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# Can the Generalized Haag Theorem Be Further Generalized?

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$L^{\dagger}$-invariant $n$-point functions of scalar field theories satisfying the Wightman axioms are considered in the framework of the recently proposed inhomogeneous U(3,1)-invariant extension which is Weierstrass analytic in both the real and imaginary parts of complex four-vectors. The algebraic variety over which the extension is analytic is investigated, and it is shown that there is a shift in the appearance of singular points from $n \geq 6$, as for the customary complex analytic extension, to $n \geq 10$. The extended analyticity domain is investigated too, and it is proved that it contains all the spacelike points of the analyticity domain of the physical $n$-point function. A procedure to reach physical timelike separation, as well as any separation, is introduced, and it is shown that the above type of Weierstrass analyticity is sufficient to determine the physical $n$-point function at any separation from its value at spacelike separation. The above results are applied to the generalized Haag theorem in order to see whether its validity can be extended to more than the first four vacuum expectation values for the considered type of field theories.

## I. INTRODUCTION

In a recent paper[1] the extension to complex four-vectors $z_k = \xi_k - i\eta_k$ of $L^{\dagger}$-invariant $n$-point functions

$$W_n(x_1, \ldots, x_n) = w_n(\xi_1, \ldots, \xi_{n-1})$$
$$= \langle 0|\phi_1(x_1) \cdots \phi_n(x_n)|0\rangle,$$
$$\xi_k = x_k - x_{k+1}, \quad k = 1, 2, \ldots, n-1 \quad (1.1)$$

of a scalar field theory satisfying the Wightman axioms was investigated from a new analyticity approach, namely, Weierstrass analyticity in both the real and imaginary parts $\xi_k$ and $\eta_k$.

A theorem was proved which essentially states that under certain restrictions all possible analytic extensions of $w_n(\xi_1, \ldots, \xi_{n-1})$ to complex four-vectors $z_k = \xi_k - i\eta_k$ are characterized as follows:

(1) There exists an extension (Bargmann-Hall-

Wightman theorem[2]), $\underline{w}_n^{(1)}(\xi_1,\eta_1;\ldots;\xi_m,\eta_m)$, $m=n-1$ which is analytic in the domain

$$\underline{D}_{2m}^{(1)}=(\xi_k\,.\,\eta_k\mid -\infty<\xi_{k\mu}<+\infty;\ \eta_k\in V_+;\quad k=1,2,\ldots,m;\ \mu=0,1,2,3) \tag{1.2}$$

and (complex) analytic in the variety

$$\underline{M}_m^{(1)}=(U_{ij}\mid U_{ij}=z_iz_j;\ i,j=1,2,\ldots,m) \tag{1.3}$$

over which the orthogonal scalar products $z_iz_j$ vary for all the $z$'s in the tube

$$\underline{T}_m^{(1)}=(z_k\mid z_k=\xi_k-i\eta_k;\ \xi_k,\eta_k\in\underline{D}_{2m}^{(1)};\ k=1,2,\ldots,m). \tag{1.4}$$

Furthermore, the extension possesses a single-valued continuation to the extended tube

$$\underline{T}'^{(1)}_m=\bigcup\underline{\Lambda}^{(1)}\underline{T}_m^{(1)},\quad \underline{\Lambda}^{(1)}\in \mathrm{L}_+(C) \tag{1.5}$$

and is invariant under the proper complex orthogonal Lorentz group $\mathrm{L}_+(C)$.

(2) There exists a new extension, $\underline{w}_n^{(2)}(\xi_1,\eta_1;\ldots;\xi_m,\eta_m)$, which is analytic in an open and connected subset $\underline{S}_{2m}^{(2)}$ of the region[3]

$$\underline{D}_{2m}^{(2)}=(\xi_k,\eta_k\mid \xi_k^{\,2}<0,\ \eta_k^{\,2}<0;\ k=1,2,\ldots,m) \tag{1.6}$$

and (real) analytic in the variety

$$\underline{M}_m^{(2)}=(V_{ij}\mid V_{ij}=\tfrac{1}{2}(z_iz_j^*+z_i^*z_j);\ i,j=1,2,\ldots,m), \tag{1.7}$$

over which the Hermitian scalar products $\frac{1}{2}(z_iz_j^*+z_i^*z_j)$ vary for all the $z$'s in the domain

$$\underline{T}_m^{(2)}=(z_k\mid z_k=\xi_k-i\eta_k;\ \xi_k,\eta_k\in\underline{S}_{2m}^{(2)};\ k=1,2,\ldots,m). \tag{1.8}$$

Furthermore, the new extension possesses a single-valued continuation to the extended domain

$$\underline{T}'^{(2)}_m=\underline{\Lambda}^{(2)}\underline{T}_m^{(2)},\quad \underline{\Lambda}^{(2)}\in \mathrm{U}(3,1), \tag{1.9}$$

and is invariant under the unitary U(3,1) group.

In a more recent paper,[4] some examples of $\underline{w}_n^{(2)}$ analytic extensions were constructed with corresponding analyticity domains $\underline{T}_m^{(2)}$ as an explicit check on the validity of the theorem.

In the same paper we introduced an algebraic procedure for constructing the analyticity domain of the physical $n$-point function without any recursion to the extended tube $\underline{T}'^{(1)}_m$, by disproving a rather popular belief that this domain cannot be constructed without the knowledge of the $\underline{w}_n^{(1)}$ extension. This ultimately proved the independent existence of the two nonequivalent extensions $\underline{w}_n^{(1)}$ and $\underline{w}_n^{(2)}$ in the framework of the assumed type of Weierstrass analyticity.

As is well known, the extension $\underline{w}_n^{(1)}$ possesses, among others, the following properties[5]:

$A^{(1)}$: The matrices $(U_{ij})$ of the variety (1.3) have rank $r\leq 4$. This property has relevant physical implications, since it ultimately implies a restriction on the first four vacuum expectation values for the validity of the generalized Haag theorem.

$B^{(1)}$: The identity $I$ can be continuously connected to the total inversion $I_{st}$ on account of the connectivity properties of the invariance group $\mathrm{L}_+(C)$. This property is of central importance in the derivation of the TCP theorem.

$C^{(1)}$: The $\mathrm{L}_+(C)$ invariance group preserves rank and order (as complex manifold) of the Lorentz group, a property which has some relevance in the expansion of the scattering amplitude.

The corresponding properties of the extension $\underline{w}_m^{(2)}$ are as follows[1]:

$A^{(2)}$: The matrices $(V_{ij})$ of the variety (1.7) have rank $r\leq 8$.

$B^{(2)}$: The identity component $I$ can be continuously connected not only to the total inversion $I_{st}$, but also to the space inversion $I_s$ and the time inversion $I_t$ on account of the connectivity properties of the U(3,1) invariance group.

$C^{(2)}$: The U(3,1) invariance group possesses order and rank larger than those of the Lorentz group and the full invariance group; the inhomogeneous IU(3,1) group presents the interesting feature of admitting the SU(3) group as a little group.

Even though the above-listed properties of the new extension are rather striking as compared to the corresponding properties of extension $\underline{w}_n^{(1)}$, obviously they do not cast a shadow on the extension of the Bargmann-Hall-Wightman theorem, which retains all its power centered on its uniqueness once the framework of complex analyticity is assumed.

Nevertheless, in our opinion, the above properties are sufficiently interesting to motivate further studies on the new extension.

It is the purpose of the present paper to investigate some aspects of the new extension $\underline{w}_n^{(2)}$ which are essential for an evaluation of the possible physical applications of properties $A^{(2)}$, $B^{(2)}$, and $C^{(2)}$ listed above.

In Sec. II we study the dimensionality, together with the singular and exceptional points of the algebraic variety $\underline{M}_m^{(2)}$.

In Sec. III we investigate the real points[4] of the extended domain $\underline{\tau}'^{(1)}_n$ in order to see whether they have equivalent properties and effectiveness as the Jost points[5] of extension $\underline{w}_n^{(1)}$.

In Sec. IV we attack the problem of how to reach real timelike points (which are outside both analyticity domains $\underline{\tau}_m^{(2)}$ and $\underline{\tau}'^{(2)}_m$) in order to see whether relations between spacelike and timelike separations can be obtained in the framework of the $\underline{w}_n^{(2)}$ extension.

In Sec. V we investigate whether the knowledge of the extension $\underline{w}_n^{(2)}$ in a (real) neighborhood of a physical spacelike point uniquely determines the $n$-point function $w_n$ at all physical points. Finally, we apply our results to the generalized Haag theorem.

## II. THE ALGEBRAIC VARIETY $\underline{M}_m^{(2)}$

As is well known,[2] the $n$-point functions $w_n(\xi_1, \ldots, \xi_m)$, $m=n-1$ are (real) analytic in the variety

$$M_m = (S_{ij} \,|\, S_{ij} = \xi_i \xi_j ;\ i,j = 1,2,\ldots,m), \tag{2.1}$$

over which the scalar products $\xi_i \xi_j$ vary for all the $\xi$'s in an open and connected subset $S_m$ of the domain for all spacelike $\xi$:

$$D_m = (\xi_k \,|\, \xi_k^2 < 0;\ k = 1,2,\ \ldots, m)\,. \tag{2.2}$$

Thus, there exist functions $h_n(S_{11}, \ldots, S_{mm})$, with $S_{ij} \in M_m$, such that

$$w_n(\xi_1, \ldots, \xi_m) = h_n(S_{11}, \ldots, S_{mm})\,. \tag{2.3}$$

$M_m$ is an algebraic variety in the $\frac{1}{2}m(m+1)$ scalar products $\xi_i\xi_j$ and is an open subset of the set of all real symmetric matrices with rank $r \leq 4$ with dimensions

$$\dim(M_m) = \begin{cases} \frac{1}{2}m\,(m+1) \text{ for } m = 1,2,3,4 \\ 4m-6 \text{ for } m > 4\,. \end{cases} \tag{2.4}$$

For points of $M_m$ with $m \leq 4$ $(n \leq 5)$ where the rank is not maximum, namely for exceptional points,[6] $M_m$ is locally an open set in a $\frac{1}{2}m(m+1)$-dimensional space, and the Weierstrass definition of real analyticity applies.

For points of $M_m$ with $m \geq 5$ $(n \geq 6)$ where $r = 4$, $M_m$ is locally an open set in a $(4m-6)$-dimensional Euclidean space, and the standard definition of real analyticity also applies.

Points of $M_m$ with $m \geq 5$ and $r < 4$ are singular points and their neighborhoods are no longer locally Euclidean. In this case analyticity simply refers to boundedness and continuity.

The variety $\underline{M}_m^{(2)}$ defined by (1.7) is also an algebraic variety in the $\frac{1}{2}m(m+1)$ scalar products $V_{ij} = \xi_i\xi_j + \eta_i\eta_j$ and is also an open subset of the set of all $m\times m$ real symmetric matrices. However, the rank of the matrices $(V_{ij})$ is $\leq 8$ (Ref. 1) while the rank of the matrices $(S_{ij})$ is $\leq 4$.

The above difference in the rank has implications for the dimensionality of the variety $\underline{M}_m^{(2)}$ as well for the location of exceptional and singular points.

As is well known, a real symmetric $m\times m$ matrix of rank $r$ can be brought, via a similarity transformation, into a form in which only the first $r$ rows do not vanish:

$$\begin{bmatrix} V_{11} & \cdots\cdots & V_{1m} \\ & V_{22} \ \cdots & V_{2m} \\ & \ddots & \vdots \\ & & \vdots \\ & & V_{mm} \end{bmatrix} \cong \begin{bmatrix} V'_{11} & \cdots\cdots & V'_{1m} \\ & V'_{22} \ \cdots & V'_{2m} \\ & \ddots & \vdots \\ & V'_{rr} \cdots & V'_{rm} \\ & 0 \ \cdots & 0 \\ & \ddots & \vdots \\ & & 0 \end{bmatrix} \tag{2.5}$$

Thus, the extension $\underline{h}_n^{(2)}(V_{11}, \ldots, V_{mm}) = \underline{w}_n^{(2)}(\xi_1, \eta_1;\ \ldots;\ \xi_m, \eta_m)$ in the $\frac{1}{2}m(m+1)$ elements $V_{ij}$ is actually defined on a variety which in general has dimension less than $\frac{1}{2}m(m+1)$ if $m > 8$ and the rank is maximal.

To find the actual dimension we must subtract from $\frac{1}{2}m(m+1)$ the number of elements $V_{ij}$ which are zero after the similarity transformation. Clearly, such a number is $\frac{1}{2}[(m-r)(m-r+1)]$.

For $M_m$ we have, putting $r = 4$,

$$\begin{aligned} \dim(M_m) &= \tfrac{1}{2}[m(m+1)] - \tfrac{1}{2}[(m-4)(m-3)] \\ &= 4m-6\,; \end{aligned} \tag{2.6}$$

similarly for $\underline{M}_m^{(2)}$ we have, putting $r = 8$,

$$\begin{aligned} \dim(\underline{M}_m^{(2)}) &= \tfrac{1}{2}[m(m+1)] - [(m-8)(m-7)] \\ &= 8m-28. \end{aligned} \tag{2.7}$$

We thus see that the extension $\underline{h}_n^{(2)}(V_{11}, \ldots, V_{mm})$ is defined on a variety $\underline{M}_m^{(2)}$ whose dimension for $m > 8$ cannot exceed $8m-28$ if the rank is maximum.

As analytic counterpart of the algebraic rule (2.7) we have the following lemma.

*Lemma 2.1.* If a function $\underline{f}^{(2)}_n(z_1,\ldots,z_m;z_1^*,\ldots,z_m^*)=\underline{w}^{(2)}_n(\xi_1,\eta_1;\ldots;\xi_m,\eta_m)$ is analytic in the extended domain $\underline{T}'^{(2)}_m$ and invariant under the U(3,1) group, then the following equations hold in $\underline{T}'^{(2)}_m$:

$$\sum_j \left( z_{j\mu}\frac{\partial \underline{f}^{(2)}_m}{\partial z_j^\nu} - z^*_{j\nu}\frac{\partial \underline{f}^{(2)}_m}{\partial z_j^{*\mu}}\right)=0\ , \tag{2.8}$$

$$\sum_j \left( z_{j\mu}\frac{\partial \underline{f}^{(2)}_m}{\partial z_j^{*\nu}} - z_{j\nu}\frac{\partial \underline{f}^{(2)}_m}{\partial z_j^{*\mu}}\right)=0, \tag{2.9}$$

$$\sum_j \left( z^*_{j\mu}\frac{\partial \underline{f}^{(2)}_m}{\partial z_j^{\nu}} - z^*_{j\nu}\frac{\partial \underline{f}^{(2)}_m}{\partial z_j^{\mu}}\right)=0, \tag{2.10}$$

with $\mu,\nu=0,1,2,3$.

Indeed, consider a one-parameter subgroup of U(3.1) characterized by the transformation $\Lambda(a)$. Differentiating the identity

$$\underline{f}^{(2)}_n(\Lambda(a)z_1,\ldots,\Lambda(a)z_m;\,z_1^+\Lambda^+(a),\ldots,z_m^+\Lambda^+(a)) = \underline{f}^{(2)}_n\,(z_1,\ldots,z_m;\,z_1^*,\ldots,z_m^*) \tag{2.11}$$

with respect to $a$ we get

$$\sum_j\left\{ \frac{\partial \underline{f}^{(2)}_m}{\partial[\Lambda(a)z_j]^\mu}\,\frac{\partial[\Lambda(a)z_j]^\mu}{\partial a} + \frac{\partial \underline{f}^{(2)}_m}{\partial[z_j^+\Lambda^+(a)]^\mu}\,\frac{\partial[z_j^+\Lambda^+(a)]^\mu}{\partial a}\right\}=0. \tag{2.12}$$

At $a=0$ we have

$$\left.\frac{\partial[\Lambda(a)z_j]^\mu}{\partial a}\right|_{a=0}=H^{\mu\nu}z_{j\nu}, \tag{2.13}$$

and

$$\left.\frac{\partial[z_j^+\Lambda^+(a)]^\mu}{\partial a}\right|_{a=0}=z^*_{j\nu}H^{*\nu\mu}, \tag{2.14}$$

where $H^{\mu\nu}$ is the infinitesimal generator of the $\Lambda(a)$ transformation.

Consider the one-parameter subgroup for which all elements of the matrix $(H^{\mu\nu})$ are zero except for a pair $H^{\mu\nu}$ and $H^{\nu\mu}$ such that $H^{\mu\nu}=-H^{*\nu\mu}=\text{const}$, $H^{\mu\nu}=H^{\nu\mu}$. Then relation (2.12) becomes

$$\sum_j\left( z_{j\nu}\frac{\partial \underline{f}^{(2)}_m}{\partial z_j^\mu} + z_{j\mu}\frac{\partial \underline{f}^{(2)}_m}{\partial z_j^\nu} - z^*_{j\nu}\frac{\partial \underline{f}^{(2)}_m}{\partial z_j^{*\mu}} - z^*_{j\mu}\frac{\partial \underline{f}^{(2)}_m}{\partial z_j^{*\nu}}\right)=0\ , \tag{2.15}$$

which can be written

$$O_{\nu\mu}\underline{f}^{(2)}_m + O_{\mu\nu}\underline{f}^{(2)}_m=0\ , \tag{2.16}$$

with

$$O_{\nu\mu}=\sum_j\left( z_{j\nu}\frac{\partial}{\partial z_j^\mu} - z^*_{j\mu}\frac{\partial}{\partial z_j^{*\nu}}\right). \tag{2.17}$$

For $\mu=\nu$, Eq. (2.16) reduces to (2.8). For $\mu\neq\nu$ recall that a function $\underline{f}^{(2)}_m$ which is invariant under U(3, 1) and analytic in $\underline{T}'^{(2)}_m$ is also a function of the scalar products[1] $V_{ls}=\frac{1}{2}(z_l z_s^* + z_l^* z_s)$, i.e.,

$$\underline{f}^{(2)}_n(z_1,\ldots,z_m;z_1^*,\ldots,z_m^*)=\underline{w}^{(2)}_n(\xi_1,\eta_1;\ldots;\xi_m,\eta_m) = \underline{h}^{(2)}_n(V_{11},\ldots,V_{mm})\,. \tag{2.18}$$

But then

$$O_{\nu\mu}\underline{f}^{(2)}_m=\sum_{ls}\frac{\partial \underline{h}^{(2)}_m}{\partial V_{ls}}\sum_j\left( z_{j\nu}\frac{\partial V_{ls}}{\partial z_j^\mu} - z^*_{j\mu}\frac{\partial V_{ls}}{\partial z_j^{*\nu}}\right)=0\ , \tag{2.19}$$

and relation (2.8) for $\mu\neq\nu$ follows from (2.16).

Consider now the function $\underline{w}^{(2)}_m$. Differentiating the identity

$$\underline{w}^{(2)}_n(\Lambda(a)\xi_1,\Lambda(a)\eta_1;\ldots;\Lambda(a)\xi_m,\Lambda(a)\eta_m) = \underline{w}^{(2)}_n(\xi_1,\eta_1;\ldots;\xi_m,\eta_m) \tag{2.20}$$

with respect to $a$, where $\Lambda(a)$ is a transformation of $L^\dagger\in$U(3, 1), and following a similar procedure as above, we get

$$\sum_i\left(\xi_{i\mu}\frac{\partial \underline{w}^{(2)}_n}{\partial \xi_i^\nu} - \xi_{i\nu}\frac{\partial \underline{w}^{(2)}_n}{\partial \xi_i^\mu} + \eta_{i\mu}\frac{\partial \underline{w}^{(2)}_n}{\partial \eta_i^\nu} - \eta_{i\nu}\frac{\partial \underline{w}^{(2)}_n}{\partial \eta_i^\mu}\right)=0\,. \tag{2.21}$$

In view of the structure of (2.18), this equation can be satisfied if and only if

$$\sum_j\left(\xi_{j\mu}\frac{\partial \underline{w}^{(2)}_n}{\partial \xi_j^\nu} - \xi_{j\nu}\frac{\partial \underline{w}^{(2)}_n}{\partial \xi_j^\mu}\right)=0, \tag{2.22}$$

and similarly for the other terms in the $\eta$'s in (2.21).

Using the substitution

$$\frac{\partial}{\partial\xi}=\frac{1}{2}\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial z^*}\right), \tag{2.23}$$

and by means of Eq. (2.8), relation (2.22) can be written

$$B_{\mu\nu}\underline{h}^{(2)}_n + B^*_{\mu\nu}\underline{h}^{(2)}_n=0\,, \tag{2.24}$$

with

$$B_{\mu\nu}=\sum_j\left(z_{j\mu}\frac{\partial}{\partial z_j^{*\nu}}-z_{j\nu}\frac{\partial}{\partial z_j^{*\mu}}\right). \qquad (2.25)$$

On account of the structure of (2.18), Eqs. (2.9) and (2.10) easily follow from (2.24).

Equations (2.8), (2.9), and (2.10) represent a set of at most 28 independent equations for $8n$ variables, the $z_i$ and $z_i^*$. Consequently the dimensionality of $\underline{M}_m^{(2)}$ for $m>8$ and $r=8$ is $8m-28$ as derived from the algebraic rule (2.7).[7]

This essentially means that for $m>8$ and when the rank is maximum there are at least $8m-28$ functionally independent Hermitian scalar products which can be formed out of the $8m$ variables $z_{j\mu}$ and $z_{j\nu}^*$.

Furthermore, in this case the tangent space of any point of $\underline{M}_m^{(2)}$ is locally Euclidean and has also dimension $8m-28$.

If for $m>8$ the rank is not maximum, then the tangent space has dimension $\frac{1}{2}[m(m+1)]$ and it is no longer locally Euclidean. Indeed, in a way equivalent to the case for $M_m$ (see Ref. 2), the tangent space at any point of $\underline{M}_m^{(2)}$ is determined by a set of linear equations in the differentials $dV_{ls}$ whose coefficients are $8\times 8$ minors of $(V_{ls})$. If all those $8\times 8$ determinants vanish, then the set of equations is satisfied by any choice of $dV_{ls}$.

For $m\leq 8$, $\underline{M}_m^{(2)}$ is clearly an open set in a (real) $\frac{1}{2}[m(m+1)]$ dimensional Euclidean space. Indeed, since there are no points for $m\leq 8$ which are singular in the sense of algebraic geometry, the tangent space is always locally Euclidean and has dimension $\frac{1}{2}[m(m+1)]$.

By identifying the dimension of the variety with the dimension of its tangent space, we can summarize the above results with the following proposition.

*Proposition 2.1.* The dimension of the algebraic variety $\underline{M}_m^{(2)}$ is $\frac{1}{2}[m(m+1)]$ for $m\leq 8$; $8m-28$ at nonsingular points for $m>8$; and $\frac{1}{2}[m(m+1)]$ at singular points for $m>8$.

The dimensions of the algebraic varieties $M_m$ and $\underline{M}_m^{(2)}$, thus, coincide for $m=1, 2, 3, 4$ and in both cases no singular point occurs. For $m=5, 6, 7, 8$ the dimension of $M_m$ is $4m-6$, while that of $\underline{M}_m^{(2)}$ is $\frac{1}{2}[m(m+1)]$ and singular points can occur only for $M_m$. For $m>8$ and maximum rank, the dimension of $M_m$ is $4m-6$ and that of $\underline{M}_m^{(2)}$ is $8m-28$. Finally, for $m>8$ the dimensions of $M_m$ and $\underline{M}_m^{(2)}$ again coincide at singular points.

An interesting (open and connected) subset of $\underline{M}_m^{(2)}$ is the variety

$$\underline{\hat{M}}_m^{(2)}=(\hat{V}_{ij}\mid \hat{V}_{ij}\in\underline{M}_m^{(2)};\ \operatorname{rank}(\hat{V}_{ij})\leq 4;\quad i,j=1,2,\ldots,m). \qquad (2.26)$$

Clearly, $\underline{M}_m^{(2)}$ behaves like $M_m$ for what concerns both the dimensionality and the location of exceptional and singular points.

## III. REAL POINTS IN THE EXTENDED DOMAIN $\underline{\tau}'^{(2)}_m$

In Ref. 4 the existence of physical (real) spacelike points in the extended domain $\underline{\tau}'^{(2)}_m$ was indicated, and the cases of two- and three-point functions were explicitly discussed.

We shall now investigate this problem in a more general framework by proving first the following lemma.

*Lemma 3.1.* The algebraic variety $M_m$ is a subset of the variety $\underline{M}_m^{(2)}$.

Consider the variety $\underline{\hat{M}}_m^{(2)}$ as defined in (2.26). Any element $\hat{V}_{ij}\in\underline{\hat{M}}_m^{(2)}$ can be written as a scalar product of real four-vectors $\alpha_i$ such that

$$\begin{aligned}\hat{V}_{ij}&=\hat{\xi}_i\hat{\xi}_j+\hat{\eta}_i\hat{\eta}_j\\&=\alpha_i\alpha_j\qquad i,j=1,2,\ldots,m.\end{aligned} \qquad (3.1)$$

Indeed, any real symmetric $m\times m$ matrix $(\hat{V}_{ij})$ of rank $r\leq 4$ and index 1 can be written, through a similarity transformation, in the form

$$(\hat{V}_{ij})=F\begin{pmatrix}G&0\\0&0\end{pmatrix}F^T, \qquad (3.2)$$

where $G=(g_{\mu\nu})$ and $F$ is a nonsingular matrix. Relation (3.1) is then proved by simply putting

$$\alpha_i\equiv(F_{i1},F_{i2},\ldots,F_{ir};0,0,\ldots,0),\qquad i=1,2,\ldots,m \qquad (3.3)$$

where $0,0,\ldots,0$ consists of $4-r$ zeros.

All four-vectors $\alpha_i$ must be spacelike by construction.[1,4] This implies that $S_m$ is a subset of the set of all four-vectors $\alpha_i$ or, equivalently,

$$M_m\subseteq\underline{\hat{M}}_m^{(2)}\subset\underline{M}_m^{(2)}. \qquad (3.4)$$

Indeed, consider a spacelike physical point $(\xi_1,\ldots,\xi_m)\in S_m$. Then there always exists a point $(\hat{\xi}_1,\hat{\eta}_1;\ldots;\hat{\xi}_m,\hat{\eta}_m)\in\underline{S}_{2m}^{(2)}$ such that

$$\begin{aligned}S_{ij}&=\xi_i\xi_j\\&=\hat{V}_{ij}\\&=\hat{\xi}_i\hat{\xi}_j+\hat{\eta}_i\hat{\eta}_j\ .\end{aligned} \qquad (3.5)$$

Choose, for instance, the point

$$\hat{\xi}_i=a\xi_i,\quad \hat{\eta}_i=b\xi_i,\quad i=1,2,\ldots,m \qquad (3.6)$$

with $a$ and $b$ real constants satisfying the restrictions

$$0<a<1,\quad 0<b<1,\quad a^2+b^2=1\ . \qquad (3.7)$$

Then Eq. (3.5) follows. Furthermore, the point

$(\hat{\xi}_1, \hat{\eta}_1; \ldots; \hat{\xi}_m, \hat{\eta}_m)$ as defined by (3.6) is a point of $\underline{S}^{(2)}_{2m}$ if conditions (3.7) are satisfied, since in this case $\hat{V}_{ij}$ is within the analyticity domain of extension $\underline{w}^{(2)}_n$ by construction.

Finally, if the variety $\underline{M}^{(2)}_m$ constructed according to (3.5), (3.6), and (3.7) does not span the entire variety $M_m$, then it constitutes a subdomain of analyticity. In this case one can perform an analytic continuation through a chain of overlapping polycircles until relation (3.4) is satisfied.

The major implication of Lemma 3.1 can be expressed in terms of the following theorem.

*Theorem 3.1.* The extended domain $\underline{\tau}'^{(2)}_m$ contains all the points of the analyticity domain $S_m$ of the physical $n$-point function.

Consider real four-vectors $\xi_i$, $i = 1, 2, \ldots, m$ according to relation (3.5). They cannot belong to $\underline{\tau}^{(2)}_m$ since $\eta_k = -\mathrm{Im}(z_k) \neq 0$ by construction for any $z_k \in \underline{\tau}^{(2)}_m$.

Real four-vectors $\xi_i$, however, do belong to the extended domain $\underline{\tau}'^{(2)}_m$. Indeed a point $(\hat{z}_1, \ldots, \hat{z}_m) \in \underline{\tau}^{(2)}_m$ defined by means of (3.6) and (3.7) can be written

$$a = \cos\alpha, \quad b = \sin\alpha, \quad \hat{z}_k = e^{-i\alpha}\xi_k . \tag{3.8}$$

But then there always exists a U(3, 1) transformation, such as the one-parameter transformation $\Lambda(\alpha) = e^{i\alpha}1$ for which

$$\Lambda\hat{z}_k = \xi_k, \quad k = 1, 2, \ldots, m . \tag{3.9}$$

This is essentially a mapping from $\hat{z}_k \in \underline{\tau}^{(2)}_m$ to points $\hat{z}'_k = \Lambda\hat{z}_k = \xi_k \in \underline{\tau}'^{(2)}_m$, and consequently shows the existence of real spacelike four-vectors in the extended domain $\underline{\tau}'^{(2)}_m$.

The fact that all physical spacelike points of $S_m$ belong to the extended domain $\underline{\tau}'^{(2)}_m$ then follows from Lemma 3.1. Since for all physical points $(\xi_1, \ldots, \xi_m) \in S_m$ there are points $(\hat{\xi}_1, \hat{\eta}_1; \ldots; \hat{\xi}_m, \hat{\eta}_m) \in \underline{S}^{(2)}_m$ for which relations (3.5), (3.6), and (3.7) hold, then, corresponding to all physical points $\xi_k \in S_m$, there are points $\hat{z}_k \in \underline{\tau}^{(2)}_m$ and transformations $\Lambda \in \mathrm{U}(3,1)$ for which relation (3.9) holds.

On account of the topology[1] of $\underline{M}^{(2)}_m$ and its invariance properties, it is not difficult to see that each real point of $\underline{\tau}'^{(2)}_m$ possesses a neighborhood of real points all in $\underline{\tau}'^{(2)}_m$.

Note that, in view of Lemma 3.1, the variety

$$T_m = (\xi_i\xi_j \,|\, \xi_i\xi_j \in M_m;\ \xi_{k0} = 0;\ i, j, k = 1, 2, \ldots, m) , \tag{3.10}$$

namely the so-called equal-time manifold, is also contained in $\underline{M}^{(2)}_m$.

It is interesting to remark that $T_m \cap \underline{M}^{(2)}_m$ lies in the singular subset of $\underline{M}^{(2)}_m$ only for $m \geq 9$, while $T_m \cap M_m$ lies in the singular subset of $M_m$ for $m \geq 5$.

From Theorem 3.1 and Proposition 2.1 it follows that all the physical points $M_m$ for $m \geq 9$ lie in a singular subset of $\underline{M}^{(2)}_m$. This is not necessarily the case for the subvariety $\underline{\hat{M}}^{(2)}_m$. In this connection, the following domains can be introduced:

$$\hat{S}^{(2)}_{2m} = (\hat{\xi}_k, \hat{\eta}_k \,|\, \hat{\xi}_k, \hat{\eta}_k \in \underline{S}^{(2)}_{2m};\ \mathrm{rank}\ (\hat{\xi}_i\hat{\xi}_j + \hat{\eta}_i\hat{\eta}_j) \leq 4;\ i, j, k = 1, 2, \ldots, m), \tag{3.11}$$

$$\underline{\hat{\tau}}^{(2)}_m = (\hat{z}_k \,|\, \hat{z}_k = \hat{\xi}_k - i\hat{\eta}_k;\ \hat{\xi}_k, \hat{\eta}_k \in \underline{\hat{S}}^{(2)}_{2m};\ k = 1, 2, \ldots, m), \tag{3.12}$$

$$\underline{\hat{\tau}}'^{(2)}_m = \cup\underline{\Lambda}^{(2)}\underline{\hat{\tau}}^{(2)}_m, \quad \underline{\Lambda}^{(2)} \in \mathrm{U}(3,1) , \tag{3.13}$$

and they all constitute analyticity subdomains for the $\underline{w}^{(2)}_n$ extension.

Since no point of $\underline{M}^{(2)}_m$ where the rank is $\geq 5$ can be a physical point, we have as a consequence that no point of $S_m$ is contained in $\underline{\tau}'^{(2)}_m - \underline{\hat{\tau}}'^{(2)}_m$.

We can thus complement Theorem 3.1 with the following corollary.

*Corollary 3.1.* All points of $S_m$ are contained in $\underline{\hat{\tau}}'^{(1)}_m$.

Let us recall that the diagonal elements $V_{ij}$ of $\underline{M}^{(2)}_m$ must be negative,[4] i.e.,

$$z_i z_i^* = \xi_i^2 + \eta_i^2 < 0, \quad i = 1, 2, \ldots, m . \tag{3.14}$$

This implies that only real spacelike, lightlike, or null four-vectors can be boundary points of $\underline{\tau}'^{(2)}_m$, while real timelike four-vectors are outside the analyticity domain $\underline{\tau}'^{(2)}_m$ and consequently they cannot be approached in the framework of the $\underline{w}^{(2)}_n$ extension.

Leaving the investigation of this problem to Sec. IV, let us now remark that the real parts $\xi'_k$ of points $z'_k \in \underline{\tau}'^{(2)}_m$ are arbitrary four-vectors and, consequently, they can be timelike. Indeed, starting from a point $z_k = \xi_k - i\eta_k \in \underline{\tau}^{(2)}_m$, where both $\xi_k$ and $\eta_k$ are spacelike, there always exists a U(3, 1) transformation $\Lambda$ for which $\xi'_k = \mathrm{Re}(\Lambda z_k) = \mathrm{Re}(z'_k)$ is a timelike vector, even though restriction (3.14) is preserved.[4]

A relevant question is whether the real parts $\hat{\xi}_k$ of points $\hat{z}_k \in \hat{\tau}'^{(2)}_m$ can also be arbitrary timelike vectors.

The answer is in the affirmative, as can be seen from the following argument. Consider an arbitrary timelike point $\hat{\xi}_1, \ldots, \hat{\xi}_m$. A corresponding point $\hat{z}_k = \hat{\xi}_k - i\hat{\eta}_k \in \underline{\tau}'^{(2)}_m$ can be constructed by putting

$$\hat{\eta}_i\hat{\eta}_j = (\alpha_{ij} - 1)\hat{\xi}_i\hat{\xi}_j, \quad i, j = 1, 2, \ldots, m \tag{3.15}$$

where $\alpha_{ij}$ are free parameters suitably chosen to ensure that $\hat{z}_k$ is within the analyticity domain. Since

$$\hat{V}_{ij} = \alpha_{ij}\,\hat{\xi}_i\,\hat{\xi}_j , \tag{3.16}$$

this implies that all diagonal terms $\alpha_{kk}$ must be negative.

To show that the matrix $(\hat{V}_{ij})$ can have rank $r \leq 4$, consider the matrix $(\hat{\xi}_i \hat{\xi}_j)$. Multiplying the first row and the first column by $\sqrt{\alpha_{11}}$, and the second row and the second column by $\sqrt{\alpha_{22}}$, we do not change the rank of the resultant matrix. Putting (without summation)

$$\alpha_{ij} = (\alpha_{ii}\,\alpha_{jj})^{1/2}\,, \tag{3.17}$$

the rank of the $(\hat{V}_{ij})$ matrix so constructed remains identical to the rank of $(\hat{\xi}_i \hat{\xi}_j)$.

## IV. THE TIMELIKE PHYSICAL POINTS

On account of the results of Sec. III, we can state that given an $n$-point function $w_n$ in a neighborhood of a point $(\xi_1, \ldots, \xi_m) \in S_m$, we can reach any other point of $S_m$ through an analytical continuation in $\underline{w}_n^{(2)}$ by means of a chain of overlapping polycircles. Similarly, we can approach any lightlike or null separation.

The same procedure, however, is not applicable in the framework of the $\underline{w}_n^{(2)}$ extension to reach timelike physical points since they are outside $\underline{\tau}'^{(2)}_m$.

We introduce now what we shall term an "auxiliary function" and show that via that function we can also approach in a unique manner any (real) timelike separation.

Consider the extension $\underline{h}_n^{(2)}(V_{11}, \ldots, V_{mm})$ of a physical $n$-point function $w_n(\xi_1, \ldots, \xi_m)$ and affix to each of the elements $V_{ij} = \xi_i \xi_j + \eta_i \eta_j \in \underline{M}_m^{(2)}$ an imaginary part $-i\epsilon_{ij}$, where the $\epsilon_{ij}$ are infinitesimal parameters independent of the $\xi_i$ and $\eta_i$ vectors. We call the function $\underline{k}_n^{(2)}(V_{11} - i\epsilon_{11}, \ldots, V_{mm} - i\epsilon_{mm})$ so constructed the "auxiliary $n$-point function."

Clearly the function $\underline{k}_n^{(2)}$ is equivalent to a first type extension, i.e.,

$$\underline{k}_n^{(2)}(V_{11} - i\epsilon_{11}, \ldots, V_{mm} - i\epsilon_{mm}) \approx \underline{h}_n^{(1)}(U_{11}, \ldots, U_{mm})\,. \tag{4.1}$$

Indeed at nonsingular points there always exist points $(\xi_1', \eta_1'; \ldots; \xi_m', \eta_m') \in \underline{S}_{2m}^{(1)}$ for arbitrary $(\xi_1, \eta_1; \ldots; \xi_m, \eta_m) \in \underline{S}_{2m}^{(2)}$ such that

$$\xi_i \xi_j + \eta_i \eta_j = \xi_i' \xi_j' - \eta_i' \eta_j'\,, \tag{4.2}$$

$$\epsilon_{ij} = 2(\xi_i' \eta_j' + \xi_j' \eta_i')\,, \tag{4.3}$$

e.g., for infinitesimal $\eta_i' \in V_+$ and spacelike $\xi_i'$.[8]

This ensures us that $\underline{k}_m^{(2)}$ is analytic in the $(V_{ij} - i\epsilon_{ij})$ complex space within a narrow strip along the $V_{ij}$ axis with the nonnegative part of the $V_{ii}$ axis constituting a cut.

Let us now stress the differences between the auxiliary function $\underline{k}_n^{(2)}$ and the extension $\underline{h}_n^{(1)}$.

The separations $\xi_i'$ in $\underline{h}_n^{(1)}$ do not generally coincide with the separations $\xi_i$ in $\underline{k}_n^{(2)}$. Indeed, for infinitesimal $\eta_i' \in V_+$, the $\xi_i'$ vectors are totally spacelike, while the corresponding $\xi_i$ in (4.2) are arbitrary four-vectors. This allows the identification of the physical separation with the $\xi_i$ vectors of $\underline{k}_n^{(2)}$ rather than with the $\xi_i'$ vectors of the $\underline{h}_n^{(1)}$ function.

Furthermore, the invariance group of the auxiliary function is the U(3, 1) group, while the invariance group of the $\underline{h}_n^{(1)}$ function is the $L_+(C)$ group. On account of the independence of the imaginary parts $\xi_{ij}$ from the real parts $V_{ij}$, the actual functional dependence of the $\underline{k}_n^{(2)}$ function is on the vectors $z_k \in \underline{\tau}'^{(2)}_m$, and the set of transformations in $\underline{\tau}'^{(2)}_m$ leaving the auxiliary function invariant is the U(3, 1) group. On the contrary, the actual dependence of the $\underline{h}_n^{(1)}$ function is on the vectors $z_k' = \xi_k' - i\eta_k' \in \underline{\tau}'^{(1)}_m$, and the set of transformations in $\underline{\tau}'^{(1)}_m$ leaving $\underline{h}_n^{(1)}$ invariant is the $L_+(C)$ group.

We are now in a position to approach a physical timelike point preserving the U(3, 1) invariance and related broader connectivity properties. Indeed, starting from a point of $S_m$ we can identify such a point with a real point of $\underline{\tau}'^{(2)}_m$. A U(3, 1) transformation or an analytic continuation will bring us into a new point of $\underline{\tau}'^{(2)}_m$ whose real part can be any timelike point. We then introduce the auxiliary function and consider the limit when the imaginary part of such a new point of $\underline{\tau}'^{(2)}_m$ goes to zero by moving inside the analyticity strip of the auxiliary function. This procedure will bring us infinitesimally close to a physical timelike point on account of the infinitesimal character of the $\epsilon$'s according to the following chain of transitions:

$$\begin{aligned}
&w_n(\xi_1, \ldots, \xi_m), \quad \xi_k \in S_m,\ \xi_k^{\,2} < 0\,;\\
&h_n(S_{11}, \ldots, S_{mm}), \quad S_{ij} = \xi_i \xi_j \in M_m;\\
&\underline{h}_n^{(2)}(V_{11}, \ldots, V_{mm}), \quad V_{ij} = \tfrac{1}{2}[(z_i z_j^* + z_i^* z_j)] = S_{ij} \in \underline{M}_m^{(2)},\ z_k \equiv \xi_k \in \underline{\tau}'^{(2)}_m\,;\\
&\underline{h}_n^{(2)}(V_{11}, \ldots, V_{mm}), \quad V_{ij} = \tfrac{1}{2}[(\bar{z}_i \bar{z}_j^* + \bar{z}_i^* \bar{z}_j)] \in \underline{M}_m^{(2)},\ \bar{z}_k = \bar{\xi}_k - i\eta_k \in \tau'^{(2)}_m,\ \bar{\xi}_k^{\,2} = (\mathrm{Re}\bar{z}_k)^2 > 0;\\
&\underline{k}_n^{(2)}(V_{11} - i\epsilon_{11}, \ldots, V_{mm} - i\epsilon_{mm}), \quad V_{ij} - i\epsilon_{ij} \in \underline{M}_m^{(1)},\ \epsilon_{ij} \approx 0;\\
&\lim_{\eta_1', \ldots, \eta_m' \to 0} \underline{k}_n^{(2)}(V_{11} - i\epsilon_{11}, \ldots, V_{mm} - i\epsilon_{mm}) = \underline{k}_n^{(2)}(\bar{\xi}_1 \bar{\xi}_1 - i\epsilon_{11}, \ldots, \bar{\xi}_m \bar{\xi}_m - i\epsilon_{mm}), \quad \epsilon_{ij} \approx 0\,.
\end{aligned} \tag{4.4}$$

When taking the limit $\eta'_1, \ldots, \eta'_m \to 0$, the boundary of the analyticity domain $\underline{T}'^{(2)}_m$ is crossed without affecting either the uniqueness or the absolute convergence of the power-series expansion of the function, since for any value of $\bar{\eta}_k$ the auxiliary function $\underline{h}^{(2)}_m$ remains within its analyticity domain.

As a side consideration, let us remark that the introduction of the auxiliary function ensures the convergence of the mass integration with respect to arbitrary tempered weight functions in the integral representation of the $n$-point function, as well as the convergence of the Laplace transform for any separation.

## V. CONCLUSIONS

In Sec. II we investigated the variety $\underline{M}^{(2)}_m$ over which the $\underline{w}^{(2)}_n$ function is (real) analytic and we stressed its differences with the variety $M_m$ for the physical $n$-point function $w_n$. We found a shift in the appearance of the exceptional points from $m=n-1\leq 4$ in $M_m$ to $m\leq 8$ in $\underline{M}^{(2)}_m$, and a corresponding shift in the appearance of singular points from $m\geq 5$ in $M_m$ to $m\geq 9$ in $\underline{M}^{(2)}_m$.

Since $M_m$ behaves like the variety $\underline{M}^{(1)}_m$ of the $\underline{w}^{(1)}_n$ function for what concerns the location of exceptional and singular points as well as for the dimensionality, the above result implies that for $5\leq m\leq 8$, singular points can occur for the algebraic variety of the $\underline{w}^{(1)}_n$ function, while they do not occur for the algebraic variety of the $\underline{w}^{(2)}_n$ functions.[9]

The physical implications of the above result can be expressed by the following theorem.

*Theorem 5.1.* For $n=1, 2, \ldots, 9$, the physical $n$-point function $w_n(\xi_1, \ldots, \xi_m)$ of a scalar[10] field theory satisfying the Wightman axioms possesses removable singularities at all points of the variety $M_m$ where the rank is less than maximum.

In Sec. III we proved that the variety $M_m$ is fully contained in $\underline{M}^{(2)}_m$, and we showed that the extended analyticity domain $\underline{T}'^{(2)}_m$ of the $\underline{w}^{(2)}_m$ extension contains all the points of the analyticity domain $S_m$ of the physical $n$-point function.[11]

Consequently, the physical $n$-point function $w_n$ at any point of $M_m$ can be identified with the $\underline{w}^{(2)}_n$ extension at the same point of $M_m \in \underline{M}^{(2)}_m$.

The proof of Theorem 5.1 then relies on a known theorem on removable singularities[12] which essentially states that if a function is analytic in a neighborhood of a point, except possibly an exceptional set of points, then the function is analytic in a complete neighborhood of that point.

In the framework of the $\underline{w}^{(1)}_n$ extension it is possible to state the analyticity of $w_n$ at exceptional points of $M_m$ only for $n=1, 2, 3, 4, 5$.[2]

Consider, for instance, the case $n=5$ $(m=4)$ and an exceptional point $E\in M_4$. Then $\dim(E) < \dim(M_4) = 10$. To see that $w_5$ has a removable singularity at $E$, we must consider a complete neighborhood of $E$, namely we must increase the dimension of the variety by considering a neighborhood $N$ of dimension 10 with the exception of a set of points where the dimension is $<10$. In this case the $\underline{w}^{(1)}_n$ and $\underline{w}^{(2)}_n$ extensions produce identical results since $E$ is an exceptional point for both $\underline{M}^{(1)}_m$ and $\underline{M}^{(2)}_m$.

Consider now the case $n=6$ $(m=5)$ and a point $S\in M_5$ where the rank is less than maximum. Then $\dim(S) < \dim(M_5) = 14$. In this case $S$ is a singular point of $\underline{M}^{(1)}_5$ and the use of the $\underline{w}^{(1)}_n$ extension does not allow the singularity at $S$ to be removable. In view of Theorem 5.1, this essentially implies that, since $\underline{M}^{(1)}_5$ preserves as complex manifold the dimension of $M_5$, a neighborhood of dimension 14 of the point $S$ is not sufficient to remove the singularity at $S$. The point $S$, however, is an exceptional point of $\underline{M}^{(2)}_5$ and $\dim(\underline{M}^{(2)}_5)=15$. A complete neighborhood of dimension 15 of $S$ is then sufficient to remove the singularity at $S$.

The cases $n=7, 8, 9$ then follow on similar ground.

As is well known from the framework of the $\underline{w}^{(1)}_n$ function, the physical $n$-point functions $w_n(\xi_1, \ldots, \xi_m)$ for $n=1, 2, 3, 4$ are uniquely determined at any separation from their values at equal-time separations.[2]

An interesting question is whether through the use of the $\underline{w}^{(2)}_n$ function we can increase the number of physical $n$-point functions which are uniquely determined from their value at equal times.

In this connection we can state the following theorem.

*Theorem 5.2.* For $n=1, 2, \ldots, 8$ the physical $n$-point function $w_n(\xi_1, \ldots, \xi_m)$ of a scalar field theory satisfying the Wightman axioms is uniquely determined at any spacelike separation from its values at equal-time separation.

Consider the equal-time manifold $T_m$ defined by (3.8) and the $n$-point function $h_n(\hat{\xi}_1\hat{\xi}_1, \ldots, \hat{\xi}_m\hat{\xi}_m)$, $n=m+1=1, 2, \ldots, 8$ in a (real) neighborhood of an equal-time point $(\hat{\xi}_i\hat{\xi}_j)\in T_m$. Then, in view of Theorem 5.1, $h_n$ in analytic in a complete neighborhood $N\in \underline{M}^{(2)}_m$ of the considered point.

Perform the transition to the $\underline{h}^{(2)}_n$ extension by adding to each element $\hat{\xi}_i\hat{\xi}_j \in T_m$ scalars $\eta_i\eta_j$ of infinitesimal value. This can be done by choosing a point $(\hat{\xi}_1, \eta_1; \ldots; \hat{\xi}_m, \eta_m)\in \underline{S}^{(2)}_{2m}$ for which the vectors $\eta_1, \ldots, \eta_m$ have infinitesimal length or finite values of their component but infinitesimal values of all their scalar products.

The new function $h_n(V_{11}, \ldots, V_{mm})$ so constructed with

$$V_{ij} = \tfrac{1}{2}[(z_i z_j^* + z_i^* z_j)]$$

$$= \hat{\xi}_i \hat{\xi}_j + \eta_i \eta_j ,$$

$$z_i = \hat{\xi}_i - i\eta_i \in \underline{\tau}'^{(2)}_m, \quad i, j = 1, 2, \ldots, m$$

also admits an absolutely convergent power-series expansion since the point $(V_{ij})$ is in the neighborhood of $(\hat{\xi}_i \hat{\xi}_j)$.

Recall that the complete neighborhood $N \in \underline{M}^{(2)}_m$ of $(\hat{\xi}_i \hat{\xi}_j)$ contains an exceptional set $E$ at which $\det(V_{ij}) = 0$ as well as a regular set $R$ at which $\det(V_{ij}) \approx 0$, but not $\equiv 0$.

Consider a set of vectors $\eta_k$ such that $(V_{ij}) \in R$. Then the customary analytical continuation by means of a chain of overlapping polycircles can be performed starting from a neighborhood $N'$ of $(V_{ij})$ with $N' \cap N \neq 0$. This will allow us to reach in a unique manner an arbitrary spacelike separation and the proof of Theorem 5.2 is completed.[13]

Both Theorems 5.1 and 5.2 remain within the analyticity domain of the $\underline{w}^{(2)}_n$ function without any recursion to points outside of $\underline{\tau}'^{(2)}_m$ such as physical timelike separations. Nevertheless, Theorem 5.2 is sufficient to determine uniquely the physical $n$-point function at any separation from their values at equal-times. This can be done, for instance, by using the well-known property in the framework of the $\underline{w}^{(1)}_n$ extension for which the physical $n$-point functions are uniquely determined at any separation from their values at spacelike separation.[2]

Leaving aside at the moment the problem of reaching timelike separations in the framework of the $\underline{w}^{(2)}_n$ extension, we can state a two-step procedure according to which the first transition from equal-time separation to any spacelike separation arises from the $\underline{w}^{(2)}_n$ extension, while the transition from spacelike separation to arbitrary separation arises from the $\underline{w}^{(1)}_n$ extension.

The application of the above results to the generalized Haag theorem is straightforward. Although the topic demands further investigations, we can state that there is relevant evidence according to which the validity of the generalized Haag theorem[2] can be extended up to the first eight vacuum expectation values of scalar field theories satisfying the Wightman axioms.

Until now we essentially used the property according to which from the knowledge of the $\underline{w}^{(2)}_n$ function in $\underline{\tau}'^{(2)}_m$ we can reach or approach any physical separation with the exception of real timelike separations, since those points are outside of $\underline{\tau}'^{(2)}_m$.

In Sec. IV we introduced the so-called "auxiliary function" which guarantees the crossing of the boundary of $\underline{\tau}'^{(2)}_m$ to reach timelike points, preserving the absolutely convergent power series expansion, the uniqueness of the transition, the existence of an integral representation of the $n$-point function with arbitrary tempered weight-functions, and, most important, the validity of the U(3, 1) invariance during the limit procedure. We shall now restrict our attention to the subvariety $\underline{\hat{M}}^{(2)}_m \in \underline{M}^{(2)}_m$ of rank $r \leq 4$ defined by (2.26) with corresponding subdomain $\hat{\underline{\tau}}'^{(2)}_m \in \underline{\tau}'^{(2)}_m$ given in (3.11).

*Theorem 5.3.* The knowledge of the $\underline{w}^{(2)}_n$ function in a real neighborhood of a real point of $\hat{\underline{\tau}}'^{(2)}_m$ uniquely determines $\underline{w}^{(2)}_n$ everywhere in $\tilde{\hat{\underline{\tau}}}'^{(2)}_m$, hence it uniquely determines the auxiliary function and the physical $n$-point function at any separation.

In Sec. III we showed that each real point $P$ of $\underline{\tau}'^{(2)}_m$ possesses a neighborhood $N$ of real points all in $\underline{\tau}'^{(2)}_m$ or equivalently, according to Corollary 3.1, all in $\hat{\underline{\tau}}'^{(2)}_m$. Clearly, all points of $N$ have rank $r \leq 4$. Consequently, the analytic continuation by means of a chain of overlapping polycircles will allow us to reach all points of $\underline{\tau}'^{(2)}_m$ with rank $r \leq 4$, namely $\hat{\underline{\tau}}'^{(2)}_m$, and the first part of the theorem follows.

The variety $\underline{\hat{M}}^{(2)}_m$ over which the function is now defined has the same dimension of $\underline{M}^{(1)}_m$ and consequently forms a real environment for the auxiliary function in $\underline{M}^{(1)}_m$. The uniqueness of the auxiliary function then follows from the knowledge of the $\underline{w}^{(2)}_n$ function in $\underline{\hat{M}}^{(2)}_m$. The physical $n$-point function at any spacelike, lightlike, or null separation arises from the $\underline{w}^{(2)}_n$ function and at any timelike separation arises from the auxiliary function.

The implications of Theorem 5.3 are rather deep. The broader assumption of Weierstrass analyticity in the real and imaginary parts of complex four-vectors, instead of the more restrictive assumption of complex analyticity,[14] allows the determination of the physical $n$-point functions at any separation from their knowledge at all spacelike separations.[15]

Furthermore, Theorem 5.3 now allows a direct link between Theorem 5.2 and the generalized Haag theorem. Indeed, for $n = 1, 2, \ldots, 8$ $(m = 1, 2, \ldots, 7)$ starting from an equal-time separation, Theorem 5.2 will allow the transition to an arbitrary real point of $\hat{\underline{\tau}}'^{(2)}_m$. Then Theorem 5.3 will determine the physical $n$-point function at any separation, from which the application to the generalized Haag theorem directly follows.

Finally, Theorem 5.3 introduces a different prospective for the possible physical applications of other properties of the $\underline{w}_n^{(2)}$ extension which have not been investigated in the present paper, such as the broader connectivity properties and the larger rank of the invariance group.

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[1]R. M. Santilli and P. Roman, Nuovo Cimento 2A, 965 (1971).

[2]D. Hall and A. S. Wightman, in *Dispersion Relations and the Abstract Approach to Field Theory*, edited by Lewis J. Klein (Gordon and Breach, New York, 1961).

[3]We use the metric $g_{00}=-g_{11}=-g_{22}=-g_{33}=1$.

[4]R. M. Santilli, P. Roman, and C. N. Ktorides, Particles and Nuclei 3, 332 (1972).

[5]R. Jost, *The General Theory of Quantized Fields* (Amer. Math. Soc. Publications, Providence, R. I., 1965); R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (Benjamin, New York, 1964).

[6]Note that $\frac{1}{2}[m(m+1)]=4m-6$ for $m=3, 4$.

[7]Note that $\frac{1}{2}[m(m+1)]=8m-28$ for $m=7, 8$.

[8]At nonsingular points for $m\geq 9$, relation (4.2) contains $8m-28$ conditions for the $8m$ variables $\xi'_{k\mu}$, $\eta'_{k\mu}$. Similarly at any point for $m<9$, relation (4.2) contains $\frac{1}{2}[m(m+1)]$ conditions for $8m$ variables and $8m > \frac{1}{2}[m(m+1)]$. Relation (4.3) then follows, for instance, by assuming that the $\eta'_k \in V_+$ are infinitesimal four-vectors, or by using the arbitrariness of the $\xi_{ij}$.

[9]To visualize the above shift in relation to the dimension of the carrier space, one can think of $M_m$ as an algebraic variety defined in terms of scalar products $\xi_i\xi_j$ of vectors $\xi_i$ in a Minkowski space $E_{3,1}$. Then for $m$ larger than the dimension of the carrier space $(m>4)$, points at which the rank is less than maximum are singular. The situation is similar for the variety $\underline{M}_m^{(1)}$ which is defined in terms of scalar products $z_i z_j$ of vectors $z_i$ in a complex Minkowski space $\underline{E}_{3,1}$. Then points of $\underline{M}_m^{(1)}$ with $m>4$ at which the rank is less than maximum are also singular. The situation is different for $\underline{M}_m^{(2)}$, since it can be defined in terms of the linear combination of scalar products $\xi_i\xi_j+\eta_i\eta_j$ of vectors $\xi_i$ and $\eta_i$ defined in two independent (real) Minkowski spaces $E_{3,1}^{(\xi)}$ and $E_{3,1}^{(\eta)}$. Then the carrier space is the eight-dimensional Kronecker product $E_{3,1}^{(\xi)}\times E_{3,1}^{(\eta)}$ and singular points can occur only for $m>8$.

[10]The case of the parity-violating $n$-point function of a scalar field theory was investigated in Ref. 1. The analytic properties of the $\underline{w}_n^{(2)}$ extension are the same as for the parity-conserving function, with the difference that the invariance group is the unimodular SU(3, 1) group. The cases of vectorial and tensorial field theories follow the same pattern since the SU(3, 1) group contains all representations of nonspinorial type (Ref. 1). Finally, the transition to distributions was also investigated in Ref. 1. The $\underline{w}_n^{(2)}$ extension for spinorial field theories, however, has not been investigated at the moment.

[11]This result should be compared with the corresponding properties of Jost points of the $\underline{w}_n^{(1)}$ extension.

[12]S. Bochner and W. T. Martin, *Several Complex Variables*, edited by M. Morse and A. W. Tucker (Princeton Univ. Press, Princeton, N. J., 1948), p. 173. This theorem on removable singularities applies to functions of both real and complex variables.

[13]The case $n=9$ $(m=8)$ is excluded from Theorem 5.2 because of lack of uniqueness in the transition from $T_8$ to $\underline{M}_8^{(2)}$. Indeed in the transition from a point $(\xi_i\xi_j)\in T_8$ to a point $(\xi_i\xi_j+\eta_i\eta_j)\in \underline{M}_8^{(2)}$, the transition of the rank is from $r=3$ up to a maximum of $r=7$. This essentially means that starting from an (equal-time) exceptional point of $\underline{M}_8^{(2)}$, we can only construct through the introduction of the infinitesimal $\eta_i\eta_j$ terms a neighborhood of that point entirely consisting of exceptional points. Consequently we cannot reach a regular point since the transition to rank $r=8$ is now impossible, and the analytic continuation from $T_8$ to $\underline{M}_8^{(2)}$ would not be unique. Note that this lack of uniqueness is not in contradiction with Theorem 5.1 which holds for $n=9$ $(m=8)$ too.

[14]Let us recall that such definition of Weierstrass analyticity implies complex analyticity as a particular case. See Appendix B of Ref. 1.

[15]Note that the starting point of Theorem 5.3 is an arbitrary point of $S_m\in \underline{\mathcal{T}}'^{(2)}_m$, namely a point $(x_1,\ldots, x_n)$ for which $\xi_k=x_k-x_{k+1}\in S_m$ $(k=1, 2,\ldots, m)$ is spacelike. This is already a generalization from the use of a "totally" spacelike point, namely a point $(x_1,\ldots, x_n)$ for which all differences $x_i-x_j$ are spacelike.