}_1 = \hat{a}\hat{\times}\hat{0} = \hat{0},$$
which is a contradiction. Therefore $\hat{I}_1 \neq \hat{0}$.

5.
$$(-\hat{I}_1)\hat{\times}\hat{g} = (-\hat{I}_1)\hat{\times}\hat{g} + \hat{0} =$$
$$= (-\hat{I}_1)\hat{\times}\hat{g} + (\hat{g} + (-\hat{g})) = ((-\hat{I}_1)\hat{\times}\hat{g} + \hat{g}) + (-\hat{g})$$
$$= \hat{g}\hat{\times}((-\hat{I}_1) + \hat{I}_1) + (-\hat{g}) = \hat{g}\hat{\times}(\hat{I}_1 + (-\hat{I}_1)) + (-\hat{g})$$
$$= \hat{g}\hat{\times}\hat{0} + (-\hat{g}) = \hat{0} + (-\hat{g}) = -\hat{g}.$$

6. $(-\hat{I}_1)\hat{\times}(-\hat{I}_1) = -(-\hat{I}_1) = \hat{I}_1$.

7.
$$-(\hat{g} - \hat{h}) = (-\hat{I}_1)\hat{\times}(\hat{g} - \hat{h}) = (-\hat{I}_1)\hat{\times}(\hat{g} + (-\hat{h}))$$
$$= (-\hat{I}_1)\hat{\times}\hat{g} + (-\hat{I}_1)\hat{\times}(-\hat{h}) = -(-\hat{h}) + (-\hat{g}) = \hat{h} - \hat{g}.$$

□

The term positive isonumber is a primary term and its content is determined by the following axioms

A1 The isozero $\hat{0}$ is not positive number.

A2 If $\hat{g} \neq \hat{0}$ then one of the isonumbers $\hat{g}$ and $-\hat{g}$ is positive isonumber.

A3 If the isoreals $\hat{g}$, $\hat{h} \in \hat{F}_{\mathbb{R}}$ are positive then the isoreals $\hat{g} + \hat{h}$ and $\hat{g}\hat{\times}\hat{h}$ are positive.

A4 If $\hat{g}, \hat{h} \in \hat{F}_{\mathbb{R}}$ and $\hat{g}$ is positive and $\hat{g} = \hat{h}$ then $\hat{h}$ is positive isonumber.

Corollary 2.0.10. *If $\hat{g} \in \hat{F}_{\mathbb{R}}$ and $\hat{g} \neq \hat{0}$ then the isonumbers $\hat{g}$ and $-\hat{g}$ can not be simultaneously positive.*

Proof. If $\hat{g}$ and $-\hat{g}$ are simultaneously positive then from axiom A3 follows that their sum is positive, but their sum is the isozero which is a contradiction with the axiom A1. □

Definition 2.0.11. *The isonumbers which are not positive and are not equal to the isozero will be called negative.*

Corollary 2.0.12. *The isonumber $\hat{I}_1$ is positive number.*

Proof. We have that $\hat{I}_1 \neq \hat{0}$. if we suppose that $-\hat{I}_1$ is positive then from the axiom A3 follows that

$$(-\hat{I}_1)\hat{\times}(-\hat{I}_1) = \hat{I}_1$$

is positive. Therefore $\hat{I}_1$ and $-\hat{I}_1$ are simultaneously positive, which is a contradiction. Consequently $\hat{I}_1$ is positive. □

Definition 2.0.13. *We will say that the isonumber $\hat{g} \in \hat{F}_{\mathbb{R}}$ is less than the isonumber $\hat{h} \in \hat{F}_{\mathbb{R}}$ and we will write*

$$\hat{g} < \hat{h}$$

if $\hat{h} - \hat{g}$ is positive.

Example 2.0.14. *Let $\hat{T}_1 = 4$. Then*

$$\hat{3}\hat{\times}\hat{x} + \hat{4} = \hat{5} \quad \Longleftrightarrow$$

$$\tfrac{3}{4}4\tfrac{x}{4} + \tfrac{4}{4} = \tfrac{5}{4} \quad \Longrightarrow$$

$$\tfrac{3}{4}x + 1 = \tfrac{5}{4} \quad \Longrightarrow$$

$$\tfrac{3}{4}x = \tfrac{5}{4} - 1 \quad \Longrightarrow$$

$$\tfrac{3}{4}x = \tfrac{1}{4} \quad \Longrightarrow$$

$$x = \tfrac{1}{3} \quad \Longrightarrow$$

$$\hat{x} = \tfrac{1}{3}\tfrac{1}{4} = \tfrac{1}{12}.$$

Example 2.0.15. *Let $\hat{T}_1 = 5$ and let us consider the equation*

$$\hat{3}\hat{x} + \hat{2} = \hat{7},$$

which is equivalent of the equation

$$\frac{3}{5}\frac{x}{5} + \frac{2}{5} = \frac{7}{5}$$

from where

$$\tfrac{3}{5}x + 2 = 7 \quad \Longrightarrow$$

$$\tfrac{3}{5}x = 5 \quad \Longrightarrow$$

$$x = \tfrac{25}{3} \quad \Longrightarrow$$

$$\hat{x} = \tfrac{25}{3}\tfrac{1}{5} = \tfrac{5}{3}.$$

Exercise 2.0.16. *Let $\hat{T}_1 = 2$. Find $\hat{x}$ such that*

$$1) \quad \hat{2}\hat{\times}\hat{x} = \hat{4}, \quad 2) \quad \hat{2}\hat{x} = \hat{4}.$$

Answer. 1) $\hat{x} = 1$, 2) $\hat{x} = 2$.

Definition 2.0.17. *We will say that the isonumber $\hat{g} \in \hat{F}_{\mathbb{R}}$ is less or equal to the isonumber $\hat{h} \in \hat{F}_{\mathbb{R}}$ we will write*

$$\hat{g} \leq \hat{h} \quad \text{or} \quad \hat{g} \leq \hat{h},$$

if $\hat{h} - \hat{g}$ is positive or it is equal to the isozero $\hat{0}$.

Corollary 2.0.18. *For every two isoreals $\hat{g}, \hat{h} \in \hat{F}_{\mathbb{R}}$ is valid one of the following relations*

1. $\hat{g} = \hat{h}$,
2. $\hat{g} > \hat{h}$,
3. $\hat{g} < \hat{h}$.

Proof. We have two posibilities $\hat{g} = \hat{h}$ or $\hat{g} \neq \hat{h}$. If $\hat{g} \neq \hat{h}$ then $\hat{g} - \hat{h} \neq \hat{0}$ and one of the isonumbers $\hat{g} - \hat{h}$, $-(\hat{g} - \hat{h})$ is positive. If $\hat{g} - \hat{h}$ is positive then we have the second relation, if $-(\hat{g} - \hat{h})$ is positive then we have the third relation. □

Corollary 2.0.19. *The isoreal $\hat{g} \in \hat{F}_{\mathbb{R}}$ is positive iff $\hat{g} > \hat{0}$.*

Proof. Using the definition $\hat{g} > \hat{0}$ iff $\hat{g} - \hat{0} = \hat{g}$ is positive. □

Corollary 2.0.20. *Let* $\hat{a}, \hat{b}, \hat{c}, \hat{d} \in \hat{F}_{\mathbb{R}}$*. If* $\hat{a} > \hat{b}$ *and* $\hat{c} > \hat{d}$ *then*

$$\hat{a} + \hat{c} > \hat{b} + \hat{d}.$$

Proof. From $\hat{a} > \hat{b}$ and $\hat{c} > \hat{d}$ follows that

$$\hat{a} - \hat{b} > \hat{0}, \quad \hat{c} - \hat{d} > \hat{0}.$$

Then from axiom A3 follows that

$$(\hat{a} - \hat{b}) + (\hat{c} - \hat{d}) > \hat{0}. \tag{2.0.4}$$

Also,

$$(\hat{a} + \hat{c}) - (\hat{b} + \hat{d}) = (\hat{a} + \hat{c}) + (-\hat{I})\hat{\times}(\hat{b} + \hat{d})$$

$$= (\hat{a} + \hat{c}) + ((-\hat{I})\hat{\times}\hat{b} + (-\hat{I})\hat{\times}\hat{d})$$

$$= ((\hat{a} + \hat{c}) + (-\hat{I})\hat{\times}\hat{b}) + (-\hat{I})\hat{\times}\hat{d}$$

$$((\hat{c} + \hat{a}) + (-\hat{I})\hat{\times}\hat{b}) + (-\hat{I})\hat{\times}\hat{d}$$

$$= (\hat{c} + (\hat{a} + (-\hat{I})\hat{\times}\hat{b})) + (-\hat{I})\hat{\times}\hat{d}$$

$$= ((\hat{a} + (-\hat{I})\hat{\times}\hat{b}) + \hat{c}) + (-\hat{I})\hat{\times}\hat{d}$$

$$= (\hat{a} + (-\hat{I})\hat{\times}\hat{b}) + (\hat{c} + (-\hat{I})\hat{\times}\hat{d})$$

$$= (\hat{a} - \hat{b}) + (\hat{c} - \hat{d})$$

From the last expression, from (2.0.4) and axiom A4 we obtain that

$$(\hat{a} + \hat{c}) - (\hat{b} + \hat{d}) > \hat{0},$$

from where

$$\hat{a} + \hat{c} > \hat{b} + \hat{d}.$$

□

Corollary 2.0.21. *1. If $\hat{g}, \hat{h} \in \hat{F}_{\mathbb{R}}$ and $\hat{g} > \hat{h}$ then $\hat{g} + \hat{l} > \hat{h} + \hat{l}$ for every $\hat{l} \in \hat{F}_{\mathbb{R}}$,*

2. If $\hat{g}, \hat{h}, \hat{l} \in \hat{F}_{\mathbb{R}}$ and $\hat{g} > \hat{h}$ and $\hat{h} > \hat{l}$ then $\hat{g} > \hat{l}$,

3. If $\hat{g}, \hat{h}, \hat{l} \in \hat{F}_{\mathbb{R}}$, $\hat{g} > \hat{0}$, $\hat{h} > \hat{l}$ then $\hat{g} \hat{\times} \hat{h} > \hat{g} \hat{\times} \hat{l}$,

4. If $\hat{g}, \hat{h}, \hat{l} \in \hat{F}_{\mathbb{R}}$, $\hat{g} < \hat{0}$, $\hat{h} > \hat{l}$ then $\hat{g} \hat{\times} \hat{h} < \hat{g} \hat{\times} \hat{l}$.

Proof. 1. We have that $\hat{g} > \hat{h}$ iff $\hat{g} - \hat{h} > \hat{0}$. On the other hand

$$(\hat{g} + \hat{l}) - (\hat{h} + \hat{l}) = (\hat{g} - \hat{h}),$$

from here and from A4 follows that

$$(\hat{g} + \hat{l}) - (\hat{h} + \hat{l}) > \hat{0},$$

and therefore $\hat{g} + \hat{l} > \hat{h} + \hat{l}$.

2. From $\hat{g} > \hat{h}$ and $\hat{h} > \hat{l}$ follows that $\hat{g} - \hat{h} > \hat{0}$ and $\hat{h} - \hat{l} > \hat{0}$. From here and *1* follows that $(\hat{g} - \hat{h}) + (\hat{h} - \hat{l}) > \hat{0}$. Since

$$\hat{g} - \hat{l} = (\hat{g} - \hat{h}) + (\hat{h} - \hat{l}),$$

using A4 we conclude that $\hat{g} - \hat{l}$ is positive and then $\hat{g} > \hat{l}$.

3. We have that $\hat{h} - \hat{l} > \hat{0}$, therefore $\hat{g} \hat{\times} (\hat{h} - \hat{l}) > \hat{0}$. Because $\hat{g} \hat{\times} (\hat{h} - \hat{l}) = \hat{g} \hat{\times} \hat{h} - \hat{g} \hat{\times} \hat{l}$ we have $\hat{g} \hat{\times} \hat{h} - \hat{g} \hat{\times} \hat{l} > \hat{0}$.

4. We have $\hat{h} - \hat{l} > \hat{0}$ and $-\hat{g} > \hat{0}$. Therefore $-\hat{g} \hat{\times} (\hat{h} - \hat{l}) = -\hat{g} \hat{\times} \hat{h} + \hat{g} \hat{\times} \hat{l} = \hat{g} \hat{\times} \hat{l} - \hat{g} \hat{\times} \hat{h}$ is positive. Consequently $\hat{g} \hat{\times} \hat{l} > \hat{g} \hat{\times} \hat{h}$, from here $\hat{g} \hat{\times} \hat{h} < \hat{g} \hat{\times} \hat{l}$. □

Definition 2.0.22. *Absolute value or modulus of the isoreal $\hat{g} \in \hat{F}_{\mathbb{R}}$ is called the larger isonumber of the isonumbers $\hat{g}$, $-\hat{g}$. We will write $|\hat{g}|$. From this definition follows that*

1. $|\hat{g}| \geq \hat{0}$ *for every* $\hat{g} \in \hat{F}_{\mathbb{R}}$, $|\hat{g}| = \hat{0}$ *iff* $\hat{g} = \hat{0}$,
2. $\hat{g} \leq |\hat{g}|$, $-\hat{g} \leq |\hat{g}|$ *for every* $\hat{g} \in \hat{F}_{\mathbb{R}}$,
3. $|\hat{g}| = |-\hat{g}|$ *for every* $\hat{g} \in \hat{F}_{\mathbb{R}}$.

Theorem 2.0.23. *Let* $\hat{g}, \hat{h}, \hat{l} \in \hat{F}_{\mathbb{R}}$. *Then*

1. $|\hat{g} + \hat{h}| \leq |\hat{g}| + |\hat{h}|$,
2. $|\hat{g} - \hat{h}| \geq |\hat{g}| - |\hat{h}|$,
3. $||\hat{g}| - |\hat{h}|| \leq |\hat{g} - \hat{h}|$,
4. $|\hat{g} \hat{\times} \hat{h}| = |\hat{g}| \hat{\times} |\hat{h}|$,
5. $|\hat{g} \hat{/} \hat{h}| = |\hat{g}| \hat{/} |\hat{h}|$, $\hat{h} \neq \hat{0}$,
6. $|\hat{g}| < \hat{h} \quad \Longleftrightarrow \quad -\hat{h} < \hat{g} < \hat{h}$, *for* $\hat{h} > \hat{0}$.

Proof. 1. If $\hat{g} + \hat{h} \geq \hat{0}$ then $|\hat{g} + \hat{h}| = \hat{g} + \hat{h} \leq |\hat{g}| + |\hat{h}|$. If $\hat{g} + \hat{h} =< \hat{0}$ then $|\hat{g} + \hat{h}| = -(\hat{g} + \hat{h}) = -\hat{g} - \hat{h} =< |\hat{g}| + |\hat{h}|$.

2. Using 1 we have $|\hat{g}| - |\hat{h}| = |\hat{g} - \hat{h} + \hat{h}| - |\hat{h}| = |(\hat{g} - \hat{h}) + \hat{h}| - |\hat{h}| =< |\hat{g} - \hat{h}| + |\hat{h}| - |\hat{h}| = |\hat{g} - \hat{h}|$.

3. If $|\hat{g}| - |\hat{h}| \geq \hat{0}$ then $||\hat{g}| - |\hat{h}|| = |\hat{g}| - |\hat{h}| =< |\hat{g} - \hat{h}|$. If $|\hat{g}| - |\hat{h}| =< \hat{0}$ then $||\hat{g}| - |\hat{h}|| = |\hat{h}| - |\hat{g}| =< |\hat{h} - \hat{g}| = |\hat{g} - \hat{h}|$.

4. If $\hat{g} \geq \hat{0}$ and $\hat{h} \geq \hat{0}$ then $\hat{g} \hat{\times} \hat{h} \geq \hat{0}$ and $|\hat{g} \hat{\times} \hat{h}| = \hat{g} \hat{\times} \hat{h} = |\hat{g}| \hat{\times} |\hat{h}|$. If $\hat{g} \geq \hat{0}$

and $\hat{h} \leq \hat{0}$ then $\hat{g}\hat{\times}\hat{h} \leq \hat{0}$ and

$$|\hat{g}\hat{\times}\hat{h}| = -(\hat{g}\hat{\times}\hat{h}) = (-\hat{I}_1)\hat{\times}(\hat{g}\hat{\times}\hat{h})$$

$$= (-\hat{I}_1)\hat{\times}(\hat{h}\hat{\times}\hat{g}) = (-\hat{I}_1\hat{\times}\hat{h})\hat{\times}\hat{g} = (-\hat{h})\hat{\times}(\hat{g})$$

$$= |\hat{h}|\hat{\times}|\hat{g}| = |\hat{g}|\hat{\times}|\hat{h}|.$$

If $\hat{g} =< \hat{0}$ and $\hat{h} \geq \hat{0}$ then $\hat{g}\hat{\times}\hat{h} =< \hat{0}$ and

$$|\hat{g}\hat{\times}\hat{h}| = -(\hat{g}\hat{\times}\hat{h}) = (-\hat{I}_1)\hat{\times}(\hat{g}\hat{\times}\hat{h}) = (-\hat{I}_1\hat{\times}\hat{g})\hat{\times}(\hat{h}) = |\hat{g}|\hat{\times}|\hat{h}|.$$

If $\hat{g} =< \hat{0}$ and $\hat{h} =< \hat{0}$ then $\hat{g}\hat{\times}\hat{h} \geq \hat{0}$ and

$$|\hat{g}\hat{\times}\hat{h}| = \hat{g}\hat{\times}\hat{h} = (-\hat{I}_1)\hat{\times}(-\hat{I}_1)\hat{\times}(\hat{g}\hat{\times}\hat{h})$$

$$= (-\hat{I}_1)\hat{\times}((-\hat{I}_1)\hat{\times}(\hat{g}\hat{\times}\hat{h})) = (-\hat{I}_1)\hat{\times}(((-\hat{I}_1)\hat{\times}\hat{g})\hat{\times}\hat{h})$$

$$= (-\hat{I}_1)\hat{\times}((-\hat{g})\hat{\times}\hat{h})$$

$$= (-\hat{I}_1)\hat{\times}(\hat{h}\hat{\times}(-\hat{g})) = ((-\hat{I}_1)\hat{\times}\hat{h})\hat{\times}(-\hat{g})$$

$$= (-\hat{h})\hat{\times}(-\hat{g}) = (-\hat{g})\hat{\times}(-\hat{h}) = |\hat{g}|\hat{\times}|\hat{h}|.$$

5. This assertion follows from the previous statement for $\hat{g}$ and $\hat{I}_1 \hat{/} \hat{h}$.

6. If $\hat{g} \geq 0$ then $|\hat{g}| = \hat{g}$ and $|\hat{g}| < \hat{h}$ is equivalent of the inequality $\hat{g} < \hat{h}$ and since every positive isonumber is greater than every negative isoreal then $|\hat{g}| < \hat{h}$ is equivalent of $-\hat{h} \leq \hat{g} = \hat{h}$. If $\hat{g} \leq \hat{0}$ then $|\hat{g}| = -\hat{g}$ and the inequality $|\hat{g}| < \hat{h}$ is equivalent of the inequality $-\hat{g} < \hat{h}$, from where $(-\hat{I}_1)\hat{\times}(-\hat{g}) > (-\hat{I}_1) > \hat{h}$ or $\hat{g} > -\hat{h}$. Also, since $\hat{g}$ is negative isoreal it is less than $\hat{h}$, i.e. the considered inequality is equivalent of the inequality $-\hat{h} < \hat{g} < \hat{h}$.

□

Example 2.0.24. *Let $\hat{T}_1 = 2$. We will solve the following inequality*

$$\hat{3}\hat{x} - \hat{4} > \hat{5}.$$

For this inequality we have

$$\hat{3}\hat{x}-\hat{4}>\hat{5} \quad \Longleftrightarrow$$

$$\tfrac{3}{2}\tfrac{x}{2}-\tfrac{4}{2}>\tfrac{5}{2} \quad \Longrightarrow$$

$$\tfrac{3}{2}x-4>5 \quad \Longrightarrow$$

$$\tfrac{3}{2}x>9 \quad \Longrightarrow$$

$$x>6 \quad \Longleftrightarrow$$

$$x\tfrac{1}{2}>6\tfrac{6}{2} \quad \Longleftrightarrow$$

$$\hat{x}>\hat{3}.$$

Example 2.0.25. *Let $\hat{T}_1=4$. We consider the inequality*

$$\hat{5}\hat{\times}\hat{x}-\hat{3}<\hat{2},$$

which is equivalent of the inequality

$$\tfrac{5}{4}4\tfrac{x}{4}-\tfrac{3}{4}<\tfrac{2}{4} \quad \Longrightarrow$$

$$5x-3<2 \quad \Longrightarrow$$

$$5x<5 \quad \Longrightarrow$$

$$x<1 \quad \Longrightarrow$$

$$x\tfrac{1}{4}<\tfrac{1}{4} \quad \Longrightarrow$$

$$\hat{x}<\tfrac{1}{4}.$$

Exercise 2.0.26. *Let $\hat{T}_1=5$. Solve the inequality*

$$\hat{3}\hat{x}-\hat{2}<4.$$

Solution. The given inequality is equivalent of the inequality

$$\frac{3}{5}\frac{x}{5}-\frac{2}{5}<4,$$

from where

$$
\begin{aligned}
&3x - 10 < 100 \quad \Longrightarrow \\
&3x < 110 \quad \Longrightarrow \\
&x < \tfrac{110}{3} \quad \Longleftrightarrow \\
&x\tfrac{1}{5} < \tfrac{110}{3}\tfrac{1}{5} \quad \Longleftrightarrow \\
&\hat{x} < \tfrac{22}{5}.
\end{aligned}
$$

Exercise 2.0.27. *Let* $\hat{T}_1 = 3$. *Solve the inequality*

$$\hat{3}\hat{\times}\hat{x} - \hat{2}\hat{\times}x - 4 > \hat{5}.$$

Answer. $\hat{x} < -\frac{17}{9}$.

Exercise 2.0.28. *Let* $\hat{T}_1 = 4$. *Solve the equation*

$$|\hat{x}| - \hat{3}\hat{x} = \hat{5}.$$

Solution. The given equation is equivalent of the equation

$$\left|\frac{x}{4}\right| - \frac{3}{4}\frac{x}{4} = \frac{5}{4},$$

from where

$$
\begin{aligned}
&|x| - \tfrac{3}{4}x = 5 \quad \Longleftrightarrow \\
&x - \tfrac{3}{4}x = 5, \quad -x - \tfrac{3}{4}x = 5 \quad \Longrightarrow \\
&x = 20, \quad x = -\tfrac{20}{7}, \quad \Longrightarrow \\
&\hat{x} = 5, \quad \hat{x} = -\tfrac{5}{7}.
\end{aligned}
$$

Definition 2.0.29. *For $\hat{x} \in \hat{F}|_{\mathbb{R}}$ we define*

$$\hat{x}^{\hat{2}} = \hat{x}\hat{\times}\hat{x},$$

$$\hat{x}^{\hat{3}} = \hat{x}\hat{\times}\hat{x}^{\hat{2}} = \hat{x}\hat{\times}\hat{x}\hat{\times}\hat{x},$$

$$\ldots$$

$$\hat{x}^{\hat{n}} = \hat{x}\hat{\times}\hat{x}^{n\hat{-}1}, \quad n \in \mathbb{N},$$

$$\hat{x}^2 = \hat{x}\hat{x},$$

$$\hat{x}^3 = \hat{x}\hat{x}^2 = \hat{x}\hat{x}\hat{x},$$

$$\ldots$$

$$\hat{x}^n = \hat{x}\hat{x}^{n-1}, \quad n \in \mathbb{N}.$$

Example 2.0.30. *Let $\hat{T}_1 = 4$. Then*

$$\hat{3}^{\hat{3}} + \hat{3}^2 + \hat{2}^{\hat{2}} = \hat{4} = \hat{3}\hat{\times}\hat{3}\hat{\times}\hat{3} + \hat{3}\hat{3} + \hat{2}\hat{\times}\hat{2}$$

$$= \tfrac{3}{4}4\tfrac{3}{4}4\tfrac{3}{4} + \tfrac{3}{4}\tfrac{3}{4} + \tfrac{2}{4}4\tfrac{2}{4}$$

$$= \tfrac{27}{4} + \tfrac{9}{16} + 1$$

$$= \tfrac{133}{16}.$$

Exercise 2.0.31. *Let $\hat{T}_1 = 3$. Solve the equation*

$$\hat{x}^{\hat{3}} + \hat{2}\hat{\times}\hat{x} - \hat{3}\hat{x} + 4 = 0.$$

Solution. We have

$$\hat{x}^{\hat{2}} = \hat{x}\hat{\times}\hat{x} = \hat{x}\hat{\times}\hat{x} = \tfrac{x}{3}3\tfrac{x}{3} = \tfrac{x^2}{3},$$

$$\hat{2}\hat{\times}\hat{x} = \tfrac{2}{3}3\tfrac{x}{3} = \tfrac{2x}{3},$$

$$\hat{3}\hat{x} = \tfrac{3}{3}\tfrac{x}{3} = \tfrac{x}{3}.$$

Then the given equation is equivalent of the equation

$$\frac{x^2}{3} + \frac{2x}{3} - \frac{x}{3} + 4 = 0$$

or

$$x^2 + x + 12 = 0,$$

which has not solutions in $\mathbb{R}$.

Exercise 2.0.32. *Simplify*

$$\hat{x}^{\hat{2}}\hat{\times}\hat{x}^{\hat{3}}\hat{x}^{4}\hat{\times}\hat{x}^{\hat{5}}.$$

Answer. $\hat{x}^{\hat{10}}\hat{x}^{4}$.

Definition 2.0.33. *A nonempty set $\hat{A}$ of isoreals will be called bounded above if there exists a isoreal $\hat{a}$ such that*

$$\hat{g} \leq \hat{a}$$

for every $\hat{g} \in \hat{A}$. In this case the isoreal $\hat{a}$ is called upper estimate of the set $\hat{A}$.

Definition 2.0.34. *A nonempty set $\hat{A}$ of isoreals will be called bounded below if there exists a isoreal $\hat{b}$ such that*

$$\hat{g} \geq \hat{b}.$$

In this case the isoreal $\hat{b}$ is called lower estimate of the set $\hat{A}$.

Definition 2.0.35. *A nonempty set $\hat{A}$ of isoreals will be called bounded if it is bounded above and bounded below.*

Definition 2.0.36. *A nonempty set $\hat{A}$ of isoreals will be called nonbounded if it is not bounded.*

Definition 2.0.37. *The least upper estimate of the nonempty bounded above set $\hat{A}$ will be called* supremum *of $\hat{A}$ and we will write*

$$\sup \hat{A} = \hat{\alpha}.$$

The supremum of the set is characterized with the following two properties:

1. $\hat{g} =< \hat{\alpha}$ For every $\hat{g} \in \hat{A}$,
2. for every $\hat{\epsilon} > \hat{0}$ there exists $\hat{h} \in \hat{A}$ so that $\hat{\alpha} - \hat{\epsilon} < \hat{h}$.

Definition 2.0.38. *The greatest lower estimate of the nonempty bounded below set $\hat{A}$ will be called infimum of $\hat{A}$ and we will write*

$$\inf \hat{A} = \hat{\beta}.$$

The infimum is characterized with the following properties

1. $\hat{g} \geq \hat{\beta}$ for every $\hat{g} \in \hat{A}$,
2. for every $\hat{\epsilon} > \hat{0}$ there exists $\hat{h} \in \hat{A}$ so that $\hat{h} < \hat{\beta} + \hat{\epsilon}$.

Theorem 2.0.39. *Every bounded above nonempty set of isoreals has supremum.*

Proof. Let $\hat{n} \in \hat{F}_{\mathbb{N}}$ is chosen so that there is not exist a isoelement $\hat{p} \in \hat{F}_{\mathbb{N}}$ so that

$$\hat{n} = \hat{p}\hat{\times}\hat{p}.$$

We define the set

$$\hat{H} = \{\hat{h} : \hat{h} \in \hat{F}_{\mathbb{Q}}, \hat{h} > \hat{0}, \hat{h}x\hat{h} =< \hat{n}\}.$$

Let us suppose that the set $\hat{H}$ has a rational supremum, namely, $\sup\hat{H} = \hat{l} \in \hat{F}_{\mathbb{Q}}$.

Then for $\hat{l}$ we have the following possibilities

1. $\hat{l}\hat{\times}\hat{l} < \hat{n}$,

2. $\hat{l}\hat{\times}\hat{l} = \hat{n}$,

3. $\hat{l}\hat{\times}\hat{l} > \hat{n}$.

1. Let $\hat{l}\hat{\times}\hat{l} < \hat{n}$. Let us consider the isonumber

$$\hat{r} = (\hat{n} + \hat{n}\hat{\times}\hat{l}) \nearrow (\hat{n} + \hat{l}).$$

For it we have $\hat{r} > \hat{0}$ and

$$\hat{n} - \hat{r}\hat{\times}\hat{r}$$

$$= \hat{n} - ((\hat{n} + \hat{n}\hat{\times}\hat{l})\hat{\times}(\hat{n} + \hat{n}\hat{\times}\hat{l})) \nearrow ((\hat{n} + \hat{l})\hat{\times}(\hat{n} + \hat{l}))$$

$$= (\hat{n}(\hat{n} + \hat{l})\hat{\times}(\hat{n} + \hat{l})) \nearrow ((\hat{n} + \hat{l})\hat{\times}(\hat{n} + \hat{l}))$$

$$-((\hat{n} + \hat{n}\hat{\times}\hat{l})\hat{\times}(\hat{n} + \hat{n}\hat{\times}\hat{l})) \nearrow ((\hat{n} + \hat{l})\hat{\times}(\hat{n} + \hat{l}))$$

$$= ((\hat{n} + \hat{n}\hat{\times}\hat{l})\hat{\times}(\hat{n} + \hat{n}\hat{\times}\hat{l})) \nearrow ((\hat{n} + \hat{l})\hat{\times}(\hat{n} + \hat{l}))$$

$$-((\hat{n} + \hat{n}\hat{\times}\hat{l})\hat{\times}(\hat{n} + \hat{n}\hat{\times}\hat{l}))) \nearrow ((\hat{n} + \hat{l})\hat{\times}(\hat{n} + \hat{l}))$$

$$= (\hat{n}\hat{\times}(\hat{n}\hat{\times}\hat{n} + \hat{n}\hat{\times}\hat{l} + \hat{n}\hat{\times}\hat{l} + \hat{l}\hat{\times}\hat{l}) - \hat{n}\hat{\times}\hat{n} - \hat{n}\hat{\times}\hat{n}\hat{\times}\hat{l}$$

$$-\hat{n}\hat{l}\hat{\times}\hat{n} - \hat{n}\hat{\times}\hat{l}\hat{\times}\hat{n}\hat{\times}\hat{l}) \nearrow ((\hat{n} + \hat{l})\hat{\times}(\hat{n} + \hat{l}))$$

$$= (\hat{n}\hat{\times}\hat{n}\hat{\times}\hat{n} + \hat{n}\hat{\times}\hat{n}\hat{\times}\hat{l} + \hat{n}\hat{\times}\hat{n}\hat{\times}\hat{l} + \hat{n}\hat{\times}\hat{l}\hat{\times}\hat{l}$$

$$-\hat{n}\hat{\times}\hat{n} - \hat{n}\hat{\times}\hat{n}\hat{\times}\hat{l} - \hat{n}\hat{\times}\hat{n}\hat{\times}\hat{l} - \hat{n}\hat{\times}\hat{n}\hat{\times}\hat{l}\hat{\times}\hat{l}) \nearrow ((\hat{n} + \hat{l})\hat{\times}(\hat{n} + \hat{l}))$$

$$= (\hat{n}\hat{\times}\hat{n}\hat{\times}(\hat{n} - \hat{I}) - \hat{n}\hat{\times}\hat{l}\hat{\times}\hat{l}\hat{\times}(\hat{n} - \hat{I})) \nearrow ((\hat{n} + \hat{l})\hat{\times}(\hat{n} + \hat{l}))$$

$$= (\hat{n}\hat{\times}(\hat{n} - \hat{I})\hat{\times}(\hat{n} - \hat{l}\hat{\times}\hat{l})) \nearrow ((\hat{n} + \hat{l})\hat{\times}(\hat{n} + \hat{l})) \geq \hat{0},$$

$$\hat{r} - \hat{l} = (\hat{n} + \hat{n}\hat{\times}\hat{l}) \nearrow (\hat{n} + \hat{l}) - \hat{l}$$

$$= (\hat{n} + \hat{n}\hat{\times}\hat{l}) \nearrow (\hat{n} + \hat{l}) - (\hat{l}\hat{\times}(\hat{n} + \hat{l})) \nearrow (\hat{n} + \hat{l})$$

$$= ((\hat{n} + \hat{n}\hat{\times}\hat{l}) - (\hat{l}\hat{\times}(\hat{n} + \hat{l}))) \nearrow (\hat{n} + \hat{l})$$

$$= (\hat{n} - \hat{l}\hat{\times}\hat{l}) \nearrow (\hat{n} + \hat{l}) > \hat{0},$$

from where we conclude that $\hat{r} \in \hat{H}$ and $\hat{r} > \hat{l}$, which is a contradiction with the definition for supremum.
2. Let $\hat{l}\hat{\times}\hat{l} = \hat{n}$. Since $\hat{I} \in \hat{H}$ and $\hat{l} = \text{isosup}\hat{H}$, there exists $\hat{p}, \hat{q} \in \hat{F}_{\mathbb{N}}$ such that $\hat{l} = \hat{p} \nearrow \hat{q}$ from where we obtain the equality

$$(\hat{p}\hat{\times}\hat{p}) \nearrow (\hat{q}\hat{\times}\hat{q}) = \hat{n},$$

from here

$$(\hat{p}\hat{\times}\hat{p}) \hat{/} \hat{q} = \hat{n}\hat{\times}\hat{q},$$

which is possible when $\hat{q} = \hat{I}$. Therefore $\hat{n} = \hat{p}\hat{\times}\hat{p}$, which is a contradiction with the definition of $\hat{n}$.
3. Let $\hat{l}\hat{\times}\hat{l} > \hat{n}$ and let

$$\hat{m} = (\hat{n} + \hat{n}\hat{\times}\hat{l}) \hat{/} (\hat{n} + \hat{l}),$$

and as in the first case we have

$$\hat{l} - \hat{m} = (\hat{l}\hat{\times}\hat{l} - \hat{n}) \hat{/} (\hat{n} + \hat{l}) > \hat{0}$$

and for every $\hat{r} \in \hat{H}$

$$\hat{m} - \hat{r} = (\hat{n}\hat{\times}(\hat{n} - \hat{I})\hat{\times}(\hat{l} - \hat{r})) \hat{/} ((\hat{n} + \hat{l})\hat{\times}(\hat{n} + \hat{r}))$$

$$+(\hat{n} - \hat{r}\hat{\times}\hat{r}) \hat{/} (\hat{n} + \hat{r}) \geq \hat{0},$$

which is a contradiction again with the definition of supremum. □

Theorem 2.0.40. *Every bounded below nonempty set of isoreals has exactly one infimum.*

Proof. Let $\hat{H}$ is an arbitrary nonempty bounded below set of isoreals. Then there exists $\hat{l} \in \hat{F}_{\mathbb{R}}$ so that $\hat{l} \leq \hat{h}$ for every $\hat{h} \in \hat{H}$. Let us put

$$\hat{M} = \{-\hat{h} : \hat{h} \in \hat{H}\}.$$

Then $\hat{M}$ is nonempty set of reals and $\hat{m} \leq -\hat{l}$ for every $\hat{m} \in \hat{M}$, i.e. the set $\hat{M}$ is a bounded above set. From here and from Theorem 2.0.38 follows that the set $\hat{M}$ has supremum $\hat{\alpha}$. Consequently

1. for every $-\hat{m} \in \hat{M}$ we have $-\hat{m} \leq \hat{\alpha}$,

2. for every $\hat{\epsilon} > \hat{0}$ there exists $-\hat{p} \in \hat{M}$ such that $-\hat{p} > \hat{\alpha} - \hat{\epsilon}$,

in other words

1. for every $\hat{m} \in \hat{H}$ we have $\hat{m} \geq -\hat{\alpha}$,

2. for every $\hat{\epsilon} > \hat{0}$ there exists $\hat{p} \in \hat{H}$ such that $\hat{p} < -\hat{\alpha} + \hat{\epsilon}$,

therefore $-\hat{\alpha}$ is supremum of $\hat{H}$. □

Definition 2.0.41. *Sets of isoreals*

$$[\hat{a}, \hat{b}] = \{\hat{x} \in \hat{F}_{\mathbb{R}} : \hat{a} \leq \hat{x} \leq \hat{b}\} \tag{2.0.5}$$

will be called closed intervals.

Definition 2.0.42. *Sets of isoreals*

$$(\hat{a}, \hat{b}) = \{\hat{x} \in \hat{F}_{\mathbb{R}} : \hat{a} < \hat{x} < \hat{b}\} \tag{2.0.6}$$

will be called open intervals.

Definition 2.0.43. *Sets of isoreals*

$$[\hat{a}, \hat{b}) = \{\hat{x} \in \hat{F}_{\mathbb{R}} : \hat{a} \leq \hat{x} < \hat{b}\} \tag{2.0.7}$$

will be called semiclosed on the left intervals.

Definition 2.0.44. *Sets of isoreals*

$$(\hat{a}, \hat{b}] = \{\hat{x} \in \hat{F}_{\mathbb{R}} : \hat{a} < \hat{x} \leq \hat{b}\} \tag{2.0.8}$$

will be called semiclosed on the right intervals.

Definition 2.0.45. *The intervals* (2.0.5), (2.0.6), (2.0.7), (2.0.8) *will be called finite intervals.*

Definition 2.0.46. *Infinite intervals are defined as follows*

$$(-\infty, \hat{a}] = \{\hat{x} \in \hat{F}_{\mathbb{R}} : \hat{x} \leq \hat{a}\},$$

$$(-\infty, \hat{a}) = \{\hat{x} \in \hat{F}_{\mathbb{R}} : \hat{x} < \hat{a}\},$$

$$[\hat{a}, \infty) = \{\hat{x} \in \hat{F}_{\mathbb{R}} : \hat{x} \geq \hat{a}\},$$

$$(\hat{a}, \infty) = \{\hat{x} \in \hat{F}_{\mathbb{R}} : \hat{x} > \hat{a}\}.$$

We will introduce the following rules

1. $\hat{a} + \infty = \infty + \hat{a} = +\infty$ for every $\hat{a} \in \hat{F}_{\mathbb{R}}$,
2. $\hat{a} - \infty = -\infty + \hat{a} = -\infty$ for every $\hat{a} \in \hat{F}_{\mathbb{R}}$,
3. $\infty + \infty = \infty$, $\infty - (-\infty) = \infty$, $\infty \hat{\times} \infty = \infty$, $\infty \hat{\times} (-\infty) = -\infty$,
4. $\hat{a} \hat{\times} \infty = \infty$, $\hat{a} \hat{\times} (-\infty) = -\infty$ if $\hat{a} > \hat{0}$,
5. $\hat{a} \hat{\times} (-\infty) = \infty$, $\hat{a} \hat{\times} \infty = -\infty$ if $\hat{a} < \hat{0}$.

Theorem 2.0.47. *Every bounded below set of isoelements of $F_{\mathbb{Z}}$ has smallest isoelement.*

Proof. Let $\hat{A}$ be an set of isoelements of $F_{\mathbb{Z}}$. We suppose that there is not smallest isoelement in $\hat{A}$. Since $\hat{A}$ is bounded below set then there exists

$$\hat{\beta} = \mathrm{isoinf}\, \hat{A}.$$

For it we have

1. $\hat{x} \geq \hat{\beta}$ for every $\hat{x} \in \hat{A}$,
2. for every $\hat{\epsilon} > \hat{0}$ there exists $\hat{x}_{\hat{\epsilon}} \in \hat{A}$ such that $\hat{x}_{\hat{\epsilon}} < \hat{\beta} + \hat{\epsilon}$.

For $\hat{\epsilon} = \hat{I}$ we have that there exists $\hat{y}_{\hat{\epsilon}} \in \hat{A}$ such that $\hat{y}_{\hat{\epsilon}} < \hat{\beta} + \hat{I}$. Since we suppose that $\hat{A}$ has not smallest isoelement then there exists $\hat{z} \in \hat{A}$ such that

$$\hat{\beta} \leq \hat{z} < \hat{y}_{\hat{\epsilon}} < \hat{\beta} + \hat{I}.$$

From here follows that

$$\hat{0} < \hat{y}_{\hat{\epsilon}} - \hat{z} < \hat{I},$$

which is a contradiction since in particular we have

$$\hat{y}_{\hat{\epsilon}} - \hat{z} \geq \hat{I}.$$

□

Theorem 2.0.48. *The set $\hat{F}_{\mathbb{N}}$ is not bounded above.*

Proof. Let us suppose that $\hat{F}_{\mathbb{N}}$ is bounded above. Then there exists

$$\hat{\alpha} = \sup \hat{F}_{\mathbb{N}}.$$

For it we have

1. $\hat{x} \leq \hat{\alpha}$ for every $\hat{x} \in \hat{F}_{\mathbb{N}}$,
2. for every $\hat{\epsilon} > \hat{0}$ there exists $\hat{x}_{\hat{\epsilon}} \in \hat{F}_{\mathbb{N}}$ such that $\hat{x}_{\hat{\epsilon}} > \hat{\alpha} - \hat{\epsilon}$.

In particular, if $\hat{\epsilon} = \hat{I}$ we have that there exists $\hat{y}_{\hat{\epsilon}} \in \hat{F}_{\mathbb{N}}$ such that

$$\hat{y}_{\hat{\epsilon}} > \hat{\alpha} - \hat{I}.$$

From where we obtain that

$$\hat{\alpha} < \hat{y}_{\hat{\epsilon}} + \hat{I}. \tag{2.0.9}$$

Since $\hat{y} \in \hat{F}_{\mathbb{N}}$, we conclude that $\hat{y} + \hat{\in} \hat{F}_{\mathbb{N}}$. From here and from the definition of $\hat{\alpha}$ follows that

$$\hat{y}_{\hat{\epsilon}} + \hat{I} \leq \hat{\alpha},$$

which is a contradiction with (2.0.9). □

Theorem 2.0.49. *Let*

$$\hat{\Delta}_{\hat{1}} \supseteq \hat{\Delta}_{\hat{2}} \supseteq \hat{\Delta}_{\hat{3}} \supseteq \cdots \supseteq \hat{\Delta}_{\hat{n}} \hat{\supset} \cdots$$

be system of closed finite intervals. Then there exists $\hat{y} \in \hat{F}_{\mathbb{R}}$ such that $\hat{y} \in \hat{\Delta}_{\hat{k}}$ for every $\hat{k} \in \hat{F}_{\mathbb{N}}$.

Proof. Let

$$\hat{\Delta}_{\hat{n}} = [\hat{a}_{\hat{n}}, \hat{b}_{\hat{n}}].$$

Let also $\hat{p}, \hat{q} \in \hat{F}_{\mathbb{N}}$. For $\hat{p}, \hat{q}$ we have the following possibilities

1. $\hat{p} < \hat{q}$,
2. $\hat{p} = \hat{q}$,
3. $\hat{p} > \hat{q}$.

If $\hat{p} < \hat{q}$ then we have

$$[\hat{a}_{\hat{p}}, \hat{b}_{\hat{p}}] \hat{\supset} [\hat{a}_{\hat{q}}, \hat{b}_{\hat{q}}],$$

from here follows that

$$\hat{a}_{\hat{p}} \leq \hat{b}_{\hat{q}}.$$

If $\hat{p} = \hat{q}$, then from the definition of finite closed interval follows that

$$\hat{a}_{\hat{p}} \leq \hat{b}_{\hat{q}}.$$

If $\hat{p} > \hat{q}$ then

$$[\hat{a}_{\hat{q}}, \hat{b}_{\hat{q}}] \supseteq [\hat{a}_{\hat{p}}, \hat{b}_{\hat{p}}],$$

from where

$$\hat{a}_{\hat{p}} \leq \hat{b}_{\hat{q}}.$$

Consequently for every naturals $\hat{p}, \hat{q}$ we have

$$\hat{a}_{\hat{p}} \leq \hat{b}_{\hat{q}}. \tag{2.0.10}$$

Let $\hat{q} \in \hat{F}_{\mathbb{N}}$ be fixed and $\hat{p} \in \hat{F}_{\mathbb{N}}$ runs the all set $\hat{F}_{\mathbb{N}}$. Then from (2.0.10) follows that the set

$$\hat{A} = \{\hat{a}_{\hat{1}}, \hat{a}_{\hat{2}}, \ldots, \hat{a}_{\hat{p}}, \ldots\}$$

is bounded above. Then there exists

$$\hat{\alpha} = \sup \hat{A}.$$

From here follows that for every $\hat{p} \in \hat{F}_{\mathbb{N}}$ we have

$$\hat{a}_{\hat{p}} \leq \hat{\alpha}.$$

Since $\hat{b}_{\hat{q}}$ is an above estimate of the set $\hat{A}$ and from the definition of $\hat{\alpha}$ we have

$$\hat{\alpha} \leq \hat{b}_{\hat{q}} \quad \forall \hat{q} \in \hat{F}_{\mathbb{N}}.$$

Since $\hat{q} \in \hat{F}_{\mathbb{N}}$ was arbitrary chosen then we have for every $\hat{p}, \hat{q} \in \hat{F}_{\mathbb{N}}$

$$\hat{a}_{\hat{p}} \leq \hat{\alpha} \leq \hat{b}_{\hat{q}}.$$

In particular, for $\hat{p} = \hat{q} = \hat{n} \in \hat{F}_{\mathbb{N}}$ we get

$$\hat{a}_{\hat{n}} \leq \hat{\alpha} \leq \hat{b}_{\hat{n}}.$$

$\square$

Advanced practical exercises

Problem 2.0.50. *Let $\hat{T}_1 = 3$, Find $\hat{x}$ such that*

$$1) \quad \hat{2}\hat{\times}\hat{x} + \hat{2} = \hat{4}, \quad 2) \quad \hat{2}\hat{x} + \hat{2} = \hat{5}.$$

Answer. 1) $\hat{x} = \frac{1}{3}$, 2) $\hat{x} = \frac{3}{2}$.

Problem 2.0.51. *Let $\hat{T}_1 = 5$. Solve the inequality*

$$2\hat{x} - \hat{4}\hat{\times}x + 3\hat{\times}\hat{x} < \hat{2}.$$

Answer. $\hat{x} < -\frac{2}{15}$.

Problem 2.0.52. *Let $\hat{T}_1 = 5$. Solve the equation*

$$\hat{x}^{\hat{2}} - \hat{4}\hat{\times}\hat{x} - \hat{12} = 0.$$

Answer. $\hat{x_1} = \frac{6}{5}$, $\hat{x_2} = -\frac{2}{5}$.

Problem 2.0.53. *Simplify*

$$\hat{x}\hat{\times}\hat{x}^{\hat{10}}\hat{\times}\hat{x}^2 x^3.$$

Answer. $\hat{x}^{\hat{12}} \hat{\times} x^3$.

Problem 2.0.54. *Simplify*

$$\hat{3} \hat{\times} \left(\hat{x}^{\hat{2}} \hat{\times} \hat{x} + \hat{4} \right) \hat{\times} \hat{x}.$$

Answer. $\hat{3} \hat{\times} \hat{x}^{\hat{3}} + \hat{12} \hat{\times} \hat{x}$.

Problem 2.0.55. *Let* $\hat{T}_1 = 6$. *Solve the equation in* $\hat{F}_{\mathbb{R}}$

$$\hat{3} \hat{\times} (\hat{x} + \hat{4}) - \hat{x}^{\hat{2}} = 4.$$

Answer. No solutions.

Problem 2.0.56. *Let* $\hat{T}_1 = 4$. *Solve the equation in* $\hat{F}_{\mathbb{R}}$

$$\hat{5} \hat{\times} (\hat{x} + \hat{5}) - \hat{x}^2 = 1.$$

Answer. No solutions.

Chapter 3

Sequences of isoreals

Let $\{\hat{T}_n\}_{n=1}^{\infty}$ is a sequence of positive reals.

Definition 3.0.57. *Let $\{a_n\}_{n=1}^{\infty}$ is a sequence of reals. The sequence*

$$\left\{\hat{a}_n = \frac{a_n}{\hat{T}_n}\right\}_{n=1}^{\infty}$$

will be called sequence of isoreals.

Example 3.0.58. *Let $\{a_n\}_{n=1}^{\infty} = \{n^2+1\}_{n=1}^{\infty}$, $\{\hat{T}_n\}_{n=1}^{\infty} = \{n+3\}_{n=1}^{\infty}$. Then the sequence $\left\{\frac{n^2+1}{n+3}\right\}_{n=1}^{\infty}$ is a sequence of isoreals.*

Definition 3.0.59. *A sequence of isoreals $\{\hat{a}_n\}_{n=1}^{\infty}$ will be called*

1. *bounded above if there exists $\hat{l} \in \hat{F}_{\mathbb{R}}$ so that $\hat{a}_n \leq \hat{l}$ for every $n \in \mathbb{N}$,*
2. *bounded below if there exists $\hat{m} \in \hat{F}_{\mathbb{R}}$ so that $\hat{a}_n \geq \hat{m}$ for every $n \in \mathbb{N}$,*
3. *bounded if it is bounded above and bounded below.*

Example 3.0.60. *Let* $\{\hat{T}_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty} = \{n^2\}_{n=1}^{\infty}$. *Then*

$$\hat{a}_n = \frac{a_n}{\hat{T}_n} = \frac{n^2}{\frac{1}{n}} = n^3$$

is unbounded above sequence.

Example 3.0.61. *Let* $\{\hat{T}_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty} = \{-n^2+1\}_{n=1}^{\infty}$. *Then*

$$\hat{a}_n = \frac{a_n}{\hat{T}_n} = \frac{-n^2+1}{n}$$

is unbounded below sequence.

Example 3.0.62. *Let* $\{\hat{T}_n\}_{n=1}^{\infty} = \{n^3+1\}_{n=1}^{\infty}$, $\{a_n = n+1\}_{n=1}^{\infty}$. *Then*

$$0 \leq \hat{a}_n = \frac{a_n}{\hat{T}_n} = \frac{n+1}{n^3+1} = \frac{1}{n^2+n+1} \leq 1.$$

Therefore the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ *is a bounded sequence.*

Exercise 3.0.63. *Let* $\{\hat{T}_n\}_{n=1}^{\infty} = \{\sin^2 n + 1\}_{n=1}^{\infty}$, $\{a_n = \cos n\}_{n=1}^{\infty}$. *Prove that the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *is a bounded sequence.*

Theorem 3.0.64. *Let* $\{\hat{T}_n\}_{n=1}^{\infty}$ *is a bounded above sequence and* $\{a_n\}_{n=1}^{\infty}$ *is a bounded below sequence. Then the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *is a bounded below sequence.*

Proof. Since $\{a_n\}_{n=1}^{\infty}$ is a bounded below sequence then there exists $m \in \mathbb{R}$ such that

$$a_n \geq m \quad \text{for} \quad \forall n \in \mathbb{N}. \tag{3.0.1}$$

Because $\{\hat{T}_n\}_{n=1}^{\infty}$ is a bounded above sequence then there exists $p \in \mathbb{R}$, $p > 0$, such that

$$\hat{T}_n \leq p \quad \text{for} \quad \forall n \in \mathbb{N}.$$

Therefore

$$\frac{1}{\hat{T}_n} \geq \frac{1}{p} \quad \text{for} \quad \forall n \in \mathbb{N}.$$

From here and (3.0.1) it follows

$$\frac{a_n}{\hat{T}_n} \geq \frac{m}{p} \quad \text{for} \quad \forall n \in \mathbb{N}.$$

Consequently the sequence $\{\hat{a}\}_{n=1}^{\infty}$ is a bounded below sequence. $\square$

Theorem 3.0.65. *Let $\{a_n\}_{n=1}^{\infty}$ is a bounded above sequence, $\{\hat{T}_n\}_{n=1}^{\infty}$ is a bounded below sequence of the positive real p. Then the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is a bounded above sequence.*

Proof. Since $\{a_n\}_{n=1}^{\infty}$ is a bounded above sequence then there exists $m \in \mathbb{R}$ such that

$$a_n \leq m \quad \text{for} \quad \forall n \in \mathbb{N}. \tag{3.0.2}$$

Because $\{\hat{T}_n\}_{n=1}^{\infty}$ is a bounded below sequence of the positive real p it follows

$$\hat{T}_n \geq p \quad \text{for} \quad \forall n \in \mathbb{N},$$

from where

$$\frac{1}{\hat{T}_n} \leq \frac{1}{p} \quad \text{for} \quad \forall n \in \mathbb{N}.$$

From here and (3.0.2) we get

$$\hat{a}_n = \frac{a_n}{\hat{T}_n} \leq \frac{m}{p} \quad \text{for} \quad \forall n \in \mathbb{N}.$$

Consequently the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is a bounded above sequence. $\square$

Remark 3.0.66. *The condition $\{\hat{T}_n\}_{n=1}^{\infty}$ to be bounded below of the positive real p is essential because if $\lim_{n\longrightarrow\infty} \hat{T}_n = 0$ then since $\{a_n\}_{n=1}^{\infty}$ is a bounded above we will have*

$$\lim_{n\longrightarrow\infty} \frac{a_n}{\hat{T}_n} = +\infty \quad \text{or} \quad \lim_{n\longrightarrow\infty} \frac{a_n}{\hat{T}_n} = -\infty,$$

i.e. the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ will be an unbounded sequence.

Theorem 3.0.67. *A sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *is bounded if and only if there exists* $\hat{q} \in \hat{F}_{\mathbb{R}}$ *such that* $|\hat{a}_n| \leq \hat{q}$ *for every* $n \in \mathbb{N}$.

Proof. 1. Let $|\hat{a}_n| \leq \hat{q}$ for every $n \in \mathbb{N}$. Then

$$-\hat{q} \leq \hat{a}_n \leq \hat{q} \quad \forall n \in \mathbb{N}.$$

Since $\hat{a}_n \leq \hat{q}$ for every $n \in \mathbb{N}$ we conclude that the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is bounded above.
From $-\hat{q} \leq \hat{a}_n$ for every $n \in \mathbb{N}$ follows that the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is bounded below.
Therefore the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is bounded.
2. Let $\{\hat{a}_n\}_{n=1}^{\infty}$ is bounded. Then it is bounded above and below. Then there exist $\hat{l}, \hat{m} \in \mathbb{R}$ so that

$$\hat{l} \leq \hat{a}_n \leq \hat{m}$$

Let $\hat{q} = \max\{|\hat{l}|, |\hat{m}|\}$. Then $|\hat{a}_n| \leq \hat{s}$. □

Definition 3.0.68. *A sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *is called unbounded if it is not bounded.*

Exercise 3.0.69. *Let* $\{\hat{T}_n\}_{n=1}^{\infty} = \left\{\frac{1}{n^8+7}\right\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$. *Prove that* $\{\hat{a}_n\}_{n=1}^{\infty}$ *is an unbounded sequence.*

In other words, a sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is unbounded if there exists $\hat{t} \in \hat{F}_{\mathbb{R}}$ and $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geq N$, we have

$$|\hat{a}_n| \geq \hat{t}.$$

Definition 3.0.70. *We will say that a sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *diverges to* $+\infty$ *if for every* $M \in \mathbb{R}$, $M \geq 0$ *there exists* $N \in \mathbb{N}$ *such that for every* $n \in \mathbb{N}$, $n \geq N$, *we have*

$$\hat{a}_n \geq M.$$

Example 3.0.71. *Let* $\{a_n\}_{n=1}^{\infty} = \{n+3\}_{n=1}^{\infty}$, $\{\hat{T}_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. *We will prove that the sequence* $\{\hat{a}_n\}_{n=1}^{\infty} = \{\frac{n+3}{\frac{1}{n}}\}_{n=1}^{\infty} = \{n(n+3)\}_{n=1}^{\infty}$ *diverges to* ∞.
Really, let $M > 0$ *be arbitrary chosen and fixed. We choose* $N \in \mathbb{N}$ *so that*

$$N \geq \frac{-3 + \sqrt{9 + 4M}}{2}.$$

Then for every $n > N$ *we have*

$$n(n+3) \geq M.$$

From here

$$\hat{a}_n \geq M \quad \text{for} \quad \forall n > N.$$

Since $M > 0$ *was arbitrary chosen we conclude that the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *diverges to* $+\infty$.

Exercise 3.0.72. *Let* $\{a_n\}_{n=1}^{\infty} = \{n^2 + 2\}_{n=1}^{\infty}$, $\{\hat{T}_n\}_{n=1}^{\infty} = \left\{\frac{1}{n^2+3}\right\}_{n=1}^{\infty}$. *Prove that the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *diverges to* $+\infty$.

Exercise 3.0.73. *Let* $\{a_n\}_{n=1}^{\infty} = \{n^4 + 3\}_{n=1}^{\infty}$, $\{\hat{T}_n\}_{n=1}^{\infty} = \left\{\frac{1}{n^4+1}\right\}_{n=1}^{\infty}$. *Prove that the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *diverges to* $+\infty$.

Definition 3.0.74. *A sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *diverges to* $-\infty$ *if for every* $P \in \mathbb{R}$, $P \leq 0$ *there exists* $N \in \mathbb{N}$ *such that for every* $n \in \mathbb{N}$, $n \geq N$, *we have*

$$\hat{a}_n \leq P.$$

Exercise 3.0.75. *Let* $\{a_n\}_{n=1}^{\infty} = \{-n^2 - 32\}_{n=1}^{\infty}$, $\{\hat{T}_n\}_{n=1}^{\infty} = \left\{\frac{1}{n^2+3}\right\}_{n=1}^{\infty}$. *Prove that the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *diverges to* $-\infty$.

Exercise 3.0.76. *Let* $\{a_n\}_{n=1}^{\infty} = \{-n + 2\}_{n=1}^{\infty}$, $\{\hat{T}_n\}_{n=1}^{\infty} = \left\{\frac{1}{n+3}\right\}_{n=1}^{\infty}$. *Prove that the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *diverges to* $-\infty$.

Definition 3.0.77. *The number* $a \in \mathbb{R}$ *is called limit of the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *if for every* $\epsilon > 0$ *there exists* $N = N(\epsilon) \in \mathbb{N}$ *such that for every* $n \in \mathbb{N}$, $n > N$ *we have*

$$|\hat{a}_n - a| < \epsilon.$$

In this case we will write $\lim_{n\longrightarrow\infty} \hat{a}_n = a$ *and we will say that the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *is convergent.*
In other words the number a *is a limit of the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *if*

$$\lim_{n\longrightarrow\infty} \frac{a_n}{\hat{T}_n} = a.$$

Example 3.0.78. *Let* $\{a_n\}_{n=1}^{\infty} = \{n+4\}_{n=1}^{\infty}$, $\{\hat{T}_n\}_{n=1}^{\infty} = \{n+5\}_{n=1}^{\infty}$. *Then*

$$\lim_{n\longrightarrow\infty} \hat{a}_n = \lim_{n\longrightarrow\infty} \frac{a_n}{\hat{T}_n} = \lim_{n\longrightarrow\infty} \frac{n+4}{n+5} = 1.$$

Exercise 3.0.79. *Let* $\{a_n\}_{n=1}^{\infty} = \{2n + 1\}_{n=1}^{\infty}$, $\{\hat{T}_n\}_{n=1}^{\infty} = \{3n + 5\}_{n=1}^{\infty}$. *Find*

$$\lim_{n\longrightarrow\infty} \hat{a}_n.$$

Answer. $\frac{2}{3}$.

Exercise 3.0.80. *Let* $\{a_n\}_{n=1}^{\infty} = \{3n^2 + 4\}_{n=1}^{\infty}$, $\{\hat{T}_n\}_{n=1}^{\infty} = \{5n^2 + 7\}_{n=1}^{\infty}$. *Find*

$$\lim_{n\longrightarrow\infty} \hat{a}_n.$$

Answer. $\frac{3}{5}$.

Theorem 3.0.81. *Let the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is convergent to $a \in \mathbb{R}$ and $a \neq 0$. Then there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > N$, we have*

$$|\hat{a}_n| > \frac{|a|}{2}.$$

Also, if $a > 0$ then there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > N$, we have

$$\hat{a}_n > \frac{a}{2},$$

if $a < 0$ then there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > N$, we have

$$\hat{a}_n < \frac{a}{2}.$$

Proof. Let $\epsilon = \frac{a}{2}$. Then there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > N$, we have

$$|\hat{a}_n - a| < \frac{a}{2}.$$

From here and from the properties of the modulus we get

$$a - \frac{a}{2} < \hat{a}_n < a - \frac{a}{2} \tag{3.0.3}$$

for every $n \in \mathbb{N}$, $n > N$.
On the other hand

$$|\hat{a}_n - a| = |a - \hat{a}_n| \geq |a| - |\hat{a}_n|.$$

Consequently for every $n \in \mathbb{N}$, $n > N$, we have

$$|a| - |\hat{a}_n| < \frac{a}{2}$$

or

$$|\hat{a}_n| > \frac{a}{2}.$$

If $a > 0$ then $|a| = a$ and from the left hand of (3.0.3) we obtain

$$\hat{a}_n > \frac{a}{2}$$

for every $n \in \mathbb{N}$, $n > N$.

If $a < 0$ then $|a| = -a$ and from the right hand of (3.0.3) we obtain

$$\hat{a}_n < \frac{a}{2}$$

for every $n \in \mathbb{N}$, $n > N$. $\square$

Theorem 3.0.82. *Let* $\lim_{n\longrightarrow\infty} \hat{a}_n = a$, $\lim_{n\longrightarrow\infty} \hat{b}_n = b$, $\hat{a}_n \leq \hat{b}_n$ *for every* $n \geq n_0$. *Then* $a \leq b$.

Proof. Let us suppose that $b < a$ and let $\epsilon = \frac{a-b}{2} > 0$. Then there exists $n_1 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > n_1$, we have

$$|\hat{a}_n - a| < \frac{\epsilon}{2}, \tag{3.0.4}$$

and there exists $n_2 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > n_2$, we have

$$|\hat{b}_n - b| < \frac{\epsilon}{2}. \tag{3.0.5}$$

Let $N = \max\{n_0, n_1, n_2\}$. Then for every $n \in \mathbb{N}$, $n > N$, we have $\hat{a}_n \leq \hat{b}_n$. From (3.0.4), (3.0.5) follows that for every $n \in \mathbb{N}$, $n > N$,

$$\hat{b}_n < b + \epsilon = \frac{a+b}{2} = a - \epsilon < \hat{a}_n,$$

which is a contradiction. Therefore $a \leq b$. $\square$

Corollary 3.0.83. *Let* $\lim_{n\longrightarrow\infty} \hat{a}_n = a$ *and let there exists* $n_0 \in \mathbb{N}$ *such that* $\hat{a}_n \leq b$ *for every* $n \geq n_0$. *Then* $a \leq b$.

Theorem 3.0.84. *Let* $\lim_{n\longrightarrow\infty} \hat{a}_n = a$, $\lim_{n\longrightarrow\infty} \hat{b}_n = a$, *and there exists* $n_0 \in \mathbb{N}$ *such that* $\hat{a}_n \leq \hat{c}_n \leq \hat{b}_n$ *for every* $n \geq n_0$. *Then* $\lim_{n\longrightarrow\infty} \hat{c}_n = a$

Proof. Let $\epsilon > 0$ is arbitrary chosen and fixed. Then there exists $n_1 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > n_1$, we have

$$|\hat{a}_n - a| < \epsilon, \tag{3.0.6}$$

and there exists $n_2 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > n_2$, we have

$$|\hat{b}_n - a| < \epsilon. \tag{3.0.7}$$

Let $N = \max\{n_0, \hat{n}_1, n_2\}$. Then for every $n \in \mathbb{N}$, $n > N$

$$\hat{a}_n \leq \hat{c}_n \leq \hat{b}_n,$$

$$a - \epsilon < \hat{a}_n \leq \hat{c}_n \leq \hat{b}_n < a + \epsilon \quad \Longrightarrow$$

$$a - \epsilon < \hat{c}_n < a + \epsilon,$$

i.e. $|\hat{c}_n - a| < \epsilon$. Consequently $\lim_{n \longrightarrow \infty} \hat{c}_n = a$. □

Theorem 3.0.85. *Let* $\lim_{n \longrightarrow \infty} \hat{a}_n = a$. *Then* $\lim_{n \longrightarrow \infty} |\hat{a}_n| = |a|$.

Proof. Let $\epsilon > 0$ is arbitrary chosen and fixed. Then there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > N$, we have

$$|\hat{a}_n - a| < \epsilon.$$

From the properties of modulus we have the inequality

$$||\hat{a}_n| - |a|| \leq |\hat{a}_n - a|.$$

Therefore for every $n \in \mathbb{N}$, $n > N$, we have

$$||\hat{a}_n| - |a|| \leq |\hat{a}_n - a| < \epsilon.$$

Consequently $\lim_{n \longrightarrow \infty} |\hat{a}_n| = |a|$. □

Corollary 3.0.86. *Let* $\lim_{n\longrightarrow\infty} \hat{a}_n = 0$. *Then* $\lim_{n\longrightarrow\infty} |\hat{a}_n| = 0$.

Theorem 3.0.87. *Every convergent sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *is bounded sequence.*

Proof. Let $\lim_{n\longrightarrow\infty} \hat{a}_n = a$ and let $\epsilon > 0$ is arbitrary chosen and fixed. Then there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > N$, we have

$$|\hat{a}_{\hat{n}} - a| < \epsilon$$

and since

$$|\hat{a}_n| - |a| \leq |\hat{a}_n - a|,$$

then for every $n \in \mathbb{N}$, $n > N$, we have

$$|\hat{a}_n| < \epsilon + |a|.$$

Let n_1 is the smallest number as an element of $\mathbb{N}$ such that $n_1 > n$. We put

$$\hat{A} = \max\{\hat{a}_1, \ldots, \hat{a}_{n_1}, \epsilon + |a|\}.$$

Then for every $n \in \mathbb{N}$ we have

$$|\hat{a}_n| \leq \hat{A}.$$

□

Theorem 3.0.88. *Let* $\lim_{n\longrightarrow\infty} \hat{a}_n = a$, $\lim_{n\longrightarrow\infty} \hat{b}_n = b$. *Then*

1. $\lim_{n\longrightarrow\infty}(\hat{a}_n \pm \hat{b}_n) = \lim_{n\longrightarrow\infty} \hat{a}_n \pm \lim_{n\longrightarrow\infty} \hat{b}_n = a \pm b$,
2. $\lim_{n\longrightarrow\infty}(\hat{a}_n \hat{\times} \hat{b}_n) = \lim_{n\longrightarrow\infty} \hat{b}_n \hat{\times} \lim_{n\longrightarrow\infty} \hat{a}_n = a \hat{\times} b$,
3. $\lim_{n\longrightarrow\infty}(\hat{a}_n \nearrow \hat{b}_n) = \lim_{n\longrightarrow\infty} \hat{a}_n \nearrow \lim_{n\longrightarrow\infty} \hat{b}_n = a \nearrow b$, *if* $\hat{b}_{\hat{n}} \neq 0$, $b \neq 0$.

Proof. 1. Let $\epsilon > 0$ is arbitrary chosen and fixed. Then there exists $n_1 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > n_1$, we have

$$|\hat{a}_n - a| < \frac{\epsilon}{2},$$

there exists $n_2 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > n_2$, we have

$$|\hat{b}_n - b| < \frac{\epsilon}{2}.$$

Let $N = \max\{n_1, n_2\}$. Then for every $n \in \mathbb{N}$, $n > N$, we have

$$|(\hat{a}_n \pm \hat{b}_n) - (a - b)| = |(\hat{a}_n - a) \mp (\hat{b}_n - b)|$$

$$\leq |\hat{a}_n - a| + |\hat{b}_n - b|$$

$$< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

2. Since the sequences $\{\hat{a}_n\}_{n=1}^{\infty}$, $\{\hat{b}_n\}_{n=1}^{\infty}$ are convergent then there exist $P, Q \in \mathbb{R}$ such that

$$|\hat{a}_n| \leq P, \quad |\hat{b}_n| \leq Q \quad \forall n \in \mathbb{N}.$$

Let $\epsilon > 0$ is arbitrary chosen and fixed. Then there exists $n_1 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > n_1$, we have

$$|\hat{a}_n - a| < \frac{\epsilon}{2P},$$

there exists $n_2 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > n_2$, we have

$$|\hat{b}_n - b| < \frac{\epsilon}{2Q}.$$

Let $N = \max\{n_1, n_2\}$. Then for every $n \in \mathbb{N}$, $n > N$, we have

$$|\hat{a}_n \hat{\times} \hat{b}_n - a \hat{\times} b| = |\hat{a}_n \hat{\times} \hat{b}_n - a \hat{\times} \hat{b}_n + a \hat{\times} \hat{b}_n - a \hat{\times} b|$$

$$\leq |\hat{a}_n \hat{\times} \hat{b}_n - a \hat{\times} \hat{b}_n| + |a \hat{\times} \hat{b}_n - a \hat{\times} b|$$

$$= |(\hat{a}_n - a) \hat{\times} \hat{b}_n| + |(\hat{b}_n - b) \hat{\times} \hat{a}_n|$$

$$= |\hat{a}_n - a| \hat{\times} |\hat{b}_n| + |\hat{b}_n - b| \hat{\times} |\hat{a}_n|$$

$$\leq Q \hat{\times} |\hat{a}_n - a| + P \hat{\times} |\hat{b}_n - b|$$

$$< Q \hat{\times} \epsilon \mathbin{\curlywedge} (2Q) + P \hat{\times} \epsilon \mathbin{\curlywedge} (2 \hat{\times} P)$$

$$= \epsilon \mathbin{\curlywedge} 2 + \epsilon \mathbin{\curlywedge} 2 = \epsilon.$$

3. There exists $n_1 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > n_1$, we have
$$|\hat{b}_n| > |b| \hat{/} 2,$$
therefore for every $n \in \mathbb{N}$, $n > n_1$, we have
$$1 \hat{/} |\hat{b}_n| < 2 \hat{/} |b|.$$
There exists $n_2 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > n_2$, we have
$$|\hat{a}_n - a| < (\epsilon \hat{\times} b \hat{\times} b) \hat{/} (2 \hat{\times} (|a + b|))$$
there exists $n_3 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > n_3$, we have
$$|\hat{b}_n - b| < (\epsilon \hat{\times} b \hat{\times} b) \hat{/} (2 \hat{\times} (|a + b|)).$$
Let $N = \max\{n_1, n_2, n_3\}$. Then for every $n \in \mathbb{N}$, $n > n$, we have
$$|\hat{a}_n \hat{/} \hat{b}_n - a \hat{/} b|$$
$$= |\hat{a}_n \hat{\times} b - a \hat{\times} b + a \hat{\times} b - \hat{b}_n \hat{\times} a| \hat{/} (|b| \hat{\times} |\hat{b}_n|)$$
$$= |(\hat{a}_n - a) \hat{\times} b + a \hat{\times} (b - \hat{b}_n)| \hat{/} (|b| \hat{\times} |\hat{b}_n|)$$
$$\leq |(\hat{a}_n - a) \hat{\times} b| \hat{/} (|b| \hat{\times} |\hat{b}_n|)$$
$$+ |a \hat{\times} (b - \hat{b}_n)| \hat{/} (|b| > |\hat{b}_n|)$$
$$= (|(\hat{a}_n - a)| \hat{\times} |b|) \hat{/} (|b| \hat{\times} |\hat{b}_n|)$$
$$+ |a| \hat{\times} |b - \hat{b}_n| \hat{/} (|b| \hat{\times} |\hat{b}_n|)$$
$$< (2 \hat{\times} |b| > \epsilon \hat{\times} b \hat{\times} b) \hat{/} (b \hat{\times} b \hat{\times} 2 \hat{\times} (|a| + |b|))$$
$$+ (2 \hat{\times} |a| \hat{\times} \epsilon \hat{\times} b \hat{\times} b) \hat{/} (b \hat{\times} b \hat{\times} 2 \hat{\times} (|a| + |b|))$$
$$= \epsilon.$$

□

Corollary 3.0.89. *Let* $\hat{\lim}_{n \hat{\longrightarrow} \infty} \hat{a}_n = a$. *Then* $\lim_{n \longrightarrow \infty} (\alpha \hat{\times} \hat{a}_n) = \alpha \hat{\times} a$ *for every* $\alpha \in \mathbb{R}$.

Exercise 3.0.90. *Find* $\lim_{n\longrightarrow\infty} \hat{a}_n$ *if*

1. $a_n = n+1$, $\hat{T}_n = n$,
2. $a_n = n^2+4$, $\hat{T}_n = n$,
3. $a_n = 2n^2+n+2$, $\hat{T}_n = n^2+1$,
4. $a_n = a_0n^k + a_1n^{k-1} + \cdots + a_k$, $\hat{T}_n = b_0n^k + b_1n^{k-1} + \cdots + b_k$, $b_0 \neq 0$,
5. $a_n = n+1000$, $\hat{T}_n = n^2+2$,
6. $a_n = 2^n+3^n$, $\hat{T}_n = 4^n$,
7. $a_n = 2^{n+1}+3^{n+1}$, $\hat{T}_n = 2^n+3^n$,
8. $a_n = a^n$, $\hat{T}_n = 1+a^n$, $a \in \mathbb{R}$,
9. $a_n = a^n$, $\hat{T}_n = 1+a^{2n}$, $a \in \mathbb{R}$,
10. $a_n = a^n - a^{-n}$, $\hat{T}^n = a^n + a^{-n}$, $a \in \mathbb{R}$.

Answer.
1) 1, 2) ∞, 3) 2, 4) $\frac{a_0}{b_0}$, 5) 0, 6) 0, 7) 3, 8) 0 for $|a| < 1$, 1 for $|a| > 1$, $\frac{1}{2}$ for $a = 1$, for $a = -1$ the sequence is not defined, 9) 0 for $|a| \neq 1$, $\frac{1}{2}$ for $a = 1$, divergent for $a = -1$, 10) for $a = 0$ the sequence is not defined, -1 for $|a| < 1$, 0 for $|a| = 1$, 1 for $|a| > 1$.

Definition 3.0.91. *The sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *is called infinite small if it is convergent and its limit is equal to* 0*.*

Corollary 3.0.92. *The sum, subtraction and multiplication of infinite small sequences of isonumbers is infinite small sequence.*

Theorem 3.0.93. *Let* $\{a_n\}_{n=1}^{\infty}$ *is an infinite small sequence and* $\{\hat{T}_n\}_{n=1}^{\infty}$ *is a bounded below sequence of positive real* p*. Then* $\{\hat{a}_n\}_{n=1}^{\infty}$ *is an infinite small sequence.*

Proof. Since $\{a_n\}_{n=1}^{\infty}$ is an infinite small sequence then

$$\lim_{n\longrightarrow\infty} |a_n| = 0. \tag{3.0.8}$$

Because $\{\hat{T}_n\}_{n=1}^{\infty}$ is a bounded below sequence of positive real p then for every $n \in \mathbb{N}$ we have

$$\hat{T}_n \geq p \quad \frac{1}{\hat{T}_n} \leq \frac{1}{p}.$$

From here and (3.0.8) it follows

$$0 \leq \lim_{n\longrightarrow\infty} \frac{|a_n|}{\hat{T}_n} \leq \lim_{n\longrightarrow\infty} \frac{|a_n|}{p} = 0.$$

Consequently the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is an infinite small sequence. □

Theorem 3.0.94. *The number* a *is limit of the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *if and only if it can be represented in the form*

$$a = \hat{a}_n - \hat{\alpha}_n,$$

where $\{\hat{\alpha}_n\}_{n=1}^{\infty}$ *is infinite small sequence.*

Proof. If $\hat{a}_n = a$ then $\hat{\alpha}_n = 0$. Let $\hat{a}_n \neq a$ and $\hat{\alpha}_n = \hat{a}_n - a$. Then a is limit of the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n > N$

$$|\hat{a}_n - a| < \epsilon \quad \Longleftrightarrow |\alpha_n| < \epsilon.$$

□

Theorem 3.0.95. *Let $\{\hat{\alpha}_n\}_{n=1}^{\infty}$ is infinite small sequence and $\{\hat{a}_n\}_{n=1}^{\infty}$ is bounded sequence. Then $\{\hat{\alpha}_n \hat{\times} \hat{a}_n\}_{n=1}^{\infty}$ is infinite small sequence.*

Proof. Since $\{\hat{a}_n\}_{n=1}^{\infty}$ is bounded sequence then there exists $M \in \mathbb{R}$, $M > 0$ so that for every $n \in \mathbb{N}$ we have

$$|\hat{a}_n| \leq M.$$

Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > N$ we have

$$|\hat{\alpha}_n| < \epsilon \rightthreetimes M.$$

From here for every $n \in \mathbb{N}$, $n > N$ we have

$$|\hat{\alpha}_n \hat{\times} \hat{a}_n| = |\hat{\alpha}_n| \hat{\times} |\hat{a}_n| < M \hat{\times} \epsilon \rightthreetimes M = \epsilon.$$

$\square$

Definition 3.0.96. *A sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is called infinite large if for every $M \in \mathbb{R}$, $M > 0$ there exists $N \in \mathbb{N}$ such that for every $n > N$ we have*

$$|\hat{a}_n| \geq M.$$

In other words a sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is an infinite large sequence if

$$\lim_{n \longrightarrow} |\hat{a}_n| = \infty.$$

Theorem 3.0.97. *Let $\{a_n\}_{n=1}^{\infty}$ is a sequence such that the sequence $\{|a_n|\}_{n=1}^{\infty}$ is a bounded below sequence of the positive real p, $\{\hat{T}_n\}_{n=1}^{\infty}$ is an infinite small sequence. Then the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is an infinite large sequence.*

Proof. Since $\{|a_n|\}_{n=1}^{\infty}$ is a bounded below sequence of the positive real p then for every $n \in \mathbb{N}$ we have

$$|a_n| \geq p.$$

From here

$$\lim_{n \longrightarrow \infty} |\hat{a}_n| = \lim_{n \longrightarrow \infty} \frac{|a_n|}{\hat{T}_n} \geq \lim_{n \longrightarrow \infty} \frac{p}{\hat{T}_n} = \infty,$$

because $\lim_{n \longrightarrow \infty} \hat{T}_n = 0$. $\square$

Theorem 3.0.98. *Let $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence, $\{\hat{T}_n\}_{n=1}^{\infty}$ is an infinite large sequence. Then the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is an infinite small sequence.*

Proof. Since the sequence $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence then there exists a constant $M > 0$ such that

$$|a_n| \leq M \quad \text{for} \quad \forall n \in \mathbb{N}.$$

Because $\{\hat{T}_n\}_{n=1}^{\infty}$ is an infinite large sequence then

$$\lim_{n \longrightarrow \infty} \hat{T}_n = \infty.$$

From here

$$0 \leq \lim_{n \longrightarrow \infty} |\hat{a}_n| = \lim_{n \longrightarrow \infty} \frac{|a_n|}{\hat{T}_n} \leq \lim_{n \longrightarrow \infty} \frac{M}{\hat{T}_n} = 0.$$

$\square$

Theorem 3.0.99. *Let $\{\hat{a}_n\}_{n=1}^{\infty}$ be bounded sequence and $\{\hat{b}_n\}_{n=1}^{\infty}$ be infinite large sequence and $\hat{b}_n \neq \hat{0}$ for every $n \in \mathbb{N}$. Then the sequence $\{\hat{a}_n \not\prec \hat{b}_n\}_{n=1}^{\infty}$ is infinite small sequence.*

Proof. Let $\epsilon > 0$ be arbitrary chosen and fixed. Since the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is bounded then there exists $M \in \mathbb{R}$, $M > 0$ such that for every $n \in \mathbb{N}$ we have

$$|\hat{a}_n| \leq M.$$

Also, there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > N$

$$|\hat{b}_n| > M \leftthreetimes \epsilon$$

or for every $n \in \mathbb{N}$, $n > N$

$$1 \leftthreetimes |\hat{b}_n| < \epsilon \leftthreetimes M.$$

Consequently for every $n \in \mathbb{N}$, $n > N$

$$|\hat{a}_n \leftthreetimes \hat{b}_{\hat{n}}| = |\hat{a}_n| \hat{\times} 1 \leftthreetimes |\hat{b}_n| < M \hat{\times} \epsilon \leftthreetimes M = \epsilon.$$

□

Theorem 3.0.100. *Let $\{|\hat{a}_n|\}_{n=1}^{\infty}$ be bounded below sequence by a positive isoreal and $\lim_{n \longrightarrow \infty} \hat{\alpha}_n = 0$ and $\hat{\alpha}_n \neq 0$ for every $n \in \mathbb{N}$. Then the sequence $\{\hat{a}_n \leftthreetimes \hat{\alpha}_n\}_{n=1}^{\infty}$ is infinite large sequence.*

Proof. There exists $\hat{K} \in \hat{F}_{\mathbb{R}}$, $\hat{K} > 0$ such that for every $n \in \mathbb{N}$

$$\hat{a}_n \geq \hat{K}.$$

Let $\hat{M} \in \hat{F}_{\mathbb{R}}$, $\hat{M} > 0$. Then there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > N$ we have

$$\hat{\alpha}_n| < \hat{K} \leftthreetimes \hat{M},$$

from where for every $n \in \mathbb{N}$, $n > N$ we have

$$1 \leftthreetimes |\hat{\alpha}_n| > \hat{M} \leftthreetimes \hat{K},$$

and

$$|\hat{a}_n \leftthreetimes \hat{\alpha}_n| = |\hat{a}_n| > 1 \leftthreetimes |\hat{\alpha}_n| > \hat{K} > \hat{M} \leftthreetimes \hat{K} = \hat{M}.$$

□

Corollary 3.0.101. *Let $\{\hat{a}_n\}_{n=1}^{\infty}$ be infinite large sequence. Then $\{1 \nwarrow \hat{a}_n\}_{n=1}^{\infty}$ is infinite small sequence.*

Corollary 3.0.102. *Let $\{\hat{a}_n\}_{n=1}^{\infty}$ be infinite small sequence. Then $\{1 \nwarrow \hat{a}_{\hat{n}}\}_{n=1}^{\infty}$ is infinite large sequence.*

Definition 3.0.103. *The sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ will be called*

1. *increasing if from $n, m \in \mathbb{N}$, $n > m$ follows that $\hat{a}_n > \hat{a}_m$,*
2. *decreasing if from $n, m \in \mathbb{N}$, $n > m$ follows that $\hat{a}_n < \hat{a}_m$,*
3. *monotonic if it is increasing or decreasing.*

If the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is increasing it is bounded below because $\hat{a}_n \geq \hat{a}_1$ for every $n \in \mathbb{N}$.
If the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is isodecresing it is bounded above because $\hat{a}_n \leq \hat{a}_1$ for every $n \in \mathbb{N}$.

Theorem 3.0.104. *Let $\{\hat{a}_n\}_{n=1}^{\infty}$ be incresing sequence and bounded above by $\hat{M} \in \hat{F}_{\mathbb{R}}$ then it is convergent.*

Proof. Since the set $\{\hat{a}_n\}_{n=1}^{\infty}$ is bounded above then it has supremum and let it is $\hat{a}$. Then

1. $\hat{a}_n \leq \hat{a}$ for every $n \in \mathbb{N}$,

2. for every $\hat{\epsilon} \in \hat{F_{\mathbb{R}}}$, $\hat{\epsilon} > 0$, there exists $n_0 \in \mathbb{N}$ such that $\hat{a} - \hat{\epsilon} < \hat{a}_{n_0}$.

Since $\hat{a} = \sup\{\hat{a}_n\}_{n=1}^{\infty}$ then $\hat{a} \leq \hat{M}$.
Then for every $\hat{\epsilon} \in \hat{F_{\mathbb{R}}}$, $\hat{\epsilon} > 0$, there exists $n_0 \in \mathbb{N}$ so that for every $n > n_0$

$$\hat{a} - \hat{\epsilon} < \hat{a}_{n_0} \leq \hat{a}_n \leq \hat{a} < \hat{a} + \hat{\epsilon} \quad \Longrightarrow$$

$$\hat{a} - \hat{\epsilon} < \hat{a}_n < \hat{a} + \hat{\epsilon} \quad \Longleftrightarrow |\hat{a}_n - \hat{a}| < \hat{\epsilon},$$

from where $\hat{\lim}_{n \hat{\longrightarrow} \infty} \hat{a}_n = \hat{a}$. □

As in above one can prove

Theorem 3.0.105. *Let $\{\hat{a}_n\}_{n=1}^{\infty}$ be decreasing sequence and bounded below by $\hat{P} \in \hat{F_{\mathbb{R}}}$ then it is convergent.*

Corollary 3.0.106. *Every bounded monotonic sequence is convergent.*

Definition 3.0.107. *A sequence $\{\hat{a}_{\hat{n}}\}_{n=1}^{\infty}$ is called fundamental if for every $\hat{\epsilon} \in \hat{F_{\mathbb{R}}}$ there exists $N \in \mathbb{N}$ such that for every $m, n > N$ we have*

$$|\hat{a}_n - \hat{a}_m| < \hat{\epsilon}.$$

Theorem 3.0.108. *If the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is convergent then it is isofundamental.*

Proof. Let $\{\hat{a}_n\}_{n=1}^{\infty}$ is convergent to $\hat{a}$.

Let $\hat{\epsilon} \in \hat{F}_{\mathbb{R}}$, $\hat{\epsilon} > 0$ is arbitrary chosen and fixed. Then there exists $N \in \mathbb{N}$ such that for every $m, n \in \mathbb{N}$, $\hat{m}, n > N$, we have

$$|\hat{a}_n - \hat{a}| < \hat{\epsilon} \,\hat{\diagup}\, \hat{2}, \quad |\hat{a}_m - \hat{a}| < \hat{\epsilon} \,\hat{\diagup}\, \hat{2} \quad \Longrightarrow$$

$$|\hat{a}_n - \hat{a}_m| = |\hat{a}_n - \hat{a} + \hat{a} - \hat{a}_m|$$

$$\leq |\hat{a}_n - \hat{a}| + |\hat{a}_m - \hat{a}|$$

$$< \hat{\epsilon} \,\hat{\diagup}\, \hat{2} + \hat{\epsilon} \,\hat{\diagup}\, \hat{2} = \hat{\epsilon}.$$

□

Definition 3.0.109. *Every isointerval $(\hat{p}, \hat{q})$ which contains the isopoint $\hat{a}$ will be called isoneighbourhood of the isopoint $\hat{a}$.*

Definition 3.0.110. *An isopoint $\hat{a}$ will be called condensation isopoint of the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ of elements of $\hat{F}_{\mathbb{R}}$ if every isoneighbourhood of $\hat{a}$ contains incountable many isoelements of $\{\hat{a}_n\}_{n=1}^{\infty}$.*

Theorem 3.0.111. *Every bounded sequence has an condensation isopoint.*

Proof. Let $\{\hat{a}_n\}_{n=1}^{\infty}$ be bounded isosequnce. From the definition for bounded sequence follows that there exists an isoclosed isofinite isointerval $\hat{\Delta}_1$ which contains this sequence. We devide the isointerval $\hat{\Delta}_1$ of two equal parts and will denote with $\hat{\Delta}_{\hat{2}}$ the half which contains uncountable many isoelements of the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$. Again we divide $\hat{\Delta}_{\hat{2}}$ of two equal parts and we will denote that its half with $\hat{\Delta}_{\hat{3}}$ which contains uncountable many isoelements of the sequence $\{\hat{a}_n\}_{\hat{n}=1}^{\infty}$ and etc. In this way we obtain an sequence of isoclosed isointervals

$$\hat{\Delta}_1 \hat{\supseteq} \hat{\Delta}_{\hat{2}} \hat{\supseteq} \hat{\Delta}_{\hat{3}} \hat{\supseteq} \cdots$$

and every one of them contains uncountable many isoelements of the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$. Since there exists $\hat{\alpha} \in \hat{F}_{\mathbb{R}}$ so that $\hat{\alpha} \in \hat{\Delta}_n$ for every $n \in \mathbb{N}$, we have that $\hat{\alpha}$ is an condensation isopoint of the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$, because the isolength of $\hat{\Delta}_n$ decreases to 0 and $\hat{\Delta}_n$ contains uncountable many isoelements of the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$. $\square$

Definition 3.0.112. *We will say that the sequence $\{\hat{a}_{n_k}\}_{k=1}^{\infty}$ is an subsequence of the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ if $n_k \in \mathbb{N}$ for every $k \in \mathbb{N}$ and*

$$n_1 < n_{\hat{2}} < n_{\hat{3}} < \cdots .$$

Theorem 3.0.113. *Let the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ be sequence which is convergent to $\hat{a}$. Then every subsequence $\{\hat{a}_{n_k}\}_{k=1}^{\infty}$ is convergent to $\hat{a}$.*

Proof. Since the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is convergent then it is bounded. From here follows that the subsequence $\{\hat{a}_{n_k}\}_{k=1}^{\infty}$ is bounded sequence. From here and from the properties of the bounded sequences follows that the subsequence $\{\hat{a}_{n_k}\}_{k=1}^{\infty}$ has an condensation isopoint $\hat{b}$. But this condensation isopoint will be isoconedensation isopoint for the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ and since it is convergent to $\hat{a}$ it has only one condensation isopoint. Therefore $\hat{a} = \hat{b}$ and the subsequence $\{\hat{a}_{n_k}\}_{k=1}^{\infty}$ has unique condensation isopoint $\hat{a}$. Consequently the subsequence $\{\hat{a}_{n_k}\}_{k=1}^{\infty}$ is convergent to $\hat{a}$. $\square$

Definition 3.0.114. *We will say that the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *is expanded of two subsequences* $\{\hat{a}_{n_k}\}_{k=1}^{\infty}$ *and* $\{\hat{a}_{m_k}\}_{k=1}^{\infty}$ *if*

$$\{n_1, n_2, \ldots\} \hat{\cup} \{m_1, m_2, m_3, \ldots\} = \mathbb{N},$$

and

$$\{n_1, n_2, \ldots\} \cap \{m_1, m_2, m_3, \ldots\} = \emptyset.$$

Theorem 3.0.115. *Let the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *be expanded of two subsequences* $\{\hat{a}_{n_k}\}_{k=1}^{\infty}$ *and* $\{\hat{a}_{m_k}\}_{k=1}^{\infty}$ *which are convergent to the isopoint* $\hat{a}$*. Then the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *is convergent to* $\hat{a}$*.*

Proof. Since every convergent sequence is bounded, then the subsequences $\{\hat{a}_{n_k}\}_{k=1}^{\infty}$ and $\{\hat{a}_{m_k}\}_{k=1}^{\infty}$ are bounded. Therefore there exists two isointervals $[\hat{\alpha}, \hat{\beta}]$ and $[\hat{\gamma}, \hat{\delta}]$ such that $\hat{a}_{n_k} \in [\hat{\alpha}, \hat{\beta}]$, $\hat{a}_{m_k} \in [\hat{\gamma}, \hat{\delta}]$ for every $\hat{k} \in \mathbb{N}$. From the properties of the convergent sequences follows that $\hat{a} \in [\hat{\alpha}, \hat{\beta}]$ and $\hat{a} \in [\hat{\gamma}, \hat{\delta}]$. Let $[\hat{p}, \hat{q}] = [\hat{\alpha}, \hat{\beta}] \hat{\cup} [\hat{\gamma}, \hat{\delta}]$. Then $\hat{a}_n \in [\hat{p}, \hat{q}]$ for every $n \in \mathbb{N}$. Therefore the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is bounded sequence. Therefore it has an condensation isopoint $\hat{b}$.

If we suppose that $\hat{b} < \hat{a}$ then for $\hat{\epsilon} = (\hat{a} - \hat{b}) \hat{\not<} \hat{2} > 0$ the isoneighbourhood $(\hat{b} - \hat{\epsilon}, \hat{b} + \hat{\epsilon})$ contains uncountable many elements of the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$, from where follows that outside of the isoneighbourhood $(\hat{a} - \hat{\epsilon}, \hat{a} + \hat{\epsilon})$ of the limit of $\hat{a}$ of the subsequences $\{\hat{a}_{n_k}\}_{k=1}^{\infty}$ and $\{\hat{a}_{m_k}\}_{k=1}^{\infty}$ there are uncountable many isoelements of one of them, which is contradiction. Therefore $\hat{b} \geq \hat{a}$. The case $\hat{b} > \hat{a}$ leads to a contradiction as in above. Consequently $\hat{a} = \hat{b}$.

Therefore we can conclude that the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ has unique condensation isopoint. From here we conclude that it is convergent to $\hat{a}$. $\square$

Corollary 3.0.116. *Let* $\{\hat{a}_n\}_{n=1}^{\infty}$ *and* $\{\hat{b}_n\}_{n=1}^{\infty}$ *be convergent sequences to the isopoint* $\hat{a}$*. Then the sequence*

$$\hat{a}_1, \hat{b}_1, \hat{a}_2, \hat{b}_2, \dots$$

is convergent sequence to $\hat{a}$*.*

Corollary 3.0.117. *Let* $\{\hat{a}_n\}_{n=1}^{\infty}$ *be sequence which is convergent to* $\hat{a}$*. Let also* $\hat{b} \in \hat{F}_{\mathbb{R}}$*. Then the sequence*

$$\hat{b}, \hat{a}_1, \hat{a}_2, \hat{a}_3, \dots$$

is convergent sequence to $\hat{a}$*.*

Corollary 3.0.118. *Let* $\{\hat{a}_n\}_{n=1}^{\infty}$ *be sequence which is convergent to* $\hat{a}$*. Let also* $\hat{b}_1\hat{b}_2, \dots, \hat{b}_k \in \hat{F}_{\mathbb{R}}$ *be finite number of isoreals. Then the sequence*

$$\hat{b}_1, \hat{b}_2, \dots, \hat{b}_k, \hat{a}_1, \hat{a}_2, \hat{a}_{\hat{3}}, \dots$$

is convergent sequence to $\hat{a}$*.*

Theorem 3.0.119. *From every infinite bounded sequence can be chosen convergent subsequence.*

Proof. Let $\{\hat{a}_n\}_{n=1}^{\infty}$ be infinite bounded sequence. Then it has condensation isopoint. Let it will be $\hat{a}$. In every isoneighbourhood of $\hat{a}$ there are uncountable many isoelements of the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$. In $(\hat{a} - 1, \hat{a} + 1)$ there are

uncountable many isoelements of the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$. Let $\hat{a}_{n_1}$ will one of them.
In the isoneighbourhood $(\hat{a} - 1 \nearrow \hat{2}, \hat{a} + 1 \nearrow \hat{2})$ there are uncountable many isoelements of the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$, let $\hat{a}_{n_{\hat{2}}}$, $\hat{a}_{n_2} \neq \hat{a}_{n_1}$ be one of them and etc. In this we construct the subsequence $\{\hat{a}_{n_k}\}_{k=1}^{\infty}$ so that

$$\hat{a} - 1 \nearrow k < \hat{a}_{n_k} < \hat{a} + 1 \nearrow k$$

and since

$$\lim_{k \longrightarrow \infty} (\hat{a} - 1 \nearrow k) = \hat{\lim}_{k \longrightarrow \infty} (\hat{a} + 1 \nearrow k) = \hat{a},$$

we conclude that

$$\lim_{k \longrightarrow \infty} \hat{a}_{n_k} = \hat{a}.$$

□

Theorem 3.0.120. *Every fundamental sequence of isoreals is convergent.*

Proof. Let $\{\hat{a}_n\}_{n=1}^{\infty}$ be isofundamental sequence. Then for every $\hat{\epsilon} > 0$ there exists $N > 0$ such that from $m, n > N$, $m, n \in \mathbb{N}$,follows that

$$|\hat{a}_m - \hat{a}_n| < \hat{\epsilon}.$$

Let $\hat{\epsilon} = 1$ and $m_1 > n$ be fixed. Then for every $n > m_1$ we have

$$|\hat{a}_n - \hat{a}_{m_1}| < 1,$$

from here and from the properties of the isomodulus follows that

$$\hat{a}_{m_1} - 1 < \hat{a}_n < \hat{a}_{m_1} + 1 \quad \forall n > m_1.$$

Let

$$\hat{l} = \min\{\hat{a}_1, \hat{a}_{\hat{2}}, \ldots, \hat{a}_{\hat{m}_1} - 1\},$$

$$\hat{L} = \max\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{m_1} + 1\}.$$

Then for every $n \in \mathbb{N}$ we have

$$\hat{l} \leq \hat{a}_n \leq \hat{L}.$$

Therefore the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is bounded. Then it has an condensation isopoint $\hat{a}$. We suppose that it has and other one isopoint $\hat{b}$. With out loss of generality we can suppose that $\hat{b} > \hat{a}$. Let

$$0 < \hat{\epsilon}_0 < (\hat{b} - \hat{a}) \hat{/} 3.$$

For this $\hat{\epsilon}_0$ we can find $n_1 > 0$ such that for every $m, n > n_1$ we have

$$|\hat{a}_m - \hat{a}_n| < \hat{\epsilon}_0, \tag{3.0.9}$$

and since $\hat{a}$ is iscondensation isopoint of the sequence $\{\hat{a}_n\}_{m=1}^{\infty}$ then there exists $m_1 > n_1$ such that

$$\hat{a} - \hat{\epsilon}_0 < \hat{a}_{m_1} < \hat{a} + \hat{\epsilon}_0.$$

As in above, there exists $N_1 > n_1$ such that

$$\hat{b} - \hat{\epsilon}_0 < \hat{a}_{N_1} < \hat{b} + \hat{\epsilon}_0.$$

Because for m_1 and N_1 we have (3.0.9) then

$$\hat{b} - \hat{a} = \hat{b} - \hat{a}_{n_1} + \hat{a}_{n_1} - \hat{a}_{m_1} + \hat{a}_{m_1} - \hat{a} < \hat{\epsilon}_0 + \hat{\epsilon}_0 + \hat{\epsilon}_0 = 3\hat{\times}\hat{\epsilon}_0,$$

which is a contradiction with the choice of $\hat{\epsilon}_x 0$.
Consequently the bounded sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ has unique condensation isopoint, therefore it is convergent. □

Definition 3.0.121. *We will say that $+\infty$ is condensation isopoint of $\{\hat{a}_n\}_{n=1}^{\infty}$ if it is isounbounded above.*

Definition 3.0.122. *We will say that $-\infty$ is condensation isopoint of $\{\hat{a}_{\hat{n}}\}_{n=1}^{\infty}$ if it is unbounded below.*

Using above definitions we can conclude that every sequence of isoreals has condensation isopoint.

Definition 3.0.123. *Let $\{\hat{a}_n\}_{n=1}^{\infty}$ be sequence of isorealls. Limit inferior and limit superior we define as follows*

$$\liminf_{n\longrightarrow\infty}\hat{a}_n = \lim_{n\longrightarrow\infty}(\inf_{\mathrm{m}\geq\mathrm{n}}\hat{\mathrm{a}}_{\mathrm{m}}),$$

$$\limsup_{n\longrightarrow\infty}\hat{a}_n = \lim_{n\longrightarrow\infty}(\sup_{\mathrm{m}\geq\mathrm{n}}\hat{\mathrm{a}}_{\mathrm{m}}),$$

respectively.

If

$$\liminf_{n\longrightarrow\infty}\hat{a}_n = \infty,$$

then

$$\lim_{n\longrightarrow\infty}\hat{a}_n = \infty.$$

If

$$\limsup_{n\longrightarrow\infty}\hat{a}_n = -\infty,$$

then

$$\lim_{n\longrightarrow\infty}\hat{a}_n = -\infty.$$

The sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ is convergent if

$$\liminf_{n\longrightarrow\infty}\hat{a}_n = \lim_{n\longrightarrow\infty}\hat{a}_n = \limsup_{n\longrightarrow\infty}\hat{a}_n.$$

Advanced practical exercises

Problem 3.0.124. *Let $\{\hat{T}_n\}_{n=1}^{\infty} = \{n^4+5\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n^2+1}\right\}_{n=1}^{\infty}$. Prove that $\{\hat{a}_n\}_{n=1}^{\infty}$ is a bounded sequence.*

Problem 3.0.125. *Let $\{\hat{T}_n\}_{n=1}^{\infty} = \left\{\frac{n+5}{n^6+7}\right\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n^2+1}\right\}_{n=1}^{\infty}$. Prove that $\{\hat{a}_n\}_{n=1}^{\infty}$ is an unbounded sequence.*

Problem 3.0.126. *Let $\{a_n\}_{n=1}^{\infty} = \{n+2\}_{n=1}^{\infty}$, $\{\hat{T}_n\}_{n=1}^{\infty} = \left\{\frac{1}{n+3}\right\}_{n=1}^{\infty}$. Prove that the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ diverges to $+\infty$.*

Problem 3.0.127. *Let $\{a_n\}_{n=1}^{\infty} = \{n^2+7\}_{n=1}^{\infty}$, $\{\hat{T}_n\}_{n=1}^{\infty} = \left\{\frac{1}{n+3}\right\}_{n=1}^{\infty}$. Prove that the sequence $\{\hat{a}_n\}_{n=1}^{\infty}$ diverges to $+\infty$.*

Problem 3.0.128. *Let* $\{a_n\}_{n=1}^{\infty} = \{-n^3 - 2\}_{n=1}^{\infty}$, $\{\hat{T}_n\}_{n=1}^{\infty} = \left\{\frac{1}{n^2+3}\right\}_{n=1}^{\infty}$. *Prove that the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *diverges to* $-\infty$.

Problem 3.0.129. *Let* $\{a_n\}_{n=1}^{\infty} = \{-n^2 + 2\}_{n=1}^{\infty}$, $\{\hat{T}_n\}_{n=1}^{\infty} = \left\{\frac{1}{n^4+1}\right\}_{n=1}^{\infty}$. *Prove that the sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *diverges to* $-\infty$.

Problem 3.0.130. *Find* $\lim_{n\longrightarrow\infty} \hat{a}_n$ *if*

1. $a_n = b_n^2 - 5b_n + 6$, $\quad \hat{T}_n = b_n^2 - 7b_n + 10$, $\lim_{n\longrightarrow\infty} b_n = 2$, $b_n \neq 2, 5$,
2. $a_n = b_n^2 - 6b_n + 8$, $\hat{T}_n = b_n^2 - 5b_n + 4$, $\lim_{n\longrightarrow\infty} b_n = 4$, $b_n \neq 1, 4$,
3. $a_n = b_n^4 + 2b_n^2 - 3$, $\hat{T}_n = b_n^2 - 3b_n + 2$, $\lim_{n\longrightarrow\infty} b_n = 1$, $b_n \neq 1, 2$,
4. $a_n = 3b_n^4 - 4b_n^3 + 1$, $\hat{T}_n = (b_n - 1)^2$, $\lim_{n\longrightarrow\infty} b_n = 1$, $b_n \neq 1$,
5. $a_n = b_n^k - 1$, $\hat{T}_n = b_n - 1$, $\lim_{n\longrightarrow\infty} b_n = 1$, $b_n \neq 1$, $k \in \mathbb{N}$,
6. $a_n = b_n^k - 1$, $\hat{T}_n = b_n^l - 1$, $\lim_{n\longrightarrow\infty} b_n = 1$, $k, l \in \mathbb{N}$, $b_n \neq 1$ *for odd* l, $|b_n| \neq 1$ *for even* l,
7. $a_n = \sqrt[3]{1 + b_n} - 1$, $\hat{T}_n = b_n$, $\lim_{n\longrightarrow\infty} b_n = 0$, $b_n \neq 0$,
8. $a_n = \sqrt[k]{1 + b_n} - 1$, $\hat{T}_n = b_n$, $\lim_{n\longrightarrow\infty} b_n = 0$, $b_n \neq 0$, $k \in \mathbb{N}$,
9. $a_n = \sqrt[3]{b_n} + 1$, $\hat{T}_n = \sqrt[5]{b_n} + 1$, $\lim_{n\longrightarrow\infty} b_n = -1$, $b_n \neq -1$,
10. $a_n = \sqrt{1 + b_n + b_n^2} - 1$,$\hat{T}_n = b_n$, $\lim_{n\longrightarrow\infty} b_n = 0$, $b_n \neq 0$,
11. $a_n = \sqrt{1 + b_n} - \sqrt{1 + b_n^2}$, $\hat{T}_n = \sqrt{1 + b_n} - 1$, $\lim_{n\longrightarrow\infty} b_n = 0$, $b_n \neq 0$,
12. $a_n = \sqrt[3]{1 + 2b_n} + 1$, $\hat{T}_n = \sqrt[3]{2 + b_n} + b_n$, $\lim_{n\longrightarrow\infty} b_n b_n = -1$, $b_n \neq -1$,
13. $a_n = (n + 2)^n$, $\hat{T}_n = n^n$,
14. $a_n = (n + 3)^n$, $\hat{T}_n = n^n$,
15. $a_n = (n + k)^n$, $\hat{T}_n = n^n$, $k \in \mathbb{N}$,
16. $a_n = (n - 1)^n$, $\hat{T}_n = n^n$,
17. $a_n = (n - 2)^n$, $\hat{T}_n = n^n$,
18. $a_n = (n - k)^n$, $\hat{T}_n = n^n$, $k \in \mathbb{N}$,
19. $a_n = (n^2 - 1)^n$, $\hat{T}_n = (n^2 - n - 6)^n$,

20. $a_n = (n^2 - 5n + 6)^n$, $\hat{T}_n = (n^2 + 5n + 6)^n$,

21. $a_n = (n^2 - 4n + 3)^n$, $\hat{T}_n = (n^2 + 3n + 2)^n$,

22. $a_n = (n^2 + n + 1)^n$, $\hat{T}_n = (n^2 + 3n + 1)^n$,

23. $a_n = (n^3 + n^2 + 3n + 1)^n$, $\hat{T}_n = (n^3 + n^2 + 2n + 1)^n$.

Answer. 1) $\frac{1}{3}$, 2) $\frac{2}{3}$, 3) -8, 4) 6, 5) k, 6) $\frac{k}{l}$, 7) $\frac{1}{3}$, 8) $\frac{1}{k}$, 9) $\frac{5}{3}$, 10) $\frac{1}{2}$, 11) 1, 12) $\frac{1}{2}$, 13) e^2, 14) e^3, 15) e^k, 16) e^{-1}, 17) e^{-2}, 18) e^{-k}, 19) e, 20) e^{-10}, 21) e^{-7}, 22) e^{-2}, 23) 1.

Chapter 4

Isofunctions-definition and properties

Let $\hat{I}_1 = \frac{1}{\hat{T}_1}$ be the isounit of $\hat{F}_{\mathbb{R}}$, $\hat{T}_1 > 0$ is a constant.
Let also D, Y, $Z \subset \mathbb{R}$ be given sets, f be a relation between D and Z, defined on all D, and $\hat{T}$ be a relation between D and Y, defined on all D, $\hat{T}(x) > 0$ for every $x \in D$. For $x \in D$ we define the operators

$$\hat{x} := \frac{x}{\hat{T}(x)}, \quad \hat{f}(x) := \frac{f(x)}{\hat{T}(x)}, \quad f^\wedge(x) := f(x\hat{T}(x)), \quad \hat{f}(\hat{x}) = \frac{f(\hat{x})}{\hat{T}(x)}.$$

Then for $x \in D$ we have

$$\hat{f}^\wedge(\hat{x}) = \frac{f\left(\hat{T}(x)\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f(x)}{\hat{T}(x)},$$

$$\hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} \quad \text{when} \quad \frac{x}{\hat{T}(x)} \in D,$$

$$f(\hat{x}) = f\left(\frac{x}{\hat{T}(x)}\right) \quad \text{when} \quad \frac{x}{\hat{T}(x)} \in D,$$

$$f^\wedge(\hat{x}) = f\left(\hat{T}(x)\frac{x}{\hat{T}(x)}\right) = f(x),$$

$$f^\wedge(x) = f(\hat{T}(x)x) \quad \text{when} \quad x\hat{T}(x) \in D.$$

Example 4.0.131. *Let $D = \mathbb{R}$, $\hat{T}(x) = x^2 + 1$, $f(x) = x$, $x \in D$. Then*

$$\hat{f}^\wedge(\hat{x}) = \tfrac{f(x)}{\hat{T}(x)} = \tfrac{x}{x^2+1},$$

$$\hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f\left(\frac{x}{x^2+1}\right)}{x^2+1} = \frac{\frac{x}{x^2+1}}{x^2+1} = \frac{x}{(x^2+1)^2},$$

$$f(\hat{x}) = f\left(\tfrac{x}{\hat{T}(x)}\right) = f\left(\tfrac{x}{x^2+1}\right) = \tfrac{x}{x^2+1},$$

$$f^\wedge(x) = f(\hat{T}(x)x) = f((x^2+1)x) = f(x^3+x) = x^3 + x.$$

Exercise 4.0.132. *Let $D = \mathbb{R}$, $\hat{T}(x) = e^{-x}$, $f(x) = x - 1$, $x \in D$. Find*

$$\hat{f}^\wedge(\hat{x}), \hat{f}(\hat{x}), f(\hat{x}), f^\wedge(x), \quad x \in D.$$

Answer.

$$\hat{f}^\wedge(\hat{x}) = (x-1)e^x, \quad \hat{f}(\hat{x}) = e^x(xe^x - 1),$$

$$f(\hat{x}) = xe^x - 1, \quad f^\wedge(x) = xe^{-x} - 1, \quad x \in D.$$

Exercise 4.0.133. *Let $D = \mathbb{R}$, $\hat{T}(x) = x^4 + 1$, $f(x) = 2x + 1$, $x \in D$. Find*

$$\hat{f}^\wedge(\hat{x}), \hat{f}(\hat{x}), f(\hat{x}), f^\wedge(x), \quad x \in D.$$

Answer.

$$\hat{f}^\wedge(\hat{x}) = \tfrac{2x+1}{x^4+1}, \quad \hat{f}(\hat{x}) = \tfrac{x^4+2x+1}{(x^4+1)^2},$$

$$f(\hat{x}) = \tfrac{x^4+2x+1}{x^4+1}, \quad f^\wedge(x) = 2x^5 + 2x + 1, \quad x \in D.$$

Definition 4.0.134. *We will tell that in the set D is defined isofunction of first kind or isomap of first kind if*

$$\hat{y} := \hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)}, \quad x \in D,$$

is a function(map). We will use the notation $\hat{f}^{\wedge\wedge}$.
The element x will be called isoargument of the isofunction of first kind or isoindependent isovariable, and its isoimage $\hat{y} = \hat{f}^{\wedge}(\hat{x})$ will be called isodependent isovariable or isovalue of the isofunction of first kind. The set

$$\{\hat{f}^{\wedge}(\hat{x}) : x \in D\}$$

will be called isocodomain of isovalues of the isofunction of first kind. The set D will be called isodomain of the isofunction of first kind. The function $\frac{f(x)}{\hat{T}(x)}$ will be called isooriginal of the isofunction of first kind.

Example 4.0.135. *Let $D = \mathbb{R}$, $\hat{T}(x) = x^2 + 1$, $f(x) = x^3 + 1$. Then*

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x^3 + 1}{x^2 + 1}.$$

Remark 4.0.136. *In the last example we saw that if f is a function on D then $\hat{f}^{\wedge\wedge}$ is a function on D. There is a possibility the original f to be not a function but the corresponding lift $\hat{f}$ to be function and the inverse. We will see this in the following examples.*

Example 4.0.137. *Let $D = [-1, 1]$,*

$$f(x) = \begin{cases} x^2 + 1 & \text{for} \quad x \in [-1, 0], \\ \\ x + 2 & \text{for} \quad x \in [0, 1], \end{cases} \qquad \hat{T}(x) = \begin{cases} 1 & \text{for} \quad x \in [-1, 0], \\ \\ 2 & \text{for} \quad x \in [0, 1]. \end{cases}$$

Then f is not a function on D because $f(0) = 1$ and $f(0) = 2$. But

$$\hat{f}^{\wedge}(\hat{x}) = \begin{cases} x^2 + 1 & \text{for} \quad x \in [-1, 0], \\ \\ \frac{x+2}{2} & \text{for} \quad x \in [0, 1], \end{cases}$$

which is a function on D because $\hat{f}^{\wedge\wedge}(\hat{0}) = 1$.

Example 4.0.138. *Let $D = [-2, 2]$, $f(x) = x^3 + 1$, and*

$$\hat{T}(x) = \begin{cases} 2 & \text{for} \quad x \in [-2, 0], \\ \\ 3 & \text{for} \quad x \in [0, 1]. \end{cases}$$

Then f is a function on D, but

$$\hat{f}^{\wedge}(\hat{x}) = \begin{cases} \frac{x^3+1}{2} & \text{for} \quad x \in [-2, 0], \\ \\ \frac{x^3+1}{3} & \text{for} \quad x \in [0, 2], \end{cases}$$

is not a function on D because $\hat{f}^{\wedge}(\hat{0}) = \frac{1}{2}$ and $\hat{f}^{\wedge}(\hat{0}) = \frac{1}{3}$.

Exercise 4.0.139. *Let $D = \mathbb{R}$, $f(x) = \frac{1}{x^2+1}$, $\hat{T}(x) = x^2 + 1$, $x \in D$. Determine*

1) *$\hat{f}^{\wedge}(\hat{x})$, $x \in D$,*

2) *isocodomain of $\hat{f}$,*

3) *if f is a function on D,*

4) *if $\hat{f}^{\wedge\wedge}$ is an isofunction on D.*

Solution.

1) Using the definition

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{\frac{1}{x^2+1}}{x^2+1} = \frac{1}{(x^2+1)^2}.$$

2) From the previous point we conclude that $\hat{f}^{\wedge}(\hat{x}) \geq 0$ for every $x \in \mathbb{R}$, from here we obtain that the isocodomain of $\hat{f}^{\wedge\wedge}$ is $\mathbb{R}_+$.

3) f is a function on D.

4) $\hat{f}^{\wedge\wedge}$ is an isofunction on D.

Exercise 4.0.140. *Let $D = \mathbb{R}$ and*

$$f(x) = \begin{cases} x+1 & \text{for} \quad x \in (-\infty, 0], \\ \\ \frac{x+2}{2} & \text{for} \quad x \in [0, \infty), \end{cases} \qquad \hat{T}(x) = \begin{cases} x^2+2 & \text{for} \quad x \in (-\infty, 0], \\ \\ x^4+5 & \text{for} \quad x \in [0, \infty). \end{cases}$$

Determine

1) *if f is a function on D,*

2) *If $\hat{f}^{\wedge\wedge}$ is an isofunction on D.*

Answer. 1) Yes, 2) No.

Exercise 4.0.141. *Let $D = [-1, 1]$ and*

$$f(x) = \begin{cases} x+1 & \text{for} \quad x \in [-1, 0], \\ x+2 & \text{for} \quad x \in [0, 1], \end{cases} \qquad \hat{T}(x) = \begin{cases} x^2+3 & \text{for} \quad x \in [-1, -\frac{1}{2}], \\ x^2+4 & \text{for} \quad x \in [-\frac{1}{2}, \frac{1}{2}], \\ x^2+6 & \text{for} \quad x \in [\frac{1}{2}, 1]. \end{cases}$$

Determine

1) $\hat{f}^{\wedge}(\hat{x})$,

2) *if f is a function on D,*

3) *if $\hat{f}^{\wedge\wedge}$ is an isofunction on D.*

Answer.

$$1) \quad \hat{f}^{\wedge}(\hat{x}) = \begin{cases} \frac{x+1}{x^2+3} & \text{for} \quad x \in [-1, -\frac{1}{2}], \\ \frac{x+1}{x^2+4} & \text{for} \quad x \in [-\frac{1}{2}, 0], \\ \frac{x+2}{x^2+4} & \text{for} \quad x \in [0, \frac{1}{2}], \\ \frac{x+2}{x^2+6} & \text{for} \quad x \in [\frac{1}{2}, 1]. \end{cases}$$

2) f is not a function on D, 3) $\hat{f}^{\wedge\wedge}$ is not an isofunction on D.

Exercise 4.0.142. *Let $D = [0, 3]$ and*

$$f(x) = \begin{cases} \frac{x+1}{x^2+2} & \text{for} \quad x \in [0, 1], \\ -\frac{1}{3}+x & \text{for} \quad x \in [1, 2], \\ -\frac{7}{3}+x^2 & \text{for} \quad x \in [2, 3], \end{cases} \qquad \hat{T}(x) = \begin{cases} x^2+1 & \text{for} \quad x \in [0, 1], \\ \frac{2}{3}(x^2+2) & \text{for} \quad x \in [1, 3]. \end{cases}$$

Determine

1) $\hat{f}^{\wedge}(\hat{x})$,

2) *if f is a function on D,*

3) *if $\hat{f}^{\wedge\wedge}$ is an function on D.*

Answer.

$$1) \quad \hat{f}^{\wedge}(\hat{x}) = \begin{cases} \frac{x+1}{(x^2+1)(x^2+2)} & \text{for} \quad x \in [0,1], \\ \frac{3}{2}\frac{x-\frac{1}{3}}{x^2+2} & \text{for} \quad x \in [1,2], \\ \frac{3}{2}\frac{x^2-\frac{7}{3}}{x^2+2} & \text{for} \quad x \in [2,3], \end{cases}$$

2) f is a function on D, 3) $\hat{f}^{\wedge\wedge}$ is an isofunction on D.

Definition 4.0.143. *We will tell that in the set D is defined isofunction of second kind or isomap of second kind if $x\hat{T}(x) \in D$ for every $x \in D$ and*

$$\hat{y} := \hat{f}^{\wedge}(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)}, \quad x \in D,$$

is a function(map). We will use the notation $\hat{f}^{\wedge}$.
The element x will be called isoargument of the isofunction of second kind or isoindependent isovariable, and its isoimage $\hat{y} = \hat{f}^{\wedge}(x)$ will be called isodependent isovariable or isovalue of the isofunction of second kind. The set

$$\{\hat{f}^{\wedge}(x) : x \in D\}$$

will be called isocodomain of isovalues of the isofunction of second kind. The set D will be called isodomain of the isofunction of second kind. The function $\frac{f(x\hat{T}(x))}{\hat{T}(x)}$ will be called isooriginal of the isofunction of second kind.

Example 4.0.144. *Let $D = [1, +\infty)$, $\hat{T}(x) = x^2 + 2$, $f(x) = x - 1$, $x \in D$. Then*

$$\hat{f}^{\wedge}(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)} = \frac{f(x(x^2+2))}{x^2+2} = \frac{f(x^3+2x)}{x^2+2} = \frac{x^3+2x-1}{x^2+2}.$$

Exercise 4.0.145. *Let $D = [1, +\infty)$, $\hat{T}(x) = x + 1$, $f(x) = x^2$, $x \in D$. Find $\hat{f}^\wedge(\hat{x})$, $\hat{f}^\wedge(x)$.*

Solution. Using the definition we have

$$\hat{f}^\wedge(\hat{x}) = \tfrac{f(x)}{\hat{T}(x)} = \tfrac{x^2}{x+1},$$

$$\hat{f}^\wedge(x) = \tfrac{f(x\hat{T}(x))}{\hat{T}(x)} = \tfrac{f(x(x+1))}{x+1} = \tfrac{f(x^2+x)}{x+1}$$

$$= \tfrac{(x^2+x)^2}{x+1} = \tfrac{x^2(x+1)^2}{x+1} = (x+1)x^2.$$

Exercise 4.0.146. *Let $D = [2, +\infty)$, $\hat{T}(x) = x - 1$, $f(x) = x^3 + x$, $x \in D$. Find $\hat{f}^\wedge(\hat{x})$, $\hat{f}^\wedge(x)$.*

Answer.

$$\hat{f}^\wedge(\hat{x}) = \frac{x^3 + x}{x - 1}, \quad \hat{f}^\wedge(x) = x^5 - 2x^4 + x^3 + x.$$

Exercise 4.0.147. *Let $f(x) = ax$, $x \in D$, $a \in \mathbb{R}$. Prove that $\hat{f}^\wedge(x) = f(x)$.*

Definition 4.0.148. *We will tell that in the set D is defined isofunction of third kind or isomap of third kind $\hat{f}$ if $\frac{x}{\hat{T}(x)} \in D$ for every $x \in D$ and*

$$\hat{y} := \hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}, \quad x \in D,$$

is a function(map). We will use the notation $\hat{\hat{f}}$.
The element x will be called isoargument of the isofunction of third kind or isoindependent isovariable, and its isoimage $\hat{y} = \hat{f}(\hat{x})$ will be called isodependent isovariable or isovalue of the isofunction of third kind. The set

$$\{\hat{f}(\hat{x}) : x \in D\}$$

will be called isocodomain of isovalues of the isofunction of third kind. The set D will be called isodomain of the isofunction of third kind. The function $\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}$ will be called isooriginal of the isofunction of third kind.

Example 4.0.149. *Let* $D = [1,3]$, $\hat{T}(x) = x^2 + 2x$, $f(x) = x^3 - 1$, $x \in D$. *Then*

$$\hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f\left(\frac{x}{x^2+2x}\right)}{x^2+2x} = \frac{f\left(\frac{1}{x+2}\right)}{x^2+2x}$$

$$= \frac{\frac{1}{(x+2)^3} - 1}{x^2+2x} = \frac{1-(x+2)^3}{x(x+2)^4} = \frac{-x^3-6x^2-12x-7}{x(x+2)^4}.$$

Exercise 4.0.150. *Let* $D = [1, +\infty)$, $\hat{T}(x) = x$, $f(x) = x + 1$. *Find* $\hat{f}(\hat{x})$.

Answer. $\frac{2}{x}$.

Exercise 4.0.151. *Let* $D = [2,5]$, $\hat{T}(x) = x + 2$, $f(x) = x^2 + 2x$, $x \in D$. *Find* $\hat{f}(\hat{x})$.

Answer. $\frac{3x^2+4x}{(x+2)^3}$.

Definition 4.0.152. *We will tell that in the set* D *is defined isofunction of fourth kind or isomap of fourth kind* $f^\wedge$ *if* $x\hat{T}(x) \in D$ *for every* $x \in D$ *and*

$$\hat{y} := f^\wedge(x) = f\left(x\hat{T}(x)\right), \quad x \in D,$$

is a function(map). We will use the notation $f^\wedge$.
The element x *will be called isoargument of the isofunction of fourth kind or isoindependent isovariable, and its isoimage* $\hat{y} = f^\wedge(x)$ *will be called isodependent isovariable or isovalue of the isofunction of fourth kind. The set*

$$\{f^\wedge(x) : x \in D\}$$

will be called isocodomain of isovalues of the isofunction of fourth kind. The set D *will be called isodomain of the isofunction of fourth kind. The function* $f\left(x\hat{T}(x)\right)$ *will be called isooriginal of the isofunction of fourth kind.*

Example 4.0.153. *Let* $D = [1, +\infty)$, $f(x) = x$, $\hat{T}(x) = x^2$, $x \in D$. *Then*

$$f^\wedge(x) = f(x\hat{T}(x)) = f(x^3) = x^3.$$

Exercise 4.0.154. *Let* $D = [0, +\infty)$, $f(x) = x + 1$, $\hat{T}(x) = x + 2$, $x \in D$. *Find* $f^{\wedge}(x)$.

Answer. $x^2 + 2x + 1$.

Exercise 4.0.155. *Let* $D = [0, \infty)$, $f(x) = x - 2$, $\hat{T}(x) = x^2 + 1$, $x \in D$. *Find* $f^{\wedge}(x)$.

Answer. $x^3 + x - 2$.

Exercise 4.0.156. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 4$, $\hat{T}(x) = x^2 + 2$. *Find*

$$\hat{1}, \hat{1}^{\wedge\wedge}, \hat{1}^{\wedge}, \hat{\hat{1}}, 1^{\wedge}.$$

Answer.

$$\hat{1} = \frac{1}{4}, \quad \hat{1}^{\wedge\wedge} = \hat{1}^{\wedge} = \hat{\hat{1}} = \frac{1}{3}, \quad 1^{\wedge} = 1.$$

Definition: An isofunction $\hat{h}$ of first, second, third or fourth kind with isooriginal $\tilde{h}$ will be called isoinjection, isosurjection or isobijection if its isooriginal $\tilde{h}$ is injection, surjection or bijection, respectively.

Example 4.0.157. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = x + 1$, $\hat{T}(x) = 3$, $x \in D$. *Then* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is injection and*

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x+1}{3} : \mathbb{R} \longrightarrow \mathbb{R}$$

is injection, i.e. $\hat{f} : \hat{F}_{\mathbb{R}} \widehat{\longrightarrow} \hat{F}_{\mathbb{R}}$ *is an isoinjection.*

Example 4.0.158. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = 1$, $\hat{T}(x) = e^{-x}$, $x \in D$. *Then* f *is not injection, but* $\hat{f}^{\wedge}(\hat{x}) = e^x$ *is an isoinjection.*

Definition: Let $\hat{a} \in \hat{F}_{\mathbb{R}}$ and $\hat{f}, \hat{g}$ are isofunctions of first, second, third or fourth kind with isooriginals $\tilde{f}$, $\tilde{g}$, respectively. Then we define

1) $\hat{a} \hat{\times} \hat{f} := a \frac{1}{\hat{T}_1} \hat{T}_1 \tilde{f} = a\tilde{f}$,

2) $\hat{a}\hat{f} := \frac{a}{\hat{T}_1} \tilde{f}$,

3) $\hat{f} \pm \hat{g} := \tilde{f} \pm \tilde{g}$.

Exercise 4.0.159. *Let* $D = \mathbb{R}$, $f(x) = x - 1$, $g(x) = x + 1$, $x \in D$, $\hat{T}_1 = 3$, $\hat{T}(x) = x^2 + 1$, $x \in D$. *Find*

1) $\hat{f}^\wedge(\hat{x})$,

2) $\hat{g}^\wedge(\hat{x})$,

3) $\hat{2}\hat{\times}\hat{f}^\wedge(\hat{x})$,

4) $\hat{3}\hat{g}^\wedge(\hat{x})$,

5) $\hat{2}\hat{\times}\hat{f}^\wedge(\hat{x}) - \hat{3}\hat{g}^\wedge(\hat{x})$.

Solution.

1) Using the definition

$$\hat{f}^\wedge(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x-1}{x^2+1}.$$

2) Using the definition

$$\hat{g}^\wedge(\hat{x}) = \frac{g(x)}{\hat{T}(x)} = \frac{x+1}{x^2+1}.$$

3) Using the definition and 1)

$$\hat{2}\hat{\times}\hat{f}^\wedge(\hat{x}) = 2\frac{1}{3}3\frac{f(x)}{\hat{T}(x)} = 2\frac{x-1}{x^2+1}.$$

4) Using the definition and 2)

$$\hat{3}\hat{g}^\wedge(\hat{x}) = 3\frac{1}{3}\frac{g(x)}{\hat{T}(x)} = \frac{x+1}{x^2+1}.$$

5) Using 3) and 4)

$$\hat{2}\hat{\times}\hat{f}^\wedge(\hat{x}) - \hat{3}\hat{g}^\wedge(\hat{x}) = 2\frac{x-1}{x^2+1} - \frac{x+1}{x^2+1} = \frac{2x-2-x-1}{x^2+1} = \frac{x-3}{x^2+1}.$$

Exercise 4.0.160. *Let* $D = [-1, 1]$, $f(x) = \sin x$, $g(x) = x$, $\hat{T}(x) = x^4 + 1$, $x \in D$, $\hat{T}_1 = 4$. *Find*

1) $\hat{f}^\wedge(\hat{x})$,

2) $\hat{g}^\wedge(\hat{x})$,

3) $\hat{3}\hat{\times}\hat{f}^\wedge(\hat{x})$,

4) $\hat{2}\hat{f}^\wedge(\hat{x})$,

5) $\hat{7}\hat{\times}\hat{g}^\wedge(\hat{x})$,

6) $\hat{4}\hat{\times}\hat{f}^\wedge(\hat{x}) - \hat{2}\hat{f}^\wedge(\hat{x}) + \hat{5}\hat{\times}\hat{g}^\wedge(\hat{x})$.

Answer.

$$1) \quad \hat{f}^\wedge(\hat{x}) = \tfrac{\sin x}{x^4+1}, \quad 2) \quad \hat{g}^\wedge(\hat{x}) = \tfrac{x}{x^4+1},$$

$$3) \quad \hat{3}\hat{\times}\hat{f}^\wedge(\hat{x}) = 3\tfrac{\sin x}{x^4+1}, \quad 4) \quad \hat{2}\hat{f}^\wedge(\hat{x}) = \tfrac{1}{2}\tfrac{\sin x}{x^4+1},$$

$$5) \quad \hat{7}\hat{\times}\hat{g}^\wedge(\hat{x}) = \tfrac{7x}{x^4+1}, \quad 6) \quad \hat{4}\hat{\times}\hat{f}^\wedge(\hat{x}) - \hat{2}\hat{f}^\wedge(\hat{x}) + \hat{5}\hat{\times}\hat{g}^\wedge(\hat{x}) = \tfrac{\frac{7}{2}\sin x+5x}{x^4+1}.$$

Definition: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $g : D \longrightarrow D$, $\hat{T}(x) > 0$ for every $x \in D$. Then we define

$$\hat{f}^\wedge\Big(\hat{g}^\wedge(\hat{x})\Big) := \frac{f(\hat{T}(x)\hat{g}^\wedge(\hat{x}))}{\hat{T}(x)} = \frac{f\left(\hat{T}(x)\frac{g\left(\hat{T}(x)\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f(g(x))}{\hat{T}(x)} \quad \text{for} \quad \forall x \in D.$$

Example 4.0.161. *Let $D = \mathbb{R}$, $\hat{T}_1 = 5$, $f(x) = x + 1$, $g(x) = x$, $\hat{T}(x) = x^2 + 1$, $x \in D$. Then*

$$\hat{f}^\wedge(\hat{g}^\wedge(\hat{x})) = \frac{f(g(x))}{\hat{T}(x)} = \frac{g(x)+1}{\hat{T}(x)} = \frac{x+1}{x^2+1}.$$

Exercise 4.0.162. *Let $D = \mathbb{R}$,$\hat{T}_1 = 4$, $f(x) = x - 1$, $g(x) = 2x + 1$, $\hat{T}(x) = x^4 + 1$, $x \in D$. Find*

1) $\hat{f}^\wedge(\hat{x})$,

2) $\hat{g}^\wedge(\hat{x})$,

3) $\hat{2}\hat{\times}\hat{f}^\wedge(\hat{x}) - \hat{3}\hat{g}^\wedge(\hat{x})$,

4) $\hat{f}^\wedge(\hat{f}^\wedge(\hat{x}))$,

5) $\hat{g}^\wedge(\hat{g}^\wedge(\hat{g}^\wedge(\hat{x})))$,

6) $\hat{f}^\wedge(\hat{g}^\wedge(\hat{x}))$,

7) $\hat{g}^\wedge(\hat{f}^\wedge(\hat{x}))$.

Solution.

1) Using the definition for isofunction we have

$$\hat{f}^\wedge(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x-1}{x^4+1}.$$

2) Using the definition of isofunction we have

$$\hat{g}^\wedge(\hat{x}) = \frac{g(x)}{\hat{T}(x)} = \frac{2x+1}{x^4+1}.$$

3)

$$\hat{2}\hat{\times}\hat{f}^\wedge(\hat{x}) - \hat{3}\hat{g}^\wedge(\hat{x}) = 2\tfrac{f(x)}{\hat{T}(x)} - \tfrac{3}{4}\tfrac{g(x)}{\hat{T}(x)}$$

$$= 2\tfrac{x-1}{x^4+1} - \tfrac{3}{4}\tfrac{2x+1}{x^4+1} = \tfrac{8x-8-6x-3}{4(x^4+1)} = \tfrac{2x-1}{4(x^4+1)}.$$

4)

$$\hat{f}^\wedge(\hat{f}^\wedge(\hat{x})) = \tfrac{f(f(x))}{\hat{T}(x)} = \tfrac{f(x)-1}{\hat{T}(x)} = \tfrac{x-1-1}{x^4+1} = \tfrac{x-2}{x^4+1}.$$

5)

$$\hat{g}^\wedge(\hat{g}^\wedge(\hat{g}^\wedge(\hat{x}))) = \tfrac{g(g(g(x)))}{\hat{T}(x)} = \tfrac{2g(g(x))+1}{\hat{T}(x)} = \tfrac{2(2g(x)+1)+1}{\hat{T}(x)}$$

$$= \tfrac{2(2(2x+1)+1)+1}{x^4+1} = \tfrac{2(4x+2+1)+1}{x^4+1} = \tfrac{8x+6+1}{x^4+1} = \tfrac{8x+7}{x^4+1}.$$

6)

$$\hat{f}^\wedge(\hat{g}^\wedge(\hat{x})) = \tfrac{f(g(x))}{\hat{T}(x)} = \tfrac{g(x)-1}{\hat{T}(x)} = \tfrac{2x+1-1}{x^4+1} = \tfrac{2x}{x^4+1}.$$

7)

$$\hat{g}^\wedge(\hat{f}^\wedge(\hat{x})) = \tfrac{g(f(x))}{\hat{T}(x)} = \tfrac{2f(x)+1}{\hat{T}(x)} = \tfrac{2(x-1)+1}{x^4+1} = \tfrac{2x-1}{x^4+1}.$$

Exercise 4.0.163. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 5$, $f(x) = 2x+3$, $\hat{T}(x) = x^2+10$, $x \in D$. *Find*

$$A := \hat{f}^\wedge(\hat{f}^\wedge(\hat{f}^\wedge(\hat{x}))) - \hat{2}\hat{\times}\hat{f}^\wedge(\hat{x}) + \hat{f}^\wedge(\hat{f}^\wedge(\hat{x})).$$

Answer. $A = \frac{8x+24}{x^2+10}$.

Definition: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$, $g : D \longrightarrow D$. Then we define

$$\hat{f}^\wedge\left(\hat{g}^\wedge(x)\right) := \frac{f(\hat{T}(x)\hat{g}^\wedge(x))}{\hat{T}(x)} = \frac{f\left(\hat{T}(x)\frac{g(x\hat{T}(x))}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f(g(x\hat{T}(x)))}{\hat{T}(x)} \quad \text{for} \quad \forall x \in D.$$

Example 4.0.164. *Let* $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x+1$, $\hat{T}(x) = x^2+1$, $x \in D$. *Then*

$$\hat{f}^\wedge\left(\hat{g}^\wedge(x)\right) = \frac{f(g(x\hat{T}(x)))}{\hat{T}(x)} = \frac{f(g(x(x^2+1)))}{x^2+1}$$

$$\frac{f(g(x^3+x))}{x^2+1} = \frac{f(2(x^3+x)+1)}{x^2+1} = \frac{f(2x^3+2x+1)}{x^2+1} = \frac{2x^3+2x+1}{x^2+1}.$$

Definition: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$ for every $x \in D$, $g : D \longrightarrow D$. Then we define

$$\hat{f}^\wedge(\hat{g}(\hat{x})) := \frac{f(\hat{T}(x)\hat{g}(\hat{x}))}{\hat{T}(x)} = \frac{f\left(\hat{T}(x)\frac{g\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f\left(g\left(\frac{x}{\hat{T}(x)}\right)\right)}{\hat{T}(x)} \quad \text{for} \quad \forall x \in D.$$

Example 4.0.165. *Let* $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x+1$, $\hat{T}(x) = x^2+1$, $x \in D$. *Then*

$$\hat{f}^\wedge(\hat{g}(\hat{x})) = \frac{f\left(g\left(\frac{x}{\hat{T}(x)}\right)\right)}{\hat{T}(x)} = \frac{f\left(g\left(\frac{x}{x^2+1}\right)\right)}{x^2+1}$$

$$= \frac{f\left(\frac{2x}{x^2+1}+1\right)}{x^2+1} = \frac{\frac{2x}{x^2+1}+1}{x^2+1} = \frac{x^2+2x+1}{(x^2+1)^2}.$$

Definition: Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$, $\hat{T}(x)g(x\hat{T}(x)) \in D$ for every $x \in D$. Then we define

$$\hat{f}^\wedge(g^\wedge(x)) := \frac{f(\hat{T}(x)g^\wedge(x))}{\hat{T}(x)} = \frac{f(\hat{T}(x)g(x\hat{T}(x)))}{\hat{T}(x)} \quad \text{for} \quad \forall x \in D.$$

Example 4.0.166. *Let $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x + 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. Then*

$$\hat{f}^\wedge(g^\wedge(x)) = \frac{f(\hat{T}(x)g(x\hat{T}(x)))}{\hat{T}(x)} = \frac{f((x^2+1)g(x(x^2+1)))}{x^2+1}$$

$$= \frac{f((x^2+1)(2(x^3+x)+1))}{x^2+1} = \frac{(x^2+1)(2x^3+2x+1)}{x^2+1} = 2x^3 + 2x + 1.$$

Definition: Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $\hat{T}(x)g(x) \in D$ for every $x \in D$. Then we define

$$\hat{f}^\wedge(g(x)) := \frac{f(\hat{T}(x)g(x))}{\hat{T}(x)} \quad \text{for} \quad \forall x \in D.$$

Example 4.0.167. *Let $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x + 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. Then*

$$\hat{f}^\wedge(g(x)) = \frac{f(\hat{T}(x)g(x))}{\hat{T}(x)} = \frac{f((x^2+1)(2x+1))}{x^2+1} = \frac{(x^2+1)(2x+1)}{x^2+1} = 2x + 1.$$

Definition: Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $\frac{g(x)}{\hat{T}(x)} \in D$ for every $x \in D$. Then we define

$$\hat{f}(\hat{g}^\wedge(\hat{x})) := \frac{f(\hat{g}^\wedge(\hat{x}))}{\hat{T}(x)} = \frac{f\left(\frac{g\left(\hat{T}(x)\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f\left(\frac{g(x)}{\hat{T}(x)}\right)}{\hat{T}(x)} \quad \text{for} \quad \forall x \in D.$$

Example 4.0.168. *Let $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x + 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. Then*

$$\hat{f}(\hat{g}^\wedge(\hat{x})) = \frac{f\left(\frac{g(x)}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f\left(\frac{2x+1}{x^2+1}\right)}{x^2+1}$$

$$= \frac{\frac{2x+1}{x^2+1}}{x^2+1} = \frac{2x+1}{(x^2+1)^2}.$$

Definition: Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$, $\frac{g(x\hat{T}(x))}{\hat{T}(x)} \in D$ for every $x \in D$. Then we define

$$\hat{f}(\hat{g}^\wedge(x)) := \frac{f(\hat{g}^\wedge(x))}{\hat{T}(x)} = \frac{f\left(\frac{g(x\hat{T}(x))}{\hat{T}(x)}\right)}{\hat{T}(x)} \quad \text{for} \quad \forall x \in D.$$

Example 4.0.169. *Let $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x + 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. Then*

$$\hat{f}(\hat{g}^\wedge(x)) = \frac{f\left(\frac{g(x\hat{T}(x))}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f\left(\frac{g(x(x^2+1))}{x^2+1}\right)}{x^2+1}$$

$$= \frac{\frac{g(x^3+x)}{x^2+1}}{x^2+1} = \frac{2(x^3+x)+1}{(x^2+1)^2} = \frac{2x^3+2x+1}{(x^2+1)^2}.$$

Definition: Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$, $\frac{g\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} \in D$ for every $x \in D$. Then we define

$$\hat{f}(\hat{g}(\hat{x})) := \frac{f(\hat{g}(\hat{x}))}{\hat{T}(x)} = \frac{f\left(\frac{g\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}\right)}{\hat{T}(x)} \quad \text{for} \quad \forall x \in D.$$

Example 4.0.170. *Let $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x + 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. Then*

$$\hat{f}(\hat{g}(\hat{x})) = \frac{f\left(\frac{g\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f\left(\frac{g\left(\frac{x}{x^2+1}\right)}{x^2+1}\right)}{x^2+1}$$

$$= \frac{f\left(\frac{\frac{2x}{x^2+1}+1}{x^2+1}\right)}{x^2+1} = \frac{f\left(\frac{x^2+2x+1}{(x^2+1)^2}\right)}{x^2+1} = \frac{\frac{x^2+2x+1}{(x^2+1)^2}}{x^2+1} = \frac{x^2+2x+1}{(x^2+1)^3}.$$

Definition: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$ $x\hat{T}(x) \in D$ for every $x \in D$, $g : D \longrightarrow D$. Then we define

$$\hat{f}(g^\wedge(x)) := \frac{f(g^\wedge(x))}{\hat{T}(x)} = \frac{f(g(x\hat{T}(x)))}{\hat{T}(x)} \quad \text{for} \quad \forall x \in D.$$

Example 4.0.171. *Let $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x + 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. Then*

$$\hat{f}(g^\wedge(x)) = \frac{f(g(x\hat{T}(x)))}{\hat{T}(x)} = \frac{f(g(x(x^2+1)))}{x^2+1} = \frac{g(x^3+x)}{x^2+1}$$

$$= \frac{2(x^3+x)+1}{x^2+1} = \frac{2x^3+2x+1}{x^2+1}.$$

Definition: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $g : D \longrightarrow D$, $\hat{T}(x) > 0$ for every $x \in D$. Then we define

$$\hat{f}(g(x)) := \frac{f(g(x))}{\hat{T}(x)} \quad \text{for} \quad \forall x \in D.$$

Example 4.0.172. *Let $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x + 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. Then*

$$\hat{f}(g(x)) = \frac{f(g(x))}{\hat{T}(x)} = \frac{g(x)}{x^2+1} = \frac{2x+1}{x^2+1}.$$

Definition: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$ for every $x \in D$, $g : D \longrightarrow D$. Then we define

$$f^\wedge(\hat{g}^\wedge(\hat{x})) := f(\hat{T}(x)\hat{g}^\wedge(\hat{x}))$$

$$= f\Big(\hat{T}(x)\frac{g\Big(\hat{T}(x)\frac{x}{\hat{T}(x)}\Big)}{\hat{T}(x)}\Big) = f(g(x)) \quad \text{for} \quad \forall x \in D.$$

Example 4.0.173. *Let $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x+1$, $\hat{T}(x) = x^2+1$, $x \in D$. Then*

$$f^\wedge(\hat{g}^\wedge(\hat{x})) = f(g(x)) = g(x) = 2x+1.$$

Definition: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$, $g : D \longrightarrow D$. Then we define

$$f^\wedge(\hat{g}^\wedge(x)) := f(\hat{T}(x)\hat{g}^\wedge(x))$$

$$= f\Big(\hat{T}(x)\frac{g(x\hat{T}(x))}{\hat{T}(x)}\Big) = f(g(x\hat{T}(x))) \quad \text{for} \quad \forall x \in D.$$

Example 4.0.174. *Let $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x+1$, $\hat{T}(x) = x^2+1$, $x \in D$. Then*

$$f^\wedge(\hat{g}^\wedge(x)) = f(g(x\hat{T}(x)))$$

$$= g(x(x^2+1)) = g(x^3+x) = 2(x^3+x)+1 = 2x^3+2x+1.$$

Definition: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$ for every $x \in D$, $g : D \longrightarrow D$. Then we define

$$f^\wedge(\hat{g}(\hat{x})) := f\Big(\hat{T}(x)\frac{g\Big(\frac{x}{\hat{T}(x)}\Big)}{\hat{T}(x)}\Big) = f\Big(g\Big(\frac{x}{\hat{T}(x)}\Big)\Big) \quad \text{for} \quad \forall x \in D.$$

Example 4.0.175. *Let* $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x + 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. *Then*

$$f^\wedge(\hat{g}(\hat{x})) = f\Big(g\Big(\tfrac{x}{\hat{T}(x)}\Big)\Big) = g\Big(\tfrac{x}{x^2+1}\Big) = \tfrac{2x}{x^2+1} + 1 = \tfrac{x^2+2x+1}{x^2+1}.$$

Definition: Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$, $\hat{T}(x)g(x\hat{T}(x)) \in D$ for every $x \in D$. Then we define

$$f^\wedge(g^\wedge(x)) := f(\hat{T}(x)g(x\hat{T}(x))) \quad \text{for} \quad \forall x \in D.$$

Example 4.0.176. *Let* $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x + 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. *Then*

$$f^\wedge(g^\wedge(x)) = f(\hat{T}(x)g(x\hat{T}(x))) = \hat{T}(x)g(x\hat{T}(x))$$

$$= (x^2 + 1)g(x(x^2 + 1)) = (x^2 + 1)g(x^3 + x) = (x^2 + 1)(2(x^3 + x) + 1)$$

$$= (x^2 + 1)(2x^3 + 2x + 1) = 2x^5 + 4x^3 + x^2 + 2x + 1.$$

Definition: Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $g(x)\hat{T}(x) \in D$ for every $x \in D$. Then we define

$$f^\wedge(g(x)) := f(\hat{T}(x)g(x)) \quad \text{for} \quad \forall x \in D.$$

Example 4.0.177. *Let* $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x + 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. *Then*

$$f^\wedge(g(x)) = f(\hat{T}(x)g(x)) = \hat{T}(x)g(x) = (x^2 + 1)(2x + 1) = 2x^3 + x^2 + 2x + 1.$$

Definition: Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $\frac{g(x)}{\hat{T}(x)} \in D$ for every $x \in D$. Then we define

$$f(\hat{g}^\wedge(\hat{x})) := f\Bigg(\frac{g\Big(\hat{T}(x)\frac{x}{\hat{T}(x)}\Big)}{\hat{T}(x)}\Bigg) = f\Big(\frac{g(x)}{\hat{T}(x)}\Big) \quad \text{for} \quad \forall x \in D.$$

Example 4.0.178. *Let* $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x + 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. *Then*

$$f(\hat{g}^\wedge(\hat{x})) = f\left(\frac{g(x)}{\hat{T}(x)}\right) = \frac{g(x)}{\hat{T}(x)} = \frac{2x+1}{x^2+1}.$$

Definition: Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$, $\frac{g(x\hat{T}(x))}{\hat{T}(x)} \in D$ for every $x \in D$. Then we define

$$f(\hat{g}^\wedge(x)) := f\left(\frac{g(x\hat{T}(x))}{\hat{T}(x)}\right) \quad \text{for} \quad \forall x \in D.$$

Example 4.0.179. *Let* $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x + 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. *Then*

$$f(\hat{g}^\wedge(x)) = f\left(\frac{g(x\hat{T}(x))}{\hat{T}(x)}\right) = \frac{g(x\hat{T}(x))}{\hat{T}(x)}$$

$$= \frac{2x\hat{T}(x)+1}{\hat{T}(x)} = \frac{2x(x^2+1)+1}{x^2+1} = \frac{2x^3+2x+1}{x^2+1}.$$

Definition: Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$, $\frac{g\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} \in D$ for every $x \in D$. Then we define

$$f(\hat{g}(\hat{x})) := f\left(\frac{g\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}\right) \quad \text{for} \quad \forall x \in D.$$

Example 4.0.180. *Let* $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x + 1$, $\hat{T}(x) = x^2 + 1$,

$x \in D$. *Then*

$$f(\hat{g}(\hat{x})) = f\Big(\frac{g\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}\Big) = \frac{g\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}$$

$$= \frac{\frac{2x}{\hat{T}(x)}+1}{\hat{T}(x)} = \frac{\frac{2x}{x^2+1}+1}{x^2+1} = \frac{x^2+2x+1}{(x^2+1)^2}.$$

Definition: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$, $g : D \longrightarrow D$. Then we define

$$f(g^{\wedge}(x)) := f(g(x\hat{T}(x))) \quad \text{for} \quad \forall x \in D.$$

Example 4.0.181. *Let* $D = \mathbb{R}$, $f(x) = x$, $g(x) = 2x + 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. *Then*

$$f(g^{\wedge}(x)) = f(g(x\hat{T}(x))) = g(x\hat{T}(x))$$

$$= 2x\hat{T}(x) + 1 = 2x(x^2 + 1) + 1 = 2x^3 + 2x + 1.$$

Definition: Let $\hat{f}$ and $\hat{g}$ be isofunctions of first, second, third or fourth kinds, $\tilde{f}$ and $\tilde{g}$ be their isooriginals, respectively. Then we define isomultiplication and multiplication as follows

$$\hat{f}\hat{\times}\hat{g} = \tilde{f}\hat{T}(x)\tilde{g} \quad (\textbf{isomultiplication})$$

$$\hat{f}\hat{g} = \tilde{f}\tilde{g} \quad (\textbf{multiplication}).$$

In particular

$$\hat{f}^2 = \hat{f}\hat{f}, \quad \hat{f}^3 = \hat{f}^2\hat{f} = \hat{f}\hat{f}\hat{f}, \ldots$$

$$\hat{f}^{\hat{2}} = \hat{f}\hat{\times}\hat{f}, \quad \hat{f}^{\hat{3}} = \hat{f}\hat{\times}\hat{f}^{\hat{2}} = \hat{f}\hat{\times}\hat{f}\hat{\times}\hat{f}, \ldots.$$

The above defined isomultiplication and multiplication satisfy commutative, associative and distributive laws.

Example 4.0.182. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) = x^2 - x$, $g(x) = x - 3$, $\hat{T}(x) = x^2 + 1$, $x \in D$. *Then*

$$\hat{f}^\wedge(\hat{x}) \hat{\times} \hat{g}^\wedge(\hat{x}) = \frac{f(x)g(x)}{\hat{T}(x)} = \frac{(x^2-1)(x-3)}{x^2+1} = \frac{x^3-4x^2+3x}{x^2+1},$$

$$\hat{f}^\wedge(\hat{x})\hat{g}^\wedge(\hat{x}) = \frac{f(x)g(x)}{\hat{T}^2(x)} = \frac{(x^2-x)(x-3)}{(x^2+1)^2} = \frac{x^3-4x^2+3x}{(x^2+1)^2},$$

$$\hat{f}^2(\hat{x}) = \frac{f^2(x)}{\hat{T}^2(x)} = \frac{(x^2-x)^2}{(x^2+1)^2},$$

$$\hat{f}^{\hat{2}}(\hat{x}) = \frac{f^2(x)}{\hat{T}(x)} = \frac{(x^2-x)^2}{x^2+1}.$$

Exercise 4.0.183. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = x - 2$, $g(x) = 2x + 1$, $\hat{T}(x) = e^{-x}$, $x \in D$. *Compute*

1) $\hat{f}^\wedge(\hat{x})$,

2) $\hat{g}^\wedge(\hat{x})$,

3) $A := \hat{2}\hat{\times}\hat{g}^\wedge(\hat{x}) + \hat{3}\hat{g}^\wedge(\hat{x}) + \hat{f}^2(\hat{x}) + \hat{2}\hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x}) - \hat{f}^\wedge(\hat{x})\hat{g}^\wedge(\hat{x}) + \hat{g}^{\hat{2}}(\hat{x})$.

Solution.

1) From the definition of isofunction we get

$$\hat{f}^\wedge(\hat{x}) = \frac{f(x)}{T\hat{(x)}} = (x-2)e^x.$$

2) Using the definition of isofunction we obtain

$$\hat{g}^\wedge(\hat{x}) = \frac{g(x)}{\hat{T}(x)} = (2x+1)e^x.$$

3) We have

$$\hat{2}\hat{\times}\hat{f}^\wedge(\hat{x}) = 2\frac{f(x)}{\hat{T}(x)} = 2(x-2)e^x = (2x-4)e^x,$$

$$\hat{3}\hat{g}^\wedge(\hat{x}) = \frac{3}{2}\frac{g(x)}{\hat{T}(x)} = \frac{3}{2}(2x+1)e^x = (3x+\frac{3}{2})e^x,$$

$$\hat{f}^2(\hat{x}) = \frac{f^2(x)}{\hat{T}^2(x)} = (x-2)^2e^{2x} = (x^2-4x+4)e^{2x},$$

$$\hat{2}\hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x}) = 2\frac{1}{2}\frac{f(x)g(x)}{\hat{T}(x)} = (x-2)(2x+1)e^x = (2x^2-3x-2)e^x,$$

$$\hat{f}^\wedge(\hat{x})\hat{g}^\wedge(\hat{x}) = \frac{f(x)g(x)}{\hat{T}^2(x)} = (x-2)(2x+1)e^{2x} = (2x^2-3x-2)e^{2x},$$

$$\hat{g}^{\hat{2}}(\hat{x}) = \frac{g^2(x)}{\hat{T}(x)} = (2x+1)^2e^x = (4x^2+4x+1)e^x.$$

Then

$$A = (2x-4)e^x + (3x+\tfrac{3}{2})e^x + (x^2-4x+4)e^{2x}$$

$$+(2x^2-3x-2)e^x - (2x^2-3x-2)e^{2x} + (4x^2+4x+1)e^x$$

$$= (6x^2+6x-\tfrac{7}{2})e^x + (-x^2-x+6)e^{2x}.$$

Exercise 4.0.184. *Let $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) = x$, $g(x) = x+1$, $\hat{T}(x) = e^x$, $x \in D$. Find*

$$1) \quad A := \hat{f}^2(\hat{x}) - \hat{f}^\wedge(\hat{x})\hat{g}^\wedge(\hat{x}),$$

$$2) \quad B := \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x}),$$

$$3) \quad C := \hat{g}^{\hat{2}}(\hat{x}) - \hat{3}\hat{\times}\hat{f}^\wedge(\hat{x}) + \hat{2}\hat{f}^2(\hat{x}),$$

$$4) \quad D := \hat{f}^\wedge(\hat{x})\hat{\times}(\hat{2}\hat{\times}\hat{f}^\wedge(\hat{x}) - \hat{4}\hat{g}^\wedge(\hat{x})).$$

Answer.

$$1) \quad A = -xe^{-2x}, \quad 2) \quad B = (x^2+x)e^{-x},$$

$$3) \quad C = (x^2-x+1)e^{-x} + \tfrac{2}{3}x^2e^{-2x},$$

$$4) \quad D = \left(\tfrac{2}{3}x^2 - \tfrac{4}{3}x\right)e^{-x}.$$

Exercise 4.0.185. *Let $\hat{f}, \hat{g}, \hat{h} : \hat{D} \widehat{\longrightarrow} \hat{Y}$, $\hat{a}, \hat{b} \in \hat{F_{\mathbb{R}}}$. Prove that*

$$1) \quad \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x}) = \hat{g}^\wedge(\hat{x})\hat{\times}\hat{f}^\wedge(\hat{x}),$$

$$2) \quad \hat{f}^\wedge(\hat{x})\hat{\times}(\hat{g}^\wedge(\hat{x}) + \hat{h}^\wedge(\hat{x})) = \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x}) + \hat{f}^\wedge(\hat{x})\hat{\times}\hat{h}^\wedge(\hat{x}),$$

$$3) \quad \hat{a}\hat{\times}(\hat{f}^\wedge(\hat{x}) + \hat{g}^\wedge(\hat{x})) = \hat{a}\hat{\times}\hat{f}^\wedge(\hat{x}) + \hat{a}\hat{\times}\hat{g}^\wedge(\hat{x}),$$

$$4) \quad (\hat{a}+\hat{b})\hat{\times}\hat{f}^\wedge(\hat{x}) = \hat{a}\hat{\times}\hat{f}^\wedge(\hat{x}) + \hat{b}\hat{\times}\hat{f}^\wedge(\hat{x}).$$

Exercise 4.0.186. *Let* $\hat{f}, \hat{g}, \hat{h} : \hat{D} \widehat{\longrightarrow} \hat{Y}$, $\hat{a}, \hat{b} \in \hat{F}_{\mathbb{R}}$. *Prove that*

$$\begin{aligned}
&1) \quad \hat{f}^{\wedge}(\hat{x})\hat{g}^{\wedge}(\hat{x}) = \hat{g}^{\wedge}(\hat{x})\hat{f}^{\wedge}(\hat{x}),\\
&2) \quad \hat{f}^{\wedge}(\hat{x})(\hat{g}^{\wedge}(\hat{x}) + \hat{h}^{\wedge}(\hat{x})) = \hat{f}^{\wedge}(\hat{x})\hat{g}^{\wedge}(\hat{x}) + \hat{f}^{\wedge}(\hat{x})\hat{h}^{\wedge}(\hat{x}),\\
&3) \quad \hat{a}(\hat{f}^{\wedge}(\hat{x}) + \hat{g}^{\wedge}(\hat{x})) = \hat{a}\hat{f}^{\wedge}(\hat{x}) + \hat{a}\hat{g}^{\wedge}(\hat{x}),\\
&4) \quad (\hat{a} + \hat{b})\hat{f}^{\wedge}(\hat{x}) = \hat{a}\hat{f}^{\wedge}(\hat{x}) + \hat{b}\hat{f}^{\wedge}(\hat{x}).
\end{aligned}$$

Exercise 4.0.187. *Let* $\hat{f}, \hat{g}, \hat{h} : \hat{D} \widehat{\longrightarrow} \hat{Y}$. *Prove that*

$$\hat{f}^{\wedge}(\hat{x})\hat{\times}(\hat{g}^{\wedge}(\hat{x})\hat{h}^{\wedge}(\hat{x})) = (\hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge}(\hat{x}))\hat{h}^{\wedge}(\hat{x}).$$

Example 4.0.188. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) = x + 1$, $\hat{T}(x) = x^2 + x + 1$, $x \in D$. *Then* $x\hat{T}(x) : D \longrightarrow D$ *and*

$$\hat{f}(x) = \frac{f(\hat{T}(x)x)}{\hat{T}(x)} = \frac{f((x^2+x+1)x)}{x^2+x+1} = \frac{f(x^3+x^2+x)}{x^2+x+1} = \frac{x^3+x^2+x+1}{x^2+x+1}.$$

Example 4.0.189. *Let* $D = [0, 1]$, $\hat{T}_1 = 2$, $f(x) = x^2 + 1$, $\hat{T}(x) = x^2 + 2$, $x \in D$. *Then* $x\hat{T}(x)$ *does not image* D *in* D *and therefore we can not consider* $\hat{f}(x)$.

Exercise 4.0.190. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = x - 2$, $g(x) = x + 2$, $\hat{T}(x) = e^{-x}$, $x \in D$. *Find*

$$\begin{aligned}
&1) \quad \hat{f}(x) + \hat{g}^{\wedge}(\hat{x}),\\
&2) \quad \hat{f}(x) - \hat{g}(x),\\
&3) \quad \hat{2}\hat{\times}\hat{f}(x) + \hat{4}\hat{g}(x).
\end{aligned}$$

Solution.

1) We have

$$\hat{f}(x) = \frac{f(\hat{T}(x)x)}{\hat{T}(x)} = \frac{f(e^{-x}x)}{e^{-x}} = e^x(xe^{-x} - 2) = x - 2e^x,$$

$$\hat{g}^{\wedge}(\hat{x}) = \frac{g(x)}{\hat{T}(x)} = \frac{x+2}{e^{-x}} = e^x(x + 2).$$

From here

$$\hat{f}(x) + \hat{g}^{\wedge}(\hat{x}) = x - 2e^x + e^x(x + 2) = x + xe^x.$$

2) We have

$$\hat{g}(x) = \frac{g(\hat{T}(x))}{\hat{T}(x)} = \frac{g(e^{-x}x)}{e^{-x}} = e^x(xe^{-x}+2) = x + 2e^x.$$

From here

$$\hat{f}(x) - \hat{g}(x) = x - 2e^x - x - 2e^x = -4e^x.$$

3) We have

$$\hat{2}\hat{\times}\hat{f}^{\wedge}(\hat{x}) = \frac{2f(x)}{\hat{T}(x)} = \frac{2(x-2)}{e^{-x}} = (2x-4)e^x,$$

also

$$\hat{4}\hat{g}(x) = \frac{4}{2}(x+2e^x) = 2x + 4e^x.$$

Therefore

$$\hat{2}\hat{\times}\hat{f}(x) + \hat{4}\hat{g}(x) = (2x-4)e^x + 2x + 4e^x = 2xe^x + 2x.$$

Exercise 4.0.191. *Let $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) = 3 - 0x$, $g(x) = x + 1$, $\hat{T}(x) = 1 + \sin^2 x$, $x \in D$. Find*

1) $A := \hat{f}^{\wedge}(\hat{x}) - \hat{2}\hat{\times}\hat{g}(x)$,

2) $B := \hat{f}^{\hat{2}}(\hat{x}) - \hat{g}^{\wedge}(\hat{x}) + \hat{g}(x)$,

3) $C := \hat{f}(x) - \hat{g}(x)$.

Answer.

$$1) \quad A = \frac{1-x}{1+\sin^2 x} - 2x, \quad 2) \quad B = \frac{x^2-6x+x\sin^2 x+9}{1+\sin^2 x}, 3) \quad C = \frac{2}{1+\sin^2 x} - 2x.$$

Example 4.0.192. *Let $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) = x$, $g(x) = x - 1$, $\hat{T}(x) = 1 + x^2$. Then*

$$\hat{f}(\hat{g}(x)) = \frac{f(g(\hat{T}(x)x))}{\hat{T}(x)} = \frac{f(\hat{T}(x)x-1)}{\hat{T}(x)} = \frac{\hat{T}(x)x-1}{\hat{T}(x)}$$

$$= x - \frac{1}{\hat{T}(x)} = x - \frac{1}{1+x^2} = \frac{x^3+x-1}{1+x^2},$$

$$\hat{f}(\hat{f}(x)) = \frac{f(f(\hat{T}(x)))}{\hat{T}(x)} = \frac{f(\hat{T}(x)x)}{\hat{T}(x)} = \frac{\hat{T}(x)x}{\hat{T}(x)} = x,$$

$$\hat{g}(\hat{g}(x)) = \frac{g(g(\hat{T}(x)x))}{\hat{T}(x)} = \frac{g(\hat{T}(x)x-1)}{\hat{T}(x)}$$

$$= \frac{\hat{T}(x)x-2}{\hat{T}(x)} = x - \frac{2}{\hat{T}(x)} = x - \frac{2}{1+x^2} = \frac{x^3+x-2}{x^2+1}.$$

Exercise 4.0.193. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = 1 + x$, $g(x) = x$, $\hat{T}(x) = 1 + x^4$, $x \in D$. *Find*

$$1) \quad A := \hat{f}(x) + \hat{f}(\hat{g}(x)),$$

$$2) \quad B := \hat{2}\hat{\times}\hat{f}(\hat{f}^{\wedge}(\hat{x})) + \hat{g}(\hat{g}(x)),$$

$$3) \quad C := \hat{f}(\hat{f}^{\wedge}(\hat{x})) + \hat{f}^{\wedge}(\hat{x}) - \hat{g}^{\wedge}(\hat{x}).$$

Solution.

1) We have

$$\hat{f}(x) = \frac{f(\hat{T}(x)x)}{\hat{T}(x)} = \frac{1 + x\hat{T}(x)}{\hat{T}(x)} = x + \frac{1}{\hat{T}(x)} = x + \frac{1}{x^4 + 1} = \frac{x^5 + x + 1}{x^4 + 1},$$

$$\hat{f}(\hat{g}(x)) = \frac{f(g(\hat{T}(x)x))}{\hat{T}(x)} = \frac{\hat{T}(x)x + 1}{\hat{T}(x)} = x + \frac{1}{\hat{T}(x)} = x + \frac{1}{x^4 + 1} = \frac{x^5 + x + 1}{x^4 + 1}.$$

Consequently

$$A = \frac{x^5 + x + 1}{x^4 + 1} + \frac{x^5 + x + 1}{x^4 + 1} = 2\frac{x^5 + x + 1}{x^4 + 1}.$$

2) We have

$$\hat{f}(\hat{f}^{\wedge}(\hat{x})) = \frac{f(f(x))}{\hat{T}(x)} = \frac{f(1 + x)}{x^4 + 1} = \frac{2 + x}{x^4 + 1},$$

$$\hat{2}\hat{\times}\hat{f}(\hat{f}^{\wedge}(\hat{x})) = \frac{2f(f(x))}{\hat{T}(x)} = \frac{2f(x + 1)}{\hat{T}(x)} = \frac{2(x + 2)}{x^4 + 1} = \frac{2x + 4}{x^4 + 1},$$

$$\hat{g}(\hat{g}(x)) = \frac{g(g(\hat{T}(x)x))}{\hat{T}(x)} = \frac{g(\hat{T}(x)x)}{\hat{T}(x)} = \frac{\hat{T}(x)x}{\hat{T}(x)} = x,$$

therefore

$$B = \frac{4 + 2x}{1 + x^4} + x = \frac{x^5 + 3x + 4}{1 + x^4}.$$

3) We have

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{1 + x}{1 + x^4},$$

$$\hat{g}^{\wedge}(\hat{x}) = \frac{g(x)}{\hat{T}(x)} = \frac{x}{1 + x^4}.$$

From here

$$C = \frac{2+x}{1+x^4} + \frac{1+x}{1+x^4} - \frac{x}{1+x^4} = \frac{3+x}{1+x^4}.$$

Exercise 4.0.194. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) = 2 + x$, $g(x) = x$, $\hat{T}(x) = 1 + x^2$, $x \in D$. *Find*

$$A := \hat{f}(\hat{f}(\hat{f}^\wedge(\hat{x}))) + \hat{g}(\hat{f}(\hat{g}(x))).$$

Answer.

$$A = \frac{x^3 + 2x + 8}{x^2 + 1}.$$

Example 4.0.195. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) = x$, $g(x) = x + 1$, $\hat{T}(x) = 1 + x^2$, $x \in D$. *Then*

$$\hat{f}(x)\hat{\times}\hat{g}(x) = \frac{f(\hat{T}(x)x)g(\hat{T}(x)x)}{\hat{T}(x)} = \frac{\hat{T}(x)x(\hat{T}(x)x+1)}{\hat{T}(x)}$$

$$= x(\hat{T}(x)x + 1) = x(x^3 + x + 1) = x^4 + x^2 + x,$$

$$\hat{f}(x)\hat{\times}\hat{g}(x) = \frac{f(\hat{T}(x)x)g(\hat{T}(x)x)}{\hat{T}^2(x)} = \frac{\hat{T}(x)x(\hat{T}(x)+1)}{\hat{T}^2(x)} = \frac{x\hat{T}(x)(\hat{T}(x)x+1)}{\hat{T}^2(x)}$$

$$= x^2 + \frac{x}{\hat{T}(x)} = x^2 + \frac{x}{1+x^2} = \frac{x^4+x^2+x}{1+x^2},$$

$$\hat{f}^{\hat{2}}(x) = \frac{f^2(\hat{T}(x)x)}{\hat{T}(x)} = \frac{\hat{T}^2(x)x^2}{\hat{T}(x)} = \hat{T}(x)x^2 = (1+x^2)x^2 = x^2 + x^4,$$

$$\hat{f}^2(x) = \frac{f^2(\hat{T}(x)x)}{\hat{T}^2(x)} = \frac{\hat{T}^2(x)x^2}{\hat{T}^2(x)} = x^2.$$

Exercise 4.0.196. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = 2x + 1$, $g(x) = 3x + 2$, $\hat{T}(x) = x^4 + 1$, $x \in D$. *Find*

1) $\hat{f}(x)$, $\hat{f}^\wedge(\hat{x})$, $\hat{g}(x)$, $\hat{g}^\wedge(\hat{x})$,

2) $A := \hat{f}(x)\hat{\times}\hat{g}(x) + \hat{3}\hat{g}^\wedge(\hat{x})$,

3) $B := \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}(x) - \hat{f}(\hat{g}^\wedge(\hat{x}))$,

4) $C := \hat{f}^{\hat{2}}(\hat{x}) - \hat{g}(x) - \hat{4}\hat{\times}\hat{f}^\wedge(\hat{x})$.

Solution.

1) We have

$$\hat{f}(x) = \frac{f(\hat{T}(x)x)}{\hat{T}(x)} = \frac{f((x^4+1)x)}{x^4+1} = \frac{f(x^5+x)}{x^4+1} = \frac{2x^5+2x+1}{x^4+1},$$

$$\hat{f}^\wedge(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{2x+1}{x^4+1},$$

$$\hat{g}(x) = \frac{g(\hat{T}(x)x)}{\hat{T}(x)} = \frac{g(x^5+x)}{x^4+1} = \frac{3x^5+3x+2}{x^4+1},$$

$$\hat{g}^\wedge(\hat{x}) = \frac{g(x)}{\hat{T}(x)} = \frac{3x+2}{x^4+1}.$$

2) We have

$$\hat{f}(x)\hat{\times}\hat{g}(x) = \frac{f(\hat{T}(x)x)g(\hat{T}(x)x)}{\hat{T}(x)} = \frac{f(x^5+x)g(x^5+x)}{x^4+1}$$

$$= \frac{(2x^5+2x+1)(3x^5+3x+2)}{x^4+1} = \frac{6x^{10}+12x^6+7x^5+6x^2+7x+2}{x^4+1},$$

$$\hat{3}\hat{g}^\wedge(\hat{x}) = \frac{3}{2}\frac{g(x)}{\hat{T}(x)} = \frac{3}{2}\frac{3x+2}{x^4+1} = \frac{\frac{9}{2}x+3}{x^4+1}.$$

Then

$$A = \frac{6x^{10}+12x^6+7x^5+6x^2+7x+2}{x^4+1} + \frac{\frac{9}{2}x+3}{x^4+1} = \frac{6x^{10}+12x^6+7x^5+6x^2+\frac{23}{2}x+5}{x^4+1}.$$

3) We have

$$\hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x}) = \frac{f(x)g(x)}{\hat{T}(x)} = \frac{(2x+1)(3x+2)}{x^4+1} = \frac{6x^2+7x+2}{x^4+1},$$

$$\hat{f}(\hat{g}^\wedge(\hat{x})) = \frac{f(g(x))}{\hat{T}(x)} = \frac{f(3x+2)}{x^4+1} = \frac{2(3x+2)+1}{x^4+1} = \frac{6x+5}{x^4+1},$$

$$B = \frac{6x^2+7x+2}{x^4+1} - \frac{6x+5}{x^4+1} = \frac{6x^2+x-3}{x^4+1}.$$

4) We have

$$\hat{f}^{\hat{2}}(\hat{x}) = \frac{f^2(x)}{\hat{T}(x)} = \frac{(2x+1)^2}{x^4+1} = \frac{4x^2+4x+1}{x^4+1},$$

$$\hat{4}\hat{\times}\hat{f}^\wedge(\hat{x}) = 4\frac{f(x)}{\hat{T}(x)} = 4\frac{2x+1}{x^4+1} = \frac{8x+4}{x^4+1},$$

$$C = \frac{4x^2+4x+1}{x^4+1} - \frac{3x^5+3x+2}{x^4+1} - \frac{8x+4}{x^4+1} = \frac{-3x^5+4x^2-7x-5}{x^4+1}.$$

Exercise 4.0.197. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 4$, $f(x) = x$, $g(x) = x + 1$, $\hat{T}(x) = x^6 + 1$, $x \in D$. *Find*

1) $\hat{f}(x)$, $\hat{f}^\wedge(\hat{x})$, $\hat{g}(x)$, $\hat{g}^\wedge(\hat{x})$,

2) $A := \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x}) - \hat{f}(x)\hat{\times}\hat{g}(x)$,

3) $B := \hat{f}^{\hat{2}}(\hat{x}) - \hat{g}^{\hat{3}}(\hat{x})$,

4) $C := \hat{f}^2(x) - \hat{f}(x)\hat{g}(x)$.

Answer.

1) $\hat{f}(x) = x, \quad \hat{f}^\wedge(\hat{x}) = \frac{x}{x^6+1}, \quad \hat{g}(x) = x + \frac{1}{x^6+1}, \quad \hat{g}^\wedge(\hat{x}) = \frac{x+1}{x^6+1},$

2) $A = \frac{x^2+x}{x^6+1} - x^8 - x^2 - x,$ 3) $B = \frac{-x^3-2x^2-3x-1}{x^6+1},$ 4) $C = -\frac{x}{x^6+1}.$

Exercise 4.0.198. *Let* $\hat{f}, \hat{g}, \hat{h} : D\widehat{\longrightarrow}Y$. *Prove*

1) $\hat{f}(x)\hat{\times}\hat{g}(x) = \hat{g}(x)\hat{\times}\hat{f}(x)$,

2) $\hat{f}(x)\hat{g}(x) = \hat{g}(x)\hat{f}(x)$,

3) $\hat{f}(x)\hat{\times}(\hat{g}(x)\hat{h}(x)) = (\hat{f}(x)\hat{\times}\hat{g}(x))\hat{h}(x)$,

4) $\hat{f}(x)\hat{\times}(\hat{g}(x) + \hat{h}(x)) = \hat{f}(x)\hat{\times}\hat{g}(x) + \hat{f}(x)\hat{\times}\hat{h}(x)$,

5) $\hat{f}(x)(\hat{g}(x) + \hat{h}(x)) = \hat{f}(x)\hat{g}(x) + \hat{f}(x)\hat{h}(x)$.

Exercise 4.0.199. *Let* $\hat{f}, \hat{g} : D\widehat{\longrightarrow}Y$, $\hat{h} : \hat{D}\widehat{\longrightarrow}\hat{Y}$. *Prove*

1) $\hat{f}(x)\hat{\times}\hat{h}^\wedge(\hat{x}) = \hat{h}^\wedge(\hat{x})\hat{\times}\hat{f}(x)$,

2) $\hat{f}(x)\hat{h}^\wedge(\hat{x}) = \hat{h}^\wedge(\hat{x})\hat{f}(x)$,

3) $\hat{f}(x)\hat{\times}(\hat{g}(x)\hat{h}^\wedge(\hat{x})) = (\hat{f}(x)\hat{\times}\hat{g}(x))\hat{h}^\wedge(\hat{x})$,

4) $\hat{f}(x)\hat{\times}(\hat{g}(x) + \hat{h}^\wedge(\hat{x})) = \hat{f}(x)\hat{\times}\hat{g}(x) + \hat{f}(x)\hat{\times}\hat{h}^\wedge(\hat{x})$,

5) $\hat{f}(x)(\hat{g}(x) + \hat{h}^\wedge(\hat{x})) = \hat{f}(x)\hat{g}(x) + \hat{f}(x)\hat{h}^\wedge(\hat{x})$.

Exercise 4.0.200. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = x$, $x \in D$. *Find* $\hat{T}(x)$, $x \in D$ *so that*

$$\hat{f}^{\hat{2}}(\hat{x}) + \hat{1} = \hat{x}.$$

Solution. The given equation is equivalent of the equation

$$\frac{f(x)}{\hat{T}(x)} + \frac{1}{2} = \frac{x}{\hat{T}(x)}$$

or

$$\frac{x}{\hat{T}(x)} + \frac{1}{2} = \frac{x}{\hat{T}(x)},$$

from where $\frac{1}{2} = 0$, which is impossible. Consequently ther is not such isotopic element $\hat{T}(x)$.

Exercise 4.0.201. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = x$, $x \in D$. *Find* $\hat{T}(x)$, $x \in D$, *such that*

$$\hat{f}^{\hat{2}}(\hat{x}) = x^2(x^2+1).$$

Solution. The given equation is equivalent of the equation

$$\frac{f^2(x)}{\hat{T}(x)} = x^2(x^2+1)$$

or

$$\frac{x^2}{\hat{T}(x)} = x^2(x^2+1),$$

from where

$$\hat{T}(x) = \frac{1}{x^2+1}.$$

Exercise 4.0.202. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) = x$, $x \in D$. *Find* $\hat{T}(x)$, $x \in D$ *so that*

$$\hat{f}(x) = x.$$

Solution. The given equation is equivalent of the equation

$$\frac{f(\hat{T}(x)x)}{\hat{T}(x)} = x$$

or

$$\frac{\hat{T}(x)x}{\hat{T}(x)} = x$$

or

$$x = x.$$

Consequently every positive function $\hat{T}$ on D will be satisfied the given equation.

Exercise 4.0.203. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 8$, $f(x) = x^2$, $x \in D$. *Find* $\hat{T}(x)$, $x \in D$ *so that*

$$\hat{f}(x) = x^2(x^6 + x^3 + 1).$$

Solution. The given equation is equivalent of the following equation

$$\frac{f(\hat{T}(x)x)}{\hat{T}(x)} = x^2(x^6 + x^3 + 1)$$

or

$$\frac{\hat{T}^2(x)x^2}{\hat{T}(x)} = x^2(x^6 + x^3 + x)$$

or

$$\hat{T}(x)x^2 = x^2(x^6 + x^3 + 1),$$

therefore

$$\hat{T}(x) = x^6 + x^3 + 1.$$

Definition: An isofunction of first, second, third or fourth kind will be called bounded below if its isooriginal is a bounded below function.

Example 4.0.204. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = x^2 + 1$, $g(x) = x^{10} + 1$, $\hat{T}(x) = x^4 + 1$, $x \in D$. *Then*

$$\hat{f}^\wedge(\hat{x}) = \tfrac{f(x)}{\hat{T}(x)} = \tfrac{x^2+1}{x^4+1},$$

$$\hat{g}^\wedge(\hat{x}) = \tfrac{g(x)}{\hat{T}(x)} = \tfrac{x^{10}+1}{x^4+1}.$$

The isofunction $\hat{f}$ *is a bounded below isofunction on* D *and* $\hat{g}$ *is not bounded below isofunction on* $\hat{D}$. *The function* f *is unbounded below function.*

Definition: An isofunction of first, second, third or fourth kind will be called bounded above if its isooriginal is a bounded above function.

Example 4.0.205. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) = e^{-x^2}$, $g(x) = e^{2x^2}$, $\hat{T}(x) = e^{x^2}$, $x \in D$. *Then*

$$\hat{f}^\wedge(\hat{x}) = \tfrac{f(x)}{\hat{T}(x)} = \tfrac{e^{-x^2}}{e^{x^2}} = e^{-2x^2},$$

$$\hat{g}^\wedge(\hat{x}) = \tfrac{g(x)}{\hat{T}(x)} = \tfrac{e^{2x^2}}{e^{x^2}} = e^{x^2}.$$

Then $\hat{f}^{\wedge}(\hat{x})$ is a bounded above isofunction on D and $\hat{g}^{\wedge}(\hat{x})$ is unbounded above isofunction.

Definition: An isofunction of first, second, third or fourth kind will be called bounded isofunction if its isooriginal is bounded function.

Example 4.0.206. *Let $D = \mathbb{R}$, $\hat{T}_1 = 4$, $f(x) = x$, $g(x) = x^7 + x$, $\hat{T}(x) = x^6 + 1$, $x \in D$. Then*

$$\hat{f}^{\wedge}(\hat{x}) = \tfrac{f(x)}{\hat{T}(x)} = \tfrac{x}{x^6+1},$$

$$\hat{g}^{\wedge}(\hat{x}) = \tfrac{g(x)}{\hat{T}(x)} = \tfrac{x^7+x}{x^6+1}.$$

Then $\hat{f}^{\wedge}(\hat{x})$ is a bounded isofunction on D and $\hat{g}$ is unbounded isofunction on D.

Theorem: Let $f : D \longrightarrow Y$ is a bounded below function, $\hat{T} : D \longrightarrow Y$ is a bounded above positive function. Then there exists a constant M such that

$$\hat{f}^{\wedge}(\hat{x}) \geq M \quad \text{for} \quad \forall x \in D.$$

In other words the considered isofunction of first kind is bounded below.

Proof. Since $f : D \longrightarrow Y$ is a bounded below function then there exists $a \in \mathbb{R}$ such that

$$f(x) \geq a \quad \text{for} \quad \forall x \in D.$$

Because $\hat{T} : D \longrightarrow Y$ is a bounded above positive function then there is $b \in \mathbb{R}$, $b > 0$ such that

$$\hat{T}(x) \leq b \quad \text{for} \quad \forall x \in D \quad \text{for} \quad \forall x \in D.$$

Therefore

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)} \geq \frac{a}{\hat{T}(x)} \geq \frac{a}{b} = M.$$

$\square$

As in above one can prove the following Theorems.

Theorem: Let $f : D \longrightarrow Y$ is a bounded below function, $\hat{T} : D \longrightarrow Y$ is a bounded above positive function, $x\hat{T}(x) \in D$ for every $x \in D$. Then there exists a constant M such that

$$\hat{f}^{\wedge}(x) \geq M \quad \text{for} \quad \forall x \in D.$$

In other words the considered isofunction of second kind is bounded below.

Theorem: Let $f : D \longrightarrow Y$ is a bounded below function, $\hat{T} : D \longrightarrow Y$ is a bounded above positive function, $\frac{x}{\hat{T}(x)} \in D$ for every $x \in D$. Then there exists a constant M such that

$$\hat{f}(\hat{x}) \geq M \quad \text{for} \quad \forall x \in D.$$

In other words the considered isofunction of third kind is bounded below.

Theorem: Let $f : D \longrightarrow Y$ is a bounded below function, $\hat{T} : D \longrightarrow Y$ is a bounded above positive function, $x\hat{T}(x) \in D$ for every $x \in D$. Then there exists a constant M such that

$$f^{\wedge}(x) \geq M \quad \text{for} \quad \forall x \in D.$$

In other words the considered isofunction of fourth kind is bounded below.

Theorem: Let $f : D \longrightarrow Y$ is a bounded above function, $\hat{T} : D \longrightarrow Y$ is a bounded below positive function. Then there exists a constant N such that

$$\hat{f}^{\wedge}(\hat{x}) \leq N \quad \text{for} \quad \forall x \in D.$$

Proof. Since $f : D \longrightarrow Y$ is a bounded above function then there exists $a \in \mathbb{R}$ such that

$$f(x) \leq a \quad \text{for} \quad \forall x \in D.$$

Because $\hat{T} : D \longrightarrow Y$ is a bounded below positive function then there is $b \in \mathbb{R}$, $b > 0$ such that

$$\hat{T}(x) \geq b \quad \text{for} \quad \forall x \in D.$$

Therefore

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)} \leq \frac{a}{\hat{T}(x)} \leq \frac{a}{b} = N \quad \text{for} \quad \forall x \in D,$$

i.e. the considered isofunction of first kind is bounded above. $\square$

As in above one can prove the following Theorems.

Theorem: Let $f : D \longrightarrow Y$ is a bounded above function, $\hat{T} : D \longrightarrow Y$ is a bounded below positive function, $x\hat{T}(x) \in D$ for every $x \in D$. Then there exists a constant N such that

$$\hat{f}^{\wedge}(x) \leq N \quad \text{for} \quad \forall x \in D.$$

Theorem: Let $f : D \longrightarrow Y$ is a bounded above function, $\hat{T} : D \longrightarrow Y$ is a bounded below positive function, $\frac{x}{\hat{T}}(x) \in D$ for every $x \in D$. Then there exists a constant N such that

$$\hat{f}(\hat{x}) \leq N \quad \text{for} \quad \forall x \in D.$$

Theorem: Let $f : D \longrightarrow Y$ is a bounded above function, $\hat{T} : D \longrightarrow Y$ is a bounded below positive function, $x\hat{T}(x) \in D$ for every $x \in D$. Then there exists a constant N such that

$$f^{\wedge}(x) \leq N \quad \text{for} \quad \forall x \in D.$$

Theorem: Let $f : D \longrightarrow Y$ is a bounded function, $\hat{T} : D \longrightarrow Y$ is a positive bounded below function. Then there exists a positive constant M such that

$$|\hat{f}^{\wedge}(\hat{x})| \leq M \quad \text{for} \quad \forall x \in D.$$

Proof. Since $f : D \longrightarrow Y$ is a bounded function then there exists $a \in \mathbb{R}$ such that

$$|f(x)| \leq a \quad \text{for} \quad \forall x \in D.$$

Because $\hat{T} : D \longrightarrow Y$ is a positive bounded below function then there exists $b \in \mathbb{R}$, $b > 0$ such that

$$\hat{T}(x) \geq b \quad \text{for} \quad \forall x \in \mathbb{R}.$$

From here

$$|\hat{f}^\wedge(\hat{x})| = \left|\frac{f(x)}{\hat{T}(x)}\right| = \frac{|f(x)|}{\hat{T}(x)}$$

$$\leq \frac{a}{\hat{T}(x)} \leq \frac{a}{b} = N \quad \text{for} \quad \forall x \in D.$$

$\square$

Corollary: Let $f : D \longrightarrow Y$ is a bounded function, $\hat{T} : D \longrightarrow Y$ is a positive bounded function. Then there exists a positive constant M such that

$$|\hat{f}^\wedge(\hat{x})| \leq M \quad \text{for} \quad \forall x \in D.$$

Theorem: Let $f : D \longrightarrow Y$ is a bounded function, $\hat{T} : D \longrightarrow Y$ is a positive bounded below function, $x\hat{T}(x) \in D$ for every $x \in D$. Then there exists a positive constant M such that

$$|\hat{f}^\wedge(x)| \leq M \quad \text{for} \quad \forall x \in D.$$

Corollary: Let $f : D \longrightarrow Y$ is a bounded function, $\hat{T} : D \longrightarrow Y$ is a positive bounded function, $x\hat{T}(x) \in D$ for every $x \in D$. Then there exists a positive constant M such that

$$|\hat{f}^\wedge(x)| \leq M \quad \text{for} \quad \forall x \in D.$$

Theorem: Let $f : D \longrightarrow Y$ is a bounded function, $\hat{T} : D \longrightarrow Y$ is a positive bounded below function, $\frac{x}{\hat{T}(x)} \in D$ for every $x \in D$. Then there exists a positive constant M such that

$$|\hat{f}(\hat{x})| \leq M \quad \text{for} \quad \forall x \in D.$$

Corollary: Let $f : D \longrightarrow Y$ is a bounded function, $\hat{T} : D \longrightarrow Y$ is a positive bounded function, $\frac{x}{\hat{T}(x)} \in D$ for every $x \in D$. Then there exists a positive constant M such that

$$|\hat{f}(\hat{x})| \leq M \quad \text{for} \quad \forall x \in D.$$

Theorem: Let $f : D \longrightarrow Y$ is a bounded function, $\hat{T} : D \longrightarrow Y$ is a positive bounded below function, $x\hat{T}(x) \in D$ for every $x \in D$. Then there exists a positive constant M such that

$$|f^{\wedge}(x)| \leq M \quad \text{for} \quad \forall x \in D.$$

Corollary: Let $f : D \longrightarrow Y$ is a bounded function, $\hat{T} : D \longrightarrow Y$ is a positive bounded function, $x\hat{T}(x) \in D$ for every $x \in D$. Then there exists a positive constant M such that

$$|f^{\wedge}(x)| \leq M \quad \text{for} \quad \forall x \in D.$$

Definition: Let f and g are isofunctions of first, second, third or fourth kind on D with isooriginals $\tilde{f}$, $\tilde{g}$, respectively, $\tilde{f} = \tilde{f}(x)$, $\tilde{g} = \tilde{g}(x)$, $x \in D$. Then isodivision of f and g we define as follows

$$f \nearrow g(x) := \frac{1}{\hat{T}(x)} \frac{\tilde{f}(x)}{\tilde{g}(x)}, \quad x \in D, \tilde{g}(x) \neq 0.$$

Then for $x \in D$, $\tilde{f}(x), \tilde{g}(x) \neq 0$, we have

$$\Big(f \nearrow g(x)\Big) \hat{\times} \Big(g \nearrow f(x)\Big) = \frac{1}{\hat{T}(x)} \frac{\tilde{f}(x)}{\tilde{g}(x)} \hat{T}(x) \frac{1}{\hat{T}(x)} \frac{\tilde{g}(x)}{\tilde{f}(x)} = \frac{1}{\hat{T}(x)}.$$

Example 4.0.207. *Let $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) = x$, $g(x) = 2x - 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. Then*

$$\hat{f}^\wedge(\hat{x}) \nearrow \hat{g}^\wedge(\hat{x}) = \tfrac{1}{\hat{T}(x)} \tfrac{\hat{f}^\wedge(\hat{x})}{\hat{g}^\wedge(\hat{x})} = \tfrac{1}{\hat{T}(x)} \tfrac{\frac{f(x)}{\hat{T}(x)}}{\frac{g(x)}{\hat{T}(x)}} = \hat{T}(x) \tfrac{f(x)}{g(x)}$$

$$= \tfrac{1}{x^2+1} \tfrac{x}{2x-1} = \tfrac{x}{(x^2+1)(2x-1)}, \quad x \neq \tfrac{1}{2}, \quad x \in D,$$

$$\hat{f}^\wedge(\hat{x}) \nearrow \hat{g}^\wedge(x) = \tfrac{1}{\hat{T}(x)} \tfrac{\hat{f}^\wedge(\hat{x})}{\hat{g}^\wedge(x)} = \tfrac{1}{\hat{T}(x)} \tfrac{\frac{f(x)}{\hat{T}(x)}}{\frac{g(x\hat{T}(x))}{\hat{T}(x)}} = \tfrac{1}{\hat{T}(x)} \tfrac{f(x)}{g(x\hat{T}(x))}$$

$$= \tfrac{1}{x^2+1} \tfrac{x}{g((x^2+1)x)} = \tfrac{x}{(x^2+1)g(x^3+x)} = \tfrac{x}{(x^2+1)(2x^3+2x-1)}, \quad 2x^3 + x - 1 \neq 0, x \in D.$$

Exercise 4.0.208. *Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $g(x) \neq 0$ for every $x \in D$. Prove*

$$\hat{f}^\wedge(\hat{x}) \nearrow \hat{g}^\wedge(\hat{x}) = \frac{1}{\hat{T}(x)} \frac{f(x)}{g(x)}, \quad x \in D.$$

Solution. For $x \in D$ we have

$$\hat{f}^\wedge(\hat{x}) \nearrow \hat{g}^\wedge(\hat{x}) = \tfrac{1}{\hat{T}(x)} \tfrac{\hat{f}^\wedge(\hat{x})}{\hat{g}^\wedge(\hat{x})} = \tfrac{1}{\hat{T}(x)} \tfrac{\frac{f(x)}{\hat{T}(x)}}{\frac{g(x)}{\hat{T}(x)}} = \tfrac{1}{\hat{T}(x)} \tfrac{f(x)}{g(x)}.$$

Exercise 4.0.209. *Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$, $g(x) \neq 0$ for every $x \in D$. Prove*

$$\hat{f}^\wedge(\hat{x}) \nearrow \hat{g}^\wedge(x) = \frac{1}{\hat{T}(x)} \frac{f(x)}{g(x\hat{T}(x))} \quad \forall x \in D.$$

Exercise 4.0.210. *Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$, $g(x) \neq 0$ for every $x \in D$. Prove*

$$\hat{f}^{\wedge}(\hat{x}) \nearrow \hat{g}(\hat{x}) = \frac{1}{\hat{T}(x)} \frac{f(x)}{g\left(\frac{x}{\hat{T}(x)}\right)} \quad \forall x \in D.$$

Exercise 4.0.211. *Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$, $g(x) \neq 0$ for every $x \in D$. Prove*

$$\hat{f}^{\wedge}(\hat{x}) \nearrow g^{\wedge}(x) = \frac{1}{\hat{T}^2(x)} \frac{f(x)}{g(x\hat{T}(x))} \quad \forall x \in D.$$

Exercise 4.0.212. *Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$, $g(x) \neq 0$ for every $x \in D$. Prove*

$$\hat{f}^{\wedge}(x) \nearrow \hat{g}^{\wedge}(\hat{x}) = \frac{1}{\hat{T}(x)} \frac{f(x\hat{T}(x))}{g(x)} \quad \forall x \in D.$$

Exercise 4.0.213. *Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$, $g(x) \neq 0$ for every $x \in D$. Prove*

$$\hat{f}^{\wedge}(x) \nearrow \hat{g}^{\wedge}(x) = \frac{1}{\hat{T}(x)} \frac{f(x\hat{T}(x))}{g(x\hat{T}(x))} \quad \forall x \in D.$$

Exercise 4.0.214. *Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$, $\frac{x}{\hat{T}(x)} \in D$, $g(x) \neq 0$ for every $x \in D$. Prove*

$$\hat{f}^{\wedge}(x) \nearrow \hat{g}(\hat{x}) = \frac{1}{\hat{T}(x)} \frac{f(x\hat{T}(x))}{g\left(\frac{x}{\hat{T}(x)}\right)} \quad \forall x \in D.$$

Exercise 4.0.215. *Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$, $g(x) \neq 0$ for every $x \in D$. Prove*

$$\hat{f}^{\wedge}(x) \nearrow g^{\wedge}(x) = \frac{1}{\hat{T}^2(x)} \frac{f(x\hat{T}(x))}{g(x\hat{T}(x))} \quad \forall x \in D.$$

Exercise 4.0.216. *Let $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$, $g(x) \neq 0$ for every $x \in D$. Prove*

$$\hat{f}(\hat{x}) \nearrow \hat{g}^{\wedge}(\hat{x}) = \frac{1}{\hat{T}(x)} \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{g(x)} \quad \forall x \in D.$$

Exercise 4.0.217. *Let* $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$, $x\hat{T}(x) \in D$, $g(x) \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{f}(\hat{x}) \nearrow \hat{g}^{\wedge}(x) = \frac{1}{\hat{T}(x)} \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{g(x\hat{T}(x))} \quad \forall x \in D.$$

Exercise 4.0.218. *Let* $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$, $g(x) \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{f}(\hat{x}) \nearrow \hat{g}(\hat{x}) = \frac{1}{\hat{T}(x)} \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{g\left(\frac{x}{\hat{T}(x)}\right)} \quad \forall x \in D.$$

Exercise 4.0.219. *Let* $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$, $x\hat{T}(x) \in D$, $g(x) \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{f}(\hat{x}) \nearrow g^{\wedge}(x) = \frac{1}{\hat{T}^2(x)} \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{g(x\hat{T}(x))} \quad \forall x \in D.$$

Exercise 4.0.220. *Let* $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$, $g(x) \neq 0$ *for every* $x \in D$. *Prove*

$$f^{\wedge}(x) \nearrow \hat{g}^{\wedge}(\hat{x}) = \frac{f(x\hat{T}(x))}{g(x)} \quad \forall x \in D.$$

Exercise 4.0.221. *Let* $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$, $g(x) \neq 0$ *for every* $x \in D$. *Prove*

$$f^{\wedge}(x) \nearrow \hat{g}^{\wedge}(x) = \frac{f(x\hat{T}(x))}{g(x\hat{T}(x))} \quad \forall x \in D.$$

Exercise 4.0.222. *Let* $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$, $x\hat{T}(x) \in D$, $g(x) \neq 0$ *for every* $x \in D$. *Prove*

$$f^{\wedge}(x) \nearrow \hat{g}(\hat{x}) = \frac{f(x\hat{T}(x))}{g\left(\frac{x}{\hat{T}(x)}\right)} \quad \forall x \in D.$$

Exercise 4.0.223. *Let* $f, g, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$, $g(x) \neq 0$ *for every* $x \in D$. *Prove*

$$f^{\wedge}(x) \nearrow g^{\wedge}(x) = \frac{1}{\hat{T}(x)} \frac{f(x\hat{T}(x))}{g(x\hat{T}(x))} \quad \forall x \in D.$$

Definition: Let $\hat{f}, \hat{g} : \hat{D} \widehat{\longrightarrow} \hat{Y}$. If for $x \in D$ we have $ax+b \in D$, $cx+d \in D$ for some a, b, c, $d \in \mathbb{R}$, then we define

$$\hat{f}^\wedge(\widehat{ax+b}) := \frac{f(ax+b)}{\hat{T}(ax+b)},$$

$$\hat{f}^\wedge(ax+b) := f((ax+b)\hat{T}(ax+b)),$$

$$\hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\widehat{ax+b}) = \frac{f(x)}{\hat{T}(x)}\hat{T}(x)\frac{g(ax+b)}{\hat{T}(ax+b)} = \frac{f(x)g(ax+b)}{\hat{T}(ax+b)},$$

$$\hat{g}^\wedge(\widehat{ax+b})\hat{\times}\hat{f}^\wedge(\hat{x}) = \frac{g(ax+b)}{\hat{T}(ax+b)}\hat{T}(x)\frac{f(x)}{\hat{T}(x)} = \frac{g(ax+b)f(x)}{\hat{T}(x)},$$

$$\hat{f}^\wedge(\widehat{ax+b})\hat{\times}\hat{g}^\wedge(\widehat{cx+d}) = \frac{f(ax+b)}{\hat{T}(ax+b)}\hat{T}(x)\frac{g(cx+d)}{\hat{T}(cx+d)},$$

$$\hat{f}^\wedge(\hat{x})\hat{g}^\wedge(\widehat{ax+b}) = \frac{f(x)}{\hat{T}(x)}\frac{g(ax+b)}{\hat{T}(ax+b)},$$

$$\hat{f}^\wedge(\widehat{ax+b})\hat{g}^\wedge(\widehat{cx+d}) = \frac{f(ax+b)}{\hat{T}(ax+b)}\frac{g(cx+d)}{\hat{T}(cx+d)},$$

$$\hat{f}^\wedge(ax+b)\hat{\times}\hat{g}^\wedge(cx+d) = f(\hat{T}(ax+b)(ax+b))\hat{T}(x)g(\hat{T}(cx+d)(cx+d)),$$

$$\hat{f}^\wedge(ax+b)\hat{g}^\wedge(cx+d) = f(\hat{T}(ax+b)(ax+b))g(\hat{T}(cx+d)(cx+d)).$$

Example 4.0.224. *Let $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = x$, $g(x) = x+1$, $\hat{T}(x) = e^{-x}$, $x \in D$. Then*

$$\hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}(\widehat{x+1}) = \frac{f(x)}{\hat{T}(x)}\hat{T}(x)\frac{g(x+1)}{\hat{T}(x+1)} = \frac{f(x)g(x+1)}{\hat{T}(x+1)}$$

$$= \frac{x(x+2)}{\hat{T}(x+1)} = \frac{x(x+2)}{e^{-(x+1)}} = x(x+2)e^{x+1} = (x^2+2x)e^{x+1},$$

$$\hat{f}^\wedge(\widehat{x-1})\hat{\times}\hat{g}(\widehat{x+1}) = \frac{f(x-1)}{\hat{T}(x-1)}\hat{T}(x)\frac{g(x+1)}{\hat{T}(x+1)}$$

$$= \frac{x-1}{e^{-x+1}}e^{-x}\frac{x+2}{e^{-x-1}} = (x-1)(x+2)e^{x} = (x^2+x-2)e^{x},$$

$$\hat{f}^\wedge(\widehat{x-1})\hat{g}(\widehat{x+1}) = \frac{f(x-1)}{\hat{T}(x-1)}\frac{g(x+1)}{\hat{T}(x+1)} = \frac{x-1}{e^{-x+1}}\frac{x+2}{e^{-x-1}} = (x^2+x-2)e^{2x},$$

$$\hat{f}^\wedge(x-1)\hat{g}(x+1) = f(\hat{T}(x-1)(x-1))g(\hat{T}(x+1)(x+1))$$

$$= f\Big(e^{-x+1}(x-1)\Big)g\Big(e^{-x-1}(x+1)\Big) = e^{-x+1}(x-1)\Big(e^{-x-1}(x+1)+1\Big),$$

$$\hat{f}^\wedge(x-1)\hat{\times}\hat{g}^\wedge(x+1) = f(\hat{T}(x-1)(x-1))\hat{T}(x)g(\hat{T}(x+1)(x+1))$$

$$= e^{-2x+1}(x-1)\Big(e^{-x-1}(x+1)+1\Big)$$

Exercise 4.0.225. *Let $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = 2x$, $g(x) = x+1$, $\hat{T}(x) = x^2+1$, $x \in D$. Find*

1) $\hat{f}^\wedge(\widehat{x+1})\hat{\times}\hat{g}^\wedge(\hat{x})$,

2) $\hat{f}^\wedge(x-1)\hat{g}(2x+2)$.

Solution.

1) $\hat{f}^\wedge(\widehat{x+1})\hat{\times}\hat{g}^\wedge(\hat{x}) = \frac{f(x+1)}{\hat{T}(x+1)}\hat{T}(x)\frac{g(x)}{\hat{T}(x)} = \frac{f(x+1)g(x)}{\hat{T}(x+1)}$

$= \frac{(2x+2)(x+1)}{(x+1)^2+1} = 2\frac{(x+1)^2}{(x+1)^2+1}$,

2) $\hat{f}^\wedge(x-1)\hat{g}^\wedge(2x+2) = \frac{f(x-1)}{\hat{T}(x-1)}\frac{g(2x+2)}{\hat{T}(2x+2)} = \frac{(2x-2)}{(x-1)^2+1}\frac{2x+3}{4(x+1)^2+1}$.

Definition: An isofunction of first, second, third or fourth kind will be called even(odd) if its isooriginal is even(odd).

Definition: An isofunction of first, second, third or fourth kind will be called increasing, decreasing and monotonic if its isooriginal is increasing, decreasing and monotonic, respectively.

Example 4.0.226. *Let $D = \mathbb{R}$, $\hat{T}_1 = 4$, $f(x) = x^2$, $\hat{T}(x) = x^4+1$, $x \in D$. Then*

$$\hat{f}^\wedge(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x^2}{x^4+1}$$

and since

$$\frac{x^2}{x^4+1} = \frac{(-x)^2}{(-x)^4+1}$$

we conclude that the considered isofunction of first kind is an even isofunction on D. Also, we have that f is an even function on D.

Example 4.0.227. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = x^4$, $\hat{T}(x) = 2 + \sin x$, $x \in D$. *Then*

$$f(-x) = (-x)^4 = x^4 \quad \text{for} \quad \forall x \in D,$$

i.e. f is an even function on D, but

$$\hat{f}^\wedge(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x^2}{2 + \sin x},$$

from here

$$\hat{f}(-\hat{x}) = \frac{f(-x)}{\hat{T}(-x)} = \frac{x^4}{2 - \sin x} \neq \hat{f}^\wedge(\hat{x}) \quad \text{for} \quad \forall \hat{x} \in \hat{D} \backslash \{0\}.$$

Consequently the considered isofunction of first kind is not even isofunction on D.

Exercise 4.0.228. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 4$, $f(x) = x$, $\hat{T}(x) = 4 + \cos x$, $x \in D$. *Check if the corresponding isofunction of first kind is an even isofunction on $\hat{D}$.*

Answer. No.

Theorem: Let $f, \hat{T} : D \longrightarrow Y$ are even functions. Then the correswponding isofunctions of first, second, third and fourth kinds are even isofunction.

Proof. Since $f, \hat{T} : D \longrightarrow Y$ are even functions then

$$f(x) = f(-x), \quad \hat{T}(x) = \hat{T}(-x) \quad \text{for} \quad \forall x \in D.$$

Therefore the considered isofunction of first kind is an even function.
As in above we can prove our assertion for the isofunctions of second, third and fourth kinds.

□

Theorem: Let $f, \hat{T} : D \longrightarrow Y$ are odd functions. Then the corresponding isofunctions of first and fourth kinds are even isofunctions and the corresponding isofunctions of second and third kinds are odd isofunction..

Proof. We will prove our assertion for the isofunctions of first kind.
Since $f, \hat{T} : D \longrightarrow Y$ are odd functions then

$$-f(x) = f(-x), \quad -\hat{T}(x) = \hat{T}(-x) \quad \text{for} \quad \forall x \in D.$$

From here the corresponding isofunction of first kind is an even isofunction.

□

Example 4.0.229. *Let $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = x$, $\hat{T}(x) = x^2 + 1$, $x \in D$. Then*

$$\hat{f}^\wedge(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x}{x^2+1}, \quad x \in D.$$

From here the corresponding isofunction of first kind is an odd isofunction.

Exercise 4.0.230. *Let $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = x^2 + x + 1$, $\hat{T}(x) = x^4 + 1$, $x \in D$. Check if the corresponding isofunction of first kind is odd or even isofunction on D.*

Solution. For $\hat{f}^\wedge(\hat{x})$ we have the representation

$$\hat{f}^\wedge(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x^2+x+1}{x^4+1}.$$

From here

$$\frac{(-x)^2 - x + 1}{(-x)^4 + 1} = \frac{x^2 - x + 1}{x^4+1}.$$

Therefore the corresponding isofunction of first kind is not even isofunction on D.
Because

$$\frac{f(x)}{\hat{T}(x)} \neq \frac{f(-x)}{\hat{T}(-x)} \quad \text{for} \quad \forall x \in D$$

then the corresponding isofunction of first kind is not odd isofunction on D.

Exercise 4.0.231. *Let $D = \mathbb{R}$, $\hat{T}_1 = 4$, $f(x) = x$, $g(x) = x^2$, $\hat{T}(x) = x^6+1$, $x \in D$. Check if the corresponding isofunctions of first kind are even or odd isofunctions on D.*

Answer. The corresponding isofunction of f is an odd isofunction on D, The corresponding isofunction of g is an even isofunction on D.

Theorem: Let $f : D \longrightarrow Y$ be an odd function, $\hat{T} : D \longrightarrow Y$ be an even positive function. Then the corresponding isofunctions of first, second and fourth kinds are odd isofunctions and the corresponding isofunction of third kind is an even isofunction.

Proof. We will prove our assertion for the isofunctions of first kind. Since $f : D \longrightarrow Y$ is an odd function then

$$f(x) = -f(-x) \quad \text{for} \quad \forall x \in D.$$

Because $\hat{T} : D \longrightarrow Y$ is an even function we have

$$\hat{T}(x) = \hat{T}(-x) \quad \text{for} \quad \forall x \in D.$$

From here

$$\frac{f(x)}{\hat{T}(x)} = \frac{-f(-x)}{\hat{T}(-x)} = -\frac{f(-x)}{\hat{T}(-x)} = \quad \forall x \in D.$$

Consequently the corresponding isofunction of first kind is an odd isofunction. □

Theorem: Let $f : D \longrightarrow Y$ be an even function, $\hat{T} : D \longrightarrow Y$ be an odd positive function. Then the corresponding isofunctions of first, second and third kinds are odd isofunctions and the corresponding isofunction of foourth kind is an even isofunction.

Proof. We will prove our assertion for isofunctions of first kind. Since $f : D \longrightarrow Y$ is an even function then

$$f(x) = f(-x) \quad \text{for} \quad \forall x \in D.$$

Because $\hat{T} : D \longrightarrow Y$ is an odd function we have

$$\hat{T}(x) = -\hat{T}(-x) \quad \text{for} \quad \forall x \in D.$$

Consequently the corresponding isofunction of first kind is an odd isofunction. □

Example 4.0.232. *Let $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = x^2 + 1$, $\hat{T}(x) = x^4 + 1$, $x \in D$. Then for $x \in D$ we have*

$$\hat{f}^\wedge(x) = \tfrac{f(\hat{T}(x)x)}{\hat{T}(x)} = \tfrac{(x\hat{T}(x))^2+1}{\hat{T}(x)} = x^2\hat{T}(x) + \tfrac{1}{\hat{T}(x)} = x^2(x^4+1) + \tfrac{1}{x^4+1},$$

$$\tfrac{f(-\hat{T}(-x)x)}{\hat{T}(-x)} = \tfrac{(-x\hat{T}(-x))^2+1}{\hat{T}(-x)} = x^2\hat{T}(-x) + \tfrac{1}{\hat{T}(-x)},$$

$$= x^2((-x)^4+1) + \tfrac{1}{(-x)^4+1} = x^2(x^4+1) + \tfrac{1}{x^4+1} = \hat{f}^\wedge(x),$$

therefore the corresponding isofunction of second kind is an even isofunction.

Example 4.0.233. *Let $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) =$, $\hat{T}(x) = x^2 + 1$, $x \in D$. Then for $x \in D$ we have*

$$\hat{f}^\wedge(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)} = \frac{x\hat{T}(x)}{\hat{T}(x)} = x,$$

consequently the considered isofunction of second kind is an odd isofunction.

Example 4.0.234. *Let $D = \mathbb{R}$, $\hat{T}_1 = 4$, $f(x) = x+1$, $\hat{T}(x) = e^x$, $x \in D$. Then for $x \in D$ we have*

$$\hat{f}^\wedge(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)} = \frac{x\hat{T}(x)+1}{\hat{T}(x)} = x + \frac{1}{\hat{T}(x)} = x + e^{-x},$$

$$\frac{f(-x\hat{T}(-x))}{\hat{T}(-x)} = \frac{-x\hat{T}(-x)+1}{\hat{T}(-x)} = -x + \frac{1}{\hat{T}(-x)} = -x + e^x,$$

from here we conclude that considered the isofunction of second kind is not even and odd isofunction.

Exercise 4.0.235. *Let $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = -x$, $g(x) = x^2$, $\hat{T}(x) = x^4 + 1$, $x \in D$. Check if the corresponding isofunctions of second kind are odd or even isofunctions.*

Answer. The corresponding isofunction of second kind of f is an odd isofunction, the corresponding isofunction of second kind of g is an even isofunction.

Definition: Let $w \in \mathbb{R}$, $w > 0$. An isofunction of first, second, third or fourth kind will be called $\hat{w}$-isoperiodic isofunction if its isooriginal is a w -periodic function.

Example 4.0.236. *Let $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) = \sin x$, $\hat{T}(x) = 2 + \sin^2 x$, $x \in D$. Then*

$$\hat{f}^\wedge(\widehat{x+2\pi}) = \frac{f(x+2\pi)}{\hat{T}(x+2\pi)} = \frac{\sin(x+2\pi)}{2+\sin^2(x+2\pi)} = \frac{\sin x}{2+\sin^2 x}$$

for every $x \in D$. Consequently the considered isofunction of first kind is a $\widehat{2\pi}$-isoperiodic isofunction.

Example 4.0.237. *Let $D = \mathbb{R}$, $\hat{T}_1 = 4$, $f(x) = 1$, $\hat{T}(x) = 2 + x^2$, $x \in D$. Then if we suppose that there exists $w \in \mathbb{R}$, $w > 0$ so that*

$$\hat{f}^\wedge(\widehat{x+w}) = \hat{f}^\wedge(\hat{x})$$

we obtain

$$\frac{f(x+w)}{\hat{T}(x+w)} = \frac{f(x)}{\hat{T}(x)} \quad \text{for} \quad \forall x \in D$$

or

$$\frac{1}{2+(x+w)^2} = \frac{1}{2+x^2} \quad \text{for} \quad \forall x \in D,$$

from where

$$(x+w)^2 = x^2 \quad \text{for} \quad \forall x \in D,$$

which is impossible. Therefore the considered isofunction of first kind is not isoperiodic isofunction.

Exercise 4.0.238. *Let $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = 1+\cos^2 x$, $\hat{T}(x) = 1+\cos^2 x$, $\hat{T}(x) = 3 + \sin^2 x$, $x \in D$. Check if the corresponding isofunction of first kind is $\hat{\pi}$ - isoperiodic isofunction.*

Exercise 4.0.239. *Let $D = \mathbb{R}$, $\hat{T}_1 = 5$, $f(x) = 2+\sin^2 x$, $\hat{T}(x) = 7+2\sin^2 x$, $x \in D$. Check if the corresponding isofunction of first kind is an isoperiodic isofunction.*

Answer. The considered isofunction of first kind is a π - isoperiodic isofunction.

Theorem 4.0.240. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ is w_1 - periodic function, $w_1 \in \mathbb{R}$, $w_1 > 0$, $\hat{T} : D \longrightarrow Y$ is positive w_2 - periodic function, $w_2 \in \mathbb{R}$, $w_2 > 0$. If there exist $k, l \in \mathbb{N}$ such that*

$$p := w_1 k = w_2 l$$

then the corresponding isofunction of first kind is a $\hat{p}$ - isoperiodic isofunction.

Proof. Because f is w_1 - periodic function we have that

$$f(x+p) = f(x+w_l l) = f(x+w_1) = f(x) \quad \text{for} \quad \forall x \in \mathbb{R}.$$

Since $\hat{T}$ is w_2 - periodic function we get

$$\hat{T}(x+p) = \hat{T}(x+w_2 l) = \hat{T}(x+w_2) = \hat{T}(x) \quad \text{for} \quad \forall x \in \mathbb{R}.$$

From here

$$\frac{f(x+p)}{\hat{T}(x+p)} = \frac{f(x)}{\hat{T}(x)} \quad \text{for} \quad \forall x \in \mathbb{R}.$$

Consequently the considered isofunction of first kind is $\hat{p}$ - isoperiodic isofunction. $\square$

Theorem: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a w_1 - periodic function, $w_1 \in \mathbb{R}$, $w_1 > 0$, $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a w_2-periodic function, $w_2 \in \mathbb{R}$, $w_2 > 0$ and $\hat{T} : \mathbb{R} \longrightarrow \mathbb{R}$ be a positive w_3 - periodic function, $w_3 \in \mathbb{R}$, $w_3 > 0$. If there exist $k, l, m \in \mathbb{N}$ such that

$$p := w_1 k = w_2 l = w_3 m$$

then if $\hat{f}$ and $\hat{g}$ are isofunctions of first kind we have

$$\hat{f} \pm \hat{g}, \quad \hat{f} \hat{\times} \hat{g}, \quad \hat{f}\hat{g}$$

are $\hat{p}$-isoperiodic isofunctions.

Proof. Since $f, g, \hat{T} : \mathbb{R} \longrightarrow \mathbb{R}$ are w_1, w_2 and w_3 - periodic functions, respectively, then

$$f(x) = f(x + w_1) = f(x + kw_1) = f(x + p),$$

$$g(x) = g(x + w_2) = g(x + lw_2) = g(x + p),$$

$$\hat{T}(x) = \hat{T}(x + w_3) = \hat{T}(x + mw_3) = \hat{T}(x + p)$$

for every $x \in \mathbb{R}$. From here

$$\hat{f}^\wedge(\hat{x}) \pm \hat{g}^\wedge(\hat{x}) = \tfrac{f(x)}{\hat{T}(x)} \pm \tfrac{g(x)}{\hat{T}(x)} = \tfrac{f(x+p)}{\hat{T}(x+p)} \pm \tfrac{g(x+p)}{\hat{T}(x+p)} = \hat{f}(\widehat{x+p}) \pm \hat{g}^\wedge(\widehat{x+p}),$$

$$\hat{f}^\wedge(\hat{x}) \hat{\times} \hat{g}^\wedge(\hat{x}) = \tfrac{f(x)g(x)}{\hat{T}(x)} = \tfrac{f(x+p)g(x+p)}{\hat{T}(x+p)} = \hat{f}(\widehat{x+p}) \hat{\times} \hat{g}(\widehat{x+p}),$$

$$\hat{f}^\wedge(\hat{x})\hat{g}^\wedge(\hat{x}) = \tfrac{f(x)g(x)}{\hat{T}^2(x)} = \tfrac{f(x+p)g(x+p)}{\hat{T}^2(x+p)} = \hat{f}(\widehat{x+p})\hat{g}(\widehat{x+p})$$

for every $x \in \mathbb{R}$. ☐

Remark 4.0.241. *The isofunctions of second, third and fourth kind can not be isoperiodic isofunction since in the general case for arbitrary $w \in \mathbb{R}$, $w > 0$, the equality*

$$f(\hat{T}(x + w)(x + w)) = f(\hat{T}(x)x)$$

or

$$f(\frac{x + w}{\hat{T}(x + w)}) = f(\frac{x}{\hat{T}(x)})$$

is not valid for every $x \in \mathbb{R}$, because the isotopic element $\hat{T}$ can not be represented in the form $\hat{T}(x) = xg(x)$ or $\hat{T}(x) = \frac{x}{g(x)}$, $g(0) \neq 0$, cause the *condition for positiveness of the isotopic element, in our case it should be valid for every $x \in \mathbb{R}$.*

Advanced practical exercises

Problem 4.0.242. *Let $D = \mathbb{R}$, $\hat{T}(x) = x^2 + 1$, $f(x) = 3x - 2$, $x \in D$. Find*

$$\hat{f}^\wedge(\hat{x}), \hat{f}^\wedge(\hat{x}), f(\hat{x}), f^\wedge(x), \quad x \in D.$$

Answer.

$$\hat{f}^\wedge(\hat{x}) = \tfrac{3x-2}{x^2+1}, \quad \hat{f}^\wedge(\hat{x}) = \tfrac{-2x^2+3x-2}{(x^2+1)^2},$$

$$f(\hat{x}) = \tfrac{-2x^2+3x-2}{x^2+1}, \quad f^\wedge(x) = 3x^3 + 3x - 2, \quad x \in D.$$

Problem 4.0.243. *Let $D = \mathbb{R}$, $\hat{T}(x) = e^x$, $f(x) = x$, $x \in D$. Find*

$$\hat{f}^\wedge(\hat{x}), \hat{f}^\wedge(\hat{x}), f(\hat{x}), f^\wedge(x), \quad x \in D.$$

Answer.

$$\hat{f}^\wedge(\hat{x}) = xe^{-x}, \quad \hat{f}^\wedge(\hat{x}) = xe^{-2x},$$

$$f(\hat{x}) = xe^{-x}, \quad f^\wedge(x) = xe^{x}, \quad x \in D.$$

Problem 4.0.244. *Find $\hat{f}^\wedge(\hat{x})$ if*

1) $\hat{T}(x) = x^2 + 1, f(x) = \sin x, D = \mathbb{R}$,

2) $\hat{T}(x) = x^2 + 5, f(x) = \cos x, D = \mathbb{R}$,

3) $\hat{T}(x) = x^2 + x + 1, f(x) = \tan x, D = \mathbb{R}$,

4) $\hat{T}(x) = x^2 + 2, f(x) = \sin x + 2\cos x, D = \mathbb{R}$,

5) $\hat{T}(x) = x^2 - x + 1, f(x) = x^2 + 2x, D = \mathbb{R}$,

6) $\hat{T}(x) = x^2 - x + 5, f(x) = x^3 + e^x, D = \mathbb{R}$,

7) $\hat{T}(x) = e^x, f(x) = e^{2x}, D = \mathbb{R}$,

8) $\hat{T}(x) = e^x + e^{2x} + 2, f(x) = x^3 - 1, D = \mathbb{R}$,

9) $\hat{T}(x) = e^x + e^{-x}, f(x) = \sin x, D = \mathbb{R}$,

10) $\hat{T}(x) = 10 + \ln^2(x^2 + x + 1), f(x) = e^x, D = \mathbb{R}$.

Answer.

$$1)\quad \hat{f}^{\wedge}(\hat{x}) = \frac{\sin x}{x^2+1},\quad 2)\quad \hat{f}^{\wedge}(\hat{x}) = \frac{\cos x}{x^2+5},$$

$$3)\quad \hat{f}^{\wedge}(\hat{x}) = \frac{\tan x}{x^2+x+1},\quad 4)\quad \hat{f}^{\wedge}(\hat{x}) = \frac{\sin x + 2\cos x}{x^2+2},$$

$$5)\quad \hat{f}^{\wedge}(\hat{x}) = \frac{x^2+2x}{x^2-x+1},\quad 6)\quad \hat{f}^{\wedge}(\hat{x}) = \frac{x^3+e^x}{x^2-x+5},$$

$$7)\quad \hat{f}^{\wedge}(\hat{x}) = e^x,\quad 8)\quad \hat{f}^{\wedge}(\hat{x}) = \frac{x^3-1}{e^x+e^{2x}+2},$$

$$9)\quad \hat{f}^{\wedge}(\hat{x}) = \frac{\sin x}{e^x+e^{-x}}.\quad 10)\quad \hat{f}^{\wedge}(\hat{x}) = \frac{e^x}{10+\ln^2(x^2+x+1)}.$$

Problem 4.0.245. *Check if $\hat{f}^{\wedge}(\hat{x})$ is an isofunction, where*

$$1)\quad f(x) = \begin{cases} 2 & \text{for} \quad x \in [0,1], \\ 3 & \text{for} \quad x \in [1,2],, \end{cases} \qquad \hat{T}(x) = x^2+1, \quad D = \mathbb{R},$$

$$2)\quad f(x) = \sin x, \quad \hat{T}(x) = \begin{cases} x^2+1 & \text{for} \quad x \in (-\infty, 0], \\ x^2+4 & \text{for} \quad x \in [0,\infty), \end{cases} \qquad D = \mathbb{R},$$

$$3)\quad f(x) = \begin{cases} x^2-1 & \text{for} \quad x \in [-1,0], \\ x^2+2 & \text{for} \quad x \in [0,1], \end{cases} \qquad \hat{T}(x) = \cos^2 x + 1, \quad D = [-1,1],$$

$$4)\quad f(x) = x, \quad \hat{T}(x) = \sin^2 x + 4, \quad D = [-2,2],$$

$$5)\quad f(x) = x^2, \quad \hat{T}(x) = \begin{cases} 1 & \text{for} \quad x \in [-3,0], \\ x^2+5 & \text{for} \quad x \in [0,3], \end{cases} \qquad D = [-3,3].$$

Answer. 1) No, 2) No, 3) No, 4) Yes, 5) No.

Problem 4.0.246. *Let $\hat{D} = [-1,1]$, $\hat{f}^{\wedge}(\hat{x}) = \frac{x+1}{x^4+5}$, $f(x) = x+1$, $x \in D$. Find $\hat{T}(x)$, $x \in D$.*

Answer. $\hat{T}(x) = x^4 + 5$.

Problem 4.0.247. *Let $D = \mathbb{R}$, $\hat{f}^{\wedge}(\hat{x}) = \frac{\sin x + e^x}{x^2+1}$, $\hat{T}(x) = x^2+1$, $x \in D$. Find $f(x)$, $x \in D$.*

Answer. $f(x) = \sin x + e^x$.

Problem 4.0.248. *Let* $D = \mathbb{R}$, $f(x) = x^2 - 1$, $g(x) = x - 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$, $\hat{T}_1 = 5$. *Find*

1) $\hat{f}^\wedge(\hat{x})$,

2) $\hat{g}^\wedge(\hat{x})$,

3) $\hat{2}\hat{\times}(\hat{3}\hat{f}^\wedge(\hat{x}) - \hat{g}^\wedge(\hat{x})) + \hat{4}\hat{\times}\hat{g}^\wedge(\hat{x})$.

Answer.

$$1) \quad \hat{f}^\wedge(\hat{x}) = \tfrac{x^2-1}{x^2+1}, \quad 2) \quad \hat{g}^\wedge(\hat{x}) = \tfrac{x-1}{x^2+1},$$

$$3) \quad \hat{2}\hat{\times}(\hat{3}\hat{f}^\wedge(\hat{x}) - \hat{g}^\wedge(\hat{x})) + \hat{4}\hat{\times}\hat{g}^\wedge(\hat{x}) = \tfrac{\frac{9}{5}x^2+x-\frac{14}{5}}{x^2+1}.$$

Problem 4.0.249. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) = x$, $g(x) = x - 1$, $\hat{T}(x) = x^2 + 2$, $x \in D$. *Find*

1) $\hat{f}^\wedge(\hat{x})$,

2) $\hat{g}^\wedge(\hat{x})$,

3) $A := \hat{f}^\wedge(\hat{f}^\wedge(\hat{x})) + \hat{f}^\wedge(\hat{g}^\wedge(\hat{x})) + \hat{g}^\wedge(\hat{f}^\wedge(\hat{x}))$,

4 $B := \hat{2}\hat{f}^\wedge(\hat{x}) - \hat{f}^\wedge(\hat{g}^\wedge(\hat{f}^\wedge(\hat{x})))$.

Answer.

$$1) \quad \hat{f}^\wedge(\hat{x}) = \tfrac{x}{x^2+2}, \quad 2) \quad \hat{g}^\wedge(\hat{x}) = \tfrac{x-1}{x^2+2},$$

$$3) \quad A = \tfrac{3x-2}{x^2+2}, \quad 4) \quad B = \tfrac{-\frac{1}{3}x+1}{x^2+2}.$$

Problem 4.0.250. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = x+1$, $g(x) = x$, $h(x) = x-1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. *Find*

$$A := \hat{f}^\wedge(\hat{x})\hat{\times}(\hat{g}^{\hat{2}}(\hat{x}) - \hat{3}\hat{\times}\hat{h}^\wedge(\hat{x})) + \hat{h}^3(\hat{x}).$$

Answer.

$$A = \frac{x^7 - 2x^6 - 4x^5 - 7x^4 - 11x^3 - 8x^2 - 6x - 3}{(x^2+1)^3}.$$

Problem 4.0.251. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) = 1 + x$, $g(x) = 2 + x$, $\hat{T}(x) = 1 + x^2$, $x \in D$. *Find*

$$A := \hat{f}^\wedge(\hat{x}) + f(\hat{x}) + \hat{2}\hat{\times}\hat{g}^\wedge(\hat{x}) - \hat{g}(x).$$

Answer.

$$A = \frac{2x+3}{1+x^2}.$$

Problem 4.0.252. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) = x + 10$, $g(x) = x$, $\hat{T}(x) = x^2 + 1$, $x \in D$. *Find*

$$A = \hat{f}(\hat{g}(\hat{f}(\hat{g}(\hat{f}(\hat{g}(x)))))).$$

Answer.

$$A = \frac{x^3 + x + 30}{x^2 + 1}.$$

Problem 4.0.253. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = x - 1$, $g(x) = x + 1$, $\hat{T}(x) = x^4 + 1$, $x \in D$. *Find*

1) $\hat{f}(x)$, $\hat{f}^\wedge(\hat{x})$, $\hat{g}(x)$, $\hat{g}^\wedge(\hat{x})$,

2) $A := \hat{f}^{\hat{2}}(\hat{x}) - \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x}) + \hat{f}(x)\hat{\times}\hat{g}(x)$.

Answer.

1) $\hat{f}(x) = \frac{x^5+x-1}{x^4+1}$, $\hat{f}^\wedge(\hat{x}) = \frac{x-1}{x^4+1}$, $\hat{g}(x) = \frac{x^5+x+1}{x^4+1}$, $\hat{g}^\wedge(\hat{x}) = \frac{x+1}{x^4+1}$,

2) $A = \frac{x^{10}+2x^6+x^2-2x+2}{x^4+1}$.

Problem 4.0.254. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 8$, $f(x) = x^3$, $x \in D$. *Find* $\hat{T}(x)$, $x \in D$, *such that*

$$\hat{f}^{\hat{2}}(x) = -x^2.$$

Answewr. No solutions.

Problem 4.0.255. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 2$, $f(x) = x^3$, $\hat{T}(x) = 4 - \sin x$, $x \in D$. *Check if* $\hat{f}$ *is an even isofunction on* $\hat{D}$.

Answer. No.

Problem 4.0.256. *Let* $D = \mathbb{R}$, $\hat{T}_1 = 3$, $f(x) = x^4$, $\hat{T}(x) = x^{10} + 2x^2 + 1$, $x \in D$. *Check if* $\hat{f}$ *is an even isofunction on* $\hat{D}$.

Answer. Yes.

Problem 4.0.257. *Let* $D = \mathbb{R}$*,* $\hat{T}_1 = 3$*,* $f(x) = x^4 + x^2$*,* $g(x) = x^3 + x$*,* $\hat{T}(x) = x^4 + 1$*,* $x \in D$*. Check if* $\hat{f}$*,* $\hat{g}$ *are odd or even isofunctions on* $\hat{D}$*.*

Answer. $\hat{f}$ is an even isofunction on $\hat{D}$, $\hat{g}$ is an odd isofunction on $\hat{D}$.

Problem 4.0.258. *Let* $D = \mathbb{R}$*,* $\hat{T}_1 = 3$*,* $f(x) = x^3$*,* $g(x) = x^4$*,* $\hat{T}(x) = x^6 + 1$*,* $x \in D$*. Check if* $\hat{f}, \hat{g} : D \widehat{\longrightarrow} Y$ *are odd or even isofunctions.*

Answer. $\hat{f}$ is an odd isofunction, $\hat{g}$ is an even isofunction.

Problem 4.0.259. *Let* $D = \mathbb{R}$*,* $\hat{T}_1 = 3$*,* $f(x) = 3 + \sin x + \cos^2 x$*,* $\hat{T}_1(x) = 4 + \cos^2(3x)$*,* $x \in D$*. Check if* $\hat{f} : \hat{F}_{\mathbb{R}} \widehat{\longrightarrow} \hat{F}_{\mathbb{R}}$ *is an isoperiodic isofunction.*

Answer. $\hat{f}$ is $\widehat{2\pi}$-isoperiodic isofunction.

Problem 4.0.260. *Let* $D = \mathbb{R}$*,* $\hat{T}_1 = 3$*,* $f(x) = 3 + x + \sin x$*,* $\hat{T}(x) = 4 + \sin^2 x$*,* $x \in D$*. Check if* $\hat{f} : \hat{F}_{\mathbb{R}} \widehat{\longrightarrow} \hat{F}_{\mathbb{R}}$ *is an isoperiodic isofunction.*

Answer. $\hat{f}$ is not isoperiodic isofunction.

Problem 4.0.261. *Let* $D = [1, +\infty)$*,* $f(x) = x$*,* $\hat{T}(x) = x^2 + 1$*,* $x \in D$*. Find* $\hat{f}^\wedge(\hat{x})$*,* $\hat{f}^\wedge(x)$*.*

Answer.

$$\hat{f}^\wedge(\hat{x}) = \frac{x}{x^2 + 1}, \quad \hat{f}^\wedge(x) = x.$$

Chapter 5

Limit of isofunctions. Continuous isofunctions

Let $D \subset \mathbb{R}$ and $\hat{f} : D \longrightarrow \mathbb{R}$ is an isofunction of first, second, third or fourth kind and $\tilde{f}$ is its isooriginal.

Definition 5.0.262. *The real a will be called left limit of $\hat{f}$ at $x_0 \in D$ if it is left limit of $\tilde{f}$ at x_0.*

Definition 5.0.263. *The real a will be called right limit of $\hat{f}$ at $x_0 \in D$ if it is right limit of $\tilde{f}$ at x_0.*

Definition 5.0.264. *The real a will be called limit of $\hat{f}$ at $x_0 \in D$ if it is limit of $\tilde{f}$ at x_0.*

Example 5.0.265. *Let* $D = [-1, 1]$,

$$f(x) = \begin{cases} x+2 & \text{for} \quad x \in [-1, 0] \\ x+4 & \text{for} \quad x \in [0, 1], \end{cases} \qquad \hat{T}(x) = \begin{cases} \frac{1}{2} & \text{for} \quad x \in [-1, 0] \\ 1 & \text{for} \quad x \in [0, 1]. \end{cases}$$

Then

$$\hat{f}^\wedge(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \begin{cases} 2(x+2) & \text{for} \quad x \in [-1, 0] \\ x+4 & \text{for} \quad x \in [0, 1]. \end{cases}$$

We have that the limit $\lim_{x \longrightarrow 0} f(x)$ *does not exists because*

$$\lim_{x \longrightarrow 0-} f(x) = 2, \qquad \lim_{x \longrightarrow 0+} f(x) = 4.$$

On the other hand, there exists

$$\lim_{x \longrightarrow 0} \hat{f}^\wedge(\hat{x}) = 4.$$

Example 5.0.266. *Let* $D = [0, 4]$, $f(x) = x^2$, $x \in D$,

$$\hat{T}(x) = \begin{cases} x+2 & \text{for} \quad x \in [0, 2] \\ 2 & \text{for} \quad x \in [2, 4]. \end{cases}$$

Then

$$\hat{f}^\wedge(\hat{x}) = \begin{cases} \frac{x^2}{x+2} & \text{for} \quad x \in [0, 2] \\ \frac{x^2}{2} & \text{for} \quad x \in [2, 4]. \end{cases}$$

From here, since

$$\lim_{x \longrightarrow 2-} \hat{f}^\wedge(\hat{x}) = 1, \qquad \lim_{x \longrightarrow 2+} \hat{f}^\wedge(\hat{x}) = 2,$$

the limit

$$\lim_{x \longrightarrow 2} \hat{f}^\wedge(\hat{x})$$

does not exist, also there exists

$$\lim_{x \longrightarrow 2} f(x) = 4.$$

Theorem 5.0.267. *Let $x_0 \in D$. Then there exists*

$$\lim_{x \longrightarrow x_0, x \in D} \hat{f}(x) = a$$

if and only if there exist $\hat{f}(x_0+0)$, $\hat{f}(x_0-0)$ and

$$\hat{f}(x_0-0) = \hat{f}(x_0+0) = a.$$

Proof.

1. Let there exists $\lim_{x \longrightarrow x_0, x \in D} \hat{f}(x) = a$. Then there exist $\hat{f}(x_0+0)$, $\hat{f}(x_0-0)$ and

$$\hat{f}(x_0-0) = \hat{f}(x_0+0) = a.$$

2. Let there exist $\hat{f}(x_0+0)$, $\hat{f}(x_0-0)$ and

$$\hat{f}(x_0-0) = \hat{f}(x_0+0) = a.$$

Let also, $\epsilon > 0$ be fixed. Then there exist $\delta_1 > 0$, $\delta_2 > 0$, such that from

$$|x - x_0| < \delta_1, x < x_0, x \in D; \quad |y - x_0| < \delta_2, y > x_0, y \in D,$$

we have

$$|\hat{f}(x) - a| < \epsilon, \quad |\hat{f}(y) - a| < \epsilon.$$

Let

$$\delta = \min\{\delta_1, \delta_2\}.$$

Then from

$$|x - x_0| < \delta, x \in D,$$

follows that

$$|\hat{f}(x) - a| < \epsilon.$$

In other words

$$\lim_{x \longrightarrow x_0, x \in D} \hat{f}(x) = a.$$

□

Theorem 5.0.268. *Let $\hat{f} : [a, b] \longrightarrow \mathbb{R}$ be monotonic isofunction. Then the left limit of $\hat{f}$ exists in every point $x_0 \in (a, b]$ and the right limit of $\hat{f}$ exists in every point $y_0 \in [a, b)$.*

Proof. We will prove the assertion for left limit.

1. Let $\hat{f}$ is increasing function. From $a \leq x \leq x_0$ follows that

$$\hat{f}(a) \leq \hat{f}(x) \leq \hat{f}(x_0).$$

Consequently the isofunction $\hat{f}$ is bounded on the interval $[a, x_0)$. From here and from continuous principle follows that there exists

$$\sup \hat{f}([a, x_0)) = \alpha.$$

From here follows that

1.1. $\hat{f}(x) \leq \alpha$ for every $x \in [a, x_0)$,
1.2. for every $\epsilon > 0$ there exists $\hat{z} \in [a, x_0)$ such that $\alpha - \epsilon < \hat{f}(z)$.

Let $\epsilon > 0$ be arbitrary chosen and let $z \in [a, x_0)$ is chosen in connection with 1.2 and let $\delta = x_0 - z$. Then from 1.1 and 1.2 and since $\hat{f}$ is increasing follows that for every $x \in [a, x_0)$,

$$-\delta < x - x_0 < 0$$

is equivalent of

$$z < x < x_0$$

and

$$\alpha - \epsilon < \hat{f}(z) \leq \hat{f}(x) \leq \alpha < \alpha + \epsilon,$$

therefore there exists $\hat{f}(x_0 - 0)$ and $\hat{f}(x_0 - 0) = \alpha$.

2. Let $\hat{f}$ is decreasing function. From $a \leq x \leq x_0$ follows that

$$\hat{f}(a) \geq \hat{f}(x) \geq \hat{f}(x_0).$$

Consequently the isofunction $\hat{f}$ is bounded on the interval $[a, x_0)$. From here and from continuous principle follows that there exists

$$\inf \hat{f}([a, x_0)) = \hat{\beta}.$$

From here follows that

2.1. $\hat{f}(x) \geq \beta$ for every $x \in [a, x_0)$,

2.2. for every $\epsilon > 0$ there exists $y \in [a, x_0)$ such that $\beta + \epsilon > \hat{f}(y)$.

Let $\epsilon > 0$ be arbitrary chosen and let $y \in [a, x_0)$ is chosen in connection with 2.2 and let $\delta = x_0 - y$. Then from 2.1 and 2.2 and since $\hat{f}$ is decreasing follows that for every $x \in [a, x_0)$,

$$-\delta < x - x_0 < 0$$

is equivalent of

$$y < x < x_0$$

and

$$\beta - \epsilon < \beta < \hat{f}(x) \leq \hat{f}(y) \leq \beta + \epsilon,$$

therefore there exists $\hat{f}(x_0 - 0)$ and $\hat{f}(x_0 - 0) = \beta$.

□

Definition 5.0.269. *We will say that the number $b \in \mathbb{R}$ is a limit of the isofunction $\hat{f}$ when $x \longrightarrow \infty$ if it is a limit of the isooriginal $\tilde{f}$ when $x \longrightarrow \infty$.*

Theorem 5.0.270. *Let the isofunction $\hat{f}$ has limit a at the point $x_0 \in D$. Then there exist a neighbourhood $U(x_0)$ and a number $b > 0$ such that for every $x \in U(x_0) \cap D$, $x \neq x_0$, we have*

$$|\hat{f}(x)| \leq b.$$

Proof. Let $\epsilon \in D$, $\epsilon \in (0, 1)$. Then there exists an neighbourhood $U(x_0)$ such that for every $x \in U(x_0) \cap D$, $x \neq x_0$, we have

$$|\hat{f}(x) - a| \leq \epsilon < 1,$$

and from the properties of the modulus follows that

$$|\hat{f}(x)| - |a| \leq 1$$

for every $x \in U(x_0) \cap D$, $x \neq x_0$. Therefore for every $x \in U(x_0) \cap D$, $x \neq x_0$,

$$|\hat{f}(x)| \leq 1 + |a| =: b.$$

□

Theorem 5.0.271. *Let* $\lim_{x \longrightarrow x_0} \hat{f}(x) = b$, $b \neq 0$. *Then*

1. *there exists a neighbourhood* $U(x_0)$ *such that for every* $x \in U(x_0) \cap D$, $x \neq x_0$, *we have*
$$|\hat{f}(x)| > \frac{|b|}{2},$$
2. *if* $b > 0$ *there exists a neighbourhood* $U(x_0)$ *such that for every* $x \in U(x_0) \cap D$, $x \neq x_0$, *we have*
$$\hat{f}(x) > \frac{b}{2},$$
3. *if* $b < 0$ *there exists a neighbourhood* $U(x_0)$ *such that for every* $x \in U(x_0) \cap D$, $x \neq x_0$, *we have*
$$\hat{f}(x) < \frac{b}{2}.$$

Proof. **1.** Since $\lim_{x \longrightarrow x_0} \hat{f}(x) = b$ then for $\epsilon = \frac{|b|}{2}$ there exists a neighbourhood $U(x_0)$ such that for every $x \in U(x_0) \cap D$, $x \neq x_0$, we have

$$|\hat{f}(x) - b| < \frac{|b|}{2}.$$

From here and from the properties of the modulus follows that for every $x \in U(x_0) \cap D$, $x \neq x_0$, we have

$$|b| - |\hat{f}(x)| \leq |\hat{f}(x) - b| < \frac{|b|}{2}, \tag{5.0.1}$$

from where for every $x \in U(x_0) \cap D$, $x \neq x_0$, we have

$$|\hat{f}(x)| > |b| - \frac{|b|}{2} = \frac{|b|}{2}.$$

2. 3. The second inequality of (5.0.1) is equivalent of :for every $x \in U(x_0) \cap D$, $x \neq x_0$, we have

$$b - \frac{|b|}{2} < \hat{f}(x) < b + \frac{|b|}{2}. \tag{5.0.2}$$

From here if $b > 0$, from the left inequality of (5.0.2) we have, for every $x \in U(x_0) \cap D$, $x \neq x_0$,

$$\hat{f}(x) > \frac{b}{2},$$

and if $b < 0$, from the right inequality (5.0.2), for every $x \in U(x_0) \cap D$, $x \neq x_0$, we have

$$\hat{f}(x) < \frac{b}{2}.$$

□

Theorem 5.0.272. *Let* $\hat{\phi} : D \longrightarrow \hat{\phi}(D)$ *and* $\lim_{x \longrightarrow x_0} \hat{f}(x) = a$, $\lim_{x \longrightarrow x_0} \hat{\phi}(x) = b$ *and* $\hat{f}(x) \leq \hat{\phi}(x)$ *for every* $x \in D$. *Then* $a \leq b$.

Proof. There exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \longrightarrow_{n \longrightarrow \infty} x_0$ and the sequnecs $\{\hat{f}(x_n)\}_{n=1}^{\infty}$, $\{\hat{\phi}(x_n)\}_{n=1}^{\infty}$ are convergent to a and b, respectively. Also, we have

$$\hat{f}(x_n) \leq \hat{\phi}(x_n).$$

From here and from the properties of the convergent sequences follows that

$$a \leq b.$$

□

Theorem 5.0.273. *Let* $\hat{\phi} : D \longrightarrow \hat{\phi}(D)$, $\hat{g} : D \longrightarrow \hat{g}(D)$ *and*

$$\lim_{x\longrightarrow x_0} \hat{f}(x) = \lim_{x\longrightarrow x_0} \hat{\phi}(x) = a,$$

and

$$\hat{f}(x) \leq \hat{g}(x) \leq \hat{\phi}(x) \quad \forall x \in D.$$

Then

$$\lim_{x\longrightarrow x_0} \hat{g}(x) = a.$$

Proof. There exists an sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \longrightarrow_{n\longrightarrow\infty} x_0$ and the sequnecs $\{\hat{f}(x_n)\}_{n=1}^{\infty}$, $\{\hat{\phi}(x_n)\}_{n=1}^{\infty}$ are convergent to a. From here and from

$$\hat{f}(x_n) \leq \hat{g}(x_n) \leq \hat{\phi}(x_n)$$

follows that there exists $\lim_{x\longrightarrow x_0} \hat{g}(x) = a$. □

Using the definition for limit of an isofunction and using the properties of the sequences we have the following properties:
Let $\hat{g} : D \longrightarrow \hat{g}(D)$ and $\hat{f}$ has isolimit at the isopoint $x_0 \in D$. Then

1. $\lim_{x\longrightarrow x_0}(\hat{f}(x) \pm \hat{g}(x)) = \lim_{x\longrightarrow x_0} \hat{f}(x) \pm \lim_{x\longrightarrow x_0} \hat{f}(x)$,
2. $\lim_{x\longrightarrow x_0}(\hat{f}(x)\hat{\times}\hat{g}(x)) = \lim_{x\longrightarrow x_0}\hat{f}(x)\hat{\times}\lim_{x\longrightarrow x_0}\hat{g}(x)$,
3. $\lim_{x\longrightarrow x_0}(\hat{f}(x)\hat{g}(x)) = \lim_{x\longrightarrow x_0}\hat{f}(x)\lim_{x\longrightarrow x_0}\hat{g}(x)$,
4. $\lim_{x\longrightarrow x_0}(\hat{f}(x)\hat{/}\hat{g}(x)) = \lim_{x\longrightarrow x_0}\hat{f}(x)\hat{/}\lim_{x\longrightarrow x_0} \hat{g}(x)$, if $\lim_{x\longrightarrow x_0}\hat{g}(x) \neq 0$,
5. $\lim_{x\longrightarrow x_0} \frac{\hat{f}(x)}{\hat{g}(x)} = \frac{\lim_{x\longrightarrow x_0}\hat{f}(x)}{\lim_{x\longrightarrow x_0} \hat{g}(x)}$, if $\lim_{x\longrightarrow x_0}\hat{g}(x) \neq 0$,
6. if $|\hat{f}(x)|$ is bounded below and $\lim_{x\longrightarrow x_0}\hat{g}(x) = 0$ then $\lim_{x\longrightarrow x_0}(\hat{f}(x)\hat{/}\hat{g}(x)) = \infty$,
7. if $\lim_{x\longrightarrow x_0}\hat{f}(x) = a$ and $\lim_{x\longrightarrow x_0}\hat{g}(x) = \infty$, then $\lim_{x\longrightarrow x_0}(\hat{f}(x)\hat{/}\hat{g}(x)) = 0$.

Exercise 5.0.274. *Let* $D = \mathbb{R}_+$. *Find* $\lim_{x\longrightarrow a} \hat{f}^{\wedge}(\hat{x})$ *if*

1. $a = 2,\ f(x) = x^2 + 2,\ \hat{T}(x) = x + 3,$
2. $a = 3,\ f(x) = x^3 + 3,\ \hat{T}(x) = x + 2,$
3. $a = 1,\ f(x) = x + 2,\ \hat{T}(x) = x + 3.$

Answer. 1) $\frac{6}{5}$, 2) 6, 3) $\frac{3}{4}$.

Exercise 5.0.275. *Let* $D = \mathbb{R}_+$. *Find* $\lim_{x\longrightarrow a} \hat{f}^{\wedge}(x)$ *if*

1. $a = 1,\ f(x) = x^2 + 2,\ \hat{T}(x) = x,$
2. $a = 2,\ f(x) = x,\ \hat{T}(x) = x^2 + 1,$
3. $a = 3,\ f(x) = x + 2,\ \hat{T}(x) = 2x + 3.$

Answer. 1) 3, 2) 2, 3) $\frac{29}{9}$.

Exercise 5.0.276. *Let* $D = \mathbb{R}_+$. *Find* $\lim_{x\longrightarrow a} \hat{f}(\hat{x})$ *if*

1. $a = 1,\ f(x) = x^2 + 2,\ \hat{T}(x) = x,$
2. $a = 2,\ f(x) = x,\ \hat{T}(x) = x^2 + 1,$
3. $a = 3,\ f(x) = x + 2,\ \hat{T}(x) = 2x + 3.$

Answer. 1) 3, 2) $\frac{2}{25}$, 3) $\frac{7}{27}$.

Exercise 5.0.277. *Let* $D = \mathbb{R}_+$. *Find* $\lim_{x\longrightarrow a} f^{\wedge}(x)$ *if*

1. $a = 1,\ f(x) = x^2 + 2,\ \hat{T}(x) = x,$
2. $a = 2,\ f(x) = x,\ \hat{T}(x) = x^2 + 1,$
3. $a = 3,\ f(x) = x + 2,\ \hat{T}(x) = 2x + 3.$

Answer. 1) 3, 2) 10, 3) 29.

Exercise 5.0.278. *Let* $D = [100, \infty)$. *Find* $\lim_{x\longrightarrow\infty} \hat{f}^{\wedge}(\hat{x})$ *if*

1. $f(x) = \ln(e^x - 1),\ \hat{T}(x) = x,$
2. $f(x) = \ln(e^x - x^2),\ \hat{T}(x) = x,$
3. $f(x) = \ln(1 + x),\ \hat{T}(x) = \ln x,$
4. $f(x) = \ln(x + \ln^2 x),\ \hat{T}(x) = \ln x,$

5. $f(x) = \ln(x^3 + x^2 + 1)$, $\hat{T}(x) = \ln(x^2 - 2x - 2)$,
6. $f(x) = \ln(x + 2^x)$, $\hat{T}(x) = \ln(x - 3)$.

Answer. 1) 1, 2) 1, 3) 1, 4) 1, 5) $\frac{3}{2}$, 6) ∞.

Theorem 5.0.279. *The limit $\lim_{x \longrightarrow x_0} \hat{f}(x) = a$ exists if and only if for every $\epsilon > 0$ there exists a neighbourhood $U(x_0)$ such that for every $x_1, x_2 \in U(x_0)$, $x_1 \neq x_0$, $x_2 \neq x_0$, we have*

$$|\hat{f}(x_1) - \hat{f}(x_2)| < \epsilon.$$

Proof. 1. Let the limit $\lim_{x \longrightarrow x_0} \hat{f}(x) = a$ exists and $\epsilon > 0$. Then there exists a neighbourhood $U(x_0)$ so that for $x_1 \in U(x_0)$, $x_1 \neq x_0$, we have

$$|\hat{f}(x_1) - a| < \frac{\epsilon}{2},$$

and for $x_2 \in U(x_0)$, $x_2 \neq x_0$, we have

$$|\hat{f}(x_1) - a| < \frac{\epsilon}{2}.$$

Therefore

$$|\hat{f}(x_1) - \hat{f}(x_2)| = |\hat{f}(x_1) - a + a - \hat{f}(x_2)|$$

$$\leq |\hat{f}(x_1) - a| + |\hat{f}(x_2) - a| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

2. Let for every $\epsilon > 0$ there exists a neighbourhood $U(x_0)$ such that for every $x_1, x_2 \in U(x_0)$, $x_1 \neq x_0$, $x_2 \neq x_0$, we have

$$|\hat{f}(x_1) - \hat{f}(x_2)| < \epsilon.$$

We fix $\epsilon > 0$ and $\hat{U}(x_0, \delta)$. Then for every sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \longrightarrow_{n \longrightarrow \infty} x_0$, there exists $N > 0$ such that for $n > N$ we have

$$|x_n - x_0| < \delta.$$

Let now $m, n > N$. Then $x_m, x_n \in U(x_0)$ and

$$|\hat{f}(x_n) - \hat{f}(x_m)| < \epsilon.$$

Therefore the sequence $\{\hat{f}(x_n)\}_{n=1}^{\infty}$ is fundametal sequence in D, therefore it is convergent. Consequently there exists $\lim_{x\longrightarrow x_0} \hat{f}(x) = a$.

$\square$

Definition 5.0.280. *The isofunction $\hat{f}$ of first, second, third or fourth kind will be called continuous at the point $x_0 \in D$ if its isooriginal is continuous function at x_0.*

From the properties of the limit of isofunctions follows the validity of the following assertions.

Proposition 5.0.281. *Let $\hat{g} : D \longrightarrow \hat{g}(D)$ and $\hat{f}$ are continuous at x_0, $x_0 \in D$. Then*

1. *$\hat{f} \pm \hat{g}$ is continuous at x_0,*
2. *$\hat{f}\hat{\times}\hat{g}$ is continuous at x_0,*
3. *$\hat{f}\hat{g}$ is continuous at x_0,*
4. *$\hat{f} \hat{/} \hat{g}$ is continuopus at x_0 if $\hat{g}(x_0) \neq 0$,*
5. *$\frac{\hat{f}}{\hat{g}}$ is continuopus at x_0 if $\hat{g}(x_0) \neq 0$*

Exercise 5.0.282. *Let $D = [2, +\infty)$. Prove that the isofunctions*

$$\hat{f}^{\wedge}(\hat{x}), \hat{f}^{\wedge}(x), \hat{f}(\hat{x}), f^{\wedge}(x)$$

are continuous functions if

1. *$f(x) = x^2 + 1$, $\hat{T}(x) = e^x$,*
2. *$f(x) = x$, $\hat{T}(x) = \ln(x + 1)$,*
3. *$f(x) = \ln(x + 2)$, $\hat{T}(x) = \ln(10 + \sin x)$.*

Theorem 5.0.283. *Let $\hat{f}$ is continuous at $x_0 \in D$. Then there exists a neighbourhood $U(x_0)$ so that in $U(x_0) \cap D$ the function $\hat{f}$ is bounded.*

Theorem 5.0.284. *Let $\hat{f}$ is continuous at $x_0 \in D$. Then*

1. *there exists a neighbourhood $U(x_0)$ such that for every $x \in U(x_0) \cap D$, $x \neq x_0$, we have*
$$|\hat{f}(x)| > \frac{|\hat{f}(x_0)|}{2},$$
2. *if $\hat{f}(x_0) > 0$ there exists a neighbourhood $U(x_0)$ such that for every $x \in U(x_0) \cap D$, $x \neq x_0$, we have*
$$\hat{f}(x) > \frac{\hat{f}(x_0)}{2},$$
3. *if $\hat{f}(x_0) < 0$ there exists a neighbourhood $U(x_0)$ such that for every $x \in U(x_0) \cap D$, $x \neq x_0$, we have*
$$\hat{f}(x) < \frac{\hat{f}(x_0)}{2}.$$

Theorem 5.0.285. *Let $\hat{f}$ is continuous at x_0, $x_0 \in D$, and $\hat{g} : \hat{f}(D) \longrightarrow \hat{g}(\hat{f}(D))$ is continous at $\hat{u}_0 = \hat{f}(x_0)$. Then the function $\hat{g} \hat{\circ} \hat{f}$ is continuous at x_0. Here $\hat{\circ}$ is the composition of the isofunctions $\hat{f}$ and $\hat{g}$, defined in Chapter 2.*

Proof. Let $\epsilon > 0$ be fixed. Since $\hat{g}$ is continuous at $\hat{u}_0$ then there exists a neighbourhood $U(u_0) = U(u_0, \eta)$, $\eta > 0$, such that from

$$|u - u_0| < \eta$$

follows that

$$|\hat{g}(u) - \hat{g}(u_0)| < \epsilon, u \in \hat{f}(D).$$

Since $\hat{f}$ is continuous at x_0 then there exists $\delta > 0$ such that from

$$|x - x_0| < \delta, x \in D,$$

follows

$$|u - u_0| = |\hat{f}(x) - \hat{f}(x_0)| < \eta$$

and from here

$$|\hat{g}\hat{\circ}\hat{f}(x) - \hat{g}\hat{\circ}\hat{f}(x_0)| < \epsilon,$$

therefore $\hat{g}\hat{\circ}\hat{f}$ is continuous at x_0. □

Definition 5.0.286. *The isofunction $\hat{f}$ of first, second, third or fourth kind will be called discontinuous at $x_0 \in D$ of first kind if there exist*

$$\hat{f}(x_0 - 0), \quad \hat{f}(x_0 + 0),$$

and

$$\hat{f}(x_0 - 0) \neq \hat{f}(x_0 + 0).$$

Definition 5.0.287. *The isofunction $\hat{f}$ will be called discontinuous of second kind at $x_0 \in D$ if one of*

$$\hat{f}(x_0 - 0), \quad \hat{f}(x_0 + 0)$$

does not exist. Here are included the cases

$$\hat{f}(x_0 - 0) = \pm\infty, \quad \hat{f}(x_0 + 0) = \pm\infty.$$

Definition 5.0.288. *We will say that the isofunction of first second, third or fourth kind is continuous in D if it is continuous at every its point.*

Below we will suppose that $K \subseteq D$ is compact isoset.

Theorem 5.0.289. *Let $\hat{f} : K \longrightarrow D$ is continuous function in K. Then it is bounded.*

Proof. Let u suppose that the isofunction $\hat{f}$ is nonbounded. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of K so that

$$|\hat{f}(x_n)| \geq n. \tag{5.0.3}$$

From the main properties of the bounded sequences follows that there exists an subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which is convergent to $x_0 \in K$, from here follows that $\{\hat{f}(x_{n_k})\}_{k=1}^{\infty}$ is convergent to $\hat{f}(x_0)$, which is a contradiction with (5.0.3). □

Advanced practical exercises

Problem 5.0.290. *Let $D = \mathbb{R}_+$. Find*

$$\lim_{\longrightarrow a} \hat{f}^{\wedge}(\hat{x}), \lim_{\longrightarrow a} \hat{f}^{\wedge}(x), \lim_{\longrightarrow a} \hat{f}(\hat{x}), \lim_{\longrightarrow a} f^{\wedge}(x),$$

if

1. $a = 2,\ f(x) = x + 4,\ \hat{T}(x) = x^2 + 2,$
2. $a = 3,\ f(x) = x^2 + x + 1,\ \hat{T}(x) = x + 1,$
3. $a = 1,\ f(x) = x + 2,\ \hat{T}(x) = x^2 + x.$

Answer. 1) 1, $\frac{8}{3}$, $\frac{13}{18}$, 16, 2) $\frac{13}{4}$, $\frac{157}{4}$, $\frac{37}{64}$, 157, 3) $\frac{3}{2}$, 2, $\frac{5}{4}$, 4.

Problem 5.0.291. *Let $D = [5, \infty)$. Find* $\lim_{x \longrightarrow \infty} \hat{f}6\wedge(\hat{x})$ *if*

1. $f(x) = x,\ \hat{T}(x) = 2^x,$

2. $f(x) = x$, $\hat{T}(x) = a^x$, $a > 1$,
3. $f(x) = x^2$, $\hat{T}(x) = a^x$, $a > 1$,
4. $f(x) = x^n$, $\hat{T}(x) = a^x$, $a > 1$, $n \in \mathbb{N}$,
5. $f(x) = x^\alpha$, $\hat{T}(x) = a^x$, $a > 1$, $\alpha \in \mathbb{R}$,
6. $f(x) = \ln x$, $\hat{T}(x) = x$,
7. $f(x) = \ln x$, $\hat{T}(x) = x^\alpha$, $\alpha > 0$,
8. $f(x) = \ln x$, $\hat{T}(x) = x^\alpha$, $\alpha < 0$,
9. $f(x) = x + \ln^2 x$, $\hat{T}(x) = x + 1$,
10. $f(x) = \ln(e^x - 10$, $\hat{T}(x) = x$,
11. $f(x) = \ln(e^x - x^2)$, $\hat{T}(x) = x$,
12. $f(x) = \ln(x + 1)$, $\hat{T}(x) = \ln x$,
13. $f(x) = \ln(x + 2^x)$, $\hat{T}(x) = \ln(x - 3)$.

Answer. 1) - 7) 0, 8) ∞, 9)-12) 1, 13) ∞.

Problem 5.0.292. *Let $D = \mathbb{R}_+$. Prove that the functions*

$$\hat{f}^\wedge(\hat{x}), \hat{f}^\wedge(x), \hat{f}(\hat{x}), f^\wedge(x)$$

are continuous functions if

1. $f(x) = x^2 + 10$, $\hat{T}(x) = \sin x + 10$,
2. $f(x) = x^6$, $\hat{T}(x) = e^x$,
3. $f(x) = \sin x + \cos(2x)$, $\hat{T}(x) = \ln x$.

Chapter 6

Isodifferentiable isofunctions

Let $D \subset \mathbb{R}$ be given set, $f : D \longrightarrow \mathbb{R}$ is enough times differentiable function, $\hat{T} : D \longrightarrow \mathbb{R}$ is positive and enough times differentiable function. Where is necessary we will suppose the additional condition $x\hat{T}(x) \in D$ for every $x \in D$ or $\frac{x}{\hat{T}(x)} \in D$ for every $x \in D$ so that to be defined the second, fourth or third kinds isofunctions, respectively.

Definition: For arbitrary isofunction h(of first or second or third or fourth kind) we define isodifferential $\hat{d}$ of h in the following way

$$\hat{d}(h) = \hat{T}(x)d(h),$$

where $d(h)$ is the first differential of h.

Using the above definition for the isodifferential of the isofunctions of first, second, third or fourth kind we have the following representations

$$\hat{d}\Big(\hat{f}^\wedge(\hat{x})\Big) = \hat{T}(x)d\Big(\hat{f}^\wedge(\hat{x})\Big)$$

$$= \hat{T}(x)d\Big(\tfrac{f(x)}{\hat{T}(x)}\Big)$$

$$= \hat{T}(x)\Big(\tfrac{f(x)}{\hat{T}(x)}\Big)'dx$$

$$= \hat{T}(x)\tfrac{f'(x)\hat{T}(x)-f(x)\hat{T}'(x)}{\hat{T}^2(x)}dx$$

$$= \Big(f'(x) - f(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx,$$

i.e.

$$\boxed{\hat{d}\Big(\hat{f}^\wedge(\hat{x})\Big) = \Big(f'(x) - f(x)\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx} \tag{6.0.1}$$

$$\hat{d}\Big(\hat{f}^\wedge(x)\Big) = \hat{T}(x)d\Big(\hat{f}^\wedge(x)\Big)$$

$$= \hat{T}(x)d\Big(\tfrac{f(x\hat{T}(x))}{\hat{T}(x)}\Big)$$

$$= \hat{T}(x)\Big(\tfrac{f(x\hat{T}(x))}{\hat{T}(x)}\Big)'dx$$

$$= \hat{T}(x)\tfrac{(f(x\hat{T}(x)))'\hat{T}(x)-f(x\hat{T}(x))\hat{T}'(x)}{\hat{T}^2(x)}dx$$

$$= \hat{T}(x)\tfrac{f'(x\hat{T}(x))(\hat{T}(x)+x\hat{T}'(x))\hat{T}(x)-f(x\hat{T}(x))\hat{T}'(x)}{\hat{T}^2(x)}dx$$

$$= \Big(f'(x\hat{T}(x))(\hat{T}(x) + x\hat{T}'(x)) - f(x\hat{T}(x))\tfrac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx,$$

i.e.

$$\boxed{\hat{d}\Big(\hat{f}^\wedge(x)\Big) = \Big(f'(x\hat{T}(x))(\hat{T}(x) + x\hat{T}'(x)) - f(x\hat{T}(x))\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx} \tag{6.0.2}$$

$$\hat{d}\Big(\hat{f}(\hat{x})\Big) = \hat{T}(x)d\Big(\frac{f\Big(\frac{x}{\hat{T}(x)}\Big)}{\hat{T}(x)}\Big)$$

$$= \hat{T}(x)\Big(\frac{f\Big(\frac{x}{\hat{T}(x)}\Big)}{\hat{T}(x)}\Big)'dx$$

$$= \hat{T}(x)\frac{\Big(f\Big(\frac{x}{\hat{T}(x)}\Big)\Big)'\hat{T}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}'(x)}{\hat{T}^2(x)}dx$$

$$= \frac{f'\Big(\frac{x}{\hat{T}(x)}\Big)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}^2(x)}\hat{T}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}'(x)}{\hat{T}(x)}dx$$

$$= \Big(f'\Big(\frac{x}{\hat{T}(x)}\Big)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}^2(x)} - f\Big(\frac{x}{\hat{T}(x)}\Big)\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx,$$

i.e.

$$\boxed{\hat{d}\Big(\hat{f}(\hat{x})\Big) = \Big(f'\Big(\frac{x}{\hat{T}(x)}\Big)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}^2(x)} - f\Big(\frac{x}{\hat{T}(x)}\Big)\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx \qquad (6.0.3)}$$

$$\hat{d}\Big(f^{\wedge}(x)\Big) = \hat{T}(x)d\Big(f^{\wedge}(x)\Big)$$

$$= \hat{T}(x)\Big(f^{\wedge}(x)\Big)'dx$$

$$= \hat{T}(x)\Big(f(x\hat{T}(x))\Big)'dx$$

$$= \hat{T}(x)f'(x\hat{T}(x))(\hat{T}(x) + x\hat{T}'(x))dx,$$

i.e.

$$\boxed{\hat{d}\Big(f^{\wedge}(x)\Big) = \hat{T}(x)f'(x\hat{T}(x))(\hat{T}(x) + x\hat{T}'(x))dx \qquad (6.0.4)}$$

In particular

$$\hat{d}\hat{x} = \hat{T}(x)d\hat{x} = \hat{T}(x)d\Big(\tfrac{x}{\hat{T}(x)}\Big)$$

$$= \hat{T}(x)\Big(\tfrac{x}{\hat{T}(x)}\Big)' dx$$

$$= \hat{T}(x)\tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx$$

$$= \Big(1 - x\tfrac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx,$$

i.e.

$$\boxed{\hat{d}\hat{x} = \Big(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx} \tag{6.0.5}$$

Example 6.0.293. *Let $D = \mathbb{R}$, $f(x) = x + 1$, $\hat{T}(x) = x^2 + 1$, $x \in D$. Then*

$$f'(x) = x, \quad \hat{T}'(x) = 2x, \quad x\hat{T}(x) = x^3 + x, \quad \tfrac{x}{\hat{T}(x)} = \tfrac{x}{x^2+1},$$

$$f(x\hat{T}(x)) = x^3 + x + 1, \quad f\Big(\tfrac{x}{\hat{T}(x)}\Big) = \tfrac{x}{x^2+1} + 1.$$

Using (6.0.1) *we have*

$$\hat{d}\Big(\hat{f}^\wedge(\hat{x})\Big) = \Big(1 - (x+1)\tfrac{2x}{x^2+1}\Big)dx$$

$$= \tfrac{-x^2-2x+1}{1+x^2}dx.$$

Using (6.0.2) *we get*

$$\hat{d}\Big(\hat{f}^\wedge(x)\Big) = \Big(x^2 + 1 + 2x^2 - (x^3 + x + 1)\tfrac{2x}{x^2+1}\Big)dx$$

$$= \tfrac{x^4+x^2-2x}{x^2+1}dx.$$

Using (6.0.3) *we obtain*

$$\hat{d}\Big(\hat{f}(\hat{x})\Big) = \Big(\tfrac{x^2+1-2x^2}{(x^2+1)^2} - \Big(\tfrac{x}{x^2+1} + 1\Big)\tfrac{2x}{x^2+1}\Big)dx$$

$$= \tfrac{-2x^3-3x^2-2x+1}{(x^2+1)^2}dx.$$

Using (6.0.4) *we have*

$$\hat{d}\Big(f^\wedge(x)\Big) = (x^2+1)(x^2+1+2x^2)dx$$

$$= (3x^4+4x^2+1)dx,$$

and using (6.0.5) *we obtain*

$$\hat{d}\hat{x} = \Big(1 - x\tfrac{2x}{x^2+1}\Big)dx = \tfrac{1-x^2}{1+x^2}dx.$$

Exercise 6.0.294. *Let* $D = \mathbb{R}$, $f(x) = x-1$, $\hat{T}(x) = e^x$, $x \in D$. *Find*

$$\hat{d}\Big(\hat{f}^\wedge(\hat{x})\Big), \quad \hat{d}\Big(\hat{f}^\wedge(x)\Big), \quad \hat{d}\Big(\hat{f}(\hat{x})\Big), \quad \hat{d}\Big(f^\wedge(x)\Big), \quad \hat{d}\hat{x}.$$

Answer.

$$\hat{d}\Big(\hat{f}^\wedge(\hat{x})\Big) = (2-x)dx, \quad \hat{d}\Big(\hat{f}^\wedge(x)\Big) = (e^x+1)dx,$$

$$\hat{d}\Big(\hat{f}(\hat{x})\Big) = e^{-x}(1-2x+e^x)dx, \quad \hat{d}\Big(f^\wedge(x)\Big) = e^{2x}(x+1)dx, \quad \hat{d}\hat{x} = (1-x)dx.$$

Definition: For arbitrary isofunction h(of first or second or third or fourth kind) we define its first isoderivatrve as follows

$$h^{\circledast} = \hat{d}(h) \nearrow \hat{d}\hat{x} := \frac{1}{\hat{T}(x)}\frac{\hat{d}(h)}{\hat{d}\hat{x}}.$$

For the first isoderivatives of isofunctions of first, second, third and fourth kind, using (6.0.1)-(6.0.5), when for $x \in D$

$$1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0,$$

we have the following representations

$$\hat{f}^{\wedge\circledast}(\hat{x}) = \frac{1}{\hat{T}^2(x)}\frac{f'(x)\hat{T}(x) - f(x)\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}} \tag{6.0.6}$$

$$\boxed{\hat{f}^{\wedge\circledast}(x) = \frac{1}{\hat{T}^2(x)} \frac{f'(x\hat{T}(x))(\hat{T}^2(x) + x\hat{T}(x)\hat{T}'(x)) - f(x\hat{T}(x))\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}} \quad (6.0.7)}$$

$$\boxed{\hat{f}^{\circledast}(\hat{x}) = \frac{1}{\hat{T}^2(x)} \frac{f'\left(\frac{x}{\hat{T}(x)}\right)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}} \quad (6.0.8)}$$

$$\boxed{f^{\wedge\circledast}(x) = \frac{f'(x\hat{T}(x))(\hat{T}(x) + x\hat{T}'(x))}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}} \quad (6.0.9)}$$

Example 6.0.295. *Let $D = [0, \infty)$, $f(x) = x$, $\hat{T}(x) = x + 1$, $x \in D$. Find*

$$\hat{f}^{\wedge\circledast}(\hat{x}), \quad \hat{f}^{\wedge\circledast}(x), \quad \hat{f}^{\circledast}(\hat{x}), \quad f^{\wedge\circledast}(x).$$

Solution. For $x \in D$ we have

$$f'(x) = 1, \quad \hat{T}(x) = 1, \quad f(x\hat{T}(x)) = f(x^2 + x) = x^2 + 1,$$

$$f\left(\tfrac{x}{\hat{T}(x)}\right) = f\left(\tfrac{x}{x+1}\right) = \tfrac{x}{x+1}.$$

Then, for $x \in D$, using (6.0.6)-(6.0.9), we get

$$\hat{f}^{\wedge\circledast}(\hat{x}) = \frac{1}{(x+1)^2} \frac{x + 1 - x}{1 - \frac{x}{x+1}} = \frac{1}{(x+1)^2} \frac{1}{\frac{1}{x+1}} = \frac{1}{x+1},$$

$$\hat{f}^{\wedge\circledast}(x) = \tfrac{1}{(x+1)^2} \tfrac{(x+1)^2 + x^2 - x^2 - 1}{1 - \frac{x}{x+1}} = \tfrac{1}{(x+1)^2} \tfrac{x^2 + 2x}{\frac{1}{x+1}} = \tfrac{1}{(x+1)^2}(x^2 + 2x)(x + 1) = \tfrac{x^2 + 2x}{x+1},$$

$$\hat{f}^{\circledast}(\hat{x}) = \tfrac{1}{(x+1)^2} \tfrac{\frac{x+1-x}{x+1} - \frac{x}{x+1}}{1 - \frac{x}{x+1}} = \tfrac{1}{(x+1)^2} \tfrac{\frac{1-x}{x+1}}{\frac{1}{x+1}} = \tfrac{1-x}{(1+x)^2},$$

$$f^{\wedge\circledast}(x) = \tfrac{1}{(x+1)^2} \tfrac{(x+1)^2((x+1) - x)}{1 - \frac{x}{x+1}} = \tfrac{1}{(x+1)^2} \tfrac{(x+1)^2}{\frac{1}{x+1}} = x + 1.$$

Exercise 6.0.296. *Let* $D=\mathbb{R}$, $f(x)=2x+1$, $\hat{T}(x)=e^x$, $x\in D$. *Find*

$$\hat{f}^{\wedge\circledast}(\hat{x}),\quad \hat{f}^{\wedge\circledast}(x),\quad \hat{f}^{\circledast}(\hat{x}),\quad f^{\wedge\circledast}(x).$$

Answer.

$$\hat{f}^{\wedge\circledast}(\hat{x})=\tfrac{1-2x}{1-x}e^{-x},\quad \hat{f}^{\wedge\circledast}(x)=\tfrac{1-4x}{1-x},$$

$$\hat{f}^{\circledast}(\hat{x})=e^{-2x}\tfrac{2-4x-e^x}{1-x},\quad f^{\wedge\circledast}(x)=2\tfrac{1+x}{1-x}e^x,\quad x\neq 1.$$

Exercise 6.0.297. *Let* $D=\mathbb{R}$, $\hat{T}_1=2$, $f(x)=x+1$, $\hat{T}(x)=x^2+1$, $x\in D$. *Find*

$$A:=\hat{f}^{\wedge\circledast}(\hat{x})+\hat{3}\hat{\times}\hat{f}^{\circledast}(\hat{x}), x\in D.$$

Solution. For $x\in D$ we have

$$f'(x)=1,\hat{T}'(x)=2x,$$

$$f(x\hat{T}(x))=f(x(x^2+1))=f(x^3+x)=x^3+x+1,$$

$$f\left(\tfrac{x}{\hat{T}(x)}\right)=f\left(\tfrac{x}{x^2+1}\right)=\tfrac{x}{x^2+1}+1=\tfrac{x^2+x+1}{1+x^2},$$

$$\hat{f}^{\wedge\circledast}(\hat{x})=\tfrac{1}{(1+x^2)^2}\tfrac{x^2+1-(x+1)2x}{1-x\frac{2x}{1+x^2}}=\tfrac{1}{(1+x^2)^2}\tfrac{(-x^2-2x+1)(1+x^2)}{1-x^2}=\tfrac{-x^2-2x+1}{1-x^4},\quad x\neq\pm1,$$

$$\hat{f}^{\circledast}(\hat{x})=\tfrac{1}{(1+x^2)^2}\tfrac{\frac{x^2+1-x2x}{1+x^2}-\frac{x^2+x+1}{1+x^2}2x}{1-x\frac{2x}{1+x^2}}=\tfrac{-2x^3-3x^2-2x+1}{(1+x^2)^2(1-x^2)},\quad x\neq\pm1,$$

$$\hat{3}\hat{\times}\hat{f}^{\circledast}(\hat{x})=3\tfrac{1}{(1+x^2)^2}\tfrac{-2x^3-3x^2-2x+1}{1-x^2}=\tfrac{-6x^3-9x^2-6x+1}{(1+x^2)^2(1-x^2)},\quad x\neq\pm1.$$

Then

$$A=\tfrac{1}{(1+x^2)^2}\left(\tfrac{-x^4-2x^3-2x+1}{1-x^2}+\tfrac{-6x^3-9x^2-6x+1}{1-x^2}\right)=\tfrac{-x^4-8x^3-9x^2-8x+4}{(1+x^2)^2(1-x^2)},\quad x\neq\pm1.$$

Exercise 6.0.298. *Let* $D=[0,\infty)$, $\hat{T}_1=4$, $f(x)=x-1$, $\hat{T}(x)=x+1$, $x\in D$. *Find*

1) $A:=\hat{f}^{\wedge}(\hat{x})+\hat{f}^{\wedge\circledast}(\hat{x})$,

2) $B:=f(\hat{x})+\hat{f}^{\wedge\circledast}(x)$,

3) $C:=f^{\wedge}(x)+\hat{f}^{\circledast}(\hat{x})$,

4) $D:=f(\hat{x})+\hat{2}\hat{\times}f^{\wedge\circledast}(x)$.

Theorem: Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$ for every $x \in D$. Then for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have

$$\widehat{f \pm g}^{\wedge \circledast}(\hat{x}) = \hat{f}^{\wedge \circledast}(\hat{x}) \pm \hat{g}^{\wedge \circledast}(\hat{x}).$$

Proof. For every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$, using (6.0.6), we have

$$\widehat{f \pm g}^{\wedge \circledast}(\hat{x}) = \frac{1}{\hat{T}^2(x)} \frac{(f(x) \pm g(x))'\hat{T}(x) - (f(x) \pm g(x))\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)} \frac{(f'(x) \pm g'(x))\hat{T}(x) - (f(x) \pm g(x))\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)} \frac{f'(x)\hat{T}(x) - f(x)\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}} \pm \frac{1}{\hat{T}^2(x)} \frac{g'(x)\hat{T}(x) - g(x)\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \hat{f}^{\wedge \circledast}(\hat{x}) \pm \hat{g}^{\wedge \circledast}(\hat{x}).$$

□

Theorem: Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Then for $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have

$$\widehat{f \pm g}^{\wedge \circledast}(x) = \hat{f}^{\wedge \circledast}(x) \pm \hat{g}^{\wedge \circledast}(x).$$

Proof. For every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$, using (6.0.7), we have

$$\widehat{f \pm g}^{\wedge\circledast}(x) = \frac{1}{\hat{T}^2(x)} \frac{(f\pm g)'(x\hat{T}(x))(\hat{T}^2(x)+x\hat{T}(x)\hat{T}'(x))-(f\pm g)(x\hat{T}(x))\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)} \frac{(f'(x\hat{T}(x))\pm g'(x\hat{T}(x))(\hat{T}^2(x)+x\hat{T}(x)\hat{T}'(x))-(f(x\hat{T}(x))\pm g(x\hat{T}(x)))\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)} \frac{f'(x\hat{T}(x))(\hat{T}^2(x)+x\hat{T}(x)\hat{T}'(x))-f(x\hat{T}(x))\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}} \pm \frac{1}{\hat{T}^2(x)} \frac{g'(x\hat{T}(x))(\hat{T}^2(x)+x\hat{T}(x)\hat{T}'(x))-g(x\hat{T}(x))\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \hat{f}^{\wedge\circledast}(x) \pm \hat{g}^{\wedge\circledast}(x).$$

□

Theorem: Let $f, g, \hat{T} \in ;\mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$ for $x \in D$. Then for $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have

$$\widehat{f \pm g}^{\circledast}(\hat{x}) = \hat{f}^{\circledast}(\hat{x}) \pm \hat{g}^{\circledast}(\hat{x}).$$

Proof. For every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$, using (6.0.8), we have

$$\widehat{f \pm g}^{\circledast}(\hat{x}) = \frac{1}{\hat{T}^2(x)} \frac{(f\pm g)'\left(\frac{x}{\hat{T}(x)}\right)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}-(f\pm g)\left(\frac{x}{\hat{T}(x)}\right)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)} \frac{\left(f'\left(\frac{x}{\hat{T}(x)}\right)\pm g'\left(\frac{x}{\hat{T}(x)}\right)\right)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}-\left(f\left(\frac{x}{\hat{T}(x)}\right)\pm g\left(\frac{x}{\hat{T}(x)}\right)\right)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)} \frac{f'\left(\frac{x}{\hat{T}(x)}\right)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}} \pm \frac{1}{\hat{T}^2(x)} \frac{g'\left(\frac{x}{\hat{T}(x)}\right)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}-g\left(\frac{x}{\hat{T}(x)}\right)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \hat{f}^{\circledast}(\hat{x}) \pm \hat{g}^{\circledast}(\hat{x}).$$

□

Theorem: Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Then for $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have

$$(f \pm g)^{\wedge\circledast}(x) = f^{\wedge\circledast}(x) \pm g^{\wedge\circledast}(x).$$

Proof. For $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$, using (6.0.9), we have

$$(f \pm g)^{\wedge\circledast}(x) = \frac{1}{\hat{T}^2(x)} \frac{(f \pm g)'(x\hat{T}(x))(\hat{T}(x) + x\hat{T}'(x))}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)} \frac{(f'(x\hat{T}(x)) \pm g'(x\hat{T}(x)))(\hat{T}(x) + x\hat{T}'(x))}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)} \frac{f'(x\hat{T}(x))(\hat{T}(x) + x\hat{T}'(x))}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}} \pm \frac{1}{\hat{T}^2(x)} \frac{g'(x\hat{T}(x))(\hat{T}(x) + x\hat{T}'(x))}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= f^{\wedge\circledast}(x) \pm g^{\wedge\circledast}(x).$$

□

Theorem: Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$ for every $x \in D$. Then for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have

$$\left(\hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge}(\hat{x})\right)^{\circledast} = \hat{f}^{\wedge\circledast}(\hat{x})\hat{\times}\hat{g}^{\wedge}(\hat{x}) + \hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge\circledast}(\hat{x})$$

$$+ \frac{1}{\hat{T}^2(x)} f(x)g(x) \frac{\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Proof. For $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have

$$\left(\hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge}(\hat{x})\right)^{\circledast} = \hat{d}(\hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge}(\hat{x})) \nearrow \hat{d}\hat{x} = \frac{1}{\hat{T}(x)}\frac{\hat{d}(\hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge}(\hat{x}))}{\hat{d}\hat{x}}$$

$$= \frac{d\left(\frac{f(x)g(x)}{\hat{T}(x)}\right)}{\left(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx} = \frac{\left(\frac{f(x)g(x)}{\hat{T}(x)}\right)'dx}{\left(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx}$$

$$= \frac{\frac{(f(x)g(x))'\hat{T}(x)-f(x)g(x)\hat{T}'(x)}{\hat{T}^2(x)}}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}} = \frac{1}{\hat{T}^2(x)}\frac{(f(x)g(x))'\hat{T}(x)-f(x)g(x)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)}\frac{f'(x)g(x)\hat{T}(x)+f(x)g'(x)\hat{T}(x)-f(x)g(x)\hat{T}(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)}\frac{f'(x)g(x)\hat{T}(x)-f(x)g(x)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}} + \frac{1}{\hat{T}^2(x)}\frac{f(x)g'(x)\hat{T}(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)}\frac{f'(x)g(x)\hat{T}(x)-f(x)g(x)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}} + \frac{1}{\hat{T}^2(x)}\frac{f(x)g'(x)\hat{T}(x)-f(x)g(x)\hat{T}'(x)+f(x)g(x)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)}\frac{f'(x)g(x)\hat{T}(x)-f(x)g(x)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}} + \frac{1}{\hat{T}^2(x)}\frac{f(x)g'(x)\hat{T}(x)-f(x)g(x)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$+\frac{1}{\hat{T}^2(x)}f(x)g(x)\frac{\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \hat{f}^{\wedge\circledast}(\hat{x})g(x) + f(x)\hat{g}^{\wedge\circledast}(\hat{x}) + \frac{1}{\hat{T}^2(x)}f(x)g(x)\frac{\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \hat{f}^{\wedge\circledast}(\hat{x})\hat{T}(x)\frac{g\left(\hat{T}(x)\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} + \frac{f\left(\hat{T}(x)\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}\hat{T}(x)\hat{g}^{\wedge\circledast}(\hat{x})$$

$$+\frac{1}{\hat{T}^2(x)}f(x)g(x)\frac{\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \hat{f}^{\wedge\circledast}(\hat{x})\hat{\times}\hat{g}^{\wedge}(\hat{x}) + \hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge\circledast}(\hat{x}) + \frac{1}{\hat{T}^2(x)}f(x)g(x)\frac{\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

□

Exercise 6.0.299. *Let* $f, g, \hat{T} \in \mathcal{C}(D)$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ *for every* $x \in D$. *Prove that for* $x \in D$ *for which* $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *we have*

$$\left(\hat{f}^{\wedge}(\hat{x}) \hat{\times} \hat{g}^{\wedge}(x)\right)^{\circledast} = \hat{f}^{\wedge\circledast}(\hat{x}) \hat{\times} \hat{g}^{\wedge}(x) + \hat{f}^{\wedge}(\hat{x}) \hat{\times} \hat{g}^{\wedge\circledast}(x) + \frac{1}{\hat{T}^2(x)} \frac{f(x)g(x\hat{T}(x))\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Solution. For $x \in D$ and for $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have

$$\Big(\hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge}(x)\Big)^{\circledast} = \hat{d}\Big(\hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge}(x)\Big) \nearrow \hat{d}\hat{x}$$

$$= \frac{1}{\hat{T}(x)} \frac{\hat{d}(\hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge}(x))}{\hat{d}\hat{x}}$$

$$= \frac{1}{\hat{T}(x)} \frac{\hat{T}(x) d\Big(\hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge}(x)\Big)}{\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx}$$

$$= \frac{d\Big(\frac{f(x)g(x\hat{T}(x))}{\hat{T}(x)}\Big)}{\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx}$$

$$= \frac{\Big(\frac{f(x)g(x\hat{T}(x))}{\hat{T}(x)}\Big)' dx}{\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx}$$

$$= \frac{\frac{(f(x)g(x\hat{T}(x)))'\hat{T}'(x) - f(x)g(x\hat{T}(x))\hat{T}'(x)}{\hat{T}^2(x)}}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)} \frac{f'(x)g(x\hat{T}(x))\hat{T}(x) + f(x)g'(x\hat{T}(x))(\hat{T}(x)+x\hat{T}'(x))\hat{T}(x) - f(x)g(x\hat{T}(x))\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)} \frac{f'(x)g(x\hat{T}(x))\hat{T}(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}} + \frac{1}{\hat{T}^2(x)} f(x) \frac{g'(x\hat{T}(x))(\hat{T}(x)+x\hat{T}'(x))\hat{T}(x) - g(x\hat{T}(x))\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)} \frac{f'(x)g(x\hat{T}(x))\hat{T}(x) - f(x)g(x\hat{T}(x))\hat{T}'(x) + \frac{1}{\hat{T}^2(x)} f(x)g(x\hat{T}(x))\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}} + f(x)\hat{g}^{\wedge\circledast}(x)$$

$$= \frac{1}{\hat{T}^2(x)} \frac{f'(x)g(x\hat{T}(x))\hat{T}(x) - f(x)g(x\hat{T}(x))\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}} + \frac{1}{\hat{T}^2(x)} \frac{f(x)g(x\hat{T}(x))\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}} + f(x)\hat{g}^{\wedge\circledast}(x)$$

$$= \hat{f}^{\wedge\circledast}(\hat{x})g(x\hat{T}(x)) + f(x)\hat{g}^{\wedge\circledast}(x) + \frac{1}{\hat{T}^2(x)} \frac{f(x)g(x\hat{T}(x))\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \hat{f}^{\wedge\circledast}(\hat{x})\hat{T}(x)\frac{g(x\hat{T}(x))}{\hat{T}(x)} + \frac{f\Big(\hat{T}(x)\frac{x}{\hat{T}(x)}\Big)}{\hat{T}(x)}\hat{T}(x)\hat{g}^{\wedge\circledast}(x) + \frac{1}{\hat{T}^2(x)} \frac{f(x)g(x\hat{T}(x))\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \hat{f}^{\wedge\circledast}(\hat{x})\hat{\times}\hat{g}^{\wedge}(x) + \hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge\circledast}(x) + \frac{1}{\hat{T}^2(x)} \frac{f(x)g(x\hat{T}(x))\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Exercise 6.0.300. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}(\hat{x})\right)^{\circledast} = \hat{f}^{\wedge\circledast}(\hat{x})\hat{\times}\hat{g}(\hat{x}) + \hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\circledast}(\hat{x}) + \frac{1}{\hat{T}^2(x)}\frac{f(x)g\left(\frac{x}{\hat{T}(x)}\right)\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Exercise 6.0.301. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}^{\wedge}(\hat{x})\hat{\times}g^{\wedge}(x)\right)^{\circledast} = \hat{f}^{\wedge\circledast}(\hat{x})\hat{\times}g^{\wedge}(x) + \hat{f}^{\wedge}(\hat{x})\hat{\times}g^{\wedge\circledast}(x)$$

$$+\frac{1}{\hat{T}^2(x)}\frac{f(x)g(x\hat{T}(x))\hat{T}(x)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Exercise 6.0.302. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}^{\wedge}(x)\hat{\times}\hat{g}^{\wedge}(x)\right)^{\circledast} = \hat{f}^{\wedge\circledast}(x)\hat{\times}\hat{g}^{\wedge}(x) + \hat{f}^{\wedge}(x)\hat{\times}\hat{g}^{\wedge\circledast}(x)$$

$$+\frac{1}{\hat{T}^2(x)}f(x\hat{T}(x))g(x\hat{T}(x))\frac{\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Exercise 6.0.303. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$, $\frac{x}{\hat{T}(x)} \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}^{\wedge}(x)\hat{\times}\hat{g}(\hat{x})\right)^{\circledast} = \hat{f}^{\wedge\circledast}(x)\hat{\times}\hat{g}(\hat{x}) + \hat{f}^{\wedge}(x)\hat{\times}\hat{g}^{\circledast}(\hat{x})$$

$$+\frac{1}{\hat{T}^2(x)}\frac{f(x\hat{T}(x))g\left(\frac{x}{\hat{T}(x)}\right)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Exercise 6.0.304. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}^{\wedge}(x)\hat{\times}g^{\wedge}(x)\right)^{\circledast} = f^{\wedge\circledast}(x)\hat{\times}\hat{g}^{\wedge}(x) + \hat{f}^{\wedge}(x)\hat{\times}g^{\wedge\circledast}(x).$$

Exercise 6.0.305. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$*,* $\hat{T}(x) > 0$*,* $\frac{x}{\hat{T}(x)} \in D$ *for every* $x \in D$*. Prove that for every* $x \in D$ *for which* $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *we have*

$$\Big(\hat{f}(\hat{x})\hat{\times}\hat{g}(\hat{x})\Big)^{\circledast} = \hat{f}^{\circledast}(\hat{x})\hat{\times}\hat{g}(\hat{x}) + \hat{f}(\hat{x})\hat{\times}\hat{g}^{\circledast}(\hat{x}) + \frac{1}{\hat{T}^2(x)}\frac{f\Big(\frac{x}{\hat{T}(x)}\Big)g\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Exercise 6.0.306. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$*,* $\hat{T}(x) > 0$*,* $x\hat{T}(x) \in D$*,* $\frac{x}{\hat{T}(x)} \in D$ *for every* $x \in D$*. Prove that for every* $x \in D$ *for which* $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *we have*

$$\Big(\hat{f}(\hat{x})\hat{\times}g^{\wedge}(x)\Big)^{\circledast} = \hat{f}^{\circledast}(\hat{x})\hat{\times}g^{\wedge}(x) + \hat{f}(\hat{x})\hat{\times}g^{\wedge\circledast}(x)$$

$$+\frac{1}{\hat{T}^2(x)}\frac{f\Big(\frac{x}{\hat{T}(x)}\Big)g(x\hat{T}(x))\hat{T}(x)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Exercise 6.0.307. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$*,* $\hat{T}(x) > 0$*,* $x\hat{T}(x) \in D$ *for every* $x \in D$*. Prove that for every* $x \in D$ *for which* $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *we have*

$$\Big(f^{\wedge}(x)\hat{\times}g^{\wedge}(x)\Big)^{\circledast} = f^{\wedge\circledast}(x)\hat{\times}g^{\wedge}(x) + f^{\wedge}(x)\hat{\times}g^{\wedge\circledast}(x)$$

$$+\frac{1}{\hat{T}^2(x)}\frac{f(x\hat{T}(x))g(x\hat{T}(x))\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Theorem: Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$ for every $x \in D$. Then for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have

$$\Big(\hat{f}^{\wedge}(\hat{x})\hat{g}^{\wedge}(\hat{x})\Big)^{\circledast} = \hat{f}^{\wedge\circledast}(\hat{x})\hat{g}^{\wedge}(\hat{x}) + \hat{f}^{\wedge}(\hat{x})\hat{g}^{\wedge\circledast}(\hat{x}).$$

Proof. For $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have

$$\left(\hat{f}^{\wedge}(\hat{x})\hat{g}^{\wedge}(\hat{x})\right)^{\circledast} = \hat{d}\left(\hat{f}^{\wedge}(\hat{x})\hat{g}^{\wedge}(\hat{x})\right) \nearrow \hat{d}\hat{x}$$

$$= \frac{1}{\hat{T}(x)} \frac{\hat{d}\left(\hat{f}^{\wedge}(\hat{x})\hat{g}^{\wedge}(\hat{x})\right)}{\hat{d}\hat{x}}$$

$$= \frac{d\left(\hat{f}^{\wedge}(\hat{x})\hat{g}^{\wedge}(\hat{x})\right)}{\left(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx}$$

$$= \frac{d\left(\frac{f(x)g(x)}{\hat{T}^2(x)}\right)}{\left(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx}$$

$$= \frac{\left(\frac{f(x)g(x)}{\hat{T}^2(x)}\right)' dx}{\left(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx}$$

$$= \frac{\frac{f'(x)g(x)\hat{T}^2(x)+f(x)g'(x)\hat{T}^2(x)-2f(x)g(x)\hat{T}(x)\hat{T}'(x)}{\hat{T}^4(x)}}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)} \frac{f'(x)g(x)+f(x)g'(x)-2f(x)g(x)\frac{\hat{T}'(x)}{\hat{T}(x)}}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)} \frac{f'(x)g(x)+f(x)g'(x)-f(x)g(x)\frac{\hat{T}'(x)}{\hat{T}(x)}-f(x)g(x)\frac{\hat{T}'(x)}{\hat{T}(x)}}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{1}{\hat{T}^2(x)} \frac{f'(x)\hat{T}(x)-f(x)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}} \frac{1}{\hat{T}(x)} g(x) + \frac{1}{\hat{T}^2(x)} f(x) \frac{1}{\hat{T}(x)} \frac{g'(x)\hat{T}(x)-g(x)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \hat{f}^{\wedge\circledast}(\hat{x}) \frac{g\left(\hat{T}(x)\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} + \frac{f\left(\hat{T}(x)\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} \hat{g}^{\wedge\circledast}(\hat{x})$$

$$= \hat{f}^{\wedge\circledast}(\hat{x})\hat{g}(\hat{x}) + \hat{f}(\hat{x})\hat{g}^{\wedge\circledast}(\hat{x}).$$

□

Exercise 6.0.308. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}^\wedge(\hat{x})\hat{g}^\wedge(x)\right)^{\circledast} = \hat{f}^{\wedge\circledast}(\hat{x})\hat{g}^\wedge(x) + \hat{f}^\wedge(\hat{x})\hat{g}^{\wedge\circledast}(x).$$

Exercise 6.0.309. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}^\wedge(\hat{x})\hat{g}(\hat{x})\right)^{\circledast} = \hat{f}^{\wedge\circledast}(\hat{x})\hat{g}(\hat{x}) + \hat{f}^\wedge(\hat{x})\hat{g}^{\circledast}(\hat{x}).$$

Exercise 6.0.310. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}^\wedge(\hat{x})g^\wedge(x)\right)^{\circledast} = \hat{f}^{\wedge\circledast}(\hat{x})g^\wedge(x) + \hat{f}^\wedge(\hat{x})g^{\wedge\circledast}(x).$$

Exercise 6.0.311. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}^\wedge(x)\hat{g}^\wedge(x)\right)^{\circledast} = \hat{f}^{\wedge\circledast}(x)\hat{g}^\wedge(x) + \hat{f}^\wedge(x)\hat{g}^{\wedge\circledast}(x).$$

Exercise 6.0.312. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$, $x\hat{T}(x) \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}^\wedge(x)\hat{g}(\hat{x})\right)^{\circledast} = \hat{f}^{\wedge\circledast}(x)\hat{g}(\hat{x}) + \hat{f}^\wedge(x)\hat{g}^{\circledast}(\hat{x}).$$

Exercise 6.0.313. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}^\wedge(x)\hat{g}^\wedge(x)\right)^{\circledast} = \hat{f}^{\wedge\circledast}(x)\hat{g}^\wedge(x) + \hat{f}^\wedge(x)\hat{g}^{\wedge\circledast}(x).$$

Exercise 6.0.314. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}(\hat{x})\hat{g}(\hat{x})\right)^{\circledast} = \hat{f}^{\circledast}(\hat{x})\hat{g}(\hat{x}) + \hat{f}(\hat{x})\hat{g}^{\circledast}(\hat{x}).$$

Exercise 6.0.315. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$, $\frac{x}{\hat{T}(x)} \in D$ *for every* $x \in D$. *Prove that for every* $x \in D$ *for which* $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *we have*

$$\Big(\hat{f}(\hat{x})g^{\wedge}(x)\Big)^{\circledast} = \hat{f}^{\circledast}(\hat{x})g^{\wedge}(x) + \hat{f}(\hat{x})g^{\wedge\circledast}(x).$$

Exercise 6.0.316. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ *for every* $x \in D$. *Prove that for every* $x \in D$ *for which* $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *we have*

$$\Big(f^{\wedge}(x)g^{\wedge}(x)\Big)^{\circledast} = f^{\wedge\circledast}(x)g^{\wedge}(x) + f^{\wedge}(x)g^{\wedge\circledast}(x).$$

Exercise 6.0.317. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $g(x) \neq 0$ *for every* $x \in D$. *Then for every* $x \in D$ *for which* $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *we have*

$$\Big(\hat{f}^{\wedge}(\hat{x}) \nearrow \hat{g}^{\wedge}(\hat{x})\Big)^{\circledast} = \Big(\hat{f}^{\wedge\circledast}(\hat{x})\hat{\times}\hat{g}^{\wedge}(\hat{x}) - \hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge\circledast}(\hat{x})\Big) \nearrow \hat{g}^{\hat{2}\wedge}(\hat{x})$$

$$-\frac{1}{\hat{T}^2(x)}\frac{f(x)}{g(x)}\frac{\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Exercise 6.0.318. *Let* $f, g, \hat{T} \in \mathcal{C}(D)$, $\hat{T}(x) > 0$, $g(x) \neq 0$, $x\hat{T}(x) \in D$ *for every* $x \in D$. *Prove that for* $x \in D$ *for which* $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *we have*

$$\Big(\hat{f}^{\wedge}(\hat{x}) \nearrow \hat{g}^{\wedge}(x)\Big)^{\circledast} = \Big(\hat{f}^{\wedge\circledast}(\hat{x})\hat{\times}\hat{g}^{\wedge}(x) - \hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge\circledast}(x)\Big) \nearrow \hat{g}^{\hat{2}\wedge}(x)$$

$$-\frac{1}{\hat{T}^2(x)}\frac{\frac{f(x)}{g(x\hat{T}(x))}\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Exercise 6.0.319. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $g(x) \neq 0$, $\frac{x}{\hat{T}(x)} \in D$ *for every* $x \in D$. *Prove that for every* $x \in D$ *for which* $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *we have*

$$\Big(\hat{f}^{\wedge}(\hat{x}) \nearrow \hat{g}(\hat{x})\Big)^{\circledast} = \Big(\hat{f}^{\wedge\circledast}(\hat{x})\hat{\times}\hat{g}(\hat{x}) - \hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\circledast}(\hat{x})\Big)\hat{g}^{\hat{2}}(\hat{x})$$

$$-\frac{1}{\hat{T}^2(x)}\frac{\frac{f(x)}{g\left(\frac{x}{\hat{T}(x)}\right)}\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Exercise 6.0.320. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $g(x) \neq 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}^{\wedge}(\hat{x}) \nearrow g^{\wedge}(x)\right)^{\circledast} = \left(\hat{f}^{\wedge\circledast}(\hat{x})\hat{\times}g^{\wedge}(x) - \hat{f}^{\wedge}(\hat{x})\hat{\times}g^{\wedge\circledast}(x)\right) \nearrow g^{\hat{2}\wedge}(x)$$

$$-\frac{1}{\hat{T}^2(x)}\frac{\frac{f(x)}{g(x\hat{T}(x))}\hat{T}(x)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Exercise 6.0.321. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $g(x) \neq 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}^{\wedge}(x) \nearrow \hat{g}^{\wedge}(x)\right)^{\circledast} = \left(\hat{f}^{\wedge\circledast}(x)\hat{\times}\hat{g}^{\wedge}(x) - \hat{f}^{\wedge}(x)\hat{\times}\hat{g}^{\wedge\circledast}(x)\right) \nearrow \hat{g}^{\hat{2}\wedge}(x)$$

$$-\frac{1}{\hat{T}^2(x)}\frac{f(x\hat{T}(x))}{g(x\hat{T}(x))}\frac{\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Exercise 6.0.322. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $g(x) \neq 0$, $\frac{x}{\hat{T}(x)} \in D$, $x\hat{T}(x) \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}^{\wedge}(x) \nearrow \hat{g}(\hat{x})\right)^{\circledast} = \left(\hat{f}^{\wedge\circledast}(x)\hat{\times}\hat{g}(\hat{x}) - \hat{f}^{\wedge}(x)\hat{\times}\hat{g}^{\circledast}(\hat{x})\right) \nearrow \hat{g}^{\hat{2}}(\hat{x})$$

$$-\frac{1}{\hat{T}^2(x)}\frac{\frac{f(x\hat{T}(x))}{g\left(\frac{x}{\hat{T}(x)}\right)}\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Exercise 6.0.323. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $g(x) \neq 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}^{\wedge}(x)\hat{\times}g^{\wedge}(x)\right)^{\circledast} = \left(f^{\wedge\circledast}(x)\hat{\times}\hat{g}^{\wedge}(x) - \hat{f}^{\wedge}(x)\hat{\times}g^{\wedge\circledast}(x)\right) \nearrow g^{\hat{2}\wedge}(x).$$

Exercise 6.0.324. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $g(x) \neq 0$, $\frac{x}{\hat{T}(x)} \in D$ for* *every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}(\hat{x}) \nearrow \hat{g}(\hat{x})\right)^{\circledast} = \left(\hat{f}^{\circledast}(\hat{x})\hat{\times}\hat{g}(\hat{x}) - \hat{f}(\hat{x})\hat{\times}\hat{g}^{\circledast}(\hat{x})\right) \nearrow \hat{g}^{\hat{2}}(\hat{x})$$

$$-\frac{1}{\hat{T}^2(x)}\frac{\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{g\left(\frac{x}{\hat{T}(x)}\right)}\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Exercise 6.0.325. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $g(x) \neq 0$, $x\hat{T}(x) \in D$, $\frac{x}{\hat{T}(x)} \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1-x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(\hat{f}(\hat{x})\hat{\times}g^{\wedge}(x)\right)^{\circledast} = \left(\hat{f}^{\circledast}(\hat{x})\hat{\times}g^{\wedge}(x) - \hat{f}(\hat{x})\hat{\times}g^{\wedge\circledast}(x)\right) \nearrow \hat{g}^{\hat{2}}(\hat{x})$$

$$-\frac{1}{\hat{T}^2(x)}\frac{\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{g(x\hat{T}(x))}\hat{T}(x)\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Exercise 6.0.326. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $g(x) \neq 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Prove that for every $x \in D$ for which $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ we have*

$$\left(f^{\wedge}(x) \nearrow g^{\wedge}(x)\right)^{\circledast} = \left(f^{\wedge\circledast}(x)\hat{\times}g^{\wedge}(x) - f^{\wedge}(x)\hat{\times}g^{\wedge\circledast}(x)\right) \nearrow g^{\hat{2}\wedge}(x)$$

$$-\frac{1}{\hat{T}^2(x)}\frac{\frac{f(x\hat{T}(x))}{g(x\hat{T}(x))}\hat{T}'(x)}{1-x\frac{\hat{T}'(x)}{\hat{T}(x)}}.$$

Definition: We will say that an isofunction of first, second, third or fourth kind increases(decreases) at the point a if its isooriginal increase(decrease) at the point a.
An isofunction of first, second, third or fourth kind will be called increasing(decreasing) in D if it increases(decreases) in every point of D.

Example 6.0.327. *Let $D = \mathbb{R}$, $f(x) = x$, $g(x) = -x$, $\hat{T}(x) = x^2 + 1$, $x \in D$. Then f is increasing function in D, g is decreasing function in D,* *the isofunction of first kind*

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x}{x^2+1}$$

is increasing function in $[-1,1]$ *and decreasing function in* $(-\infty,-1] \cup [1,\infty)$. *The isofunction*

$$g^{\wedge}(x) = g(x\hat{T}(x)) = g(x^3+x) = -x^3 - x$$

is decreasing function in D.

Theorem: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$ for every $x \in D$. Let also the functions f and $\hat{T}$ be differentiable functions at the point $x_0 \in D$. If

$$f'(x_0) \geq (\leq) f(x_0)\frac{\hat{T}'(x_0)}{\hat{T}(x_0)} \tag{6.0.10}$$

then the isofunction $\hat{f}^{\wedge\wedge}$ of first kind increases(decreases) at x_0.

Proof. Since f and $\hat{T}$ are differentiable functions at the point x_0 we have

$$\Big(\hat{f}^{\wedge}(\hat{x_0})\Big)' = \Big(\frac{f(x_0)}{\hat{T}(x_0)}\Big)' = \frac{f'(x_0)\hat{T}(x_0) - f(x_0)\hat{T}'(x_0)}{\hat{T}^2(x_0)}. \tag{6.0.11}$$

From (6.0.10) because $\hat{T}(x_0) > 0$ we get

$$f'(x_0)\hat{T}(x_0) - f(x_0)\hat{T}'(x_0) \geq (\leq) 0,$$

from here, using that $\hat{T}^2(x_0) > 0$,

$$\frac{f'(x_0)\hat{T}(x_0) - f(x_0)\hat{T}'(x_0)}{\hat{T}^2(x_0)} \geq (\leq) 0.$$

From the last inequality and (6.0.11) it follows

$$\Big(\hat{f}^{\wedge}(\hat{x_0})\Big)' \geq (\leq) 0.$$

Consequently $\hat{f}^{\wedge\wedge}$ increases(decreases) at the point x_0. $\square$

As in above one can prove the following theorems.

Theorem: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Let also the functions f and $\hat{T}$ be differentiable functions at the points $x_0\hat{T}(x_0)$, $x_0 \in D$, respectively. If

$$f'(x_0\hat{T}(x_0))(\hat{T}(x_0) + x_0\hat{T}'(x_0))\hat{T}(x_0) \geq (\leq) f(x_0\hat{T}(x_0))\hat{T}'(x_0)$$

then the isofunction $\hat{f}^{\wedge}$ of second kind increases(decreases) at the point x_0.

Theorem: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$ for every $x \in D$. Let also the functions f and $\hat{T}$ be differentiable functions at the points $\frac{x_0}{\hat{T}(x_0)}$, $x_0 \in D$, respectively. If

$$f'\Big(\frac{x_0}{\hat{T}(x_0)}\Big)(\hat{T}(x_0) - x_0\hat{T}'(x_0)) \geq (\leq) f\Big(\frac{x_0}{\hat{T}(x_0)}\Big)\hat{T}'(x_0)\hat{T}(x_0)$$

then the isofunction $\hat{\hat{f}}$ of third kind increases(decreases) at the point x_0.

Theorem: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Let also the functions f and $\hat{T}$ be differentiable functions at the points $x_0\hat{T}(x_0)$, $x_0 \in D$, respectively. If

$$f'(x_0\hat{T}(x_0))(\hat{T}(x_0) + x_0\hat{T}'(x_0)) \geq (\leq) 0$$

then the isofunction $f^{\wedge}$ of fourth kind increases(decreases) at the point x_0.

Definition: We will say that an isofunction of first, second, third or fourth kind has local extremum(local maximum, local minimum) in a point of D if its isooriginal has local extremum at the same point.

Theorem: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$ for every $x \in D$. Let also the functions f and $\hat{T}$ be differentiable functions at the point $x_0 \in D$. If the isofunction $\hat{f}^{\wedge\wedge}$ of first kind has local extremum at the point x_0 then

$$f'(x_0)\hat{T}(x_0) = f(x_0)\hat{T}'(x_0).$$

Proof. Because the isofunction $\hat{f}^{\wedge\wedge}$ of first kind has local extremum at the point x_0 then its isooriginal

$$\frac{f(x)}{\hat{T}(x)}$$

has local extremum at the point x_0. From here

$$\left(\frac{f(x)}{\hat{T}(x)}\right)'_{x=x_0} = 0 \quad \Longleftrightarrow$$

$$\frac{f'(x_0)\hat{T}(x_0) - f(x_0)\hat{T}'(x_0)}{\hat{T}^2(x_0)} = 0 \quad \Longleftrightarrow$$

$$f'(x_0)\hat{T}(x_0) = f(x_0)\hat{T}'(x_0).$$

□

The proofs of the following theorems we left to the reader.

Theorem: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Let also the functions f and $\hat{T}$ be differentiable functions at the points $x_0\hat{T}(x_0)$, $x_0 \in D$, respectively. If the isofunction $\hat{f}^{\wedge}$ of second kind has local extremum at the point x_0 then

$$f'(x_0\hat{T}(x_0)(\hat{T}(x_0) + x_0\hat{T}'(x_0))\hat{T}(x_0) = f(x_0\hat{T}(x_0))\hat{T}'(x_0).$$

Theorem: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in D$ for every $x \in D$. Let also the functions f and $\hat{T}$ be differentiable functions at the points $\frac{x_0}{\hat{T}(x_0)}$, $x_0 \in D$, respectively. If the isofunction $\hat{\hat{f}}$ has local extremum at the point x_0 then

$$f'\Big(\frac{x_0}{\hat{T}(x_0)}\Big)(\hat{T}(x_0) - x_0\hat{T}'(x_0)) = f\Big(\frac{x_0}{\hat{T}(x_0)}\Big)\hat{T}'(x_0)\hat{T}(x_0).$$

Theorem: Let $f, \hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in D$ for every $x \in D$. Let also the functions f and $\hat{T}$ be differentiable functions at the points $x_0\hat{T}(x_0)$, $x_0 \in D$, respectively. If the isofunction $f^\wedge$ of fourth kind has local extremum at the point x_0 then

$$f'(x_0\hat{T}(x_0))(\hat{T}(x_0) + x_0\hat{T}'(x_0)) = 0.$$

Below with (a, b) and $[a, b]$ will be denoted intervals in $\mathbb{R}$.

Theorem: Let $f, \hat{T} \in \mathcal{C}([a, b])$, $f, \hat{T} \in \mathcal{C}^1((a, b))$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in (a, b)$, $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that

$$\Big(\hat{f}^\wedge(\hat{c})\Big)^{\circledast} = -\frac{f(c)\hat{T}'(c)}{\hat{T}(c)(\hat{T}(c) - c\hat{T}'(c))}.$$

Proof. Since $f, \hat{T} \in \mathcal{C}^1((a, b))$ and $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in (a, b)$ then there exists $\Big(\hat{f}^\wedge(\hat{x})\Big)^{\circledast}$ for every $x \in (a, b)$ and

$$\Big(\hat{f}^\wedge(\hat{x})\Big)^{\circledast} = \frac{f'(x) - f(x)\frac{\hat{T}'(x)}{\hat{T}(x)}}{\hat{T}(x) - x\hat{T}'(x)} \tag{6.0.12}$$

for every $x \in (a, b)$.

Because $f \in \mathcal{C}([a,b])$, $f \in \mathcal{C}^1((a,b))$, $f(a) = f(b)$, it follows from the Theorem of Rolle that there exists $c \in (a,b)$ such that

$$f'(c) = 0.$$

From here, after we put $x = c$ in (6.0.12), we get

$$\left(\hat{f}^\wedge(\hat{c})\right)^{\circledast} = \frac{-f(c)\frac{\hat{T}'(c)}{\hat{T}(c)}}{\hat{T}(c) - c\hat{T}'(c)} = -\frac{f(c)\hat{T}'(c)}{\hat{T}(c)(\hat{T}(c) - c\hat{T}'(c))}.$$

□

Theorem: Let $f, \hat{T} \in \mathcal{C}([a,b])$, $f, \hat{T} \in \mathcal{C}^1((a,b))$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in (a,b)$, $\hat{T}(a) = \hat{T}(b)$. Then there exists $c \in (a,b)$ such that

$$\left(\hat{f}^\wedge(\hat{c})\right)^{\circledast} = \frac{f'(c)}{\hat{T}(c)} = \widehat{f'}^\wedge(\hat{c}). \tag{6.0.13}$$

Proof. Because $f, \hat{T} \in \mathcal{C}^1((a,b))$, $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in (a,b)$ then there exists $\left(\hat{f}^\wedge(\hat{x})\right)^{\circledast}$ and (6.0.12) hold for every $x \in (a,b)$.
From $\hat{T} \in \mathcal{C}([a,b])$, $\hat{T} \in \mathcal{C}^1((a,b))$, $\hat{T}(a) = \hat{T}(b)$ and from the Theorem of Rolle it follows that there exists $c \in (a,b)$ such that $\hat{T}'(c) = 0$. From here, after we put $x = c$ in (6.0.12) we get (6.0.13).

□

Theorem: Let $f \in \mathcal{C}^1([a,b])$, $\hat{T} \in \mathcal{C}([a,b]) \bigcap \mathcal{C}^1((a,b))$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in [a,b]$, $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in [a,b]$, $\hat{T}(a) = \hat{T}(b)$. Then there exists $c \in (a,b)$ such that

$$\left(\hat{f}^\wedge(c)\right)^{\circledast} = f'(c\hat{T}(c)) = f'^\wedge(c). \tag{6.0.14}$$

Proof. From $f, \hat{T} \in \mathcal{C}((a,b))$, $x\hat{T}(x) \in [a,b]$, $\hat{T}'(x) - x\hat{T}(x) \neq 0$ for every $x \in (a,b)$, it follows that there exists $\left(\hat{f}^\wedge(x)\right)^{\circledast}$ for every $x \in (a,b)$ and

$$\left(\hat{f}^\wedge(x)\right)^{\circledast} = \frac{f'(x\hat{T}(x))(\hat{T}(x) + x\hat{T}'(x)) - f(x\hat{T}(x))\frac{\hat{T}'(x)}{\hat{T}(x)}}{\hat{T}(x) - x\hat{T}'(x)} \tag{6.0.15}$$

for every $x \in (a,b)$.
From $\hat{T} \in \mathcal{C}([a,b]) \bigcap \mathcal{C}^1((a,b))$, $\hat{T}(a) = \hat{T}(b)$, it follows from the Theorem of Rolle that there exists $c \in (a,b)$ such that $\hat{T}'(c) = 0$. From here, after we put $x = c$ in (6.0.15), we get (6.0.14). □

Theorem: Let $f, \hat{T} \in \mathcal{C}^1([a,b])$, $x\hat{T}(x) : [a,b] \longrightarrow [a,b]$ is bijection, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in [a,b]$, $f(a) = f(b)$. Then there exists $c \in [a,b]$ such that

$$\left(\hat{f}^\wedge(c)\right)^{\circledast} = -\frac{f(c\hat{T}(c))\hat{T}'(c)}{\hat{T}(c)(\hat{T}(c) - c\hat{T}'(c))}. \tag{6.0.16}$$

Proof. From $f, \hat{T} \in \mathcal{C}^1([a,b])$, $x\hat{T}(x) \in [a,b]$, $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in [a,b]$, it follows that there exists $\left(\hat{f}^\wedge(x)\right)^{\circledast}$ for every $x \in [a,b]$ and (6.0.15) hold for every $x \in [a,b]$.
From $f \in \mathcal{C}^1([a,b])$, $f(a) = f(b)$, it follows from the Theorem of Rolle that there exists $c_1 \in (a,b)$ such that $f'(c_1) = 0$. Since $x\hat{T}(x) : [a,b] \longrightarrow [a,b]$ is bijection we conclude that there exists $c \in [a,b]$ such that

$$c_1 = c\hat{T}(c).$$

From here, after we put $x = c$ in (6.0.15), we get (6.0.16). □

Theorem 6.0.328. *Let $f, \hat{T} \in \mathcal{C}^1([a,b])$, $\frac{x}{\hat{T}(x)} : [a,b] \longrightarrow [a,b]$ is a bijection, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in [a,b]$, $f(a) = f(b)$. Then there exists $c \in (a,b)$ such that*

$$\left(\hat{f}(\hat{c})\right)^{\circledast} = -\frac{f\left(\frac{c}{\hat{T}(c)}\right)\hat{T}'(c)}{\hat{T}(c)(\hat{T}(c) - c\hat{T}'(c))}. \tag{6.0.17}$$

Proof. Since $f, \hat{T} \in \mathcal{C}^1([a,b])$, $\frac{x}{\hat{T}(x)} \in [a,b]$, $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in [a,b]$, then there exists $\left(\hat{f}(\hat{x})\right)^{\circledast}$ for every $x \in [a,b]$ and

$$\left(\hat{f}(\hat{x})\right)^{\circledast} = \frac{f'\left(\frac{x}{\hat{T}(x)}\right)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} - f\left(\frac{x}{\hat{T}(x)}\right)\frac{\hat{T}'(x)}{\hat{T}(x)}}{\hat{T}(x) - x\hat{T}'(x)}. \tag{6.0.18}$$

for every $x \in [a,b]$.
From $f \in \mathcal{C}^1([a,b])$, $f(a) = f(b)$, it follows from the Theorem of Rolle that there exists $c_1 \in (a,b)$ such that $f'(c_1) = 0$. From $\frac{x}{\hat{T}(x)} : [a,b] \longrightarrow [a,b]$ is a bijection we can choose $c \in [a,b]$ such that

$$c_1 = \frac{c}{\hat{T}(c)}.$$

After we put $x = c$ in (6.0.18) we obtain (6.0.19). □

Theorem 6.0.329. *Let $f, \hat{T} \in \mathcal{C}^1([a,b])$, $\frac{x}{\hat{T}(x)} : [a,b] \longrightarrow [a,b]$ is a bijection, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in [a,b]$, $\hat{T}(a) = \hat{T}(b)$. Then there exists $c \in (a,b)$ such that*

$$\left(\hat{f}(\hat{c})\right)^{\circledast} = \frac{f\left(\frac{c}{\hat{T}(c)}\right)}{\hat{T}(c)(\hat{T}(c) - c\hat{T}'(c))}. \tag{6.0.19}$$

Proof. Since $f, \hat{T} \in \mathcal{C}^1([a,b])$, $\frac{x}{\hat{T}(x)} \in [a,b]$, $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in [a,b]$, then there exists $\left(\hat{f}(\hat{x})\right)^{\circledast}$ for every $x \in [a,b]$ and (6.0.18) hold for every $x \in [a,b]$.
From $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(a) = \hat{T}(b)$, it follows from the Theorem of Rolle, that there exists $c \in [a,b]$ such that $\hat{T}'(c) = 0$. From here, after we put $x = c$ in (6.0.18), we get (6.0.19). □

Theorem 6.0.330. *Let $f, \hat{T} \in \mathcal{C}^1([a,b])$, $x\hat{T}(x) : [a,b] \longrightarrow [a,b]$ is a bijection, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in [a,b]$, $f(a) = f(b)$. Then there exists $c \in [a,b]$ such that*

$$\left(f^{\wedge}(c)\right)^{\circledast} = 0. \tag{6.0.20}$$

Proof. Since $f, \hat{T} \in \mathcal{C}^1([a,b])$, $x\hat{T}(x) \in [a,b]$, $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in [a,b]$ it follows that there exists $\left(f^{\wedge}(x)\right)^{\circledast}$ for every $x \in [a,b]$ and

$$\left(f^{\wedge}(x)\right)^{\circledast} = \frac{\hat{T}(x) f'(x\hat{T}(x))(\hat{T}(x) + x\hat{T}'(x))}{\hat{T}(x) - x\hat{T}'(x)} \tag{6.0.21}$$

for every $x \in [a,b]$.
From $f \in \mathcal{C}^1([a,b])$, $f(a) = f(b)$ and from the Theorem of Rolle it follows that there exists $c_1 \in (a,b)$ such that $f'(c_1) = 0$. From $x\hat{T}(x) : [a,b] \longrightarrow [a,b]$ is a bijection we can find $c \in [a,b]$ such that

$$c_1 = c\hat{T}(c).$$

After we put $x = c$ in (6.0.21) we get (6.0.20). □

Theorem 6.0.331. *Let $f, \hat{T} \in \mathcal{C}^1([a,b])$, $x\hat{T}(x) \in [a,b]$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in [a,b]$, $\hat{T}(a) = \hat{T}(b)$. Then there exists $c \in [a,b]$ such that*

$$\left(f^{\wedge}(c)\right)^{\circledast} = \hat{T}(c) f'(c\hat{T}(c)). \tag{6.0.22}$$

Proof. Since $f, \hat{T} \in \mathcal{C}^1([a,b])$, $x\hat{T}(x) \in [a,b]$, $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in [a,b]$ it follows that there exists $\Big(f^\wedge(x)\Big)^{\circledast}$ for every $x \in [a,b]$ and (6.0.21) hold for every $x \in [a,b]$.
Because $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(a) = \hat{T}(b)$ and from the Theorem of Rolle it follows that there exists $c \in [a,b]$ such that $\hat{T}'(c) = 0$. After we put $x = c$ in (6.0.21) we get (6.0.22). □

Exercise 6.0.332. *Let $f, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \neq 0$ for every $x \in D$. Prove that $\Big(\hat{f}^\wedge(\hat{x})\Big)^{\circledast} = 0$ if and only if $f(x) = c\hat{T}(x)$, $c = \mathrm{const}$, $x \in D$.*

Exercise 6.0.333. *Let $D = [0,\infty)$, $\hat{T}(x) = x^2 + 1$, $x \in D$. Find*

$$\lim_{x\longrightarrow 0} \hat{f}^\wedge(\hat{x})$$

if

$$1)\quad f(x) = \tfrac{x^2\cos x}{\cos x - 1},\qquad 2)\quad f(x) = \tfrac{e^x - e^{-x}}{\ln(1+x)},$$

$$3)\quad f(x) = \tfrac{e^{x+1} - (1+x)^{\frac{1}{x}}}{x},\qquad 4)\quad f(x) = \tfrac{x\sin(\sin x) - \sin^2 x}{x^6},$$

$$4)\quad f(x) = \tfrac{\sin x - x\cos x}{\sin^3 x},\qquad 6)\quad f(x) = \tfrac{\arcsin(2-x)}{\sqrt{x^2-3x+2}},$$

$$7)\quad f(x) = \tfrac{\arcsin(2x) - 2\arcsin x}{x^3}.$$

Answer.

$$1)\quad -2,\quad 2)\quad 2,\quad 3)\quad -\tfrac{e}{2},\quad 4)\quad \tfrac{1}{18},\quad 5)\quad \tfrac{1}{3},\quad 6)\quad 0,\quad 7)\quad 1.$$

Exercise 6.0.334. *Let $D = [0,\infty)$, $\hat{T}(x) = \frac{x+2}{x+4}$, $x \in D$. Find*

$$\lim_{x\longrightarrow\infty} \hat{f}^\wedge(\hat{x})$$

if

$$1)\quad f(x) = \tfrac{x + 2\ln x}{x},\qquad 2)\quad f(x) = \tfrac{e^x + \sin x}{x + \sin x},$$

$$3)\quad f(x) = \tfrac{x+\sin x}{x - \sin x},\qquad 4)\quad f(x) = \tfrac{\ln(1+e^x)}{a+bx},\qquad b \neq 0,$$

$$5)\quad f(x) = (\pi - 2\arctan x)\ln x,\qquad 6)\quad f(x) = \arcsin\tfrac{x-a}{a}\cot(x-a),$$

$$7)\quad f(x) = \ln x\ln(1-x),\qquad 8)\quad f(x) = (\sin x - 1)e^{\tan x}.$$

Answer.

$$1)\quad 1,\quad 2)\quad \infty,.\quad 3)\quad 1,\quad 4)\quad \tfrac{1}{6},\quad 5)\quad 0,$$

$$6)\quad \tfrac{1}{a},\quad 7)\quad 0,\quad 8)\quad \infty.$$

Definition: An isofunction of first, second, third or fourth kind will be called isoconvex(isoconcave) if its isooriginal is convex(concave) function.

Example 6.0.335. *Let $D = \mathbb{R}$, $\hat{T}(x) = e^{-x}$, $f(x) = x^2 + x + 1$, $x \in D$. Then*

$$\hat{f}^{\wedge}(\hat{x}) = \tfrac{f(x)}{\hat{T}(x)} = \tfrac{x^2+x+1}{e^{-x}} = e^x(x^2 + x + 1),$$

$$\Big(e^x(x^2 + x + 1)\Big)' = e^x(x^2 + x + 1 + 2x + 1) = e^x(x^2 + 3x + 2),$$

$$\Big(e^x(x^2 + x + 1)\Big)'' = \Big(e^x(x^2 + 3x + 2)\Big)' = e^x(x^2 + 3x + 2 + 2x + 3)$$

$$= e^x(x^2 + 5x + 5).$$

Since $x^2 + 5x + 5 > 0$ for every $x \in \mathbb{R}$ then the isofunction $\hat{f}^{\wedge}(\hat{x})$ of first kind is an isoconvex isofunction in D.

Exercise 6.0.336. *Let $D = [1, +\infty)$, $\hat{T}(x) = \frac{1}{x+1}$, $f(x) = x^2 + x$, $x \in D$. Prove that $\hat{f}^{\wedge}$ is isoconvex isofunction in D.*

Definition 6.0.337. *Second isoderivative of an isofunction f of first, second, third or fourth kind is defined as follows*

$$f^{2\circledast} = (f^{\circledast})^{\circledast},$$

third isoderivative

$$f^{3\circledast} = ((f^{\circledast})^{\circledast})^{\circledast},$$

and etc.

Example 6.0.338. *Let $D = [1, +\infty)$, $\hat{T}(x) = e^{-x}$, $f(x) = x$. Then*

$$\hat{f}^\wedge(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x}{e^{-x}} = xe^x,$$

$$\left(\hat{f}^\wedge(\hat{x})\right)^{\circledast} = \hat{d}\hat{f}^\wedge(\hat{x}) \nearrow \hat{d}\hat{x} = \tfrac{1}{\hat{T}(x)} \tfrac{\hat{d}\hat{f}^\wedge(\hat{x})}{\hat{d}\hat{x}} = \tfrac{1}{\hat{T}(x)} \tfrac{\hat{T}(x) d\hat{f}^\wedge(\hat{x})}{\left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx} = \tfrac{d(xe^x)}{(1+x)dx} = e^x,$$

$$\left(\left(\hat{f}^\wedge(\hat{x})\right)^{\circledast}\right)^{\circledast} = \hat{d}\left(\left(\hat{f}^\wedge(\hat{x})\right)^{\circledast}\right) \nearrow \hat{d}\hat{x} = \tfrac{1}{\hat{T}(x)} \tfrac{\hat{d}\left(\hat{f}^\wedge(\hat{x})\right)^{\circledast}}{\hat{d}\hat{x}} - \tfrac{1}{\hat{T}(x)} \tfrac{\hat{T}(x) d\left(\hat{f}^\wedge(\hat{x})\right)^{\circledast}}{\left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx}$$

$$= \tfrac{d(e^x)}{(x+1)dx} = \tfrac{e^x}{x+1}.$$

Definition: Let f is an isofunction of first, second, third or fourth kind, which is infinite number isodifferentiable in D. For $x_0 \in D$ the isoseries

$$f(x_0) + \frac{f^{\circledast}(x_0)}{1!} \hat{\times} (\widehat{x - x_0}) + \frac{f^{\circledast\circledast}(x_0)}{2!} \hat{\times} (\widehat{x - x_0})^{\hat{2}} + \cdots$$

is called iso- Taylor isoseries of the isofunction f at x_0. When $x_0 = 0$ it is called iso-Macleurin isoseries of f.

Example 6.0.339. *Let $D = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $f(x) = e^x$, $\hat{T}(x) = e^{-x}$. Then*

$$\left(\hat{f}^\wedge(\hat{x})\right)^{\circledast} = \tfrac{e^{2x}}{1-x}, \quad \left(\hat{f}^\wedge(0)\right)^{\circledast} = 1,$$

$$\left(\left(\hat{f}^\wedge(\hat{x})\right)^{\circledast}\right)^{\circledast} = \tfrac{e^{2x}(3-2x)}{(1-x)^3}, \quad \left(\left(\hat{f}^\wedge(0)\right)^{\circledast}\right)^{\circledast} = 3,$$

and the corresponding iso-Macleurin isoseries is

$$1 + \widehat{x} + \frac{3}{2} \hat{\times} \hat{x}^{\hat{2}} + \cdots$$

Advanced practical exercises

Problem 6.0.340. *Let $D = [0, \infty)$, $f(x) = 2x - 4$, $\hat{T}(x) = x + 1$, $x \in D$. Find*

$$\hat{d}\left(\hat{f}^\wedge(\hat{x})\right), \quad \hat{d}\left(\hat{f}^\wedge(x)\right), \quad \hat{d}\left(\hat{f}(\hat{x})\right), \quad \hat{d}\left(f^\wedge(x)\right), \quad \hat{d}\hat{x}.$$

Answer.

$$\hat{d}\Big(\hat{f}^{\wedge}(\hat{x})\Big) = \tfrac{6}{x+1}dx, \quad \hat{d}\Big(\hat{f}^{\wedge}(x)\Big) = \tfrac{2x+6}{(x+1)^2}dx,$$

$$\hat{d}\Big(\hat{f}(\hat{x})\Big) = -\tfrac{2}{x+1}, \quad \hat{d}\Big(f^{\wedge}(x)\Big) = (4x^2+6x+2)dx, \quad \hat{d}\hat{x} = \tfrac{1}{x+1}dx.$$

Problem 6.0.341. *Let $D=[0,\infty)$, $f(x)=2x$, $\hat{T}(x)=2x+1$, $x\in D$. Find*

$$\hat{f}^{\wedge\circledast}(\hat{x}), \quad \hat{f}^{\wedge\circledast}(x), \quad \hat{f}^{\circledast}(\hat{x}), \quad f^{\wedge\circledast}(x).$$

Answer.

$$\hat{f}^{\wedge\circledast}(\hat{x}) = 4x+2, \quad \hat{f}^{\wedge\circledast}(x) = -32x^3-24x^2+2,$$

$$\hat{f}^{\circledast}(\hat{x}) = 2-4x, \quad f^{\wedge\circledast}(x) = 64x^4+112x^3+72x^2+20x+2.$$

Problem 6.0.342. *Let $D=[0,\infty)$, $\hat{T}_1=4$, $f(x)=x$, $\hat{T}(x)=2x+1$, $x\in D$. Find*

$$A := \hat{f}^{\wedge}(\hat{x}) + \hat{f}(\hat{x}) + \hat{f}^{\wedge\circledast}(\hat{x}).$$

Answer. $A = \frac{6x^2+6x+1}{2x+1}$.

Problem 6.0.343. *Let $D=[2,+\infty)$, $\hat{T}(x)=x$, $f(x)=x^2+1$, $x\in D$. Prove that $f^{\wedge}$ is an isoconvex isofunction in D.*

Chapter 7

Isointegrals

Let $D = [a, b]$ and $\hat{T} \in \mathcal{C}^1([a, b])$, $\hat{T}(x) > 0$ for every $x \in [a, b]$.

Definition 7.0.344. *Let $\hat{f}$ be isofunction of first, second, third or fourth kind. With $\tilde{f}$ we will denote its isooriginal, which is defined and integrable on $[a, b]$. Isointegral or isoprimitive of $\hat{f}$ will be called*

$$\hat{\int} \hat{f}(x)\hat{\times}\hat{d}\hat{x} := \int \hat{T}^{-1}(x)\tilde{f}(x)\hat{\times}\hat{d}\hat{x}$$

and iso-Cauchy isointegral of $\hat{f}$ on $[a, b]$ will be called

$$\hat{\int}_a^b \hat{f}(x)\hat{\times}\hat{d}\hat{x} := \int_a^b \hat{T}^{-1}(x)\tilde{f}(x)\hat{\times}\hat{d}\hat{x}.$$

Below we will suppose that $\hat{f}$, $\hat{g}$ are isofunctions with isooriginals $\tilde{f}$ and $\tilde{g}$, respectively, which will be supposed defined and integrable on $[a, b]$.

From the definition of isointegral it follows

1)

$$\left(\hat{\int}\hat{f}(x)\hat{\times}\hat{d}\hat{x}\right)^{\circledast}=\hat{d}\left(\hat{\int}\hat{f}(x)\hat{\times}\hat{d}\hat{x}\right)\nearrow\hat{d}\hat{x}$$

$$=\hat{T}^{-1}(x)\frac{\hat{d}\hat{\int}\tilde{f}(x)\hat{\times}\hat{d}\hat{x}}{\hat{d}\hat{x}}$$

$$=\hat{T}^{-1}(x)\frac{\hat{T}(x)d\hat{\int}\tilde{f}(x)\hat{\times}\hat{d}\hat{x}}{\hat{d}\hat{x}}$$

$$=\frac{d\int\hat{T}^{-1}(x)\tilde{f}(x)\hat{\times}\hat{d}\hat{x}}{\hat{d}\hat{x}}$$

$$=\frac{d\int\hat{T}^{-1}\tilde{f}(x)\hat{T}(x)\hat{d}\hat{x}}{\hat{d}\hat{x}}$$

$$=\frac{d\int\tilde{f}(x)\hat{d}\hat{x}}{\hat{d}\hat{x}}$$

$$=\frac{\tilde{f}(x)\hat{d}\hat{x}}{\hat{d}\hat{x}}$$

$$=\tilde{f}(x)\quad\forall x\in[a,b].$$

2)

$$\hat{\int}\hat{d}\hat{x}=\hat{\int}1\hat{d}\hat{x}=\int\hat{T}^{-1}(x)\hat{d}\hat{x}=\int\hat{T}^{-1}(x)\hat{T}(x)d\hat{x}=\int d\hat{x}=\hat{x}\quad\forall x\in[a,b].$$

3)

$$\hat{\int}\hat{d}\hat{x}=\hat{\int}1\hat{d}\hat{x}=\int\hat{T}^{-1}(x)\hat{d}\hat{x}$$

$$=\int\hat{T}^{-1}(x)\hat{T}(x)d\hat{x}=\int d\hat{x}=\hat{x}\quad\forall x\in[a,b].$$

4)

$$\hat{\int}\hat{d}\hat{f}(x)=\int\hat{T}^{-1}\hat{d}\hat{f}(x)$$

$$=\int\hat{T}^{-1}(x)\hat{T}(x)d\hat{f}(x)$$

$$=\int d\hat{f}(x)=\hat{f}(x)+C\quad\forall x\in[a,b],\quad C\in\mathbb{R}.$$

5) For $a\in\hat{F}_{\mathbb{R}}$ we have for every $x\in[a,b]$

$$\hat{\int}\hat{a}\hat{\times}\hat{f}(x)\hat{\times}\hat{d}\hat{x}=\hat{a}\hat{\times}\hat{\int}\hat{f}(x)\hat{\times}\hat{d}\hat{x}.$$

Proof.

$$\hat{\int} \hat{a} \hat{\times} \hat{f}(x) \hat{\times} \hat{d}\hat{x} = \int \hat{T}^{-1}(x) \hat{a} \hat{\times} \hat{f}(x) \hat{\times} \hat{d}\hat{x}$$

$$= \int \hat{T}^{-1}(x) \tfrac{a}{\hat{T}_1} \hat{T}_1 \hat{f}(x) \hat{T}(x) \hat{d}\hat{x}$$

$$= \int a \hat{f}(x) \hat{T}(x) d\hat{x}$$

$$= a \int \hat{f}(x) \hat{T}(x) d\hat{x}$$

$$= a \int \hat{f}(x) \hat{d}\hat{x}$$

$$= \tfrac{a}{T_1} T_1 \int \hat{f}(x) \hat{d}\hat{x}$$

$$= \hat{a} \hat{\times} \int \hat{f}(x) \hat{d}\hat{x}$$

$$= \hat{a} \hat{\times} \int \hat{T}^{-1}(x) \hat{f}(x) \hat{T}(x) \hat{d}\hat{x}$$

$$= \hat{a} \hat{\times} \hat{\int} \hat{f}(x) \hat{\times} \hat{d}\hat{x} \quad \forall x \in [a, b].$$

□

6) For every $x \in [a, b]$ we have

$$\hat{\int} \Big(\hat{f}(x) + \hat{g}(x) \Big) \hat{\times} \hat{d}\hat{x} = \hat{\int} \hat{f}(x) \hat{\times} \hat{d}\hat{x} + \hat{\int} \hat{g}(x) \hat{\times} \hat{d}\hat{x}.$$

Proof. For $x \in [a, b]$ we have

$$\hat{\int} \Big(\hat{f}(x) + \hat{g}(x) \Big) \hat{\times} \hat{d}\hat{x} = \int \hat{T}^{-1}(x) \Big(\hat{f}(x) + \hat{g}(x) \Big) \hat{\times} \hat{d}\hat{x}$$

$$= \int \hat{T}^{-1}(x) \Big(\hat{f}(x) + \hat{g}(x) \Big) \hat{T}(x) \hat{d}\hat{x}$$

$$= \int \Big(\hat{f}(x) + \hat{g}(x) \Big) \hat{T}(x) d\hat{x}$$

$$= \int f(x) \hat{T}(x) d\hat{x} + \int g(x) \hat{T}(x) d\hat{x}$$

$$= \int \hat{f}(x)\hat{d}\hat{x} + \int \hat{g}(x)\hat{d}\hat{x}$$

$$= \int \hat{T}^{-1}(x)\hat{f}(x)\hat{T}(x)\hat{d}\hat{x} + \int \hat{T}^{-1}(x)\hat{g}(x)\hat{T}(x)\hat{d}\hat{x}$$

$$= \hat{\int} \hat{f}(x)\hat{\times}\hat{d}\hat{x} + \hat{\int} \hat{g}(x)\hat{\times}\hat{d}\hat{x}.$$

$\square$

For the isointegrals of isofunctions of first, second, third and fourth kind we have the following representations:

1) isofunctions of first kind

$$\hat{\int} \hat{f}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} = \int \hat{T}^{-1}(x)\frac{f(x)}{\hat{T}(x)}\hat{T}(x)\left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx = \int f(x)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)}dx.$$

2) isofunctions of second kind

$$\hat{\int} \hat{f}^\wedge(x)\hat{\times}\hat{d}\hat{x} = \int \hat{T}^{-1}(x)\frac{f(x\hat{T}(x))}{\hat{T}(x)}\left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx$$

$$= \int f(x\hat{T}(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)}dx.$$

3) isofunctions of third kind

$$\hat{\int} \hat{f}(\hat{x})\hat{\times}\hat{d}\hat{x} = \int \hat{T}^{-1}(x)\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}\hat{T}(x)\left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx$$

$$= \int f\left(\frac{x}{\hat{T}(x)}\right)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)}dx.$$

4) isofunctions of fourth kind

$$\hat{\int} f^\wedge(x)\hat{\times}\hat{d}\hat{x} = \int \hat{T}^{-1}(x)f(x\hat{T}(x))\hat{T}(x)\left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx$$

$$= \int f(x\hat{T}(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}dx.$$

Example 7.0.345. *Let $D = [a, b]$, $a, b \in \mathbb{R}$, $a < b$, a and b are arbitrary* *chosen, let also* $f(x)=x^2-x$, $\hat{T}(x)=e^{-x}$. *Then*

$$\hat{\int}\hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{d}\hat{x}=\int f(x)\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}^2(x)}dx$$

$$=\int(x^2-x)\tfrac{e^{-x}+xe^{-x}}{e^{-2x}}dx$$

$$=\int(x^2-x)(1+x)e^xdx$$

$$=\int(x^3-x)e^xdx$$

$$=\int(x^3-x)de^x$$

$$=(x^3-x)e^x-\int(3x^2-1)e^xdx$$

$$=(x^3-x)e^x-\int(3x^2-1)de^x$$

$$=(x^3-x)e^x-(3x^2-1)e^x+6\int xe^xdx$$

$$=(x^3-3x^2-x+1)e^x+6\int xde^x$$

$$=(x^3-3x^2-x+1)e^x+6xe^x-6\int e^xdx$$

$$=(x^3-3x^2+5x+1)e^x-6e^x+C$$

$$=(x^3-3x^2+5x-5)e^x+C,\quad C\in\mathbb{R},$$

$$\hat{\int}\hat{f}^{\wedge}(x)\hat{\times}\hat{d}\hat{x}=\int f(x\hat{T}(x))\tfrac{e^{-x}+xe^{-x}}{e^{-2x}}dx$$

$$=\int f\Big(xe^{-x}\Big)e^x(x+1)dx$$

$$=\int\Big(\Big(xe^{-x}\Big)^2-xe^{-x}\Big)e^x(x+1)dx$$

$$=\int\Big(x^2e^{-2x}-xe^{-x}\Big)e^x(1+x)dx$$

$$=\int(x^3+x^2)e^{-x}dx-\int xdx-\int x^2dx$$

$$=-\int(x^3+x^2)de^{-x}-\tfrac{x^2}{2}-\tfrac{x^3}{3}$$

$$= (-x^3 - x^2)e^{-x} + \int (3x^2 + 2x)e^{-x}dx - \tfrac{x^2}{2} - \tfrac{x^3}{3}$$

$$= (-x^3 - x^2)e^{-x} - \int (3x^2 + 2x)de^{-x} - \tfrac{x^2}{2} - \tfrac{x^3}{3}$$

$$= (-x^3 - x^2)e^{-x} - (3x^2 + 2x)e^{-x} + \int (6x + 2)e^{-x}dx - \tfrac{x^2}{2} - \tfrac{x^3}{3}$$

$$= (-x^3 - 4x^2 - 2x)e^{-x} + \int (6x + 2)de^{-x} - \tfrac{x^2}{2} - \tfrac{x^3}{3}$$

$$= (-x^3 - 4x^2 - 2x)e^{-x} - (6x + 2)e^{-x} + 6\int e^{-x}dx - \tfrac{x^2}{2} - \tfrac{x^3}{3}$$

$$= (-x^3 - 4x^2 - 8x - 2)e^{-x} - 6e^{-x} - \tfrac{x^2}{2} - \tfrac{x^3}{3} + C$$

$$= (-x^3 - 4x^2 - 8x - 8)e^{-x} - \tfrac{x^2}{2} - \tfrac{x^3}{3} + C, \quad C \in \mathbb{R},$$

$$\hat{\int} \hat{f}(\hat{x}) \hat{\times} \hat{d}\hat{x} = \int f\left(\tfrac{x}{\hat{T}(x)}\right) \tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx$$

$$= \int f\left(xe^x\right) \tfrac{(1+x)e^{-x}}{e^{-2x}} dx$$

$$= \int \left(\left(xe^x\right)^2 - xe^x\right) e^x (x+1) dx$$

$$= \int \left(x^2 e^{2x} - xe^x\right) e^x (1+x) dx$$

$$= \int (x^3 + x^2)e^{3x}dx - \int (x^2 + x)e^{2x}dx$$

$$= \tfrac{1}{3}(x^3 + x^2)e^{3x} - \tfrac{1}{3}\int (3x^2 + 2x)e^{3x}dx - \tfrac{1}{2}(x^2 + x)e^{2x} + \tfrac{1}{2}\int (2x + 1)e^{2x}dx$$

$$= \tfrac{1}{3}(x^3 + x^2)e^{3x} - \tfrac{1}{2}(x^2 + x)e^{2x} - \tfrac{1}{9}\int (3x^2 + 2x)de^{3x} + \tfrac{1}{4}\int (2x + 1)de^{2x}$$

$$= \tfrac{1}{3}(x^3 + x^2)e^{3x} - \tfrac{1}{2}(x^2 + x)e^{2x} - \tfrac{1}{9}(3x^2 + 2x)e^{3x} + \tfrac{1}{4}(2x + 1)e^{2x}$$

$$+ \tfrac{1}{9}\int (6x + 2)e^{3x}dx - \tfrac{1}{2}\int e^{2x}dx$$

$$= \left(\tfrac{x^3}{3} - \tfrac{2}{9}x\right)e^{3x} + \left(-\tfrac{1}{2}x^2 + \tfrac{1}{4}\right)e^{2x} + \tfrac{1}{27}\int (6x + 2)de^{3x} - \tfrac{1}{4}e^{2x}$$

$$= \left(\tfrac{x^3}{3} - \tfrac{2}{9}x\right)e^{3x} - \tfrac{1}{2}x^2 e^{2x} + \tfrac{1}{27}(6x + 2)e^{3x} - \tfrac{6}{27}\int e^{3x}dx$$

$$= \left(\frac{x^3}{3} + \frac{2}{27}\right)e^{3x} - \frac{x^2}{2}e^{2x} - \frac{2}{27}e^{3x} + C$$

$$= \frac{x^3}{3}e^{3x} - \frac{x^2}{2}e^{2x} + C, \quad C \in \mathbb{R},$$

$\hat{\int} f^{\wedge}(x) \hat{\times} \hat{d}\hat{x} = \int f(x\hat{T}(x)) \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} dx$

$= \int f\left(xe^{-x}\right) \frac{e^{-x} + xe^{-x}}{e^{-x}} dx$

$= \int \left(\left(xe^{-x}\right)^2 - xe^{-x}\right)(1+x)dx$

$= \int \left(x^2 e^{-2x} - xe^{-x}\right)(x+1)dx$

$= \int (x^3 + x^2)e^{-2x}dx - \int (x^2 + x)e^{-x}dx$

$= -\frac{1}{2}\int (x^3 + x^2)de^{-2x} + \int (x^2 + x)de^{-x}$

$= -\frac{1}{2}(x^3 + x^2)e^{-2x} + \frac{1}{2}\int (3x^2 + 2x)e^{-2x}dx - (x^2 + x)e^{-x} + \int (2x+1)e^{-x}dx$

$= -\frac{1}{2}(x^3 + x^2)e^{-2x} - \frac{1}{4}\int (3x^2 + 2x)de^{-2x} - (x^2 + x)e^{-x} - \int (2x+1)de^{-x}$

$= -\frac{1}{2}(x^3 + x^2)e^{-2x} - \frac{1}{4}(3x^2 + 2x)e^{-2x} + \frac{1}{4}\int (6x+2)e^{-2x}dx$

$-(x^2 + x)e^{-x} - (2x+1)e^{-x} + 2\int e^{-x}dx$

$= \left(-\frac{x^3}{2} - \frac{5}{4}x^2 - \frac{x}{2}\right)e^{-2x} - \frac{1}{4}(3x+1)e^{-2x} + \frac{3}{4}\int e^{-2x}dx - (x^2 + 3x + 3)e^{-x}$

$= \left(-\frac{x^3}{2} - \frac{5}{4}x^2 - \frac{5}{4}x - \frac{1}{4}\right)e^{-2x} - \frac{3}{8}e^{-2x} - (x^2 + 3x + 3)e^{-x} + C$

$= \left(-\frac{x^3}{2} - \frac{5}{4}x^2 - \frac{5}{4}x - \frac{5}{8}\right)e^{-2x} - (x^2 + 3x + 3)e^{-x} + C, \quad C \in \mathbb{R}.$

Exercise 7.0.346. *Let* $D = [1,2]$*,* $f(x) = x$*,* $\hat{T}(x) = x^2$*. Find*

$$\hat{\int} \hat{f}^{\wedge}(\hat{x}) \hat{\times} \hat{d}\hat{x}, \quad \hat{\int} \hat{f}^{\wedge}(x) \hat{\times} \hat{d}\hat{x}, \quad \hat{\int} \hat{f}(\hat{x}) \hat{\times} \hat{d}\hat{x}, \quad \hat{\int} f^{\wedge}(x) \times \hat{d}\hat{x}.$$

Answer.

$$-\ln x + C, \quad -\frac{x^2}{2} + C, \quad \frac{1}{x} + C, \quad -\frac{x^4}{4} + C, \quad C \in \mathbb{R}.$$

Exercise 7.0.347. *Let* $D = [2, 3]$, $f(x) = \hat{T}(x) = x^2$. *Find*

$$\hat{\int}_2^3 \hat{f}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x}, \quad \hat{\int}_2^3 \hat{f}^\wedge(x)\hat{\times}\hat{d}\hat{x}, \quad \hat{\int}_2^3 \hat{f}(\hat{x})\hat{\times}\hat{d}\hat{x}, \quad \hat{\int}_2^3 f^\wedge(x)\times\hat{d}\hat{x}.$$

Answer.

$$-1, \quad -\frac{211}{5}, \quad -\frac{19}{648}, \quad \frac{2059}{7}.$$

Theorem 7.0.348. *Let* $f, \hat{T} \in \mathcal{C}^1(D)$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$, $\hat{T}(x) > 0$ *for every* $x \in D$. *Then*

$$\hat{\int} \hat{f}^{\wedge\circledast}(\hat{x})\hat{d}\hat{x} = \hat{f}^\wedge(\hat{x})$$

for every $x \in D$.

Proof. For $x \in D$, using the representation of the first isoderivative of iso-functions of first kind, we have

$$\hat{\int} \hat{f}^{\wedge\circledast}(\hat{x})\hat{d}\hat{x} = \int \hat{T}^{-1}(x)\hat{f}^{\wedge\circledast}(\hat{x})\hat{d}\hat{x}$$

$$= \int \hat{T}^{-1}(x)\frac{1}{\hat{T}^2(x)}\frac{f'(x)\hat{T}(x) - f(x)\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}\hat{\times}\hat{d}\hat{x}$$

$$= \int \frac{1}{\hat{T}^3(x)}\frac{f'(x)\hat{T}(x) - f(x)\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}\hat{T}(x)\left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx$$

$$= \int \frac{f'(x)\hat{T}(x) - f(x)\hat{T}'(x)}{\hat{T}^2(x)}dx$$

$$= \int d\left(\frac{f(x)}{\hat{T}(x)}\right)'$$

$$= \frac{f(x)}{\hat{T}(x)} = \hat{f}^\wedge(\hat{x}).$$

∎

Theorem 7.0.349. *Let* $f, \hat{T} \in \mathcal{C}^1(D)$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$, $\hat{T}(x) > 0$ *for every* $x \in D$. *Then*

$$\hat{\int} \hat{f}^{\wedge\circledast}(x) \hat{\times} \hat{d}\hat{x} = \hat{f}^{\wedge}(x)$$

for every $x \in D$.

Proof. Using the representation of first isoderivative of an isofunction of second kind we get

$\hat{\int} \hat{f}^{\wedge\circledast}(x) \hat{\times} \hat{d}\hat{x}$

$= \int \hat{T}^{-1}(x) \frac{1}{\hat{T}^2(x)} \frac{f'(x\hat{T}(x))(\hat{T}^2(x) + x\hat{T}(x)\hat{T}'(x)) - f(x\hat{T}(x))\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}} \hat{T}(x) \Big(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big) dx$

$= \int \frac{(f(x\hat{T}(x)))'\hat{T}(x) - f(x\hat{T}(x))\hat{T}'(x)}{\hat{T}^2(x)} dx$

$= \int d\Big(\frac{f(x\hat{T}(x))}{\hat{T}(x)}\Big)'$

$= \frac{f(x\hat{T}(x))}{\hat{T}(x)} = \hat{f}^{\wedge}(x).$

□

Exercise 7.0.350. *Let* $f, \hat{T} \in \mathcal{C}^1(D)$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$, $\hat{T}(x) > 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} \hat{f}^{\circledast}(\hat{x}) \hat{\times} \hat{d}\hat{x} = \hat{f}(\hat{x})$$

for every $x \in D$.

Exercise 7.0.351. *Let* $f, \hat{T} \in \mathcal{C}^1(D)$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$, $\hat{T}(x) > 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} f^{\wedge\circledast}(x) \hat{\times} \hat{d}\hat{x} = f^{\wedge}(x)$$

for every $x \in D$.

Theorem 7.0.352. *(integration by parts) Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$, $\hat{T}(x) > 0$ for every $x \in D$. Then*

$$\hat{\int} \hat{f}^{\wedge \circledast}(\hat{x}) \hat{g}^{\wedge}(\hat{x}) \hat{\times} \hat{d}\hat{x} = \hat{f}^{\wedge}(\hat{x}) \hat{g}^{\wedge}(\hat{x}) - \hat{\int} \hat{f}^{\wedge}(\hat{x}) \hat{g}^{\wedge \circledast}(\hat{x}) \hat{\times} \hat{d}\hat{x}.$$

for every $x \in D$.

Proof. Using the main the definition for isointegral, the representation of the first isoderivative of isofunction of first kind and the representation for $\hat{d}\hat{x}$ we get

$$\hat{\int} \hat{f}^{\wedge \circledast}(\hat{x}) \hat{g}^{\wedge}(\hat{x}) \hat{\times} \hat{d}\hat{x}$$

$$= \int \hat{T}^{-1}(x) \frac{f'(x)\hat{T}(x) - f(x)\hat{T}'(x)}{\hat{T}^2(x)\left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)} \frac{g(x)}{\hat{T}(x)} \hat{T}(x) \left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right) dx$$

$$= \int \frac{f'(x)\hat{T}(x) - f(x)\hat{T}'(x)}{\hat{T}^2(x)} \frac{g(x)}{\hat{T}(x)} dx$$

$$= \int \frac{g(x)}{\hat{T}(x)} d\left(\frac{f(x)}{\hat{T}(x)}\right)$$

$$= \frac{f(x)}{\hat{T}(x)} \frac{g(x)}{\hat{T}(x)} - \int \frac{f(x)}{\hat{T}(x)} \left(\frac{g(x)}{\hat{T}(x)}\right)' dx$$

$$= \hat{f}^{\wedge}(\hat{x}) \hat{g}^{\wedge}(\hat{x}) - \int \frac{f(x)}{\hat{T}(x)} \frac{g'(x)\hat{T}(x) - g(x)\hat{T}'(x)}{\hat{T}^2(x)} dx$$

$$= \hat{f}^{\wedge}(\hat{x}) \hat{g}^{\wedge}(\hat{x}) - \int \hat{f}^{\wedge}(\hat{x}) \frac{g'(x)\hat{T}(x) - g(x)\hat{T}'(x)}{\hat{T}^2(x)\left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)} \left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right) dx$$

$$= \hat{f}^{\wedge}(\hat{x}) \hat{g}^{\wedge}(\hat{x}) - \int \hat{f}^{\wedge}(\hat{x}) \hat{g}^{\wedge \circledast}(\hat{x}) \hat{d}\hat{x}$$

$$= \hat{f}^{\wedge}(\hat{x}) \hat{g}^{\wedge}(\hat{x}) - \int \hat{f}^{\wedge}(\hat{x}) \hat{g}^{\wedge \circledast}(\hat{x}) \frac{1}{\hat{T}(x)} \hat{T}(x) \hat{d}\hat{x}$$

$$= \hat{f}^{\wedge}(\hat{x}) \hat{g}^{\wedge}(\hat{x}) - \hat{\int} \hat{f}^{\wedge}(\hat{x}) \hat{g}^{\wedge \circledast}(\hat{x}) \hat{\times} \hat{d}\hat{x}.$$

□

Exercise 7.0.353. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} \hat{f}^{\wedge\circledast}(x)\hat{g}^{\wedge}(\hat{x})\hat{\times}\hat{d}\hat{x} = \hat{f}^{\wedge}(x)\hat{g}^{\wedge}(\hat{x}) - \hat{\int} \hat{f}^{\wedge}(x)\hat{g}^{\wedge\circledast}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.354. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} \hat{f}^{\circledast}(\hat{x})\hat{g}^{\wedge}(\hat{x})\hat{\times}\hat{d}\hat{x} = \hat{f}(\hat{x})\hat{g}^{\wedge}(\hat{x}) - \hat{\int} \hat{f}(\hat{x})\hat{g}^{\wedge\circledast}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.355. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} f^{\wedge\circledast}(x)\hat{g}^{\wedge}(\hat{x})\hat{\times}\hat{d}\hat{x} = f^{\wedge}(x)\hat{g}^{\wedge}(\hat{x}) - \hat{\int} f^{\wedge}(x)\hat{g}^{\wedge\circledast}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.356. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} \hat{f}^{\wedge\circledast}(x)\hat{g}^{\wedge}(x)\hat{\times}\hat{d}\hat{x} = \hat{f}^{\wedge}(x)\hat{g}^{\wedge}(x) - \hat{\int} \hat{f}^{\wedge}(x)\hat{g}^{\wedge\circledast}(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.357. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} \hat{f}^{\circledast}(\hat{x})\hat{g}^{\wedge}(x)\hat{\times}\hat{d}\hat{x} = \hat{f}(\hat{x})\hat{g}^{\wedge}(x) - \hat{\int} \hat{f}(\hat{x})\hat{g}^{\wedge\circledast}(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.358. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} f^{\wedge\circledast}(x)g^{\wedge}(x)\hat{\times}\hat{d}\hat{x} = f^{\wedge}(x)g^{\wedge}(x) - \hat{\int} f^{\wedge}(x)g^{\wedge\circledast}(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.359. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} \hat{f}^{\circledast}(\hat{x})\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} = \hat{f}(\hat{x})\hat{g}(\hat{x}) - \hat{\int} \hat{f}(\hat{x})\hat{g}^{\circledast}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.360. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} \hat{f}^{\circledast}(\hat{x}) g^{\wedge}(x) \hat{\times} \hat{d}\hat{x} = \hat{f}(\hat{x}) g^{\wedge}(x) - \hat{\int} \hat{f}(\hat{x}) g^{\wedge\circledast}(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.361. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} f^{\wedge\circledast}(x) g^{\wedge}(x) \hat{\times} \hat{d}\hat{x} = f^{\wedge}(x) g^{\wedge}(x) - \hat{\int} f^{\wedge}(x) g^{\wedge\circledast}(x) \hat{\times} \hat{d}\hat{x}.$$

Theorem 7.0.362. *(integration by parts) Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Then*

$$\hat{\int} \hat{f}^{\wedge\circledast}(\hat{x}) \hat{\times} \hat{g}^{\wedge}(\hat{x}) \hat{d}\hat{x} = \hat{f}^{\wedge}(\hat{x}) \hat{g}^{\wedge}(\hat{x}) - \hat{\int} \hat{f}^{\wedge}(\hat{x}) \hat{\times} \hat{g}^{\wedge}(\hat{x}) \hat{d}\hat{x}$$

for every $x \in D$.

Proof.

$$\hat{\int} \hat{f}^{\wedge\circledast}(\hat{x}) \hat{\times} \hat{g}^{\wedge}(\hat{x}) \hat{d}\hat{x}$$

$$= \int \hat{T}^{-1}(x) \frac{f'(x)\hat{T}(x) - f(x)\hat{T}'(x)}{\hat{T}^2(x)\left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)} \hat{T}(x) \frac{g(x)}{\hat{T}(x)} \left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right) dx$$

$$= \int \frac{f'(x)\hat{T}(x) - f(x)\hat{T}'(x)}{\hat{T}^2(x)} \frac{g(x)}{\hat{T}(x)} dx$$

$$= \int \left(\frac{f(x)}{\hat{T}(x)}\right)' \frac{g(x)}{\hat{T}(x)} dx$$

$$= \frac{f(x)}{\hat{T}(x)} \frac{g(x)}{\hat{T}(x)} - \int \frac{f(x)}{\hat{T}(x)} \frac{g'(x)\hat{T}(x) - g(x)\hat{T}'(x)}{\hat{T}^2(x)} dx$$

$$= \hat{f}^{\wedge}(\hat{x})\hat{g}^{\wedge}(\hat{x}) - \int \hat{T}^{-1}(x) \frac{f(x)}{\hat{T}(x)} \hat{T}(x) \frac{g'(x)\hat{T}(x) - g(x)\hat{T}'(x)}{\hat{T}^2(x)\left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)} \left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right) dx$$

$$= \hat{f}^{\wedge}(\hat{x})\hat{g}^{\wedge}(\hat{x}) - \hat{\int} \hat{f}^{\wedge}(\hat{x}) \hat{\times} \hat{g}^{\wedge}(\hat{x}) \hat{d}\hat{x} \quad \forall x \in D.$$

$\square$

Exercise 7.0.363. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ for every $x \in D$. Prove*

$$\hat{\int} \hat{f}^{\wedge\circledast}(x) \hat{\times} \hat{g}^{\wedge}(\hat{x}) \hat{d}\hat{x} = \hat{f}^{\wedge}(x)\hat{g}^{\wedge}(\hat{x}) - \hat{\int} \hat{f}^{\wedge}(x) \hat{\times} \hat{g}^{\wedge\circledast}(\hat{x}) \hat{d}\hat{x}.$$

Exercise 7.0.364. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ for every $x \in D$. Prove*

$$\hat{\int} \hat{f}^{\circledast}(\hat{x}) \hat{\times} \hat{g}^{\wedge}(\hat{x}) \hat{d}\hat{x} = \hat{f}(\hat{x})\hat{g}^{\wedge}(\hat{x}) - \hat{\int} \hat{f}(\hat{x}) \hat{\times} \hat{g}^{\wedge\circledast}(\hat{x}) \hat{d}\hat{x}.$$

Exercise 7.0.365. *Let $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ for every $x \in D$. Prove*

$$\hat{\int} f^{\wedge\circledast}(x) \hat{\times} \hat{g}^{\wedge}(\hat{x}) \hat{d}\hat{x} = f^{\wedge}(x)\hat{g}^{\wedge}(\hat{x}) - \hat{\int} f^{\wedge}(x) \hat{\times} \hat{g}^{\wedge\circledast}(\hat{x}) \hat{d}\hat{x}.$$

Exercise 7.0.366. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} \hat{f}^{\wedge\circledast}(x)\hat{\times}\hat{g}^{\wedge}(x)\hat{d}\hat{x} = \hat{f}^{\wedge}(x)\hat{g}^{\wedge}(x) - \hat{\int} \hat{f}^{\wedge}(x)\hat{\times}\hat{g}^{\wedge\circledast}(x)\hat{d}\hat{x}.$$

Exercise 7.0.367. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} \hat{f}^{\circledast}(\hat{x})\hat{\times}\hat{g}^{\wedge}(x)\hat{d}\hat{x} = \hat{f}(\hat{x})\hat{g}^{\wedge}(x) - \hat{\int} \hat{f}(\hat{x})\hat{\times}\hat{g}^{\wedge\circledast}(x)\hat{d}\hat{x}.$$

Exercise 7.0.368. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} f^{\wedge\circledast}(x)\hat{\times}g^{\wedge}(x)\hat{d}\hat{x} = f^{\wedge}(x)g^{\wedge}(x) - \hat{\int} f^{\wedge}(x)\hat{\times}g^{\wedge\circledast}(x)\hat{d}\hat{x}.$$

Exercise 7.0.369. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} \hat{f}^{\circledast}(\hat{x})\hat{\times}\hat{g}(\hat{x})\hat{d}\hat{x} = \hat{f}(\hat{x})\hat{g}(\hat{x}) - \hat{\int} \hat{f}(\hat{x})\hat{\times}\hat{g}^{\circledast}(\hat{x})\hat{d}\hat{x}.$$

Exercise 7.0.370. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} \hat{f}^{\circledast}(\hat{x})\hat{\times}g^{\wedge}(x)\hat{d}\hat{x} = \hat{f}(\hat{x})g^{\wedge}(x) - \hat{\int} \hat{f}(\hat{x})\hat{\times}g^{\wedge\circledast}(x)\hat{d}\hat{x}.$$

Exercise 7.0.371. *Let* $f, g, \hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} \neq 0$ *for every* $x \in D$. *Prove*

$$\hat{\int} f^{\wedge\circledast}(x)\hat{\times}g^{\wedge}(x)\hat{d}\hat{x} = f^{\wedge}(x)g^{\wedge}(x) - \hat{\int} f^{\wedge}(x)\hat{\times}g^{\wedge\circledast}(x)\hat{d}\hat{x}.$$

Theorem 7.0.372. *Let* f *is integrable function on* $[a, b]$, $f(x) \leq 0$ *for every* $x \in [a, b]$, $\hat{T} \in \mathcal{C}^1([a, b])$, $\hat{T}(x) > 0$ *for every* $x \in [a, b]$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ *for every* $x \in [a, b]$. *Then*

$$\hat{\int_a^b} \hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{d}\hat{x} \geq 0.$$

Proof. Since

$$f(x) \le 0, \quad \hat{T}(x) - x\hat{T}'(x) \le 0 \quad \forall x \in [a,b]$$

then

$$f(x)\Big(\hat{T}(x) - x\hat{T}'(x)\Big) \ge 0 \quad \forall x \in [a,b].$$

From here and from $\hat{T}^2(x) > 0$ for every $x \in [a,b]$ it follows that

$$f(x)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} \ge 0 \quad \forall x \in [a,b].$$

We integrate the last inequality on $[a,b]$ and we obtain

$$\int_a^b f(x)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)}dx \ge 0$$

which is equivalent of, using the definition for iso-Cauchy isointegral of iso-function of first kind,

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \ge 0.$$

$\square$

Exercise 7.0.373. *Let f is integrable function on $[a,b]$, $f(x) \le 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$ for every $x \in [a,b]$, $\hat{T}(x) - x\hat{T}'(x) \le 0$ for every $x \in [a,b]$. Then*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{d}\hat{x} \ge 0.$$

Exercise 7.0.374. *Let f is integrable function on $[a,b]$, $f(x) \le 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$ for every $x \in [a,b]$, $\hat{T}(x) - x\hat{T}'(x) \le 0$ for every $x \in [a,b]$. Then*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}d\hat{x} \ge 0.$$

Exercise 7.0.375. *Let f is integrable function on $[a, b]$, $f(x) \le 0$ for every $x \in [a, b]$, $\hat{T}$ is integrable on $[a, b]$ $\hat{T}(x) > 0$ for every $x \in [a, b]$. Then*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times} dx \le 0.$$

Exercise 7.0.376. *Let f is integrable function on $[a, b]$, $f(x) \le 0$ for every $x \in [a, b]$, $\hat{T}$ is integrable on $[a, b]$, $\hat{T}(x) > 0$ for every $x \in [a, b]$. Then*

$$\int_a^b \hat{f}^\wedge(\hat{x})\hat{\times} dx \le 0.$$

Exercise 7.0.377. *Let f is integrable function on $[a, b]$, $f(x) \le 0$ for every $x \in [a, b]$, $\hat{T}$ is integrable on $[a, b]$, $\hat{T}(x) > 0$ for every $x \in [a, b]$. Then*

$$\int_a^b \hat{f}^\wedge(\hat{x}) dx \le 0.$$

Exercise 7.0.378. *Let f is integrable function on $[a, b]$, $f(x) \ge 0$ for every $x \in [a, b]$, $\hat{T} \in \mathcal{C}^1([a, b])$, $\hat{T}(x) > 0$ for every $x \in [a, b]$, $\hat{T}(x) - x\hat{T}'(x) \ge 0$ for every $x \in [a, b]$. Then*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{d}\hat{x} \ge 0.$$

Exercise 7.0.379. *Let f is integrable function on $[a, b]$, $f(x) \le 0$ for every $x \in [a, b]$, $\hat{T} \in \mathcal{C}^1([a, b])$, $\hat{T}(x) > 0$ for every $x \in [a, b]$, $\hat{T}(x) - x\hat{T}'(x) \le 0$, $x\hat{T}(x) \in [a, b]$ for every $x \in [a, b]$. Then*

$$\hat{\int}_a^b \hat{f}^\wedge(x)\hat{\times}\hat{d}\hat{x} \ge 0.$$

Exercise 7.0.380. *Let f is integrable function on $[a, b]$, $f(x) \ge 0$ for every $x \in [a, b]$, $\hat{T} \in \mathcal{C}^1([a, b])$, $\hat{T}(x) > 0$ for every $x \in [a, b]$, $\hat{T}(x) - x\hat{T}'(x) \ge 0$, $x\hat{T}(x) \in [a, b]$ for every $x \in [a, b]$. Then*

$$\hat{\int}_a^b \hat{f}^\wedge(x)\hat{\times}\hat{d}\hat{x} \ge 0.$$

Exercise 7.0.381. *Let f is integrable function on $[a, b]$, $f(x) \le 0$ for every $x \in [a, b]$, $\hat{T} \in \mathcal{C}^1([a, b])$, $\hat{T}(x) > 0$ for every $x \in [a, b]$, $\hat{T}(x) - x\hat{T}'(x) \le 0$, $\frac{x}{\hat{T}(x)} \in [a, b]$ for every $x \in [a, b]$. Then*

$$\hat{\int}_a^b \hat{f}(\hat{x})\hat{\times}\hat{d}\hat{x} \ge 0.$$

Exercise 7.0.382. *Let f is integrable function on $[a,b]$, $f(x)\geq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1([a,b])$, $\hat{T}(x)>0$ for every $x\in[a,b]$, $\hat{T}(x)-x\hat{T}'(x)\geq 0$, $\frac{x}{\hat{T}(x)}\in[a,b]$ for every $x\in[a,b]$. Then*

$$\hat{\int_a^b}\hat{f}(\hat{x})\hat{\times}\hat{d}\hat{x}\geq 0.$$

Exercise 7.0.383. *Let f is integrable function on $[a,b]$, $f(x)\leq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1([a,b])$, $\hat{T}(x)>0$ for every $x\in[a,b]$, $\hat{T}(x)-x\hat{T}'(x)\leq 0$, $x\hat{T}(x)\in[a,b]$ for every $x\in[a,b]$. Then*

$$\hat{\int_a^b}f^{\wedge}(x)\hat{\times}\hat{d}\hat{x}\geq 0.$$

Exercise 7.0.384. *Let f is integrable function on $[a,b]$, $f(x)\geq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1([a,b])$, $\hat{T}(x)>0$ for every $x\in[a,b]$, $\hat{T}(x)-x\hat{T}'(x)\geq 0$, $x\hat{T}(x)\in[a,b]$ for every $x\in[a,b]$. Then*

$$\hat{\int_a^b}f^{\wedge}(x)\hat{\times}\hat{d}\hat{x}\geq 0.$$

Exercise 7.0.385. *Let f is integrable function on $[a,b]$, $f(x)\geq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1([a,b])$, $\hat{T}(x)>0$, $\hat{T}(x)-x\hat{T}'(x)\leq 0$ for every $x\in[a,b]$. Then*

$$\hat{\int_a^b}\hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{d}\hat{x}\leq 0.$$

Exercise 7.0.386. *Let f is integrable function on $[a,b]$, $f(x)\leq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1([a,b])$, $\hat{T}(x)>0$, $\hat{T}(x)-x\hat{T}'(x)\geq 0$ for every $x\in[a,b]$. Then*

$$\hat{\int_a^b}\hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{d}\hat{x}\leq 0.$$

Exercise 7.0.387. *Let f is integrable function on $[a,b]$, $f(x)\geq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1([a,b])$, $\hat{T}(x)>0$, $\hat{T}(x)-x\hat{T}'(x)\leq 0$, $x\hat{T}(x)\in[a,b]$ for every $x\in[a,b]$. Then*

$$\hat{\int_a^b}\hat{f}^{\wedge}(x)\hat{\times}\hat{d}\hat{x}\leq 0.$$

Exercise 7.0.388. *Let f is integrable function on $[a,b]$, $f(x)\leq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1([a,b])$, $\hat{T}(x)>0$, $\hat{T}(x)-x\hat{T}'(x)\geq 0$, $x\hat{T}(x)\in[a,b]$ for every $x\in[a,b]$. Then*

$$\hat{\int_a^b}\hat{f}^{\wedge}(x)\hat{\times}\hat{d}\hat{x}\leq 0.$$

Exercise 7.0.389. *Let f is integrable function on $[a,b]$, $f(x)\geq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1([a,b])$, $\hat{T}(x)>0$, $\hat{T}(x)-x\hat{T}'(x)\leq 0$, $\frac{x}{\hat{T}(x)}\in[a,b]$ for every $x\in[a,b]$. Then*

$$\hat{\int}_a^b \hat{f}(\hat{x})\hat{\times}\hat{d}\hat{x}\leq 0.$$

Exercise 7.0.390. *Let f is integrable function on $[a,b]$, $f(x)\leq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1([a,b])$, $\hat{T}(x)>0$, $\hat{T}(x)-x\hat{T}'(x)\geq 0$, $\frac{x}{\hat{T}(x)}\in[a,b]$ for every $x\in[a,b]$. Then*

$$\hat{\int}_a^b \hat{f}(\hat{x})\hat{\times}\hat{d}\hat{x}\leq 0.$$

Exercise 7.0.391. *Let f is integrable function on $[a,b]$, $f(x)\geq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1([a,b])$, $\hat{T}(x)>0$, $\hat{T}(x)-x\hat{T}'(x)\leq 0$, $x\hat{T}(x)\in[a,b]$ for every $x\in[a,b]$. Then*

$$\hat{\int}_a^b f^{\wedge}(x)\hat{\times}\hat{d}\hat{x}\leq 0.$$

Exercise 7.0.392. *Let f is integrable function on $[a,b]$, $f(x)\leq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1([a,b])$, $\hat{T}(x)>0$, $\hat{T}(x)-x\hat{T}'(x)\geq 0$, $x\hat{T}(x)\in[a,b]$ for every $x\in[a,b]$. Then*

$$\hat{\int}_a^b f^{\wedge}(x)\hat{\times}\hat{d}\hat{x}\leq 0.$$

Theorem 7.0.393. *Let f and g are integrable functions on $[a,b]$, $f(x)\leq g(x)$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1([a,b])$, $\hat{T}(x)>0$, $\hat{T}(x)-x\hat{T}'(x)\leq 0$ for every $x\in[a,b]$. Then*

$$\hat{\int}_a^b \hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{d}\hat{x}\geq\hat{\int}_a^b \hat{g}^{\wedge}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Proof. Since

$$f(x)\leq g(x),\quad \hat{T}(x)-x\hat{T}'(x)\leq 0\quad \forall x\in[a,b]$$

we get

$$f(x)(\hat{T}(x) - x\hat{T}'(x)) \geq g(x)(\hat{T}(x) - x\hat{T}'(x)) \quad \forall x \in [a,b]$$

and because $\hat{T}^2(x) > 0$ for every $x \in [a,b]$ we obtain

$$f(x)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} \geq g(x)\frac{\hat{T}(x) - x\hat{T}'(x))}{\hat{T}^2(x)} \quad \forall x \in [a,b].$$

We integrate the last inequality from a to b and we get

$$\int_a^b f(x)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)}dx \geq \int_a^b g(x)\frac{\hat{T}(x) - x\hat{T}'(x))}{\hat{T}^2(x)}dx \quad \forall x \in [a,b],$$

which is equivalent of

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \geq \hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

□

Exercise 7.0.394. *Let f and g are integrable functions on $[a,b]$, $f(x) \leq g(x)$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Prove*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.395. *Let f and g are integrable functions on $[a,b]$, $f(x) \leq g(x)$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Prove*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{d}\hat{x} \geq \hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{d}\hat{x}.$$

Exercise 7.0.396. *Let f and g are integrable functions on $[a,b]$, $f(x) \leq g(x)$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Prove*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})d\hat{x} \geq \hat{\int}_a^b \hat{g}^\wedge(\hat{x})d\hat{x}.$$

Exercise 7.0.397. *Let f and g are integrable functions on $[a,b]$, $f(x) \leq g(x)$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $x\hat{T}(x) \in [a,b]$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Prove*

$$\hat{\int}_a^b \hat{f}^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.398. *Let f and g are integrable functions on $[a,b]$, $f(x) \leq g(x)$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $x\hat{T}(x) \in [a,b]$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Prove*

$$\hat{\int}_a^b \hat{f}^\wedge(x)\hat{\times}\hat{d}\hat{x} \geq \hat{\int}_a^b \hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.399. *Let f and g are integrable functions on $[a,b]$, $f(x) \leq g(x)$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\frac{x}{\hat{T}(x)} \in [a,b]$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Prove*

$$\hat{\int}_a^b \hat{f}(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.400. *Let f and g are integrable functions on $[a,b]$, $f(x) \leq g(x)$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\frac{x}{\hat{T}(x)} \in [a,b]$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Prove*

$$\hat{\int}_a^b \hat{f}(\hat{x})\hat{\times}\hat{d}\hat{x} \geq \hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.401. *Let f and g are integrable functions on $[a,b]$, $f(x) \leq g(x)$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $x\hat{T}(x) \in [a,b]$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Prove*

$$\hat{\int}_a^b f^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.402. *Let f and g are integrable functions on $[a,b]$, $f(x) \leq g(x)$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $x\hat{T}(x) \in [a,b]$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Prove*

$$\hat{\int}_a^b f^\wedge(x)\hat{\times}\hat{d}\hat{x} \geq \hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.403. *Let f is integrable function on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Prove*

$$\Big|\hat{\int_a^b} \hat{f}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x}\Big| \leq \hat{\int_a^b} |\hat{f}^\wedge(\hat{x})|\hat{d}\hat{x}.$$

Exercise 7.0.404. *Let f is integrable function on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Prove*

$$\Big|\hat{\int_a^b} \hat{f}^\wedge(x)\hat{\times}\hat{d}\hat{x}\Big| \leq \hat{\int_a^b} |\hat{f}^\wedge(x)|\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.405. *Let f is integrable function on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Prove*

$$\Big|\hat{\int_a^b} \hat{f}(\hat{x})\hat{\times}\hat{d}\hat{x}\Big| \leq \hat{\int_a^b} |\hat{f}(\hat{x})|\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.406. *Let f is integrable function on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Prove*

$$\Big|\hat{\int_a^b} f^\wedge(x)\hat{\times}\hat{d}\hat{x}\Big| \leq \hat{\int_a^b} |f^\wedge(x)|\hat{\times}\hat{d}\hat{x}.$$

Theorem 7.0.407. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{M}\hat{\times}\hat{\int_a^b} \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{\int_a^b} \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{m}\hat{\times}\hat{\int_a^b} \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Proof. Since f is integrable function on $[a,b]$ then it is a bounded function. Therefore there exist $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$ and we have the following inequalities

$$m \leq f(x) \leq M \quad \text{for} \quad \forall x \in [a,b].$$

Since $g(x) \geq 0$ for every $x \in [a,b]$, from the last inequality we obtain

$$mg(x) \leq f(x)g(x) \leq Mg(x) \quad \text{for} \quad \forall x \in [a,b].$$

From here and $\hat{T}(x) - x\hat{T}'(x) \leq 0$, $\hat{T}^2(x) > 0$ for every $x \in [a,b]$, we get the inequality

$$Mg(x)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} \leq f(x)g(x)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} \leq Mg(x)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)},$$

for every $x \in [a,b]$, which we integrate from a to b and we get

$$M \int_a^b g(x)\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}^2(x)}dx \leq \int_a^b f(x)g(x)\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}^2(x)}dx$$

$$\leq M \int_a^b g(x)\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}^2(x)}dx,$$

which is equivalent of

$$\hat{M}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{m}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

$\square$

Exercise 7.0.408. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.409. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.410. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{M}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{m}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.411. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.412. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.413. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{M}\hat{\times}\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{m}\hat{\times}\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.414. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.415. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.416. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{M}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{m}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.417. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.418. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.419. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{M}\hat{\times}\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{f}^\wedge(x)\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{m}\hat{\times}\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.420. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int_a^b}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{\int_a^b}\hat{f}^\wedge(x)\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int_a^b}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.421. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int_a^b}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{\int_a^b}\hat{f}^\wedge(x)\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int_a^b}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.422. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{M}\hat{\times}\hat{\int_a^b}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int_a^b}\hat{f}^\wedge(x)\hat{\times}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{m}\hat{\times}\hat{\int_a^b}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.423. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int_a^b}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int_a^b}\hat{f}^\wedge(x)\hat{\times}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int_a^b}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.424. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int_a^b}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int_a^b}\hat{f}^\wedge(x)\hat{\times}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int_a^b}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.425. *Let f and g are integrable functions on $[a,b]$, $g(x)\geq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1(D)$, $\hat{T}(x)>0$, $\hat{T}(x)-x\hat{T}'(x)\leq 0$ for every $x\in[a,b]$. Then*

$$\hat{M}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x}\leq\hat{\int}_a^b \hat{f}^\wedge(x)\hat{\times}g^\wedge(x)\hat{\times}\hat{d}\hat{x}\leq\hat{m}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m=\inf_{x\in[a,b]}f(x)$, $M=\sup_{x\in[a,b]}f(x)$.

Exercise 7.0.426. *Let f and g are integrable functions on $[a,b]$, $g(x)\leq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1(D)$, $\hat{T}(x)>0$, $\hat{T}(x)-x\hat{T}'(x)\leq 0$ for every $x\in[a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x}\leq\hat{\int}_a^b \hat{f}^\wedge(x)\hat{\times}g^\wedge(x)\hat{\times}\hat{d}\hat{x}\leq\hat{M}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m=\inf_{x\in[a,b]}f(x)$, $M=\sup_{x\in[a,b]}f(x)$.

Exercise 7.0.427. *Let f and g are integrable functions on $[a,b]$, $g(x)\geq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1(D)$, $\hat{T}(x)>0$, $\hat{T}(x)-x\hat{T}'(x)\geq 0$ for every $x\in[a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x}\leq\hat{\int}_a^b \hat{f}^\wedge(x)\hat{\times}g^\wedge(x)\hat{\times}\hat{d}\hat{x}\leq\hat{M}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m=\inf_{x\in[a,b]}f(x)$, $M=\sup_{x\in[a,b]}f(x)$.

Exercise 7.0.428. *Let f and g are integrable functions on $[a,b]$, $g(x)\geq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1(D)$, $\hat{T}(x)>0$, $\hat{T}(x)-x\hat{T}'(x)\leq 0$ for every $x\in[a,b]$. Then*

$$\hat{M}\hat{\times}\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}\leq\hat{\int}_a^b \hat{f}(\hat{x})\hat{\times}\hat{g}(x)\hat{\times}\hat{d}\hat{x}\leq\hat{m}\hat{\times}\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x},$$

where $m=\inf_{x\in[a,b]}f(x)$, $M=\sup_{x\in[a,b]}f(x)$.

Exercise 7.0.429. *Let f and g are integrable functions on $[a,b]$, $g(x)\leq 0$ for every $x\in[a,b]$, $\hat{T}\in\mathcal{C}^1(D)$, $\hat{T}(x)>0$, $\hat{T}(x)-x\hat{T}'(x)\leq 0$ for every $x\in[a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}\leq\hat{\int}_a^b \hat{f}(\hat{x})\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}\leq\hat{M}\hat{\times}\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x},$$

where $m=\inf_{x\in[a,b]}f(x)$, $M=\sup_{x\in[a,b]}f(x)$.

Exercise 7.0.430. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int_a^b}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{\int_a^b}\hat{f}(\hat{x})\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int_a^b}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.431. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{M}\hat{\times}\hat{\int_a^b}g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int_a^b}\hat{f}(\hat{x})\hat{\times}g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{m}\hat{\times}\hat{\int_a^b}g^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.432. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int_a^b}g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int_a^b}\hat{f}(\hat{x})\hat{\times}g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int_a^b}g^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.433. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int_a^b}g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int_a^b}\hat{f}(\hat{x})\hat{\times}g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int_a^b}g^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.434. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{M}\hat{\times}\hat{\int_a^b}g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int_a^b}f^\wedge(x)\hat{\times}g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{m}\hat{\times}\hat{\int_a^b}g^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.435. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b f^\wedge(x)\hat{\times}g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Exercise 7.0.436. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $\hat{T} \in \mathcal{C}^1(D)$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\hat{m}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b f^\wedge(x)\hat{\times}g^\wedge(x)\hat{\times}\hat{d}\hat{x} \leq \hat{M}\hat{\times}\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x},$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$.

Theorem 7.0.437. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} = \hat{\mu}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Proof. We have

$$\hat{M}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \leq \hat{m}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x}. \tag{7.0.1}$$

From $g(x) \geq 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$ it follows that

$$\hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \leq 0.$$

From here and (7.0.1) we get

$$m \leq \frac{\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x}}{\hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x}} \leq M.$$

Let

$$\mu = \frac{\hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{\times} \hat{g}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x}}{\hat{\int}_a^b \hat{g}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x}}.$$

Therefore

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{\times} \hat{g}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b \hat{g}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x}$$

□

Exercise 7.0.438. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{\times} \hat{g}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b \hat{g}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.439. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{\times} \hat{g}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b \hat{g}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.440. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{\times} \hat{g}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b \hat{g}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.441. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{\times} \hat{g}^\wedge(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b \hat{g}^\wedge(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.442. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$,*

$\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ *for every* $x \in [a, b]$. *Then there exists* $\mu \in \mathbb{R}$ *such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} = \hat{\mu}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.443. *Let* f *and* g *are integrable functions on* $[a, b]$, $g(x) \leq 0$ *for every* $x \in [a, b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a, b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ *for every* $x \in [a, b]$. *Then there exists* $\mu \in \mathbb{R}$ *such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} = \hat{\mu}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.444. *Let* f *and* g *are integrable functions on* $[a, b]$, $g(x) \geq 0$ *for every* $x \in [a, b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a, b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ *for every* $x \in [a, b]$. *Then there exists* $\mu \in \mathbb{R}$ *such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} = \hat{\mu}\hat{\times}\hat{\int}_a^b \hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.445. *Let* f *and* g *are integrable functions on* $[a, b]$, $g(x) \geq 0$ *for every* $x \in [a, b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a, b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ *for every* $x \in [a, b]$. *Then there exists* $\mu \in \mathbb{R}$ *such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} = \hat{\mu}\hat{\times}\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.446. *Let* f *and* g *are integrable functions on* $[a, b]$, $g(x) \leq 0$ *for every* $x \in [a, b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a, b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ *for every* $x \in [a, b]$. *Then there exists* $\mu \in \mathbb{R}$ *such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} = \hat{\mu}\hat{\times}\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.447. *Let* f *and* g *are integrable functions on* $[a, b]$, $g(x) \leq 0$ *for every* $x \in [a, b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a, b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ *for every* $x \in [a, b]$. *Then there exists* $\mu \in \mathbb{R}$ *such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} = \hat{\mu}\hat{\times}\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.448. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{\times} \hat{g}(\hat{x}) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b \hat{g}(\hat{x}) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.449. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{\times} g^\wedge(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b g^\wedge(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.450. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{\times} g^\wedge(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b g^\wedge(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.451. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{\times} g^\wedge(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b g^\wedge(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.452. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{\times} g^\wedge(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b g^\wedge(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.453. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}^\wedge(x) \hat{\times} \hat{g}^\wedge(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b \hat{g}^\wedge(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.454. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} \hat{f}^\wedge(x)\hat{\times}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} = \hat{\mu}\hat{\times}\hat{\int_a^b}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.455. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} \hat{f}^\wedge(x)\hat{\times}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} = \hat{\mu}\hat{\times}\hat{\int_a^b}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.456. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} \hat{f}^\wedge(x)\hat{\times}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} = \hat{\mu}\hat{\times}\hat{\int_a^b}\hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.457. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} \hat{f}^\wedge(x)\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} = \hat{\mu}\hat{\times}\hat{\int_a^b}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.458. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} \hat{f}^\wedge(x)\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} = \hat{\mu}\hat{\times}\hat{\int_a^b}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.459. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} \hat{f}^\wedge(x)\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} = \hat{\mu}\hat{\times}\hat{\int_a^b}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.460. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} \hat{f}^\wedge(x) \hat{\times} \hat{g}(\hat{x}) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int_a^b} \hat{g}(\hat{x}) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.461. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} \hat{f}^\wedge(x) \hat{\times} g^\wedge(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int_a^b} g^\wedge(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.462. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} \hat{f}^\wedge(x) \hat{\times} g^\wedge(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int_a^b} g^\wedge(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.463. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} \hat{f}^\wedge(x) \hat{\times} g^\wedge(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int_a^b} g^\wedge(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.464. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} \hat{f}^\wedge(x) \hat{\times} g^\wedge(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int_a^b} g^\wedge(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.465. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} \hat{f}(\hat{x}) \hat{\times} \hat{g}(\hat{x}) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int_a^b} \hat{g}(\hat{x}) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.466. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}(\hat{x}) \hat{\times} \hat{g}(\hat{x}) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b \hat{g}(\hat{x}) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.467. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}(\hat{x}) \hat{\times} \hat{g}(\hat{x}) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b \hat{g}(\hat{x}) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.468. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}(\hat{x}) \hat{\times} \hat{g}(\hat{x}) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b \hat{g}(\hat{x}) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.469. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}(\hat{x}) \hat{\times} g^{\wedge}(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b g^{\wedge}(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.470. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}(\hat{x}) \hat{\times} g^{\wedge}(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b g^{\wedge}(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.471. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int}_a^b \hat{f}(\hat{x}) \hat{\times} g^{\wedge}(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int}_a^b g^{\wedge}(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.472. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} \hat{f}(\hat{x}) \hat{\times} g^{\wedge}(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int_a^b} g^{\wedge}(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.473. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} f^{\wedge}(x) \hat{\times} g^{\wedge}(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int_a^b} g^{\wedge}(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.474. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} f^{\wedge}(x) \hat{\times} g^{\wedge}(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int_a^b} g^{\wedge}(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.475. *Let f and g are integrable functions on $[a,b]$, $g(x) \leq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} f^{\wedge}(x) \hat{\times} g^{\wedge}(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int_a^b} g^{\wedge}(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.476. *Let f and g are integrable functions on $[a,b]$, $g(x) \geq 0$ for every $x \in [a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then there exists $\mu \in \mathbb{R}$ such that*

$$\hat{\int_a^b} f^{\wedge}(x) \hat{\times} g^{\wedge}(x) \hat{\times} \hat{d}\hat{x} = \hat{\mu} \hat{\times} \hat{\int_a^b} g^{\wedge}(x) \hat{\times} \hat{d}\hat{x}.$$

Theorem 7.0.477. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\left(\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x}\right)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\wedge\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}\hat{\times}\hat{\int}_a^b \hat{g}^{\wedge\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Proof. Let $\hat{\lambda} \in \hat{F}_{\mathbb{R}}$ is arbitrary chosen. Then

$$\begin{aligned}
&\hat{\int}_a^b \left(\hat{f}^\wedge(\hat{x}) + \hat{\lambda}\hat{\times}\hat{g}^\wedge(\hat{x})\right)^{\hat{2}}\hat{\times}\hat{d}\hat{x}\\
&= \int_a^b (f(x) + \lambda g(x))^2 \tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx\\
&= \int_a^b \Big((f^2(x) + 2\lambda f(x)g(x) + \lambda^2 g^2(x)\Big) \tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx \qquad (7.0.2)\\
&= \int_a^b f^2(x) \tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx + 2\lambda \int_a^b f(x)g(x) \tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx\\
&+\lambda^2 \int_a^b g^2(x) \tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx.
\end{aligned}$$

Let

$$\alpha = \int_a^b g^2(x) \tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx, \quad \beta = \int_a^b f(x)g(x) \tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx,$$

$$\gamma = \int_a^b f^2(x) \tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx.$$

From here and (7.0.2) we get

$$\hat{\int}_a^b \left(\hat{f}^\wedge(\hat{x}) + \hat{\lambda}\hat{\times}\hat{g}^\wedge(\hat{x})\right)^{\hat{2}}\hat{\times}\hat{d}\hat{x} = \alpha^2\lambda^2 + 2\beta\lambda + \gamma.$$

Since

$$\hat{T}(x) - x\hat{T}'(x) \leq 0 \quad \text{for} \quad \forall x \in [a,b]$$

and $\lambda \in \hat{F}_{\mathbb{R}}$ was arbitrary chose we get that

$$\hat{\int}_a^b \left(\hat{f}^\wedge(\hat{x}) + \hat{\lambda}\hat{\times}\hat{g}^\wedge(\hat{x})\right)^{\hat{2}}\hat{\times}\hat{d}\hat{x} \leq 0$$

and the inequality

$$\alpha\lambda^2 + 2\beta\lambda + \gamma \leq 0$$

is valid for every $\lambda \in \mathbb{R}$. Therefore, using that $\alpha \leq 0$, we obtain

$$\beta^2 \leq \alpha\gamma,$$

from where

$$\Big(\int_a^b f(x)g(x)\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}^2(x)}dx\Big)^2 \leq \Big(\int_a^b f^2(x)\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}^2(x)}dx\Big)\Big(\int_a^b g^2(x)\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}^2(x)}dx\Big),$$

which is equivalent of

$$\Big(\hat{\int}_a^b \hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge}(\hat{x})\hat{\times}\hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\wedge\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}\hat{\times}\hat{\int}_a^b \hat{g}^{\wedge\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

□

Exercise 7.0.478. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b \hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge}(\hat{x})\hat{\times}\hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\wedge\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}\hat{\times}\hat{\int}_a^b \hat{g}^{\wedge\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.479. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b \hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge}(x)\hat{\times}\hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\wedge\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}\hat{\times}\hat{\int}_a^b \hat{g}^{\wedge\hat{2}}(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.480. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \leq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b \hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\wedge\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}\hat{\times}\hat{\int}_a^b \hat{g}^{\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.481. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b \hat{f}^{\wedge}(\hat{x})\hat{\times}g^{\wedge}(x)\hat{\times}\hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\wedge\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}\hat{\times}\hat{\int}_a^b g^{\wedge\hat{2}}(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.482. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{\times} g^\wedge(x) \hat{\times} \hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\wedge\hat{2}}(\hat{x}) \hat{\times} \hat{d}\hat{x} \hat{\times} \hat{\int}_a^b g^{\wedge\hat{2}}(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.483. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b \hat{f}^\wedge(x) \hat{\times} \hat{g}^\wedge(x) \hat{\times} \hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\wedge\hat{2}}(x) \hat{\times} \hat{d}\hat{x} \hat{\times} \hat{\int}_a^b \hat{g}^{\wedge\hat{2}}(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.484. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b \hat{f}^\wedge(x) \hat{\times} \hat{g}^\wedge(x) \hat{\times} \hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\wedge\hat{2}}(x) \hat{\times} \hat{d}\hat{x} \hat{\times} \hat{\int}_a^b \hat{g}^{\wedge\hat{2}}(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.485. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b \hat{f}^\wedge(x) \hat{\times} \hat{g}(\hat{x}) \hat{\times} \hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\wedge\hat{2}}(x) \hat{\times} \hat{d}\hat{x} \hat{\times} \hat{\int}_a^b \hat{g}^{\hat{2}}(\hat{x}) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.486. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b \hat{f}^\wedge(x) \hat{\times} \hat{g}(\hat{x}) \hat{\times} \hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\wedge\hat{2}}(x) \hat{\times} \hat{d}\hat{x} \hat{\times} \hat{\int}_a^b \hat{g}^{\hat{2}}(\hat{x}) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.487. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b \hat{f}^\wedge(x) \hat{\times} g^\wedge(x) \hat{\times} \hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\wedge\hat{2}}(x) \hat{\times} \hat{d}\hat{x} \hat{\times} \hat{\int}_a^b g^{\wedge\hat{2}}(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.488. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b \hat{f}^\wedge(x) \hat{\times} g^\wedge(x) \hat{\times} \hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\wedge\hat{2}}(x) \hat{\times} \hat{d}\hat{x} \hat{\times} \hat{\int}_a^b g^{\wedge\hat{2}}(x) \hat{\times} \hat{d}\hat{x}.$$

Exercise 7.0.489. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b \hat{f}(\hat{x})\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}\hat{\times}\hat{\int}_a^b \hat{g}^{\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.490. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b \hat{f}(\hat{x})\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}\hat{\times}\hat{\int}_a^b \hat{g}^{\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.491. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b \hat{f}(\hat{x})\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}\hat{\times}\hat{\int}_a^b \hat{g}^{\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.492. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b \hat{f}(\hat{x})\hat{\times}g^{\wedge}(x)\hat{\times}\hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b \hat{f}^{\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}\hat{\times}\hat{\int}_a^b g^{\wedge\hat{2}}(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.493. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b f^{\wedge}(x)\hat{\times}g^{\wedge}(x)\hat{\times}\hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b f^{\wedge\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}\hat{\times}\hat{\int}_a^b g^{\wedge\hat{2}}(x)\hat{\times}\hat{d}\hat{x}.$$

Exercise 7.0.494. *Let f and g are integrable functions on $[a,b]$, $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in [a,b]$. Then*

$$\Big(\hat{\int}_a^b f^{\wedge}(x)\hat{\times}g^{\wedge}(x)\hat{\times}\hat{d}\hat{x}\Big)^{\hat{2}} \leq \hat{\int}_a^b f^{\wedge\hat{2}}(\hat{x})\hat{\times}\hat{d}\hat{x}\hat{\times}\hat{\int}_a^b g^{\wedge\hat{2}}(x)\hat{\times}\hat{d}\hat{x}.$$

Theorem 7.0.495. *Let f is an integrable function on $[a, b]$ and differentiable at $x_0 \in [a, b]$. Then the isofunction*

$$\hat{F}(x) = \hat{\int}_a^x \hat{f}^\wedge(\hat{t}) \hat{\times} \hat{d}\hat{t}$$

is isodifferentiable at the point x_0 and

$$\hat{d}\Big(\hat{\int}_a^x \hat{f}^\wedge(\hat{t}) \hat{\times} \hat{d}\hat{t}\Big) \nearrow \hat{d}\hat{x}\Big|_{x=x_0} = \hat{f}^\wedge(\hat{x}_0).$$

Proof. We have

$$\hat{d}\Big(\hat{\int}_a^x \hat{f}^\wedge(\hat{t}) \hat{\times} \hat{d}\hat{t}\Big) \nearrow \hat{d}\hat{x} = \frac{1}{\hat{T}(x)} \frac{\hat{d}\Big(\hat{\int}_a^x \hat{f}^\wedge(\hat{t}) \hat{\times} \hat{d}\hat{t}\Big)}{\hat{d}\hat{x}}$$

$$= \frac{1}{\hat{T}(x)} \frac{\hat{T}(x) d\Big(\int_a^x f(t) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}^2(t)} dt\Big)}{\Big(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big) dx}$$

$$= \frac{\Big(\int_a^x f(t) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}^2(t)}\Big)' dx}{\Big(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big) dx}$$

$$= \frac{f(x) \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)}}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}$$

$$= \frac{f(x)}{\hat{T}(x)} = \hat{f}^\wedge(\hat{x}).$$

From here

$$\hat{d}\Big(\hat{\int}_a^x \hat{f}^\wedge(\hat{t}) \hat{\times} \hat{d}\hat{t}\Big) \nearrow \hat{d}\hat{x}\Big|_{x=x_0} = \hat{f}^\wedge(\hat{x}_0).$$

□

Exercise 7.0.496. *Let f is an integrable function on $[a, b]$ and differentiable* *at $x_0 \in [a,b]$. Prove that the isofunction*

$$\hat{F}(x) = \hat{\int}_a^x \hat{f}^\wedge(t)\hat{\times}\hat{d}\hat{t}$$

is isodifferentiable at the point x_0 and

$$\hat{d}\Big(\hat{\int}_a^x \hat{f}^\wedge(t)\hat{\times}\hat{d}\hat{t}\Big) \nearrow \hat{d}\hat{x}\Big|_{x=x_0} = \hat{f}^\wedge(x_0).$$

Exercise 7.0.497. *Let f is an integrable function on $[a,b]$ and differentiable at $x_0 \in [a,b]$. Prove that the isofunction*

$$\hat{F}(x) = \hat{\int}_a^x \hat{f}(\hat{t})\hat{\times}\hat{d}\hat{t}$$

is isodifferentiable at the point x_0 and

$$\hat{d}\Big(\hat{\int}_a^x \hat{f}(\hat{t})\hat{\times}\hat{d}\hat{t}\Big) \nearrow \hat{d}\hat{x}\Big|_{x=x_0} = \hat{f}(\hat{x}_0).$$

Exercise 7.0.498. *Let f is an integrable function on $[a,b]$ and differentiable at $x_0 \in [a,b]$. Prove that the isofunction*

$$\hat{F}(x) = \hat{\int}_a^x f^\wedge(t)\hat{\times}\hat{d}\hat{t}$$

is isodifferentiable at the point x_0 and

$$\hat{d}\Big(\hat{\int}_a^x f^\wedge(t)\hat{\times}\hat{d}\hat{t}\Big) \nearrow \hat{d}\hat{x}\Big|_{x=x_0} = f^\wedge(x_0).$$

Definition 7.0.499. *(isoinetgrable isofunction) When we say that an isofunction of first, second, third or fourth kind is an isointegrable function on $[a,b]$ we will have in mind the the isointegral*

$$\hat{\int}_a^b \hat{f}(x)\hat{\times}\hat{d}\hat{x}$$

exists.

Definition 7.0.500. *Let* $\hat{f} : [a, b) \longrightarrow \mathbb{R}$ *is an isointegrable isofunction of first, second, third or fourth kind in every interval* $[a, \eta] \subset [a, b)$ *and eventual unbounded in a neighbourhood of* b. *Then the isointegral*

$$\hat{\int}_a^\eta \hat{f}(x) \hat{\times} \hat{d}\hat{x}$$

exists for every $\eta \in [a, b)$. *If there exists*

$$\lim_{\eta \longrightarrow b-0} \hat{\int}_a^\eta \hat{f}(x) \hat{\times} \hat{d}\hat{x} \tag{7.0.3}$$

then it is called improper isointegral of $\hat{f}$ *in* $[a, b]$ *and it will be denoted with*

$$\hat{\int}_a^b \hat{f}(x) \hat{\times} \hat{d}\hat{x}.$$

Definition 7.0.501. *Let* $\hat{f} : (a, b] \longrightarrow \mathbb{R}$ *is an isointegrable isofunction of first, second, third or fourth kind in every interval* $[\eta, b] \subset (a, b]$ *and eventual unbounded in a neighbourhood of* a. *Then the isointegral*

$$\hat{\int}_\eta^b \hat{f}(x) \hat{\times} \hat{d}\hat{x}$$

exists for every $\eta \in (a, b]$. *If there exists*

$$\lim_{\eta \longrightarrow a+0} \hat{\int}_\eta^b \hat{f}(x) \hat{\times} \hat{d}\hat{x} \tag{7.0.4}$$

then it is called improper isointegral of $\hat{f}$ *in* $[a, b]$ *and it will be denoted with*

$$\hat{\int}_a^b \hat{f}(x) \hat{\times} \hat{d}\hat{x}.$$

Definition 7.0.502. *Let $\hat{f} : [a, \infty) \longrightarrow \mathbb{R}$ is an isointegrable isofunction of first, second, third or fourth kind in every interval $[a, \eta] \subset [a, \infty)$. Then the isointegral*

$$\hat{\int}_a^\eta \hat{f}(x) \hat{\times} \hat{d}\hat{x}$$

exists for every $\eta \in [a, \infty)$. If there exists

$$\lim_{\eta \longrightarrow \infty} \hat{\int}_a^\eta \hat{f}(x) \hat{\times} \hat{d}\hat{x}$$

then it is called improper isointegral of $\hat{f}$ in $[a, \infty)$ and it will be denoted with

$$\hat{\int}_a^\infty \hat{f}(x) \hat{\times} \hat{d}\hat{x}.$$

Analogously, it is defined the improper isointegral

$$\hat{\int}_{-\infty}^a \hat{f}(x) \hat{\times} \hat{d}\hat{x}$$

and we define

$$\hat{\int}_{-\infty}^\infty \hat{f}(x) \hat{\times} \hat{d}\hat{x} = \hat{\int}_a^\infty \hat{f}(x) \hat{\times} \hat{d}\hat{x} + \hat{\int}_{-\infty}^a \hat{f}(x) \hat{\times} \hat{d}\hat{x}.$$

From the above definitions it follows that every cases of improper isointegrals is reduced to (7.0.3), (7.0.4). To consider every one of them simultaneously we will say the symbol

$$\hat{\int}_a^b \hat{f}(x) \hat{\times} \hat{d}\hat{x} \tag{7.0.5}$$

improper isointegral of $\hat{f}$ with unique singular point b if

1) $\hat{f}$ is isointegrable on every $[a, \eta]$ for every $\eta \in [a, b)$ and $\hat{f}$ or $\hat{T}(x) - x\hat{T}'(x)$ is unbounded at b

or

2) $\hat{f}$ is an isointegrable on $[a, \eta]$ for every $\eta \in [a, +\infty)$.

Exercise 7.0.503. *Prove that the improper isointegral satisfies the linear property.*

Theorem 7.0.504. *Let $\hat{T}$ is positive and differentiable on $[a,b]$. If f and $\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}^2(x)}$ are integrable on $[a,b]$ then there exists $\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x}$.*

Proof. The proof follows from the fact that the multiplication of two integrable functions is an integrable function. □

Exercise 7.0.505. *Let $\hat{T}$ is positive and differentiable on $[a,b]$, $x\hat{T}(x) \in [a,b]$ for every $x \in [a,b]$. If f and $\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}^2(x)}$ are integrable on $[a,b]$ then there exists $\hat{\int}_a^b \hat{f}^\wedge(x)\hat{\times}\hat{d}\hat{x}$.*

Exercise 7.0.506. *Let $\hat{T}$ is positive and differentiable on $[a,b]$, $\frac{x}{\hat{T}(x)} \in [a,b]$ for every $x \in [a,b]$. If f and $\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}^2(x)}$ are integrable on $[a,b]$ then there exists $\hat{\int}_a^b \hat{f}(\hat{x})\hat{\times}\hat{d}\hat{x}$.*

Exercise 7.0.507. *Let $\hat{T}$ is positive and differentiable on $[a,b]$, $x\hat{T}(x) \in [a,b]$ for every $x \in [a,b]$. If f and $\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}$ are integrable on $[a,b]$ then there exists $\hat{\int}_a^b f^\wedge(x)\hat{\times}\hat{d}\hat{x}$.*

Definition 7.0.508. *The improper isointegral* (7.0.5) *will be called absolutely convergent isointegral if*

$$\hat{\int}_a^b |\hat{f}(x)|\hat{\times}|\hat{d}\hat{x}| < \infty.$$

Theorem 7.0.509. *Let* (7.0.5) *is absolutely convergent. Then it is convergent.*

Proof.

$$\Big|\hat{\int}_a^b \hat{f}(x)\hat{\times}\hat{d}\hat{x}\Big| \leq \hat{\int}_a^b |\hat{f}(x)|\hat{\times}|\hat{d}\hat{x}| < \infty.$$

□

Theorem 7.0.510. *Let the improper isointegrals*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \tag{7.0.6}$$

and

$$\hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \tag{7.0.7}$$

have the same singular point b. Let also for every $x \in [a, b]$

$$0 \leq f(x) \leq g(x), \hat{T} \in \mathcal{C}^1([a, b]), \hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \leq 0.$$

Then from the convergence of (7.0.7) *it follows the convergence of* (7.0.6) *and from the divergence of* (7.0.6) *it follows the divergence of* (7.0.7).

Proof. Since

$$0 \leq f(x) \leq g(x), \quad \hat{T}(x) - x\hat{T}'(x \leq 0 \quad \forall x \in [a, b],$$

it follows

$$g(x)(\hat{T}(x) - x\hat{T}'(x)) \leq f(x)(\hat{T}(x) - x\hat{T}'(x)) \leq 0 \quad \text{for} \quad \forall x \in [a, b],$$

from where

$$\frac{g(x)(\hat{T}(x) - x\hat{T}'(x))}{\hat{T}^2(x)} \leq \frac{f(x)(\hat{T}(x) - x\hat{T}'(x))}{\hat{T}^2(x)} \leq 0 \quad \text{for} \quad \forall x \in [a, b],$$

Therefore

$$\int_a^b \frac{g(x)(\hat{T}(x) - x\hat{T}'(x))}{\hat{T}^2(x)} dx \le \int_a^b \frac{f(x)(\hat{T}(x) - x\hat{T}'(x))}{\hat{T}^2(x)} dx \le 0 \quad \text{for} \quad \forall x \in [a, b],$$

or

$$\hat{\int}_a^b \hat{g}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x} \le \hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x} \le 0. \tag{7.0.8}$$

If

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x} = -\infty,$$

then from (7.0.8), it follows

$$\hat{\int}_a^b \hat{g}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x} = -\infty.$$

If

$$\hat{\int}_a^b \hat{g}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x} > -\infty,$$

using (7.0.8), we get

$$-\infty < \hat{\int}_a^b \hat{f}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x} \le 0.$$

□

Exercise 7.0.511. *Let the improper isointegrals* (7.0.6) *and* (7.0.7) *have the same singular point* b. *Let also for every* $x \in [a, b]$

$$0 \le f(x) \le g(x), \hat{T} \in \mathcal{C}^1([a, b]), \hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \ge 0.$$

Then from the convergence of (7.0.7) *it follows the convergence of* (7.0.6) *and from the divergence of* (7.0.6) *it follows the divergence of* (7.0.7).

Exercise 7.0.512. *Let the improper isointegrals*

$$\hat{\int}_a^b \hat{f}^\wedge(x) \hat{\times} \hat{d}\hat{x} \tag{7.0.9}$$

and

$$\hat{\int}_a^b \hat{g}^\wedge(x) \hat{\times} \hat{d}\hat{x} \tag{7.0.10}$$

has the same singular point b. Let also for every $x \in [a,b]$

$$0 \le f(x) \le g(x), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0, x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \le 0.$$

Then from the convergence of (7.0.9) *it follows the convergence of* (7.0.10) *and from the divergence of* (7.0.9) *it follows the divergence of* (7.0.10).

Exercise 7.0.513. *Let the improper isointegrals* (7.0.9) *and* (7.0.10) *have the same singular point b. Let also for every $x \in [a,b]$*

$$0 \le f(x) \le g(x), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0, x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \ge 0.$$

Then from the convergence of (7.0.10) *it follows the convergence of* (7.0.9) *and from the divergence of* (7.0.9) *it follows the divergence of* (7.0.10).

Exercise 7.0.514. *Let the improper isointegrals*

$$\hat{\int}_a^b \hat{f}(\hat{x})\hat{\times}\hat{d}\hat{x} \tag{7.0.11}$$

and

$$\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \tag{7.0.12}$$

has the same singular point b. Let also for every $x \in [a,b]$

$$0 \le f(x) \le g(x), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0, \frac{x}{\hat{T}(x)} \in [a,b], \hat{T}(x) - x\hat{T}'(x) \le 0.$$

Then from the convergence of (7.0.11) *it follows the convergence of* (7.0.12) *and from the divergence of* (7.0.11) *it follows the divergence of* (7.0.12).

Exercise 7.0.515. *Let the improper isointegrals* (7.0.11) *and* (7.0.12) *have the same singular point b. Let also for every $x \in [a,b]$*

$$0 \le f(x) \le g(x), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0, \frac{x}{\hat{T}(x)} \in [a,b], \hat{T}(x) - x\hat{T}'(x) \ge 0.$$

Then from the convergence of (7.0.12) *it follows the convergence of* (7.0.11) *and from the divergence of* (7.0.11) *it follows the divergence of* (7.0.12).

Exercise 7.0.516. *Let the improper isointegrals*

$$\hat{\int}_a^b f^{\wedge}(x)\hat{\times}\hat{d}\hat{x} \tag{7.0.13}$$

and

$$\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x} \tag{7.0.14}$$

has the same singular point b. Let also for every $x \in [a,b]$

$$0 \le f(x) \le g(x), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0, x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \le 0.$$

Then from the convergence of (7.0.13) *it follows the convergence of* (7.0.14) *and from the divergence of* (7.0.13) *it follows the divergence of* (7.0.14).

Exercise 7.0.517. *Let the improper isointegrals* (7.0.13) *and* (7.0.14) *have the same singular point b. Let also for every* $x \in [a,b]$

$$0 \le f(x) \le g(x), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0, x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \ge 0.$$

Then from the convergence of (7.0.14) *it follows the convergence of* (7.0.13) *and from the divergence of* (7.0.13) *it follows the divergence of* (7.0.14).

Exercise 7.0.518. *Let the improper isointegrals*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \tag{7.0.15}$$

and

$$\hat{\int}_a^b \hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} \tag{7.0.16}$$

has the same singular point b. Let also for every $x \in [a,b]$

$$0 \le f(x) \le g(x\hat{T}(x)), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0,$$

$$x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \le 0.$$

Then from the convergence of (7.0.15) *it follows the convergence of* (7.0.16) *and from the divergence of* (7.0.15) *it follows the divergence of* (7.0.16).

Exercise 7.0.519. *Let the improper isointegrals* (7.0.15) *and* (7.0.16) *have the same singular point b. Let also for every* $x \in [a,b]$

$$0 \le f(x) \le g(x\hat{T}(x)), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0,$$

$$x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \ge 0.$$

Then from the convergence of (7.0.16) *it follows the convergence of* (7.0.15) *and from the divergence of* (7.0.15) *it follows the divergence of* (7.0.16).

Exercise 7.0.520. *Let the improper isointegrals*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \tag{7.0.17}$$

and

$$\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \tag{7.0.18}$$

has the same singular point b. Let also for every $x \in [a,b]$

$$0 \leq f(x) \leq g\Big(\frac{x}{\hat{T}(x)}\Big), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0,$$

$$\frac{x}{\hat{T}(x)} \in [a,b], \hat{T}(x) - x\hat{T}'(x) \leq 0.$$

Then from the convergence of (7.0.17) *it follows the convergence of* (7.0.18) *and from the divergence of* (7.0.17) *it follows the divergence of* (7.0.18)*.*

Exercise 7.0.521. *Let the improper isointegrals* (7.0.17) *and* (7.0.18) *have the same singular point b. Let also for every $x \in [a,b]$*

$$0 \leq f(x) \leq g\Big(\frac{x}{\hat{T}(x)}\Big), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0,$$

$$\frac{x}{\hat{T}(x)} \in [a,b], \hat{T}(x) - x\hat{T}'(x) \geq 0.$$

Then from the convergence of (7.0.18) *it follows the convergence of* (7.0.17) *and from the divergence of* (7.0.17) *it follows the divergence of* (7.0.18)*.*

Exercise 7.0.522. *Let the improper isointegrals*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \tag{7.0.19}$$

and

$$\hat{\int}_a^b g^\wedge(x)\hat{\times}\hat{d}\hat{x} \tag{7.0.20}$$

has the same singular point b. Let also for every $x \in [a,b]$

$$0 \leq \frac{f(x)}{\hat{T}(x)} \leq g(x\hat{T}(x)), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0,$$

$$x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \leq 0.$$

Then from the convergence of (7.0.19) *it follows the convergence of* (7.0.20) *and from the divergence of* (7.0.19) *it follows the divergence of* (7.0.20)*.*

Exercise 7.0.523. *Let the improper isointegrals* (7.0.19) *and* (7.0.20) *have the same singular point* b. *Let also for every* $x \in [a, b]$

$$0 \le \frac{f(x)}{\hat{T}(x)} \le g(x\hat{T}(x)), \hat{T} \in \mathcal{C}^1([a, b]), \hat{T}(x) > 0,$$

$$\frac{x}{\hat{T}(x)} \in [a, b], \hat{T}(x) - x\hat{T}'(x) \ge 0.$$

Then from the convergence of (7.0.19) *it follows the convergence of* (7.0.20) *and from the divergence of* (7.0.19) *it follows the divergence of* (7.0.20).

Exercise 7.0.524. *Let the improper isointegrals*

$$\hat{\int}_a^b \hat{f}^\wedge(x) \hat{\times} \hat{d}\hat{x} \tag{7.0.21}$$

and

$$\hat{\int}_a^b \hat{g}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x} \tag{7.0.22}$$

has the same singular point b. *Let also for every* $x \in [a, b]$

$$0 \le f(x\hat{T}(x)) \le g(x), \hat{T} \in \mathcal{C}^1([a, b]), \hat{T}(x) > 0,$$

$$x\hat{T}(x) \in [a, b], \hat{T}(x) - x\hat{T}'(x) \le 0.$$

Then from the convergence of (7.0.21) *it follows the convergence of* (7.0.22) *and from the divergence of* (7.0.21) *it follows the divergence of* (7.0.22).

Exercise 7.0.525. *Let the improper isointegrals* (7.0.21) *and* (7.0.22) *have the same singular point* b. *Let also for every* $x \in [a, b]$

$$0 \le f(x\hat{T}(x)) \le g(x), \hat{T} \in \mathcal{C}^1([a, b]), \hat{T}(x) > 0,$$

$$x\hat{T}(x) \in [a, b], \hat{T}(x) - x\hat{T}'(x) \ge 0.$$

Then from the convergence of (7.0.21) *it follows the convergence of* (7.0.22) *and from the divergence of* (7.0.21) *it follows the divergence of* (7.0.22).

Exercise 7.0.526. *Let the improper isointegrals*

$$\hat{\int}_a^b \hat{f}^\wedge(x) \hat{\times} \hat{d}\hat{x} \tag{7.0.23}$$

and

$$\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \tag{7.0.24}$$

has the same singular point b. Let also for every $x \in [a, b]$

$$0 \leq f(x\hat{T}(x)) \leq g\Big(\frac{x}{\hat{T}(x)}\Big), \hat{T} \in \mathcal{C}^1([a, b]), \hat{T}(x) > 0,$$

$$\frac{x}{\hat{T}(x)} \in [a, b], x\hat{T}(x) \in [a, b], \hat{T}(x) - x\hat{T}'(x) \leq 0.$$

Then from the convergence of (7.0.23) *it follows the convergence of* (7.0.24) *and from the divergence of* (7.0.23) *it follows the divergence of* (7.0.24).

Exercise 7.0.527. *Let the improper isointegrals* (7.0.23) *and* (7.0.24) *have the same singular point b. Let also for every* $x \in [a, b]$

$$0 \leq f(x\hat{T}(x)) \leq g\Big(\frac{x}{\hat{T}(x)}\Big), \hat{T} \in \mathcal{C}^1([a, b]), \hat{T}(x) > 0,$$

$$\frac{x}{\hat{T}(x)} \in [a, b], x\hat{T}(x) \in [a, b], \hat{T}(x) - x\hat{T}'(x) \geq 0.$$

Then from the convergence of (7.0.23) *it follows the convergence of* (7.0.24) *and from the divergence of* (7.0.23) *it follows the divergence of* (7.0.24).

Exercise 7.0.528. *Let the improper isointegrals*

$$\hat{\int}_a^b \hat{f}^{\wedge}(x)\hat{\times}\hat{d}\hat{x} \tag{7.0.25}$$

and

$$\hat{\int}_a^b g^{\wedge}(x)\hat{\times}\hat{d}\hat{x} \tag{7.0.26}$$

has the same singular point b. Let also for every $x \in [a, b]$

$$0 \leq \frac{f(x\hat{T}(x))}{\hat{T}(x)} \leq g(x\hat{T}(x)), \hat{T} \in \mathcal{C}^1([a, b]),$$

$$\hat{T}(x) > 0, x\hat{T}(x) \in [a, b], \hat{T}(x) - x\hat{T}'(x) \leq 0.$$

Then from the convergence of (7.0.25) *it follows the convergence of* (7.0.26) *and from the divergence of* (7.0.25) *it follows the divergence of* (7.0.26).

Exercise 7.0.529. *Let the improper isointegrals* (7.0.25) *and* (7.0.26) *have the same singular point* b. *Let also for every* $x \in [a,b]$

$$0 \leq \frac{f(x\hat{T}(x))}{\hat{T}(x)} \leq g(x\hat{T}(x)), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0,$$

$$x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \geq 0.$$

Then from the convergence of (7.0.25) *it follows the convergence of* (7.0.26) *and from the divergence of* (7.0.25) *it follows the divergence of* (7.0.26).

Exercise 7.0.530. *Let the improper isointegrals*

$$\hat{\int}_a^b \hat{f}(\hat{x}) \hat{\times} \hat{d}\hat{x} \tag{7.0.27}$$

and

$$\hat{\int}_a^b \hat{g}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x} \tag{7.0.28}$$

has the same singular point b. *Let also for every* $x \in [a,b]$

$$0 \leq f\left(\frac{x}{\hat{T}(x)}\right) \leq g(x), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0,$$

$$\frac{x}{\hat{T}(x)} \in [a,b], \hat{T}(x) - x\hat{T}'(x) \leq 0.$$

Then from the convergence of (7.0.27) *it follows the convergence of* (7.0.28) *and from the divergence of* (7.0.27) *it follows the divergence of* (7.0.28).

Exercise 7.0.531. *Let the improper isointegrals* (7.0.27) *and* (7.0.28) *have the same singular point* b. *Let also for every* $x \in [a,b]$

$$0 \leq f\left(\frac{x}{\hat{T}(x)}\right) \leq g(x), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0,$$

$$\frac{x}{\hat{T}(x)} \in [a,b], \hat{T}(x) - x\hat{T}'(x) \geq 0.$$

Then from the convergence of (7.0.27) *it follows the convergence of* (7.0.28) *and from the divergence of* (7.0.27) *it follows the divergence of* (7.0.28).

Exercise 7.0.532. *Let the improper isointegrals*

$$\hat{\int}_a^b \hat{f}(\hat{x}) \hat{\times} \hat{d}\hat{x} \tag{7.0.29}$$

and

$$\hat{\int_a^b} \hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} \tag{7.0.30}$$

has the same singular point b. Let also for every $x \in [a, b]$

$$0 \leq f\left(\frac{x}{\hat{T}(x)}\right) \leq g(x\hat{T}(x)), \hat{T} \in \mathcal{C}^1([a, b]), \hat{T}(x) > 0,$$

$$x\hat{T}(x) \in [a, b], \frac{x}{\hat{T}(x)} \in [a, b], \hat{T}(x) - x\hat{T}'(x) \leq 0.$$

Then from the convergence of (7.0.29) *it follows the convergence of* (7.0.30) *and from the divergence of* (7.0.29) *it follows the divergence of* (7.0.30).

Exercise 7.0.533. *Let the improper isointegrals* (7.0.29) *and* (7.0.30) *have the same singular point b. Let also for every $x \in [a, b]$*

$$0 \leq f\left(\frac{x}{\hat{T}(x)}\right) \leq g(x\hat{T}(x)), \hat{T} \in \mathcal{C}^1([a, b]), \hat{T}(x) > 0,$$

$$x\hat{T}(x) \in [a, b], \frac{x}{\hat{T}(x)} \in [a, b], \hat{T}(x) - x\hat{T}'(x) \geq 0.$$

Then from the convergence of (7.0.29) *it follows the convergence of* (7.0.30) *and from the divergence of* (7.0.29) *it follows the divergence of* (7.0.30).

Exercise 7.0.534. *Let the improper isointegrals*

$$\hat{\int_a^b} \hat{f}(\hat{x})\hat{\times}\hat{d}\hat{x} \tag{7.0.31}$$

and

$$\hat{\int_a^b} \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \tag{7.0.32}$$

has the same singular point b. Let also for every $x \in [a, b]$

$$0 \leq f\left(\frac{x}{\hat{T}(x)}\right) \leq g\left(\frac{x}{\hat{T}(x)}\right), \hat{T} \in \mathcal{C}^1([a, b]),$$

$$\hat{T}(x) > 0, \frac{x}{\hat{T}(x)} \in [a, b], \hat{T}(x) - x\hat{T}'(x) \leq 0.$$

Then from the convergence of (7.0.31) *it follows the convergence of* (7.0.32) *and from the divergence of* (7.0.31) *it follows the divergence of* (7.0.32).

Exercise 7.0.535. *Let the improper isointegrals* (7.0.31) *and* (7.0.32) *have the same singular point b. Let also for every* $x \in [a, b]$

$$0 \leq f\left(\frac{x}{\hat{T}(x)}\right) \leq g\left(\frac{x}{\hat{T}(x)}\right), \hat{T} \in \mathcal{C}^1([a, b]),$$

$$\hat{T}(x) > 0, \frac{x}{\hat{T}(x)} \in [a, b], \hat{T}(x) - x\hat{T}'(x) \geq 0.$$

Then from the convergence of (7.0.31) *it follows the convergence of* (7.0.32) *and from the divergence of* (7.0.31) *it follows the divergence of* (7.0.32).

Exercise 7.0.536. *Let the improper isointegrals*

$$\hat{\int}_a^b \hat{f}(\hat{x}) \hat{\times} \hat{d}\hat{x} \tag{7.0.33}$$

and

$$\hat{\int}_a^b g^{\wedge}(x) \hat{\times} \hat{d}\hat{x} \tag{7.0.34}$$

has the same singular point b. Let also for every $x \in [a, b]$

$$0 \leq f\left(\frac{x}{\hat{T}(x)}\right) \frac{1}{\hat{T}(x)} \leq g(x\hat{T}(x)), \hat{T} \in \mathcal{C}^1([a, b]), \hat{T}(x) > 0,$$

$$x\hat{T}(x) \in [a, b], \frac{x}{\hat{T}(x)} \in [a, b], \hat{T}(x) - x\hat{T}'(x) \leq 0.$$

Then from the convergence of (7.0.33) *it follows the convergence of* (7.0.34) *and from the divergence of* (7.0.33) *it follows the divergence of* (7.0.34).

Exercise 7.0.537. *Let the improper isointegrals* (7.0.33) *and* (7.0.34) *have the same singular point b. Let also for every* $x \in [a, b]$

$$0 \leq f\left(\frac{x}{\hat{T}(x)}\right) \frac{1}{\hat{T}(x)} \leq g(x), \hat{T} \in \mathcal{C}^1([a, b]), \hat{T}(x) > 0,$$

$$x\hat{T}(x) \in [a, b], \frac{x}{\hat{T}(x)} \in [a, b], \hat{T}(x) - x\hat{T}'(x) \geq 0.$$

Then from the convergence of (7.0.33) *it follows the convergence of* (7.0.34) *and from the divergence of* (7.0.33) *it follows the divergence of* (7.0.34).

Exercise 7.0.538. *Let the improper isointegrals*

$$\hat{\int}_a^b f^{\wedge}(x) \hat{\times} \hat{d}\hat{x} \tag{7.0.35}$$

and

$$\hat{\int}_a^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \tag{7.0.36}$$

has the same singular point b. Let also for every $x \in [a,b]$

$$0 \le f\Big(x\hat{T}(x)\Big) \le g(x), \hat{T} \in \mathcal{C}^1([a,b]),$$

$$\hat{T}(x) > 0, x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \le 0.$$

Then from the convergence of (7.0.35) *it follows the convergence of* (7.0.36) *and from the divergence of* (7.0.35) *it follows the divergence of* (7.0.36).

Exercise 7.0.539. *Let the improper isointegrals* (7.0.35) *and* (7.0.36) *have the same singular point b. Let also for every* $x \in [a,b]$

$$0 \le f\Big(x\hat{T}(x)\Big) \le g(x), \hat{T} \in \mathcal{C}^1([a,b]),$$

$$\hat{T}(x) > 0, x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \ge 0.$$

Then from the convergence of (7.0.35) *it follows the convergence of* (7.0.36) *and from the divergence of* (7.0.35) *it follows the divergence of* (7.0.36).

Exercise 7.0.540. *Let the improper isointegrals*

$$\hat{\int}_a^b f^\wedge(x)\hat{\times}\hat{d}\hat{x} \tag{7.0.37}$$

and

$$\hat{\int}_a^b \hat{g}^\wedge(x)\hat{\times}\hat{d}\hat{x} \tag{7.0.38}$$

has the same singular point b. Let also for every $x \in [a,b]$

$$0 \le f\Big(x\hat{T}(x)\Big) \le g(x\hat{T}(x)), \hat{T} \in \mathcal{C}^1([a,b]),$$

$$\hat{T}(x) > 0, x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \le 0.$$

Then from the convergence of (7.0.37) *it follows the convergence of* (7.0.38) *and from the divergence of* (7.0.37) *it follows the divergence of* (7.0.38).

Exercise 7.0.541. *Let the improper isointegrals* (7.0.37) *and* (7.0.38) *have the same singular point b. Let also for every $x \in [a,b]$*

$$0 \leq f\Big(x\hat{T}(x)\Big) \leq g(x\hat{T}(x)), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0,$$

$$x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \geq 0.$$

Then from the convergence of (7.0.37) *it follows the convergence of* (7.0.38) *and from the divergence of* (7.0.37) *it follows the divergence of* (7.0.38).

Exercise 7.0.542. *Let the improper isointegrals*

$$\hat{\int_a^b} f^{\wedge}(x)\hat{\times}\hat{d}\hat{x} \tag{7.0.39}$$

and

$$\hat{\int_a^b} g^{\wedge}(x)\hat{\times}\hat{d}\hat{x} \tag{7.0.40}$$

has the same singular point b. Let also for every $x \in [a,b]$

$$0 \leq f\Big(x\hat{T}(x)\Big)\frac{1}{\hat{T}(x)} \leq g(x\hat{T}(x)), \hat{T} \in \mathcal{C}^1([a,b]),$$

$$\hat{T}(x) > 0, x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \leq 0.$$

Then from the convergence of (7.0.39) *it follows the convergence of* (7.0.40) *and from the divergence of* (7.0.39) *it follows the divergence of* (7.0.40).

Exercise 7.0.543. *Let the improper isointegrals* (7.0.39) *and* (7.0.40) *have the same singular point b. Let also for every $x \in [a,b]$*

$$0 \leq f\Big(x\hat{T}(x)\Big)\frac{1}{\hat{T}(x)} \leq g(x\hat{T}(x)), \hat{T} \in \mathcal{C}^1([a,b]),$$

$$\hat{T}(x) > 0, x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \geq 0.$$

Then from the convergence of (7.0.39) *it follows the convergence of* (7.0.40) *and from the divergence of* (7.0.39) *it follows the divergence of* (7.0.40).

Exercise 7.0.544. *Let the improper isointegrals*

$$\hat{\int_a^b} f^{\wedge}(x)\hat{\times}\hat{d}\hat{x} \tag{7.0.41}$$

and

$$\hat{\int}_a^b \hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x} \tag{7.0.42}$$

has the same singular point b. Let also for every $x \in [a,b]$

$$0 \le f\Big(x\hat{T}(x)\Big) \le g\Big(\tfrac{x}{\hat{T}(x)}\Big), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0,$$

$$\tfrac{x}{\hat{T}(x)} \in [a,b], x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \le 0.$$

Then from the convergence of (7.0.41) *it follows the convergence of* (7.0.42) *and from the divergence of* (7.0.41) *it follows the divergence of* (7.0.42).

Exercise 7.0.545. *Let the improper isointegrals* (7.0.41) *and* (7.0.42) *have the same singular point b. Let also for every $x \in [a,b]$*

$$0 \le f\Big(x\hat{T}(x)\Big) \le g\Big(\tfrac{x}{\hat{T}(x)}\Big), \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0,$$

$$\tfrac{x}{\hat{T}(x)} \in [a,b], x\hat{T}(x) \in [a,b], \hat{T}(x) - x\hat{T}'(x) \ge 0.$$

Then from the convergence of (7.0.41) *it follows the convergence of* (7.0.42) *and from the divergence of* (7.0.41) *it follows the divergence of* (7.0.42).

Theorem 7.0.546. *Let the improper integrals* (7.0.6), (7.0.7) *have unique singular point at b. Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), \hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \le 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x)}{g(x)} = A > 0.$$

Then the isointegrals (7.0.6), (7.0.7) *are simultaneously divergent or convergent.*

Proof. From $\lim_{x\longrightarrow b-0} \frac{f(x)}{g(x)} = A$ it follows that for every $\epsilon \in (0,A)$ there exists $c \in [a,b)$ such that

$$\Big|\frac{f(x)}{g(x)} - A\Big| < \epsilon \quad \Longleftrightarrow A - \epsilon < \frac{f(x)}{g(x)} < A + \epsilon$$

for every $x \in (c, b)$, from here, since $g(x) > 0$ for every $x \in [a, b)$, we get

$$(A - \epsilon)g(x) \le f(x) \le (A + \epsilon)g(x) \quad \text{for} \quad \forall x \in (c, b),$$

and since

$$\hat{T}(x) - x\hat{T}'(x) \le 0 \quad \text{for} \quad \forall x \in [a, b),$$

we obtain the inequalities

$$(A+\epsilon)g(x)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} \le f(x)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} \le (A-\epsilon)g(x)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)}$$

for every $x \in (c, b)$. After we integrate the last inequalities from c to b we get

$$\widehat{(A+\epsilon)}\hat{\times}\hat{\int}_c^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \le \hat{\int}_c^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x} \le \widehat{(A-\epsilon)}\hat{\times}\hat{\times}\hat{\int}_c^b \hat{g}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x}$$

and since the improper isointegrals (7.0.6) and (7.0.7) have unique singular point at b, from the last inequalities we conclude that they are simultaneously divergent or convergent. $\square$

Exercise 7.0.547. *Let the improper integrals* (7.0.6), (7.0.7) *have unique singular point at b. Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a, b]), \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \ge 0 \quad \forall x \in [a, b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x)}{g(x)} = A > 0.$$

Then the isointegrals (7.0.6), (7.0.7) *are simultaneously divergent or convergent.*

Exercise 7.0.548. *Let the improper integrals* (7.0.9), (7.0.10) *have unique singular point at b. Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a, b]), x\hat{T}(x) \in [a, b],$$

$$\hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \le 0 \quad \forall x \in [a, b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x\hat{T}(x))}{g(x\hat{T}(x))} = A > 0.$$

Then the isointegrals (7.0.9), (7.0.10) *are simultaneously divergent or convergent.*

Exercise 7.0.549. *Let the improper integrals* (7.0.9), (7.0.10) *have unique singular point at* b. *Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b],$$

$$\hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \geq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x\hat{T}(x))}{g(x\hat{T}(x))} = A > 0.$$

Then the isointegrals (7.0.9), (7.0.10) *are simultaneously divergent or convergent.*

Exercise 7.0.550. *Let the improper integrals* (7.0.11), (7.0.12) *have unique singular point at* b. *Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), \tfrac{x}{\hat{T}(x)} \in [a,b], \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \leq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{g\left(\frac{x}{\hat{T}(x)}\right)} = A > 0.$$

Then the isointegrals (7.0.11), (7.0.12) *are simultaneously divergent or convergent.*

Exercise 7.0.551. *Let the improper integrals* (7.0.11), (7.0.12) *have unique singular point at* b. *Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), \tfrac{x}{\hat{T}(x)} \in [a,b], \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \geq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{g\left(\frac{x}{\hat{T}(x)}\right)} = A > 0.$$

Then the isointegrals (7.0.11), (7.0.12) *are simultaneously divergent or convergent.*

Exercise 7.0.552. *Let the improper integrals* (7.0.13), (7.0.14) *have unique singular point at* b. *Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b], \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \leq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x\hat{T}(x))}{g(x\hat{T}(x))} = A > 0.$$

Then the isointegrals (7.0.13), (7.0.14) *are simultaneously divergent or convergent.*

Exercise 7.0.553. *Let the improper integrals* (7.0.13), (7.0.14) *have unique singular point at* b. *Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b], \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \geq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x\hat{T}(x))}{g(x\hat{T}(x))} = A > 0.$$

Then the isointegrals (7.0.13), (7.0.14) *are simultaneously divergent or convergent.*

Exercise 7.0.554. *Let the improper integrals* (7.0.15), (7.0.16) *have unique singular point at* b. *Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b],$$

$$\hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \leq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x)}{g(x\hat{T}(x))} = A > 0.$$

Then the isointegrals (7.0.15), (7.0.16) *are simultaneously divergent or convergent.*

Exercise 7.0.555. *Let the improper integrals* (7.0.15), (7.0.16) *have unique singular point at* b. *Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b],$$

$$\hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \geq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x)}{g(x\hat{T}(x))} = A > 0.$$

Then the isointegrals (7.0.15), (7.0.16) *are simultaneously divergent or convergent.*

Exercise 7.0.556. *Let the improper integrals* (7.0.17), (7.0.18) *have unique singular point at* b. *Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), \tfrac{x}{\hat{T}(x)} \in [a,b],$$

$$\hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \leq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x)}{g\left(\frac{x}{\hat{T}(x)}\right)} = A > 0.$$

Then the isointegrals (7.0.17), (7.0.18) *are simultaneously divergent or convergent.*

Exercise 7.0.557. *Let the improper integrals* (7.0.17), (7.0.18) *have unique singular point at* b. *Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), \tfrac{x}{\hat{T}(x)} \in [a,b],$$

$$\hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \geq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x)}{g\left(\frac{x}{\hat{T}(x)}\right)} = A > 0.$$

Then the isointegrals (7.0.17), (7.0.18) *are simultaneously divergent or convergent.*

Exercise 7.0.558. *Let the improper integrals* (7.0.19), (7.0.20) *have unique singular point at b. Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b], \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \le 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x)}{g(x\hat{T}(x))\hat{T}(x)} = A > 0.$$

Then the isointegrals (7.0.19), (7.0.20) *are simultaneously divergent or convergent.*

Exercise 7.0.559. *Let the improper integrals* (7.0.19), (7.0.20) *have unique singular point at b. Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b], \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \ge 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x)}{g(x\hat{T}(x))\hat{T}(x)} = A > 0.$$

Then the isointegrals (7.0.19), (7.0.20) *are simultaneously divergent or convergent.*

Exercise 7.0.560. *Let the improper integrals* (7.0.21), (7.0.22) *have unique singular point at b. Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b], \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \le 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x\hat{T}(x))}{g(x)} = A > 0.$$

Then the isointegrals (7.0.21), (7.0.22) *are simultaneously divergent or convergent.*

Exercise 7.0.561. *Let the improper integrals* (7.0.21), (7.0.22) *have unique singular point at b. Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b], \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \geq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x\hat{T}(x))}{g(x)} = A > 0.$$

Then the isointegrals (7.0.21), (7.0.22) *are simultaneously divergent or convergent.*

Exercise 7.0.562. *Let the improper integrals* (7.0.23), (7.0.24) *have unique singular point at b. Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b], \tfrac{x}{\hat{T}(x)} \in [a,b],$$

$$\hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \leq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x\hat{T}(x))}{g\left(\frac{x}{\hat{T}(x)}\right)} = A > 0.$$

Then the isointegrals (7.0.23), (7.0.24) *are simultaneously divergent or convergent.*

Exercise 7.0.563. *Let the improper integrals* (7.0.23), (7.0.24) *have unique singular point at b. Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b], \tfrac{x}{\hat{T}(x)} \in [a,b], \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \geq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x\hat{T}(x))}{g\left(\frac{x}{\hat{T}(x)}\right)} = A > 0.$$

Then the isointegrals (7.0.23), (7.0.24) *are simultaneously divergent or convergent.*

Exercise 7.0.564. *Let the improper integrals* (7.0.25), (7.0.26) *have unique singular point at* b. *Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b], \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \leq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x\hat{T}(x))}{\hat{T}(x)g(x\hat{T}(x))} = A > 0.$$

Then the isointegrals (7.0.25), (7.0.26) *are simultaneously divergent or convergent.*

Exercise 7.0.565. *Let the improper integrals* (7.0.25), (7.0.26) *have unique singular point at* b. *Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b], \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \geq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f(x\hat{T}(x))}{\hat{T}(x)g(x\hat{T}(x))} = A > 0.$$

Then the isointegrals (7.0.25), (7.0.26) *are simultaneously divergent or convergent.*

Exercise 7.0.566. *Let the improper integrals* (7.0.27), (7.0.28) *have unique singular point at* b. *Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), \tfrac{x}{\hat{T}(x)} \in [a,b],$$

$$\hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \leq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{g(x)} = A > 0.$$

Then the isointegrals (7.0.27), (7.0.28) *are simultaneously divergent or convergent.*

Exercise 7.0.567. *Let the improper integrals* (7.0.27), (7.0.28) *have unique singular point at* b. *Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), \frac{x}{\hat{T}(x)} \in [a,b],$$

$$\hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \geq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{g(x)} = A > 0.$$

Then the isointegrals (7.0.27), (7.0.28) *are simultaneously divergent or convergent.*

Exercise 7.0.568. *Let the improper integrals* (7.0.29), (7.0.30) *have unique singular point at* b. *Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), \frac{x}{\hat{T}(x)} \in [a,b], x\hat{T}(x) \in [a,b],$$

$$\hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \leq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{g(x\hat{T}(x))} = A > 0.$$

Then the isointegrals (7.0.29), (7.0.30) *are simultaneously divergent or convergent.*

Exercise 7.0.569. *Let the improper integrals* (7.0.29), (7.0.30) *have unique singular point at* b. *Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), \frac{x}{\hat{T}(x)} \in [a,b], x\hat{T}(x) \in [a,b], \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \geq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{g(x\hat{T}(x))} = A > 0.$$

Then the isointegrals (7.0.29), (7.0.30) *are simultaneously divergent or convergent.*

Exercise 7.0.570. *Let the improper integrals* (7.0.31), (7.0.32) *have unique singular point at b. Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), \frac{x}{\hat{T}(x)} \in [a,b], \hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \leq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{g\left(\frac{x}{\hat{T}(x)}\right)} = A > 0.$$

Then the isointegrals (7.0.31), (7.0.32) *are simultaneously divergent or convergent.*

Exercise 7.0.571. *Let the improper integrals* (7.0.31), (7.0.32) *have unique singular point at b. Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), \frac{x}{\hat{T}(x)} \in [a,b], \hat{T}(x) - x\hat{T}'(x) \geq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{g\left(\frac{x}{\hat{T}(x)}\right)} = A > 0.$$

Then the isointegrals (7.0.31), (7.0.32) *are simultaneously divergent or convergent.*

Exercise 7.0.572. *Let the improper integrals* (7.0.33), (7.0.34) *have unique singular point at b. Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), \frac{x}{\hat{T}(x)} \in [a,b], x\hat{T}(x) \in [a,b],$$

$$\hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \leq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x) g(x\hat{T}(x))} = A > 0.$$

Then the isointegrals (7.0.33), (7.0.34) *are simultaneously divergent or convergent.*

Exercise 7.0.573. *Let the improper integrals* (7.0.33), (7.0.34) *have unique singular point at b. Let also*

$$f(x) > 0, g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), \frac{x}{\hat{T}(x)} \in [a,b], x\hat{T}(x) \in [a,b], \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \geq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x\longrightarrow b-0}\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)g(x\hat{T}(x))}=A>0.$$

Then the isointegrals (7.0.33), (7.0.34) *are simultaneously divergent or convergent.*

Exercise 7.0.574. *Let the improper integrals* (7.0.35), (7.0.36) *have unique singular point at b. Let also*

$$f(x)>0, g(x)>0, \hat{T}\in\mathcal{C}^1([a,b]), x\hat{T}(x)\in[a,b],$$

$$\hat{T}(x)>0, \hat{T}(x)-x\hat{T}'(x)\leq 0 \quad \forall x\in[a,b),$$

there exists

$$\lim_{x\longrightarrow b-0}\frac{f\left(x\hat{T}(x)\right)}{g(x)}=A>0.$$

Then the isointegrals (7.0.35), (7.0.36) *are simultaneously divergent or convergent.*

Exercise 7.0.575. *Let the improper integrals* (7.0.35), (7.0.36) *have unique singular point at b. Let also*

$$f(x)>0, g(x)>0, \hat{T}\in\mathcal{C}^1([a,b]), x\hat{T}(x)\in[a,b], \hat{T}(x)>0,$$

$$\hat{T}(x)-x\hat{T}'(x)\geq 0 \quad \forall x\in[a,b),$$

there exists

$$\lim_{x\longrightarrow b-0}\frac{f\left(x\hat{T}(x)\right)}{g(x)}=A>0.$$

Then the isointegrals (7.0.35), (7.0.36) *are simultaneously divergent or convergent.*

Exercise 7.0.576. *Let the improper integrals* (7.0.37), (7.0.38) *have unique singular point at b. Let also*

$$f(x)>0, \quad g(x)>0, \hat{T}\in\mathcal{C}^1([a,b]), x\hat{T}(x)\in[a,b],$$

$$\hat{T}(x)>0, \hat{T}(x)-x\hat{T}'(x)\leq 0 \quad \forall x\in[a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f\left(x\hat{T}(x)\right)}{g(x\hat{T}(x))} = A > 0.$$

Then the isointegrals (7.0.37), (7.0.38) *are simultaneously divergent or convergent.*

Exercise 7.0.577. *Let the improper integrals* (7.0.37), (7.0.38) *have unique singular point at b. Let also*

$$f(x) > 0, \quad g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b], \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \geq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f\left(x\hat{T}(x)\right)}{g(x\hat{T}(x))} = A > 0.$$

Then the isointegrals (7.0.37), (7.0.38) *are simultaneously divergent or convergent.*

Exercise 7.0.578. *Let the improper integrals* (7.0.39), (7.0.40) *have unique singular point at b. Let also*

$$f(x) > 0, \quad g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b],$$

$$\hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \leq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f\left(x\hat{T}(x)\right)}{g(x\hat{T}(x))} = A > 0.$$

Then the isointegrals (7.0.39), (7.0.40) *are simultaneously divergent or convergent.*

Exercise 7.0.579. *Let the improper integrals* (7.0.39), (7.0.40) *have unique singular point at b. Let also*

$$f(x) > 0, \quad g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b], \hat{T}(x) > 0,$$

$$\hat{T}(x) - x\hat{T}'(x) \geq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f\Big(x\hat{T}(x)\Big)}{g(x\hat{T}(x))} = A > 0.$$

Then the isointegrals (7.0.39), (7.0.40) *are simultaneously divergent or convergent.*

Exercise 7.0.580. *Let the improper integrals* (7.0.41), (7.0.42) *have unique singular point at* b. *Let also*

$$f(x) > 0, \quad g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b],$$

$$\tfrac{x}{\hat{T}(x)} \in [a,b], \hat{T}(x) > 0, \hat{T}(x) - x\hat{T}'(x) \leq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f\Big(x\hat{T}(x)\Big)}{g\Big(\frac{x}{\hat{T}(x)}\Big)} = A > 0.$$

Then the isointegrals (7.0.41), (7.0.42) *are simultaneously divergent or convergent.*

Exercise 7.0.581. *Let the improper integrals* (7.0.41), (7.0.42) *have unique singular point at* b. *Let also*

$$f(x) > 0, \quad g(x) > 0, \hat{T} \in \mathcal{C}^1([a,b]), x\hat{T}(x) \in [a,b], \hat{T}(x) > 0,$$

$$\tfrac{x}{\hat{T}(x)} \in [a,b], \hat{T}(x) - x\hat{T}'(x) \geq 0 \quad \forall x \in [a,b),$$

there exists

$$\lim_{x \longrightarrow b-0} \frac{f\Big(x\hat{T}(x)\Big)}{g\Big(\frac{x}{\hat{T}(x)}\Big)} = A > 0.$$

Then the isointegrals (7.0.41), (7.0.42) *are simultaneously divergent or convergent.*

Theorem 7.0.582. *Let $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $|\hat{T}(x) - x\hat{T}'(x)| \leq M$ for every $x \in [a,b]$, where M is a positive constant. Then if $\alpha < 1$ the integrals*

$$\int_a^b \frac{1}{(b-x)^\alpha} \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx, \quad \int_a^b \frac{1}{(b-x)^\alpha} \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} dx$$

are convergent.

Proof. Since $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$ for every $x \in [a,b]$, then there exists a positive constant m such that

$$\hat{T}(x) \geq m, \quad \hat{T}^2(x) \geq m \quad \text{for} \quad \forall x \in [a,b].$$

Therefore

$$\left|\int_a^b \tfrac{1}{(b-x)^\alpha} \tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx\right| \leq \int_a^b \tfrac{1}{(b-x)^\alpha} \left|\tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)}\right| dx$$

$$\leq \tfrac{M}{m} \int_a^b \tfrac{1}{(b-x)^\alpha} dx = \tfrac{M}{m} \tfrac{(b-a)^{1-\alpha}}{1-\alpha} < \infty,$$

and from here

$$\left|\int_a^b \tfrac{1}{(b-x)^\alpha} \tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} dx\right| \leq \int_a^b \tfrac{1}{(b-x)^\alpha} \left|\tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}\right| dx$$

$$\leq \tfrac{M}{m} \int_a^b \tfrac{1}{(b-x)^\alpha} dx < \infty.$$

□

Theorem 7.0.583. *Let $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq N > 0$ for every $x \in [a,b]$. If $\alpha \geq 1$ the integrals*

$$\int_a^b \frac{1}{(b-x)^\alpha} \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx, \quad \int_a^b \frac{1}{(b-x)^\alpha} \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} dx$$

are divergent.

Proof. Since $\hat{T} \in \mathcal{C}^1([a,b])$ and $\hat{T}(x) > 0$ for every $x \in [a,b]$, then there exists $n > 0$ such that

$$\hat{T}(x) \leq n, \quad \hat{T}^2(x) \leq n \quad \text{for} \quad \forall x \in [a,b].$$

From here

$$\int_a^b \frac{1}{(b-x)^\alpha} \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx \geq \frac{N}{n} \int_a^b \frac{dx}{(b-x)^\alpha} dx = \infty,$$

$$\int_a^b \frac{1}{(b-x)^\alpha} \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} dx \geq \frac{N}{n} \int_a^b \frac{dx}{(b-x)^\alpha} = \infty.$$

□

Exercise 7.0.584. *Let $\hat{T} \in \mathcal{C}^1([a,\infty))$, $\hat{T}(x) > 0$, $|\hat{T}(x) - x\hat{T}'(x)| \leq M$ for every $x \in [a,\infty)$, where M is a positive constant, $a \neq 0$. Then if $\alpha > 1$ the integrals*

$$\int_a^\infty \frac{1}{x^\alpha} \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx, \quad \int_a^\infty \frac{1}{x^\alpha} \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} dx$$

are convergent.

Exercise 7.0.585. *Let $\hat{T} \in \mathcal{C}^1([a,\infty))$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq N > 0$ for every $x \in [a,\infty)$, $a \neq 0$. If $\alpha \leq 1$ the integrals*

$$\int_a^\infty \frac{1}{x^\alpha} \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}^2(x)} dx, \quad \int_a^\infty \frac{1}{x^\alpha} \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} dx$$

are divergent.

Exercise 7.0.586. *Let $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $|\hat{T}(x) - x\hat{T}'(x)| \leq M$ for every $x \in [a,b]$, where M is a positive constant. Then if $\alpha < 1$ and there exists*

$$\lim_{x \longrightarrow b-0} |f(x)|(b-x)^\alpha = A > 0$$

prove that

$$\hat{\int_a^b} \hat{f}^\wedge(\hat{x}) \hat{\times} \hat{d}\hat{x}$$

is convergent.

Exercise 7.0.587. *Let $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq N > 0$ for every $x \in [a,b]$. If $\alpha \geq 1$ and there exists*

$$\lim_{x \longrightarrow b-0} |f(x)|(b-x)^{\alpha} = A > 0$$

prove that

$$\hat{\int}_a^b \hat{f}^{\wedge}(\hat{x}) \hat{\times} \hat{d}\hat{x}$$

is divergent

Exercise 7.0.588. *Let $\hat{T} \in \mathcal{C}^1([a,\infty))$, $\hat{T}(x) > 0$, $|\hat{T}(x) - x\hat{T}'(x)| \leq M$ for every $x \in [a,\infty)$, where M is a positive constant, $a \neq 0$. Then if $\alpha > 1$ and there exists*

$$\lim_{x \longrightarrow \infty} |f(x)|x^{\alpha} = A > 0$$

prove that

$$\hat{\int}_a^{\infty} \hat{f}^{\wedge}(\hat{x}) \hat{\times} \hat{d}\hat{x}$$

is convergent.

Exercise 7.0.589. *Let $\hat{T} \in \mathcal{C}^1([a,\infty))$, $\hat{T}(x) > 0$, $\hat{T}(x) - x\hat{T}'(x) \geq N > 0$ for every $x \in [a,\infty)$, $a \neq 0$. If $\alpha \leq 1$ and there exists*

$$\lim_{x \longrightarrow \infty} |f(x)|x^{\alpha} = A > 0$$

prove that

$$\hat{\int}_a^{\infty} \hat{f}^{\wedge}(\hat{x}) \hat{\times} \hat{d}\hat{x}$$

is divergent.

Exercise 7.0.590. *Let $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in [a,b]$, $|\hat{T}(x) - x\hat{T}'(x)| \leq M$ for every $x \in [a,b]$, where M is a positive constant. Then if $\alpha < 1$ and there exists*

$$\lim_{x \longrightarrow b-0} |f(x\hat{T}(x))|(b-x)^{\alpha} = A > 0$$

prove that

$$\hat{\int}_a^b \hat{f}^{\wedge}(x) \hat{\times} \hat{d}\hat{x}$$

is convergent.

Exercise 7.0.591. *Let* $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in [a,b]$, $\hat{T}(x) - x\hat{T}'(x) \geq N > 0$ *for every* $x \in [a,b]$. *If* $\alpha \geq 1$ *and there exists*

$$\lim_{x \longrightarrow b-0} |f(x\hat{T}(x))|(b-x)^{\alpha} = A > 0$$

prove that

$$\hat{\int}_a^b \hat{f}^{\wedge}(x)\hat{\times}\hat{d}\hat{x}$$

is divergent

Exercise 7.0.592. *Let* $\hat{T} \in \mathcal{C}^1([a,\infty))$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in [a,\infty)$, $|\hat{T}(x) - x\hat{T}'(x)| \leq M$ *for every* $x \in [a,\infty)$, *where* M *is a positive constant,* $a \neq 0$. *Then if* $\alpha > 1$ *and there exists*

$$\lim_{x \longrightarrow \infty} |f(x\hat{T}(x))|x^{\alpha} = A > 0$$

prove that

$$\hat{\int}_a^{\infty} \hat{f}^{\wedge}(x)\hat{\times}\hat{d}\hat{x}$$

is convergent.

Exercise 7.0.593. *Let* $\hat{T} \in \mathcal{C}^1([a,\infty))$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in [a,\infty)$, $\hat{T}(x) - x\hat{T}'(x) \geq N > 0$ *for every* $x \in [a,\infty)$, $a \neq 0$. *If* $\alpha \leq 1$ *and there exists*

$$\lim_{x \longrightarrow \infty} |f(x\hat{T}(x))|x^{\alpha} = A > 0$$

prove that

$$\hat{\int}_a^{\infty} \hat{f}^{\wedge}(x)\hat{\times}\hat{d}\hat{x}$$

is divergent.

Exercise 7.0.594. *Let* $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in [a,b]$, $|\hat{T}(x) - x\hat{T}'(x)| \leq M$ *for every* $x \in [a,b]$, *where* M *is a positive constant. Then if* $\alpha < 1$ *and there exists*

$$\lim_{x \longrightarrow b-0} \left| f\left(\frac{x}{\hat{T}(x)}\right) \right| (b-x)^{\alpha} = A > 0$$

prove that

$$\hat{\int}_a^b \hat{f}(\hat{x})\hat{\times}\hat{d}\hat{x}$$

is convergent.

Exercise 7.0.595. *Let* $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in [a,b]$, $\hat{T}(x) - x\hat{T}'(x) \geq N > 0$ *for every* $x \in [a,b]$. *If* $\alpha \geq 1$ *and there exists*

$$\lim_{x \longrightarrow b-0} \left| f\left(\frac{x}{\hat{T}(x)}\right) \right| (b-x)^{\alpha} = A > 0$$

prove that

$$\hat{\int}_a^b \hat{f}(\hat{x}) \hat{\times} \hat{d}\hat{x}$$

is divergent

Exercise 7.0.596. *Let* $\hat{T} \in \mathcal{C}^1([a,\infty))$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in [a,\infty)$, $|\hat{T}(x) - x\hat{T}'(x)| \leq M$ *for every* $x \in [a,\infty)$, *where* M *is a positive constant,* $a \neq 0$. *Then if* $\alpha > 1$ *and there exists*

$$\lim_{x \longrightarrow \infty} \left| f\left(\frac{x}{\hat{T}(x)}\right) \right| x^{\alpha} = A > 0$$

prove that

$$\hat{\int}_a^{\infty} \hat{f}(\hat{x}) \hat{\times} \hat{d}\hat{x}$$

is convergent.

Exercise 7.0.597. *Let* $\hat{T} \in \mathcal{C}^1([a,\infty))$, $\hat{T}(x) > 0$, $\frac{x}{\hat{T}(x)} \in [a,\infty)$, $\hat{T}(x) - x\hat{T}'(x) \geq N > 0$ *for every* $x \in [a,\infty)$, $a \neq 0$. *If* $\alpha \leq 1$ *and there exists*

$$\lim_{x \longrightarrow \infty} \left| f\left(\frac{x}{\hat{T}(x)}\right) \right| x^{\alpha} = A > 0$$

prove that

$$\hat{\int}_a^{\infty} \hat{f}(\hat{x}) \hat{\times} \hat{d}\hat{x}$$

is divergent.

Exercise 7.0.598. *Let* $\hat{T} \in \mathcal{C}^1([a,b])$, $\hat{T}(x) > 0$, $x\hat{T}(x) \in [a,b]$, $|\hat{T}(x) - x\hat{T}'(x)| \leq M$ *for every* $x \in [a,b]$, *where* M *is a positive constant. Then if* $\alpha < 1$ *and there exists*

$$\lim_{x \longrightarrow b-0} |f(x\hat{T}(x))| (b-x)^{\alpha} = A > 0$$

prove that

$$\hat{\int}_a^b f^{\wedge}(x) \hat{\times} \hat{d}\hat{x}$$

is convergent.

Exercise 7.0.599. *Let* $\hat{T} \in \mathcal{C}^1([a,b])$*,* $\hat{T}(x) > 0$*,* $x\hat{T}(x) \in [a,b]$*,* $\hat{T}(x) - x\hat{T}'(x) \geq N > 0$ *for every* $x \in [a,b]$*. If* $\alpha \geq 1$ *and there exists*

$$\lim_{x \longrightarrow b-0} |f(x\hat{T}(x))|(b-x)^{\alpha} = A > 0$$

prove that

$$\hat{\int}_a^b f^{\wedge}(x) \hat{\times} \hat{d}\hat{x}$$

is divergent

Exercise 7.0.600. *Let* $\hat{T} \in \mathcal{C}^1([a,\infty))$*,* $\hat{T}(x) > 0$*,* $x\hat{T}(x) \in [a,\infty)$*,* $|\hat{T}(x) - x\hat{T}'(x)| \leq M$ *for every* $x \in [a,\infty)$*, where* M *is a positive constant,* $a \neq 0$*. Then if* $\alpha > 1$ *and there exists*

$$\lim_{x \longrightarrow \infty} |f(x\hat{T}(x))|x^{\alpha} = A > 0$$

prove that

$$\hat{\int}_a^{\infty} f^{\wedge}(x) \hat{\times} \hat{d}\hat{x}$$

is convergent.

Exercise 7.0.601. *Let* $\hat{T} \in \mathcal{C}^1([a,\infty))$*,* $\hat{T}(x) > 0$*,* $x\hat{T}(x) \in [a,\infty)$*,* $\hat{T}(x) - x\hat{T}'(x) \geq N > 0$ *for every* $x \in [a,\infty)$*,* $a \neq 0$*. If* $\alpha \leq 1$ *and there exists*

$$\lim_{x \longrightarrow \infty} |f(x\hat{T}(x))|x^{\alpha} = A > 0$$

prove that

$$\hat{\int}_a^{\infty} f^{\wedge}(x) \hat{\times} \hat{d}\hat{x}$$

is divergent.

Advanced practical exercises

Problem 7.0.602. *Let* $D = [1,2]$*,* $f(x) = \hat{T}(x) = x^3$*. Find*

$$\hat{\int}_1^2 \hat{f}^{\wedge}(\hat{x}) \hat{\times} \hat{d}\hat{x}, \quad \hat{\int}_1^2 \hat{f}^{\wedge}(x) \hat{\times} \hat{d}\hat{x}, \quad \hat{\int}_1^2 \hat{f}(\hat{x}) \hat{\times} \hat{d}\hat{x}, \quad \hat{\int}_1^2 f^{\wedge}(x) \times \hat{d}\hat{x}.$$

Answer.

$$-2, \quad 21, \quad -\frac{255}{1024}, \quad -\frac{4054}{3}.$$

Problem 7.0.603. *Investigate for divergence and convergence the isointegral*

$$\hat{\int}_a^b \hat{f}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x}$$

if $\hat{T}(x) = \frac{x}{x+1}$ *and*

1) $a = \frac{1}{2}, b = 1, f(x) = \frac{1}{(1-x)(2x-1)}$,

2) $a = 0, b = +\infty, f(x) = \frac{x}{(x+1)^3}$,

3) $a = 3, b = +\infty, f(x) = \frac{2x}{(x^2-1)^2}$,

4) $a = 0, b = +\infty, f(x) = \frac{1}{c^2x^2+d^2}, c \neq 0, b \neq 0$,

5) $a = 0, b = +\infty, f(x) = \frac{x}{x^2+4x+3}$,

6) $a = 2, b = +\infty, f(x) = \frac{3x-1}{x^2+5x-7}$,

7) $a = 0, b = +\infty, f(x) = x^2 5^{-x}$,

8) $a = 1, b = +\infty, f(x) = \ln x$,

9) $a = 0, b = 1, f(x) = \ln x$,

10) $a = 0, b = \frac{1}{2}, f(x) = \frac{1}{x \ln^2 x}$.

Answer. !) divergent, 2)-4) convergent, 5), 6) divergent, 7) convergent, 8) divergent, 9), 10) convergent.

Chapter 8

Appendix:Elements of isodual mathematics

8.1 Isodual reals

Let $F = F(a, +, \cdot)$ be the field of real numbers a with convenbtional sum $+$ and convential product $\cdot$.
Santlli's isodual field is the ring $F^d = F^d(a^d, +^d, \times^d)$ with elements given by isodual reals

$$a^d = -a, \quad a \in F,$$

with associative and commutative isodual sum

$$a^d +^d b^d = -a + (-b) = -(a + b) = (a + b)^d,$$

(below we will use the notation $+$ instead $+^d$), associative and distributive isodual product

$$a^d \times^d b^d = (-a)(-1)(-b) = -(ab) = (ab)^d,$$

additive isodual unit $0^d = 0$,

$$a^d + 0^d = 0^d + a^d = a^d,$$

and multiplicative isodual unit $I^d = -1$,

$$a^d \times^d I^d = I^d \times^d a^d = a^d.$$

The proof of the following properties are elementary.

Lemma 8.1.1. *lemma1 Isodual fields are fields, i.e. if A is a field, its image A^d under the isodual map is also a field.*

Lemma 8.1.2. *lemma2 Fields A and their isodual images A^d are anti-isomorphic to each other.*

Lemma 8.1.2 and Lemma 8.1.2 illustrate the origin of the name "isodual mathematics". In fact, to represent antimatter the needed mathematics must be a suitable "dual" of conventional mathematics, while the prefix "iso" is used in its Greek meaning of preserving the original axioms.

Example 8.1.3. $3^d = -3$, $(-4)^d = 4$.

Exercise 8.1.4. *Compute*

$$A = 3^d + (-4) - 5^d.$$

Solution. We have

$$3^d == -3, \quad 5^d = -5.$$

Then

$$A = -3 + (-4) - (-5) = -7 + 5 = -2 = 2^d.$$

Exercise 8.1.5. *Compute*

1) $A = 2^d \times^d (-5)^d$,

2) $B = 3^d(-2)^d \times^d 2^d + (-4)^d$.

Answer. 1) $A = (-10)^d$, $B = 8^d$.
All operations of real numbers must be subjected to isoduality when dealing with isodual numbers. This implies

1) isodual isopowers

$$(a^d)^{n^d} = \underbrace{a^d \times^d a^d \times^d a^d \times^d \cdots \times^d a^d}_{n},$$

2) isodual powers

$$(a^d)^n = \underbrace{a^d a^d \cdots a^d}_{n},$$

3) isodual iso-n-th root

$$a^{d^{(1/n)d}} = -\sqrt[n]{-a},$$

4) isodual n-th root

$$a^{d(1/n)} = \sqrt[n]{-a},$$

5) isodual isoquotient

$$a^d /^d b^d = -\frac{a^d}{b^d} = -\frac{-a}{-b} = -\frac{a}{b} = (a/b)^d.$$

6) isodual quotient

$$a^d / b^d = \frac{a^d}{b^d} = \frac{-a}{-b} = \frac{a}{b} = a/b.$$

Example 8.1.6. *Let us compute*

$$A = (3^d)^{2^d} - 4^d \times^d 2^d.$$

We have

$$(3^d)^{2^d} = 3^d \times^d 3^d = (-3)(-1)(-3) = -9, \quad 4^d = -4, \quad 2^d = -2,$$

$$4^d \times^d 2^d = -4(-1)(-2) = -8.$$

Then

$$A = -9 - (-8) = -1 = 1^d.$$

Lemma 8.1.7. *Isodual fields have a nonnegative definite norm, called isodual norm*

$$|a^d|^d = -|-a| = -|a| \leq 0.$$

Lemma 8.1.8. *All quantities that are positive-definite when referred to positive units and related fields of matter (such as mass, energy, angular momentum, density, temperature, time, etc.) become negative-definite when referred to isodual units and related isodual fields of antimatter.*

8.2 Isodual sequences

Definition 8.2.1. *Let* $\{a_n\}_{n=1}^{\infty}$ *is a sequence of reals. The sequence*

$$\left\{a_n^d = -a_n\right\}_{n=1}^{\infty}$$

will be called isodual sequence.

Example 8.2.2. *Let* $\{a_n\}_{n=1}^{\infty} = \{n+1\}_{n=1}^{\infty}$. *Then the sequence* $\Big\{-n-1\Big\}_{n=1}^{\infty}$ *is an isodual sequence.*

Definition 8.2.3. *An isodual sequence* $\{a_n^d\}_{n=1}^{\infty}$ *will be called*

1. *bounded above if there exists* $l^d \in F^d$ *so that* $a_n^d \leq l^d$ *for every* $n \in \mathbb{N}$,

2. *bounded below if there exists* $m^d \in F^d$ *so that* $a_n^d \geq m^d$ *for every* $n \in \mathbb{N}$,

3. *bounded if it is bounded above and bounded below.*

Example 8.2.4. *Let* $\{a_n\}_{n=1}^{\infty} = \{n^3\}_{n=1}^{\infty}$. *Then*

$$a_n^d = -n^3$$

is unbounded below isodual sequence and bounded above isodual sequence. The sequence $\{a_n\}_{n=1}^{\infty}$ *is a bounded below sequence and an unbounded above sequence.*

We have the following properties.

1) If $\{a_n\}_{n=1}^{\infty}$ is a bounded above sequence then the isodual sequence $\{a_n^d\}_{n=1}^{\infty}$ is bounded below and the inverse.

2) If $\{a_n\}_{n=1}^{\infty}$ is a bounded below sequence then the isodual sequence $\{a_n^d\}_{n=1}^{\infty}$ is bounded above and the inverse.

3) If $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence then the isodual sequence $\{a_n^d\}_{n=1}^{\infty}$ is bounded and the inverse.

Definition 8.2.5. *An isodual sequence* $\{a_n^d\}_{n=1}^{\infty}$ *is called unbounded if it is not bounded.*

In other words, a sequence $\{a_n^d\}_{n=1}^{\infty}$ is unbounded if there exists $t^d \in F^d$ and $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geq N$, we have

$$|a_n^d| \geq t$$

or

$$|a_n^d|^d \leq t^d.$$

Definition 8.2.6. *We will say that a sequence $\{a_n^d\}_{n=1}^{\infty}$ diverges to $+\infty$ if for every $M \in \mathbb{R}$, $M \geq 0$ there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geq N$, we have*

$$a_n^d \geq M.$$

Definition 8.2.7. *A sequence $\{a_n^d\}_{n=1}^{\infty}$ diverges to $-\infty$ if for every $P \in \mathbb{R}$, $P \leq 0$ there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geq N$, we have*

$$a_n^d \leq P.$$

We have the following properties.

1) If $\{a_n\}_{n=1}^{\infty}$ diverges to $-\infty$ then the isodual sequence $\{a_n^d\}_{n=1}^{\infty}$ diverges to $+\infty$ and the inverse.

2) If $\{a_n\}_{n=1}^{\infty}$ diverges to $+\infty$ then the isodual sequence $\{a_n^d\}_{n=1}^{\infty}$ diverges to $-\infty$ and the inverse.

Exercise 8.2.8. *Let $\{a_n\}_{n=1}^{\infty} = \{-n^4 - 8\}_{n=1}^{\infty}$. Prove that the sequence $\{a_n^d\}_{n=1}^{\infty}$ diverges to $+\infty$.*

Exercise 8.2.9. *Let $\{a_n\}_{n=1}^{\infty} = \{n + 12\}_{n=1}^{\infty}$. Prove that the sequence $\{a_n^d\}_{n=1}^{\infty}$ diverges to $-\infty$.*

Definition 8.2.10. *The isodual real $a^d \in F^d$ is called limit of the isodual sequence $\{a_n^d\}_{n=1}^{\infty}$ if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > N$ we have*

$$|a_n^d - a^d| < \epsilon$$

or

$$|a_n^d - a^d|^d > \epsilon^d.$$

In this case we will write $\lim_{n\longrightarrow\infty} a_n^d = a^d$ and we will say that the sequence $\{a_n^d\}_{n=1}^{\infty}$ is convergent.
In other words the number a^d is a limit of the sequence $\{a_n^d\}_{n=1}^{\infty}$ if

$$\lim_{n\longrightarrow\infty} a_n = a.$$

Example 8.2.11. *Let $\{a_n\}_{n=1}^{\infty} = \left\{\frac{n+4}{n+2}\right\}_{n=1}^{\infty}$. Then*

$$\lim_{n\longrightarrow\infty} a_n^d = -\lim_{n\longrightarrow\infty} a_n = -\lim_{n\longrightarrow\infty} \frac{n+4}{n+2} = -1 = 1^d.$$

Exercise 8.2.12. *Let $\{a_n\}_{n=1}^{\infty} = \left\{\frac{2n+1}{3n+5}\right\}_{n=1}^{\infty}$. Find*

$$\lim_{n\longrightarrow\infty} a_n^d.$$

Answer. $\left(\frac{2}{3}\right)^d$.

Theorem 8.2.13. *Let the sequence* $\{a_n\}_{n=1}^{\infty}$ *is convergent to* $a \in \mathbb{R}$ *and* $a \neq 0$. *Then there exists* $N \in \mathbb{N}$ *such that for every* $n \in \mathbb{N}$, $n > N$, *we have*

$$|a_n^d|^d < \left(\frac{|a|}{2}\right)^d.$$

Also, if $a > 0$ *then there exists* $N \in \mathbb{N}$ *such that for every* $n \in \mathbb{N}$, $n > N$, *we have*

$$a_n^d < \left(\frac{a}{2}\right)^d,$$

if $a < 0$ *then there exists* $N \in \mathbb{N}$ *such that for every* $n \in \mathbb{N}$, $n > N$, *we have*

$$a_n^d > \left(\frac{a}{2}\right)^d.$$

Theorem 8.2.14. *Let* $\lim_{n\longrightarrow\infty} a_n^d = a^d$, $\lim_{n\longrightarrow\infty} b_n^d = b^d$, $a_n^d \leq b_n^d$ *for every* $n \geq n_0$. *Then* $a^d \leq b^d$.

Corollary 8.2.15. *Let* $\lim_{n\longrightarrow\infty} a_n^d = a^d$ *and let there exists* $n_0 \in \mathbb{N}$ *such that* $a_n^d \geq b^d$ *for every* $n \geq n_0$. *Then* $a^d \geq b^d$.

Theorem 8.2.16. *Let* $\lim_{n\longrightarrow\infty} a_n^d = a^d$, $\lim_{n\longrightarrow\infty} b_n^d = a^d$, *and there exists* $n_0 \in \mathbb{N}$ *such that* $a_n^d \leq c_n^d \leq b_n^d$ *for every* $n \geq n_0$. *Then* $\lim_{n\longrightarrow\infty} c_n^d = a^d$

Theorem 8.2.17. *Let* $\lim_{n\longrightarrow\infty} a_n^d = a^d$. *Then* $\lim_{n\longrightarrow\infty} |a_n^d|^d = |a^d|^d$.

Corollary 8.2.18. *Let* $\lim_{n\longrightarrow\infty} a_n^d = 0$*. Then* $\lim_{n\longrightarrow\infty} |a_n^d|^d = 0$*.*

Theorem 8.2.19. *Every convergent isodual sequence* $\{\hat{a}_n\}_{n=1}^{\infty}$ *is bounded sequence.*

Theorem 8.2.20. *Let* $\lim_{n\longrightarrow\infty} a_n^d = a^d$, $\lim_{n\longrightarrow\infty} b_n^d = b^d$*. Then*

1. $\lim_{n\longrightarrow\infty}(a_n^d \pm b_n^d) = \lim_{n\longrightarrow\infty} a_n^d \pm \lim_{n\longrightarrow\infty} b_n^d = a^d \pm b^d$,
2. $\lim_{n\longrightarrow\infty}(a_n^d \times^d b_n^d) = \lim_{n\longrightarrow\infty} b_n^d \times^d \lim_{n\longrightarrow\infty} a_n^d = a^d \times^d b^d$,
3. $\lim_{n\longrightarrow\infty}(a_n^d /^d b_n^d) = \lim_{n\longrightarrow\infty} a_n^d /^d \lim_{n\longrightarrow\infty} b_n^d = a^d /^d b^d$*, if* $b_{\hat{n}}^d \neq 0$, $b^d \neq 0$.

Corollary 8.2.21. *Let* $\lim_{n\hat{\longrightarrow}\infty} a_n^d = a^d$*. Then* $\lim_{n\longrightarrow\infty}(\alpha^d \times^d a_n^d) = \alpha^d \times^d a^d$ *for every* $\alpha^d \in F^d$*.*

Definition 8.2.22. *The isodual sequence* $\{a_n^d\}_{n=1}^{\infty}$ *is called infinite small if it is convergent and its limit is equal to* 0*.*

Corollary 8.2.23. *The sum, subtraction and multiplication of infinite small isodual sequences is infinite small isodual sequence.*

Theorem 8.2.24. *Let $\{a_n\}_{n=1}^{\infty}$ is an infinite small sequence of reals. Then $\{a_n^d\}_{n=1}^{\infty}$ is an infinite small isodual sequence.*

Theorem 8.2.25. *The number a^d is limit of the isodual sequence $\{a_n^d\}_{n=1}^{\infty}$ if and only if it can be represented in the form*

$$a^d = a_n^d - \alpha_n^d,$$

where $\{\alpha_n^d\}_{n=1}^{\infty}$ is infinite small isodual sequence.

Theorem 8.2.26. *Let $\{\alpha_n^d\}_{n=1}^{\infty}$ is infinite small isodual sequence and $\{a_n^d\}_{n=1}^{\infty}$ is bounded isodual sequence. Then $\{\alpha_n^d \times^d a_n^d\}_{n=1}^{\infty}$ is infinite small isodual sequence.*

Definition 8.2.27. *A sequence* $\{a_n^d\}_{n=1}^{\infty}$ *is called infinite large if for every* $M \in \mathbb{R}$, $M > 0$ *there exists* $N \in \mathbb{N}$ *such that for every* $n > N$ *we have*

$$|a_n^d| \geq M$$

or

$$|a_n^d|^d \leq M^d.$$

In other words an isodual sequence $\{a_n^d\}_{n=1}^{\infty}$ *is an infinite large isodual sequence if*

$$\lim_{n \longrightarrow} |a_n^d| = \infty$$

or

$$\lim_{n \longrightarrow} |a_n^d|^d = -\infty.$$

Theorem 8.2.28. *Let* $\{a_n\}_{n=1}^{\infty}$ *is an infinite large sequence of reals then the isodual sequence* $\{a_n^d\}_{n=1}^{\infty}$ *is an infinite large isodual sequence and the inverse.*

Theorem 8.2.29. *Let* $\{a_n^d\}_{n=1}^{\infty}$ *be bounded isodual sequence and* $\{b_n^d\}_{n=1}^{\infty}$ *be infinite large isodual sequence. Then the isodual sequence* $\{a_n^d /^d b_n^d\}_{n=1}^{\infty}$ *is infinite small isodual sequence.*

Theorem 8.2.30. *Let* $\{|a_n^d|\}_{n=1}^{\infty}$ *be bounded below sequence by a positive isodual and* $\lim_{n\longrightarrow\infty} \alpha_n^d = 0$ *and* $\alpha_n^d \neq 0$ *for every* $n \in \mathbb{N}$. *Then the isodual sequence* $\{a_n^d /^d \alpha_n^d\}_{n=1}^{\infty}$ *is infinite large isodual sequence.*

Corollary 8.2.31. *Let* $\{a_n^d\}_{n=1}^{\infty}$ *be infinite large isodual sequence. Then* $\{1^d/^d a_n^d\}_{n=1}^{\infty}$ *is infinite small isodual sequence.*

Corollary 8.2.32. *Let* $\{a_n^d\}_{n=1}^{\infty}$ *be infinite small isodual sequence. Then* $\{1^d/^d a_n^d\}_{n=1}^{\infty}$ *is infinite large isodual sequence.*

Definition 8.2.33. *The sequence* $\{a_n^d\}_{n=1}^{\infty}$ *will be called*

1. *increasing if from* $n, m \in \mathbb{N}$, $n > m$ *follows that* $a_n^d > a_m^d$,
2. *decreasing if from* $n, m \in \mathbb{N}$, $n > m$ *follows that* $a_n^d < a_m^d$,
3. *monotonic if it is increasing or decreasing.*

If the sequence $\{a_n^d\}_{n=1}^{\infty}$ is increasing it is bounded below because $a_n^d \geq \hat{a}_1^d$ for every $n \in \mathbb{N}$.

If the sequence $\{a_n^d\}_{n=1}^{\infty}$ is decresing it is bounded above because $a_n^d \leq a_1^d$ for every $n \in \mathbb{N}$.

Theorem 8.2.34. *Let* $\{a_n^d\}_{n=1}^{\infty}$ *be incresing sequence and bounded above by* $M^d \in F^d$ *then it is convergent.*

Theorem 8.2.35. *Let* $\{a_n^d\}_{n=1}^{\infty}$ *be decreasing isodual sequence and bounded below by* $P^d \in F^d$ *then it is convergent.*

Corollary 8.2.36. *Every bounded monotonic isodual sequence is convergent.*

Definition 8.2.37. *An isodual sequence* $\{a_{\hat{n}}^d\}_{n=1}^{\infty}$ *is called fundamental if for every* $\epsilon^d \in F^d$, $\epsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that for every* $m, n > N$ *we have*

$$|a_n^d - a_m^d|^d > \epsilon^d.$$

Theorem 8.2.38. *If the sequence* $\{a_n^d\}_{n=1}^{\infty}$ *is convergent then it is fundamental.*

Definition 8.2.39. *Every isointerval* (p^d, q^d) *which contains the isopoint* a^d *will be called isoneighbourhood of the isopoint* a^d.

Definition 8.2.40. *An isopoint* a^d *will be called condensation isopoint of the sequence* $\{a_n^d\}_{n=1}^{\infty}$ *of elements of* F^d *if every isoneighbourhood of* a^d *contains incountable many isoelements of* $\{a_n^d\}_{n=1}^{\infty}$.

Theorem 8.2.41. *Every bounded isodual sequence has a condensation isopoint.*

Definition 8.2.42. *We will say that the isodual sequence* $\{a_{n_k}^d\}_{k=1}^{\infty}$ *is an subsequence of the isodual sequence* $\{a_n^d\}_{n=1}^{\infty}$ *if* $n_k \in \mathbb{N}$ *for every* $k \in \mathbb{N}$ *and*

$$n_1 < n_{\hat{2}} < n_{\hat{3}} < \cdots .$$

Theorem 8.2.43. *Let the isodual sequence* $\{a_n^d\}_{n=1}^{\infty}$ *be sequence which is convergent to* a^d. *Then every subsequence* $\{a_{n_k}^d\}_{k=1}^{\infty}$ *is convergent to* a^d.

Definition 8.2.44. *We will say that the isodual sequence* $\{a_n^d\}_{n=1}^{\infty}$ *is expanded of two subsequences* $\{a_{n_k}^d\}_{k=1}^{\infty}$ *and* $\{a_{m_k}^d\}_{k=1}^{\infty}$ *if*

$$\{n_1, n_2, \ldots\} \cup \{m_1, m_2, m_3, \ldots\} = \mathbb{N},$$

and

$$\{n_1, n_2, \ldots\} \cap \{m_1, m_2, m_3, \ldots\} = \emptyset.$$

Theorem 8.2.45. *Let the isodual sequence* $\{a_n^d\}_{n=1}^{\infty}$ *be expanded of two subsequences* $\{a_{n_k}^d\}_{k=1}^{\infty}$ *and* $\{a_{m_k}^d\}_{k=1}^{\infty}$ *which are convergent to the isopoint* a^d*. Then the sequence* $\{a_n^d\}_{n=1}^{\infty}$ *is convergent to* a^d*.*

Corollary 8.2.46. *Let* $\{a_n^d\}_{n=1}^{\infty}$ *and* $\{b_n^d\}_{n=1}^{\infty}$ *be convergent sequences to the isopoint* a^d*. Then the isodual sequence*

$$a_1^d, b_1^d, a_2^d, b_2^d, \ldots$$

is convergent isodual sequence to a^d*.*

Corollary 8.2.47. *Let* $\{a_n^d\}_{n=1}^{\infty}$ *be isodual sequence which is convergent to* a^d*. Let also* $b^d \in F^d$*. Then the sequence*

$$b^d, a_1^d, a_2^d, a_3^d, \ldots$$

is convergent isodual sequence to a^d*.*

Corollary 8.2.48. *Let* $\{a_n^d\}_{n=1}^{\infty}$ *be isodual sequence which is convergent to* a^d*. Let also* $b_1^d, b_2^d, \ldots, b_k^d \in F^d$ *be finite number of isodual reals. Then the isodual sequence*

$$b_1^d, b_2^d, \ldots, b_k^d, a_1^d, a_2^d, a_3^d, \ldots$$

is convergent isodual sequence to a^d*.*

Theorem 8.2.49. *From every infinite bounded isodual sequence can be chosen convergent isodual subsequence.*

Theorem 8.2.50. *Every fundamental isodual sequence is convergent.*

Definition 8.2.51. *We will say that* $+\infty$ *is condensation isopoint of* $\{a_n^d\}_{n=1}^{\infty}$ *if it is unbounded above.*

Definition 8.2.52. *We will say that* $-\infty$ *is condensation isopoint of* $\{a_{\hat{n}}^d\}_{n=1}^{\infty}$ *if it is unbounded below.*

Using above definitions we can conclude that every isodual sequence of reals has condensation isopoint.

Definition 8.2.53. *Let* $\{a_n^d\}_{n=1}^{\infty}$ *be isodual sequence. Limit inferior and limit superior we define as follows*

$$\liminf_{n\longrightarrow\infty} a_n^d = \lim_{n\longrightarrow\infty}(\mathrm{inf}_{\mathrm{m}\geq\mathrm{n}}\, \mathrm{a}_{\mathrm{m}}^{\mathrm{d}}),$$

$$\limsup_{n\longrightarrow\infty} a_n^d = \lim_{n\longrightarrow\infty}(\mathrm{sup}_{\mathrm{m}\geq\mathrm{n}}\, \mathrm{a}_{\mathrm{m}}^{\mathrm{d}}),$$

respectively.

If

$$\liminf_{n\longrightarrow\infty} a_n^d = \infty,$$

then

$$\lim_{n\longrightarrow\infty} a_n^d = \infty.$$

If

$$\limsup_{n\longrightarrow\infty} a_n^d = -\infty,$$

then

$$\lim_{n\longrightarrow\infty} a_n^d = -\infty.$$

The sequence $\{a_n^d\}_{n=1}^{\infty}$ is convergent if

$$\liminf_{n\longrightarrow\infty} a_n^d = \lim_{n\longrightarrow\infty} a_n^d = \limsup_{n\longrightarrow\infty} a_n^d.$$

8.3 Isodual functions

Here we will suppose that $D \subset \mathbb{R}$.

Definition 8.3.1. *We will say that in the set D is defined isodual function of first kind if from $x \in D$ it follows that $-x \in D$ and*

$$f^d(x^d) = -f(-x), x \in D,$$

is a function(map). We will use the notation f^{ddd}.
The set

$$\{f^d(x^d) : x \in D\}$$

will be called isodual codomain of isodual values of the isodual function f^{ddd} of first kind. The function $-f(-x)$ will be called isodual original of the isodual function f^{ddd} of first kind. The element x will be called isodual argument or isodual independent variable of f^{ddd} and its isodual image $f^d(x^d)$ will be called isodual dependent or isodual value of f^{ddd}.

Example 8.3.2. *Let $D = \mathbb{R}$, $f(x) = x^2+x+1$. Then from $x \in D$ it follows that $-x \in D$ and*

$$f^d(x^d) = -f(-x) = -\Big((-x)^2 - x + 1\Big) = -\Big(x^2 - x + 1\Big) = -x^2 + x - 1.$$

Exercise 8.3.3. *Let* $D = [-3, 3]$, $f(x) = \frac{x+1}{x+5}$. *Find*

$$f^d(x^d) + 2f(x).$$

Solution. Firstly, we will note that from $x \in D$ it follows that $-x \in D$. Also,

$$f^d(x^d) = -f(-x) = -\frac{-x+1}{-x+5} = \frac{x-1}{5-x}.$$

Then

$$f^d(x^d) + 2f(x) = \frac{x-1}{5-x} + \frac{2x+2}{x+5} = \frac{x^2-12x-5}{x^2-25}.$$

Definition 8.3.4. *We will say that in the set* D *is defined isodual function of second kind if for* $x \in D$ *we have*

$$f^d(x) = -f(x)$$

is a function(map). We will use the notation f^{dd}.
The set

$$\{f^d(x) : x \in D\}$$

will be called isodual codomain of isodual values of the isodual function f^{dd} *of second kind. The function* $-f(x)$ *will be called isodual original of the isodual function* f^{dd} *of second kind. The element* x *will be called isodual argument or isodual independent variable of* f^{dd} *and its isodual image* $f^d(x)$ *will be called isodual dependent or isodual value of* f^{dd}.

Example 8.3.5. *Let* $D = \mathbb{R}$ *and* $f(x) = \cos x + \sin x$. *Then*

$$f^d(x) = -f(x) = -\cos x - \sin x, \quad x \in D.$$

Definition 8.3.6. *We will say that in the set D is defined isodual function of third kind if from $x \in D$ it follows that $-x \in D$ and*

$$f(x^d) = f(-x), x \in D,$$

is a function(map). We will use the notation f^d.
The set

$$\{f(x^d) : x \in D\}$$

will be called isodual codomain of isodual values of the isodual function f^d of third kind. The function $f(-x)$ will be called isodual original of the isodual function f^d of third kind. The element x will be called isodual argument or isodual independent variable of f^d and its isodual image $f(x^d)$ will be called isodual dependent or isodual value of f^d.

Example 8.3.7. *Let $D = [-1, 1]$, $f(x) = e^x - \sin x$, $x \in D$. Then*

$$f(x^d) = f(-x) = e^{-x} - \sin(-x) = e^{-x} + \sin x, \quad x \in D.$$

Exercise 8.3.8. *Let $D = [-2, 2]$,*

$$f(x) = x^2 + x + 1, \quad g(x) = x + 2, \quad h(x) = x - 1, \quad x \in D.$$

Find

$$A(x) = 2^d \times^d f^d(x) - 3g^d(x^d) + 2^d \times^d (f(x^d) + h^d(x)).$$

Solution. Firstly we will note that

$$f^{dd}, f^d, g^{ddd}, h^{dd}$$

are defined in D. Also,

$$f^d(x) = -f(x) = -(x^2 + x + 1) = -x^2 - x - 1,$$

$$2^d \times^d f^d(x) = (-2)(-1)(-x^2 - x - 1) = -2x^2 - 2x - 2,$$

$$g^d(x^d) = -g(-x) = -(-x + 2) = x - 2,$$

$$3g^d(x^d) = 3(x - 2) = 3x - 6,$$

$$f(x^d) = f(-x) = (-x)^2 + (-x) + 1 = x^2 - x + 1,$$

$$h^d(x) = -h(x) = -x + 1,$$

$$f(x^d) + h^d(x) = x^2 - x + 1 - x + 1 = x^2 - 2x + 2,$$

$$2^d \times^d (f(x^d) + h^d(x)) = -2(-1)(x^2 - 2x + 2) = 2x^2 - 4x + 4.$$

From here

$$A(x) = -2x^2 - 2x - 2 - (3x - 6) + 2x^2 - 4x + 4 = -9x + 8.$$

Exercise 8.3.9. *Let* $D = [-5, 5]$, $f(x) = \frac{x+3}{x^2+5}$. *Find*

$$g^{ddd}, \quad g^{dd}, \quad g^d$$

such that

$$g^{ddd} \equiv f, \quad g^{dd} \equiv f, \quad g^d \equiv f \quad \text{in} \quad D.$$

Solution. We have

$$\tfrac{x+3}{x^2+5} - \left(-\tfrac{x+3}{x^2+5}\right) = -\tfrac{-x-3}{x^2+5} = -\tfrac{x^d+3^d}{-(x^d)^{2^d}-5^d},$$

therefore

$$g^d(x^d) = \left(\frac{x^d + 3^d}{-(x^d)^{2^d} - 5^d}\right)^d.$$

From

$$\frac{x+3}{x^2+5} = -\left(-\frac{x+3}{x^2+5}\right)$$

it follows that

$$g^d(x) = \left(-\frac{x+3}{x^2+5}\right)^d.$$

Also,

$$\tfrac{x+3}{x^2+5} = \tfrac{-(-x)-(-3)}{-(-x)(-1)(-x)-(-5)} = \tfrac{-x^d-3^d}{-(x^d)^{2^d}-5^d},$$

consequently

$$g(x^d) = \frac{-x^d - 3^d}{-(x^d)^{2^d} - 5^d}.$$

Now we suppose that f^{ddd}, f^{dd}, f^d, g^{ddd}, g^{dd}, g^d are defined in D. Then for $x \in D$ we have

1) $f^d((g^d(x^d))^d) = -f(-(g^d(x^d))) = -f(-(-g(-x))) = -f(g(-x))$,

2) $f^d(g^d(x^d)) = -f(-g(-x))$,

3) $f((g^d(x^d))^d) = f(-(-g(-x))) = f(g(-x))$,

4) $f^d((g^d(x))^d) = -f(-(-g(x))) = -f(g(x))$,

5) $f^d(g^d(x)) = -f(-g(x))$,

6) $f((g^d(x))^d) = f(-(-g(x))) = f(g(x))$,

7) $f^d((g(x^d))^d) = -f(-g(-x))$,

8) $f^d(g(x^d)) = -f(g(-x))$,

9) $f((g(x^d))^d) = f(-g(-x))$.

Example 8.3.10. *Let $D = \mathbb{R}$, $f(x) = x + 2$, $g(x) = x + 3$. Then*

$$f^d((g^d(x^d))^d) = -f(g(-x)) = -f(-x+3) = -((-x+3)+2) = -(-x+5) = x-5.$$

Exercise 8.3.11. *Let $D = \mathbb{R}$, $f(x) = x^2 + x$, $g(x) = x + 5$. Find*

$$f^d((g(x^d))^d).$$

Solution. We have

$$f^d((g(x^d))^d) = -f(-g(-x)) = -f(-(-x+5))$$

$$= -f(x-5) = -((x-5)^2 + x - 5) = -(x^2 - 9x + 20) = -x^2 + 9x - 20.$$

Definition 8.3.12. *An isodual function of first, second or third kind, defined in D, will be called bounded below if its isodual original is bounded below in D.*

Definition 8.3.13. *An isodual function of first, second or third kind, defined in D, will be called bounded above if its isodual original is bounded above in D.*

Definition 8.3.14. *An isodual function of first, second or third kind, defined in D, will be called bounded if it is bounded above and below in D.*

Definition 8.3.15. *An isodual function of first, second or third kind, defined in D, will be called even isodual function if its isodual original is even function in D.*

Definition 8.3.16. *An isodual function of first, second or third kind, defined in D, will be called odd isodual function if its isodual original is odd function in D.*

Definition 8.3.17. *An isodual function of first, second or third kind, defined in D, will be called $\omega^d \in F^d$, $\omega > 0$, -periodic isodual function if its isodual original is ω- periodic function in D.*

Definition 8.3.18. *An isodual function of first, second or third kind, defined in D, will be called monotonic increasing isodual function if its isodual original is monotonic increasing function in D.*

Definition 8.3.19. *An isodual function of first, second or third kind, defined in D, will be called monotonic decreasing isodual function if its isodual original is monotonic decreasing function in D.*

Definition 8.3.20. *An isodual function of first, second or third kind, defined in D, will be called monotonic isodual function if its isodual original is monotonic increasing function in D.*

8.4 Limit of isodual functions. Continuous isodual functions

Here we suppose that $D \subset \mathbb{R}$, F is an isodual function of first, second or third kind, defined in D, and with $\tilde{f}$ we will denote its isodual original.

Definition 8.4.1. *The real a will be called left limit of F at $x_0 \in D$ if it is a left limit of $\tilde{f}$ at x_0.*

Definition 8.4.2. *The real a will be called right limit of F at $x_0 \in D$ if it is a right limit of $\tilde{f}$ at x_0.*

Definition 8.4.3. *The real a will be called limit of F at $x_0 \in D$ if it is a limit of $\tilde{f}$ at x_0.*

Example 8.4.4. *Let $D = \mathbb{R}$, $f(x) = x^2 + 3$. Then*

$$\lim_{x^d \longrightarrow 1^d} f^d(x^d) = \lim_{x \longrightarrow 1}(-f(-x))$$

$$= \lim_{x \longrightarrow 1}(-((-x)^2 + 3)) = \lim_{x \longrightarrow 1}(-x^2 - 3) = -4.$$

Example 8.4.5. *Let $D = \mathbb{R}$, $f(x) = \frac{x^2+4}{x^2+5}$. Then*

$$\lim_{x \longrightarrow 1^d} f^d(x^d) = \lim_{x \longrightarrow -1}(-f(-x))$$

$$= -\lim_{x \longrightarrow -1} \frac{(-x)^2+4}{(-x)^2+5} = -\lim_{x \longrightarrow -1} \frac{x^2+4}{x^2+5} = -\frac{5}{6}.$$

Definition 8.4.6. *We will say that the isodual function F is a continuous at $x_0 \in D$ if its isodual original $\tilde{f}$ is a continuous function at x_0.*

Definition 8.4.7. *We will say that the isodual function F is a continuous in D if it is continuous in every point in D.*

Example 8.4.8. *Let* $D = \mathbb{R}$*,*

$$f(x) = \begin{cases} x+2 & \text{for} \quad x \in (-\infty, 1] \\ 4-x & \text{for} \quad x \in [1, +\infty). \end{cases}$$

Then f *is a continuous function at* $x = 1$*. But*

$$f^d(x^d) = \begin{cases} x-2 & \text{for} \quad x \in (-\infty, 1] \\ -4-x & \text{for} \quad x \in [1, +\infty), \end{cases}$$

which is not continuous ate $x = 1$ *because*

$$\lim_{x \longrightarrow 1-} f^d(x^d) = -1, \lim_{x \longrightarrow 1+} f^d(x^d) = -5.$$

Remark 8.4.9. *We will note if* f*,* f^{ddd}*,* f^{dd}*,* f^d *are defined in* D *and* f *is a continuous function at* $x_0 \in D$ *then there is a possibility some of the isodual functions* f^{ddd}*,* f^{dd} *or* f^d *to be not continuous at* x_0 *and the inverse.*

8.5 Isodual differential calculas

Definition 8.5.1. *Isodual differential is defined as follows*

$$d^d(\cdot) = -d(\cdot).$$

Using the above definition we have

1) $d^d x = -dx$,

2) $d^d x^d = -dx^d = -d(-x) = dx$.

Let F be isodual function of first, second or third kind, defined in $D \subset \mathbb{R}$ and its isodual original is differentiable function. Then we have the following possibilities.

1) isodual derivative of first kind $d^dF/^dd^dx^d$.

2) isodual derivative of second kind $d^dF/^ddx^d$.

3) isodual derivative of third kind $d^dF/^dd^dx$.

4) isodual derivative of fourth kind $d^dF/^ddx$.

5) isodual derivative of fifth kind d^dF/d^dx^d.

6) isodual derivative of sixth kind d^dF/dx^d.

7) isodual derivative of seventh kind d^dF/d^dx.

8) isodual derivative of eighth kind d^dF/dx.

9) isodual derivative of ninth kind dF/d^dx^d.

10) isodual derivative of tenth kind dF/dx^d.

11) isodual derivative of eleventh kind dF/d^dx.

12) isodual derivative of twelfth kind dF/dx.

13) isodual derivative of thirtieth kind $dF/^dd^dx^d$.

14) isodual derivative of fourteenth kind $dF/^ddx^d$.

15) isodual derivative of fifteenth kind $dF/^dd^dx$.

16) isodual derivative of sixteenth kind $dF/^ddx$.

Below we will give explicit expressions of the isodual derivatives for every class isodual functions.

8.5.1 Isodual functions of first kind

Here $F = f^{ddd}$. Then for $x \in D$ we have

1) isodual derivative of first kind

$$d^d f^d(x^d)/^d d^d x^d = -\frac{d^d f^d(x^d)}{d^d x^d} = -\frac{-df^d(x^d)}{-dx^d}$$

$$= -\frac{d(-f(-x))}{d(-x)} = -\frac{df(-x)}{dx} = f'(-x).$$

2) isodual derivative of second kind

$$d^d f^d(x^d)/^d dx^d = -\frac{d^d f^d(x^d)}{dx^d} = -\frac{-df^d(x^d)}{dx^d}$$

$$= \frac{d(-f(-x))}{d(-x)} = \frac{df(-x)}{dx} = -f'(-x).$$

3) isodual derivative of third kind

$$d^d f^d(x^d)/^d d^d x = -\frac{d^d f^d(x^d)}{d^d x} = -\frac{-df^d(x^d)}{-dx}$$

$$= -\frac{d(-f(-x))}{dx} = \frac{df(-x)}{dx} = -f'(-x).$$

4) isodual derivative of fourth kind

$$d^d f^d(x^d)/^d dx = -\frac{d^d f^d(x^d)}{dx} = \frac{df^d(x^d)}{dx}$$

$$= \frac{d(-f(-x))}{dx} = -\frac{df(-x)}{dx} = f'(-x).$$

5) isodual derivative of fifth kind

$$d^d f^d(x^d)/d^d x^d = \frac{-df^d(x^d)}{-dx^d} = \frac{d(-f(-x))}{d(-x)} = \frac{df(-x)}{dx} = -f'(-x).$$

6) isodual derivative of sixth kind

$$d^d f^d(x^d)/dx^d = \frac{-df^d(x^d)}{dx^d} = -\frac{d(-f(-x))}{d(-x)} = -\frac{df(-x)}{dx} = f'(-x).$$

7) isodual derivative of seventh kind

$$d^d f^d(x^d)/d^d x = \frac{-df^d(x^d)}{-dx} = \frac{d(-f(-x))}{dx} = -\frac{df(-x)}{dx} = f'(-x).$$

8) isodual derivative of eighth kind

$$d^d f^d(x^d)/dx = \frac{-df^d(x^d)}{dx} = -\frac{d(-f(-x))}{dx} = \frac{df(-x)}{dx} = -f'(-x).$$

9) isodual derivative of ninth kind

$$df^d(x^d)/d^d x^d = \frac{d(-f(-x))}{-dx^d} = \frac{df(-x)}{d(-x)} = f'(-x).$$

10) isodual derivative of tenth kind

$$df^d(x^d)/dx^d = \frac{d(-f(-x))}{d(-x)} = -\frac{df(-x)}{d(-x)} = -f'(-x).$$

11) isodual derivative of eleventh kind

$$df^d(x^d)/d^d x = \frac{d(-f(-x))}{-dx} = \frac{df(-x)}{dx} = -f'(-x).$$

12) isodual derivative of twelfth kind

$$df^d(x^d)/dx = \frac{d(-f(-x))}{dx} = -\frac{df(-x)}{dx} = f'(-x).$$

13) isodual derivative of thirtieth kind

$$df^d(x^d)/^d d^d x^d = -\frac{df^d(x^d)}{d^d x^d} = -\frac{d(-f(-x))}{-dx^d} = -\frac{df(-x)}{d(-x)} = -f'(-x).$$

14) isodual derivative of fourteenth kind

$$df^d(x^d)/^d dx^d = -\frac{df^d(x^d)}{dx^d} = -\frac{d(-f(-x))}{d(-x)} = -\frac{df(-x)}{dx} = f'(-x).$$

15) isodual derivative of fifteenth kind

$$df^d(x^d)/^d d^d x = -\frac{df^d(x^d)}{d^d x} = -\frac{d(-f(-x))}{-dx} = -\frac{df(-x)}{dx} = f'(-x).$$

16) isodual derivative of sixteenth kind

$$df^d(x^d)/^d dx = -\frac{df^d(x^d)}{dx} = -\frac{d(-f(-x))}{dx} = \frac{df(-x)}{dx} = -f'(-x).$$

8.5.2 Isodual functions of second kind

Here $F = f^{dd}$. Then for $x \in D$ we have

1) isodual derivative of first kind

$$d^d f^d(x)/^d d^d x^d = -\tfrac{d^d f^d(x)}{d^d x^d} = -\tfrac{-df^d(x)}{-dx^d} = -\tfrac{d(-f(x))}{d(-x)} = -f'(x).$$

2) isodual derivative of second kind

$$d^d f^d(x)/^d dx^d = -\frac{d^d f^d(x)}{dx^d} = -\frac{-df^d(x)}{d(-x)} = -\frac{d(-f(x))}{dx} = f'(x).$$

3) isodual derivative of third kind

$$d^d f^d(x)/^d d^d x = -\frac{d^d f^d(x)}{d^d x} = -\frac{-df^d(x)}{-dx} = -\frac{d(-f(x))}{dx} = f'(x).$$

4) isodual derivative of fourth kind

$$d^d f^d(x)/^d dx = -\frac{d^d f^d(x)}{dx} = -\frac{-df^d(x)}{dx} = \frac{d(-f(x))}{dx} = -f'(x).$$

5) isodual derivative of fifth kind

$$d^d f^d(x)/d^d x^d = \frac{-df^d(x)}{-dx^d} = \frac{d(-f(x))}{d(-x)} = f'(x).$$

6) isodual derivative of sixth kind

$$d^d f^d(x)/dx^d = \frac{-df^d(x)}{d(-x)} = \frac{d(-f(x))}{dx} = -f'(x).$$

7) isodual derivative of seventh kind

$$d^d f^d(x)/d^d x = \frac{-df^d(x)}{-dx} = \frac{d(-f(x))}{dx} = -f'(x).$$

8) isodual derivative of eighth kind

$$d^d f^d(x)/dx = -\frac{df^d(x)}{dx} = -\frac{d(-f(x))}{dx} = f'(x).$$

9) isodual derivative of ninth kind

$$df^d(x)/d^d x^d = \frac{df^d(x)}{-dx^d} = \frac{d(-f(x))}{-d(-x)} = -f'(x).$$

10) isodual derivative of tenth kind

$$df^d(x)/dx^d = \frac{d(-f(x))}{d(-x)} = f'(x).$$

11) isodual derivative of eleventh kind

$$df^d(x)/d^d x = \frac{d(-f(x))}{-dx} = \frac{df(x)}{dx} = f'(x).$$

12) isodual derivative of twelfth kind

$$df^d(x)/dx = \frac{d(-f(x))}{dx} = -f'(x).$$

13) isodual derivative of thirteenth kind

$$df^d(x)/^d d^d x^d = -\frac{df^d(x)}{d^d x^d} = -\frac{d(-f(x))}{-dx^d} = -\frac{df(x)}{d(-x)} = f'(x).$$

14) isodual derivative of fourteenth kind

$$df^d(x)/^d dx^d = -\frac{df^d(x)}{dx^d} = -\frac{d(-f(x))}{d(-x)} = -f'(x).$$

15) isodual derivative of fifteenth kind

$$df^d(x)/^d d^d x = -\frac{df^d(x)}{d^d x} = -\frac{d(-f(x))}{-dx} = -f'(x).$$

16) isodual derivative of sixteenth kind

$$df^d(x)/^d dx = -\frac{df^d(x)}{dx} = -\frac{d(-f(x))}{dx} = f'(x).$$

8.5.3 Isodual functions of third kind

Here $F = f^d$. Then for $x \in D$ we have

1) isodual derivative of first kind

$$d^d f(x^d)/^d d^d x^d = -\frac{d^d f(x^d)}{d^d x^d} = -\frac{-df(x^d)}{-dx^d} = -\frac{df(-x)}{d(-x)} = -f'(-x).$$

2) isodual derivative of second kind

$$d^d f(x^d)/^d dx^d = -\frac{d^d f(x^d)}{dx^d} = -\frac{-df(x^d)}{dx^d} = \frac{df(-x)}{d(-x)} = f'(-x).$$

3) isodual derivative of third kind

$$d^d f(x^d)/^d d^d x = -\frac{d^d f(x^d)}{d^d x} = -\frac{-df(x^d)}{-dx} = -\frac{df(-x)}{dx} = f'(-x).$$

4) isodual derivative of fourth kind

$$d^d f(x^d)/^d dx = -\frac{d^d f(x^d)}{dx} = \frac{df(-x)}{dx} = -f'(-x).$$

5) isodual derivative of fifth kind

$$d^d f(x^d)/d^d x^d = \frac{-df(x^d)}{-dx^d} = \frac{df(-x)}{d(-x)} = f'(-x).$$

6) isodual derivative of sixth kind

$$d^d f(x^d)/dx^d = \frac{-df(x^d)}{dx^d} = -\frac{df(-x)}{d(-x)} = -f'(-x).$$

7) isodual derivative of seventh kind

$$d^d f(x^d)/d^d x = \frac{-df(x^d)}{-dx} = \frac{df(-x)}{dx} = -f'(-x).$$

8) isodual derivative of eighth kind

$$d^d f(x^d)/dx = \frac{-df(x^d)}{dx} = -\frac{df(-x)}{dx} = f'(-x).$$

9) isodual derivative of ninth kind

$$df(x^d)/d^d x^d = \frac{df(-x)}{-dx^d} = -\frac{df(-x)}{d(-x)} = -f'(-x).$$

10) isodual derivative of tenth kind

$$df(x^d)/dx^d = \frac{df(-x)}{d(-x)} = f'(-x).$$

11) isodual derivative of eleventh kind

$$df(x^d)/d^d x = \frac{df(-x)}{-d(x)} = f'(-x).$$

12) isodual derivative of twelfth kind

$$df(x^d)/dx = \frac{df(-x)}{dx} = -f'(-x).$$

13) isodual derivative of thirteenth kind

$$df(x^d)/^d d^d x^d = -\frac{df(x^d)}{d^d x^d} = -\frac{df(-x)}{-dx^d} = \frac{df(-x)}{d(-x)} = f'(-x).$$

14) isodual derivative of fourteenth kind

$$df(x^d)/^d dx^d = -\frac{df(x^d)}{dx^d} = -\frac{df(-x)}{d(-x)} = -f'(-x).$$

15) isodual derivative of fifteenth kind

$$df(x^d)/^d d^d x = -\frac{df(x^d)}{d^d x} = -\frac{df(-x)}{-dx} = \frac{df(-x)}{dx} = -f'(-x).$$

16) isodual derivative of sixteenth kind

$$df(x^d)/^d dx = -\frac{df(x^d)}{dx} = -\frac{df(-x)}{dx} = f'(-x).$$

8.6 Isodual inetgrals

We suppose that F is an isodual function of first, second or third kind, defined in D and its isodual original is integrable in D. Then we have the following possibilities.

1) isodual integral of first kind $\int^d F \times^d d^d x^d = -\int F \times^d d^d x^d$.

2) isodual integral of second kind $\int^d F \times^d d^d x = -\int F \times^d d^d x$.

3) isodual integrals of third kind $\int^d F \times^d dx^d = -\int F \times^d dx^d$.

4) isodual integral of fourth kind $\int^d F \times^d dx = -\int F \times^d dx$.

5) isodual integral of fifth kind $\int^d F d^d x^d = -\int F d^d x^d$.

6) isodual integral of sixth kind $\int^d F d^d x = -\int F d^d x$.

7) isodual integral of seventh kind $\int^d F dx^d = -\int F dx^d$.

8) isodual integral of eighth kind $\int^d F dx = -\int F dx$.

9) isodual integral of ninth kind $\int F \times^d d^d x^d$.

10) isodual integral of tenth kind $\int F \times^d d^d x$.

11) isodual integral of eleventh kind $F \times^d dx^d$.

12) isodual integral of twelfth kind $\int F \times^d dx$.

13) isodual integral of thirteenth kind $\int F d^d x^d$.

14) isodual integral of fourteenth kind $\int F d^d x$.

15) isodual integral of fifteenth kind $\int F dx^d$.

16) isodual integral of sixteenth kind $\int F dx$.

Below we will give explicit expressions of the isodual integrals for every class of isodual functions.

8.6.1 Isodual functions of first kind

Here $F = f^{ddd}$. Then for $x \in D$ we have

1) isodual integral of first kind

$$\int^d f^d(x^d) \times^d d^d x^d = -\int (-f(-x))(-1)(-d(-x)) = -\int f(-x)dx.$$

2) isodual integral of second kind

$$\int^d f^d(x^d) \times^d d^d x = -\int (-f(-x))(-1)(-dx) = \int f(-x)dx.$$

3) isodual integral of third kind

$$\int^d f^d(x^d) \times^d dx^d = -\int (-f(-x))(-1)d(-x) = \int f(-x)dx.$$

4) isodual integral of fourth kind

$$\int^d f^d(x^d) \times^d dx = -\int (-f(-x))(-1)dx = -\int f(-x)dx.$$

5) isodual integrals of fifth kind

$$\int^d f^d(x^d) d^d x^d = -\int (-f(-x))(-d(-x)) = \int f(-x)dx.$$

6) isodual integrals of seventh kind

$$\int^d f^d(x^d)d^d x = -\int (-f(-x))(-d(x)) = -\int f(-x)dx.$$

7) isodual integrals of seventh kind

$$\int^d f^d(x^d)dx^d = -\int (-f(-x))d(-x) = -\int f(-x)dx.$$

8) isodual integrals of eight kinds

$$\int^d f^d(x^d)dx = -\int (-f(-x))dx = \int f(-x)dx.$$

9) isodual integrals of ninth kind

$$\int f^d(x^d) \times^d d^d x^d = \int (-f(-x))(-1)(-d(-x)) = \int f(-x)dx.$$

10) isodual integral of tenth kind

$$\int f^d(x^d) \times^d d^d x = \int (-f(-x))(-1)(-dx) = -\int f(-x)dx.$$

11) isodual integrals of eleventh kind

$$\int f^d(x^d) \times^d dx^d = \int (-f(-x))(-1)d(-x) = -\int f(-x)dx.$$

12) isodual integrals of twelfth kind

$$\int f^d(x^d) \times^d dx = \int (-f(-x))(-1)dx = \int f(-x)dx.$$

13) isodual integrals of thirteenth kind

$$\int f^d(x^d)d^d x^d = \int (-f(-x))(-d(-x)) = -\int f(-x)dx.$$

14) isodual integrals of fourteenth kind

$$\int f^d(x^d)d^d x = \int (-f(-x))(-dx) = \int f(-x)dx.$$

15) isodual integrals of fifteenth kind

$$\int f^d(x^d)dx^d = \int (-f(-x))d(-x) = \int f(-x)dx.$$

16) isodual integrals of sixteenth kind

$$\int f^d(x^d)dx = -\int f(-x)dx.$$

8.6.2 Isodual functions of second kind

Here $F = f^{dd}$. Then for $x \in D$ we have

1) isodual integral of first kind

$$\int^d f^d(x) \times^d d^d x^d = -\int (-f(x))(-1)(-d(-x)) = -\int f(x)dx.$$

2) isodual integral of second kind

$$\int^d f^d(x) \times^d d^d x = -\int (-f(x))(-1)(-dx) = \int f(x)dx.$$

3) isodual integral of third kind

$$\int^d f^d(x) \times^d dx^d = -\int (-f(x))(-1)d(-x) = \int f(x)dx.$$

4) isodual integral of fourth kind

$$\int^d f^d(x) \times^d dx = -\int (-f(x))(-1)dx = -\int f(x)dx.$$

5) isodual integrals of fifth kind

$$\int^d f^d(x)d^d x^d = -\int (-f(x))(-d(-x)) = \int f(x)dx.$$

6) isodual integrals of seventh kind

$$\int^d f^d(x)d^d x = -\int (-f(x))(-d(x)) = -\int f(x)dx.$$

7) isodual integrals of seventh kind

$$\int^d f^d(x)dx^d = -\int(-f(x))d(-x) = -\int f(x)dx.$$

8) isodual integrals of eight kinds

$$\int^d f^d(x)dx = -\int(-f(x))dx = \int f(x)dx.$$

9) isodual integrals of ninth kind

$$\int f^d(x) \times^d d^d x^d = \int(-f(x))(-1)(-d(-x)) = \int f(x)dx.$$

10) isodual integral of tenth kind

$$\int f^d(x) \times^d d^d x = \int(-f(x))(-1)(-dx) = -\int f(x)dx.$$

11) isodual integrals of eleventh kind

$$\int f^d(x) \times^d dx^d = \int(-f(x))(-1)d(-x) = -\int f(x)dx.$$

12) isodual integrals of twelfth kind

$$\int f^d(x) \times^d dx = \int(-f(x))(-1)dx = \int f(x)dx.$$

13) isodual integrals of thirteenth kind

$$\int f^d(x)d^d x^d = \int(-f(x))(-d(-x)) = -\int f(x)dx.$$

14) isodual integrals of fourteenth kind

$$\int f^d(x)d^d x = \int(-f(x))(-dx) = \int f(x)dx.$$

15) isodual integrals of fifteenth kind

$$\int f^d(x)dx^d = \int(-f(x))d(-x) = \int f(x)dx.$$

16) isodual integrals of sixteenth kind

$$\int f^d(x)dx = -\int f(x)dx.$$

8.6.3 Isodual functions of third kind

Here $F = f^d$. Then for $x \in D$ we have

1) isodual integral of first kind

$$\int^d f(x^d) \times^d d^d x^d = -\int f(-x)(-1)(-d(-x)) = \int f(-x)dx.$$

2) isodual integral of second kind

$$\int^d f(x^d) \times^d d^d x = -\int f(-x)(-1)(-dx) = -\int f(-x)dx.$$

3) isodual integral of third kind

$$\int^d f(x^d) \times^d dx^d = -\int f(-x)(-1)d(-x) = -\int f(-x)dx.$$

4) isodual integral of fourth kind

$$\int^d f(x^d) \times^d dx = -\int f(-x)(-1)dx = \int f(-x)dx.$$

5) isodual integrals of fifth kind

$$\int^d f(x^d)d^d x^d = -\int f(-x)(-d(-x)) = -\int f(-x)dx.$$

6) isodual integrals of seventh kind

$$\int^d f(x^d)d^d x = -\int f(-x)(-d(x)) = \int f(-x)dx.$$

7) isodual integrals of seventh kind

$$\int^d f(x^d)dx^d = -\int f(-x)d(-x) = \int f(-x)dx.$$

8) isodual integrals of eight kinds

$$\int^d f(x^d)dx = -\int f(-x)dx = -\int f(-x)dx.$$

9) isodual integrals of ninth kind

$$\int f(x^d) \times^d d^d x^d = \int f(-x)(-1)(-d(-x)) = -\int f(-x)dx.$$

10) isodual integral of tenth kind

$$\int f(x^d) \times^d d^d x = \int f(-x)(-1)(-dx) = \int f(-x)dx.$$

11) isodual integrals of eleventh kind

$$\int f(x^d) \times^d dx^d = \int f(-x)(-1)d(-x) = \int f(-x)dx.$$

12) isodual integrals of twelfth kind

$$\int f(x^d) \times^d dx = \int f(-x)(-1)dx = -\int f(-x)dx.$$

13) isodual integrals of thirteenth kind

$$\int f(x^d) d^d x^d = \int f(-x)(-d(-x)) = \int f(-x)dx.$$

14) isodual integrals of fourteenth kind

$$\int f(x^d) d^d x = \int f(-x)(-dx) = -\int f(-x)dx.$$

15) isodual integrals of fifteenth kind

$$\int f(x^d) dx^d = \int f(-x)d(-x) = -\int f(-x)dx.$$

16) isodual integrals of sixteenth kind

$$\int f(x^d) dx = \int f(-x)dx.$$

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