Modern Physics Letters A, Vol. 13, No. 4 (1998) 327–335

ORIGIN, PROBLEMATIC ASPECTS AND INVARIANT FORMULATION OF q-, k- AND OTHER QUANTUM DEFORMATIONS

RUGGERO MARIA SANTILLI*
Institute for Basic Research, P. O. Box 1577, Palm Harbor, FL 34682, USA

Received 4 November 1997
Revised 27 January 1998

In this letter we indicate some of the first references on quantum deformations, we point out their physical problematic aspects and outline their invariant formulation which appears to be unknown in the rather vast literature in the field.

1. Origin of Quantum Deformations

The first deformations of the quantum mechanical (QM) product $[A, B] = AB - BA$ of type

$$(A, B) = pAB - qBA\,, \tag{1}$$

(where p, q and $p \pm q$ are non-null *parameters*; A, B are Hermitian operators; and pA, AB, etc., are conventional associative products), was published by Santilli in 1967 (Eq. (8) of Ref. 1a) as part of his Ph.D. Thesis. The first known infinitesimal and finite forms of the deformed time evolution

$$i\frac{dA}{dt} = (A, H) = pAH - qHA\,, \tag{2a}$$

$$A(t) = UA(0)U^{\dagger} = e^{iHqt}A(0)e^{-itpH}\,, \tag{2b}$$

was identified in the subsequent paper.[1b] The first known classical counterpart was identified in Ref. 1c via the deformation of Hamilton's equations (here expressed for the simple case with $m = 1$ for which $p = mv = v$)

$$\frac{dx}{dt} = p\frac{\partial H}{\partial v}\,, \qquad \frac{dv}{dt} = -q\frac{\partial}{\partial x}\,. \tag{3}$$

The first known more general deformations of QM with product

$$(A, B) = APB - BQA = (AMB - BMA) + (ANB + BNA)\,, \tag{4}$$

PACS Nos.: 03.65.-w, 03.65.Bz, 03.65.Fd
*E-mail: ibr@gte.net

where $P = M + N$, $Q = M - N$ and $P \pm Q$ are nonsingular *matrices* or *operators* admitting p and q *parameters* as particular cases, was published by Santilli in 1978 (Eq. (4.14.11) of Ref. 2a) jointly with the related infinitesimal and finite forms of the deformed time evolution (Eqs. (4.15.34) and (4.18.11) of Ref. 2a)

$$i\frac{dA}{dt} = (A, H) = APH - HQA\,, \tag{5a}$$

$$A(t) = e^{iHQt}A(0)e^{-itPH}\,. \tag{5b}$$

The transition from Eqs. (2) to (5) was rendered necessary by the fact that time evolution (2) is evidently *nonunitary*, $UU^\dagger \neq I$, when formulated on a conventional Hilbert space. The image of Eqs. (2) under their own action then yields Eqs. (5) with $A' = UAU^\dagger$, $H' = UHU^\dagger$, $P = p(UU^\dagger)^{-1}$ and $Q = q(UU^\dagger)^{-1}$, as one can easily verify.

The first known classical counterpart of Eqs. (5) was identified by Santilli[2b] also in 1978 as consisting of the historical Hamilton's equations, those *with external terms*,

$$\frac{dx}{dt} = \frac{\partial H}{\partial v}\,, \qquad \frac{dv}{dt} = -\frac{\partial H}{\partial x} + F(t, x, v, \ldots)\,, \tag{6}$$

only rewritten in such a form to admit an algebra (i.e. their brackets verify the left and right distributive and scalar laws),

$$\frac{dA}{dt} = (A, H) = \frac{\partial A}{\partial b^\mu} S^{\mu\nu} \frac{\partial H}{\partial b^\nu}\,, \qquad b = (x, v)\,, \tag{7a}$$

$$S^{\mu\nu} = \omega^{\mu\nu} + s^{\mu\nu}\,, \qquad (s^{\mu\nu}) = \text{diag.}\left(0, \frac{F}{\partial H/\partial v}\right)\,, \tag{7b}$$

where $\omega^{\mu\nu}$ is the canonical Lie tensor.

The first known (P, Q)-operator deformations of the SU(2) algebra, evidently admitting the (p, q)-parameter deformations as particular cases, was presented by Santilli at the conference *Differential Geometric Methods in Theoretical Physics*, held at the University of Clausthal, Germany, in 1980, and published in 1981.[2c]

A comprehensive presentation of the above deformations was then done in monographs.[3c,3d,3f]

As is well known, primary emphasis is given in QM in the identification of the *algebra* characterized by basic product of the time evolution, the celebrated Lie product $[A, B] = AB - BA$. Along similar lines, primary emphasis was given in Refs. 1–3 for the identification of the algebra characterized by the new products $(A, B) = APB - BQA$ which, after laborious research in the specialized mathematical literature, resulted to be of Albert's[4] *Lie-admissible and Jordan-admissible type*, i.e. such that the antisymmetric and symmetric components of the product, $[A, B] = (A, B) - (B, A) = AMB - BMA$ and $\{A, B\} = (A, B) + (B, A) = ANB + BNA$, $M = (P + Q)/2$, $N = (P - Q)/2$, are Lie and Jordan, respectively.[1–3]

Note the case with $P = Q = T$ and product $[A, B] = ATB - BTA$ preserving the original Lie axioms, which was first introduced by Santilli in Refs. 2 and 3 under the name of *Lie-isotopic theory*, including the isotopies of the basic unit, enveloping associative algebras, Lie algebras, Lie groups, Lie symmetries, transformations and representation theories (see Ref. 5 for independent studies).

The q-deformations subsequently introduced in 1989 by Biedenharn[6a] and, independently, by Macfarlane[6b]

$$(A, B) = AB - qBA\,, \tag{8}$$

are an evident *particular case* of deformations (Eq. (1)) for $p = 1$ or of deformations (Eq. (4)) for $P = 1$ and $Q =$ parameter. The same is the case for vast number of subsequent papers on q-deformations (see, e.g., Ref. 7 and references therein), k-deformations,[8] and other deformations, e.g., certain quantum groups,[9] here generically called "quantum deformations".

The first motivation of this letter is that the origination of quantum deformations in Refs. 1–3 is generally unknown in the literature.[6–9] The communication of any reference on product (Eq. (1)) prior to 1967 or on product (Eq. (4)) prior to 1978 would be appreciated.

2. Physical Shortcomings of Quantum Deformations

The quantum deformations of the preceding section, as well as other generalizations of QM, have a rather interesting *mathematical* structure which deserves studies. However, when formulated on conventional Hilbert spaces over conventional fields, they have rather serious *physical* shortcomings which have been studied in detail by Okubo,[10a] Lopez,[10b] Jannussis and Skaltzas,[10c] Jannussis, Mignani and Santilli,[10d] Schuch,[10e] Santilli,[10f] and others.

The second motivation of this letter is to indicate these physical shortcomings because they too are generally unknown in the literature.[6–10] This task can be effectively done via the following theorem which is a broader version of the results of Ref. 10:

Theorem 1. All deformations of quantum mechanics with a nonunitary time evolution defined on conventional Hilbert spaces over conventional fields have the following physical shortcomings:

(a) Lack of invariance of the basic units of space and time with consequential lack of physically acceptable applications to actual measurements.
(b) Lack of preservation of the original hermiticity in time, with consequential lack of physically acceptable observables.
(c) General lack of unique and invariant numerical predictions.
(d) General violation of causality and probability laws; and
(e) General violation of the axioms of the special relativity.

As one can see, the above problem of physical consistency are rather serious indeed and, as such, they should be quoted in the literature in quantum deformations as well as addressed. The proof of the theorem is a direct consequence of the *nonunitary* character of the time evolutions with consequential lack of invariance of space and time units, hermiticity, etc. as the reader can verify. We should note that Theorem 1 includes:

(a) (p,q)-, q-, (P,Q)- and k-deformations[1–3,6–8];

(b) *dissipative models in nuclear physics* see; e.g. Ref. 11 represented with an "imaginary potential" $H = H_0 + iV \neq H^\dagger$, and infinitesimal time evolution characterized by the triple system

$$A(t) = UA(0)U^\dagger = e^{iht}A(0)e^{-itH^\dagger} \approx -i(AH^\dagger - HA) = -i(A, H, H^\dagger)\,, \quad (9)$$

whose nonunitary character is self-evident;

(c) *theories in statistical mechanics with external terms* (see Ref. 12 and references therein) characterized by the dynamical equations for the density matrix

$$i\frac{d\rho}{dt} = (\rho, H) = \rho H - H\rho + C\,, \quad (10)$$

which have no finite time evolution, let alone a nonunitary one, and violate the conditions for the product (ρ, H) to characterize an algebra (scalar and distributive laws);

(d) the so-called *star model*[13] with lifting of the associative product

$$A * B = ATB = \text{iso-associative}\,, \quad (11)$$

which is *exactly* that of the Lie-isotopic theory[2,3,5] indicated in Sec. 1;

(e) *Weinberg's*[14a] *nonlinear theory with non-associative envelopes*

$$A * B = \text{non-associative Lie-admissible}\,, \quad (12)$$

which admits no unit at all, violates *Okubo's no-go theorem on quantization*[10a] (implying lack of equivalence between Heisenberg-, and Schrödinger-type representations under a non-associative envelope) and has other shortcomings (see Ref. 10d for details), while the reformulation of Ref. 14b *coincides* with the Lie-isotopic theory via the basic rule of Ref. 2a $[A,B] = (A,B) - (B,A) = ATB - BTA$, $T = P + Q$, which turns the *non-associative* envelope with product (A,B) into the iso-associative envelope with product $A * B = ATB$, T fixed;

(f) the statistics by Prigogine *et al.*[15] which also has a nonunitary structure;

(g) the Lie-admissible model of black holes by Ellis, Mavromatos and Nanopoulos[16];

(h) the so-called *squeezed states theories*[17];

(i) noncanonical time theories[18];

(j) *supersymmetric theories*[19] because their product is an evident particular case of the joint Lie-admissible and Jordan-admissible product (4) for M and N constants with consequential nonunitary time evolution;

(k) *Kac–Moody superalgebras*[20] because they are also a particular case of product (Eq. (4)) with M a constant and N representing a matrix of phase factors $(-1)^{ji}$ (see the original notion of Lie-admissibility in Ref. 2b in which the P and Q matrices depend on the generators), thus having a nonunitary time evolution;

and any other possible generalization of QM with a nonunitary structure.

In regard to the so-called "quantum groups"[9] we should mention that, when formulated in terms of the Hopf algebras, their enveloping algebra is generally isomorphic to the conventional quantum envelope. This generally implies *unitary* time evolutions and the consequential *lack* of activation of Theorem 1. However, there are quantum groups characterizing nontrivial *deformations of the structure constants* of space–time symmetries which are also nonunitary, thus implying the activation of Theorem 1.

We should indicate that a classical counterpart of Theorem 1 has also been formulated in Ref. 10f. It essentially includes all *noncanonical* theories on conventional spaces over conventional fields, including deformations of Euclidean or Minkowskian geometries, as well as various generalizations of Hamiltonian mechanics[21] which have resulted not to leave invariant the basic units of space and time, thus lacking clear applicability to measurements.

We should finally indicate that, by no means, the representation of open-nonconservative systems necessarily requires nonunitary time evolutions. In fact, a number (although not all) of open systems can indeed be represented via generalized, yet Hermitian Hamiltonians, in which case the time evolution is unitary and Theorem 1 is then inapplicable although in the latter case other problems arise (for details, and literature see for brevity Refs. 4e and 4f).

3. Lie-Admissible Invariant Formulation of Quantum Deformations

The third motivation for this letter is to indicate the on-going efforts for the resolution of the physical inconsistencies of Theorem 1, which are also generally unknown in the literature on generalized theories.

A technical understanding of the topic requires the knowledge that our joint Lie-admissible and Jordan-admissible (P, Q)-operator deformations on each bracket (Eq. (4)) and time evolution (5) are "directly universal", in the sense that they admit as particular cases *all* infinitely possible generalizations of quantum mechanics with a consistent algebraic structure indicated in Sec. 2 (universality), and the representation occurs in the fixed coordinates of the observer without the use of the transformation theory (direct universality).

The above occurrence is important because axiomatic problems can only be studied for the unified representation (4), (5) of quantum deformations, rather than for a large variety of seemingly different cases, such as q-deformations, dissipative nuclear models, supersymmetric models, Kac–Moody algebras, etc.

The physical inconsistencies of Theorem 1 occur when quantum deformations are formulated and elaborated via the *conventional mathematics* of QM. In fact, Santilli has insisted through the years on the need for *a new mathematics specifically constructed for the treatment of quantum deformations*, which was originally proposed under the name of *genotopic mathematics* or *genomathematics* for short[2b,3b,3d] and has now reached sufficient maturity for physical applications.[10f,22] *Santilli's Lie-admissible formulations* are generally referred to in the literature (see, e.g., Ref. 5 and references therein) to deformations (Eq. (4)) or (5) when entirely formulated and elaborated with the new genomathematics. The *Lie-isotopic formulations* are referred to theories entirely formulated with the *isotopic* subcase of genomathematics called *isomathematics* [*loc. cit.*].

As a specific illustration, if one deforms the associative product $AB \to A * B = qAB$ while defining it on conventional spaces over conventional fields with the conventional unit I (as done in Refs. 6–8, 13 and in most other papers in q-deformations), it is evident that the *old unit* I of AB is no longer invariant under a theory based on the *new product* $A * B$, and no meaningful physical application is possible, to our knowledge (shortcoming (a) of Theorem 1).

Ironically, by the time of the appearance of the q-deformations of Refs. 6 in 1989, Santilli had long abandoned their formulation on conventional spaces over conventional fields because of the above inconsistencies. It is therefore unfortunate that Refs. 1–3 had not propagated to Refs. 6 and to the subsequent literature in the field.

The new genomathematics is based on the following *two* different generalizations of the *unit* of QM, one on each ordered product,

$$^{<}I = 1/P\,, \qquad A < B = APB\,, \qquad ^{<}I < A = A < {^{<}I} = A\,, \tag{13a}$$

$$I^{>} = 1/Q\,, \qquad B > A = BQA\,, \qquad I^{>} > A = A > I^{>} = A\,, \tag{13b}$$

which require *two* different reconstructions of conventional mathematics, one on each ordered product and related unit. When all products are ordered to the right, $A > B$ (to the left, $A < B$), the theory is used for the representation of *motion forward in time* (*motion backward in time*). Deformations (5) are, therefore, intrinsically irreversible, that is, they are irreversible even for reversible Hamiltonians. Moreover, they can *only* be used to characterize *open-nonconservative systems* (as it is evidently the case also for q-deformations), because of the rule $i\,dH/dt = H(P - Q)H \neq 0$.

The representation of closed-isolated systems with an irreversible internal structure is instead possible for the Lie-isotopic formulations for which $i\,dH/dt = HTH - HTH \equiv 0$,[2,3] in which case irreversibility can be represented via the operator $T(t) \neq T(-t)$ under a fully reversible and conserved Hamiltonian $H(t) \equiv H(-t)$.

The main lines of the resolution of the physical shortcomings of Theorem 1 are the following.[10f,22] First, two generalizations of conventional fields F (real R, complex C or quaternionic fields Q) are constructed, $F^>$ and ${}^<F$, *called genofields to the right and to the left*, with corresponding units $I^>$ and ${}^<I$, elements $n^> = nI^>$ and ${}^<n = {}^<In$, called *genonumbers*, and ordered products $n^> > m^> = (nm)I^>$ and ${}^<n < {}^<m = {}^<I(nm)$. Note that for real or complex numbers we have the commutativity properties $n > m = m > n$ and $n < m = m < n$. Nevertheless, the two ordered products are different, $n > m \neq n < m$. Note also that each genofield preserves all axioms of ordinary fields.

Then, the entire original mathematics must be reconstructed over $F^>$ and, separately, over ${}^<F$, in such a way to admit $I^>$ and, separately, ${}^<I$ as the left and right units. This implies the construction of two genospaces, genodifferential calculus, genospecial functions, genogeometries, genotopologies, etc. The emerging new mathematics then assures the invariance of the units by conception and construction, thus resolving inconsistency (a) of Theorem 1.

After that, one must select one given ordered product, say, that to the right $A > B$. The following *forward genotopy of the Schrödinger's representation* is then constructed

$$i^> > \partial_t^> |\,\rangle = iQ_t^{-1}\partial_t |\,\rangle = H > |\,\rangle = HQ_S |\,\rangle = E^> > |\,\rangle = E |\,\rangle\,, \qquad (14)$$

where we have used the time and space decomposition $Q = \mathrm{diag}(Q_t, Q_s)$ for non-relativistic settings, which is defined over the *geno-Hilbert space* $\mathcal{H}^>$ with product

$$\langle |Q_S| \rangle I_S^> \in C^>\,. \qquad (15)$$

It is easy to see that the *forward hermiticity* coincides with the conventional hermiticity and, thus, all quantities which are Hermitian-observables on $\mathcal{H}$ over C remain so on $\mathcal{H}^>$ over $C^>$. This yields the first *hermiticity of a nonconserved Hamiltonian* known to this author. It is also easy to prove that the genoeigenvalues of genohermitian operators are genoreal (i.e. $E^> = EI^>$ where E is real), and that the genohermiticity is conserved under the motion forward in time characterized by Eq. (14). This resolves inconsistency (b) of Theorem 1.

Note the new invariance law whenever Q_S is independent of the integration variables, $\langle |Q_S| \rangle I_S^> \equiv \langle | \rangle I$. Therefore, the Lie-admissible formulations are merely characterized by a *"hidden" degree of freedom of the conventional Hilbert spaces.* The emerging theory is then of the form "completion" of quantum mechanics much along the historical argument by Einstein, Podolsky and Rosen.[10f] These occurrences have remained undetected until recently because they required the prior discovery of *new numbers*, those with ordered product to the right or to the left and the corresponding arbitrary units.

The preservation of a Hilbert structure then permits the resolution of inconsistencies (c) and (d) of Theorem 1 (see Ref. 22b for details).

The theory is then automatically form-invariant, because nonunitary transforms $UU^\dagger = I^> \neq I$ can be identically rewritten as *genounitary transforms* on $\mathcal{H}^>$

over $C^{>}$, $U = U^{>}Q_S^{1/2}$, $UU^{\dagger} = U^{>} > U^{>\dagger} = U^{>\dagger} > U^{>} = I^{>}$ under which $I'^{>} = U^{>} > I^{>} > U^{>\dagger} \equiv I^{>}$, $U^{>} > (A > B) > U^{>\dagger} = A' > B'$, etc. Note not only the *invariance* of the generalized unit, but also *the preservation of its numerical value.* Note also the *uniqueness* of the above Lie-admissible formulations, in the sense that any other *invariant* formulations of open (closed) quantum deformations is necessarily isomorphic to the Lie-admissible (Lie-isotopic) formulations (for detail, see Ref. 10f).

The reconstruction of special relativity is under study on certain bimodule over bifields in which all axioms of the special relativity can apparently be reconstructed as being *exact for open-nonconservative conditions.*[22b]

Needless to say, the above studies are at their initiation and so much remains to be done. Nevertheless, a conclusion which can be drawn following studies conducted during the past three decades is that the majestic axiomatic consistency and invariance of QM is lost whenever the mechanics is deformed, while preserving the original mathematics, that is, the original numbers, fields, spaces, geometries, topologies, etc. On the contrary, realistic possibilities for the preservation of the original axiomatic consistency and invariance under open or closed quantum deformations exist via the use of *new mathematics* specifically constructed for that purpose.

References

1. (a) R. M. Santilli, *Nuovo Cimento* **51**, 570 (1967); (b) *Suppl. Nuovo Cimento* **6**, 1225 (1968); (c) *Meccanica* **1**, 3 (1969).
2. (a) R. M. Santilli, *Hadronic J.* **1**, 574 (1978); (b) *ibid.* **1**, 228 (1978); (c) *ibid.* **4**, 1166 (1981); (d) *Phys. Rev.* **D20**, 555 (1979).
3. (a) R. M. Santilli, *Foundations of Theoretical Mechanics* (Springer, 1978), Vol. 1; (b) *ibid.* (Springer, 1983), Vol. 2; (c) *Lie-Admissible Approach to the Hadronic Structure* (1978), Vols. I; (d) *ibid.* (1982), Vol. II; (e) *Elements of Hadronic Mechanics* (Ukraine Academy of Sciences, 1995), Vol. I; (f) *ibid.*, Vol. II.
4. A. A. Albert, *Trans. Amer. Math. Soc.* **64**, 552 (1948).
5. D. M. Norris *et al.*, *Tomber's Bibliography and Index in Nonassociative Algebras* (Hadronic Press, 1984); A. K. Aringazin, A. Jannussis, D. F. Lopez, M. Nishioka and B. Veljanosky, *Santilli's Lie-Isotopiac Generalization of Galileiÿs and Einstein's Relativities* (Kostarakis Publisher, Athens, 1990); J. V. Kadeisvili, *Santilli's Isotopies of Contemporary Algebras, Geometries and Relativities* (Hadronic Press, 1991): *ibid* (Ukraine Academy of Sciences, 1997), 2nd edition; D. S. Sourlas and G. T. Tsagas *Mathematical Foundations of the Lie–Santilli Theory* (Ukraine Akademy of Sciences 1993); J. Lohmus, E. Paal and L. Sorgsepp, *Nonassociative Algebras in Physics* (Hadronic Press, 1994); J. V. Kadeisvili, *Math. Methods Appl. Sci.* **19**, 1349 (1996)
6. (a) L. C. Biedernharn, *J. Phys.* **A22**, L873 (1989); (b) A. J. Macfarlane, *ibid.* **A22** L4581 (1989).
7. V. Dobrev, in *Proceedings of the Second Wigner Symposium*, Clausthal 1991 (World Scientific, 1991); J. Lukierski, A. Novicki, H. Ruegg and V. Tolstoy, *Phys. Lett.* **B261** 331 (1991); O. Ogivetski, W. B. Schmidke, J. Wess and B. Zumino, *Commun. Math. Phys.* **50**, 495 (1992); S. Giller, J. Kunz, P. Kosinky, M. Majewskil and P. Maslanka *Phys. Lett.* **B286**, 57 (1992).

8. J. Lukierski, A. Nowiski and H. Ruegg, *Phys. Lett.* **B293**, 344 (1992); J. Lukierski, H. Ruegg and W. Ruhl, *ibid.* **B313**, 357 (1993); J. Lukierski and H. Ruegg, *ibid.* **B329**, 189 (1994); S. Majid and H. Ruegg, *ibid.* **B334**, 348 (1994).
9. Eds. T. L. Curtis, B. Fairlie and Z. K. Zachos, *Quantum Groups* (World Scientific, 1991); *Introduction to Quantum Groups and Integrable Massive Models of Quantum Field Theory*, eds. M.-L. Ge and B. H. Zhao (World Scientific, 1991); *Proc. of the Fifth Workshop Symmetry Methods in Physics*, eds. Yu. F. Smirnov and R. M. Asherova (JINR, 1992).
10. (a) S. Okubo, *Hadronic J.* **5**, 1667 (1982); D. F. Lopezz, in *Symmetry Methods in Physics*, eds. A. N. Sissakian, G. S. Pogosyan and X. I. Vinitsky (JINR, 1994); (b) *Hadronic J.* **16**, 429 (1993); (c) A. Jannussis and D. Skaltsas, *Ann. Fond. L. de Broglie* **18**, 137 (1993); (d) A. Jannussis, R. Mignani and R. M. Santilli, *Ann. Fond. L. de Broglie* **18**, 371 (1993); (e) D. Schuch, *Phys. Rev.* **A55**, 955 (1997); (f) R. M. Santilli, *Found. Phys.* **27**, 625 (1997).
11. J. M. Blatt and V. F. Weiskopf, *Theoretical Nuclear Physics* (J. Wiley & Sons, 1964).
12. P. Roman and R. M. Santilli, *Lett. Nuovo Cimento* **2**, 449 (1969).
13. F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowitz and D. Sternheimer, *Ann. Phys.* **111**, 61; 111 (1978).
14. (a) S. Weinberg, *Ann. Phys.* **194**, 336 (1989); (b) T. F. Jordan, *ibid.* **225**, 83 (1993).
15. Cl. George, F. Henin, F. Mayne and I. Prigogine, *Hadronic J.* **1**, 520 (1978).
16. J. Ellis, N. E. Mavromatos and D. V. Nanopoulos, in *Proceedings of the Erice Summer School, 31st Course: From Superstrings to the Origin of Space–Time* (World Scientific, 1996).
17. S. Sebawe Abdallah *et al.*, *Physica* **A163**, 822 (1990); 202; 301 (1994); *Phys. Rev.* **A48**, 1526; 3174 (1993); *J. Modern Optics* **39**, 771; 1067 (1992); **40**, 441; 1351; 1369 (1993); *Phys. Lett.* **A181**, 341 (1993).
18. A. Jannussis *et al.*, *Lett. Nuovo Cimento* **29**, 427 (1980); *Nuovo Cimento* **B103**, 537 (1989).
19. R. N. Mahopatra, *Unification and Spersymmetries: The Frontirs of Quark–Lepton Physics* (Springer-Verlag, 1992), 2nd edition.
20. V. G. Kac, *Infinite Dimensional Lie Algebras* (Birkhäuser, 1983).
21. (a) G. D. Birkhoff, *Dynamical Systems* (AMS, 1927); (b) C. G. Sudarshan and N. Mukunda, *Classical Dynamics, A Modern Perspective* (Wiley-Interscience, 1974).
22. (a) R. M. Santilli, *Rendiconti Circolo Matematico Palermo, Suppl.* **42**, 7 (1996); (b) *Found. Phys.* **27**, 1157 (1997); (c) "Invariant Lie-admissible formulation of dissipative nuclear models", submitted for publication.