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An introduction to iso-, geno- and hyper-mathematics and their applications in physics, chemistry and biology

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Abstract

In this work, the author reviews recent advances on isomathematics, genomathematics and hypermathematics that were initiated by the Italian-American scientist Ruggero Maria Santilli for the invariant representation of reversible and irreversible systems of extended constituents in conditions of deep mutual entanglement with conventional linear, local and potential interactions represented by the conventional Hamiltonian H plus non-linear, non-local and non-potential interactions represented by the Santillian S in the axiom-preserving iso-, geno- and hyper-completion $A\hat{\times}B = ASB$ of the millenary old associative product of operators $A \times B = AB$. We then review and present, apparently for the first time, new advances in physics, chemistry and biology which are permitted by iso-, geno- and hyper-mathematics, but impossible for 20th century applied mathematics.

Keywords: Lie-isotopic, Lie-admissible, hyperstructural mathematic, biological structures, hadron mechanics

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1 Foreword

Many important advances in science and technology have been generated by advances in mathematics. Consider, for example, the fundamental importance of the development of differential calculus for our modern knowledge or the discovery of complex numbers, without which there would be no contemporary electronics, or quantum mechanics.

By using a language accessible to the general scientific audience, in this paper we present an introduction to isomathematics, genomathematics and hypermathematics that were initiated by the Italian-American scientist Ruggero Maria Santilli (Biographical Notes http://www.i-b-r.org/Sir-Santilli-bionotes-05-15-21.pdf) during his Ph. D, studies in the 1960's at the University of Torino, Italy [1]-[3], continued in the late 1970's at Harvard University under DOE support [4]-[7], completed at the Institute for Basic Research, Palm Harbor, Florida [8]-[15], studied by numerous scholars [16]-[28] at a number of international workshops and conferences [29]-[37] and recently applied for significant advances [81]-[73] in mathematics, physics, chemistry and biology that would not have been possible with 20th century mathematics.

The primary aim of the studies is the representation of microscopic systems, such as particles, molecules or cells, composed by *extended* constituents in condition of partial mutual penetration/entanglement with ensuing conventional/potential as well as contact/non-potential interactions, said systems having the following classification for conditions of increasing complexity:

1.1. Closed/isolated, thus time reversal invariant systems, such as stable nuclei, molecules or cells, that can be represented via the single-valued *Santilli's Lie-isotooic mathematics* [17] [26] also known as *isomathematics,* where the prefix "iso" indicates in its Greek meaning the preservation of the abstract axioms of 20th century applied mathematics.

1.2. Open/non-conservative, thus time irreversible systems, such as nuclear fusions, chemical combustions or embryos, which can be represented via the single-valued *Santilli's Lie-admissible mathematics* [23] [26], also known as *genomathematics,* where the prefix "geno" indicates in its Greek meaning the generalization of the abstract axioms of 20th century applied mathematics.

1.3. Biological systems, such as a cell or a DNA, which can be represented via the multi-valued *Santill-Vougiouklis Lie-admissible hyperstructural mathematics* [1]- [10] [45]-[47], also known as *hypermathematics* for short, where the prefix "hyper" indicates in its Greek meaning the use of generalized multi-valued operations.

Following a rudimentary review of the above new mathematics, we present intriguing new applications in physics, chemistry and biology.

2 Isomathematics

2.1 Elements of isotopic methods

Isomathematics was initiated in 1978 by R. M. Santilli [4] [5] at Harvard University for the representation of time reversal invariant systems with Hamiltonian plus contact, zero-range non-Hamiltonian interactions between extended constituents (particles, molecules or cells). The representation of the additional contact non-Hamiltonian interactions was achieved via a generalization of the conventional associative product of a set of elements $\xi : \{A, B, ...; AB = A \times B; 1\}$ into the associativity preserving, thus isototoic form called isoproduct [5]

$$A\hat{\times}B = A \times \hat{S} \times B,$$
$$A\hat{\times}(B\hat{\times}C) = (A\hat{\times}B)\hat{\times}C, \tag{1}$$

with compatible generalization of the right and left multiplicative unit 1 of 20th century mathematics into the form $\hat{1} = 1/\hat{S}$ called the isounit

$$\hat{1}\hat{\times}A = A\hat{\times}\hat{1} \equiv A \,\forall\, A \;\in\; \xi, \tag{2}$$

where the positive- definite quantity $\hat{S}$, called the *Santillian* [81] [82] [72] [73] represents the extended size of particles, molecules or cells and their non-Hamiltonian interactions with realizations for a two-body system of the type

$$\hat{S} = 1/\hat{1} =$$
$$= \Sigma_{k=1,2} Diag.(1/n_k, 1/n_k, 1/n_k, 1/n_k) \times e^{-\Gamma(t,r,p,\psi,...)} \ll 1, \tag{3}$$
$$k = 1, 2,\ \Gamma > 0,$$

where n_k^2, n_k^2, n_k^2 represent the semi-axes, n_k^2 the density and Γ the non-potential interactions of the extended constituents.

Santilli isomathematics [25] is characterized by the isotopies of the totality of 20th for century applied mathematics with no known exceptions because any treatment mixing the use of the isoproduct $A\hat{\times}B$ with the conventional product $A \times B$ is afflicted by catastrophic mathematical and physical inconsistencies [52].

As an example, the isoproduct can be obtained via a simple non-unitary transformation of the conventional product

$$U1U^{\dagger} = \hat{1} \neq 1,$$
$$U(AB)U^{\dagger} = (UAU^{\dagger})(UU^{\dagger})^{-1}(UBU^{\dagger}) = A'\hat{\times}B', \tag{4}$$
$$\hat{S} = (UU^{\dagger})^{-1},$$

and the same holds for the map of a conventional theory into its isotopic image.

However, the resulting *non-unitary theory* is structurally inconsistent when formulated on conventional spaces over conventional fields [52] and has to be reformulated with the full use of isomathematics, beginning with the reformulation of the non- unitary into an *iso-unitary transformation*

$$U = \hat{U} \times \hat{S}^{1/2},$$
$$UU^{\dagger} = \hat{U}\hat{\times}U^{\dagger} = \hat{U}^{\dagger}\hat{\times}U = \hat{1}, \tag{5}$$

and then passing to the entire formulation in terms of the isoproduct, isonumbers, isospaces, isodifferential calculus, etc.

In the following subsections, we present an introduction to the various branches of isomathemtatics that are needed for correct formulations. At this moment, we mention the important *Lie-Santilli isotheory* [5] [17], which includes the isotopies of an N-dimensional Lie algebra with Hermitean generators $\xi : \{X_i, X_j \times X_j; 1\}$

$$[X_i \hat{,} X_j] = X_i\hat{\times}\hat{X}_j - X_j\hat{\times}\hat{X}_i = C_{ij}^k\hat{\times}X_k, \tag{6}$$

and the isotopies of the corresponding Lie transformation group

$$A\left(t\right) = \Pi_k^{iX_k \times \hat{S} \times w} \times A\left(0\right) \times e^{-iw \times \hat{S} \times X_k}. \tag{7}$$

As an illustration of the significance of Santilli's isotopies, we mention that, for the case in which the parameter w represents an infinitesimal period of time t and the generator X_k is the Hamiltonian H, isotransformation isogroup (4) implies the *Heisenberg-Santilli time evolution* [5]

$$\hat{I}\frac{dA}{dt} = [A\hat{,}H] = A\hat{\times}H - H\hat{\times}A = A\hat{S}H - H\hat{S}A, \tag{8}$$

where conventional potential interactions are represented by the Hamiltonian H, while the extended size of the constituents and their non-potential interactions are represented by the Santillian $\hat{S}$. Note that Eq. (6) is invariant under anti-

Hermiticity, and therefore, can solely represent variationally nonselfadjoint (VNS) [4] time reversal invariant systems such as stable nuclei or stable molecules.

We should also note that isomathematics has permitted the progressive recovering of Einstein's determinism under strong interactions [81] [82], the first known numerically exact and time invariant representation of nuclear data [95] and nuclear stability [96], the first known attractive force between the identical electrons of a valence bond despite their extreme Coulomb repulsion [13] with ensuing first known exact representation of molecular data [65] [66], the first known mathematical representation of life [72] and other applications.

Additional applications of isomathematics in physics, chemistry and biology are outlined in Sections 8, 10 and 11, respectively.

2.2 Classification of isotopies

The characteristics of all possible topologically different isounits, first done by J. V. Kadeisvili [19] identifies all possible new mathematics:

Class I: the isounity is a positive defined Hermitian matrix.
Class II: the isounity is a negative defined Hermitian matrix.
Class III: includes preceding classes and isounits of undefined sign.
Class IV: includes previous classes and the case $\hat{1} = 0$
Class V: includes the previous classes and isounities with arbitrary characteristics thus being distribution, functions, et al.

Evidently, all the above isounits have one single value and the same holds for the isoproduct. Later on, for applications to biology, the above classification has to be extended to include multi-valued isounits and isoproducts.

2.3 Isofields

As it is well known, experimental verifications of a given theory produce numbers that, for consistency, have to be elements of the numeric field on which the theory is defined. The generalization of the basic unit 1 into the isounit $\hat{1}$ implies the loss of conventional fields with ensuing lack of consistent experimental verifications of isotopic theories.

In 1993, while visiting the Joint Institute for Nuclear Research in Dubna, Russia, Santilli [42] proved that the abstract axiom of a numeric field do not necessarily require the multiplicative unit to be the number 1 because it can be an arbitrary quantity $\hat{1}$ provided it is invertible. In this way Santilli introduced the new *isonumbers* and *isofields* that verify all axioms of a numeric field, thus being suitable for experimental verifications.

Let $F = F(a, +, \times, 0, 1)$ be a field with additive unit 0 and multiplicative unit 1. An "isofield" $\hat{F} = \hat{F}(\hat{a}, +, \hat{\times}, 0, \hat{1})$ [42] (see also monograph [20]) is a ring characterized by the operations $(+, \times)$ where $+$ is the conventional sum with unit 0 and $\hat{\times}$ is the new multiplication with isounit $\hat{1}$ whose elements $\hat{n} = n \times \hat{1}$, called isonumbers, obey the isoproduct

$$\hat{n}\hat{\times}\hat{m} = \hat{n} \times \hat{S} \times \hat{m} = (n \times m) \times 1 = c \in \hat{F}, \tag{9}$$

where $\hat{S} = \hat{1}^{-1}$ is the same Santillian of Eq. (9). Consequently, all conventional operations on numbers are similarly generalized [42] [8] [**?**].

It should be noted that, being an axiom-preserving generalization of the standard number theory, Santilli's isonumber theory is straightforward, although conceptually it requires a conceptual effort in understanding that, when projected in our three-dimensional space over the Reals, *Santilli's isonumbers* $\hat{n} = n \times \hat{1}$ *represent volumes,* as evident from the realization of the isounit from Santillian (5)

$$\hat{n} = n \times \hat{1} = n\Sigma_{k=1,2}Diag.(n_k^2, n_k^2, n_k^2). \tag{10}$$

Rather than being a mathematical curiosity, the indicated conception of isonumbers is necessary for the invariant representation of the dimensions, shape and density of the constituents to such an extent that said feature has to be carried over at all levels of treatments.

The isofields considered above are of Class I with $\hat{1} > 0$ and are at the foundation of isomathematics. Class II isofields have a negative-definite isounit $\hat{1} < 0$, characterize the *isodual isomathematics* for the description of antimatter [15] which is interconnected with isomathematics by an anti-Hermitean map called *isoduality* and indicated with an upper index d, e.g.,

$$\hat{1} > 0 \ \rightarrow \ \hat{1}^d = -\hat{1} < 0. \tag{11}$$

The isodual isofields $\hat{F}^d = \hat{F}^d(\hat{n}^d, +, \hat{\times}^d)$ are characterized by isodual numbers $\hat{n}^d = -\hat{n}^\dagger$ and isodual isoproduct with isodual Santillian $\hat{S}^d = -\hat{S}$.

2.4 Isospaces

An Euclidean space $E\,(x, \delta, R)$ with metric $\delta = Diag.(1, 1, 1) \ \in \ R$ is characterized by the Euclidean distance as an invariant quantity between two points x and y over the field of real numbers R

$$(x - y)^2 = \left(x^i - y^i\right)\delta_{ij}\left(x^j - y^j\right) \in R\,(n, +, \times)\,. \tag{12}$$

An *iso-Euclidean isospace,* also called *an Euclid-Santilli isospace,*[43] $\hat{E}\left(\hat{x}, \hat{\Delta}, \hat{R}\right)$ [43] (see also monographs [8] [25]) is an N-dimensional metric space defined on a Class III isoreal isofield $\hat{R}\,(\hat{n},+,\hat{x})$ with an $N \times N$-dimensional isounit $\hat{1}$, isocoordinates $\hat{x} = x \times \hat{1}$ and isometric $\Delta = \delta\hat{1} \in \hat{R}$ with *isodistance* between two isopoints

$$(\hat{x}-\hat{y})^2 = \left[\left(\hat{x}^i - \hat{y}^i\right)\hat{\times}\hat{\Delta}_{ij}\hat{\times}\left(\hat{x}^j - \hat{y}^j\right)\right] \times \hat{1} \in \hat{R}, \tag{13}$$

characterized by the isometric

$$\hat{\delta} = \hat{S} \times \delta. \tag{14}$$

An *iso-Minkowskian isospace,* also called a *Minkowski-Santilli isospace* first introduced in the 1983 Ref. [43], is characterized by the *isometric* $\hat{\Omega} = \hat{\eta} \times \hat{1} = (\hat{S} \times \eta)\hat{1} \in \hat{R}$ where η is the familiar Minkowskian metric $\eta = (1,1,1,-1)$ and it is at the foundation of relativistic treatments of systems of extended particles with potential and non-potential interactions.

Recall that, to characterize metric spaces over the reals, the elements of metrics δ and η must be real numbers $n \in R$. Similarly, to characterize isospaces over the isoreals, the elements of isometrics Δ and Ω must be isoreal isonumbers $\hat{n} \in \hat{R}$.

2.5 Isodifferential calculus

Let us recall that, by the mid 1990's, the virtual entirety of 20th century applied mathematics had been isotopically generalized. Nevertheless, the resulting physical theories were afflicted by serious inconsistencies with particular reference to the inability to achieve a representation invariant over time of the dimensions, shape and density of particles, molecules and cells as requested by experimental verifications.

Following years of trials and errors, Santilli finally identified in 1995 the origin of the inconsistencies as being due to the use of the conventional Newton-Leibnitz differential calculus within the context of isotopic theories because the former can only be defined at isolated points while isotopic theories are conceived to represent volumes.

This observation led Santilli to the discovery of the *isodifferential calculus,* first presented in the 1995 monograph [8], applied in the subsequent monograph [9] to the generalization of quantum into *hadronic mechanics* and extensively presented in the 1996 memoir [44] which, in essence, is a calculus defined on volumes with evident applications in all quantitative sciences, including engineering.

Let dx be the conventional differential of an Euclidean coordinate x. Santilli isodifferential calculus is characterized by all possible generalized differentials $\hat{d}\hat{x}$ of an iso-Euclidean isocoordinate $\hat{x} = x \times \hat{1}(x,...)$, whose isounit $\hat{1}$ has an explicit dependence on the local coordinate x, such to verify the limit condition

$$Lim\ \hat{d}_{\hat{1}\to 1}\hat{x} = dx. \tag{15}$$

Let us define as *isofunction* the quantity $\hat{f}(\hat{x}) = f(\hat{x}) \times \hat{1}$ with value in $\hat{F}$. Then, an infinite succession $\hat{f}_1, \hat{f}_2, \hat{f}_3, ...$ is strongly isoconvergent to a value $\hat{f}$ when:

$$lim_{k \to \infty} \hat{|} \hat{f}_k - \hat{f} \hat{|} = 0. \tag{16}$$

A strongly convergent series is also strongly isoconverging. However, a strongly isoconverging series is not necessarily strongly converging.

The isofield $\hat{E}\left(\hat{x}, \hat{\delta}, \hat{R}\right)$ with coordinates $\hat{x} = \left\{\hat{x}^k\right\},\ k = 1, ..., N$ and isometric $\hat{\delta} = \hat{S} \times \delta$ on the isoreals, admits per isounit a matrix $N \times N$ of class III, $\hat{1} = \left(\hat{I}_i^j\right) = \hat{S}^{-1} = \left(\hat{S}_i^j\right)^{-1}$ with line element

$$\hat{x}^i \hat{\delta_{ij}} \hat{x}^j = \hat{x}^i \hat{S}_i^j \delta_{jm} \hat{x}^m = \hat{x}_i \hat{\delta^{ij}} \hat{x}_j = \hat{x}^k \hat{x}_k = \hat{x}_k \hat{x}^k. \tag{17}$$

The isodifferentials of first order of covariant and contravariant coordinates are given respectivey by [8] [44] [24])

$$\begin{aligned} \hat{d}\hat{x}^k &= \hat{1}_k^i \left(x, ...\right) dx^i, \\ \hat{d}\hat{x}_k &= S_k^j \left(x, ...\right) dx_i. \end{aligned} \tag{18}$$

The isodifferentials expressed in this way allow the calculation of the corresponding isoderivatives:

$$\begin{aligned} \hat{f}'\left(\hat{a}^k\right) &= \frac{\hat{d}\hat{f}(\hat{x})}{\hat{d}\hat{x}^k}|_{\hat{x}^k = \hat{a}^k} = \hat{S}_k^i \frac{df(x)}{dx^i}|_{\hat{x}^k = \hat{a}^k}, \\ \hat{f}'\left(\hat{a}_k\right) &= \frac{\hat{d}\hat{f}(\hat{x})}{\hat{d}\hat{x}_k}|_{\hat{x}_k = \hat{a}_k} = \hat{I}_k^i \frac{df(x)}{dx_i}|_{\hat{x}_k = \hat{a}_k}, \end{aligned} \tag{19}$$

which are at the foundation of all applications of isomathematics.

Recall that integration is defined as the inverse of the differentiation, e.g., $\int dx = x$. Consequently, isointegration can be defined as being the isoinverse of isodifferentiation

$$\hat{\int} \hat{d}\hat{x} = (\int \hat{1})(S d\hat{x}) = \hat{x}. \tag{20}$$

Nineteen years later, the mathematician Svetlin Georgiev discovered Santilli's isodifferential calculus [44] and published a monumental volume of works in the field, including eight monographs [24] [25] [26].

2.6 Iso-Euclidean geometry

An *isoline* [8] [26] is defined as the image of the ordinary line on the reals under the lifting $R\left(n, +, \times\right) \to \hat{R}\left(\hat{n}, +, \hat{x}\right)$ and the position of its isopoints (images in the isospace of the points of the line) is described by the isocoordinates $\hat{x} = \hat{I}x$ whose isoorigin is $\hat{0} = \hat{I}0$

The isodistance from two isopoints is then given by

$$\hat{1}\left(\hat{x} - \hat{x'}\right)\hat{|} = \hat{1}\left(\hat{x} - \hat{x'}\right) \times \hat{S} \times \left(\hat{x} - \hat{x'}\right)\hat{|}^{\frac{1}{2}} \times \hat{1}. \tag{21}$$

The value of the distance between two points is different for the line and isoline. This result leads to a number of intriguing implications in mathematics and physics such as a newly developed propulsion method called geometric propulsion [15], which is based on the movement of an isopoint from one isocoordinate to another by altering the underlying geometry rather than the conventional motion or the point in space.

Let us consider now the three-dimensional Euclidean vector space $V[r, +, \odot,$-$R(n, +, \times)]$. The Class I isotopies of the three-dimensional Euclidean vector space, called iso-Euclidean isovector isospaces, are given by the same set of contravariant vectors $r = (x, y, z)$ reformulated as "isovectors"

$$r = (x, y, z) \to \hat{r} = r \times \hat{1} = (\hat{x}, \hat{y}, \hat{z}) = (x \times \hat{I}, y \times \hat{I}, z \times \hat{I}). \tag{22}$$

In the conversion from Euclidean vector space to iso-Euclidean isovector isospace, the following quantity is invariant [9]:

$$Length \times Unity = Isolength \times Isounity. \tag{23}$$

In the 3-dimensional space, the isounit is a matrix 3×3

$$\hat{1} = \hat{S}^{-1} = Diag\left(b_1^{-2}, b_2^{-2}, b_3^{-2}\right). \tag{24}$$

Furthermore, isoseparation coincides with conventional separation:

$$\hat{r}^2 = \left(r_k \times r^k\right) \times \hat{I} = \left(r^i \times \hat{S} \times \delta_{ij} \times r^j\right) \times \hat{1} = r^2 \times \hat{S} \times \hat{S}^{-1} = r^2. \tag{25}$$

If length and unit were invariant, instead of their product, it would be a different geometry from the isogeometry, because $\delta \to \delta \times \hat{1}$ rather than $\delta \to \delta \times \hat{S}$.

The isodistance [8] [26] from two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the isoscalar

$$\hat{D}_{12} = \hat{1}\left(\hat{r}_1 - \hat{r}_2\right)\hat{|} = \left[\left(\hat{x}_1 - \hat{x}_2\right)^2 b_1^2 + \left(\hat{y}_1 - \hat{y}_2\right)^2 b_2^2 + \left(\hat{z}_1 - \hat{z}_2\right)^2 b_3^2\right]^{\frac{1}{2}} \times \hat{I} \in \hat{R} \tag{26}$$

where $\hat{r}_1$ and $\hat{r}_2$ represent the isovectors from the origin to P_1 and P_2.

By indicating with d the Euclidean distance between two points and with $\hat{D} = D \times \hat{1}$ the corresponding isoeuclidean distance between two isounits, the following rule is valid:

$$\begin{aligned} D > d \longleftrightarrow det\hat{1} < 1, \\ D < d \longleftrightarrow det\hat{1} > 1. \end{aligned} \tag{27}$$

Consequently, an object has different dimensions and shape in Euclidean and Isoeuclidean geometries. An internal observer to the isoeuclidean isospace may appear to the external observer arbitrarily smaller or larger. Also, the isodimensions can vary in time and can be arbitrarily moved in the future or past of the external observer, as well as be in motion without the application of any force.

For example [15], the enormous distance of stars in Euclidean space can be made as small as desired in an isoeuclidean isospace.

The equation of an isoret isoline is given by one of the following forms:

$$\hat{a}\hat{\times}\hat{x}+\hat{b}\hat{\times}\hat{y}+\hat{c}\hat{\times}\hat{z}+\hat{d}=(a\times x+b\times y+c\times z+d)\times\hat{I}=0,$$

$$\left\{\begin{array}{c}\hat{x}-\hat{x}_1-\hat{p}\times\hat{a}_1=(x-x_1-pa_1)\times\hat{1}=0\\ \\ \hat{y}-\hat{y}_1-\hat{p}\times\hat{a}_2=(y-y_1-pa_2)\times\hat{I}=0,\\ \\ \hat{z}-\hat{z}_1-\hat{p}\times\hat{a}_3=(z-z_1-pa_3)\times\hat{I}=0,\end{array}\right\} \tag{28}$$

with $a,b,c,d\in R$ and $\hat{a},\hat{b},\hat{c},\hat{d}\in\hat{R}$.

The representation of an isoline in isovectoral notation is given by

$$\hat{r}_k-\hat{r}_{1k}-\hat{n}\hat{\times}\hat{a}=(r_k-r_{1k}-na_k)\times\hat{I}. \tag{29}$$

The isogeometric propulsion [15] moves a point $P_1(x_1,y_!,z_1)$ into a point $P_2(x_2,y_2,z_2)$ which lies on the same line connecting $P_1(x_1,y_!,z_1)$ to the origin by means of the following steps:

1) The geometry underlying point P_1 is isotopically raised with the resulting isodistance to another point P_2.

2) The isotopy is chosen according to the law:

$$D_{01}=\left(x_1^2b_1^2+y_1^2b_2^2\right)\times\hat{1}=d_{02}\times\hat{1}, \tag{30}$$

with the simplest possible solution

$$b_1^2=\frac{x_2^2}{x_1^2};b_2^2=\frac{y_2^2}{y_1^2};b_3^2=\frac{z_2^2}{z_1^2};Det.\hat{1}>1. \tag{31}$$

3) The geometry is then returned to the original Euclidean form.

The isogeometric propulsion shown here is a purely mathematical notion, although its possible actual realization has been studied by R. M. Santilli [15] via means which can alter the fundamental units of space and time.

In the iso-Euclidean geometry we can introduce the isoangle $\hat{a}$ as a generalization of the ordinary angle with isotrigonometric isocoordinates [8]

$$isocos\hat{a}=\frac{x_1b_1^2x_2+y_1b_2^2y_2}{(x_1b_1^2x_1+y_1b_2^2y_1+x_2b_1^2x_2+y_2b_2^2y_2)^{\frac{1}{2}}}. \tag{32}$$

By introducing the angular Santillian $\hat{S}_a = b_1 b_2$ and the angular isounit $\hat{1}_a = b_1^{-1} b_2^{-1}$ the formula of the isoangle is given by

$$\hat{a} = b_1 b_2 a = \hat{S}_a a = \hat{1}_a^{-1} a. \tag{33}$$

2.7 Isoaxioms of the iso-Euclidean geometry

The above isogeometric properties allow the identification of the isoaxioms of the isoeuclidean isogeometry (see also [8] [26]):

Isoaxiom I: There exists one and only one isostraight isoline from one isopoint to another isopoint.

Isoaxiom II: An isosegment can be prolonged continuously into an isostraight line from each end.

Isoaxioms III: For any given center and isoradius there is one and only one isosphere.

Isoaxioms IV: All isoright isoangles are equivalent.

Isoaxioms V: For each given isosegment between two isopoints there exist only two isoparallel lines, one per each isopoint, which are isoperpendicular to that isosegment.

2.8 Isodual iso-Euclidean geometries

Class I iso-Euclidean isogeometry is used in physics for characterizing matter [8].

Class II isodual isoeuclidean isogeometry is used in physics for the characterization of antimatter [15].

Class III isogeometry seems particularly suitable for applications in theoretical biology [14].

The reversal of the sign of time on conventional spaces over conventional fields leads to the violation of causality. The only known alternative to avoid this problem in the study of antimatter is Santilli's map [15] of the entire geometry into an anti-automorphic image called isoduality:

$$1 \longrightarrow 1^d = -1. \tag{34}$$

The isodual iso-Euclidean isospaths are obtained via the the isodual map of the original isovectors.

The fundamental invariants of Euclidean or iso-Euclidean geometries are iso-selfdual, that is invariant under isoduality

$$r^2 = r^{d2d},$$
$$\hat{x}^2 = \hat{x}^{d2d}. \tag{35}$$

This property has important physical implications for the representation of antimatter because it implies that its isodual space is hidden but co-existing within our space.

The equations for isoret and isodual angle are respectively given by:

$$\hat{a}^d \hat{\times}^d \hat{x}^d + \hat{b}^d \hat{\times}^d \hat{y}^d + \hat{c}^d \hat{\times}^d \hat{z}^d + \hat{d}^d = (a \times x + b \times y + c \times z + d) \times \hat{I}^d = 0,$$
$$\hat{a}^d = b_1^d b_2^d a^d = -\hat{a}. \tag{36}$$

Isoduality can represent the movement back in time in a causal way allowing for the causal representation of antimatter with Dirac's negative energies. This representation is also useful for the study of biological structures.

2.9 Representation of biological structures in isospaces

Biological structures can be represented by isospheres in isoeuclidean spaces [14]. The isosphere in a three-dimensional Euclidean space $\hat{E}\left(\hat{r}, \hat{\delta}, \hat{R}\right)$ with diagonal isounity is the isotopic image of an ordinary sphere with equation

$$\hat{r}^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = \hat{R}^2. \tag{37}$$

Similarly the isodual isosphere in space $\hat{E}^{d2d}\left(\hat{r}^{d2d}\hat{\delta}^{d2d}\hat{R}^{d2d}\right)$ has the equation

$$\hat{r}^{d2d} = \hat{x}^{d2d} + \hat{y}^{d2d} + \hat{z}^{d2d} = \hat{R}^{d2d}, \tag{38}$$

of which a particular case is the isodual sphere, namely, the image of the sphere of Euclidean space

$$\hat{r}^{d2d} = \left(-\hat{x}^2 - \hat{y}^2 - \hat{z}^2\right) \times I^d = R^2 \times I^d. \tag{39}$$

The perfect sphericity in Euclidean space is given by

$$I_x = I_y = I_z = +1, \tag{40}$$

but in general an iso-Euclidean space admits the property

$$I_x = b_1^{-2} \neq I_y = b_2^{-2} \neq I_z = b_3^{-2} \neq +1. \tag{41}$$

Spinning hadrons are represented as isospherical in the isospace although their projection on ordinary spaces have an arbitrary shape with a compact topology, thus including all possible deformations which maintain the basic rotational symmetry [9].

Anti-hadrons are instead represented as isodual isospherical because all the characteristics have changed sign, while conventionally anti-hadrons are represented with the same geometry used for hadrons, thus being insufficient for their distinction.

For the simple case of an isocircle of radius 1 in the isoplane, the coordinates are given by [8]:

$$x = isocos\hat{\theta} = b_1^{-1}cos\hat{\theta}, \tag{42}$$

$$y = isosin\hat{\theta} = b_2^{-1}sin\hat{\theta}, \tag{43}$$

$$\hat{\theta} = b_1 b_2 \theta, \tag{44}$$

$$r^2 = xb_1^2x + yb_2^2y = b_1^2 isocos^2\hat{\theta} + b_2^2 isosin^2\hat{\theta} = cos^2\hat{\theta} + sin^2\hat{\theta} = 1. \tag{45}$$

The Class I isosphere unifies the sphere and all its ellipsoidal deformations and the same holds for the Class II isodual isosphere. The Class III isosphere unifies the sphere and all quadrants. Currently known physical applications are restricted to class I and II isospheres since there is no known physical phenomenon which can alter, for example, ellipsoids in paraboloids.

3 Connections with non-Euclidean geometries

The first non-Euclidean property of iso-Euclidean isospace is that it is curved unless the isometric is independent from local coordinates, but dependent on remaining variables $\hat{\delta} = \hat{\delta}\left(t, \dot{r}, \ddot{r}\right)$.

In fact a given n-dimensional iso-Euclidean space admits the non-null Christoffel symbols [53]

$$\Gamma^l_{hk} = \frac{1}{2}\hat{\delta}^{ij}\left(\frac{\partial\hat{\delta}_{kj}}{\partial r^h} + \frac{\partial\hat{\delta}_{jh}}{\partial r^k} - \frac{\partial\hat{\delta}_{hk}}{\partial r^j}\right), \tag{46}$$

that characterize quantities such as the curvature tensor

$$R^j_{lh} = \frac{\partial\Gamma^l_{rh}}{\partial r^k} - \frac{\partial\Gamma^l_{rk}}{\partial r^h} - \Gamma^j_{qk}\Gamma^q_{lh} - \Gamma^l_{qh}\Gamma^q_{lk}, \tag{47}$$

which is identically null when the isometric is independent of local coordinates.

It should be noted that the above notion of curvature emerges when iso-Euclidean spaces are projected on our spaces over conventional fields because, isospaces over isofields are *isoflat* due to their topologically equivalence to conventional spaces.

The emergence of curvature on an isopath was unexpected, so this feature was identified 12 years after the identification of the iso-Euclidean geometry. This property allows a geometric unification of special and general relativities into the unit of relativistic quantum theories [54].

Lobacevski geometries are projections of iso-Euclidean geometries in the Euclidean space. The following transformations

$$x' = \frac{x + a}{1 + ax}, \tag{48}$$

$$y' = \frac{y\left(1 - a^2\right)^{\frac{1}{2}}}{1 + ax}, \tag{49}$$

map lines into lines and circles into circles. Hence, the Lobacevski geometry is a special case of the broad class of isotropies.

A similar situation exists for the Minkowskian, Riemannian, symplectic and other geometries which can all be described as special cases of iso-Euclidean geometries.

In summary, the geometries studied in this section are the following [8] [9]:

1) Iso-Euclidean geometry of Class I, used for the characterization of biological structures evolving forward in time;

2) Isodual Isoeuclidean geometry of Class II, used for the characterization of biological structures evolving backward in time.

3) Isogeometry of Class III, used for the characterization of biological structures requiring time-inversions.

4) Iso-Euclidean geometries of Class IV, used for biological structures with singularities.

5) Iso-Euclidean geometry of Class V, used for the most general possible isotopic representation of biological structures.

4 Isotopies of classical methods

The fundamental equations of contemporary mechanics are Newton's equations for a system of N particles with non-null masses in the second-order form [4]

$$m_a \frac{dv_{ka}}{dt} = F^{SA}\left(t, r, v\right) + F_{ka}^{NSA}\left(t, r, v\right), \tag{50}$$

$$a = 1, 2, 3...N; k = 1, 2, 3(= x, y, z); v_{ka} = \frac{dr_{ka}}{dt}; m_a \neq 0. \tag{51}$$

But classical and quantum methods are structurally insufficient for the representation of biological systems.

To begin, all contemporary classical and quantum formulations are solely applicable to local-differential systems, as demanded by the underlying topology and geometry, while we are primarily interested in non-local integral systems. Also, the above methods are solely applicable to particles represented as perennial and immutable massive points, while we are interested in representing particles as extended, non-spherical and deformable. Finally, the contemporary formulations of Lagrange's and Hamilton's equations can only represent a rather small class of Newton's equations.

Fundamental analytic problem: Identify generalized analytic methods which achieve a direct universality for the representation of all possible well behaved nonlinear, nonlocal and non-Hamiltonian systems [5].

The first solution to the above problem was achieved by the creators of analytical mechanics, Lagrange and Hamilton themselves, because they formulated their famous equations, not in the form currently used in mathematics, physics and biology, but the one with external terms that represent precisely the forces F^{NSA}, i.e. the contact forces that cannot be represented by potentials.

It should be recalled that in Newtonian mechanics the potential $U(t,r,v)$ must be linear in the velocities to avoid a redefinition of the mass,

$$U\left(t,x,v\right)=U_{k}\left(t,x\right)v^{k}+U_{0}\left(t,x\right), \tag{52}$$

and the Newton equation becomes

$$\begin{aligned}&\left\{m\tfrac{dv_k}{dt}-\tfrac{d}{dt}\tfrac{\partial U(t,x,v)}{\partial v^k}+\tfrac{\partial U(t,x,v)}{\partial x^k}-F_k^{NSA}\left(t,x,v\right)\right\}^{NSA}\\ &=\left\{m\tfrac{dv_k}{dt}-\tfrac{\partial U(t,x,v)}{\partial x^s}\tfrac{dv^s}{dt}+\tfrac{\partial U_0(t,x)}{\partial x^k}-F_k^{NSA}\left(t,x,v\right)\right\}^{NSA}=0,\end{aligned} \tag{53}$$

namely, they are not in general derivable from Lagrange's or Hamilton's equations.

The first step in the application of all isotopies is the identification of the independent variables and related basic generalized units. The independent variables in Newton's equations are time t, coordinates r and velocities v.

So, we have time isounit, space isounit and velocity isounit

$$\hat{I}_t=\hat{T}_t^{-1}, \tag{54}$$

$$\hat{I}_r=\hat{T}_r^{-1}, \tag{55}$$

with total isounit

$$\hat{I}_{tot}=\hat{I}_t\times\hat{I}_v\times\hat{I}_r. \tag{56}$$

The isotopic lifting of Newton's equations in isospace $\hat{S}(t,r,v)$ of Class III known under the name of *Newton-Santilli isoequations* [44] are given by

$$\begin{gathered}\hat{\Gamma}_k\left(\hat{t},\hat{r},\hat{v}\right)=\hat{m}\frac{\hat{d}\hat{v}_k}{\hat{d}\hat{t}}-\frac{\hat{d}}{\hat{d}\hat{t}}\frac{\partial\hat{U}\left(\hat{t},\hat{r},\hat{v}\right)}{\partial\hat{v}^k}+\frac{\partial\hat{U}\left(\hat{t},\hat{r},\hat{v}\right)}{\partial\hat{r}^k}=\\ =\hat{m}\frac{\hat{d}\hat{v}_k}{\hat{d}\hat{t}}-\frac{\partial\hat{U}_k\left(\hat{t},\hat{r}\right)}{\partial\hat{r}^l}\frac{\hat{d}\hat{x}^l}{\hat{d}\hat{t}}+\frac{\partial\hat{U}_0\left(\hat{t},\hat{r}\right)}{\partial\hat{r}^k}=0,\end{gathered}\tag{57}$$

where $\hat{m}$ is the "isotopic mass", the image of the Newtonian mass in the isospace and $\hat{d}$ is the isodifferential.

The important properties of Eqs. (57) are their direct universality for the representation of all possible Newton's equations (53) directly in the frame of the experimenter, and Eqs. (57) are indeed derivable from an isotopic variational principle [44].

The studies here presented introduce a representation of the actual extended, non-spherical and deformable forms of particles at the primitive Newtonian level, which then persists in classical analytical representations as well as in maps up to the operator form. Newton-Santilli isoequations actually achieve these goals by laying down the foundations for possible new advances in various fields. The goal is achieved through the new degrees of freedom of the generalized unity of theory that are evidently absent in conventional Newtonian, classical and quantum formulations.

As a simple example, suppose that the body considered is a rigid spheroidal ellipsoid with semiaxes $n_1^2, n_2^2, n_3^2 =$ constants. This form is directly represented by the Santillian of the theory in the simple diagonal form $\hat{T}=diag\left(n_1^2,n_2^2,n_3^2\right)$

$$m\hat{T}_k^l\frac{dv_i}{dt}-\hat{T}_k^l\frac{d}{dt}\frac{\partial U_l\left(t,r\right)}{\partial v^l}v^s+\hat{T}_k^l\frac{\partial U_0\left(t,r\right)}{\partial v^l}=0.\tag{58}$$

Moreover, the nonspherical character of the shape emerges only in the projection in ordinary spaces, because at the isotopic level all particles are represented via the isosphere, i.e., the perfect sphere in isospace

$$\hat{r}^2=\left(x^1n_1^{-2}x^1+x^2n_2^{-2}x^2+x^3n_3^{-2}x^3\right)\times\hat{I}\in\hat{R}\left(\hat{n},+,\hat{\times}\right).\tag{59}$$

The representation of shapes more complex than the spheroidal ellipsoids is possible with non-diagonal isounits. The representation of the deformations of the original shape due to motion within resistive media or other reasons, can be achieved via a suitable functional dependence of the Santillian $\hat{T}_k^i$ in velocities, pressure, etc.

In particular, Newton-Santilli isoequations permit a novel representation of nonpotential, variationally nonselfadjoint forces of the type

$$m\frac{dv_k}{dt}-F_k^{NSA}\left(t,r,v\right)=\hat{I}_k^l m\frac{d\hat{T}_l^j v_j}{dt},\tag{60}$$

while leaving unchanged the representation of conventional self-adjoint forces.

5 Isotopies of Hamiltonian mechanics

Newton-Santilli isoequations can be derived from a first-order isovariational principle [4] [9].

$$\hat{A} = \int_{t_1}^{t_2} L(t, x, v, a...)dt = \int_{\hat{t}_1}^{\hat{t}_2} L(\hat{t}, \hat{x}, \hat{v})\hat{d}\hat{t}, \tag{61}$$

along an isodifferentiable path

$$\begin{aligned} \hat{\delta}\hat{A}\,(P) &= \hat{\int}_{t_1}^{\hat{t}_2} (\hat{\delta}\hat{r}^k \tfrac{\hat{\partial}}{\hat{\partial}\hat{r}^k} + \hat{\delta}\hat{v}^k \tfrac{\hat{\partial}}{\hat{\partial}\hat{v}^k})\hat{L}(\hat{P})\hat{d}\hat{t} = \\ &= \hat{\int}_{\hat{t}_1}^{\hat{t}_2} \left(\tfrac{\hat{\partial}\hat{L}}{\hat{\partial}\hat{r}^k} - \tfrac{\hat{d}}{\hat{d}\hat{t}} \tfrac{\hat{\partial}\hat{L}}{\hat{\partial}\hat{v}^k} \right) \hat{L}\left(\hat{P}\right)(P)\,\hat{\delta}\hat{r}^k \hat{d}\hat{t}. \end{aligned} \tag{62}$$

By introducing the following isodifferentials into the isospace $\hat{S}\,(t, \hat{r}, \hat{p}) = \hat{E}\,(t) \times \hat{E}\left(r, \delta, \hat{R}\right) \times \hat{E}\left(p, \delta, \hat{R}\right)$

$$\hat{d}\hat{t} = I_t dt; \hat{d}\hat{r}^k = \hat{I}^l_k dr^l; \frac{\partial r^l}{\partial r_j} = \delta^l_j; \hat{d}\hat{p}^k = \hat{I}^l_k dp^l, \tag{63}$$

with isounits

$$I_2 = diag\left(\hat{I}_t, \hat{I}, \hat{T}\right); T_2 = diag\left(\hat{T}_t, \hat{T}, \hat{I}\right), \tag{64}$$

and the isocanonical momentum

$$\hat{p} = \frac{\hat{\partial}\hat{L}\left(\hat{t}, \hat{r}, \hat{v}\right)}{\hat{\partial}\hat{v}} = \hat{m}\hat{v}_k - \hat{U}_k\left(\hat{t}, \hat{r}\right), \tag{65}$$

the iso-Hamilton equations, also called Hamilton-Santilli isoequations [17]-[24] are given by

$$\frac{\partial \hat{r}^k}{\partial \hat{t}} = \frac{\partial H\left(\hat{t}, \hat{r}, \hat{p}\right)}{\partial \hat{p}_k}, \tag{66}$$

$$\frac{\partial \hat{p}_k}{\partial \hat{t}} = \frac{\partial H\left(\hat{t}, \hat{r}, \hat{p}\right)}{\partial \hat{r}^k}. \tag{67}$$

6 Isotopies of quantum mechanics

The isotopy of the simple quantization via a variational principle led Santilli to the identification of the isotopies of the Schrödinger equations at the foundation of hadronic mechanics first achieved in he 1983 monograph [5] which can be reviewed as follows.

Consider the isounitary operator $\hat{U}$ which describes the evolution over time of a hadronic isostate

$$\hat{\psi}' = \hat{U}\hat{\times}\hat{\psi} = \hat{e}^{i\hat{H}t}\hat{\times}\hat{\psi} = \hat{e}^{iHSt} \times \hat{I}\hat{\times}\hat{\psi} = \hat{e}^{i\hat{H}St} \times \hat{\psi}, \tag{68}$$

$$\hat{U}\hat{\times}\hat{U}^{\dagger} = \hat{U}^{\dagger}\hat{\times}\hat{U} = \hat{I}, \tag{69}$$

resulting in the *Schrödinger-Santilli isoequation*

$$\hat{H}\hat{\times}|\hat{\psi}>= \hat{E}\hat{\times}|\hat{\psi}>= E|\hat{\psi}>, \quad \hat{E} == E\hat{1} \in \hat{R}, \quad E \in R, \tag{70}$$

with isoexpectation values of an observable $\hat{H}$

$$< \hat{H} >= \frac{< \hat{\psi}|\hat{\times}\hat{H}\hat{\times}|\hat{\psi} >}{< \hat{\psi}|\hat{\times}|\hat{\psi} >}. \tag{71}$$

Note that the same Hamiltonian H has basically different eigenvalues in quantum mechanics (qm) and hadronic mechanics (hm), as transparent from the presence of the isotopic element in hadronic expressions

$$H|\psi >= E_{qm}|\psi > \hat{H}S|\hat{\psi} >= E|\hat{\psi} >, \quad E_{qm} \neq E_{hm}. \tag{72}$$

7 Elements of genotopic methods

While isotopic formulations are naturally set to represent total conservation laws under nonconservative internal effects, genotopic formulations generally admit no conserved quantity, because they have been conceived [56] to characterize time-rate-of-variations of a given quantity of which conservation is an evident particular case. Genotopies are therefore ideally suited to represent the nonconservative events such as nuclear fusions, chemical combustions or the growth of biological structures.

Santilli genotopies are based on a further extension of numbers: the genonumbers. These are characterized by the two alternating multiplications to the right and left, denoted by the symbols $>$ and $<$ respectively [56].

The ordering is compatible with other properties and axioms of number theory, such as commutativity, associativity, etc.

You can then use ordered fields to the left to represent motion backward in time $^{<}F\,(a,+,<)$ and ordered fields to the right to represent motion forward in time $F^{>}\,(a,+,>)$.

The genodifferential calculation is based on the following forward and backward genodifferentials

$$\hat{d}^{>}r_k = \hat{I}_k^{>}\hat{l}dr_l \tag{73}$$

$$^{<}\hat{d}r^k =^{<} \hat{I}_l^k dr^l. \tag{74}$$

The related genotopic elements

$$\frac{\partial^{>}}{\partial r^{k>}} = S_k^l \frac{\partial}{\partial r^l}, \tag{75}$$

$$\frac{^{<}\partial}{^{<}\partial r^k} = R_k^l \frac{\partial}{\partial r^l}, \tag{76}$$

are called the forward and backward Santillians, from which we obtain the Heisenberg-Santilli genoequations [5] for the representation of the time rate of variation of observables

$$\begin{gathered} i\tfrac{dA}{dt} = (A,H) = A < H - H > H = A^{<}SH - HS^{>}A, \\ i\tfrac{dH}{dt} = H(^{<}R - S^{>})H) \neq 0. \end{gathered} \tag{77}$$

8 Element of hyperstructural methods

Hyperstructures are some of the most complex mathematical structures conceived by mathematicians until now. Part of their complexity is due to their virtually endless variety of formulations and realizations which evidently multiply the difficulties for their selection and realization into a form suitable for applications. The need for the hyperstructures in theoretical biology is unavoidable because the isotopic and genotopic methods are effective up to a certain complexity of the systems considered.

A central mathematical property of the hyperstructures is that of being multi-valued, that is, products which traditionally have only one value, may assume a series of ordered different values. This central feature of hyperstructure has a clear potential for new frontiers in theoretical biology, because it is particularly suited to represent, say, the birth of a new cell in which the original number of entities was one and the final number of entities is two, which is precisely a realization of the notion of multivaluedness.

While the generalized units of isotopies and genotopies are given by simple single-valued quantities, hyperunits have the form

$$(I^{>}) = \left(I_1^{>}\left(r, t, \dot{r}, \ddot{r}\right) I_2^{>}\left(r, t, \dot{r}, \ddot{r}\right) ...\right), \tag{78}$$

from which we derive the hyperfield $\left(\hat{F}^{>}\right)\left(\left(\hat{a}^{>}\right), +, (>)\right)$ defined from a conventional numerical field $F\left(a, +, \times\right)$ with elements

$$(\hat{a}^{>}) = a \times (\hat{I}^{>}), \tag{79}$$

and related hypermultiplication

$$(\hat{\alpha}^{>})(>)(\hat{\beta}^{>}) = (\hat{\alpha}^{>}) \times (S_1, S_2...) \times (\hat{\beta}^{>}), \tag{80}$$

where the forward hyper-Santillian is given by $(S) = (S_1(t, r, \dot{r}...), S_2(t, r, \dot{r}...)...)$.

The forward hyperdifferential calculus, apparently introduced here for the first time, can be defined via the forward hyperdifferentials with corresponding forward hyperderivatives with the corresponding ordered multiplications.

$$\left(\hat{d}^{>}\right) r^k = \left(\hat{I}_i^{>k}\right) \times dr^i. \tag{81}$$

The first and perhaps most important implication of the hyperstructures is the generalization of the four directions of time [15] into multivalued forms of the type

1) Forward hypertime $\left(\hat{t}^{>}\right) = t \times \left(\hat{I}_t^{>}\right); \left(\hat{I}_t^{>}\right) = \left(\hat{S}_t^{-1}\right)$

2) Conjugated forward hypertime $\left({}^{<}\hat{t}\right) = t \times \left({}^{<}\hat{I}_t\right); \left({}^{<}\hat{I}_t\right) = \left(\hat{R}_t^{-1}\right) = \left(\hat{I}_t^{>}\right)^{\dagger}$

3) Backward hypertime $\left(\hat{t}^{>}\right)^d = -\left(\hat{t}^{>}\right); \left(\hat{I}^{>}\right)^d = -\left(\hat{I}^{>}\right)$

4) Conjugated backward hypertime $\left({}^{<}\hat{t}\right)^d = -\left({}^{<}\hat{t}\right); \left({}^{<}\hat{I}\right)^d = -\left({}^{<}\hat{I}\right)$

In addition to the invariant

$$Length \times Unity = Hyperlength \times Hyperunity, \tag{82}$$

another invariant characterizing time in biological systems is given by

$$(t_2 - t_1) \times \hat{T} \times (t_2 - t_1) \times \hat{I} = (t_2 - t_1)^2 \times 1, \tag{83}$$

$$(t_2 - t_1) \times \hat{T}^d \times (t_2 - t_1) \times \hat{I}^d = (t_2 - t_1)^2 \times 1, \tag{84}$$

$$
\begin{gathered}
(t_2 - t_1) \times \hat{T}^{>} \times (t_2 - t_1) \times \hat{I}^{>} = \\
(t_2 - t_1) \times^{<} \hat{T} \times (t_2 - t_1) \times^{<} \hat{I} = \\
(t_2 - t_1) \times \hat{T}^{>d} \times (t_2 - t_1) \times \hat{I}^{>d} = \\
(t_2 - t_1) \times^{<} \hat{T}^{d} \times (t_2 - t_1) \times^{<} \hat{I}^{d} = \\
(t_2 - t_1)^2 \times 1.
\end{gathered} \tag{85}
$$

By means of hyperunity one can generalize the action as a hyperaction and define the forward hypernewton equations

$$
(\hat{m}^{>}) \frac{\left(\hat{d}^{>}\right)(\hat{v}_k^{>})}{\hat{d}\hat{t}^{>}} - (F_k)\left(\hat{t}^{>}, \hat{r}^{>}, \hat{v}^{>}\right); k = x, y, z. \tag{86}
$$

With an analogous logic, we also arrive at the equations hyperlagrange and hyperhamilton equations.

9 Applications in physics

As we have seen, the implications of hadronic mechanics in physics are numerous and can only be examined to a limited extent in this work, including:

9.1. Bohm's hidden variables

Hadronic mechanhcs was conceived in 1978 [83] (see also the subsequent studies [5] [9]) to provide explicit and concrete realizatons of Bohm's hidden variables [57] [58] via the Santillian S of isoproduct (1) and realizations of type (5) [9] [59] [60], as a result of which *Bohm's variables are hidden in the associative product,* by therefore confirming Bohm's view on the possible recovering of classical determinism for operator formulations.

9.2. Entanglements

The main objective of hadronic mechanics is the representation of extended wave packets in conditions of deep mutual entanglements with ensuing non-linear, non-local and non-potential interactions that, evidently, are not representable via the Hamiltonian.

R. M. Santilli [61] has first proved the exact character of Einstein's criticism of "quantum entanglements" (i.e., the representation of entanglements via quantum mechanics) because, when the entangled particles are at large mutual distances r, there is no known interconnecting interaction derivable from a potential $V(r)$.

In this case, the Hamiltonian of quantum mechanics $H = K + V$ reduces to the kinetic energy $H = K+V = K = \Sigma_{k=1,2}(1/2m_k)\, p_k^2$ and consequently, *quantum mechanics can only represent entangled particles as being free*.

Santilli has then shown that the non-Hamiltonian interactions of particle entanglements admit a full quantitative representation via the Santillian S of the isoassociative product (1) [61] [73], by therefore achieving a quantitative representation of particle entanglements in reversible, irreversible and multi-valued conditions at the foundation of seminal advances in physics, chemistry and biology [62].

9.3. Isorelativities

Isotopies allow axiom-preserving generalizations of Galileo and Einstein special relativity into broader relativities for the representation of particles in deep mutual entanglement, nowadays known as *iso-Galilean and isospecial relativities* or *isorelativities* for short, that were first proposed by R. M. Santilli in paper [63] of 1983 on the Lorentz-Santilli generalization $\hat{SO}(3.1)$ for extended particles of the Lorentz symmetry $SO(3.1)$ for point-like particles of special relativity. Santilli then studied isospecial isorelativity in numerous works, including Refs. [11] [12] [54] and others.

The basic isoaxioms for the relativistic motion of extended particles in the k-direction of inhomogeneous and anisotropoic physical media with local density n_{4k}, are:

ISOAXIOM-I: The speed of light within (transparent) physical media is given by:

$$C_k = \frac{c}{n_{4k}}. \tag{87}$$

ISOAXIOM-II: The maximal causal speed within physical media is given by:

$$V_{max,k} = c\frac{n_k}{n_{4k}}. \tag{88}$$

ISOAXIOM-III: The addition of speeds within physical media follows the isotopic law:

$$V_{tot,k} = \frac{\frac{V_{1k}}{n_k} + \frac{V_{2k}}{n_k}}{1 + \frac{V_1 V_2}{c^2}\frac{n_{4k}^2}{n_k^2}}. \tag{89}$$

ISOAXIOM-IV: The dilation of time, the contraction of lengths, the variation of mass and the energy equivalence within physical media follow the isotopic laws:

$$t'_k = \hat{\gamma}_k t, \tag{90}$$

$$\ell'_k = \hat{\gamma}_k^{-1}\ell, \tag{91}$$

$$m'_k = \hat{\gamma}_k m, \tag{92}$$

$$\hat{E}_k = mV^2_{max,k} = mc^2\frac{n_k^2}{n_{4k}^2}, \tag{93}$$

where:

$$\hat{\beta}_k = \frac{\frac{v_k}{n_k}}{\frac{c}{n_{4k}}}, \tag{94}$$

and

$$\hat{\gamma}_k = \frac{1}{\sqrt{1-\hat{\beta}_k^2}}. \tag{95}$$

ISOAXIOM-V: The frequency shift within physical media follows the isotopic law (for null aberration)

$$\omega_k = \hat{\gamma}\left(1 - \hat{\beta}_k cos\left(\hat{\alpha}\right)\right)\omega. \tag{96}$$

For experimental verifications of the above isoaxioms in transparent media, including the first known causal representation of the superlumuinal electrons originating the Cherenkov light, see [54] and [62].

Santilli's isotopic formulation of gravitation, including a geometric representation of interior and exterior gravitation and their apparent resolution of some of century-old open problems of Einstein's general relativity are presented in Sections 8.2 and 8.3 of [54] (see also the problem of the origin of gravitation outlined in point 3 of Sect. 9.4).

9.4. Isocosmology

Isoaxiom V has been verified by a series of accurate experiments in Earth's atmosphere done in the USA and Europe (see [74] and papers quoted therein), where Einstein's relativity is inapplicable because the axioms of special relativity require our space-time to be homogeneous and isotropic, while our atmosphere is inhomogeneous and anisotropic, thus implying the inapplicability of the geometrical foundations of special relativity in favor of the Minkowski-Santilli isogeometry and related isotopic covering of the Poincaré symmetry $\hat{\mathcal{P}}(3.1)$ [53].

The Doppler-Santilli Isoaxiom V admits the expansion for the frequency shift of light propagating in our atmosphere

$$\Delta\omega = \pm\frac{v_k}{c} \pm H_k d, \tag{97}$$

where: H_k is (approximately) a constant; d is the distance covered by light; $\pm\frac{v}{c}$ is the standard Doppler effect, and $\pm Hd$ is the new Santilli isoshift describing the decrease (increase) of the frequency for light propagating within a gaseous medium at low (high) temperature without any relative motion between the source, the medium and the observers.

In particular, the latter effect known as isoredshift (isoblueshift) essentially consists of the release of energy by light to (acquisition of energy by light from) the medium, thus requiring no relative motion, and these new effects have received a rather wide experimental verification, firstly, for individual laser lights, and secondly, for the solar spectrum [62].

Santilli's isoredshift phenomenon is of fundamental importance for our understanding of nature and the cosmos because it provides a consistent alternative explanation of the cosmological redshift to the current interpretation via the expansion of the universe. In fact, said phenomenon suggests that light loses energy on its way to the solar system due to interaction with the intergalactic medium mostly composed by extremely cold Hydrogen gas.

Within the indicated experimental setting, the background microwave radiation results to be due to the continuous release by intergalactic Hydrogen of the energy continuously acquired by intergalactic light during its travel to reach Earth.

Independently from the experimenta verification on Earth [74], Santilli's isoredshift representation of the cosmological redshift of galactic light and related cosmological isosymmetry $\hat{\mathcal{P}}(\ni.\infty$, also known as *Santilli's isocosmology* [64], are preferable over the representation via the expansion of the universe because the latter implies the Big Bang theory, namely, a theory with a multitude of problems still unresolved after decades of studies [110].

It should also be recalled that Einstein, Hubble, Hoyle, Zwicky, Fermi, de Broglie and other famous scientists died without accepting the expansion of the universe because the representation of Hubble's law on the cosmological redshift of galactic light with the Doppler axiom of special relativity,$z = Hd = \frac{v}{c}$ (where now H is Hubble's constant and d is the travel of galactic light to reach us) implies a return to the Middle Ages with Earth at the center of the universe due to the dependence of the redshift from all possible *radial directions from Earth* [62] [110].

9.5. Heisenberg s uncertainty principle

R. M. Santilli has addittionally shown that Heisenberg's uncertainty principle for electromagnetic interactions is inapplicable for extended nucleons under strong nuclear interactions by therefore creating the grounds for advances in nuclear physics that would be otherwise impossible.

Santilli first submitted the hypothesis of the inapplicability of Heisenberg's uncertainty principle in nuclear physics in the 1981 paper [75] written at Harvard University under DOE support because, in view of the strength of nuclear forces, the standard deviations for nuclear constituents must be *smaller* than the corresponding deviations for atomic electrons orbiting in vacuum around their nuclei, which hypothesis was confirmed in the 1994 paper [76] on the uncertainties under gravitational collapse. The progressive recovering of Einstein's determinism with the increase of the nuclear density was proved in the 2019 paper [77] and elaborated in details in Refs. [78]-[80] (see independent reviews [81] [82]), resulting in a new principle originally submitted under the name of *Einstein's isodeterminism* [77] and nowadays known as *hadronic isodeterminism*.

In summary, the recovery of determinism is evidenced by the formula

$$\begin{aligned} \Delta r \Delta p \simeq \frac{1}{2} < \hat{\psi}\left(\hat{r}\right) \mid * \, [\hat{r}, \hat{p}] * \, |\hat{\psi}\left(\hat{r}\right) >= \\ = \frac{1}{2} < \hat{\psi}\left(\hat{r}\right) |\hat{S}[\hat{r}, \hat{p}]\hat{S}|\hat{\psi}\left(\hat{r}\right) >= \frac{1}{2} \int_{-\infty}^{+\infty} \hat{\psi}\left(\hat{r}\right)^{\dagger} \hat{S} \hat{\psi}\left(\hat{r}\right) \hat{d}\hat{r} = \frac{1}{2} S \end{aligned} \tag{98}$$

generalising Heisenberg's uncertainty principle in conditions where quantum mechanics is unapplicable.

9.6. Pauli's excliusion principle

An important contribution by R. M. Santilli in need of experimental verification has been the apparent lack of universal validity of Pauli's exclusion principle in physics.

The studies in the field initiated with the 1978 paper [83] written at Harvard University under DOE support indicating the need for experimental verifications of Pauli's exclusion principle beginning with its title. The studies were then continued in the 1990 paper [84] and completed in the 1998 paper [85] on the generalized structure of the notion of spin for extended particles under contact non-Hamiltonian interactions (see also [79]).

The inapplicability of Pauli's exclusion principle in nuclear physics was readily suggested on axiomatic grounds from the inapplicability of quantum mechanics for nucleons under deep entanglements, with ensuing non-linear, non-local and non-potential interactions [49], under which conditions there is no possibility of a consistent formulation of said principle.

The progressive inapplicability of Pauli's exclusion principle in atomic structures is predicted from the increase of the entanglement of orbital electrons with increase of the atomic number due to the experimentally established decrease of inter-orbital distances.

Said entanglements of orbital electrons cause a structural generalization of the $SU(2)$-spin Lie symmetry into the Lie-Santilli $\hat{SU}(2)$-spin isosymmetry [85] whose spin for the fundamental representation remains $1/2$, but the isostates are generalized due to an explicit and concrete realization of Bohm's hidden variables by the Santillian S which has to be included in *all* possible products [84], with ensuing lack of general antisymmetric character of the isostates.

For example, to represent the entanglement of orbital electrons, the antisymmetric state $|\psi(x, y)\rangle = \Sigma_{x,y} N(x, y)|x, y\rangle, \;\; N(x, y) = -N(y, x)$ representing

Pauli's exclusion principle on a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$, has to be mapped into the isotopic form $|\hat{\psi}(\hat{x},\hat{y})\rangle = \Sigma_{\hat{x},\hat{y}} M(\hat{x},\hat{y}) S|\hat{x},\hat{y}\rangle, \;\; \hat{x} = x\hat{1}, \;\hat{y} = y\hat{1}$ on an iso-Hilbert isospace $\hat{\mathcal{H}}$ over the isofield $\hat{\mathcal{C}}$, where the isocoefficient M may remain antisymmetric on $\hat{\mathcal{H}}$ over $\hat{\mathcal{C}}$, but the projection of the term MS on our space-time is, in general, no longer antisymmetric.

9.7. The structure of unstable particles

It should be indicated that, thanks to the new mathematics, relativities and physical laws, hadronic mechanics has permitted basically new quantitative *structure models* of unstable particles which: 1) Have physical constituents produced free in their decays; 2) Admit mechanisms for the spontaneous character of the decays; and 3) Achieve compatibility with conventional $SU(3)$-type *classifications* of particles, including:

1) The representation of all characteristics of the π^0 meson, including charge radius, mass, spin, charge, magnetic moment, mean life and spontaneous decays, in which the π^0 is a hadronic bound state of an electron e^- and a positron e^+ ("compressed positronium" down to $1\ fm$ of mutual distances), whose spontaneous decay is due to electron-positron annihilation (see the original proposal in Section 5.1 of the 1978 paper [83], recent update [80] and papers quoted therein). The $\pi^\pm$ mesons are characterized by a hadronic bound state of a π^0 meson and an $e^\pm$ electron totally immersed/entangled in its structure, with ensuing Lie-isotopic angular momentum equal but opposite to the spin for stability and null total angular momentum [83] [80]. The remaining mesons result to be hadronic bound states of $e^\pm$, π^0 and $\pi^\pm$ in conditions of total mutual immersion/entanglement.

2) The representation of all characteristics of the $\mu^\pm$ leptons, including charge radius, mass, spin, charge, *anomalous* magnetic moment, mean life and spontaneous decay, in which the $\mu^\pm$ leptons are bound states of an electron-positron pair and an $e^\pm$ at mutual distances bigger than $1\ fm$, i.e., being such for the ground state of the angular momentum to be null, resulting in a state with spin $1/2$ [83] [80]. The first and only known numerically exact and time invariant representation of the *anomalous magnetic moment of the muons* via relativistic hadronic mechanics is presented in the recent paper [86]. The structure of the remaining leptons follows bootstrap guidelines similar to those for mesons.

3) Stars initiate their lives as aggregates of Hydrogen that increase by accretion during their travel in Hydrogen rich intergalactic spaces. When the pressure in their interior reaches sufficient values, stars synthesize the neutron n as a "compressed Hydrogen atom" [87], i.e. as an electron e^- compressed inside the hyperdense proton p^+. Thereafter, Stars initiate the synthesis of Deuterons D from a proton and a neutron, after which they initiate the production of light.

In view of the above fundamental character, R. M. Santilli dedicated decades of research to the neutron synthesis and did achieve the first and only known, exact and invariant representation of *all* characteristics of the neutron as a hadronic bound state of an electron and a proton at the non-relativistic and relativistic levels (see the review in [62] and original references quoted therein).

Jointly, Santilli dedicated an additional decade of experimentations on the laboratory synthesis of the neutron from a commercially available Hydrogen gas [89], resulting in the production and sale by the U.S. publicly traded company Thunder Energies Corporation of the Directional Neutron Source (DNS), namely, an equipment producing on demand a flux of thermal neurons synthesized from a Hydrogen gas in the desired direction and with the desired intensity [90].

By recalling that stars initiate their lives as being aggregates of Hydrogen and, thanks to the new hadronic uncertainty principle, Santilli achieved a rigorous reduction of all matter in the universe to protons and electrons in conditions of increasing complexities, thus requiring more advanced mathematics [44].

Additionally, in his 1974 paper [92] while being in the faculty of the Institute for Theoretical Physics at MIT, Santilli proved the Poincaré hypothesis that the exterior gravitational field $G_{\mu\nu} = R_{\mu\nu} = \frac{1}{2}Rg_{\mu\nu}$ of a *neutral* massive body is entirely representable via the electromagnetic *tensor* $T^{int}_{\mu\nu}$ characterized by the rotating interior changes.

Note that the Poincaré-Santilli origin of gravitation cannot be consistently studied via Einstein's general relativity because: the former deals with the interior gravitational problem while the latter solely deals with the exterior gravitational problem; the former requires a "source" of the gravitational field which is prohibited by the latter as a necessary condition io represent gravitation via the curvature of space; the former entirely reduces the mass/energy of a "neutral" body to the interior electromagnetic field $G_{\mu\nu} = KT^{int}_{\mu\nu}$, while the latter do admit the field $G_{\mu\nu} = R_{\mu\nu} = \frac{1}{2}Rg_{\mu\nu} = K'T^{Ext}_{\mu\nu}$ in the right-hand-side of the field equations, but only for the *total charge* of the body whose contribution to gravitational field, e.g., of Earth, is of the order of $T^{ext}_{\mu\nu} \approx 10^{-35}G_{\mu\nu}$, thus being ignorable for the study of the origin of gravitation; and for other insufficiencies of general relativity studied in works [93] [94].

In view of the above momentous implications, the supporters of the conjecture that unidentifiable point-like quarks are the physical constituents of the hyperdense neutron, to achieve credibility, should prove that at the time of the neutron synthesis the permanently stable proton and electron disappear to be replaced by the hypothetical quarks and additionally, at the time of the spontaneous decay of the neutron, the hypothetical quarks shuld disappear and be replaced by the permanently stable proton and electron [88].

9.8. The structure of stable nuclei

Thanks to all preceding mathematical, theoretical and experimental advances, Santilli additionally achieved the first and only known exact and invariant hadronic representation of *all* characteristics of the Deuteron [95], as well as its stability [96], despite the natural instability of the neutron and the strongly repulsive protonic Coulomb forces, resulting in a model which is extendable to all stable nuclei [97].

9.9. HyperFusions

As it is well known, the repulsive Coulomb force between natural, thus positively charged nuclei, acquires such extreme values at mutual distances of the order of $1\ fm$ to prohibit the achievement of sustainable and controllable nuclear fusions.

To initiate studies toward the resolution of the above Coulomb barrier, Santilli proposed the synthesis via specially designed hadronic reactors of pseudo-nuclei, which are hadronic bound states of electrons and nuclei that, for sufficiently small values of the atomic number, are negatively charged, thus being attracted by natural positively charged nuclei, resulting in inevitable nuclear fusions called HyperFusions [98]- [99].

The existence of Santilli's pseudo-nuclei has been confirmed by experiments [100] which also provide a disproof of Heisenberg's uncertainty principle in nuclear physics. The sustainable production of excess thermal energy over the used energy has been independently certified in Refs. [101]-[104] jointly with the lack of harmful radiations.

10 Applications in chemistry

The application of hadronic mechanics to chemistry has naturally opened the way to new advances, marking the beginning of "hadronic chemistry" [13], with new interpretations and predictions in the field of chemical bonds as well as phenomena such as superconductivity.

10.1. Molecules

Following the achievement of maturity in 1996 of isomathematics [44] and related isomechanical branch of hadronic mechanics [8] [9], Santilli confronted the central open problem of quantum chemistry: the lack of representation in one century of the *attraction* between identical electron pairs in valence bonds as existing in nature, with ensuing lack of a quantitative representation of molecular structures. In fact, the *repulsive* Coulomb force between identical electron pairs at valence mutual distances reaches such astronomical values (of about $230\ Newtons$) that cannot be overcome by any existing, quantum chemical valence bond.

Santilli frst wrote a series of papers reviewed from Chapter 4 on of the 2001 monograph [13] on the generalization of the quantum notion of point like *particles* under the Poincaré symmetry $P(3.1)$ into extended *isoparticles* in deep mutual entanglement under the spinorial covering of the Poincaré symmetry $\hat{\mathcal{P}}(3.1)$ [53], Santilli then achieved the first and only known *attraction* between the identical electrons of a valence pair [13] reviewed in details in Appendix B, resulting in the new notion of valence bonds called *strong isovalence bonds* which are characterized by very small values of the Santillian S, with ensuing rapid convergence of perturbative series.

Thereafter, Santilli and the chemist Donald Shillady achieved the first known exact and invariant representation of binding energies as well as electric and magnetic moments of the Hydrogen [65] and water [66] molecules.

10.2. Magnecules

Following the achievement of a quantitative representation of molecular structures, Santilli confronted a second, century old, open problem in chemistry particularly important for fuel combustions and clean environmental usages: the rather general belief that the valence bond and related chemical species of *molecules* can represent the bond of all possible clusters existing in the universe.

Following due mathematical, theoretical and experimental studies, after two centuries of dominance in chemistry by the chemical species of molecules, Santilli introduced in the 1998 memoir [107] the new chemical species of *molecules* which are stable clusters comprising individual atoms (such as H, O, C), dimers (like $H-O, C-H$), and conventional molecules (e.g., $H_2, C-O, H-O-H$) bonded together by opposing magnetic polarities of toroidal polarizations of atomic orbitals, either existing in nature, e.g., in biological structures, or industrially induced in spherical atomic distributions (see [13] and quoted references). Note that the term "magnecules" was introduced by Santilli to distinguish them from conventional "molecules" as well to emphasize the magnetic origin of their bond.

To substantiate the existence of magnecules, Santilli conducted systematic measurements on gases created by DC arcs submerged in liquids [107] and the use of a gas chromatograph-mass spectrometer (GC-MS) equipped with an infrared detector (IRD). The key findings from these experiments include the detection of peaks with mass up to 1,000 atomic mass units (a.m.u.) that cannot be explained via molecular bonds, all detected peaks having no IRD signature against the existence of distinct IR signatures for molecules.

Additionally, magnecules demonstrated stability at ambient temperatures and pressures and exhibited properties not accountable via molecular bonds, such as: anomalous combustion temperatures; anomalous energy content; anomalous adhesion to substances; and other features.

10.3. Liquids

Following the achievement of a quantitative representation of molecular structures and the discovery of the new chemical species of magnecules, Santilli confronted a third, century old, open basic problem in chemistry: the attractive force between water molecules in their liquid state.

The transition of the water states from liquid to gas indicates that the attractive force between water molecules ceases to exist at the boiling temperature, by therefore suggesting that liquid water has a magnecular structure [70] and, more specifically, that $100^0\ C$ is the Curie temperature of the water magnecular bond.

An inspection of the water molecule $H-O-H$ indicated that the two H-atoms of the water molecule have precisely the toroidal distribution of magnecular structures, thus having magnetic polarities North-South along the two symmetry axes that can bond with opposite polarities resulting in the conception of liquid water as a state in which each water molecule is bonded to four other water molecules.

Needless to say, the confirmation of the magnecular structure of liquid water would imply that all liquids have the same structure [70]. As an example, the elementary constituents of ordinary gasoline are gasoline molecules while the gasoline liquid state would be characterized by magnecular bonds of gasoline molecules.

10.4. Magnecular fuels

The above mathematical, theoretical and experimental studies lead to the industrial development and sale by the U. S. publicly traded company Magnegas Corporation and others of a variety of new gaseous fuels with magnecular structure synthesized from liquid feedstock among which we mention: the new gaseous fuel magnegas with complete combustion (no combustible contaminants in the exhaust, i.e., no C-O, no HC, etc.) [13]; the new chemical species of Magne-Hydrogen whose specific weight and energy content are a multiple those of the conventional Hydrogen $H-H$ [67]; the new gaseous and combustible form of water (called by Santilli the HHO gas) which contains all needed oxygen for combustion [68]; and others (for details, see monograph [108]).

10.5. HyperCombustion

The combination of the studies reported in this and in the preceding section lead Santilli to the proposal of the new *hypercombustion* [71] which consists in the clean combustion of fossil fuels via specially designed sparks suitable to activate the HyperFusions of Sect. 9.9 in parts per millions without harmful radiations, by therefore multiplying the power output of fossil fuels, whose sole gaseous exhaust is Carbon DioXide CO_2 that can be collected and recycled into Carbon and Oxygen via the hadronic reactors of Sect. 9.9.

11 Applications in biology

The biological studies reported in this section were initiated in the 1990's by the Australian conciologist C. Illert [27] with his discovery that the time evolution of sea shells cannot be consistently represented in the Euclidean space of our sensory perception because requiring a generalized space with two values for each coordinate (see Appendix A for details).

Independently from C. Illert, R. M. Santilli introduced in the early 1980's iso-, geno- and hyper- mathematics for novel physical advances and proposed in 1994 their applications to biology [14].

Following initial studies by a number of biologists, the next important event in the field occurred with the first and only known quantitative representation of life intended as the difference between inorganic and organic molecules by R. M. Santilli and the Greek mathematician Thomas Vougiouklis [72].

The next significant event has been the application to biology by E. Velardo and F. Inglese of the lifelong physical studies on entanglements by R. M. Santilli [111] (see also the related studies on correlations of Appendix C), which studies have shown that a number of anomalous actions by biological entities, which are generally considered to be supernatural, can in fact be quantitatively represented as due to the entanglement of the wave packets of biological constituents.

Additionally, Santilli's magnecules (Sect. 10.3) have important applications in biology due to their crucial role for the liquid state of water, and therefore, for the entire body of a living organism.

Additionally, magnecules have an important role for expected new advances in other biological aspects. For instance, it has been shown that a human DNA does not have six billions valence electrons necessary to bond its billion atoms into a collection of molecules. Consequently, in Santilli's view, the DNA cannot be consistently qualified as being a "molecule" because it appears to have an extremely complex chemical composition including, of course, molecules, but also magnecules as outlined in Sect. 10, as well as additional, more complex chemical species currently under study, such as that of the multivalued "hypermagnecules".

Also, A. L. Kalcker has advanced the hypothesis that HHO magnecule plays a fundamental role in physiology in the oxidation and energy production of the organism [69].

Needless to say, Santilli's iso geno- and hyper-mathematics are expected to have intriguing applications to our notion of time. Consider, for instance, our instinctive perception of time. When investigating any physical, chemical or biological structure, we instinctively assume that they have our own time.

Biological structures represented with isotopic, genotopic and hyperstructural methods generally have their own/intrinsic time which is generally different than our own perception of time [14], both in its rate of flow and in its direction, as well as being of progressively increasing complexity depending on the complexity of the system considered.

Assume the isotopic representation of certain biological structures. The problem which has to be addressed and resolve in due time is whether we are referring to a purely mathematical property or to an actual intrinsic behavior of nature.

For these studies, consider the following isoinvariant [8]

$$D \times I_t = \hat{D} \times \hat{I}_t, \tag{99}$$

where: D is the considered time interval; I_t is our unit of time (e.g., $I_t = 1\ sec$); $\hat{D}$ and $\hat{I}_f$ are the corresponding isotopic forms.

From this invariant it is possible to derive the time interval of the organism considered because, in the event the unit of time becomes an isounit with values $\hat{I}_t = 10 \times I_t$. Then, the intrinsic time flow of the considered biological structure must decrease tenfold with respect to our time flow as a necessary condition for the entity to be perceived by our senses as evolving with our time.

The new isomathematics suggests that there are infinite dimensions of time and that time appears one-dimensional only when projected into our sensory perception.

Studies have shown that isotopic representations of biological structures may perform a closed loop inside the forward light cone, i.e., the capability of initiating at one point t in space-time, move arbitrarily forward or backward in time and then returning to the original time t. The emergence of multivalued hyperstructures is simply unavoidable if one reflects a moment on the fact that the cells of a complex organism are all generally different, thus requiring different isounits of space and time.

A similar argument is valid with regard to space, since space itself can be different from one organism to another. Thus, the perceived shapes can differ depending on the observer as well as the dimensions (an object can be perceived as very small or very large in different isogeometries) [14] and the evolution over time of the shape goes against our intuition.

For example, an object of 0.03 cm in diameter expressed in a unit of 1 cm appears to be 300 cm in a unit of 0.0001 cm.

Hyperstructures can cause changes in size such as isotopie, and also an increase or decrease of the same dimensionality. We are not sure of the correctness of this description but we believe that our perception of space is insufficient for quantitative scientific studies in biology.

We should also not forget that simple observations of the behavior of plants indicate quite clearly the existence of anomalous space behaviors, that is, space behavior anomalous with respect to our perception of space. We have all observed plants from ordinary seeds grown inside bottles which stop their growth without reaching the walls, or the rather peculiar growth of plants along directions compatible with their environment. Since plants do not have eyes, the existence of a space behavior in biological structures beyond our perception appears to be evident. The only issue which is open on scientific grounds is its correct quantitative representation.

The implications for locomotion are interesting. Biological structures represented with isotopic, genotopic and hyperstructural methods can perform internal motion according to Santilli "geometric propulsion" [15] that is, the possibility of locomotion from one point to another via the alteration of the geometry itself, without any actual motion or application of a Newtonian force.

Appendix A: Isotopic representation of sea shells

C. Illert [27] has shown that a graphic or computer representation of sea shell growth require multivalued Euclidean spaces subsequently formulated by R. M. Santilli as hyperspaces [14]. In fact, the use of the Euclidean geometry alone does not allow to represent sea shell growth because they are first distorted and then break in computer simulations.

Therefore Illert has shows the possibility of obtaining a correct representation of sea shell Angaria delphinus by doubling the number of dimensional axes.

The equations describing the shell are:

$$x = ae^{\alpha\Phi}\left(1 + e^{\Phi}cos\left(2\gamma\Phi\right)\right)cos\Phi, \tag{100}$$

$$y = ae^{\alpha\Phi}\left(1 + e^{\Phi}cos\left(2\gamma\Phi\right)\right)sin\Phi, \tag{101}$$

$$z = be^{\beta\Phi}sin\left(\gamma\Phi\right). \tag{102}$$

The Lagrangian must have a structure of the type:

$$L = K_1\left(\psi \times \psi\right)^n + K_2\left(\psi \times \xi\right)^m + K_2\left(\xi \times \xi\right)^m. \tag{103}$$

The growth of the shell is simulated correctly with the isolagrangian:

$$\hat{L} = \left[K_1\left(\frac{\hat{d}\xi}{\hat{d}t}\hat{\times}\frac{\hat{d}\xi}{\hat{d}t}\right)^{\hat{n}} + K_2\left(\frac{\hat{d}\xi}{\hat{d}t}\times\xi\right)^{\tilde{m}} + K_2\left(\xi \times \xi\right)^{\hat{p}}\right]\hat{I}. \tag{104}$$

Similar representations are possible for all sea shells.

Appendix B: Isotopic representation the valence bonds

Quantum mechanical isotopies (hadronic mechanics) has been able to represent the attraction between the two identical electrons in valence bonds outlined in Sect. 10. This result can be reviewed as follows.

Thanks to the use of hadronic mechanics, Santilli has first shown in the 1978 memoir [83] the existence of a new electron-positron bound state in relation to positronium (the only bound state predicted by quantum mechanics) which is identifiable with π^0 meson (Sect. 9.7) due to the non-local and non-Hamiltonian interaction of the wave packets. These interactions can be represented via the Hulten potential, which is predominant over the Coulomb interaction when distances are very short.

The model therefore provides a quantitative representation of the transition of the positronium into the π^0-meson

$$\left(e_{\uparrow}^{-}, e_{\downarrow}^{+}\right)_{QM} \longrightarrow \pi_0 = \left(e_{\uparrow}^{-}, e_{\downarrow}^{+}\right)_{HM}. \tag{105}$$

This phenomenon also provides a quantitative representation of the Cooper pairs in superconducting materials and studies conducted by Animalu and Santilli [109] show full compatibility of this hypothesis with experimental tests of superconductivity.

A Cooper Pair (CP) can then be represented via a structure similar to that of valence bonds

$$CP = \left(e_{\uparrow}^{-}, e_{\downarrow}^{-}\right)_{HM}, \tag{106}$$

the model has also been able to predict the experimental behavior of the material $YBa_2Cu_3O_6$, giving further scientific credibility to the model.

The modified Schrödinger equation is then given by

$$\left(\frac{1}{2m}\hat{p}_k S\hat{p}^k + \frac{e^2}{r}\hat{I} - z\frac{e^2}{r}\right) S\hat{\psi}_{\uparrow}(t,r) = E\hat{\psi}_{\uparrow}(t,r), \tag{107}$$

where the term $-z\frac{e^2}{r}$ denotes the potential of the electric field of the atomic nucleus, since the valence electrons are not isolated in space.

An explicit expression of the Santillian was first located in [83] and it is given by:

$$\hat{I} = i/\hat{S} = e^{-<\hat{\psi}_{\uparrow}|\hat{\psi}_{\downarrow}>} \simeq 1 - <\hat{\psi}_{\uparrow}|\hat{\psi}_{\downarrow}> \frac{\psi_{\uparrow}}{\hat{\psi}_{\downarrow}} + ... \tag{108}$$

$$\hat{S} = e^{+<\hat{\psi}_{\uparrow}|\hat{\psi}_{\downarrow}>} \simeq 1 + <\hat{\psi}_{\uparrow}|\hat{\psi}_{\downarrow}> \frac{\psi_{\uparrow}}{\hat{\psi}_{\downarrow}} + ... \tag{109}$$

leading to the Hulten potential, which expresses the contact potential:

$$V_0 \frac{e^{-\frac{r}{R}}}{1 - e^{-\frac{r}{R}}} \simeq V_0 \frac{R}{r}, \tag{110}$$

with $V_0 = e^2 <\hat{\psi}_{\uparrow}|\hat{\psi}_{\downarrow}>$.

It is a great achievement that the isotopic model of the pion π^0 correctly predicts all meson characteristics such as resting energy, charge, radius, magnetic moment and even decay time.

The result becomes all the more remarkable when it has been extended to unstable mesons and unstable baryons, including the representation of neutron synthesis from protons and electrons, a process of great importance that occurs in stars. The new isotopic structure of the neutron $n = (p+, e-)_{HM}$ synthesized in stars should have significant repercussions also in theoretical biology.

In conclusion, we observe that quantum mechanics can be considered a special case of hadronic mechanics in which $< \hat{\psi}_\uparrow | \hat{\psi}_\downarrow >= 0; \hat{I} = 1$, and occurs when the superposition of the wave packets of two electrons (at the origin of the non-local interaction) becomes negligible.

Appendix C: Isotopic representastion of correlations

In particle physics the Bose-Einstein correlation has been experimentally detected in the proton-antiproton annihilation at both high and low energies. It may be recommendable to briefly outline the latter correlation because its reformulation in theoretical biology is straightforward. In the Bose-Einstein correlation, the $p - \bar{p}$ fuse together into a state called the fireball, which then decays rapidly into various particles whose end results are mesons (obeying the Bose-Einstein statistics) which, even though at large distances, are correlated.

In the 1992 memoir [112], R. M. Santilli has shown that quantum mechanics cannot consistently represent the Bose- Einstein correlation (see also paper [113]) in view of the following insufficiencies. Protons and antiprotons are not ideal spheres with points in them, but are instead constituted by some of the densest media measured in laboratory by mankind until now. Being the result of the mutual penetration of the hyperdense protons and antiprotons, the fireball is therefore one of the most general known non-local integral systems. But nonlocal-integral interactions are non-Hamiltonian both conceptually and technically. It then follows again that quantum mechanics is not expected to be exactly valid for the Bose-Einstein correlation in view of its central requirement of representing everything with one single quantity, the Hamiltonian.

But even assuming that the above insufficiency is by-passed via not so infrequent machinations to preserve old knowledge (e.g., the addition in the Hamiltonian of the "nonlocal-integral potential" (which, has no mathematical or physical sense), quantum mechanics still remains structurally unable to represent correlation in an exact way.

This is due to the limitations of its very axioms as compared to the experimental evidence of the correlation. As an example, the two-body quantum mechanical axiom of expectation value of the n- point correlation function is given by

$$C_n = \sum_k (< k, a|k, a > + < k.b|k, b >,) \quad (111)$$

thus lacking the cross terms $< k, a|k, b >$ representing the correlation. By comparison, the axiom of isoexpectation values of hadronic mechanics is given by

$$\hat{C}_n = \sum_{k,i,j} \left(< k, a|\hat{S}_{kk}|k, a > + < k.b|\hat{S}_{kk}|k, b > + < i, a|\hat{S}_{ij}|j, b >, \right) \quad (112)$$

and exhibits precisely the cross terms needed for a description of correlation from first axiomatic principles.

In current ”semiphenomenological models”, the cross terms are introduced via a number of artificial methods which, however, violate the quantum axiom of expectation values, thus confirming Santilli's view [112] that quantum mechanics cannot provide an axiomatically and physically consistent description of the Bose-Einstein correlation.

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