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Necessary and Sufficient Conditions for the Existence of a Lagrangian in Field Theory. III. Generalized Analytic Representations of Tensorial Field Equations*

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In this paper we first study the equivalence transformations of class $\mathscr{C}^2$, regular, tensorial, quasi-linear systems of field equations which (a) preserve the continuity, regularity, and quasi-linear structure of the systems; and (b) occur within a fixed system of Minkowski coordinates and field components. We identify, among the transformations of this class, those which either induce or preserve a self-adjoint structure of the field equations and we term them genotopic and isotopic transformations, respectively. We then give the necessary and sufficient conditions for an equivalence transformation of the above type to be either genotopic or isotopic. By using this methodology, we then extend the theorem on the necessary and sufficient condition for the existence of ordered direct analytic representations introduced in the preceding paper to the case of ordered indirect analytic representations in terms of the conventional Lagrange equations; we introduce a method for the construction of a Lagrangian, when it exists, in this broader context; and we explore some implications of the underlying methodology for the problem of the structure of the Lagrangian capable of representing interactions within the framework of the indirect analytic representations. Some of the several aspects which demand an inspection prior to the use of this analytic approach in actual models are pointed out. In particular, we indicate a possible deep impact in the symmetries and conservation laws of the system generated by the use of the concept of indirect analytic representation. As a preparatory step prior to the analysis of these problems, we study some methodological aspects which underlie the generalized Lagrange equations postulated in the first paper of this series for the case when they are regular, namely, when they are simple equivalence transformations of the conventional Lagrange equations. We first introduce a generalization of the action principle capable of inducing the generalized as well as the conventional equations. In this way we establish that the former equations are "bona fide" analytic equations. Finally, as our most general analytic framework for the case of unconstrained field equations, we work out the necessary and sufficient condition for the existence of ordered direct analytic representations of quasi-linear systems in terms of the generalized analytic equations and study their relationship to the conventional representations.

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1. Introduction

In the preceding paper, II [1] we proved a theorem according to which a necessary and sufficient condition for a class $\mathscr{C}^2$, regular, tensorial, quasi-linear system of field equations to admit an ordered direct analytic representation in terms of the conventional Lagrange equations is that the system be self-adjoint.

A problem which immediately surfaces is that such an analytic framework is considerably *restrictive* because the class of field equations under consideration is generally non-self-adjoint and, therefore, a Lagrangian for their ordered direct analytic representation *does not*, in general, exist. This is the case, for instance, of the simple equation for the real scalar field with nonlinear self-couplings

$$[(\Box + m^2)\,\varphi + \lambda\varphi^2 + (1/\varphi)\,\varphi^{;\mu}\varphi^{;}{}_{\mu}]_{\mathrm{NSA}} = 0. \tag{1.1}$$

The objective of this paper is to explore a broadening of the analytic framework of paper II capable of allowing the analytic representation of a larger class of field equations.

Let us recall from paper I that, according to our definition, an analytic representation occurs whenever the Lagrange equations coincide with the field equations up to equivalence transformations. It is precisely such a concept of analytic representation which will allow us to achieve our objective.

By closely following, again, the analysis conducted by this author in the forthcoming monographs on the Newtonian aspect of the problem [2], we shall first study, in Section 2, the equivalence transformations of the systems of field equations considered which

(*a*) preserve the continuity, regularity, and linearity of the field equations in the field "accelerations"; and
(*b*) occur within a fixed system of Minkowski coordinates and field components;

i.e., equivalence transformations of the type

$$\{h_a{}^b(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)[A^{\mu\nu}_{bd}(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)\,\phi^{d;}{}_{\mu\nu} + B_b(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)]^{\mathscr{C}^2,R}\}^{\mathscr{C}^2,R} = 0,$$
$$a, b, c = 1, 2, \ldots, n; \quad \mu, \nu, \alpha = 0, 1, 2, 3. \tag{1.2}$$

We shall then term such transformations *genotopic*, when they *induce* a self-adjoint structure, i.e., when the original system is non-self-adjoint but the equivalent system is self-adjoint; *isotopic*, when they *preserve* a self-adjoint structure, i.e., when the original system is self-adjoint and the equivalent system is also self-adjoint.

Finally, we shall give in Section 2 the necessary and sufficient conditions for a transformation of type (1.2) to be either genotopic or isotopic.

The above concepts will then allow us in Section 3 to (a) extend the analytic framework of paper II to the case of ordered *indirect* analytic representations in terms of the conventional Lagrange equations

$$\left[d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{a;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^a}\right]^{\mathscr{C}^2,R}_{\mathrm{SA}} \equiv [h_a{}^b(A^{\mu\nu}_{bd}\phi^{d;}{}_{\mu\nu} + B_b)^{\mathscr{C}^2,R}]^{\mathscr{C}^2,R}_{\mathrm{SA}} = 0, \qquad a = 1, 2, \ldots, n; \tag{1.3}$$

(b) give a method for the construction of the Lagrangian from a given system of field equations and of factor terms $h_a{}^b$, when it exists; and (c) explore the significance of this broader framework for the problem of the structure of the Lagrangian capable of representing interactions.

The problem of "universality," namely, whether a Lagrangian for the analytic representations of type (1.3) *always* exists, is touched in Appendix A. In this appendix we shall further broaden the concept of *indirect* analytic representations with the inclusion of (class $\mathscr{C}^2$, invertible, single-valued) transformations of the Minkowski coordinates and of the field components [3]

$$x \to x' = x'(x); \qquad \phi \to \phi' = \phi'(x, \phi) \tag{1.4}$$

according to the structure

$$\left[d_\mu \frac{\partial \mathscr{L}}{\partial \phi'^{a;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi'^a}\right]_{\mathrm{SA}}^{\mathscr{C}^2, R}$$
$$\equiv \{h'^b_a(x_\alpha', \phi'^c, \phi'^{c;}{}_\alpha)[A'^{\mu\nu}_{bd}(x_\alpha', \phi'^c, \phi'^{c;}{}_\alpha)\, \phi'^{d;}{}_{\mu\nu} + B_b'(x_\alpha', \phi'^c, \phi'^{c;}{}_\alpha)]^{\mathscr{C}^2, R}\}_{\mathrm{SA}}^{\mathscr{C}^2, R} = 0$$
$$a = 1, 2, \ldots, n. \tag{1.5}$$

In essence, such further enlargement is advisable whenever the n^2 factor terms $h_a{}^b$ of transformations (1.2) are insufficient to induce (or to preserve) a self-adjoint structure. And indeed, the inclusion of the transformations of type (1.4) into the concept of indirect analytic representations adds further degrees of freedom which may be useful in achieving the consistency of the underlying overdetermined system of partial differential equations for the characterization of a Lagrangian.

The analytic representations of type (1.5) constitute, to the best knowledge of this author, the broadest analytic framework which can be formulated in terms of the conventional (unconstrained) Lagrange equations.

Despite that, the underlying methodology is still incomplete. This is due to the fact that, while the methodology which underlies the use of transformations of type (1.4) is well established, there exist several methodological aspects related to the use of the factor terms $h_a{}^b$ which have not previously been explored and which must be explored prior to their use in actual models. It is sufficient to note in this respect that equivalence transformations of type (1.2) may deeply affect the symmetries and conservation laws of the system because the factor terms $h_a{}^b$ have, in general, a dependence in the field components ϕ^c, their partial derivatives $\phi^{c;}{}_\alpha$, *and* the Minkowski coordinates x_α [4].

As a preparatory step prior to our analysis of some of these problems (which will be conducted in paper IV) we shall explore in the remaining part of this paper some methodological aspects related to the *generalized Lagrange equations* postulated in in paper I, i.e. [5],

$$g_a{}^b \mathscr{L}_b(\phi) \equiv g_\alpha{}^L(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)\left[d_\mu \frac{\partial \mathscr{L}(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)}{\partial \phi^{b;}{}_\mu} - \frac{\partial \mathscr{L}(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)}{\partial \phi^b}\right] = 0,$$
$$a, b, c = 1, 2, \ldots, n; \quad \mu, \alpha = 0, 1, 2, 3. \tag{1.6}$$

Let us recall from paper I that the above equations might be significant in field theory because of the following properties.

(1) Equations (1.6) can be formulated in a way consistent with the customary condition of Lorentz covariance for a suitable selection of the $(n^2 + 1)$-densities which characterize such equations, namely, the Lagrangian $\mathscr{L}$ and the n^2 factor terms $g_a{}^b$. We shall often refer to Eqs. (1.6) with the symbol $(g_a{}^b, \mathscr{L})$;

(2) Eq. (1.6) contain the conventional Lagrange equations, trivially, at the limit when the factor terms $g_a{}^b$ became the Kronecker $\delta_a{}^b$. We shall often refer to the conventional Lagrange equations with the symbol $(\delta_a{}^b, \mathscr{L})$. The transition from the generalized to the conventional Lagrange equations, or vice versa, can be then indicated with the notation

$$(\delta_a{}^b, \mathscr{L}) \leftrightarrow (g_a{}^b, \mathscr{L}). \tag{1.7}$$

(3) When the matrix $(g_a{}^b)$ of the factor terms is regular in the region of definition of the Lagrange equations [6], Eqs. (1.6) constitute simple equivalence transformations of type (1.2), but now applied to the analytic equations rather than to the field equations. Notice that, as was the case of transformations (1.2), we exclude a dependence of the factor terms from the second-order derivatives $\phi^{c;}{}_{\alpha\beta}$. This guarantees the preservation of the quasi-linear structure of the Lagrange equations. The additional condition that the terms $g_a{}^b$ must be of at least class $\mathscr{C}^2$ in the same region of definition of the Lagrange equations then completes the equivalence between the conventional and the generalized Lagrange equations and we shall symbolically write

$$\{g_a{}^b[\mathscr{L}_b(\phi)]_{\mathrm{SA}}^{\mathscr{C}^2,R}\}^{\mathscr{C}^2,R} = 0. \tag{1.8}$$

Throughout this paper we shall only consider generalized Lagrange equations which satisfy the above conditions.

(4) Equations (1.8) transparently exhibit the fact that the knowledge of only one density, the Lagrangian density, is generally *insufficient* to characterize an analytic representation because n^2 additional densities, the factor terms $g_a{}^b$ in our formulation, are needed. This property is somewhat hidden in the analytic representations (1.3) with the conventional Lagrange equations. More explicitly, the concept of equivalence transformation which is rooted in our definition of analytic representation can be applied either to the field equations or to the analytic equations. As a consequence, the generalized analytic equations (1.8) are implicit in our concept of analytic representations. And indeed, in view of the assumed regularity of the matrix $(h_a{}^b)$, the ordered *indirect* representations (1.3) in terms of the *conventional* Lagrange equations can be trivially turned into the ordered *direct* analytic representations in terms of the *generalized* Lagrange equations

$$\begin{gathered}\{g_a{}^b[\mathscr{L}_b(\phi)]_{\mathrm{SA}}^{\mathscr{C}^2,R}\}^{\mathscr{C}^2,R} \equiv [A_{ad}^{\mu\nu}\phi^{d;}{}_{\mu\nu} + B_a]^{\mathscr{C}^2,R} = 0 \\ a = 1, 2,..., n; \quad (g_a{}^b) \equiv (h_a{}^b)^{-1}.\end{gathered} \tag{1.9}$$

A simple example may be useful here to illustrate this concept. Consider the familiar Lagrangian for the free complex scalar field

$$\mathscr{L} = \bar{\varphi}^{;}{}_{\mu}\varphi^{;\mu} - m^2\bar{\varphi}\varphi \tag{1.10}$$

with underlying field equations

$$\begin{aligned} (\Box + m^2)\,\varphi &= 0, \\ (\Box + m^2)\,\bar{\varphi} &= 0. \end{aligned} \tag{1.11}$$

Then we can write either the ordered *indirect* analytic representation in terms of the conventional Lagrange equations

$$\begin{pmatrix} \mathscr{L}_{\varphi} \\ \mathscr{L}_{\bar{\varphi}} \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (\Box + m^2)\,\varphi \\ (\Box + m^2)\,\bar{\varphi} \end{pmatrix} = 0 \tag{1.12}$$

or, equivalently, the ordered *direct* analytic representation in terms of the generalized Lagrange equations

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathscr{L}_{\varphi} \\ \mathscr{L}_{\bar{\varphi}} \end{pmatrix} \equiv \begin{pmatrix} (\Box + m^2)\,\varphi \\ (\Box + m^2)\,\bar{\varphi} \end{pmatrix} = 0, \tag{1.13}$$

where the equivalence of the representations (1.12) and (1.13) trivially follows from the regularity of the factor matrix which, in this case, simply characterizes a permutation. This simple example also illustrates the fact that the Lagrangian (1.10) alone is insufficient to characterize an analytic representation even though the interactions are absent. And indeed, jointly with Lagrangian (1.10) the four elements of the factor matrix are neeced in either representation (1.12) or (1.13).

(5) Equations (1.8) are generally non-self-adjoint [7] and they can be written, in general,

$$\{ g_a{}^b [\mathscr{L}_b(\phi)]_{\mathrm{SA}}^{\mathscr{C}^2, R} \}_{\mathrm{NSA}}^{\mathscr{C}^2, R} = 0. \tag{1.14}$$

This property is significant for the problem of the analytic representations of the field equations. Indeed, it removes the restriction that (class $\mathscr{C}^2$, regular) Lagrange equations must always be self-adjoint by therefore broadening the underlying methodology. The significance of this property becomes transparent if we recall, as pointed out earlier, that the class of field equations considered is generally non-self-adjoint. And indeed, while the conventional Lagrange equations generally fail to produce a direct analytic representation because of their self-adjointness, this is not the case for Eqs. (1.14) and representations (1.9) can be more specifically written

$$\{ g_a{}^b [\mathscr{L}_b(\phi)]_{\mathrm{SA}}^{\mathscr{C}^2, R} \}_{\mathrm{NSA}}^{\mathscr{C}^2, R} \equiv [A_{ad}^{\mu\nu} \phi^{d;}{}_{\mu\nu} + B_a]_{\mathrm{NSA}}^{\mathscr{C}^2, R} = 0, \qquad a = 1, 2, \ldots, n. \tag{1.15}$$

(6) Equations (1.14) can be "bona fide" analytic equations in precisely the same measure as the conventional equations are if they can be derived by an action principle.

(7) Equations (1.14) are significant for the transformation theory of the Lagrangian density. Indeed, for a given Lagrangian, there may exist a nontrivial set set of factor terms $g_a{}^b$ for which they become self-adjoint, in which case we shall write

$$\{g_a{}^b[\mathscr{L}_b(\phi)]_{\mathrm{SA}}^{\mathscr{C}^2,R}\}_{\mathrm{SA}}^{\mathscr{C}^2,R} = 0. \tag{1.16}$$

But the factor terms preserve the quasi-linear structure of the equations by assumption due to their independence from the derivatives $\phi^{c;}{}_{\alpha\beta}$. Therefore, Eqs. (1.16) can be explicitly written

$$\begin{aligned} g_a{}^b\mathscr{L}_b(\phi) &\equiv \Big[\frac{1}{2}\, g_a{}^b \Big(\frac{\partial^2\mathscr{L}}{\partial\phi^{b;}{}_\mu\, \partial\phi^{c;}{}_\nu} + \frac{\partial^2\mathscr{L}}{\partial\phi^{b;}{}_\nu\, \partial\phi^{c;}{}_\mu}\Big)\Big]\, \phi^{c;}{}_{\mu\nu} \\ &\quad + \Big[g_a{}^b \Big(\frac{\partial^2\mathscr{L}}{\partial\phi^{b;}{}_\mu\, \partial\phi^{c}}\, \phi^{c;}{}_\mu + \frac{\partial^2\mathscr{L}}{\partial\phi^{b;}{}_\mu\, \partial x^\mu} - \frac{\partial\mathscr{L}}{\partial\phi^b}\Big)\Big] \\ &\equiv A_{ac}^{*\mu\nu}\phi^{c;}{}_{\mu\nu} + B_a{}^* = 0. \end{aligned} \tag{1.17}$$

As a result, whenever structure (1.16) occurs, it can be interpreted as an ordinary self-adjoint quasi-linear system; our main Theorem II.2.1 holds; and a *new* Lagrangian $\mathscr{L}^*$ for the ordered direct analytic representation

$$[\mathscr{L}_a{}^*(\phi)]_{\mathrm{SA}}^{\mathscr{C}^2,R} \equiv \{g_a{}^b[\mathscr{L}_b(\phi)]_{\mathrm{SA}}^{\mathscr{C}^2,R}\}_{\mathrm{SA}}^{\mathscr{C}^2,R} = 0 \tag{1.18}$$

exists and can be computed from Corollary II.2.1B. The two Lagrangians $\mathscr{L}$ and $\mathscr{L}^*$ represent, by construction, the *same* system and, as such, they are equivalent. But such an equivalence transformation cannot be derived with any of the conventional transformations (1.4) because it occurs *by construction* within a fixed system of Minkowski coordinates and field components. Therefore the equivalence mapping

$$\mathscr{L} \leftrightarrow \mathscr{L}^* \tag{1.19}$$

constructed with the above method characterizes a third identifiable layer of the transformation theory of the Lagrangian density. Such mappings have been termed in paper I *isotopic transformations*. The existence of such equivalence transformations does not, obviously, exclude the use of the transformations induced by the conventional mappings (1.4). Therefore, if we denote with the symbols $'$ and † the equivalence transformations induced by (class $\mathscr{C}^2$, regular, single-valued) transformations of the Minkowski coordinates and of the field components, respectively, and with the symbol $*$ those induced by an isotopic transformation, a general equivalence mapping of the Lagrangian can be written

$$\mathscr{L} \to [(\mathscr{L}')^\dagger]^* \tag{1.20}$$

or in any permuted order of the three indicated types of transformations. It should be stressed, to avoid possible misinterpretations, that the eventual *invariance* of the

Lagrangian under transformations of type (1.4) is here *ignored* and only the *equivalence* aspect is taken into consideration.

A detailed analysis of all the above indicated aspects related to the generalized analytic equations clearly goes beyond the objectives of this paper. Therefore, in the last sections of this paper we shall restrict our analysis only to the representational capability of such equations.

In Section 4, we shall first introduce a generalized action principle which induces the generalized as well as the conventional Lagrange equations. This will establish point (6) above, namely, that the former equations are "bona fide" analytic equations in exactly the same measure as the latter equations are. Then in Section 5 we shall generalize the analytic framework of paper II to the case of the generalized representations (1.15).

As a result of our analysis, it appears that at least three layers of analytic representations exist in field theory. They are

(A) the representations in terms of the *conventional* Lagrange equations with a *conventional* structure of the Lagrangian density, i.e.,

$$d_\mu \frac{\partial \mathscr{L}}{\partial \phi^a{}_{,\mu}} - \frac{\partial \mathscr{L}}{\partial \phi^a} = 0, \qquad \mathscr{L}_{\mathrm{Tot}} = \sum_{1}^{\mu}{}_a \mathscr{L}^{(a)}_{\mathrm{Free}} + \mathscr{L}_{\mathrm{Int}}\,; \tag{1.21}$$

(B) the representations in terms of the *conventional* Lagrange equations with a *generalized* structure of the Lagrangian density (Corollary II.2.1.E),

$$d_\mu \frac{\partial \mathscr{L}}{\partial \phi^a{}_{,\mu}} - \frac{\partial \mathscr{L}}{\partial \phi^a} = 0, \qquad \mathscr{L}^{\mathrm{Gen}}_{\mathrm{Tot}} = \sum_{1}^{\mu}{}_a \mathscr{L}^{(a)}_{\mathrm{Int.I}} \mathscr{L}^{(a)}_{\mathrm{Free}} + \mathscr{L}_{\mathrm{Int.II}}\,; \tag{1.22}$$

(C) the representations in terms of the *generalized* Lagrange equations with a *generalized* structure of the Lagrangian density, i.e.,

$$g_a{}^b \left(d_\mu \frac{\partial \mathscr{L}}{\partial \phi^b{}_{,\mu}} - \frac{\partial \mathscr{L}}{\partial \phi^b}\right) = 0, \qquad \mathscr{L}^{\mathrm{Gen}}_{\mathrm{Tot}} = \sum_{1}^{n}{}_a \mathscr{L}^{(a)}_{\mathrm{Int.I}} \mathscr{L}^{(a)}_{\mathrm{Free}} + \mathscr{L}_{\mathrm{Int.II}}\,. \tag{1.23}$$

We are now in a position to review our program from a perspective viewpoint.

In paper I we considered the conventional analytic framework (A) and analyzed it from a variational approach to self-adjointness.

In paper II we introduced the generalized analytic framework (B) within the context of the necessary and sufficient conditions for the existence of a Lagrangian.

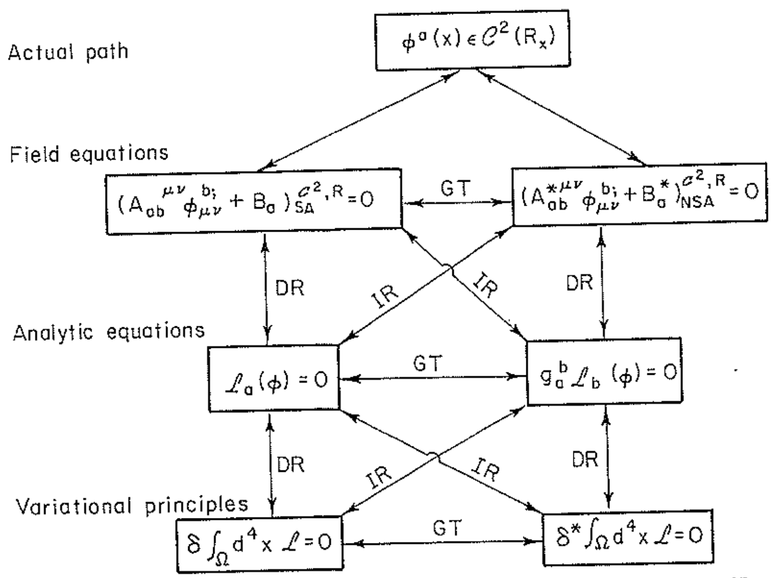
Restated from a somewhat different profile, the major objectives of Sections 4 and 5 are to indicate the existence of the generalized analytic framework of type (C) and then to explore its relationship to the other frameworks.

Our findings are summarized in Table I.

The study of other aspects, such as the problem of symmetries and conservation laws and the canonical formulations of frameworks (B) and (C) or the extension of our findings to spinorial fields, are contempleted as subsequent steps.

TABLE I

A Schematic View of Our Approach to the Lagrangian Representations of Class $\mathscr{C}^2$ Regular Quasi-linear Tensorial Systems of Field Equations[a]



[a] SA = self-adjoint; NSA = non-Self-Adjoint; DR = direct representation; IR = indirect representation; GT = genotopic transformation.

2. Isotopic and Genotopic Transformations of the Field Equations

Consider the set of fields $\hat{\phi}^a(x)$, $a = 1, 2, \ldots, n$, which are the solutions of a class $\mathscr{C}^2$, regular, quasi-linear system of tensorial field equations. It is a known fact that, while the "actual path" $\hat{\phi}^a$ is unique, the system capable of representing it is not. And indeed, there exist infinite varieties of systems of partial differential equations all admitting the same actual path $\hat{\phi}^a(x)$ or an equivalent path $\hat{\phi}'^a(x')$. When the more general latter case occurs, we shall say that the underlying transition [8]

$$F_a(\phi) \equiv [A^{\mu\nu}_{ab}\phi^{b;}{}_{\mu\nu} + B_a]^{\mathscr{C}^2,R} = 0 \xrightarrow{T_{Eq}} F_a'(\phi') \equiv [A'^{\mu\nu}_{ab}\phi'^{b;}{}_{\mu\nu} + B_a']^{\mathscr{C}^2,R} = 0 \quad (2.1)$$

is an *equivalence transformation* of the field equations.

From the viewpoint of self-adjointness, equivalence transformations (2.1) can be classified into the following four classes, depending on whether the original or final systems are self-adjoint or non-self-adjoint.

$$[F_a(\phi)]_{SA} = 0 \xrightarrow{T_{Eq}^{(1)}} [F_a'(\phi')]_{SA} = 0, \tag{2.2a}$$

$$[F_a(\phi)]_{SA} = 0 \xrightarrow{T_{Eq}^{(2)}} [F_a'(\phi')]_{NSA} = 0, \tag{2.2b}$$

$$[F_a(\phi)]_{NSA} = 0 \xrightarrow{T_{Eq}^{(3)}} [F_a'(\phi')]_{NSA} = 0, \tag{2.2c}$$

$$[F_a(\phi)]_{NSA} = 0 \xrightarrow{T_{Eq}^{(4)}} [F_a'(\phi')]_{SA} = 0. \tag{2.2d}$$

Among all possible realizations of equivalence transformations (2.2) we now restrict our analysis to those which are induced by a set of n^2 densities $h_a{}^b(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)$ whose matrix $(h_a{}^b)$ is of (at least) class $\mathscr{C}^2$ and regular in the same region of definition of the field equations according to the mapping

$$[A^{\mu\nu}_{ab}\phi^{b;}{}_{\mu\nu} + B_a]^{\mathscr{C}^2,R} = 0 \xrightarrow{T_{Eq}} [h_a{}^b(A^{\mu\nu}_{bc}\phi^{c;}{}_{\mu\nu} + B_b)^{\mathscr{C}^2,R}]^{\mathscr{C}^2,R}$$

$$\equiv A^{*\mu\nu}_{ac}\phi^{c;}{}_{\mu\nu} + B_a{}^* = 0, \tag{2.3a}$$

$$A^{*\mu\nu}_{ac} = h_a{}^b A^{\nu}_{bc}\,; \qquad B_a{}^* = h_a{}^b B_b\,, \tag{2.3b}$$

$$h_a{}^b \in \mathscr{C}^2(R); \qquad |\,h_a{}^b\,|\,(\bar{R}) \neq 0. \tag{2.3c}$$

The transition, for instance, from Eqs. (1.11) to Eqs. (1.12) can be interpreted as an equivalence transformation of type (2.2a) and so is the inverse transition. Less trivial examples will be given later on.

It should be stressed here that the selected type of equivalence transformations (2.3) satisfies, by construction, the conditions:

(a) it preserves the continuity and regularity of the original system;

(b) it does not alter the quasi-linear structure of the system in view, trivially, of the assumed independence of the factor terms $h_a{}^b$ from the second order derivatives $\phi^{b;}{}_{\mu\nu}$;

(c) it occurs within a fiexed coordinate system $x \to x' \equiv x$ and "gauge" $\phi \to \phi' \equiv \phi$.

As a result, the initial and final systems of equivalence transformations (2.3) admit the *same* actual path $\hat{\phi}^a(x)$ by construction.

The role of self-adjointness for the problem of the existence of a Lagrangian is clearly indicated by Theorem II.2.1. Among all possible transformations (2.3) we shall therefore study at this time only those which are of either type (2.2a) or (2.2d).

For convenience, we shall term *self-adjoint transformations* all equivalence transformations of type (2.3) which either preserve or induce a self-adjoint structure. Such self-adjoint transformations are here classified into *isotopic transformations* [9] when they *preserve* a self-adjoint structure, i.e., when they are of type (2.2a); *genotopic transformations* [10] when they *induce* a self-adjoint structure [11], i.e., they are of type (2.2d). The factor densities $h_a{}^b$ of mapping (2.3) will then be termed *self-adjoint, isotopic, or genotopic factors* depending on whether the induced equivalence transformation is either, in general, self-adjoint or isotopic or genotopic in particular.

It is a matter of simple use of Theorem I.6.2 to prove the following

THEOREM 2.1[12]. *Given a class $\mathscr{C}^2$, regular, tensorial, quasi-linear system of field equations*

$$[A^{\mu\nu}_{ab}(x_\alpha, \phi; \phi^{;}{}_\alpha)\, \phi^{b;}{}_{\mu\nu} + B_a(x_\alpha, \phi, \phi^{;}{}_\alpha)]^{\mathscr{C}^2, R}_{\cdot} = 0, \qquad a = 1, 2, \ldots, n, \qquad (2.4)$$

in a region R_{QL} of points $(x_a, \phi, \phi^{;}{}_\alpha)$, the necessary and sufficient condition for the equivalence transformation

$$[A^{\mu\nu}_{ab}\phi^{b;}{}_{\mu\nu} + B_a]^{\mathscr{C}^2, R} = 0 \to [h_a{}^b(x_\alpha, \phi, \phi^{;}{}_\alpha)(A^{\mu\nu}_{bc}\phi^{c;}{}_{\mu\nu} + B_b)^{\mathscr{C}^2, R}]^{\mathscr{C}^2, R}$$
$$\equiv A^{*\mu\nu}_{ac}\phi^{c;}{}_{\mu\nu} + B^*{}_a = 0 \qquad (2.5)$$

to be self-adjoint in R_{QL} is that each of the following equations in the factors $h_a{}^b$

$$A^{*\mu\nu}_{ab} = A^{*\nu\mu}_{ba} = A^{*\nu\mu}_{ab}, \qquad (2.6\text{a})$$

$$A^{*\alpha\nu;\mu}_{ac\ b} + A^{*\nu\alpha;\mu}_{bc\ a} = A^{*\mu\overline{\nu;\alpha}}_{ba\ c}, \qquad (2.6\text{b})$$

$$A^{*\overline{\alpha\beta};\overline{\mu;\nu}}_{\underline{ad\ b\ c}} = A^{*\overline{\alpha\beta};\overline{\mu;\nu}}_{\underline{bd\ a\ c}}, \qquad (2.6\text{c})$$

$$B^*{}_a{}^{;\mu}_{\ b} + B^*{}_b{}^{;\mu}_{\ a} = 2\{\partial_\nu + \phi^{c;}{}_\nu(\partial/\partial\phi^c)\}\, A^{*\mu\nu}_{ab}, \qquad (2.6\text{d})$$

$$B^*{}_a{}^{;}_{\ b} - B^*{}_b{}^{;}_{\ a} = \tfrac{1}{2}\{\partial_\nu + \phi^{c;}{}_\nu(\partial/\partial\phi^c)\}(B^*{}_a{}^{;\nu}_{\ b} - B^*{}_b{}^{;\nu}_{\ a}), \qquad (2.6\text{e})$$

$$A^{*\mu\nu;\rho}_{ab\ c} \equiv \partial A^{*\mu\nu}_{ab}/\partial\phi^{c;}{}_\rho\,; \qquad B^*{}_a{}^{;}_{\ b} \equiv \partial B_a{}^*/\partial\phi^b; \qquad \text{etc.}, \qquad (2.6\text{f})$$

$$A^{*\mu\overline{\nu;\rho}}_{ab\ c} \equiv A^{*\mu\nu;\rho}_{ab\ c} + A^{*\mu\rho;\nu}_{ab\ c}\,; \qquad \text{etc.}, \qquad (2.6\text{g})$$

$$A^{*\mu\nu}_{ab} \equiv h_a{}^c A^{\mu\nu}_{cb}\,; \qquad B^*{}_a \equiv h_a{}^c B_c, \qquad (2.6\text{h})$$

$$a, b, c, d = 1, 2, \ldots, n; \qquad \mu, \nu, \alpha, \beta, \rho = 0, 1, 2, 3,$$

holds in every bounded domain in the interior of R_{QL}.

Notice that the conditions of self-adjointness of Theorem I.6.2, i.e., Eqs. (I.6.14), are *conditions on the terms $A^{\mu\nu}_{ab}$* and B_a of the field equations. The conditions of self-adjointness of Theorem 2.1, i.e., Eqs. (2.6), are on the contrary *conditions on the factors $h_a{}^b$* (i.e., they are the necessary and sufficient conditions for the factors $h_a{}^b$ to be self-adjoint in our terminology) while the terms $A^{\mu\nu}_{ab}$ and B_a are fixed.

It should be stressed here that if a set of factors $h_a{}^b$ is self-adjoint for a given system, the *same* set is not necessarily self-adjoint for another system. To avoid confusion, we shall say that, when Theorem 2.1 holds, the factor terms $h_a{}^b$ are *self-adjoint with respect to the given system* (2.4).

Notice that conditions (2.6) are insensitive as to whether the original system is self-adjoint or not. It is precisely this property which has allowed the formulation of Theorem 2.1 for self-adjoint transformations in general rather than isotopic or genotopic transformations in particular..

We therefore trivially have

COROLLARY 2.1A. *When the condition of Theorem* 2.1 *holds, transformations* (2.5) *are either isotopic or genotopic depending on whether the original system is self-adjoint or non-self-adjoint, respectively.*

Consider now the particular case when

$$A^{\mu\nu}_{ab} \equiv A_{ab} \otimes g^{\mu\nu}. \tag{2.7}$$

Then the class of equivalence transformations under consideration includes the transition from the conventional (semilinear) form of field equations

$$\Box\phi_a - f_a(x_\alpha, \phi^c, \phi^{c;}{}_\alpha) = g^{\mu\nu}\phi_a{}^{;}{}_{\mu\nu} - f_a = 0 \tag{2.8}$$

to the more general quasi-linear form (2.4). Indeed, for case (2.7) mapping (2.5) becomes

$$h_a{}^b\delta_{bc} \otimes g^{\mu\nu}\phi^{c;}{}_{\mu\nu} - h_a{}^b f_b \equiv A^{*\mu\nu}_{ac}\phi^{c;}{}_{\mu\nu} + B^*{}_a = 0. \tag{2.9}$$

The inverse transition can be recovered, trivially, by using a set of factors $h^{-1}{}_a{}^b$ such that

$$(h^{-1}{}_a{}^b) \equiv (A^*{}_{ac})^{-1} = (h_a{}^b)^{-1}. \tag{2.10}$$

A point which we would like to recall from paper I is that, when Eqs. (2.8) are nonlinear in the derivative couplings, they cannot be self-adjoint. In this case one can use a genotopic transformation of type (2.9), when it exists, to render it self-adjoint because the conditions of self-adjointness (2.6) do allow nonlinear derivative couplings.

Another point which we would like to recall from paper II, is that the transition from Eqs. (2.8) to the form (2.9), while trivial within the context of the theory of partial differential equations, it is nontrivial from the viewpoint of their Lagrangian representation. Furthermore, such a transition implies the appearance of a third class of couplings, namely, that of the "acceleration" couplings (due to the nonnull values of the elements $h_a{}^b$), in addition to the "coordinate" and "velocity" couplings already present in form (2.8).

3. A THEOREM ON ORDERED INDIRECT ANALYTIC REPRESENTATIONS IN TERMS OF THE CONVENTIONAL LAGRANGE EQUATIONS

Consider a (class $\mathscr{C}^2$, regular, Lorentz-covariant, tensorial) quasi-linear or semilinear system of field equations which is *non-self-adjoint*. Then, according to Theorem II.2.1, its ordered *direct* analytic representation in terms of the conventional Lagrange equations *does not* exist.

In this case one can attempt to identify a Lagrangian by searching for a *self-adjointness inducing* equivalence transformation of the field equations of type (2.5), namely, a *genotopic transformation* in our terminology. And indeed, if such a transformation exists, then Theorem II.2.1 holds for the equivalent rather than the original system by producing in this way what we have termed an ordered *indirect* analytic representation.

Thus, the concept of genotopic transformations is sufficient to generalize the analytic framework of paper II to the case of indirect representations of non-self-adjoint systems.

Consider now a quasi-linear or semilinear system which is *self-adjoint*. Then, according to Theorem II.2.1, a Lagrangian for its ordered direct analytic representation does exist. But, as pointed out in Section 2, the equivalence transformations of type (2.5) are insensitive as to whether the original system is self-adjoint or not. Therefore, one can still seek an equivalence transformation of the field equations which is *isotopic*, i.e., of *self-adjointness-preserving* type. If such a transformation exists, then a Lagrangian for its ordered *indirect* analytic representation exists.

Therefore, the concept of isotopic tsansformations is sufficient for the generalization of the analytic framework of paper II to the case of indirect representations of self-adjoint systems.

The above two cases can be unified within the context of the *self-adjoint equivalence transformations* because, as introduced in Section 2, they can be, as particular cases, either of isotopic or of genotopic type.

The simple remark that the proof of Theorem II.2.1 holds irrespective of whether the represented system is in its original form or in one of its equivalent forms allows us to formulate the above concepts in a more rigorous way according to

THEOREM 3.1 [12]. *A necessary and sufficient condition for a Lorentz-covariant, tensorial, quasi-linear system of field equations*

$$A^{\mu\nu}_{ab}(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)\, \phi^{b;}{}_{\mu\nu} + B_a(x_\alpha, \phi^c, \phi^{c;}{}_\alpha) = 0, \qquad a, b, c = 1, 2, \ldots, n; \quad \mu, \nu, \alpha = 0, 1, 2, 3, \tag{3.1}$$

which is of (at least) class $\mathscr{C}^2$ and regular in a region R of the variables $(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)$ to admit an ordered indirect analytic representation in terms of the conventional Lagrange equations

$$d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{a;}{}_\mu} - \frac{\partial \mathscr{L}}{\phi \partial^a} \equiv h_a{}^b(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)[A^{\mu\nu}_{bd}\phi^{d;}{}_{\mu\nu} + B_b] = 0$$

is that the factor terms $h_a{}^b$ induce a self-adjoint equivalence transformation

$$[h_a{}^b(A^{\mu\nu}_{bd}\phi^{d;}{}_{\mu\nu} + B_b)^{\mathscr{C}^2, R}]^{\mathscr{C}^2, R}_{SA} = 0 \tag{3.3}$$

in a star-shaped region R^* of points $(x_\alpha, \phi'^c, \phi'^{c;}{}_\alpha)$ with

$$\phi'^c = \tau\phi^c, \ \phi'^{c;}{}_\alpha = \tau\phi^{c;}{}_\alpha, \qquad 0 \leqslant \tau \leqslant 1.$$

We have again formulated the above theorem on a star-shaped rather than an ordinary region as a precuationary measure, as we did in Theorem II.2.1. In essence, when the self-adjoint equivalence transformation (3.3) exists in a star-shaped rather than an ordinary region, this guarantees the applicability of the converse of the Generalized Poincaré Lemma. The conditions of self-adjointness (2.6) then become the integrability conditions for the existence of a Lagrangian. For more details see paper II.

In practice, to seek for an ordered indirect analytic representation of a given system (3.1), one must first seek for a solution of the system of partial differential equations (2.6) in the unknowns $h_a^{\ b}$. Once such a solution is identified, to verify the "star-shaped criterion" it is sufficient to verify that the equivalent system is well defined for all the values $0 \leqslant \phi^c, \phi^{c;}{}_\alpha \leqslant 1$. If this is not the case, one can attempt to verify the "star-shaped criterion" through a redefinition of the fields which removes possible divergencies in the above-indicated minimal set of values of the local variables.

Corollary II.2.1A now becomes

COROLLARY 3.1A. *If the ordering of Theorem 3.1 is relaxed, then the condition of the theorem is only sufficient for the existence of an indirect analytic representation.*

Again, the significance of this corollary is to emphasize the importance of the concept of ordering within the above analytic context. Let us recall from paper I that such an ordering refers only to the external index "a" of representation (3.2) and not to the sum with respect to the internal index "b." This implies that different orderings of the original systems may lead to different factor terms $h_a^{\ b}$ to satisfy Eqs. (2.6).

For the reader's convenience, we now reformulate Corollary II.2.1B on the methodology of constructing a Lagrangian when Theorem 3.1 holds.

COROLLARY 3.1B. *A Lagrangian density for the ordered indirect analytic representation of quasi-linear systems of field equations according to Theorem* 3.1

$$d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{a;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^a} \equiv A^{*\mu\nu}_{\ ad} \phi^d{}_{\mu\nu} + B^*{}_a = 0, \tag{3.4a}$$

$$A^{*\mu\nu}_{\ ad} \equiv h_a^{\ b} A^{\mu\nu}_{bd}\,; \qquad B^*{}_a \equiv h_a^{\ b} B_b\,, \tag{3.4b}$$

is given by

$$\mathscr{L}(x_\alpha\,, \phi^c, \phi^{c;}{}_\alpha) = K(x_\alpha\,, \phi^c, \phi^{c;}{}_\alpha) + D_a^{\ \mu}(x_\alpha\,, \phi^c)\, \phi^{a;}{}_\mu + C(x_\alpha\,, \phi^c), \tag{3.5}$$

where the $(4n + 2)$*-densities* F, $D_a^{\ \mu}$, *and* C *are a solution of the linear, generally overdetermined system of partial differential equations*

$$\tfrac{1}{2}(K^{;\mu;\nu}_{\ a\ b} + K^{;\nu;\mu}_{\ a\ b}) = A^{*\mu\nu}_{\ ab}\,; \tag{3.6a}$$

$$D^{\mu;}_{a\ b} - D^{\mu;}_{b\ a} = \tfrac{1}{2}(B^*{}_a{}^{;}{}_b - B^*{}_b{}^{;}{}_a) + (K^{;}{}_a{}^{;\mu}{}_b - k^{;}{}_b{}^{;\mu}{}_a) \equiv Z^{*\mu}_{\ ab}(x_\alpha\,, \phi^c); \tag{3.6b}$$

$$C^{;}_{a} = D^{\mu;}_{a\ \mu} - B_a{}^* - K^{;}_{a} - K^{;\mu;}_{a\ \mu} + [K^{;}_{a}{}^{;\mu}_{b} + \tfrac{1}{2}(B_a{}^{;\mu}_{b} - B_b{}^{;\mu}_{a})]\ \phi^{b;}{}_{\mu}$$
$$\equiv W_a{}^*(x_\alpha, \phi^c); \qquad (3.6c)$$
$$K^{;}_{a} \equiv \partial k/\partial\phi^a; \qquad K^{;\mu}_{a} \equiv \partial K/\partial\phi^{a;}{}_{\mu}; \qquad \text{etc.} \qquad (3.6d)$$

given by

$$K = \phi^{a;}{}_{\mu} \int_0^1 d\tau' \left[\phi^{b;}{}_{\nu} \int_0^1 d\tau\ A^{*\mu\nu}_{ab}(x_\alpha, \phi^c, \tau\phi^{c;}{}_{\alpha})\right] (\tau'\phi^{c;}{}_{\alpha}),$$
$$D_a{}^{\mu} = \phi^b \int_0^1 d\tau\ \tau Z^{*\mu}_{ab}(x_\alpha, \tau\phi^c), \qquad (3.7)$$
$$C = \phi^a \int_0^1 d\tau\ W_a{}^*(x_\alpha, \tau\phi^c).$$

The restriction that the self-adjoint equivalence transformation (3.3) be well defined in a star-shaped region then guarantees the existence of the integrals of solutions (3.7). When such integrals do not exist, this does not affect the existence of conditions (2.6) and Eqs. (3.6). Therefore, in this case one can attempt to solve Eqs. (3.6) with methods other than that of the Converse of the Poincaré Lemma and of its generalization. See, again in this respect, paper II.

Finally, the interpretation of the above results from the viewpoint of the structure of the Lagrangian capable of representing tensorial fields coupled according to the most general method allowed by Theorem 3.1 leads, as for Corollary II.2.1E, to the following

COROLLARY 3.1C. *A general structure of the total Lagrangian density for the ordered indirect analytic representation of quasi-linear systems of tensorial field equations according to Theorem* 3.1

$$d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{a;}{}_{\mu}} - \frac{\partial \mathscr{L}}{\partial \phi^a} \equiv A^*{}_{ad}\phi^{d;}{}_{\mu\nu} + B^*{}_{a} = 0, \qquad (3.8a)$$
$$A^{*\mu\nu}_{ad} \equiv h_a{}^b A^{\mu\nu}_{bd}; \qquad B^*{}_a \equiv h_a{}^b B_b \qquad (3.8b)$$

is characterized by $(n+1)$ *interaction terms*: n*-multiplicative and one additive term to the Lagrangian for the free fields according to the structure*

$$\mathscr{L}^{\text{Gen}}_{\text{Tot}}(x_\alpha, \phi^c, \phi^{c;}{}_{\alpha})$$
$$= \sum_{1}^{n}{}_a\ \mathscr{L}^{(a)}_{\text{Int,I}}(x_\alpha, \phi^c, \phi^{c;}{}_{\alpha})\ \mathscr{L}^{(a)}_{\text{Free}}(\phi^a, \phi^{a;}{}_{\alpha}) + \mathscr{L}_{\text{Int,II}}(x_\alpha, \phi^c, \phi^{c;}{}_{\alpha}), \qquad (3.9)$$

where the terms $\mathscr{L}^{(a)}_{\text{Free}}$, $\mathscr{L}^{(a)}_{\text{Int,I}}$, *and* $\mathscr{L}_{\text{Int,II}}$ *admit the decompositions*

$$\mathscr{L}^{(a)}_{\text{Free}} = \tfrac{1}{2}(\phi_a{}^{;\mu}\phi^{a;}{}_{\mu} - m^2(a)\ \phi_a\phi^a) \qquad \text{(no summation)}, \qquad (3.10a)$$
$$\mathscr{L}^{(a)}_{\text{Int,I}} = K_{\text{I}}^{(a)} + D_{\text{I}}^{(a)\mu}{}_{b}\phi^{b;}{}_{\mu} + C_{\text{I}}^{(a)}, \qquad (3.10b)$$
$$\mathscr{L}_{\text{Int,II}} = K_{\text{II}} + D_{\text{II}}{}^{\mu}_{b}\phi^{b;}{}_{\mu} + C_{\text{II}}, \qquad (3.10c)$$

and they can be expressed in terms of the solutions (3.7) *by means of the identifications*

$$K = \tfrac{1}{2}K_{\mathrm{I}}^{(a)}\phi_a{}^{;\mu}\phi^{a;}{}_{\mu} - \tfrac{1}{2}K_{\mathrm{I}}^{(a)}m^2(a)\,\phi_a\phi^a + \tfrac{1}{2}D_{\mathrm{I}}^{(a)}{}_b^{\mu}\phi^{b;}{}_{\mu}\phi_a{}^{;\alpha;}\phi^{a;}{}_{\alpha} + \tfrac{1}{2}C_{\mathrm{I}}^{(a)}\phi^{a;}{}_{\mu}\phi_a{}^{;\mu} + K_{\mathrm{II}}, \tag{3.11a}$$

$$D_b{}^{\mu} = -\tfrac{1}{2}m^2(a)\,\phi_a\phi^a D_{\mathrm{I}}^{(a)}{}_b^{\mu} + D_{\mathrm{II}}{}_b^{\mu}, \tag{3.11b}$$

$$C = -\tfrac{1}{2}m^2(a)\,\phi^a\phi_a C_{\mathrm{I}}^{(a)} + C_{\mathrm{II}}. \tag{3.11c}$$

Let us recall from paper II that the interaction terms of structure (3.9) are nontrivial when the Lagrange equations in $\sum_{1\ a}^{n} \mathscr{L}_{\mathrm{Free}}^{(a)}$ and those in $\mathscr{L}_{\mathrm{Tot}}^{\mathrm{Gen}}$ *are not* equivalent. Indeed this guarantees a modification of the actual path due to nontrivial couplings and therefore the existence of "bona fide" interactions. Let us also recall that a Lagrangian, when it exists, is not unique and, similarly, structure (3.9) is not unique.

On conceptual grounds Corollary 3.1C above is equivalent to Corollary II.2.1E. In essence, the broadest method of coupling tensorial fields which is admissible within a Lagrangian representation is any combination of

(a) generally nonlinear couplings in the field "coordinates" ϕ^c,

(b) generally nonlinear couplings in the field "velocities" $\phi^{c;}{}_{\alpha}$,

(c) linear couplings in the field "accelerations" $\phi^{c;}{}_{\alpha\beta}$

which preserves the continuity, regularity, and self-adjointness of the field equations, where the couplings of types (a) and (b) are represented by the dependence of the $A_{ab}^{\mu\nu}$ and B_a terms in the ϕ^c and $\phi^{c;}{}_{\alpha}$ variables, while the couplings of type (c) are expressed by the generally nonnull value of the elements of the matrix $(A_{ab}^{\mu\nu})$. Again, the presence of "acceleration" couplings is vital for the analytic representation of the above generalized way of coupling tensorial fields and, in turn, such couplings reflect in the appearance of multiplicative interaction terms to the Lagrangian for the free fields. Alternatively, the presence of multiplicative interaction terms is vital for the analytic representation of the above generalized way of coupling tensorial fields. And indeed, when all multiplicative interaction terms reduce to the unity, all couplings are derivable from an additive interaction term, namely; they became conventional. This is, in essence the result of our analysis of paper II.

The point which we would now like to indicate is that, on methodological grounds, Corollaries 3.1C and II.2.1E are not equivalent. In essence, within the context of the *direct* analytic representations of Corollary II.2.1E the forms of coupling tensorial fields are still restricted because of the lack of freedom in the modification of the field equations. Such a restriction is now removed within the context of the *indirect* analytic representations of Corollary 3.1C above. Indeed, within such a broader context, all possible self-adjoint equivalence transformations of the field equations are now admissible.

This added degree of freedom has a twofold significance:

When the field equations are non-self-adjoint, a genotopic transformation induces a form of coupling which is admissible within a Lagrangian representation.

When the field equations are self-adjoint, an isotopic transformation allows the study of the (analytically) equivalent ways of coupling tensorial fields.

This latter aspect will be studied within the context of paper IV.

A few examples are now useful for illustrating our results. Consider again the one-dimensional self-coupled semilinear system (1.1), i.e.,

$$[(\square + m^2)\,\varphi + \lambda\varphi^3 + (1/\varphi)\,\varphi^{;\mu}\varphi^{;}{}_{\mu}]_{\mathrm{NSA}} = 0. \tag{3.12}$$

Since the system is non-self-adjoint, a Lagrangian for its direct analytic [13] representation *does not* exist. Notice that the system is non-self-adjoint because it is of the semilinear type with a derivative coupling of nonlinear form and, as stressed in paper I, the conditions of self-adjointness for the semilinear form prohitit the presence of nonlinear coupling of this type.

In order to identify a Lagrangian, we must first seek a self-adjointness-inducing equivalence transformation, i.e., a genotopic transformation of the type (3.3),

$$\{h[(\square + m^2)\,\varphi + \lambda\varphi^3 + (1/\varphi)\,\varphi^{;\mu}\varphi^{;\mu}]_{\mathrm{NSA}}\}_{\mathrm{SA}} = 0. \tag{3.13}$$

This is achieved by using Theorem 2.1. In this case the conditions of self-adjointness (2.6) reduce to

$$\begin{aligned} \frac{\partial h}{\partial\varphi^{;}{}_{\mu}}\, g^{\alpha\nu} &= \frac{1}{2}\left(\frac{\partial h}{\partial\varphi^{;}{}_{\alpha}}\, g^{\mu\nu} + \frac{\partial h}{\partial\varphi^{;}{}_{\nu}}\, g^{\mu\alpha}\right), \\ 2\,\frac{h}{\varphi}\,\varphi^{;}{}_{\mu} + \frac{1}{\varphi}\,\varphi^{;\alpha}\varphi^{;}{}_{\alpha}\,\frac{\partial h}{\partial\varphi^{;}{}_{\mu}} &= \left\{\partial_\mu + \varphi^{;}{}_{\mu}\,\frac{\partial}{\partial\varphi}\right\} h, \end{aligned} \tag{3.14}$$

and they admit the trivial solution

$$h = \varphi^2 \tag{3.15}$$

This is due to the fact, also stressed in paper I, that the conditions of self-adjointness for the quasi-linear form do admit nonlinear derivative couplings while a genotopic transformation for system (3.12) has precisely the net effect of transforming the system from the semilinear to the quasi-linear form. Thus, Theorem 2.1 gives rise to the equivalent form

$$[\varphi^2(\square + m^2)\,\varphi + \lambda\varphi^5 + \varphi(\varphi^{;}{}_{\mu}\varphi^{;\mu}]_{\mathrm{SA}} = 0, \tag{3.16}$$

which is now self-adjoint. At this point a simple inspection indicates that the equivalent system (3.16) is well defined in a star-shaped region. Thus, Theorem 3.1 holds and the indirect analytic representation

$$d_\mu(\partial\mathscr{L}/\partial\varphi^{;}{}_{\mu}) - (\partial\mathscr{L}/\partial\varphi) \equiv \varphi^2(\square + m^2)\,\varphi + \lambda\varphi^5 + \varphi(\varphi^{;}{}_{\mu}\varphi^{;\mu}) = 0 \tag{3.17}$$

exists. Corollary 3.1B then allows the identification of a Lagrangian by means of solutions (3.7)

$$\mathscr{L} = \tfrac{1}{2}\varphi^2\varphi_{;\mu}\varphi^{;\mu} - \tfrac{1}{4}m^2\varphi^4 - (\lambda/6)\,\varphi^6 \tag{3.18}$$

and, finally, Corollary 3.1C allows the interpretation of such Lagrangians from the viewpoint of the interaction which, in this case, reads

$$\mathscr{L}^{\text{Gen}}_{\text{Tot}} = \mathscr{L}_{\text{Int,I}}\mathscr{L}_{\text{Free}} + \mathscr{L}_{\text{Int,II}}, \tag{3.19a}$$

$$\mathscr{L}_{\text{Free}} = \tfrac{1}{2}(\varphi_{;\mu}\varphi^{;\mu} - m^2\varphi^2), \tag{3.19b}$$

$$\mathscr{L}_{\text{Int,I}} = \varphi^2, \tag{3.19c}$$

$$\mathscr{L}_{\text{Int,II}} = \tfrac{1}{4}m^2\varphi^4 - \tfrac{1}{6}\varphi^6. \tag{3.19e}$$

We reach in this way, not surprisingly, a Lagrangian structure of chiral type [14]. Notice that the presence of the multiplicative interaction term in structure (3.19) is *necessary* for the representation of the system within a fixed "gauge" (i.e., without considering field transformations $\varphi \to \varphi'(\varphi)$).

In more than one dimension, one of the most significant examples of ordered indirect analytic representations is constituted by the field equations for gauge theories

$$F_a(\phi) \equiv \begin{pmatrix} -\Box A_\mu + ie[(\bar\varphi_{;\mu} + ieA_\mu\bar\varphi)\,\varphi - \bar\varphi(\varphi_{;\mu} - ieA_\mu)] \\ (\Box + m^2 - e^2A_\mu A^\mu)\,\varphi - 2ieA^\mu\varphi_{;\mu} \\ (\Box + m^2 - e^2A_\mu A^\mu)\,\bar\varphi + 2ieA^\mu\bar\varphi_{;\mu} \end{pmatrix}_{\text{NSA}} = 0, \tag{3.20}$$

which have been considered in detail in papers I and II. Let us reinspect such a case within the methodology of this paper. Equations (3.20) are non-self-adjoint (Appendix C of paper I) and of the semilinear type. Therefore a Lagrangian for their ordered direct analytic representation *does not* exist. To identify a Lagrangian we again apply Theorem 2.1 by seeking for the genotopic transformation

$$\{h_a{}^b(x_\alpha, \phi^c, \phi^c{}_{;\alpha})[F_b(\phi)]^{\mathscr{C}^\infty,R}_{\text{NSA}}\}^{\mathscr{C}^2,R}_{\text{SA}} = 0. \tag{3.21}$$

A rather tedious but straightforward calculation then shows that a solution of Eqs. (2.6) is constituted by

$$(h_a{}^b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.22}$$

namely, the simple permutation of the equations in φ and $\bar\varphi$ constitutes an example of *genotopic* transformations. And indeed, the equivalent system

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -\Box A_\mu + ie[(\bar\varphi_{;\mu} + ieA_\mu\bar\varphi)\,\varphi - \bar\varphi(\varphi_{;\mu} - ieA\,\varphi)] \\ (\Box + m^2 - e^2A_\mu A^\mu)\,\varphi - 2ieA^\mu\varphi_{;\mu} \\ (\Box + m^2 - e^2A_\mu A^\mu)\,\bar\varphi + 2ieA^\mu\bar\varphi_{;\mu} \end{pmatrix}_{\text{NSA}}\right\}_{\text{SA}} = 0 \tag{3.23}$$

is self-adjoint (see also Appendix C of paper I and Section 2 of paper II). Theorem 3.1 now holds and one recovers the well-known Lagrangian

$$\mathscr{L} = -\tfrac{1}{2}A^{\mu;}{}_{\nu}A^{\mu;\nu} + \bar{\varphi}^{;}{}_{\mu}\varphi^{;\mu} - m^2\bar{\varphi}\varphi + e^2A_{\mu}A^{\mu}\bar{\varphi}\varphi + ie(\bar{\varphi}\varphi^{;}{}_{\mu} - \bar{\varphi}^{;}{}_{\mu}\varphi)\,A^{\mu} \quad (3.24)$$

by using Corollary 3.1B (see paper II for explicit calculations).

Again, the ultimate structure of the analytic representations of gauge theories is of the quasi-linear and *not* of the semilinear type. This is an indication that the type of couplings admitted by such theories is considerably broad. And indeed, a simple inspection indicates that a combination of couplings of type (a) and (b) is present, although of the linear type, while the couplings of type (c) are absent in view of the fact that the $A^{\mu\nu}_{ab}$ terms only represent a permutation of the equations in φ and $\bar{\varphi}$ (see paper II).

The additional result which emerges from the analysis of this paper is that the structure (3.22) is only *one* (trivial) solution of the conditions (2.6) for a genotopic transformation and that additional solutions are possible. Clearly, the existence of different solutions will inevitably imply the existence of Lagrangians which are equivalent (by construction) but *different* than the familiar form (3.24). This is a typical case of isotopic transformations.

For the sake of clarity let us indicate that what we are referring here to is the study of particular solutions of the overdetermined system of differential equations (2.6) (i.e., the conditions of self-adjointness) in the genotopic functions $h_a{}^b$ applied to system (2.21) where $F_a(\phi)$ represents the (fixed) system of field equations (3.20). Furthermore, the particular solutions in which we are interested are those *other* than the trivial solution (3.22), if they exist.

It is now easy to anticipate that if such solutions exist by preserving the structure of matrix (3.22) and by simply relaxing the constancy of its elements, i.e., if such solutions are of the type

$$(h_a{}^b) = \begin{pmatrix} h_1{}^1(\phi, \phi^{;}{}_{\alpha}) & 0 & 0 \\ 0 & 0 & h_3{}^2(\phi, \phi^{;}{}_{\alpha}) \\ 0 & h_2{}^3(\phi, \phi^{;}{}_{\alpha}) & 0 \end{pmatrix} \quad (3.25)$$

then "acceleration" couplings will still be absent. On the contrary, if there exist solutions with more than three nonnull elements of matrix (3.25), then "accelaration" couplings will inevitably appear. In turn, such "acceleration" couplings will induce multiplicative interaction terms in the Lagrangian structure. The net effect is that, if such solutions exist, the new emerging Lagrangians will be analytically equivalent but structurally different than the familiar Lagrangian (3.24).

This potential change in the structure of the Lagrangian for gauge theories, if it exists, is not only formal. And indeed, it might affect the symmetry of the system (e.g., it might produce a gauge invariance breaking at the level of the Lagrangian irrespective of that of the vacuum). This point may be seen from the fact that the appearance of "acceleration" couplings with related multiplicative interaction terms will inevitably transform the Lagrangian (3.24) into a new structure which is more of the *chiral* rather than of the gauge type.

A basic aspect which must be stressed is that during this entire process the underlying dynamics is unchanged in the sense that the solutions of the field equations remain unchanged by construction.

A preliminary analysis of these problems will be attempted in subsequent paper IV within the context of our study of the isotopic transformations of the Lagrangian density postulated in paper I.

A problem of considerable significance (as well as complexity) which remains open is that of "universality," namely, whether a Lagrangian for the representations of Theorem 3.1 *always* exists. This problem is briefly discussed in Appendix A.

4. A Generalized Action Principle

Consider the action functional

$$A(\phi) \equiv \int_\Omega d^4x\, \mathscr{L}(x_\alpha, \phi^c, \phi^{c;}{}_\alpha) \equiv \int_{x_1^0}^{x_2^0} dx^0 \int_{-\infty}^{+\infty} d^3x\, \mathscr{L}, \tag{4.1a}$$

$$\mathscr{L} \in \mathscr{C}^4(R_{\mathrm{LE}}); \qquad \left| \frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\mu\, \partial \phi^{b;}{}_\nu} \right| (\bar{R}_{\mathrm{LE}}) \neq 0. \tag{4.1b}$$

The conventional Lagrange equations are customarily derived from the conditions that the first-order variation δA of A, when computed along an actual path $\hat{\phi}^a$, is identically null for all variations of the fields with fixed end points. More explicitly, consider the variations [15]

$$\delta\phi^a(x) \equiv w\eta^a(x); \qquad \delta\phi^{a;}{}_\mu(x) = d_\mu\, \delta\phi^a(x) = w\eta^{a;}{}_\mu\,; \qquad w \in O_\epsilon \tag{4.2}$$

where the functions $\eta^a(x)$ satisfy the fixed-end-point conditions

$$\eta^a(x_1^0, \mathbf{x}) = \eta^a(x_2^0, \mathbf{x}) = 0, \qquad a = 1, 2,\ldots, n. \tag{4.3}$$

Then the first-order variation $\delta A(\phi)$ can be written [16]

$$\begin{aligned} \delta A(\phi) &= w \int_\Omega d^4x \left(\frac{\partial \mathscr{L}}{\partial \phi^a}\, \eta^a + \frac{\partial \mathscr{L}}{\partial \phi^{a;}{}_\mu}\, \eta^{a;}{}_\mu \right) \\ &= -w \int_\Omega d^4x \left(d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{a;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^a} \right) \eta^a. \end{aligned} \tag{4.4}$$

Under the assumed continuity and regularity conditions of the Lagrangian, the condition [17]

$$\delta A(\phi)|_{\phi=\hat{\phi}} = -w \int_\Omega d^4x \left(d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{a;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^a} \right)\Big|_{\phi=\hat{\phi}} \eta^a = 0 \tag{4.5}$$

is then equivalent to the conventional Lagrange equations. This is the well known action principle for continuous sytems.

Let us here indicate that principle (4.5), as a purely variational algorithm, holds for an arbitrary structure and dimensionality of the Lagrangian density. Therefore, principle (4.5) is equivalent to the conventional Lagrange equations not only when the Lagrangian density has the conventional structure

$$\mathscr{L}_{\mathrm{Tot}} \equiv \sum_{1}^{\mu}{}_{a} \mathscr{L}^{(a)}_{\mathrm{Free}}(\phi^{a}, \phi^{a;}{}_{\alpha}) + \mathscr{L}_{\mathrm{Int}}(x_{\alpha}, \phi, \phi^{;}{}_{\alpha}) \tag{4.6}$$

but also when $\mathscr{L}$ has the generalized structure

$$\mathscr{L}^{\mathrm{Gen}}_{\mathrm{Tot}} \equiv \sum_{1}^{\mu}{}_{a} \mathscr{L}^{(a)}_{\mathrm{Int.I}}(x_{\alpha}, \phi, \phi_{\alpha}) \mathscr{L}^{(a)}_{\mathrm{Free}}(\phi^{a}, \phi^{a}{}_{\alpha}) + \mathscr{L}_{\mathrm{Int.II}}(x_{\alpha}, \phi, \phi_{\alpha}). \tag{4.7}$$

As a consequence, whenever a (class $\mathscr{C}^2$, regular, tensorial) system of field equations admits a (direct or indirect, ordered or unordered) analytic representation, such a system can be equivalently represented with principle (4.5).

With reference to our remarks of Section 1, we can thus say that the representations of classes I and II are "bona fide" analytic representations in the sense that the underlying analytic equations can be derived by a variational principle.

The objective of this section is to show the existence of a generalization of principle (4.5) capable of inducing the generalized analytic equations (1.5) as well as the conventional equations. As a consequence, the representations of class III are also "bona fide" analytic representations.

Let us begin by noting that the variations $\delta\phi^a(x)$ of the fields $\phi^a(x)$, besides the restriction of possessing the same continuity properties of the fields and of satisfying the fixed-end-point conditions (4.3), are arbitrary functions of the Minkowski coordinates.

The form (4.2) of the variations, which is customarily used in physics, is termed *weak variation* in the Calculus of Variations. In particular, we can say that form (4.2) is a *realization* of the "abstract" variations $\delta\phi^a(x)$.

Without entering at this point into an analysis of more refined forms of variations, it is sufficient for our purpose to point out the existence of infinite varieties of possible realizations of the variations which are consistent with the above continuity assumptions and fixed-end-point conditions.

Among all such possible realizations we select the form

$$\delta^*\phi^b(x) = \phi^{\prime b}(x) - \phi^b(x) = w\eta^{*b}(x) \equiv w\eta^a(x)\, g_a{}^b(x_a, \phi, \phi^{;}{}_{\alpha}) = \delta\phi^a(x)\, g_a{}^b, \tag{4.8a}$$

$$\delta^*\phi^{b;}{}_{\mu}(x) = \phi^{\prime b;}{}_{\mu}(x) - \phi^{b;}{}_{\mu}(x) = d^{\mu}(\delta^*\phi^b) = w\eta^{*b;}{}_{\mu}(x), \tag{4.8b}$$

where

(1) the functions $\eta^a(x)$ are of at least class $\mathscr{C}^2$ in Ω and satisfy the fixed-end-point conditions (4.3);

(2) the functions $g_a{}^b$ are of at least of class $\mathscr{C}^2$ in the same region of definition R_{LE} of the Lagrangian density and the matrix of their elements is everywhere regular in every bounded domain in the interior of $R_{\rm LE}$;

(3) the parameter w is again defined in the neighborhood of the value zero, i.e.. $w \in O_\epsilon$.

As a consequence, the above variations $\delta^*\phi^b(x)$ possess at least the same continuity properties of the fields $\phi^b(x)$; they satisfy the fixed-end-point conditions (4.3); and they are weak variations in exactly the same sense as the ordinary variations $\delta\phi^b(x)$. Therefore, the generalized form (4.8) of the variations is equivalent to the ordinary form within a variational context.

We reach in this way the following *generalization of the conventional action principle* [18].

$$\delta^* A(\phi)|_{\phi=\hat\phi} = w \int_\Omega d^4x \left(\frac{\partial\mathscr{L}}{\partial\phi^b}\,\eta^{*b} + \frac{\partial\mathscr{L}}{\partial\phi^{b;}{}_\mu}\,\eta^{*b;}{}_\mu\right)\Big|_{\phi=\hat\phi}$$

$$= -w \int_\Omega d^4x \,\Big|\, g_a{}^b \left(d_\mu \frac{\partial\mathscr{L}}{\partial\phi^{b;}{}_\mu} - \frac{\partial\mathscr{L}}{\partial\phi^b}\right)\Big|_{\phi=\hat\phi} \eta^a = 0, \qquad (4.9a)$$

$$\mathscr{L} \in \mathscr{C}^4(R_{\rm LE}); \qquad g_a{}^b \in \mathscr{C}^2(R_{\rm LE}), \qquad (4.9b)$$

$$\left| \frac{\partial^2\mathscr{L}}{\partial\phi^{a;}{}_\mu\,\partial\phi^{b;}{}_\nu} \right| (\bar R_{\rm LE}) \neq 0; \qquad |\, g_a{}^b \,|\, (\bar R_{\rm LE}) \neq 0, \qquad (4.9c)$$

which clearly induces the generalized analytic equations (1.5) rather than the conventional ones.

Trivially, at the limit $g_a{}^b \to \delta_a{}^b$ principle (4.9) reduces to principle (4.5), which is therefore recovered as a particular case.

A few comments are now in order. Within the framework of our variational approach to self-adjointness a first significant aspect principle (4.9) is to broaden the somewhat restrictive character of principle (4.5) according to which the underlying field equations are always self-adjoint (Theorem I.7.1). And indeed, the field equations of principle (4.9), i.e., Eqs. (1.6), as indicated earlier, are in general non-self-adjoint.

A second significant aspect of principle (4.9) is that it implies a broadening of the methodology for the representations of field equations in terms of variational principles as we shall see better in Section 5. As a result, the framework (C) of Section 1 is a "bona fide" analytic framework.

A third significant aspect of principle (4.9) is that it implies a broadening of the transformation theory of the action functional.

Basically we can say that, in addition to *the degrees of freedom of the coordinates*, e.g., the arena of the Poincaré transformations; and *the degrees of freedom of the fields* within a fixed coordinate system, e.g., the arena of the gauge transformations; we have, as a third identifiable layer of the transformation theory, *the degrees of freedom of the variations* within a fixed coordinate system and "gauge."

Within the context of our variational approach to self-adjointness, the above third class of degrees of freedom induces a classification of the action principles for a given system into *self-adjoint* and *non-self-adjoint* depending on whether the underlying analytic equations are self-adjoint or not, respectively, according to scheme

$$\delta^* A(\phi)|_{SA} \equiv - \int_\Omega d^4x \left[g_a{}^b \left(d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{b;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^b} \right)_{SA} \right]_{SA} d\phi^a, \tag{4.10a}$$

$$\delta^* A(\phi)|_{NSA} \equiv - \int_\Omega d^4x \left[g_a{}^b \left(d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{b;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^b} \right)_{SA} \right]_{NSA} \delta\phi^a. \tag{4.10b}$$

In paper IV we shall show that the self-adjoint case (4.10a) induces what we have termed *isotopic transformations*. Therefore, the concept of isotopic transformations occurs within the arena of the degrees of freedom of the variations with a fixed coordinate system and "gauge." An anticipation of the results of paper IV is useful here to point out certain aspects.

When case (4.10a) occurs, then a *new* Lagrangian $\mathscr{L}^*$ for the identifications

$$d_\mu \frac{\partial \mathscr{L}^*}{\partial \phi^{a;}{}_\mu} - \frac{\partial \mathscr{L}^*}{\partial \phi^a} \equiv \left[g_a{}^b \left(d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{b;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^b} \right) \right]_{SA} = 0 \tag{4.11}$$

exists from Theorem 3.1. The underlying action principle will then be termed *isotopically mapped*. In this case the property

$$\delta^* \int_\Omega d^4x\, \mathscr{L} \equiv \delta \int_\Omega d^4x\, \mathscr{L}^* \tag{4.12}$$

occurs, namely, the isotopically mapped action principle in a given Lagrangian coincides with the conventional action principle in the isotopically mapped Lagrangian.

Other properties can be easily identified. For instance, suppose that several isotopic mappings exist for one given Lagrangian. Then, property (4.12) can be generalized according to the symbolic notations for the case of two subsequent mappings

$$\begin{aligned} (\delta^{*_1})^{*_2} \int_\Omega d^4x\, \mathscr{L} &\equiv \delta^{*_1} \int_\Omega d^4x\, \mathscr{L}^{*_1} \\ &\equiv \delta \int_\Omega d^4x\, (\mathscr{L}^{*_1})^{*_2}. \end{aligned} \tag{4.13}$$

If, in particular, the two mappings are inverse of each other (namely the matrices $(g^1{}_a{}^b)$ and $(g^2{}_a{}^b)$ are inverse of each other) then we have the property

$$\delta \int_\Omega d^4x\, \mathscr{L} \equiv \delta^* \int_\Omega d^4x\, \mathscr{L}^{(*)^{-1}}. \tag{4.14}$$

The above property is not trivial because it indicates that one can directly transform the conventional principle (4.5) into the generalized principle (4.9) through an isotopic transformation (when it exists). Explicitly, property (4.14) reads [19]

$$\begin{aligned}\delta \int_\Omega d^4x\, \mathscr{L} &= -\int_\Omega d^4x \left(d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{a;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^a}\right)\delta\phi^a \\ &= -\int_\Omega d^4x \left(d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{b;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^b}\right)\delta_a{}^b\, \delta\phi^a \\ &= -\int_\Omega d^4x\, g_a^{(-1)b} \left(d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{b;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^b}\right) g^a{}_c\, \delta\phi^c \\ &= -\int_\Omega d^4x\, g_a^{(-1)b} \left(d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{b;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^b}\right)\delta^*\phi^a \\ &= -\int_\Omega d^4x \left(d_\mu \frac{\partial \mathscr{L}^{(*)-1}}{\partial \phi^{a;}{}_\mu} - \frac{\partial \mathscr{L}^{(*)-1}}{\partial \phi^a}\right)\delta^*\phi^a \\ &= \delta^* \cdot \int_\Omega d^4x\, \mathscr{L}^{(*)-1}.\end{aligned} \tag{4.15}$$

Needless to say, the results of this section must be considered as purely introductory and several additional methodological aspects must be explored prior to any final assessment of the possible significance of principle (4.9).

We are not only referring here to topics within the context of field theory (e.g., symmetries and conservation laws) but also to an analysis of the significance of the variations (4.8) within the context of the calculus of variations at large (e.g., the study of the extremal aspect with variations (4.8)).

5. Analytic Representations in Terms of the Generalized Lagrange Equations

As indicated earlier, isotopic and genotopic transformations (or, in general, self-adjoint equivalence) transformations of the field equations have several methodological implications. For instance, they may affect the symmetries and conservation laws in view of the functional dependence of the factor terms $h_a{}^b$, in general, on all the variables x_α ϕ^c, and $\phi^{c;}{}_\alpha$.

It is therefore significant to study in addition to the *ordered indirect analytic representations in terms of the conventional Lagrange equations* explored in Section 3, i.e.,

$$\left[d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{a;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^a}\right]_{\mathrm{SA}}^{\mathscr{C}^2,R} \equiv [h_a{}^b (A^{\nu}_{bc}\phi^{c;}{}_{\mu}) + B_b)^{\mathscr{C}^2,R}]_{\mathrm{SA}}^{\mathscr{C}^2,R} = 0 \tag{5.1}$$

the *ordered direct analytic representations in terms of the generalized Lagrange equations*

$$\left[g_a{}^b \left(d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{b;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^b}\right)\right]^{\mathscr{C}^2,R} \equiv [A^{\mu\nu}_{ac}\phi^{c;}{}_{\mu\nu} + B_a]^{\mathscr{C}^2,R} = 0. \tag{5.2}$$

The methodological implications of the transition from analytic representations (5.1) to those of Eqs. (5.2) and vice versa are not trivial. This is due to the fact that in the former the impact of the factor terms $h_a^{\ b}$ is incorporated in the *Lagrangian* while in the latter such impact is incorporated in the *analytic equations* and not necessarily in the related Lagrangian. As a result, the generalized variational framework introduced in Section 4 might be significant to the study of the problem of the symmetries and conservation laws of *equivalent* systems of field equations. This aspect will be explored in paper IV.

In this section we shall restrict our analysis to the study of the representational capability of the generalized Lagrange equations (1.8).

On technical grounds the transition from representations (5.1) to (5.2) is trivial. Indeed in view of the assumed regularity of the matrix of the vector terms $h_a^{\ b}$ and $g_a^{\ b}$ such a transition *always* exists and is characterized simply by

$$(g_a^{\ b}) \equiv (h_a^{\ b})^{-1}. \tag{5.3}$$

We then have the following

LEMMA 5.1. *A necessary and sufficient condition for the existence of an ordered direct analytic representation in terms of the generalized Lagrange equations is that an ordered indirect analytic representation in terms of the conventional Lagrange equations exists and vice versa.*

The "mechanics" of representations (5.2) is also quite simple. Suppose that the assigned system is *non-self-adjoint*. Then the conventional Lagrange equations produce an *equivalent* self-adjoint form of the field equations while the factor terms $g_a^{\ b}$ reproduce the original non-self-adjoint form in the given ordering according to the scheme

$$\begin{aligned}\left[g_a^{\ b}\left(d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{b;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^b}\right)_{\mathrm{SA}}\right]_{\mathrm{NSA}} &\equiv [g_a^{\ b}(A^{*\mu\nu}_{bc}\phi^{c;}{}_{\mu\nu} + B^*{}_b)_{\mathrm{SA}}]_{\mathrm{NSA}} \\ &\equiv (A^{\mu\nu}_{ac}\phi^{c;}{}_{\mu\nu} + B_a)_{\mathrm{NSA}} = 0. \end{aligned} \tag{5.4}$$

It is precisely the latter property which, in the ultimate analysis, allows the *direct* representation of *non-self-adjoint* systems.

Generalized analytic representations (5.2) can also be studied per se, i.e., independently from representations (5.1). In this respect, the following theorem is again a direct consequence of Theorem II.2.1.

THEOREM 5.1 [12]. *A necessary and sufficient conditions for a Lorentz-covariant, tensorial, quasi-linear system of field equations*

$$\begin{gathered} A^{\mu\nu}_{ab}(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)\, \phi^{b;}{}_{\mu\nu} + B_a(x_\alpha, \phi^c, \phi^{c;}{}_\alpha) = 0, \\ a, b, c = 1, 2,\ldots, n, \qquad \mu, \nu, \alpha = 0, 1, 2, 3, \end{gathered} \tag{5.5}$$

which is of (at least) class $\mathscr{C}^2$ and regular in a region R of the variables $(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)$ to admit an ordered direct analytic representation interms of the generalized Lagrange equations

$$g_a{}^b\left(d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{b;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^b}\right) \equiv A_{ac}^{\alpha\beta}\phi^{c;}{}_{\alpha\beta} + B_a = 0, \qquad a = 1, 2,..., n, \tag{5.6}$$

is that the inverse matrix $(h_a{}^b)$ of the matrix of the factor terms $(g_a{}^b)$ induces a self-adjoint equivalence transformation of the field equations

$$[h_a{}^b(A_{bc}^{\mu\nu}\phi^{c;}{}_{\mu\nu} + B_b)^{\mathscr{C}^2,R}]_{SA}^{\mathscr{C}^2,R} = 0; \qquad (h_a{}^b) \equiv (g_a{}^b)^{-1} \tag{5.7}$$

in a star-shaped region R^* of points $(x_\alpha, \phi'^c, \phi'^{c;}{}_\alpha)$ with $\phi'^c = \tau\phi^c$, $\phi'^{c;}{}_\alpha = \tau\phi^{c;}{}_\alpha$, $0 \leqslant \tau \leqslant 1$.

The action principle which underlies the above theorem is the generalized form (4.9) introduced in Section 4, i.e.,

$$\begin{aligned} \delta^* A(\phi)|_{\phi=\hat\phi} &= -\int_\Omega d^4x \left| g_a{}^b \left(d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{b;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^b}\right)\right|_{\phi=\hat\phi} \delta\phi^b \\ &\equiv -\int_\Omega d^4x\,(A_{ac}^{\alpha\beta}\phi^{c;}{}_{\alpha\beta} - B_a)|_{\phi=\hat\phi}\,\delta\phi^a = 0. \end{aligned} \tag{5.8}$$

Alternatively, one can start with the *ordinary* action principle in an *equivalent* self-adjoint form of the equations of motion and then transform such a framework into the generalized principle (4.9) by means of rule (4.12), when it applies, according to

$$\begin{aligned} \delta A^*(\phi)|_{\phi=\hat\phi} &= -\int_\Omega d^4x\,\delta_a{}^b \left(d_\mu \frac{\partial \mathscr{L}^*}{\partial \phi^{b;}{}_\mu} - \frac{\partial \mathscr{L}^*}{\partial \phi^b}\right)\Big|_{\phi=\hat\phi} \delta\phi^a \\ &= -\int_\Omega d^4x \left| g_a^{(-)b} \left(d_\mu \frac{\partial \mathscr{L}^*}{\partial \phi^{b;}{}_\mu} - \frac{\partial \mathscr{L}^*}{\partial \phi^b}\right) g_c{}^a \right|_{\phi=\hat\phi} \delta\phi^c \\ &= -\int_\Omega d^4x \left| g_c{}^a \left(d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{a;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^b}\right)\right|_{\phi=\hat\phi} \delta\phi^c \\ &= \delta^* A(\phi)|_{\phi=\hat\phi} \\ &\equiv -\int_\Omega d^4x\,(A_{cd}^{\mu\nu}\phi^{d;}{}_{\mu\nu} + B_c)|_{\phi=\hat\phi}\,\delta\phi^c = 0. \end{aligned} \tag{5.9}$$

Theorem 5.1 with the underlying variational formulation (5.8) or (5.9) constitutes the most general framework of our analytic approach to classe $\mathscr{C}^2$, regular, unconstrained, tensorial field equations. Indeed, such framework contains, as a particular case, our main Theorem II.2.1, trivially, when the factor terms $g_a{}^b$ reduce to the Kronecker $\delta_a{}^b$. In this was the case of ordered direct analytic representations in terms of the conventional Lagrange equations is recovered. In addition, the above framework incorporate through Lemma 5.1 the case of the indirect analytic representations.

It should also be stressed that Theorem 5.1 holds irrespective of whether the assigned system (5.5) is self-adjoint or non-self-adjoint. More specifically, if the assigned system is non-self-adjoint, Theorem 5.1 provides the methodology for their *direct* representations without recurring to equivalence transformations of the equations of motion. If the asigned system is self-adjoint, Theorem 5.1 provides the methodology for analytic representation both with trivial and nontrivial factor terms $(\delta_a{}^b)$ and $(h_a{}^b)$, $h_a{}^b \neq \delta_a{}^b$, respectively. As we shall see in paper IV, this latter case is particularly useful for the study of the isotopic transformations of the Lagrangian density and the related implications for symmetries and conservation laws.

Furthermore, since the Lagrangian of Theorem 5.1 is unrestricted in its structure, such a theorem provides the necessary methodology for the representations of Class (C) introduced in Section 1; the generalized variational principle (5.8) or (5.9) then establishes that such representations are truly "analytic" in the same measure as the representations of Class (A) and (B).

We are now in a position to comment on the relationship among the representations of Classes (A), (B), and (C) of Section 1.

The analytic representations of Class (A) (i.e., those with the conventional Lagrange equations and the conventional structure of the Lagrangian) contain the great majority of field theoretical models considered until now. Furthermore, the method of their quantization is fully established, although the problem of renormalization has been successfully solved only in special cases, such as in quantum electrodynamics or in the unified gauge theory of weak and electromagnetic interactions [20].

The analytic representations of Class (B) (i.e., those with the conventional Lagrange equations and a generalized structure of the Lagrange) constitute, as pointed out in paper II, a true generalization of those of Class (A) because they incorporate the broadest collection of couplings which is admissible within a Lagrangian representation. The method of their quantization, however, is not established at this time and, to the best knowledge of this author, the most significant case which has been confronted in the existing literature is that with multiplicative interaction terms depending only on the fields, i.e., the known chiral models [14].

The analytic representations of Class (C) (i.e., those with the generalized Lagrange equations and a generalized structure of the Lagrangian) are equivalent to those Class (B), as clearly pointed out by Lemma 5.1, whenever the representations of the latter class are indirect. And indeed, in this case any representation of Class (B) can always be turned into one of Class (C) and vice versa. It is precisely such equivalence which renders the representations of Class (C) attractive for practical applications. For instance, as we shall see better in the subsequent paper IV, the representations of Class (C) are particularly useful for the study of the transformation theory in general and the isotopic transformations in particular.

The significance of the representations of Class (C), however, might go beyond the framework of the transformation theory. It is rather tempting to express the hope at this point that the representations of Class (C) can be useful for the problem of quantization of arbitrary forms of couplings at least as an alternative to the quantization of the representations of Class (B).