

Necessary and Sufficient Conditions for the Existence of a Lagrangian in Field Theory

II. Direct Analytic Representations of Tensorial Field Equations*

RUGGERO MARIA SANTILLI[†]

*Laboratory for Nuclear Science and Department of Physics,
 Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

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By using a characterization of the concept of analytic representation and a variational approach to self-adjointness introduced in a preceding paper, we prove a theorem, according to which a necessary and sufficient condition for a class \mathcal{C}^2 , regular, tensorial, quasi-linear system of field equations to admit an ordered direct analytic representation in terms of the Lagrange equations in a region R of its variables is that the system is self-adjoint in R . We point out as a first corollary that if the ordering requirement is removed from the definition of analytic representation, then the condition of self-adjointness of the field equations is only sufficient for the existence of a Lagrangian density. We then provide as a second corollary a methodology for the computation of the Lagrangian density for the representation of self-adjoint quasi-linear tensorial field equations. This methodology is also particularized for ordinary semilinear systems of tensorial field equations through a third corollary. The above results are interpreted from the viewpoint of interactions. We first recover, through a fourth corollary, the conventional structure of the total Lagrangian density $\mathcal{L}_{\text{Tot}} = \sum_1^n \mathcal{L}_{\text{Free}}^{(a)} + \mathcal{L}_{\text{Int}}$ for the semilinear form of the field equations, and then introduce through a fifth corollary a generalized structure of the type $\mathcal{L}_{\text{Tot}} = \sum_1^n \mathcal{L}_{\text{Int}, I}^{(a)} \mathcal{L}_{\text{Free}}^{(a)} + \mathcal{L}_{\text{Int}, \Pi}$ for the representations of the field equations in the quasi-linear form. Therefore, our analysis seems to indicate that a general form of representing interacting fields is characterized by $(n+1)$ -interaction terms in the Lagrangian: n multiplicative terms and one additive term to the Lagrangian for the free fields.

1. INTRODUCTION

In a preceding paper [1], we have studied class \mathcal{C}^2 , regular, Lorentz-covariant, tensorial field equations in (a) the *nonlinear form*:

$$\begin{aligned}
 F_{a_1}(x_\alpha, \phi^a, \phi^{a; \alpha}, \phi^{a; \alpha\beta}) &= 0, \\
 a_1, a &= 1, 2, \dots, n, \quad \alpha, \beta = 0, 1, 2, 3, \\
 \phi^{a; \alpha} &\equiv \frac{\partial \phi^a}{\partial x^\alpha}, \quad \phi^{a; \alpha\beta} \equiv \frac{\partial^2 \phi^a}{\partial x^\alpha \partial x^\beta}; \quad (1.1)
 \end{aligned}$$

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[†] Permanent address: Boston University, Physics Department, Boston, Mass. 02215.

(b) the *quasi-linear form*:

$$A_{a_1 a_2}^{\mu\nu}(x_\alpha, \phi^a, \phi^{a_i}) \phi^{a_2 i}_{\mu\nu} + B_{a_1}(x_\alpha, \phi^a, \phi^{a_i}) = 0, \quad (1.2)$$

$$a_1, a_2, a = 1, 2, \dots, n, \quad \alpha, \mu, \nu = 0, 1, 2, 3;$$

(c) the *semilinear form*:

$$g^{\mu\nu} \phi_{a_1 i}_{\mu\nu} - \rho_{a_1 a_2}^\mu(x_\alpha, \phi^a) \phi^{a_2 i}_\mu - \sigma_{a_1}(x_\alpha, \phi^a) = 0,$$

$$(g^{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad g^{\mu\nu} \phi_{a i}_{\mu\nu} = \square \phi_a, \quad (1.3)$$

$$a_1, a_2, a = 1, 2, \dots, n, \quad \mu, \nu, \alpha = 0, 1, 2, 3;$$

and we have identified the necessary and sufficient conditions for the above forms to be *self-adjoint* [1, Theorems 6.1, 6.2, 6.3], namely, to be such that their systems of equations of variation coincide with the adjoint systems for all admissible variations.

Such necessary and sufficient conditions result in being certain systems of quasi-linear overdetermined [2] systems of partial differential equations, which we have termed *conditions of self-adjointness*, and which are given

(a) for the *nonlinear form*, by

$$\begin{cases} F_{a_1 a_2}^{i\mu\nu} = F_{a_2 a_2}^{i\mu\nu} = F_{a_1 a_2}^{i\nu\mu}, & (1.4a) \\ F_{a_1 a_2}^{i\mu} + F_{a_2 a_2}^{i\mu} = 2d_\nu F_{a_2 a_1}^{i\mu\nu}, & (1.4b) \\ F_{a_1 a_2}^{i\mu} - F_{a_2 a_1}^{i\mu} = \frac{1}{2}d_\mu(F_{a_1 a_2}^{i\mu} - F_{a_2 a_1}^{i\mu}), & (1.4c) \end{cases}$$

$$a_1, a_2 = 1, 2, \dots, n, \quad \mu, \nu = 0, 1, 2, 3; \quad (1.4d)$$

$$F_{a_1 a_2}^{i\mu\nu} \equiv (\partial F_{a_1} / \partial \phi^{a_2 i}_{\mu\nu}), \quad (1.4e)$$

$$F_{a_1 a_2}^{i\mu} \equiv (\partial F_{a_1} / \partial \phi^{a_2 i}_\mu), \quad (1.4f)$$

$$F_{a_1 a_2}^{i\mu} \equiv (\partial F_{a_1} / \partial \phi^{a_2}), \quad (1.4g)$$

$$d_\mu \equiv \partial_\mu + \phi^{a_i}_\mu (\partial / \partial \phi^a) + \phi^{a_i}_{\mu\alpha} (\partial / \partial \phi^{a_i}_\alpha) + \phi^{a_i}_{\mu\alpha\beta} (\partial / \partial \phi^{a_i}_{\alpha\beta});$$

(b) for the *quasi-linear form*, by

$$A_{a_1 a_2}^{\mu \nu} = A_{a_2 a_1}^{\nu \mu} = A_{a_1 a_2}^{\nu \mu}, \quad (1.5a)$$

$$A_{a_1 a_3}^{\nu \alpha ; \mu} + A_{a_2 a_3}^{\nu \alpha ; \mu} = A_{a_2 a_1}^{\mu \nu ; \alpha}, \quad (1.5b)$$

$$A_{a_1 a_4}^{\alpha \beta ; \mu ; \nu} = A_{a_2 a_4}^{\alpha \beta ; \mu ; \nu}, \quad (1.5c)$$

$$B_{a_1 a_2}^{\mu ; \nu} + B_{a_2 a_1}^{\mu ; \nu} = 2\{\partial_\nu + \phi^{a_3 ; \nu}(\partial/\partial\phi^{a_3})\} A_{a_1 a_2}^{\mu \nu}, \quad (1.5d)$$

$$B_{a_1 a_2}^{\mu ; \nu} - B_{a_2 a_1}^{\mu ; \nu} = \frac{1}{2}\{\partial_\mu + \phi^{a_3 ; \mu}(\partial/\partial\phi^{a_3})\}(B_{a_1 a_2}^{\mu ; \nu} - B_{a_2 a_1}^{\mu ; \nu}), \quad (1.5e)$$

$$a_1, a_2, a_3, a_4 = 1, 2, \dots, n, \quad \mu, \nu = 0, 1, 2, 3;$$

$$A_{a_1 a_2 a_3}^{\mu \nu ; \alpha} \equiv \frac{\partial A_{a_1 a_2}^{\mu \nu}}{\partial \phi^{a_3 ; \alpha}}, \quad A_{a_1 a_2 a_3 a_4}^{\mu \nu ; \alpha ; \beta} = \frac{\partial^2 A_{a_1 a_2}^{\mu \nu}}{\partial \phi^{a_3 ; \alpha} \partial \phi^{a_4 ; \beta}}, \quad (1.5f)$$

$$B_{a_1 a_2}^{\mu ; \nu} \equiv \frac{\partial B_{a_1}}{\partial \phi^{a_2 ; \mu}}, \quad B_{a_1 a_2}^{\mu ; \nu} \equiv \frac{\partial B_{a_1}}{\partial \phi^{a_2}}, \quad (1.5g)$$

where the horizontal bar denotes symmetrization of the indicated indices, e.g.,

$$A_{a_1 a_2 a_3}^{\mu \nu ; \alpha} \equiv A_{a_1 a_2 a_3}^{\mu \nu ; \alpha} + A_{a_1 a_2 a_3}^{\mu \alpha ; \nu}, \quad (1.6a)$$

$$A_{a_1 a_2 a_3 a_4}^{\mu \nu ; \alpha ; \beta} \equiv A_{a_1 a_2 a_3 a_4}^{\mu \nu ; \alpha ; \beta} + B_{a_1 a_4 a_3 a_2}^{\mu \nu ; \alpha ; \beta}; \quad (1.6b)$$

(c) for the *semilinear form*, by

$$\rho_{a_1 a_2}^{\mu} + \rho_{a_2 a_1}^{\mu} = 0, \quad (1.7a)$$

$$\rho_{a_1 a_2}^{\mu ; a_3} + \rho_{a_2 a_1}^{\mu ; a_2} + \rho_{a_2 a_3}^{\mu ; a_1} = 0, \quad (1.7b)$$

$$\partial_\mu \rho_{a_1 a_2}^{\mu} = \sigma_{a_1}^{\mu ; a_2} - \sigma_{a_2}^{\mu ; a_1}, \quad (1.7c)$$

$$a_1, a_2 = 1, 2, \dots, n, \quad \mu = 0, 1, 2, 3;$$

$$\rho_{a_1 a_2}^{\mu ; a_3} \equiv (\partial \rho_{a_1 a_2}^{\mu} / \partial \phi^{a_3}) \quad (1.7d)$$

$$\sigma_{a_1 a_2}^{\mu} \equiv (\partial \sigma_1 / \partial \phi^{a_2}) \quad (1.7e)$$

In [1] we studied the conventional *Lagrange equations* for classical field theories

$$\begin{aligned} \mathcal{L}_{a_1}(\phi) &\equiv d_\mu \frac{\partial \mathcal{L}}{\partial \phi^{a_1; \mu}} - \frac{\partial \mathcal{L}}{\partial \phi^{a_1}} \\ &= \frac{\partial^2 \mathcal{L}}{\partial \phi^{a_1; \mu} \partial \phi^{a_2; \nu}} \phi^{a_2; \mu\nu} + \frac{\partial^2 \mathcal{L}}{\partial \phi^{a_1; \mu} \partial \phi^{a_2}} \phi^{a_2; \mu} + \frac{\partial^2 \mathcal{L}}{\partial \phi^{a_1; \mu} \partial X^\mu} - \frac{\partial \mathcal{L}}{\partial \phi^{a_1}} \quad (1.8a) \\ &= \frac{1}{2}(\mathcal{L}_{a_1 a_2}^{\mu\nu} + \mathcal{L}_{a_1 a_2}^{\nu\mu}) \phi^{a_2; \mu\nu} + \mathcal{L}_{a_1 a_2}^{\mu} \phi^{a_2; \mu} + \mathcal{L}_{a_1}^{\mu; \mu} - \mathcal{L}_{a_1}^{\mu} = 0, \end{aligned}$$

$$\mathcal{L}_{a_1}^{\mu} \equiv \frac{\partial \mathcal{L}}{\partial X^\mu}, \quad \mathcal{L}_{a_1}^{\mu} \equiv \frac{\partial \mathcal{L}}{\partial \phi^a}, \quad \mathcal{L}_{a_1}^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial \phi^{a; \mu}}, \quad \text{etc.}, \quad (1.8b)$$

$$a_1, a_2 = 1, 2, \dots, n, \quad \mu, \nu = 0, 1, 2, 3$$

in a *Lagrangian density* $\mathcal{L} = \mathcal{L}(x_\alpha, \phi^a, \phi^{a; \alpha})$ and proved that, for class \mathcal{C}^4 and regular Lagrangians, they are always self-adjoint [1, Theorem 7.1].

Since the Lagrange operator $\{d_\mu \partial / \partial \phi^{a; \mu} - \partial / \partial \phi^a\}$ is self-adjoint in the conventional sense used in the theory of linear operators [3], [1]'s analysis essentially provides a variational approach to self-adjointness.

Under the assumed continuity and regularity conditions, the Lagrange equations can then be written in terms of the symbolic notation

$$[\mathcal{L}_{a_1}(\phi)]_{SA}^{\mathcal{C}^2, R} = 0. \quad (1.9)$$

Equations (1.1), (1.2), and (1.3) are termed *self-adjoint* when *all* the corresponding conditions of self-adjointness, (1.4), (1.5), and (1.7), are satisfied. Then we shall symbolically write

$$\begin{aligned} [F_{a_1}]_{SA}^{\mathcal{C}^2, R} &= 0, \\ [A_{a_1 a_2}^{\mu\nu} \phi^{a_2; \mu\nu} + B_{a_1}]_{SA}^{\mathcal{C}^2, R} &= 0, \quad (1.10) \\ [g^{\mu\nu} \phi_{a_1; \mu\nu} - \rho_{a_1 a_2}^{\mu} \phi^{a_2; \mu} - \sigma_{a_1}]_{SA}^{\mathcal{C}^1} &= 0, \end{aligned}$$

where we have reduced the continuity assumptions of Eq. (1.3) because conditions $\rho_{a_1 a_2}^{\mu}, \sigma_{a_1} \in \mathcal{C}^2$ are, in this case, redundant.

Equations (1.1), (1.2), and (1.3) are termed *non-self-adjoint* when at least *one* of the corresponding conditions of self-adjointness (1.4), (1.5), and (1.7) is violated. We shall symbolically write, in this case,

$$\begin{aligned} [F_{a_1}]_{NSA}^{\mathcal{C}^2, R} &= 0, \\ [A_{a_1 a_2}^{\mu\nu} \phi^{a_2; \mu\nu} + B_{a_1}]_{NSA}^{\mathcal{C}^2, R} &= 0, \quad (1.11) \\ [g^{\mu\nu} \phi_{a_1; \mu\nu} - \rho_{a_1 a_2}^{\mu} \phi^{a_2; \mu} - \sigma_{a_1}]_{NSA}^{\mathcal{C}^1} &= 0. \end{aligned}$$

In [2] we introduced a definition of analytic representation of a class \mathcal{C}^2 , regular, covariant, tensorial system of field equations $F_{a_1}(\phi) = 0$ in terms of the Lagrange equations $\mathcal{L}_{a_1}(\phi) = 0$, which occurs when there exists a class \mathcal{C}^2 regular $n \times n$ matrix (h) with elements $h_{a_1}^{a_2} = h_{a_1}^{a_2}(x_\alpha, \phi^a, \phi^{a;\alpha})$ such that

$$[\mathcal{L}_{a_1}(\phi)]_{SA}^{\mathcal{C}^2, R} \equiv \{h_{a_1}^{a_2} [F_{a_2}(\phi)]^{\mathcal{C}^2, R}\}^{\mathcal{C}^2, R} = 0, \quad a_1, a_2 = 1, 2, \dots, n. \quad (1.12)$$

The above definition was then specialized into *ordered direct analytic representations*

$$[\mathcal{L}_{a_1}(\phi)]_{SA}^{\mathcal{C}^2, R} \equiv [F_{a_1}(\phi)]^{\mathcal{C}^2, R} = 0, \quad a_1 = 1, 2, \dots, n, \quad (1.13)$$

and *ordered indirect analytic representations*

$$[\mathcal{L}_{a_1}(\phi)]_{SA}^{\mathcal{C}^2, R} \equiv \{h_{a_1}^{a_2} [F_{a_2}(\phi)]^{\mathcal{C}^2, R}\}^{\mathcal{C}^2, R} \quad (1.14)$$

$$h_{a_1}^{a_2} \neq \delta_{a_1}^{a_2}, \quad a_1, a_2 = 1, 2, \dots, n.$$

A variational analysis of the above analytic representations, when they existed, was also conducted in [1] with the result that the concept of ordering ensured the identity not only of the Lagrange equations with the field equations but also of the corresponding equations of variation and related adjoint systems, e.g., for case (1.13)

Lagrange equations	$\mathcal{L}_{a_1}(\phi) \equiv F_{a_1}(\phi)$	field equations;
Jacobi equations	$\Omega_{a_1}(\eta) \equiv M_{a_1}(\eta)$	equations of variation;
adjoint system of the Jacobi equations	$\tilde{\Omega}_{a_1}(\tilde{\eta}) \equiv \tilde{M}_{a_1}(\tilde{\eta})$	adjoint system of the equations of variation.

(1.15)

To avoid possible misinterpretation, let us also recall from [1]'s analysis that any quasi-linear system can always be transformed into an equivalent system for which the symmetry properties

$$A_{a_1 a_2}^{\mu \nu} = A_{a_1 a_2}^{\nu \mu} \quad (1.16)$$

are verified. Indeed, if this is not the case, we can always write, from the symmetry properties $\phi^{a;\nu\mu} = \phi^{a;\mu\nu}$,

$$A_{a_1 a_2}^{\mu \nu} \phi^{a_2;\nu\mu} + B_{a_1} \equiv \frac{1}{2}(A_{a_1 a_2}^{\mu \nu} + A_{a_1 a_2}^{\nu \mu}) \phi^{a_2;\nu\mu} + \frac{1}{2}(A_{a_1 a_2}^{\mu \nu} - A_{a_1 a_2}^{\nu \mu}) \phi^{a_2;\nu\mu} + B_{a_1}$$

$$\equiv \frac{1}{2}(A_{a_1 a_2}^{\mu \nu} + A_{a_1 a_2}^{\nu \mu}) \phi^{a_2;\nu\mu} + B_{a_1} = 0, \quad (1.17)$$

in which case symmetries (1.16) hold for the last form of Eqs. (1.17), i.e., for the redefined terms $A_{a_1 a_2}^{\mu \nu} = \frac{1}{2}(A_{a_1 a_2}^{\mu \nu} + A_{a_1 a_2}^{\nu \mu})$.

In the present paper we shall tacitly assume that all considered quasi-linear systems satisfy symmetry properties (1.16). This implies in particular that the conditions of self-adjointness (1.5) will be referred to systems obeying such properties.

The above symmetrization procedure also applies to the Lagrange equations. Indeed, if one uses symmetry properties (1.16) for the Lagrange equations in the third form of Eqs. (1.8), the relations

$$\frac{\partial^2 \mathcal{L}}{\partial \phi^{a_1 i}{}_{\mu} \partial \phi^{a_2 i}{}_{\nu}} = \frac{\partial^2 \mathcal{L}}{\partial \phi^{a_1 i}{}_{\nu} \partial \phi^{a_2 i}{}_{\mu}},$$

$$\mu, \nu = 0, 1, 2, 3, \quad a_1, a_2 = 1, 2, \dots, n \quad (1.18)$$

might result. The point which we would like to recall from [1] is that within the context of our analysis, which is ultimately based on an arbitrary structure of the Lagrangian density, properties (1.18) are *not* implied by the continuity assumption $\mathcal{L} \in \mathcal{C}^4$, and they are in general *erroneous*.

Again, symmetry properties (1.16) must be applied to the Lagrange equations in their symmetrized form, i.e., the last form of Eqs. (1.8), in which case they trivially hold for the terms

$$A_{a_1 a_2}^{\mu \nu} = \frac{1}{2} \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^{a_1 i}{}_{\mu} \partial \phi^{a_2 i}{}_{\nu}} + \frac{\partial^2 \mathcal{L}}{\partial \phi^{a_1 i}{}_{\nu} \partial \phi^{a_2 i}{}_{\mu}} \right), \quad (1.19)$$

irrespective of any continuity condition of the Lagrangian.

In the present paper we shall tacitly assume that the Lagrange equations are written in the above indicated symmetrized form and that the conditions of self-adjointness (1.5) are always referred to and identically verified for class \mathcal{C}^2 and regular Lagrange equations in this symmetrized form.

The objectives of this paper are to identify the necessary and sufficient conditions for a given class \mathcal{C}^2 , regular, Lorentz covariant, tensorial, quasi-linear system of field equations to admit an ordered direct analytic representation in terms of the Lagrange equations; to provide a method for the construction of a Lagrangian density, when it exists, from given field equations; and to explore the "structure" of the Lagrangian capable of representing tensorial fields with arbitrary forms of coupling.

The case of ordered *indirect* analytic representations is treated in subsequent paper III. We plan to study the same problems for other type of field equations (e.g., spinorial or degenerate) as well as to explore some initial significance of the underlying methodology in Field Theory at a later time.

2. A THEOREM ON THE EXISTENCE OF A LAGRANGIAN DENSITY FOR ORDERED DIRECT ANALYTIC REPRESENTATIONS

Let us recall that quasi-linear systems (1.2) and the Lagrange equations (1.8) in the Lagrangian densities $\mathcal{L}(x_\alpha, \phi^a, \phi^{a;\alpha})$ are defined in a region R of the variables x_α , ϕ^a , and $\phi^{a;\alpha}$ ($a = 1, 2, \dots, n$, $\alpha = 0, 1, 2, 3$), where the dependence of these equations in the terms $\phi^{a;\mu\nu}$ is ignored due to their linearity. Here the term "region" means an open and connected point set of the values of the indicated variables.

The condition that field equations (1.2) are of (at least) class \mathcal{C}^2 can thus be reduced to the condition that the terms $A_{\sigma_1 \sigma_2}^{\mu_1 \mu_2}$ and B_{α_1} possess continuous partial derivatives with respect to all of their variables (x_α , ϕ^a , $\phi^{a;\alpha}$) in the considered region R .

It should be indicated that within this context the variables ϕ^a are not necessarily the solutions of system (1.2). As a matter of fact, our approach to the problem of identifying the necessary and sufficient conditions for the existence of a Lagrangian does not demand the knowledge of the solutions of the field equations. This aspect is essential in view of the generally nonlinear nature of the considered field equations.

A region R of the variables (x_α , ϕ^a , $\phi^{a;\alpha}$) is here termed a *domain* when it is perfect, internally connected, and each of its points is a point of accumulation of interior points. Then, if R is a region, $\bar{R} = R \cup \partial R$ (where ∂R is the boundary of R) is a domain.

In principle, a domain of definition of Eqs. (1.2) and (1.8) can be arbitrarily selected and, since it is closed, it may consist of the entire set of possible values of the variables x_α , ϕ^a , and $\phi^{a;\alpha}$, thus including the points at infinity. This domain is, however, redundant for the problem under consideration. Besides, the behavior of the conditions of self-adjointness at infinity is quite delicate to handle.

This raises the question, which is an effective region of definition of field equations (1.2) for the problem of the existence of their Lagrangian representation.

The answer to this question is provided in the Appendices, particularly Appendix B. A region R of the variables (x_α , ϕ^a , $\phi^{a;\alpha}$) is termed a *star-shaped region* and denoted with R^* when it contains, jointly with a given open and connected set of points (x_α , ϕ^a , $\phi^{a;\alpha}$), all points (x_α , $\tau\phi^a$, $\tau\phi^{a;\alpha}$) for $0 \leq \tau \leq 1$. Notice that we assume no restriction on the behavior of the Minkowski coordinates x_α , for reasons which will appear self-evident later on. Notice also that all star-shaped regions contain the (local) origin $\phi^a = 0$, $\phi^{a;\alpha} = 0$, $a = 1, 2, \dots, n$, $\alpha = 0, 1, 2, 3$. Again, if R^* is a star-shaped region, $\bar{R}^* = R^* \cup \partial R^*$ is a domain.

Our analysis of the problem of the existence of a Lagrangian will be conducted on a star-shaped rather than a conventional region. The reason is that such regions R^* are needed for the formulation of the Converse of the Poincaré Lemma and its generalization given in Appendix B, which, as is well known, constitute

effective tools for the study, in general, of all integrability conditions. In view of their redundancy (as well as the delicate nature of their technical implications) we shall tacitly assume that all considered star-shaped domains *do not* contain points at infinity.

Our minimal region of definition of Eqs. (1.2) for the problem of the existence of their Lagrangian representation will then be a star-shaped domain \bar{R}^* of the variables $(x_\alpha, \phi^a, \phi^{a; \alpha})$ whose boundary ∂R^* consists of the unit "circle" around the origin of the variables ϕ^a and $\phi^{a; \alpha}$, together with an arbitrary (but bounded) region in the Minkowski coordinates. Under this assumption the distinction between the regions R and R^* becomes purely formal.

Our analysis of the integrability conditions for the existence of a Lagrangian will be conducted within the framework of the ordinary calculus of differential forms in the local coordinates ϕ^a and its extension to the case of local coordinates $\phi^{a; \alpha}$, which are outlined, for the reader's convenience, in Appendix A and B.

This raises the question, of which is an effective form of the conditions of self-adjointness (1.5) within the framework of differentiable manifolds with local coordinates ϕ^a and $\phi^{a; \alpha}$.

This question is explored in Appendix C, resulting in the set of integrability conditions

$$\delta_{b_1 b_2 \nu_1 \nu_2}^{a_1 a_2 \mu_1 \mu_2} A_{a_1 a_2}^{\nu_1 \nu_2} = 0, \tag{2.1a}$$

$$\delta_{b_1 b_2 b_3 \nu_1 \nu_2 \nu_3}^{a_1 a_2 a_3 \mu_1 \mu_2 \mu_3} A_{a_1 a_2 a_3}^{\nu_1 \nu_2 \nu_3} = 0, \tag{2.1b}$$

$$\delta_{b_1 b_2 b_3 b_4 \nu_1 \nu_2 \nu_3 \nu_4}^{a_1 a_2 a_3 a_4 \mu_1 \mu_2 \mu_3 \mu_4} A_{a_1 a_2 a_3 a_4}^{\nu_1 \nu_2 \nu_3 \nu_4} = 0, \tag{2.1c}$$

$$\delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} B_{a_1 a_2 a_3}^{; \mu} = 0, \tag{2.1d}$$

$$B_{a_1 a_2 a_3}^{; \mu; \nu} - B_{a_2 a_1 a_3}^{; \mu; \nu} = 2(A_{a_1 a_3}^{\mu; \nu; a_2} - A_{a_2 a_3}^{\mu; \nu; a_1}), \tag{2.1e}$$

$$b_1, \dots, b_A = 1, 2, \dots, n,$$

$$\mu_1, \dots, \mu_A, \mu, \nu = 0, 1, 2, 3,$$

where $A_{a_1 a_2}^{\mu_1 \mu_2}$ and B_{a_1} are characterized by the given system of field equations (1.2) and

$$\delta_{b_1 \dots b_\mu \nu_1 \dots \nu_\mu}^{a_1 \dots a_\mu \mu_1 \dots \mu_\mu} \equiv \begin{vmatrix} \delta_{b_1}^{a_1} & \dots & \delta_{b_n}^{a_1} \\ \dots & \dots & \dots \\ \delta_{b_1}^{a_n} & \dots & \delta_{b_n}^{a_n} \end{vmatrix} \times \begin{vmatrix} \delta_{\nu_1}^{\mu_1} & \dots & \delta_{\nu_n}^{\mu_1} \\ \dots & \dots & \dots \\ \delta_{\nu_1}^{\mu_n} & \dots & \delta_{\nu_n}^{\mu_n} \end{vmatrix}. \tag{2.2}$$

Let us state, using Appendix C, that Eqs. (2.1) are implied by the conditions of self-adjointness (1.5) in the sense that when *all* conditions (1.5) hold in a given

star-shaped region R^* of the variables $(x_\mu; \phi^a, \phi^{a;\mu})$ then all Eqs. (2.1) are identically verified in R^* . In essence, Eqs. (2.1) are a suitably selected linear combination of Eqs. (1.5).

In the following, we shall freely use either conditions (1.5) or (2.1), depending on the case at hand.

We are now equipped to formulate and prove

THEOREM 2.1 [4]. *Necessary and sufficient condition for a Lorentz covariant, tensorial, quasi-linear system of field equations*

$$A_{a_1 a_2}^{\mu_1 \mu_2}(x_\mu, \phi^a, \phi^{a;\mu}) \phi^{a_2;\mu_1 \mu_2} + B_{a_1}(x_\mu, \phi^a, \phi^{a;\mu}) = 0, \\ a, a_1, a_2 = 1, 2, \dots, n, \quad \mu, \mu_1, \mu_2 = 0, 1, 2, 3, \tag{2.3}$$

which is defined, of (at least) class \mathcal{C}^2 and is regular in a star-shaped region R^* of the variables $(x_\mu, \phi^a, \phi^{a;\mu})$, to admit an ordered direct analytic representation in terms of the Lagrange equations in R^*

$$d_\mu \frac{\partial \mathcal{L}}{\partial \phi^{a_1;\mu}} - \frac{\partial \mathcal{L}}{\partial \phi^{a_1}} \equiv A_{a_1 a_2}^{\mu_1 \mu_2} \phi^{a_2;\mu_1 \mu_2} + B_{a_1} = 0 \tag{2.4}$$

is that the system of field equations is self-adjoint in every bounded domain in the interior of R^* .

Proof. Since the system of field equations is of (at least) class \mathcal{C}^2 and is regular in R^* , a necessary condition for the existence of analytic representation (2.4) is that the Lagrangian \mathcal{L} be of (at least) class \mathcal{C}^4 and be regular in R^* . Then, Theorem 7.1 of [1] applies and the Lagrange equations are self-adjoint in R^* . The condition of self-adjointness of the system of field equations is then *necessary* for the existence of the ordered identification (2.4) in view of the self-adjointness of their lhs.

To prove *sufficiency*, we shall show that under the conditions of self-adjointness of the system of field equations in R^* a Lagrangian always exists.

From the condition of regularity it follows that a general structure of the Lagrangian density is given by

$$\mathcal{L}(x_\mu, \phi^a, \phi^{a;\mu}) \\ = K(x_\mu, \phi^a, \phi^{a;\mu}) + D_a{}^\mu(x_\mu, \phi^a) \phi^{a;\mu} + C(x_\mu, \phi^a), \tag{2.5}$$

where the “kinetic” density K is nonlinear in the derivative terms and all the densities K , $D_a{}^\mu$, and C are of (at least) class \mathcal{C}^4 in the star-shaped region of their respective variables.

By substituting form (2.5) in identifications (2.4) and from the quasi-linear structure of the Lagrange equations (1.8) we first reach the two sets of identities,

$$\frac{1}{2}(K_{a_1 a_2}^{i\mu_1\mu_2} + K_{a_1 a_2}^{i\mu_2\mu_1}) \equiv A_{a_1 a_2}^{\mu_1\mu_2}, \quad (2.6a)$$

$$(D_{a_1 a_2}^\mu - D_{a_2 a_1}^\mu) \phi^{a_2 i \mu} + (D_{a_1 \mu}^\mu - C_{a_1}^i) \equiv B_{a_1} + K_{a_1}^i - K_{a_1 \mu}^{i\mu} - (K_{a_1 a_2}^{i\mu}) \phi^{a_2 i \mu}, \quad (2.6b)$$

$$a_1, a_2 = 1, 2, \dots, n, \quad \mu, \mu_1, \mu_2 = 0, 1, 2, 3,$$

which must individually hold for a Lagrangian to exist, where we are again using, for convenience, the derivative notation of type (1.8b).

Equations (2.6a) constitute an independent system of conditions for the existence of the K density.

The assumption that such equations are solved first, allows us to consider all terms in the K density of Eqs. (2.6b) as known. For this reason they are written in the rhs jointly with the assigned B terms.

By writing Eqs. (2.6b) in the a_1 and a_2 indices, by differentiating with respect to $\phi^{a_2 i \mu}$ and $\phi^{a_1 i \mu}$, respectively, and by subtracting we reach the equations (see Appendix C for more details)

$$D_{a_1 a_2}^\mu - D_{a_2 a_1}^\mu \equiv \frac{1}{2}(B_{a_1 a_2}^{i\mu} - B_{a_2 a_1}^{i\mu}) + (K_{a_1 a_2}^{i\mu} - K_{a_1 a_2}^{i\mu}), \quad (2.7)$$

$$a_1, a_2 = 1, 2, \dots, n, \quad \mu = 0, 1, 2, 3,$$

which constitute, for a given K , an independent set of conditions for the existence of the D densities.

By assuming that such D densities exist and are computed, we now substitute Eqs. (2.7) in (2.6b) by reaching in this way the equations

$$C_{a_1}^i \equiv D_{a_1 \mu}^\mu - B_{a_1} - K_{a_1}^i + K_{a_1 \mu}^{i\mu} + [K_{a_2 a_1}^{i\mu} + \frac{1}{2}(B_{a_2 a_2}^{i\mu} - B_{a_2 a_1}^{i\mu})] \phi^{a_2 i \mu}, \quad (2.8)$$

$$a_1, a_2 = 1, 2, \dots, n, \quad \mu = 0, 1, 2, 3,$$

which constitute, for given K and D_a^μ , an independent set of conditions for the existence of the C density.

The combined set of conditions (2.6a), (2.7), and (2.8), i.e.,

$$\frac{1}{2}(K_{a_1 a_2}^{i\mu_1\mu_2} + K_{a_1 a_2}^{i\mu_2\mu_1}) \equiv A_{a_1 a_2}^{\mu_1\mu_2}, \quad (2.9a)$$

$$D_{a_1 a_2}^\mu - D_{a_2 a_1}^\mu \equiv \frac{1}{2}(B_{a_1 a_2}^{i\mu} - B_{a_2 a_2}^{i\mu}) + (K_{a_1 a_2}^{i\mu} - K_{a_2 a_1}^{i\mu}), \quad (2.9b)$$

$$C_{a_1}^i \equiv D_{a_1 \mu}^\mu - B_{a_1} - K_{a_1}^i + K_{a_1 \mu}^{i\mu} + [K_{a_2 a_1}^{i\mu} + \frac{1}{2}(B_{a_1 a_2}^{i\mu} - B_{a_2 a_1}^{i\mu})] \phi^{a_2 i \mu}, \quad (2.9c)$$

$$a_1, a_2 = 1, 2, \dots, n, \quad \mu, \mu_1, \mu_2 = 0, 1, 2, 3$$

constitutes a generally overdetermined system of partial differential equations in the $(4n + 2)$ unknown densities K , D_a^μ , and C which characterize a Lagrangian according to Eqs. (2.5).

Our proof of sufficiency will consist in showing that *the conditions of self-adjointness (1.5) or (2.1) are the integrability conditions of Eqs. (2.9).*

1. *Integrability Conditions of Eqs. (2.9a)*

Introduce the quantities

$$T_a^\mu \equiv K_a^\mu \tag{2.10}$$

and consider the system of first-order partial differential equations

$$T_{a_1 a_2}^{\mu_1 \mu_2} - A_{a_1 a_2}^{\mu_1 \mu_2} = 0 \tag{2.11}$$

with underlying (2, 2) form (see Appendix A)

$$T^{(2,2)} = (T_{a_1 a_2}^{\mu_1 \mu_2} - A_{a_1 a_2}^{\mu_1 \mu_2}) d\phi_{\mu_1}^{a_1} \wedge d\phi_{\mu_2}^{a_2}. \tag{2.12}$$

But, from the Converse of the Generalized Poincaré Lemma (see Appendix B), the conditions of self-adjointness (2.1b) are the integrability conditions of Eqs. (2.12). Thus, under the assumptions of the theorem, a solution of Eqs. (2.11) exists. However, this solution is not necessarily consistent with Eqs. (2.9a), due to the lack of symmetrization. This demands that, together with Eqs. (2.11), the equations

$$T_{a_1 a_2}^{\mu_1 \mu_2} - A_{a_1 a_2}^{\mu_2 \mu_1} = 0 \tag{2.13}$$

hold with underlying (2, 2) form

$$T'^{(2,2)} = (T_{a_1 a_2}^{\mu_1 \mu_2} - A_{a_1 a_2}^{\mu_2 \mu_1}) d\phi_{\mu_1}^{a_1} \wedge d\phi_{\mu_2}^{a_2}. \tag{2.14}$$

The consistency condition of Eqs. (2.11) and (2.13) reads

$$(A_{a_1 a_2 a_3}^{\mu_1 \mu_2 \mu_3} - A_{a_1 a_3 a_2}^{\mu_1 \mu_2 \mu_3}) d\phi_{\mu_1}^{a_1} \wedge d\phi_{\mu_2}^{a_2} \wedge d\phi_{\mu_3}^{a_3} = 0. \tag{2.15}$$

Conditions of self-adjointness (2.1a) and (2.1c) then guarantee the validity of Eqs. (2.15).

It follows that a solution of the equations

$$T_{a_1 a_2}^{\mu_1 \mu_2} - \frac{1}{2}(A_{a_1 a_2}^{\mu_1 \mu_2} + A_{a_2 a_1}^{\mu_2 \mu_1}) = 0 \tag{2.16}$$

exists and is given, from Eq. (B.30), by

$$T_{a_1}^{\mu_1} = 2 \left(\int_0^1 d\tau \tau A_{a_1 a_2}^{\mu_1 \mu_2}(\tau \phi^{a_i}_{\mu}) \right) \phi^{a_2}_{\mu_2}. \quad (2.17)$$

The second step is to consider Eqs. (2.10), i.e.,

$$K_{a_1}^{\mu_1} - T_{a_1}^{\mu_1} = K_{a_1}^{\mu_1} - 2 \left[\int_0^1 d\tau \tau A_{a_1 a_2}^{\mu_1 \mu_2}(\tau \phi^{a_i}_{\mu}) \right] \phi^{a_2}_{\mu_2} = 0. \quad (2.18)$$

The related integrability conditions are

$$\begin{aligned} \delta_{b_1 b_2 \nu_1 \nu_2}^{a_1 a_2 \mu_1 \mu_2} T_{a_1 a_2}^{\nu_1 \nu_2} &= 2 \int_0^1 d\tau \tau \delta_{b_1 b_2 \nu_1 \nu_2}^{a_1 a_2 \mu_1 \mu_2} A_{a_1 a_2}^{\nu_1 \nu_2}(\tau \phi^{a_i}_{\mu}) \\ &+ \left[\int_0^1 d\tau \tau^2 \delta_{b_1 b_2 b_3 \nu_1 \nu_2 \nu_3}^{a_1 a_2 a_3 \mu_1 \mu_2 \mu_3} A_{a_1 a_2 a_3}^{\nu_1 \nu_2 \nu_3}(\tau \phi^{a_i}_{\mu}) \right] \phi^{b_3}_{\mu_3} = 0, \end{aligned} \quad (2.19)$$

and they identically hold in view of Eqs. (2.1a) and (2.1b).

Therefore, under the assumptions of the theorem, a solution of Eqs. (2.16) exists and is given by

$$K(x_\mu, \phi^a, \phi^{a_i}_{\mu}) = 2 \phi^{a_1}_{\mu_1} \int_0^1 d\tau' \left[\phi^{a_2}_{\mu_2} \int_0^1 d\tau \tau A_{a_1 a_2}^{\mu_1 \mu_2}(x_\mu, \phi^a, \tau \phi^{a_i}_{\mu}) \right] (\tau' \phi^{a_i}_{\mu}), \quad (2.20)$$

where the square bracket indicates that the function of derivative terms resulting after integration with respect to τ must be computed along $\tau' \phi^{a_i}_{\mu}$ prior to the integration with respect to τ' .

This completes the first part of our proof and shows that *the conditions of self-adjointness (2.1a), (2.1b), and (2.1c) are the integrability conditions of Eqs. (2.9a) [6].*

2. Integrability Conditions of Eqs. (2.9b) and (2.9c)

We consider now Eqs. (2.9b) and (2.9c), which we write

$$D_{a_1 a_2}^{\mu} - D_{a_2 a_1}^{\mu} - Z_{a_1 a_2}^{\mu} = 0, \quad (2.21a)$$

$$C_{a_1}^i - W_{a_1} = 0, \quad (2.21b)$$

with a self-explanatory definition of the terms $Z_{a_1 a_2}^{\mu}$ and W_{a_1} . The underlying differential forms are the collection of $(\mu, 2)$, $\mu = 0, 1, 2, 3$ and 1 forms, respectively (see Appendix A),

$$Z^{(\mu, 2)} = Z_{a_1 a_2}^{\mu} d\phi^{a_1} \wedge d\phi^{a_2} \quad (2.22a)$$

$$W^{(1)} = W_{a_1} d\phi^{a_1}, \quad (2.22b)$$

with integrability conditions

$$\delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} Z_{a_1 a_2 a_3}^{\mu \nu} = 0, \quad (2.23a)$$

$$\delta_{b_1 b_2}^{a_1 a_2} W_{a_1 a_2}^{\nu} = 0, \quad (2.23b)$$

respectively.

By using the explicit form of the $Z_{a_1 a_2 a_3}^{\mu \nu}$ and $W_{a_1 a_2}^{\nu}$ terms from the rhs of Eqs. (2.9b) and (2.9c), respectively, conditions (2.23) can be written (see Appendix C for more details)

$$\begin{aligned} \delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} B_{a_1 a_2 a_3}^{\mu \nu} &= 0, \\ \frac{1}{2} (\delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} B_{a_1 a_2 a_3}^{\mu \nu}) \phi^{b_3 \nu} &= 0, \end{aligned} \quad (2.24)$$

and they identically hold in view of Eqs. (2.1d).

Therefore, under the conditions of self-adjointness the solutions of Eqs. (2.21) exist and are given by

$$\begin{aligned} D_{a_1}^{\mu}(x_{\alpha}, \phi^a) &= \phi^{a_2} \int_0^1 d\tau \tau Z_{a_1 a_2}^{\mu \nu}(x_{\alpha}, \tau \phi^a), \\ C(x_{\alpha}, \phi^a) &= \phi^{a_2} \int_0^1 d\tau W_{a_1}^{\nu}(x_{\alpha}, \tau \phi^a). \end{aligned} \quad (2.25)$$

This completes the second part of our proof and shows that *the conditions of self-adjointness (2.1d) are the integrability conditions of Eqs. (2.9b) and (2.9c).*

3. Consistency of Eqs. (2.9)

To complete our proof, we must first show that, for consistency, the rhs of Eqs. (2.9b) and (2.9c) is independent of derivative terms.

By differentiating Eqs. (2.9b) and (2.9c) with respect to $\phi^{a_3 \nu}$, we reach the respective conditions

$$(B_{a_1 a_2 a_3}^{\mu \nu \nu} - B_{a_2 a_1 a_3}^{\mu \nu \nu}) - 2(A_{a_1 a_3}^{\mu \nu \nu} - A_{a_2 a_3}^{\mu \nu \nu}) = 0, \quad (2.26)$$

$$[(B_{a_1 a_2 a_3}^{\mu \nu \nu} - B_{a_3 a_2 a_1}^{\mu \nu \nu}) - 2(A_{a_1 a_2}^{\mu \nu \nu} - A_{a_2 a_3}^{\mu \nu \nu})] \phi^{a_3 \nu} = 0,$$

which clearly hold in view of Eqs. (2.1e).

This proves that conditions of self-adjointness (2.1e) guarantee the independence of the rhs of Eqs. (2.9a) and (2.9b) from the derivative terms.

Our proof will be completed by showing, for consistency, that Eqs. (2.9) are compatible among themselves.

Since Eqs. (2.9a) must be solved first, the problem of compatibility of Eqs. (2.9) can be reduced to the proof that Eqs. (2.9b) and (2.9c), under identifications (2.9a), are compatible among themselves.

Let us rewrite these equations in the form

$$D_{a_1 a_2}^\mu - D_{a_2 a_1}^\mu = Z_{a_1 a_2}^\mu, \quad (2.27a)$$

$$D_{a_1 \mu}^\mu - C_{a_1}^\mu = W'_{a_1}, \quad (2.27b)$$

where $W'_{a_2} = D_{a_1 \mu}^\mu - W_a$.

After partial differentiation with respect to x^μ and ϕ^{a_2} respectively, we can write

$$D_{a_1 a_2 \mu}^\mu = Z_{a_1 a_2 \mu}^\mu + D_{a_2 a_1 \mu}^\mu, \quad (2.28)$$

$$D_{a_1 \mu a_2}^\mu = W'_{a_1 a_2} + C_{a_1 a_2}^\mu.$$

Thus, the *necessary* conditions for the consistency of Eqs. (2.27) are

$$Z_{a_1 a_2 \mu}^\mu = W'_{a_1 a_2} - W'_{a_2 a_1}, \quad (2.29)$$

where we have used Eqs. (2.27b).

To prove that conditions (2.29) are also *sufficient* for the consistency of Eqs. (2.27), consider such equations for fixed values of the indices $a_1 = a_1^0$ and $a_2 = a_2^0 (\neq a_1^0)$. Then the existence theorems for linear partial differential equations [8] apply (in view of the continuity properties of the $Z_{a_1 a_2}^\mu$ and W'_{a_1} functions) and a solution $D_{a_1^0}^\mu$, $D_{a_2^0}^\mu$, and C exists.

We now substitute such solutions into Eqs. (2.27) in the form

$$D_{a_2 a_1^0}^\mu = D_{a_1^0 a_2}^\mu - Z_{a_1^0 a_2}^\mu, \quad (2.30)$$

$$D_{a_2 \mu}^\mu = W'_{a_2} + C_{a_2}^\mu.$$

Such equations are compatible provided that

$$D_{a_1^0 a_2 \mu}^\mu - Z_{a_1^0 a_2 \mu}^\mu = W'_{a_2 a_1^0} + C_{a_2 a_1^0}^\mu. \quad (2.31)$$

But the above conditions reduce to Eqs. (2.29) after use of Eqs. (2.27b).

Thus, Eqs. (2.29) are the necessary and sufficient conditions for the consistency of Eqs. (2.27).

We must now inspect Eqs. (2.29) by using the explicit form of the $Z_{a_1 a_2}^\mu$ and W'_{a_1} terms. From the rhs of Eqs. (2.9b) and (2.9c) (and by recalling that $W'_{a_1} = D_{a_1 \mu}^\mu - W_a$), Eqs. (2.27) become

$$B_{a_1 a_2}^\mu - B_{a_2 a_1}^\mu - \frac{1}{2}(B_{a_2 a_1}^{\mu} - B_{a_2 a_1}^{\mu})^\mu + [(B_{a_2 a_3}^{\mu} - B_{a_3 a_2}^{\mu})^\mu_{a_1} + (B_{a_3 a_1}^{\mu} - B_{a_1 a_3}^{\mu})^\mu_{a_2}] \phi^{a_3 \mu} = 0, \quad (2.32)$$

which, by using conditions of self-adjointness (1.5d), can be written

$$\frac{1}{2}(\delta_{b_1 b_3}^{a_1 a_3} B_{a_1 a_2 a_3} ;^{\mu}) \phi^{b_3 ; \mu} = 0, \tag{2.33}$$

and they identically hold in view of conditions (2.1d).

This completes the third part of our proof and shows that *Eqs. (2.1d) are not only the integrability conditions for Eqs. (2.9b) and (2.9c), but also the necessary and sufficient conditions for their consistency.*

Thus, when the conditions of self-adjointness (1.5) or (2.1) hold, Eqs. (2.9) always admit solutions $K, D_a^\mu,$ and C and a Lagrangian (2.5) always exists. Q.E.D.

A few comments are now in order. Theorem 2.1 and its proof clearly indicate the effectiveness of our variational approach to self-adjointness for the problem of the existence of a Lagrangian density in classical Field Theory. Let us recall from Appendix C that *all* conditions of self-adjointness (1.5) enter into the construction of the integrability conditions (2.1). Therefore, without redundancy, all conditions of self-adjointness enter into the proof of the theorem.

This latter remark must be kept in mind for practical applications. Indeed, if the assigned system of field equations violates only *one* of the conditions of self-adjointness, then, according to our terminology, it is non-self-adjoint and a Lagrangian for the ordered direct identification (2.4) *does not* exist.

In this case, however, one can seek for an ordered "indirect" analytic representation. This aspect will be investigated in subsequent paper III.

The significance of the concept of "ordering" in the statement and proof of the theorem demands some elaboration.

Let us remark that the conditions of self-adjointness for the case of *ordinary* differential equations were rather generally considered to be both necessary and sufficient for the existence of a Lagrangian for a "direct analytic representation" without any reference as far as the ordering is concerned [5].

This author, however, identified [5] (apparently for the first time) a counterexample to the above position concerning the *necessity* of the conditions of self-adjointness.

This counterexample essentially consists of the identification of the variational properties of the Morse-Feshbach method [9] for representing certain non-conservative systems. Explicitly, the system of second-order ordinary differential equations (see [1, Appendix C] for more details)

$$\begin{aligned} \begin{pmatrix} m\ddot{q}_1 + b\dot{q}_1 + kq_1 \\ m\ddot{q}_2 - b\dot{q}_2 + kq_2 \end{pmatrix} &= \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ &= (c_{a_1 a_2} \ddot{q}_2) + (b_{a_1 a_2} \dot{q}_{a_2}) + (a_{a_1 a_2} q_{a_2}) = 0 \end{aligned} \tag{2.34}$$

is *non-self-adjoint* because it violates the conditions of self-adjointness

$$\begin{aligned} c_{a_1 a_2} - c_{a_2 a_1} &= 0, \\ b_{a_1 a_2} + b_{a_2 a_1} &= 2(d/dt) c_{a_1 a_2}, \\ a_{a_1 a_2} - a_{a_2 a_1} &= \frac{1}{2}(d/dt)(b_{a_1 a_2} - b_{a_2 a_1}) \end{aligned} \quad (2.35)$$

in the $b_{a_1 a_2}$ terms. Nevertheless, a Lagrangian for their analytic representation exists and is given by [9]

$$L = m\dot{q}_1 \dot{q}_2 + \frac{1}{2}b(q_1 \dot{q}_2 - \dot{q}_1 q_2) - kq_1 q_2. \quad (2.36)$$

The solution of the above rather puzzling situation [5] is easily found by noting that the Lagrange equations in the Lagrangian (2.36) *do not* reproduce Eqs. (2.34), but rather the same system in the inverted order, i.e.,

$$\begin{aligned} \left(\begin{array}{c} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} \end{array} \right) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m\ddot{q}_1 + b\dot{q}_1 + kq_1 \\ m\ddot{q}_2 - b\dot{q}_2 + kq_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ &= (c'_{a_1 a_2} \ddot{q}_2) + (b'_{a_1 a_2} \dot{q}_2) + (a'_{a_1 a_2} q_2) = 0. \end{aligned} \quad (2.37)$$

A simple inspection then indicates that the equations of motion in the inverted order (2.37) satisfy all Eqs. (2.35), and thus, *are* self-adjoint. Incidentally, this also confirms the self-adjointness of the Lagrange equations in the Lagrangian (2.36).

To summarize, for the case of the equations of motion under consideration, the permutation of the *order* in which these equations are assigned changes the system from non-self-adjoint to self-adjoint or vice versa.

In the transition to Field Theory the above framework remains conceptually unchanged. Indeed, by performing the transition

$$\begin{aligned} m^{1/2} q_1 &\rightarrow \varphi, & m^{1/2} q_2 &\rightarrow \bar{\varphi}, \\ b &\rightarrow 2mieA_\mu{}^e(x), & k &\rightarrow m(m_0^2 - e^2 A_\mu{}^e A^{e\mu}), \end{aligned} \quad (2.38)$$

the Morse-Feshbach Lagrangian (2.36) becomes that of the complex scalar field in interaction with an (external) electromagnetic field

$$\begin{aligned} L \rightarrow \mathcal{L} &= \bar{\varphi}^i{}_\mu \varphi^{i\mu} - m_0^2 \bar{\varphi} \varphi + e^2 A_\mu{}^e A^{e\mu} \bar{\varphi} \varphi \\ &\quad - ie(\bar{\varphi}^i{}_\mu \varphi - \bar{\varphi} \varphi^i{}_\mu) A^{e\mu}, \end{aligned} \quad (2.39)$$

with underlying field equations

$$\begin{aligned} (\square + m_0^2 - e^2 A^e_{\mu} A^{e\mu}) \varphi - 2ie A^{e\mu} \varphi_{;\mu} &= 0, \\ (\square + m_0^2 - e^2 A^e_{\mu} A^{e\mu}) \bar{\varphi} + 2ie A^{e\mu} \bar{\varphi}_{;\mu} &= 0. \end{aligned} \quad (2.40)$$

Again, system (2.40) in the ordering $(\phi^{a_1} = \varphi, \phi^{a_2} = \bar{\varphi})$ is *non-self-adjoint* (see [1, Appendix C]). However, if the same system is written in the reverse ordering produced by a permutation of the indices, then it is *self-adjoint*. And indeed, as the reader can verify with a simple inspection and as is the case in general for all Lagrangians with gauge invariant terms of the type $\bar{\varphi}\varphi$, the Lagrange equations in Lagrangian (2.39) produce the field equation (2.41) in their inverted ordering, i.e., in their self-adjoint form.

The above remarks illustrate the breakdown of the necessity of the condition of self-adjointness when the ordering of the field equations is ignored. Theorem 2.1 restores the necessity of the conditions of self-adjointness through a properly selected definition of the analytic representations they are referred to, namely, that of an "ordered direct analytic representation" with underlying structure (1.15).

This state of affairs can be summarized with

COROLLARY 2.1A. *If the ordering criterion is relaxed in Theorem 2.1 the condition of self-adjointness of the field equations is only sufficient for the existence of an analytic representation.*

Notice that the methodology which underlies the formulation and proof of Theorem 2.1 is purely variational in nature. In this respect the following remarks are in order.

Let us recall from [1] that the concept of self-adjointness for systems of *ordinary* differential equations originated within the framework of the so-called *Inverse Problem of the Calculus of Variations*. In our extension of this framework to Field Theory the extremal aspect of the problem has been ignored.

Basically, the single-integral path functionals

$$A(q) = \int_{t_1}^{t_2} dt L(t, q, \dot{q}) \quad (2.41)$$

constitute well-defined variational problems, provided that the Legendre (in essence, our condition of regularity) as well as other conditions are verified. Then, the inverse problem is well defined too. Intensive investigations were conducted in this respect but, regrettably, only up to the first part of this century [1, 5].

In the transition to Lorentz-covariant Field Theories, the above framework is

considerably altered because the extremal problem for *hyperbolic* multiple integral path functions

$$A(\phi) = \int_R d^4x \mathcal{L}(x_\mu, \phi^a, \phi^{a;\mu}) \quad (2.42)$$

is generally vacuous despite the use of boundary conditions [10].

This is, in ultimate analysis, a consequence of the Lorentz covariance of the theory, which imposes the hyperbolic nature of the problem; this hyperbolic nature implies, in view of the indefinite nature of the underlying metric, the impossibility of satisfying the Legendre condition despite the verification of the regularity condition; and this, in turn, implies the lack, in general, of either a maximum or a minimum for the functional (2.42).

Incidentally, this might be a reason for the lack of investigation, to the best knowledge of this author, on the inverse problem of hyperbolic multiple integral path functionals.

Of course, the above difficulties are absent for a Field Theory on a Euclidean space. Indeed, in this case the definite nature of the underlying metric does imply the possibility of satisfying the Legendre condition. Then, the conventional extremal problem as well as its inverse are well defined.

The point which we would like to stress here is that the extremal aspect of the problem is immaterial for the methodology to identify a Lagrangian. Therefore, for the case of Lorentz-covariant Field Theories, even though the extremal aspect is problematic, the question of the existence of a Lagrangian is well defined. This is in line with our presentation and proof of Theorem 2.1, where we have ignored the extremal aspect and used only the variational techniques for the identification of the conditions of self-adjointness.

To indicate the variational nature of our treatment, let us remark that the conventional techniques nowadays used in Field Theory such as Hamilton's Principle, Noether Theorem, etc., are, from the viewpoint of the Calculus of Variations, only first-order techniques. This is due to the fact that they arise within the context of the first-order variations of the action functional.

Another point which we would like to stress is that, to the best knowledge of this author, such first-order techniques are insufficient to provide the necessary methodology for the existence of a Lagrangian. And indeed, our proof of Theorem 2.1 demands the use of both first- and second-order variational techniques.

More specifically, with reference to the explicit structure (1.15) of an analytic representation, the first line arises within the framework of first-order techniques. This is, in essence, the derivation of the Lagrange equations and related identifications (2.4) from Hamilton's Principle. The other lines of structure (1.15) can be derived only within the framework of second-order variations which, in the Calculus of Variations, are related to the so-called "accessory extremal problem."

Therefore, our analysis is ultimately variational in nature because the concept of self-adjointness demands the use of first- and second-order variations.

Alternatively, and by also referring to structure (1.15), we can say that, together with the use of the Lagrange equations, our treatment demands the use of the related Jacobi equations which, again, are of second-order variational nature.

It should be noted, incidentally, that the joint use of Lagrange and the related Jacobi equations might also be of some significance for other aspects of Field Theory, particularly in relation to nonlinear theories. Indeed, when the solutions of the Lagrange equations cannot be computed, the solutions of the related Jacobi equations *can* always be computed (under the necessary continuity and regularity requirements) because they are always linear irrespective of the linearity or non-linearity of the Lagrange equations.

Our analysis and the above remarks seem to indicate that, despite a rather general belief to the contrary, the methodology of the Calculus of Variations at large might have a rather profound impact in Theoretical Physics which goes considerably beyond the framework of Hamilton's Principle and its applications (e.g., the Noether Theorem).

As a final point, we would like to stress that our proof of Theorem 2.1 is the simplest that this author has been able to formulate and, as such, it makes the most economical use possible of the methodology of the calculus of variations. What we want to recall here is that such methodology is rather vast indeed, and it includes tools such as [10-11] the Weierstrass' function, the formulation in terms of Hilbert's invariant integral, Weyl's theory, Charthéodory theory, etc., and topics of geometrical nature [12]. It is not inconceivable that several variational methods may have direct significance for the problem of the existence of the Lagrangian as well as for other aspects of Field Theories (e.g., the study of nonlinear theories, or, more generally, the study of arbitrary forms of Lorentz-covariant couplings). These profiles are here left to the imagination of the individual reader. For an analysis of some alternative methods for the inverse problem of single integral path functionals see [5].

3. A METHOD FOR THE CONSTRUCTION OF A LAGRANGIAN DENSITY AND AN ANALYSIS OF ITS STRUCTURE

Our proof of Theorem 2.1 provides not only the system of partial differential equations (2.9) for the construction of a Lagrangian density, when it exists, but also its solution.

This result, in essence, originates from the use of the calculus of differential forms in general, and the Converse of the Poincaré Lemma in particular.

And indeed it is a matter of a simple restatement of the proof of Theorem 2.1 to reach

COROLLARY 2.1B. *A Lagrangian density for the ordered direct analytic representation of Lorentz-covariant, tensorial, quasi-linear systems of field equations*

$$A_{a_1 a_2}^{\mu_1 \mu_2}(x_\mu, \phi^a, \phi^{a; \mu}) \phi^{a; \mu_1 \mu_2} + B_{a_1}(x_\mu, \phi^a, \phi^{a; \mu}) = 0, \\ a, a_1, a_2 = 1, 2, \dots, n, \quad \mu, \mu_1, \mu_2 = 0, 1, 2, 3, \quad (3.1)$$

which are of (at least) class \mathcal{C}^2 , are regular, and are self-adjoint in a star-shaped region R^* of points $(x_\mu, \phi^a, \phi^{a; \mu})$, is given by

$$\mathcal{L}(x_\mu, \phi^a, \phi^{a; \mu}) = K(x_\mu, \phi^a, \phi^{a; \mu}) + D_{a_1}^{\mu_1}(x_\mu, \phi^a) \phi^{a; \mu_1} + C(x_\mu, \phi^a), \quad (3.2)$$

where the $(4n + 2)$ densities K , $D_{a_1}^{\mu_1}$, and C are a solution of the linear, generally overdetermined system of partial differential equations

$$\frac{1}{2}(K_{a_1 a_2}^{\mu_1 \mu_2} + K_{a_1 a_2}^{\mu_2 \mu_1}) = A_{a_1 a_2}^{\mu_1 \mu_2}(x_\mu, \phi^a, \phi^{a; \mu}), \quad (3.3a)$$

$$D_{a_1 a_2}^{\mu_1} - D_{a_2 a_1}^{\mu_1} = \frac{1}{2}(B_{a_1 a_2}^{\mu_1} - B_{a_2 a_1}^{\mu_1}) + (K_{a_1 a_2}^{\mu_1} - K_{a_2 a_1}^{\mu_1}) \\ \equiv Z_{a_1 a_2}^{\mu_1}(x_\mu, \phi^a), \quad (3.3b)$$

$$C_{a_1}^{\mu_1} = D_{a_1 \mu_1}^{\mu_1} - B_{a_1} - K_{a_1}^{\mu_1} - K_{a_1 \mu_1}^{\mu_1} + [K_{a_1 a_2}^{\mu_1} + \frac{1}{2}(B_{a_1 a_2}^{\mu_1} - B_{a_2 a_1}^{\mu_1})] \phi^{a; \mu_1} \\ \equiv W_{a_1}(x_\mu, \phi^a), \quad (3.3c) \\ a, a_1, a_2 = 1, 2, \dots, n, \quad \mu, \mu_1, \mu_2 = 0, 1, 2, 3.$$

given by

$$K = 2\phi^{a; \mu_1} \int_0^1 d\tau' \left[\phi^{a; \mu_2} \int_0^1 d\tau \tau A_{a_1 a_2}^{\mu_1 \mu_2}(x_\mu, \phi^a, \tau \phi^{a; \mu}) \right] (\tau' \phi^{a; \mu}), \\ D_{a_1}^{\mu_1} = \phi^{a; \mu_1} \int_0^1 d\tau \tau Z_{a_1 a_2}^{\mu_1}(x_\mu, \tau \phi^a), \\ C = \phi^{a; \mu_1} \int_0^1 d\tau W_{a_1}(x_\mu, \tau \phi^a). \quad (3.4)$$

A few comments are now in order prior to illustrating the above corollary with some examples. First of all let us remark that, under the assumed continuity and regularity conditions, there is no need to verify the consistency of system (3.3) when the assigned system of field equations is self-adjoint in R^* . And indeed, the proof of sufficiency of Theorem 2.1 is precisely centered on the property that the conditions of self-adjointness are the integrability conditions of Eqs. (3.3).

Therefore, for practical applications, one must verify the continuity, regularity, and self-adjointness requirements of the assigned system of field equations in a star-shaped region R^* . When such requirements are verified, a solution of Eqs. (3.3) exists.

Notice that no knowledge of the solutions of the field equations is required for the construction of the Lagrangian according to the method of Corollary 2.1B.

To avoid possible misinterpretation, let us indicate that Eqs. (3.3) must be solved in the assigned order, namely, one must first solve Eqs. (3.3a); the knowledge of the "kinetic" density K as well as of the assigned terms B_a then allows the solution of Eqs. (3.3b) in the D_a^μ densities; and, finally, the knowledge of the K and D_a^μ densities allows the solution of Eqs. (3.3) in the C density.

Notice also that the integrals of solutions (3.4) are insensitive to the dependence of the integrands other than those indicated. For instance, the $A_{a_1 a_2}^{\mu_1 \mu_2}$ terms of integrals (3.1a) depend, in general, on the x_μ and ϕ^a variables as well as the derivative terms $\phi^{a;\mu}$. Nevertheless, the inclusion of the τ variable is done as indicated, in the derivative terms only. Furthermore, as indicated in the proof of Theorem 2.1, the bracket of Eq. (3.4a) indicates that the function of $\phi^{a;\mu}$ resulting from the integration with respect to τ must be computed along $\tau' \phi^{a;\mu}$ prior to the integration with respect to τ' .

The reader should keep in mind, as indicated in Appendix C, that if K is a particular solution of Eqs. (3.3a), its general solution is precisely given by the structure (3.2) of the Lagrangian. This point can also be derived from the "degrees of freedom" of primitive forms of type (B. 12).

The reader should also keep in mind that the solutions (3.4) are local in nature, as it is the case, in general, for all applications of the Converse of the Poincaré Lemma [7].

We now come to a crucial as well as delicate point of our method for computing a Lagrangian. This is constituted by the fact that the solutions (3.4) apply iff their integrals are well defined. In turn, this point is intimately linked to the requirement that the field equations are well defined in a star-shaped rather than an ordinary region.

Before commenting on this point, for the sake of clarity let us note, from the Appendices and from Section 2, that on practical grounds, one can ignore the distinction between ordinary and star-shaped regions and work on the "minimal domain" \bar{R}_{Min} whose boundary is constituted by unit circles around the origin in the "plane" of local coordinates $(\phi^a, \phi^{a;\mu})$. A requirement of Theorem 2 and of Corollary 2.1B is then that the field equations are well defined at least in such domain \bar{R}_{Min} . Alternatively and on more pragmatic grounds, one can simply verify that the field equations are well defined for all values $0 \leq \phi^a \leq 1$ and $0 \leq \phi^{a;\mu} \leq 1$ ($a = 1, 2, \dots, n, \mu = 0, 1, 2, 3$). If this is the case, then integrals (3.4) are well defined, too.

Now, the above requirements can clearly be violated in practical applications. This is the case when the field equations incorporate terms such as $\log \phi$, $\operatorname{cosec} \phi$, etc.

But, within the context of solutions (3.4), the variables ϕ^a and $\phi^{a; \mu}$ are local in nature. This allows the redefinition of these variables aiming at the removal of the divergences in R_{Min} . For $\operatorname{cosec} \phi$ a trivial redefinition is $\phi' = \phi + \text{const}$, in which case one can use Eqs. (3.3) and solutions (3.4) in the redefined rather than the original fields.

In conclusion, and as we shall illustrate later on, solutions (3.4) generally hold, up to redefinition of the local variables, for Field Theories in Minkowski space.

Despite that, the reader should be alerted that counterexamples are conceivable. Furthermore, the extension of the method of Corollary 2.1B to Field Theories in a Riemannian or pseudo-Riemannian manifold demands considerable care, particularly when the local coordinates are the elements $g^{\mu\nu}$ of the metric tensor and their covariant derivatives.

It should be indicated that, to the best knowledge of this author, the case when integrals of type (3.4) fail to exist is not yet known within the context of the calculus of differential forms.

We shall therefore not enter into this aspect at this time and content ourselves with the obtained solutions (3.4).

Another point which the reader should keep in mind is that Eqs. (3.4) ultimately constitute only *one* method of solving Eqs. (3.3), and other methods are conceivable. Therefore, if solutions (3.4) fail to hold, this does not prohibit the possibility of solving Eqs. (3.3) with methods other than that of the Converse of the Poincaré Lemma and its generalization as presented in Appendix B.

To clarify this point, let us first note that the system of differential equations (3.3) for the characterization of a Lagrangian exists irrespective of the type of region of definition of the field equations. This is the spirit of their derivation in the first part of the proof of Theorem 2.1. Therefore, if solutions (3.4) do not exist, this *does not* necessarily imply that system (3.3) is also not defined in a non-star-shaped region (e.g., $0 < \phi^a$, $\phi^{a; \mu} < 1$, or $1 < \phi^a$, $\phi^{a; \mu}$).

On similar grounds, the conditions of self-adjointness, as derived in [1], *do not* necessarily need a star-shaped region to be well defined.

The possibility referred to above is thus constituted by the case when the integrals of solutions (3.4) do not exist, but system (3.3) and the conditions of self-adjointness are well defined in a non-star-shaped region. It is under such circumstances that other methods of integrating Eqs. (3.3) are conceivable.

Stated in somewhat different terms, if counterexamples to the universality of solutions (3.4) up to redefinition of the local variables for all (tensorial) Field Theories in Minkowski space do exist, they *do not* necessarily imply the breakdown of system (3.3) and of self-adjointness in some ordinary region of the variables and, therefore, a Lagrangian may still exist.

In any case, this point demands specific supplementary investigation. It is for this reason that we have formulated Theorem 2.1 and Corollary 2.1B, as a precautionary measure, on a star-shaped rather than an ordinary region.

The case of nonlinear systems of field equations (1.1) is excluded by Theorem 2.1 in line with the fact that the most general system of field equations which can be represented with the Lagrange equations is, from their structure (1.8), of the quasi-linear type (1.2). We shall therefore ignore from now on the nonlinear form (1.1).

The semilinear form (1.3) of the field equations is, on the contrary, significant. This is due to the fact, already stressed in [1] that the great majority of tensorial field equations considered until now are of the semilinear form

$$\square \phi_{a_1} - f_{a_1}(x_\mu, \phi^a, \phi^{a;\mu}) = 0. \tag{3.5}$$

However, the necessary and sufficient conditions for Eqs. (3.5) to be self-adjoint are that they are linear in the derivative terms $\phi^{a;\mu}$, i.e., they are of form (1.3), and all conditions of self-adjointness (1.7) are satisfied [1, Theorem 6.3]. Therefore, we shall now restrict our analysis to semilinear systems of the reduced form (1.3).

The problem of the existence of an ordered direct analytic representation for systems (1.3) is clearly a particular case of Theorem 2.1 with

$$A_{a_1 a_2}^{\mu_1 \mu_2} = \delta_{a_1 a_2} \otimes g^{\mu_1 \mu_2}, \tag{3.6a}$$

$$B_{a_1} = -\rho_{a_1 a_2}^{\mu_1} (x_\mu, \phi^a) \phi^{a;\mu_1} - \sigma(x_\mu, \phi^a). \tag{3.6b}$$

It is, however, an instructive exercise for the interested reader to again prove Theorem 2.1 for identifications (3.6). This proof is considerably simpler, because Eqs. (3.3a) reduce, in this case, to

$$\frac{1}{2}(\bar{K}_{a_1 a_2}^{\mu_1 \mu_2} + \bar{K}_{a_2 a_1}^{\mu_2 \mu_1}) = \delta_{a_1 a_2} \otimes g^{\mu_1 \mu_2}. \tag{3.7}$$

From Eq. (3.4a) we then recover the well-known kinetic term

$$\begin{aligned} \bar{K} &= 2\phi^{a_1;\mu_1} \int_0^1 d\tau' \left[\phi^{a_2;\mu_2} \int_0^1 d\tau \tau \delta_{a_1 a_2} \otimes g^{\mu_1 \mu_2} \right] (\tau' \phi^{a;\mu}) \\ &= 2\phi^{a_1;\mu_1} \int_0^1 d\tau' \left[\phi_{a_1}^{;\mu_1} \int_0^1 d\tau \tau \right] (\tau' \phi^{a;\mu}) \\ &= 2\phi^{a_1;\mu_1} \int_0^1 d\tau' \frac{1}{2} \tau' \phi_{a_1}^{;\mu_1} \\ &= \frac{1}{2} \phi^{a_1;\mu_1} \phi_{a_1}^{;\mu_1}. \end{aligned} \tag{3.8}$$

Then the conditions of self-adjointness (1.7a) and (1.7b) coincide with the integrability and consistency conditions of Eqs. (3.3b) and (3.3c) (parts 2 and 3 of the proof of Theorem 2.1).

For the reader's convenience, the following corollary summarizes the methodology to compute a Lagrangian in this simpler case.

COROLLARY 2.1C. *A Lagrangian density for the ordered direct analytic representation of Lorentz-covariant, tensorial, semilinear systems of field equations*

$$g^{\mu\nu} \phi_{a_1; \mu_1 \mu_2} - \rho_{a_1 a_2}^{\mu_1} (x_\mu; \phi^a) \phi^{a_2; \mu_1} - \sigma_{a_1} (x_\mu, \phi^a) = 0, \\ a, a_1, a_2 = 1, 2, \dots, n, \quad \mu, \mu_1, \mu_2 = 0, 1, 2, 3, \quad (3.9)$$

which are of (at least) class \mathcal{C}^1 and self-adjoint in a star-shaped region R^* of points (x_μ, ϕ^a) is given by

$$\mathcal{L}(x_\mu; \phi^a; \phi^{a; \mu}) = \frac{1}{2} \phi^{a; \mu} \phi_a^{; \mu} + D_{a_1}^{\mu_1} (x_\mu, \phi^a) \phi^{a_2; \mu_1} + C(x_\mu, \phi^a), \quad (3.10)$$

where the $(4n + 1)$ densities $D_{a_1}^{\mu_1}$ and C are a solution of the linear, generally over-determined system of partial differential equations

$$D_{a_1 a_2}^{\mu_1} - D_{a_2 a_1}^{\mu_1} = -\rho_{a_1 a_2}^{\mu_1}, \\ C_{; a_1} = D_{a_1 a_2}^{\mu_1} - \sigma_{a_1}, \quad (3.11)$$

given by

$$D_{a_1}^{\mu_1} = \phi^{a_2} \int_0^1 d\tau \tau \rho_{a_1 a_2}^{\mu_1} (x_\mu, \tau \phi^a), \\ C = \phi^{a_1} \int_0^1 d\tau [D_{a_1 a_2}^{\mu_1} + \sigma_{a_1}] (x_\mu, \tau \phi^a). \quad (3.12)$$

Notice that now the minimal continuity conditions are that the terms $\rho_{a_1 a_2}^{\mu_1}$ and σ_{a_1} are of class \mathcal{C}^1 rather than of class \mathcal{C}^2 as in Corollary 2.1B. This is due to the fact that the conditions of self-adjointness (1.7) imply only first-order partial derivatives. Also, the condition of regularity has been dropped because Eqs. (3.9) are always everywhere regular.

Again we must stress the point that while system (3.11) and conditions of self-adjointness (1.7) apply irrespective of the type of region of definition of system (3.9), the solutions (3.12) apply only when the related integrals exist. Counter-examples when the integrals of Eqs. (3.12) do not exist might be conceivable, but in this case other methods of integration of Eqs. (3.11) are equally conceivable.

Let us also recall from [1] that the conditions of self-adjointness (1.7a) and (1.7b) imply that the $\rho_{a_1 a_2}^{\mu_1}$ term has the curl of structure

$$-\rho_{a_1 a_2}^{\mu} = \Gamma_{a_1 a_2}^{\mu_1} - \Gamma_{a_2 a_1}^{\mu_1}, \tag{3.13}$$

where the $4n$ terms $\Gamma_{a_1}^{\mu}$ are, in general, functions of (x_μ, ϕ^a) .

Under structure (3.13), i.e., for the system

$$g^{\mu_1 \mu_2} \phi_{a_1 \mu_1 \mu_2}^i + [I_{a_1}^{\mu_1}(x_\mu, \phi^a)^i_{a_2} - I_{a_2}^{\mu_1}(x_\mu, \phi^a)^i_{a_1}] \phi^{a_2}_{\mu_1} - \sigma_{a_1}(x_\mu, \phi^a) = 0, \tag{3.14}$$

the conditions of self-adjointness (1.7) reduce only to Eq. (1.7c); the densities $D_{a_1}^{\mu}$ coincide with $\Gamma_{a_1}^{\mu}$, i.e., the solutions (3.12a) read (see Appendix B)

$$D_{a_1}^{\mu_1} = \int_0^1 d\tau (d/d\tau)(\tau \Gamma_{a_1}^{\mu_1}) = \Gamma_{a_1}^{\mu_1}, \tag{3.15}$$

and the Lagrangian (3.10) takes the form

$$\mathcal{L}(x_\mu, \phi^a, \phi^{a; \mu}) = \frac{1}{2} \phi^{a; \mu} \phi_{a; \mu} + \Gamma_{a; \mu}(x_\alpha, \phi^b) \phi^{a; \mu} + C(x_\alpha, \phi^b), \tag{3.16}$$

where the density C is given by

$$C = \phi^a \int_0^1 d\tau [I_{a; \mu}^{\mu} + \sigma_a](x_\alpha, \tau \phi^b). \tag{3.17}$$

As a final remark, let us stress that the Lagrangian computed with the above method, when it exists, is *not* unique. This is for several reasons, including the “degrees of freedom” of type (B.12) of the primitive forms of the Converse of the Poincare Lemma and of its generalization.

It is for this reason that throughout our analysis we have always referred to the existence of *a* Lagrangian rather than *the* Lagrangian.

The study of the “degrees of freedom” of a Lagrangian for a given system satisfying the requirements of Theorem 2.1 is contemplated as a subsequent step.

A few examples are now in order. First, let us consider the simple case of the real scalar free field

$$(\square + m^2) \varphi \equiv g^{\mu_1 \mu_2} \varphi_{\mu_1 \mu_2} + m^2 \varphi = 0. \tag{3.18}$$

As we know from [1], this system is self-adjoint. By inspection we then see that all requirements of Theorem 2.1, or Corollary 2.1C, are satisfied. Equation (3.11) reduces to

$$C_{a_1}^i = -m^2 \varphi. \tag{3.19}$$

The solution (3.12b) becomes

$$C = -\varphi \int_0^1 d\tau m^2 \tau \varphi = \frac{1}{2} m^2 \varphi^2, \quad (3.20)$$

and the familiar Lagrangian

$$\mathcal{L} = \frac{1}{2} (\varphi^i{}_{;\mu} \varphi^{i;\mu} - m^2 \varphi^2) \quad (3.21)$$

is then recovered from Eq. (3.10).

The extension to the case of self-coupled fields

$$\square \varphi + m^2 \varphi + \lambda \varphi^3 = 0 \quad (3.22)$$

is trivial. Indeed, Eqs. (3.20) become

$$\begin{aligned} C &= -\varphi \int_0^1 d\tau [m^2(\tau\varphi) + \lambda(\tau\varphi)^3] \\ &= -\frac{1}{2} m^2 \varphi^2 - \frac{1}{4} \lambda \varphi^4, \end{aligned} \quad (3.23)$$

with the familiar Lagrangian

$$\mathcal{L} = \frac{1}{2} \varphi^i{}_{;\mu} \varphi^{i;\mu} - \frac{1}{2} m^2 \varphi^2 - \frac{1}{4} \lambda \varphi^4. \quad (3.24)$$

For the Sine-Gordon equations

$$\square \varphi + \sin \varphi = 0, \quad (3.25)$$

which is also self-adjoint, we have

$$\begin{aligned} C &= -\varphi \int_0^1 d\tau \sin \tau \varphi = [\cos \tau \varphi]_{\tau=0}^{\tau=1} \\ &= \cos \varphi - 1, \end{aligned} \quad (3.26)$$

and the known Lagrangian

$$\mathcal{L} = \frac{1}{2} \varphi^i{}_{;\mu} \varphi^{i;\mu} + [\cos \varphi - 1] \quad (3.27)$$

is thus recovered.

In more than one dimension one of the most significant cases is that of the gauge invariant theories. Consider, in this respect, Eqs. (2.40) in the self-adjoint form

$$\begin{aligned} &\begin{pmatrix} 0 & g^{\mu_1 \mu_2} \\ g^{\mu_1 \mu_2} & 0 \end{pmatrix} \begin{pmatrix} \varphi^i{}_{;\mu_1 \mu_2} \\ \bar{\varphi}^i{}_{;\mu_1 \mu_2} \end{pmatrix} + \begin{pmatrix} 0 & 2ieA^\mu \\ -2ieA^\mu & 0 \end{pmatrix} \begin{pmatrix} \varphi^i{}_{;\mu} \\ \bar{\varphi}^i{}_{;\mu} \end{pmatrix} + \begin{pmatrix} 0 & (m^2 - e^2 A_\mu A^\mu) \\ (m^2 - e^2 A_\mu A^\mu) & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \bar{\varphi} \end{pmatrix} \\ &= (A_{\alpha_1 \alpha_2}^{\mu_1 \mu_2} \phi^{\alpha_1 \alpha_2}{}_{\mu_1 \mu_2}) + (B_{\alpha_1}) = 0. \end{aligned} \quad (3.28)$$

For the kinetic term we then have, from Eq. (3.4a),

$$\begin{aligned}
 K &= 4\bar{\varphi}^i{}_{\mu_1} \int_0^1 d\tau' \left[\varphi^{i\mu_1} \int_0^1 d\tau \tau \right] (\tau\varphi^{i\mu_1}) \\
 &= \bar{\varphi}^i{}_{\mu_1} \varphi^{i\mu_1}.
 \end{aligned}
 \tag{3.29}$$

Equations (3.3b) become

$$\begin{aligned}
 D_{\varphi}^{\mu i} &= 2ieA^\mu \equiv Z_{\varphi\bar{\varphi}}^\mu, \\
 D_{\bar{\varphi}}^{\mu i} &= -2ieA^\mu \equiv -Z_{\varphi\bar{\varphi}}^\mu,
 \end{aligned}
 \tag{3.30}$$

with solutions, from Eq. (3.4b),

$$\begin{aligned}
 D_{\varphi}^{\mu} &= \bar{\varphi} \int_0^1 \tau Z_{\varphi\bar{\varphi}}^{\mu} d\tau = ieA^\mu \bar{\varphi}, \\
 D_{\bar{\varphi}}^{\mu} &= -\varphi \int_0^1 \tau Z_{\varphi\bar{\varphi}}^{\mu} d\tau = -ieA^\mu \varphi.
 \end{aligned}
 \tag{3.31}$$

Equations (3.3c) now read

$$\begin{aligned}
 \begin{pmatrix} C^i_{\varphi} \\ C^i_{\bar{\varphi}} \end{pmatrix} &= \begin{pmatrix} -B_{\varphi} \\ -B_{\bar{\varphi}} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} (B_{\varphi}^{\mu}{}_{\bar{\varphi}}) \bar{\varphi}^i{}_{\mu} - (B_{\bar{\varphi}}^{\mu}{}_{\varphi}) \bar{\varphi}^i{}_{\mu} \\ (B_{\bar{\varphi}}^{\mu}{}_{\varphi}) \varphi^i{}_{\mu} - (B_{\varphi}^{\mu}{}_{\bar{\varphi}}) \varphi^i{}_{\mu} \end{pmatrix} \\
 &= \begin{pmatrix} 0 & (-m^2 + e^2 A_{\mu} A^{\mu}) \\ (-m^2 + e^2 A_{\mu} A^{\mu}) & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \bar{\varphi} \end{pmatrix} = \begin{pmatrix} (-m^2 + e^2 A_{\mu} A^{\mu}) \bar{\varphi} \\ (-m^2 + e^2 A_{\mu} A^{\mu}) \varphi \end{pmatrix} \equiv \begin{pmatrix} W_{\varphi} \\ W_{\bar{\varphi}} \end{pmatrix}.
 \end{aligned}
 \tag{3.32}$$

Notice the lack of dependence of the rhs of the above expression from derivative terms in accordance with part 3 of Theorem 2.1.

The solution (3.4c) then becomes

$$\begin{aligned}
 C &= \varphi \int_0^1 d\tau W_{\varphi}(\tau\bar{\varphi}) + \bar{\varphi} \int_0^1 d\tau W_{\bar{\varphi}}(\tau\varphi) \\
 &= (-m^2 + e^2 A^{\mu} A_{\mu}) \varphi \bar{\varphi}.
 \end{aligned}
 \tag{3.33}$$

The densities (3.29), (3.31), and (3.33) then reproduce the known Lagrangian (2.39) through structure (3.2).

An element emerging from our analysis which is significant for the problem of the coupling of tensorial fields is that the structure of the field equations for gauge invariant theories *is not* of the semilinear form (3.5) but rather of the more general quasi-linear form (3.1) with the terms $A_{\alpha_1}^{\mu_1 \mu_2}$ incorporating a permutation of the Latin indices. This point will be further elaborated later on.

Another significant example in more than one dimension is given by the (self-adjoint) chiral equations

$$G_{a_1 a_2}(\varphi) g^{\mu_1 \mu_2} \varphi^{a_2; \mu_1 \mu_2} + \frac{\partial G_{a_1 a_2}(\varphi^a)}{\partial \varphi^{a_2}} \varphi^{a_2; \mu} \varphi^{a_3; \mu} - \frac{1}{2} \frac{\partial G_{a_2 a_3}(\varphi^a)}{\partial \varphi^{a_1}} \varphi^{a_2; \mu} \varphi^{a_3; \mu} + \frac{\partial R(\varphi)}{\partial \varphi^{a_1}} = A_{a_1 a_2}^{\mu_1 \mu_2} \varphi^{a_2; \mu_1 \mu_2} + B_{a_1} = 0, \\ G_{a_1 a_2} = G_{a_2 a_1}. \quad (3.34)$$

Then, solution (3.4a) becomes

$$K = 2G_{a_1 a_2} \varphi^{a_1; \mu} \int_0^1 d\tau' \left[\varphi^{a_2; \mu} \int_0^1 d\tau \tau \right] (\tau' \phi^{a_2; \mu}) \\ = \frac{1}{2} \varphi^{a_1; \mu} G_{a_1 a_2}(\varphi^a) \varphi^{a_2; \mu}. \quad (3.35)$$

Equations (3.3b) yield

$$D_{a_1}^{\mu; a_2} - D_{a_2}^{\mu; a_1} \equiv 0, \quad (3.36)$$

with a particular solution $D_a^\mu = 0$. Equations (3.3c) are trivial. One then recovers from the kinetic term (3.35) the familiar chiral Lagrangian

$$\mathcal{L} = \frac{1}{2} \varphi^{a_1; \mu} G_{a_1 a_2}(\varphi^a) \varphi^{a_2; \mu} + R(\varphi^a). \quad (3.37)$$

Let us stress again that the field equations in this case too are of the quasi-linear form (3.1), and *not* of the semilinear form (3.5).

This completes the illustration of our method of computing a Lagrangian. The interested reader can work out other examples with similar procedures.

We must now reinspect the above results from the viewpoint of interactions.

The problem in which we are interested at this time is the following: What is a general form of modification of the Lagrangian density for free tensorial fields capable of representing the same fields in interaction when expressed in a class \mathcal{C}^2 , regular, Lorentz-covariant, tensorial, and self-adjoint, but otherwise arbitrary *quasi-linear* form? Or, alternatively, which is a general form of coupling tensorial fields in a way compatible with Theorem 2.1?

In this paper we have established the necessary methodology to answer this question, at least on formal grounds.

An analysis of the Newtonian counterpart of this problem is conducted in [5]. A few remarks within this framework are useful to illustrate certain points.

First, let us review the conventional approach to the problem. Consider, for simplicity, an unconstrained Newtonian system of particles of unit masses with generalized coordinates q_1, \dots, q_n ($n = 3N$) representing a collection of conventional (e.g., Cartesian) coordinates. Let $\sum_{1 \leq a}^n L_{\text{Free}}^{(a)} = \sum_{1 \leq a}^n \frac{1}{2} (\dot{q}_a)^2$ be the Lagrangian

of this system in the absence of external forces. Then we can say that this system of particles is in interaction when the Lagrangian contains the free term $\sum_1^n L_{\text{Free}}^{(a)}$ and a nontrivial *additive* interaction term L_{Int} , i.e.,

$$L_{\text{Tot}}(q, \dot{q}) = \sum_1^n L_{\text{Free}}^{(a)}(\dot{q}) + L_{\text{Int}}(q, \dot{q}). \quad (3.38)$$

It should be indicated that the term L_{Int} is nontrivial when the Lagrange equations in $\sum_1^n L_{\text{Free}}^{(a)}$ and those in L_{Tot} are not equivalent. This ensures the existence of a *modification* of the actual path due to the acting forces and eliminates possible equivalence (i.e., gauge) transformations of $\sum_1^n L_{\text{Free}}^{(a)}$ induced by L_{Int} .

In more conventional notation one writes $\sum_1^n L_{\text{Free}}^{(a)} = T = \text{Kinetic Energy}$ and $-L_{\text{Int}} = U = \text{potential energy}$.

The extension of the above well-known concept to Field Theory is straightforward and equally well known. Its derivation within the context of Theorem 2.1 reads

COROLLARY 2.1D. *A total Lagrangian density for the ordered direct analytic representation of Lorentz-covariant, tensorial, semilinear systems of coupled field equations*

$$(\square + m^2(a_1)) \phi_{a_1} = [\Gamma_{a_1}^{\mu}(\phi^a)_{; a_2} - \Gamma_{a_2}^{\mu}(\phi^a)_{; a_1}] \phi^{a_2}_{; \mu} + \Lambda_{a_1}(\phi^a), \quad (3.39)$$

which are of (at least) class \mathcal{C}^1 and are self-adjoint in a star-shaped region R^* of points ϕ^a is given by

$$\begin{aligned} \mathcal{L}_{\text{Tot}}(\phi^1, \dots, \phi^n, \phi^{1; \mu}, \dots, \phi^{n; \mu}) \\ = \sum_1^n \mathcal{L}_{\text{Free}}^{(a)}(\phi^a, \phi^{a; \mu}) + \mathcal{L}_{\text{Int}}(\phi^1, \dots, \phi^n, \phi^{1; \mu}, \dots, \phi^{n; \mu}), \end{aligned} \quad (3.40)$$

where

$$\mathcal{L}_{\text{Free}}^{(a)} = \frac{1}{2}(\phi^{a; \mu} \phi_a{}^{; \mu} - m^2(a) \phi^a \phi_a), \quad (3.41a)$$

$$\mathcal{L}_{\text{Int}} = \Gamma_a^{\mu}(\phi^b) \phi^{a; \mu} + \Xi(\phi^b). \quad (3.41b)$$

there is no summation with respect to the a index (only) in Eqs. (3.39) and (3.41a), and the density Ξ is given by

$$\Xi = \phi^a \int_0^1 d\tau [\Gamma_{a; \mu}^{\mu} + \Lambda_a](\tau\phi). \quad (3.42)$$

Again, whenever the Lagrange equations in $\sum_1^n \mathcal{L}_{\text{Free}}^{(a)}$ and \mathcal{L}_{Tot} are not equivalent, the term \mathcal{L}_{Int} of the conventional structure (3.40) represents a bona fide interaction or coupling of the considered fields.

One of the implications of our analysis is that the conventional structure (3.40) of the total Lagrangian density is not exhaustive and more general structures are conceivable.

It is at this point that an inspection of the Newtonian framework, which is ultimately the arena of most of our intuitions, may be effective.

Consider again an unconstrained Newtonian system of n free particles of unit masses in the coordinates q_1, \dots, q_n . The equations of motions are, trivially,

$$\ddot{q}_a = 0, \quad a = 1, 2, \dots, n. \quad (3.43)$$

A general form of coupling the above system is constituted by the superposition of at least three different types of couplings, i.e., (I) coordinate couplings, (II) velocity couplings, (III) acceleration couplings.

To stress the physical significance of the use of these couplings it is sufficient to note that if some of them are ignored, the considered equations of motion may only be an approximation of the physical reality.

An example is needed to clarify this point. Consider, as a first step, only linear couplings. Then the type I couplings applied to Eqs. (3.43) produce the familiar form of coupled oscillators, i.e.,

$$\ddot{q}_{a_1} + c_{a_1 a_2} q_{a_2} = 0, \quad c_{a_1 a_2} = \text{const}, \quad a_1, a_2 = 1, 2, \dots, n. \quad (3.44)$$

This *conservative* system is, however, insufficient to represent an actual system in our environment, due to the inevitable presence of dissipative forces. One then introduces type II couplings, obtaining in this way the familiar form of the system of coupled and damped oscillators

$$\ddot{q}_{a_1} + b_{a_1 a_2} \dot{q}_{a_2} + c_{a_1 a_2} q_{a_2} = 0, \quad b_{a_1 a_2} = \text{const}. \quad (3.45)$$

This implementation, however, is still insufficient because, as is well known in the theory of coupled oscillators, type III couplings also occur. One then obtains the familiar form of the linear equations of motion for coupled oscillators

$$\begin{aligned} a_{a_1 a_2} \ddot{q}_{a_2} + b_{a_1 a_2} \dot{q}_{a_2} + c_{a_1 a_2} q_{a_2} &= 0, \\ a_{a_1 a_2} &= \text{const}, \quad \det(a_{a_1 a_2}) \neq 0. \end{aligned} \quad (3.46)$$

Notice that the “acceleration couplings” occur because the off-diagonal as well as the diagonal terms of the matrix $(a_{a_1 a_2})$ are generally nonnull.

Equations (3.46) still constitute an approximation of the physical reality due to the linear nature of the considered couplings as well as the constancy of the elements $a_{a_1 a_2}$, $b_{a_1 a_2}$, and $c_{a_1 a_2}$. And indeed, Eqs. (3.46), as is well known, are customarily valid only for the case of small oscillations.

The removal of the linearity of couplings I and II and of the constancy of the coefficients then brings Eqs. (3.46) into the fundamental form of Newton's equations of motion

$$A_{a_1 a_2}(t, q, \dot{q}) \ddot{q}_{a_2} + B_{a_1}(t, q, \dot{q}) = 0, \quad (3.47)$$

where external forces, which are essential to preserve the motion for the desired period of time, can now be included in the B terms.

Even though they are somewhat hidden, the three indicated classes of couplings are still present in the general structure (3.47). And indeed, the coordinate and velocity couplings are represented by the q and \dot{q} dependence, respectively, of the $A_{a_1 a_2}$ and B terms, while the acceleration couplings are represented by the nonnull values of the off-diagonal elements of the matrix $(A_{a_1 a_2})$. The central difference with Eqs. (3.46) is that for the general form (3.47) the coordinates and velocity couplings are not necessarily linear. However, the acceleration couplings are always linear in the accelerations in order to preserve the Newtonian structure of the equations of motion. For an elaboration of this point see [5].

The point which we would like to stress is that, irrespective of our interpretation and classification of the forms of couplings, when an accurate description of the physical reality is needed, structures of type (3.44) and (3.45) must be abandoned and the fundamental form (3.47) of the equations of motions must be adopted.

At this point one can argue that, under the condition of regularity, i.e.,

$$\det(A_{a_1 a_2}) \neq 0, \quad (3.48)$$

Eqs. (3.4) can always be reduced to the semilinear form

$$\ddot{q}_a - f_a(t, q, \dot{q}) = 0, \quad (3.49)$$

where the implicit functions f_a are trivially given by

$$f_a = -A_{ab}^{-1} B_b, \quad (A_{ab}^{-1}) \equiv (A_{ab})^{-1}. \quad (3.50)$$

Thus, the acceleration couplings are not essential to represent the motion.

It is precisely in this respect that our analysis of the necessary and sufficient conditions for the existence of a Lagrangian becomes crucial.

And indeed, the statement that class \mathcal{C}^2 regular systems (3.47) can always be reduced to form (3.49) is true from the Theorem on Implicit Functions. Similarly, the statement that Eqs. (3.49) without acceleration couplings can equivalently represent the motion is equally true.

However, within the framework of a Lagrangian representation of the equations of motion the situation is considerably different. Indeed, Newton's equations of motion in the form (3.49) are non-self-adjoint (unless trivial forms of couplings