

Nonlinear, nonlocal and noncanonical isotopies of the Poincaré symmetry

R M Santilli†

The Institute for Basic Research, P O Box 1577, Palm Harbor, FL 34682, USA

Received 14 July 1994

Abstract. We first outline the nonlinear-nonlocal-noncanonical, axiom-preserving isotopies of: fields, metric and Hilbert spaces, transformation theory, Lie's theory and quantum mechanics. We then apply these novel techniques to the construction of the isotopies of the Poincaré symmetry $P(3.1)=SO(3.1)\times T(3.1)$ as well as of its spinorial covering $\mathcal{P}(3.1)=SL(2.C)\times T(3.1)$. We finally point out a number of preliminary, yet intriguing applications and experimental verifications in nuclear physics, particle physics, superconductivity and other fields.

1. Statement of the problem

As is well known, the fundamental symmetries of contemporary theoretical physics, the Lorentz symmetry [1] $O(3.1)$, the Poincaré symmetry [2] $P(3.1) = SO(3.1) \times T(3.1)$ and the spinorial covering [3] $\mathcal{P}(3.1) = SL(2.C) \times T(3.1)$, are linear, local and canonical.

According to impressive experimental evidence, these symmetries have resulted to be *exactly* verified under well known physical conditions which can be identified, classically and quantum mechanically, with those of the *exterior dynamical problem*, i.e., point-like particles moving in the homogeneous and isotropic vacuum under action-at-a-distance interactions. In fact, the point-like character of the particles ensures the exact validity of the underlying local-differential geometry, while their potential character ensures the exact applicability of Lie's theory in canonical realization.

The physical conditions studied in this paper are those of the more general *interior dynamical problem*, which consists of *extended*, and therefore *deformable* particles while moving within *inhomogeneous* and *anisotropic* physical media, thus resulting in the most general known systems which are: *nonlinear*, in the coordinates x and in their derivatives \dot{x}, \ddot{x}, \dots as well as in the wavefunctions ψ and in their derivatives $\partial\psi, \partial\partial\psi, \dots$; *nonlocal*, in the sense of having a generally integral dependence on all of the preceding quantities; and *noncanonical*, i.e., violating the integrability conditions for the existence of a Lagrangian or a Hamiltonian, the *conditions of variational selfadjointness* [4].

The distinction between exterior and interior problems was identified by the

† E-mail: ibrms@pinet.aip.org

founders of analytic mechanics, such as Lagrange, Hamilton and Jacobi (see, e.g., [4], and kept up to the early part of this century (one may inspect the care with which Schwartzschild [5] presented his metrics in two separate papers, one for the exterior and one for the interior problem), although the distinction was abandoned in more recent times.

In this paper we shall return to the teaching of the founders of analytic mechanics, because recent studies have proved the impossibility of reducing interior systems to the exterior form. This is due to the so-called *No-reduction theorems* [4], which essentially state that an interior system such as a satellite during re-entry with monotonically decaying angular momentum, simply cannot be decomposed into a finite collection of elementary constituents each of which has conserved angular momentum. Vice versa, the latter cannot possibly reproduce the former.

It is evident that the linear-local-canonical Poincaré symmetry is *inapplicable* (and not 'violated') for nonlinear-nonlocal-noncanonical interior systems on a number of independent counts of topologic, geometric, algebraic and analytic nature. The objective of this paper is therefore to identify the generalization of the Poincaré symmetry $\mathcal{P}(3.1)$ and of its spinorial form $\mathcal{P}(3.1)$ verifying the following conditions:

(A) The generalized *transformations* are structurally nonlinear, nonlocal and noncanonical so as to be directly applicable to the invariance of the systems considered.

(B) The generalized *symmetry* is locally isomorphic to the conventional symmetry, so as to preserve the structural axioms of contemporary physics.

(C) The generalized symmetry admits the conventional Poincaré symmetry as a particular case, so that the former can qualify as a mathematical and physical covering of the latter.

We should therefore indicate from the outset that, by no means, we want to 'abandon' the structural axioms of the Poincaré symmetry, because our objective is merely that of realizing them in their most general possible form.

We should also indicate that this presentation is merely preliminary and still far from the needed mathematical and physical maturity, because the studies are just at the beginning and so much remains to be done, mathematically and physically.

We would like finally indicate that the line of inquiry of this paper is definitely not new, having been initiated by numerous physicists, most notably, by Blochintsev (see, e.g., [6]) and his group at the JINR in Dubna, Russia. More recently, another line of inquiry has been initiated via the so-called *q-deformations* and *quantum groups* by numerous authors (see, e.g., [7]). We apologize for our inability to review other generalizations for brevity.

A novelty of our studies rests in the presentation of the structural axioms of the Poincaré symmetry, thus permitting a number of developments, such as a *causal* description of *nonlocal* interactions. In fact, the Poincaré axiomatic structure is generally lost in the *q*-deformations and other approaches. However, as shown by Lopez [8], the *q*-deformations are particular cases of the techniques used in this paper. Thus, most (but not all) of the existing studies on the *q*-deformation of the Poincaré symmetry can be reformulated in our axiom-preserving form.

2. Isotopies and isodualities of contemporary mathematical structures

The methods which permit the achievement of the objectives identified in the preceding section were introduced by this author [9] back in 1978 (when at the Department of

Mathematics of Harvard University with support from the US Department of Energy), under the name of *isotopies*, from the Greek *ἴσος τοπος*, meaning 'same configuration' and interpreted as 'axiom-preserving'.

The fundamental isotopy from which the entire content of this paper can be derived, is the lifting of the n -dimensional unit $I = \text{diag.}(1, 1, \dots, 1)$ of Lie's theory into an n -dimensional matrix \hat{I} whose elements have the most general known dependence indicated in section 1,

$$\begin{aligned} I &= \text{diag.}(1, 1, \dots, 1) \rightarrow \\ \hat{I} &= \hat{I}(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \partial\partial\psi, \partial\partial\psi^\dagger, \dots) \end{aligned} \tag{2.1}$$

under the condition (necessary for an isotopy) of preserving the original axioms of I , i.e., nonsingularity, Hermiticity and positive-definiteness.

The isotopies of the unit demand, for consistency, a corresponding, compatible lifting of all associative products AB among generic quantities A, B , into the *isoproduct*

$$\begin{aligned} AB &\Rightarrow A * B = ATB \quad T = \text{fixed and inv.} \\ IA = AI &\equiv A \rightarrow I * A = A * I \equiv A \end{aligned} \tag{2.2}$$

whose isotopic character is ensured by the preservation of associativity, $A(BC) = (AB)C \rightarrow A*(B*C) = (A*B)*C$. Under the above conditions, \hat{I} is called the *isounit* and T the *isotopic element*.

One should recall the *necessity*, e.g., in number theory, of changing the multiplication whenever the unit is changed and vice versa. Note also that in q -deformations of associative algebras, $AB \rightarrow qAB$, the multiplication is changed, but the conventional unit of Lie's theory is preserved. The reformulation of these q -deformations in terms of the isotopies, $qAB = A * B$, $\hat{I} = q^{-1}$, permits their generalization and axiomatization into the most general possible integro-differential operator T (which, for this reason is sometimes denoted Q) [8].

The isotopies of the unit $I \Rightarrow \hat{I}$ and of the product $AB \Rightarrow A * B$ are mathematically and physically nontrivial, inasmuch as they imply the necessary lifting of *all* mathematical structures of contemporary physics into an isotopic form admitting of \hat{I} as the left and right unit (see, e.g., [10-15, 18-41]). For brevity, we can here only touch the most salient aspects.

To begin our outline, the conventional fields $F(a, +, \times)$ of real numbers R , complex number C , and quaternions Q with elements a , conventional sum $+$ and product $a \times b := ab$, must be lifted into the so-called *isofields*

$$\begin{aligned} F(a, +, \times) &\rightarrow \hat{F}(\hat{a}, +, *) \quad \hat{a} = a\hat{I} \\ \hat{a} * \hat{b} &= \hat{a}T\hat{b} = (ab)\hat{I} \quad \hat{I} = T^{-1} \end{aligned} \tag{2.3}$$

with elements \hat{a} called *isonumbers*, conventional sum $+$ and isoproduct (2.2), under the condition (again necessary for an isotopy) of preserving the original axioms of F . All operations in F must then be generalized for \hat{F} . We then have, e.g., *isosquares* $\hat{a}^2 = \hat{a} * \hat{a} = \hat{A}T\hat{a} = a^2\hat{I}$, *isoquotient* $\hat{a}/\hat{b} = (a/b)\hat{I}$, *isosquare roots* $\hat{a}^{1/2} = a^{1/2}\hat{I}$, etc. Note that $\hat{a} * A \equiv aA$ (see the recent studies [10] for details).

One can begin to understand the inapplicability of conventional mathematical thinking for isotopic formulations by noting that statements such as 'two multiplied by two equals four' are generally incorrect under isotopies. In fact, for $\hat{I} = 3$, 'two multiplied by two equals twelve', with the understanding that the very notion of integer number is lost in favour of an integro-differential generalization, e.g.,

$$\hat{2} = 2 \exp \left\{ N \int dx \psi^\dagger(x)\phi(x) \right\}.$$

Liftings $I \Rightarrow \hat{I}$, $AB \Rightarrow A * B$ and $F \Rightarrow \hat{F}$ then require the isotopies of carrier spaces, evidently because they centrally depend on the field in which they are defined. For example, a real metric/pseudo-metric space $S(x, g, R)$ must be subjected to the liftings into the so-called *isospaces* (first introduced in [11] as the foundations of the isolorentz symmetry)

$$\begin{aligned} S(x, g, R) &\Rightarrow \hat{S}(x, \hat{g}, \hat{R}) & \hat{g} &= Tg \\ \hat{I} = T^{-1} & & x^{\hat{2}} &= (x^\dagger \hat{g} x) \hat{I} \in \hat{R} \end{aligned} \tag{2.4}$$

under the condition, again, of preserving the original axioms of $S(x, g, R)$, with similar isotopies for complex spaces. In particular, *the basis of a metric (or, more generally, vector) space is preserved under isotopies*, thus including the preservation of the basis of a Lie algebra. This results in nonlinear (in \hat{x}, \hat{x}, \dots), nonlocal and noncanonical generalizations of the Euclidean, symplectic and Riemannian geometries called *isogeometries* [12, 13], with intriguing novel possibilities.

Another understanding of the inapplicability of conventional mathematical thinking for isotopic formulation can be reached by noting that all familiar notions (such as that of angles) are inapplicable in isospaces, trivially, because they are spaces with the most general known curvature, that depending also on the velocities and accelerations, thus implying the loss of straight intersecting lines. A novel aspect of the isogeometries is that they are isotopic, that is, they preserve the original geometric axioms, thus permitting the recovering of the original notions, although in a generalized form [12, 13].

These features have permitted the identification of a new branch of functional analysis called *functional isoanalysis* [14] which begins with a classification of the isounits into five, topologically different classes with corresponding classification of s , isospaces, etc herein adopted. In this paper we shall mainly use class I (for isounits which are smooth, bounded, nowhere singular, Hermitean and positive-definite), and class II (the same except that the isounits are negative-definite), with only marginal comments on the remaining classes for brevity.

The next necessary lifting is that of the conventional linear transformations in S , into the so-called *isotransformations* in \hat{S} [9]

$$\begin{aligned} x' = U(w)x & \quad w \in F \rightarrow x' = \hat{U}(\hat{w}) * x = \hat{U}(\hat{w})Tx \\ T = \text{fixed} & \quad \hat{w} \in \hat{F} \end{aligned} \tag{2.5}$$

which are *isolinear* in \hat{S} , i.e., verify the conventional axioms of linearity merely expressed in the appropriate isotopic form

$$\hat{U} * (\hat{n} * r + \hat{n}' * r') = \hat{n} * (\hat{U} * r) + \hat{n}' * (\hat{U} * r') \quad \hat{n}, \hat{n}' \in \hat{R} \tag{2.6a}$$

$$(\hat{n} * \hat{U} + \hat{n}' * \hat{U}') * r = \hat{n} * (\hat{U} * r) + \hat{n}' * (\hat{U}' * r) \tag{2.6b}$$

$$\hat{U} * (\hat{U}' * r) = (\hat{U} * \hat{U}') * r.$$

However, the same transforms are highly nonlinear when projected in the original space S , i.e., $x' = \hat{A}(w)Tx = \hat{A}(w)T(x, \hat{x}, \hat{x}, \psi, \psi^\dagger, \dots)x$. Note that nonlinear transformations can always be cast into an *identical* isolinear form. We learn in this way that *isotopic methods can turn notoriously difficult nonlinear problems into identical more manageable isolinear forms*.

A similar occurrence holds for locality, because isotransforms $x' = \hat{U}(\hat{w}) * x$ are *isolocal*, i.e., verify the condition of locality in \hat{S} , but the same transforms are generally nonlocal-integral when projected in the original space S . Yet a similar occurrence holds for canonicity. In fact, the theories herein considered are called *isocanonical* in the sense that they are derivable from conventional variational principles formulated in \hat{S} , although the same theories are not derivable from a first-order variational principle in S . The objectives of this paper (section 1) can now be formulated by saying that *we shall seek isolinear, isolocal and isocanonical realizations of the Lorentz and Poincaré symmetries.*

The preceding liftings demand a corresponding compatible lifting of all branches of Lie's theory [16, 17]). In fact, the universal enveloping associative algebra $\xi(g)$ of a Lie algebra g [17] with generic product AB , must be lifted into the isotopes $\hat{\xi}(g)$ with isounit $\hat{I} = T^{-1}$ and isoproduct (2.2), first introduced in [9] jointly with the isotopy of the Poincaré-Birkhoff-Witt theorem. These studies permitted the identification of the infinite-dimensional *isobasis* of $\hat{\xi}$ in terms of the original (ordered) basis X of g and the *isoeponentiation* (see [18] for nonassociative envelopes)

$$\begin{aligned} \hat{\xi}: \hat{I} \quad X_i * X_j \quad (i \leq j) \\ X_i * X_j * X_k \quad (i \leq j \leq k), \dots \quad i, j, k = 1, 2, \dots, n \end{aligned} \tag{2.7a}$$

$$\begin{aligned} e_{\hat{\xi}}^{i\hat{w}*X} &= \hat{I} + (i\hat{w} * X)/1! + (i\hat{w} * X) * (i\hat{w} * X)/2! + \dots \\ &= \{e^{iX^T w}\} \hat{I} = \hat{I} \{e^{i\hat{w}TX}\}. \end{aligned} \tag{2.7b}$$

One can therefore see that all notions based on the conventional exponentiation need a suitable isotopic generalization. For instance, the Dirac δ -function must be lifted into the expression called *isoDirac function* [15, 19, 20]

$$\hat{\delta}(x) = (1/2\pi) \int_{-\infty}^{+\infty} dy T e_{\hat{\xi}}^{ixy} = (1/2\pi) \int_{-\infty}^{+\infty} dy e^{ixTy} \tag{2.8}$$

the conventional Fourier transform must be lifted into the *isoFourier transform* [15, 20]

$$\begin{aligned} f(x) &= (1/2\pi) \int_{-\infty}^{+\infty} dk g(k) * e_{\hat{\xi}}^{ikx} \\ g(k) &= (1/2\pi) \int_{-\infty}^{+\infty} dx f(x) * e_{\hat{\xi}}^{ikx} \end{aligned} \tag{2.9}$$

with consequential loss of the notion of Gaussian into the form (here re-expressed in terms of the conventional exponentiation for clarity) [15, 20]

$$\psi(x) = N e^{-x^2 T/2a^2} \quad \phi(k) = N' e^{-k^2 T a^2/2}. \tag{2.10}$$

The nontriviality of functional isoanalysis can then be easily seen by noting that the above isoGaussians imply the following predictable generalization of Heisenberg's uncertainties: $\Delta x \Delta k \approx a^{-1} a = 1 \rightarrow a^{-1} T^{-1/2} a T^{-1/2} = \hat{I}$ evidently for particles in interior conditions (see also next section). The need for the isotopies of all remaining special functions transforms and distribution follows.

The preceding liftings evidently require that of Lie algebra [16, 17] $g \approx [\xi(g)]^-$ with familiar Lie theorems, e.g., $[X_i, X_j]_\xi = X_i X_j - X_j X_i = C_{ij}^k X_k$, into the *Lie-isotopic algebras* $\hat{g} \approx [\hat{\xi}(g)]^- \not\approx g$ first submitted in [9] with the *Lie-isotopic theorems*, e.g.,

$$\begin{aligned} \hat{g}: [X_i, X_j]_{\hat{\xi}} &= X_i * X_j - X_j * X_i = X_i T X_j - X_j T X_i \\ &= \hat{C}_{ij}^k(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \dots) * X_k \end{aligned} \quad (2.11)$$

where the \hat{C} s are called *structure functions*, and are restricted by the Lie-isotopic third theorem [9]. Note the presentation of the Lie algebra axioms by the isotopic product $[AB]_{\hat{\xi}}$.

The preceding isotopies then yield the (connected) *Lie-isotopic transformation groups* [9]

$$\begin{aligned} \hat{U}(0) &= \hat{I} & \hat{U}(\hat{w}) * \hat{U}(\hat{w}') &= \hat{U}(\hat{w}') * \hat{U}(\hat{w}) = \hat{U}(\hat{w} + \hat{w}') \\ \hat{U}(\hat{w}) * \hat{U}(-\hat{w}) &= \hat{I} \end{aligned} \quad (2.12a)$$

$$\begin{aligned} \hat{G}: x' &= \hat{U} * x \\ \hat{U}(w) &= \prod_k e_{\hat{\xi}}^{i\hat{w}_k * X_k} = \hat{I} \left\{ \prod_k e^{i\omega_k T X_k} \right\} = \left\{ \prod_k e^{iX_k T \omega_k} \right\} \hat{I} \end{aligned} \quad (2.12b)$$

$$\begin{aligned} \{e_{\hat{\xi}}^{iX_1}\} * \{e_{\hat{\xi}}^{iX_2}\} &= e_{\hat{\xi}}^{iX_3} \\ X_3 &= X_1 + X_2 + [X_1, X_2]_{\hat{\xi}} + [(X_1 + X_2)^\dagger, [X_1, X_2]_{\hat{\xi}}]_{\hat{\xi}}/12 + \dots \end{aligned} \quad (2.12c)$$

where one should note the appearance of the isotopic element T directly in the iso-exponent, thus ensuring the desired nonlinear, nonlocal and noncanonical character. The above isogroups are turned into *isosymmetries* via the following

Theorem [21]. Let G be an N -dimensional Lie group of isometries of an m -dimensional, metric or pseudo-metric, and real or complex space $S(x, g, F)$ over a field $F(= R \text{ or } C)$,

$$\begin{aligned} G: x' &= A(w)x \\ (x' - y')^\dagger A^\dagger g A (x' - y') &\equiv (x - y)^\dagger g (x - y) \\ A^\dagger g A &= A g A^\dagger = g. \end{aligned} \quad (2.13)$$

Then, the infinitely possible isotopes \hat{G} of G of class III characterized by the same generators and parameters of G and new isounits \hat{I} (isotopic elements T), leave invariant the isocomposition on the isospaces $\hat{S}(x, \hat{g}, \hat{F})$, $\hat{g} = Tg$, $\hat{I} = T^{-1}$,

$$\begin{aligned} \hat{G}: x' &= \hat{A}(w) * x \\ (x' - y')^\dagger * \hat{A}^\dagger \hat{g} \hat{A} * (x' - y') &= (x - y)^\dagger \hat{g} (x - y) \\ \hat{A}^\dagger \hat{g} \hat{A} &= \hat{A} \hat{g} \hat{A}^\dagger = \hat{I} \hat{g} \hat{I}. \end{aligned} \quad (2.14)$$

This yields the 'direct universality' of the Lie-isotopic symmetries, i.e., their capability of providing the invariance of all infinitely possible deformations $\hat{g} = Tg$

of the original metric g (universality), directly in the frame of the experimenter (direct universality). Note also the simplicity of the explicit construction of the desired isosymmetries via rule (2.12b) where w and X are those of G and T is derived from the deformed metric $\hat{g} = Tg$.

It is easy to prove that $\hat{G} \approx G$ for all class isotopies ($\hat{I} > 0$). This property identifies one of the primary applications of isosymmetries, the reconstruction of exact symmetries when believed to be conventional broken. In fact, we have: the reconstruction of the exact rotational symmetry at the isotopic level $\hat{O}(3) \approx O(3)$ for all ellipsoidal deformations of the sphere [21]; the reconstruction of the exact Lorentz and Poincaré symmetries at the isotopic level $\hat{P}(3.1) \approx P(3.1)$ for all signature preserving ($T > 0$) deformations of the Minkowski metric $\hat{\eta} = T\eta$ [11]; the reconstruction of the *exact* isospin symmetry at the isotopic level $\hat{SU}(2) \approx SU(2)$ under *weak and ELM interactions*; while other cases are under study (e.g., the possible reconstruction of the *exact* parity for *weak* interactions at the isotopic level).

Despite the isomorphism $\hat{G} \approx G$, Lie and Lie-isotopic symmetries are inequivalent on numerous counts, such as:

- (1) G is customarily linear-local-canonical, while \hat{G} is nonlinear-nonlocal-noncanonical;
- (2) the mathematical structures underlying \hat{G} and G (fields, spaces, etc) are structurally different;
- (3) \hat{G} can be derived from G via *nonunitary* transformations under which

$$\begin{aligned} UU^\dagger &= \hat{I} \neq I & U(AB - BA)U^\dagger &= A'TB' - B'TA' \\ T &= (UU^\dagger)^{-1} = T^\dagger & A' &= UAU^\dagger & B' &= UBU^\dagger. \end{aligned} \tag{2.15}$$

Visible differences also emerge in the *isorepresentation theory* [22, 23], e.g., because weights, Cartan tensors, etc acquire a nonlinear-nonlocal-noncanonical dependence on the base manifold. The irreducible isorepresentations of Lie-isotopic algebras \hat{g} have been preliminarily classified into [22]: (1) *regular isorepresentations*, when the structure constants of g and \hat{g} coincide, in which case the eigenvalues of g and \hat{g} differ by suitable *multiplicative* functions on the base manifolds; (2) *irregular isorepresentations*, when the original structure constant of g are turned into *structure functions*, in which case at least one of the eigenvalues of \hat{g} and g is not related by a multiplicative factor; and (3) *standard isorepresentations*, when both the structure constants and eigenvalues \hat{g} coincide with those of g , even though T is a nontrivial isotopic element.

In closing this section we should also recall that isotopies introduce rather naturally the antihomomorphic conjugation called *isoduality* first identified in [21] $\hat{I} \rightarrow \hat{I}^d = -\hat{I}$, $A * B \rightarrow A *^d B = -A * B$, with consequential isoduality of all the preceding structures. In fact, isoreals \hat{R} are mapped into the *isodual isoreals* [10] $\hat{R}^d(\hat{n}^d, +, *^d)$, $\hat{n}^d = n\hat{I}^d = -\hat{n}$, $*^d = \times T^d \times = -*$, with rather intriguing properties, such as negative-definite norm $|\hat{n}^d|^d = -|\hat{n}| < 0$, $\hat{n} \neq 0$. The isocomplex $\hat{C}(\hat{c}, +, *)$ are mapped into the *isodual isocomplex* fields $\hat{C}^d(\hat{c}^d, *, d)$ with conjugation $\hat{c} \rightarrow \hat{c}^d = -\hat{c}^\dagger$. Similarly, isospaces $\hat{S}(x, \hat{g}, \hat{R})$ are mapped into the *isodual isospaces* [12, 13] $\hat{S}^d(x, \hat{g}^d, \hat{R}^d) : \hat{g}^d = -\hat{g}$, which evolve *backward in time*, have *negative-definite* physical quantities such as energy, etc. Yet the isodual separation $x^{\hat{2}d}$ coincides with the isoseparation $x^{\hat{2}}$, $x^{\hat{2}d} = (x^t \hat{g}^d x) \hat{I}^d \equiv x^{\hat{2}} = (x^t \hat{g} x) \hat{I}$, thus permitting an intriguing novel interpretation of antiparticles from their known origin in the negative-energy solutions of conventional relativistic equations [24].

Similarly, Lie-isotopic algebras \hat{g} and groups \hat{G} admit the *isodual algebras* \hat{g}^d and *groups* [12, 13] \hat{G}^d on $\hat{S}^d(x, \hat{g}^d, \hat{R}^d)$ over \hat{F}^d in which the generators, the parameters and the isotopic element change sign

$$\begin{aligned} \hat{g}^d : [X^d_i, X^d_j]_{\hat{\xi}^d} &= -X^d_i T^d X^d_j - X^d_j T^d X^d_i \\ &= \hat{C}^d_{ij^k}(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \dots) T^d X^d_k \end{aligned} \quad (2.16a)$$

$$\begin{aligned} \hat{G}^d : x' &= \hat{U}^d(\hat{w}) T^d x \\ \hat{U}^d(\hat{w}) &= \prod_k e_{\hat{\xi}^d}^{i^d \hat{w}^d \cdot X^d_k} = \left\{ \prod_k e^{i X_k T w_k} \right\} \hat{I}^d \end{aligned} \quad (2.16b)$$

thus leading to a new *universal invariance law under sodality* [13]. Similar isodualities occur for the isoDirac function, the isoFourier transform, etc. See monograph [25] for a detailed study.

3. Isotopies and isodualities of quantum mechanics

The isotopic methods were proposed by this author for the specific purpose of constructing the isotopies of quantum mechanics (QM), originally submitted under the name of *hadronic mechanics* (HM) [26] also called *isotopic completion of quantum mechanics* [27].

The original proposal was studied by numerous authors in the ensuing years. A detailed presentation of the state of the art in the construction of this new discipline is available in monographs [28], with the understanding that these too are at the beginning and so much remains to be done. Here, we can evidently touch only some of the most essential aspects.

The objective is to construct a covering discipline capable of quantitative studies suitable for experimental verifications of the old legacy that particle interactions have a nonlinear-nonlocal-noncanonical component due to mutual overlapping of their wavepackets-wavelengths-charge distributions. Along conditions (A), (B), (C) of section 1, the covering discipline is constructed in such a way as to permit a direct representation of interior conditions in a way preserving the abstract axioms of QM for exterior conditions, as well as admitting the latter as a particular case.

By recalling that Lie's theory in operator realization characterizes the structure of QM, one can easily see that the Lie-isotopic theory in operator realization characterizes the structure of HM. In fact, the essential structural elements of HM can already be seen in the preceding section, e.g., in the isoDirac delta (2.8) because the original singularity at $x = 0$ can be spread over the region of space occupied by the particle. In turn, we should expect the capability of HM to remove the divergencies of QM for suitable values of the isotopic element $|T| \ll 1$ [15, 28].

In order to achieve an operator realization of the Lie-isotopic theory, the fundamental isotopy of HM is that of the underlying space, the Hilbert space \mathcal{H} with inner product $\langle \psi | \phi \rangle \in C$, into the so-called *isoHilbert space* $\hat{\mathcal{H}}$ with *isoinner product* and *isonormalization* first introduced in [19]

$$\begin{aligned} \hat{\mathcal{H}} : \langle \hat{\psi} | \hat{\phi} \rangle &= \langle \hat{\psi} | T | \hat{\phi} \rangle \hat{I} \in \hat{C} \\ \langle \hat{\psi} | \hat{\psi} \rangle &= \hat{I} \quad (\text{or } \langle \hat{\psi} | * | \hat{\psi} \rangle = 1). \end{aligned} \quad (3.1)$$

Note that the isoinner product remains inner for isotopies of class I (i.e., $\hat{\mathcal{H}}$ is still Hilbert) because of the positive-definiteness of T . For future needs to understand the isotopic realization of 'hidden variables', note that \mathcal{H} and $\hat{\mathcal{H}}$ coincide at the abstract level and that, for T independent of the integration variables (or a constant), $\langle \hat{\psi} | \hat{\phi} \rangle = \langle \hat{\psi} | \hat{\phi} \rangle T \hat{I} \equiv \langle \hat{\psi} | \hat{\psi} \rangle$.

An important implication of the isotopy $\mathcal{H} \rightarrow \hat{\mathcal{H}}$ is that operators X on the enveloping algebra ξ over $F(= R, C)$ which are Hermitean in \mathcal{H} , remain Hermitean in $\hat{\mathcal{H}}$ when reinterpreted on the isoenvelope $\hat{\xi}$ over \hat{F} [29, 30]. This yields the important property that *relativistic QM observables remain observable for HM*, thus including the observability of all conventional quantities, such as energy, linear momentum, angular momentum, spin parity, etc.

The liftings of the Hilbert space require corresponding isotopies of all conventional operations [15, 29, 30]. We here mention the *isounitary transformations*

$$\hat{U} * \hat{U}^\dagger = \hat{U}^\dagger * \hat{U} = \hat{I} \tag{3.2}$$

which can turn conventionally nonunitary transformations (2.15) into a form axiomatically equivalent to the unitary ones; the *isoeigenvalue equations*

$$H * | \hat{\psi} \rangle = HT | \hat{\psi} \rangle = \hat{E} * | \hat{\psi} \rangle \quad \hat{E} \in \hat{R}. \tag{3.3}$$

The *isodeterminant* of a matrix A (see [28-30] for additional properties)

$$\hat{\text{Det}}A = [\text{Det}(AT)] \hat{I} \in \hat{R} \text{ or } \hat{C}. \tag{3.4}$$

We are now equipped to outline the basic axiomatic structure of HM [15, 28] on isoEuclidean spaces $\hat{E}(r, \hat{\delta}, \hat{R})$, $\hat{\delta} = T\delta$, for isounits of class I without gravitational content (i.e., for $\partial \hat{I} / \partial r \equiv 0$):

Fundamental assumptions. (A) integro-differential generalization $\hat{h} = \hat{I} = T^{-1}$ of Planck's unit $\hbar = 1$; (B) reconstruction of the entire QM formalism to admit \hat{I} as the correct left and right unit; and (C) representation of all local-potential forces with the Hamiltonian $H = K + V$ and all nonlocal-nonpotential interactions with the isounit \hat{I} (or isotopic element T), as per the preceding isotopic methods and following physical axioms:

Isoaxiom I. The states are elements of a isoHilbert space $\hat{\mathcal{H}}$ interpreted as (left or right) isomodule with isoeigenvalue equations and isonormalization

$$H * | \hat{\psi} \rangle = HT | \hat{\psi} \rangle = \hat{E} * | \hat{\psi} \rangle \quad \langle \hat{\psi} | \hat{\psi} \rangle = \hat{I} = T^{-1}. \tag{3.5}$$

Isoaxiom II. Measurable quantities are represented by isocommuting isoHermitean operators on $\hat{\mathcal{H}}$ whose eigenvalues are conventional real numbers,

$$\begin{aligned} H^\dagger &\equiv H^\dagger & H * | \hat{\psi} \rangle &= \hat{E} * | \hat{\psi} \rangle \equiv E | \hat{\psi} \rangle \\ \hat{E} &\in \hat{R}, & E &\in R. \end{aligned} \tag{3.6}$$

Isoaxiom III. The fundamental dynamical operators, coordinates r^k and momenta p_k , are characterized by isoeigenvalue equations and isocommutation rules (in momentum representation)

$$\begin{aligned} p_k * | \hat{\psi} \rangle &= -i \hat{I}^k \nabla_k | \hat{\psi} \rangle \\ r^k_{op} * | \hat{\psi} \rangle &= \hat{r}^k * | \hat{\psi} \rangle \equiv r^k | \hat{\psi} \rangle \quad \hat{r} \in \hat{R} \quad r \in R \end{aligned} \tag{3.7a}$$

$$\begin{aligned} [a^\mu, a^\nu]_{\hat{\xi}} &= a^\mu T a^\nu - a^\nu T a^\mu = i \omega^{\mu\alpha} \hat{I}_\alpha^\nu \\ a &= (p, r) \quad (\hat{I}_\alpha^\nu) = \text{diag.}(T^{-1}, T^{-1}). \end{aligned} \quad (3.7b)$$

where $\omega^{\mu\nu}$ is the 6×6 canonical-Lie tensor.

Isoaxiom IV. The time evolution of states is characterized by isounitary transformations with the (isoHermitean) Hamiltonian as generator

$$|\hat{\psi}(t)\rangle = \hat{U}(t, t_0) * |\hat{\psi}(t_0)\rangle = \left\{ e_{\hat{\xi}}^{iH(t-t_0)} \right\} * |\hat{\psi}(t_0)\rangle \equiv e^{iHT(t-t_0)} |\hat{\psi}(t_0)\rangle. \quad (3.8)$$

while the time evolution of operators is characterized by an equivalent, one-dimensional, Lie-isotopic group of isounitary transformations with the same Hamiltonian as generators, expressible in the finite form

$$A(t) = \hat{U} * A(t_0) * \hat{U}^\dagger = \left\{ e_{\hat{\xi}}^{iH(t-t_0)} \right\} * A(t_0) * \left\{ e_{\hat{\xi}}^{-i(t-t_0)H} \right\} \quad (3.9)$$

with infinitesimal version provided by the isoHeisenberg equations

$$i \frac{\hat{d}A}{\hat{d}t} = [A, H]_{\hat{\xi}} = ATH - HTA \quad (3.10)$$

where $\hat{d}/\hat{d}t = \hat{I}_t d/dt$ is the isotopic derivative and \hat{I}_t is the isounit of time [10, 30].

Isoaxiom V. The values expected in measurements of observables are given by the isoexpectation values

$$\hat{\langle A \rangle} = \frac{\langle \hat{\psi} | * A * | \hat{\psi} \rangle}{\langle \hat{\psi} | * | \hat{\psi} \rangle} = \frac{\langle \hat{\psi} | T A T | \hat{\psi} \rangle}{\langle \hat{\psi} | T | \hat{\psi} \rangle} \in R \quad (3.11)$$

which reduce under isonormalization $\langle \hat{\psi} | * | \hat{\psi} \rangle = 1$ to $\hat{\langle A \rangle} = \langle \hat{\psi} | * A * \hat{\psi} \rangle \in R$.

The *isodual* HM for antiparticles can be constructed via the techniques of section 2. In particular, *isoduality results to be a geometric reformulation of charge conjugation* [15, 24, 28].

The above isoaxioms imply the following properties of the isounits [10, 30]: (1) \hat{I} is isoidempotent of arbitrary (finite) degree, $\hat{I}^{\hat{n}} = \hat{I} * \hat{I} * \dots * \hat{I} \equiv \hat{I}$; (2) the isoquotient of \hat{I} by itself is \hat{I} , $\hat{I}/\hat{I} \equiv \hat{I}$; (3) the isosquare root of \hat{I} is \hat{I} , $\hat{I}^{\hat{1}/2} \equiv \hat{I}$; (4) \hat{I} isocommutes with all operators, $[A, \hat{I}]_{\hat{\xi}} = A - A \equiv 0$; (5) \hat{I} is left invariant by isounitary transformations, $\hat{U} * \hat{I} * \hat{U}^\dagger \equiv U * \hat{U}^\dagger = \hat{I}$; (6) \hat{I} is conserved in time, $i \hat{d}\hat{I}/\hat{d}t \equiv [\hat{I}, H]_{\hat{\xi}} \equiv 0$; and (7) all infinitely possible isounits \hat{I} admit as isoeigenvalues the ordinary number 1, $\hat{\langle \hat{I} \rangle} = \langle \hat{\psi} | T T^{-1} T | \hat{\psi} \rangle / \langle \hat{\psi} | T | \hat{\psi} \rangle \equiv 1$. The following primary consequences then hold:

(A) *Quantum and hadronic mechanics coincide, by construction, at the abstract, realization-free level.* In fact, at the abstract level, all distinctions cease to exist between \hat{F} and F , $\hat{E}(r, \hat{\delta}, R)$ and $E(r, \delta, R)$, $\hat{\xi}$ and ξ , $\hat{\mathcal{H}}$ and \mathcal{H} etc. A subtle implication is that criticisms on the above axiomatization may eventually result to be criticisms on the axiomatic structure of quantum mechanics itself.

(B) *Hadronic mechanics is form-invariant under its own transformation theory, the isounitary transformations.* This can be seen from the fact that isocommutators are invariant under isounitary transformations, $\hat{U} * [A, B]_{\hat{\xi}} * \hat{U}^\dagger = [A', B']_{\hat{\xi}}$, or the

invariance of eigenvalues and isoexpectation values under isounitary transformations, etc. This form-invariance should be compared with the general lack of invariance of q -deformations under their time evolution [8].

(C) *Hadronic mechanics provides a fully <causal> treatment of <nonlocal> interactions.* This property originates from the embedding of *all* nonlocal interactions in the isounit of the theory. Causality then follows from the isoexpectation values of all admissible nonlocal isounits $\langle \hat{I} \rangle = 1$. Causality can also be proved in a number of other ways, e.g., from the fact that HM implies an axiom-preserving isotopy of the conventional causal treatment [9]. The above causal description of nonlocal interactions in a way embedded in the basic axioms of the theory should be compared with the loss of causality for conventional treatments of nonlocal interactions.

(D) *The property $\langle \hat{I} \rangle = 1$ implies the reconstruction of Planck's constant $\hbar = 1$ at the level of measurements.* In different terms, the integro-differential generalization of Planck's constant (1.1) which is at the foundation of HM holds only within its mathematical structure, but the conventional value $\hbar = 1$ is reconstructed in the measurement theory.

(E) *Currently available experimental measures cannot distinguish between quantum and hadronic mechanics, that is, they cannot identify whether the interactions are local or nonlocal.* In fact, the differentiation can be best tested experimentally via the verification of the novel predictions of HM, that is, predictions beyond the descriptive capacities of QM.

(F) *Hadronic mechanics permits an axiomatization of discrete-time theories via their embedding in the isounit \hat{I} , which therefore result to be 'hidden' in, and compatible with the conventional axioms of QM.* In fact, isounits of Kadeisvili's class V [14] have precisely a discrete structure. Yet their isoeigenvalues remains the same as those of class I, $\langle \hat{I} \rangle = 1$. This implies in particular that the isogeometries admit discrete-time realizations based on the same axioms of conventional continuous geometries.

(G) *Total physical quantities of isolated systems are conserved under isotopies.* This is due, first, to the preservation of Hermiticity / observability under isotopies, and then to the invariance of the basis of a vector space under isotopies up to renormalization factors [7-9]. In particular, the generators of conventional and Lie-isotopic symmetries coincide. This implies that currently available centre-of-mass measures on *total* quantities of interacting and / or bound states, by no means, can ascertain whether the internal forces are of quantum or hadronic type, i.e., of local-canonical or nonlocal-noncanonical.

4. Isotopies and isodualities of the Lorentz and Poincarè symmetries

Consider the Minkowski space $M(x, \eta, \mathcal{R})$ with local coordinates $x = \{x, x^4\}$, $x^4 = c_0 t$, $c_0 =$ speed of light in vacuum, metric $\eta = \text{diag.}(1, 1, 1, -1)$, separation $x^2 = x^\mu \eta_{\mu\nu} x^\nu$ and invariant measure $ds^2 = -dx^\mu \eta_{\mu\nu} dx^\nu$. Its group of linear-local-canonical isometries is the ten-dimensional Poincarè group $P(3.1) = O(3.1) \times T(3.1)$ characterized by the (ordered sets of) parameters $w = \{\theta, v, a\}$ (Euler's angles θ_k , speed parameter v_k and translation parameters a), and generators, say, for a system of particles with non-null masses m_a , $X = \{X_k\} = \{M_{\mu\nu}\} = \sum_a (x_{a\mu} p_{a\nu} - x_{a\nu} p_{a\mu})$, $P_\mu = \sum_a p_{a\mu}$, $\mu, \nu = 1, 2, 3, 4$, $a = 1, 2, \dots, N$, in their known adjoint (fundamental) representation [3].

Three realizations of the ten-dimensional isotopic covering $\hat{P}(3.1)$ of $P(3.1)$ have been constructed via the Lie-isotopic theory, the classical [13], operator [30] and abstract [11] ones. We now present the construction of the abstract realization of $\hat{P}(3.1)$ via the following five steps, and then the operator one, while that of the isospinorial covering will be presented in section 8.

Step 1 is the identification of the fundamental isotopic element T (which can be interpreted as 4×4 matrix generalization of q -number-deformations [7]) via its fundamental implication, the deformation of the Minkowski metric η into the most general known metric $\hat{\eta} = T\eta$ which is nonlinear, nonlocal-integral, and noncanonical in all variables, wavefunctions and their derivatives, as well as density μ of the interior medium considered, its local temperature τ , the local index of refraction n , and any other needed physical quantity

$$\hat{\eta} = T(s, x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, \dots)\eta \quad (4.1)$$

here assumed to be of Kadeisvili class III [14] (smooth, bounded, nowhere singular and Hermitean, but not necessarily positive or negative-definite). Under the assumed conditions, the T -matrix can always (but not necessarily) be diagonalized in the form

$$T = \text{diag.}(g_{11}, g_{22}, g_{33}, g_{44}) = T^\dagger \quad \text{Det } T \neq 0. \quad (4.2)$$

The isosymmetry $\hat{P}(3.1)$ is then constructed with respect to the isounit $\hat{I} = T^{-1}$.

Step 2 is the lifting of the conventional field $R(n, +, \times)$ of real numbers n into the isofield $\hat{R}(\hat{n}, +, *)$ of isoreal numbers $\hat{n} = n\hat{I}$, $\hat{I} = T^{-1}$.

Step 3 is the lifting of space $M(x, \eta, R)$ on the field R into the *isoMinkowski* space $\hat{M}(x, \hat{\eta}, \hat{R})$ on the isofield \hat{R} with isoseparation [6]

$$(x - y)^{\hat{2}} = [(x^\mu - y^\mu)\hat{\eta}_{\mu\nu}(s, x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, \dots)(x^\nu - y^\nu)]\hat{I} \in \hat{R}. \quad (4.3)$$

Step 4 identifies the basic isotransformations leaving invariant (4.3)

$$\begin{aligned} x' &= \hat{\Lambda}(w) * x & \hat{\Lambda}^\dagger \hat{\eta} \hat{\Lambda} &= \hat{\Lambda} \hat{\eta} \hat{\Lambda}^\dagger = \hat{I} \hat{\eta} \hat{I} \\ \text{Det } \hat{\Lambda} \left[\text{Det}(\hat{\Lambda}T) \right] &= \pm \hat{I} & x' &= x + A \end{aligned} \quad (4.4)$$

where the quantity A will be identified shortly. The connected component $\hat{P}^0(3.1) = S\hat{O}(3.1) \times \hat{T}(3.1)$ is characterized by $\text{Det } \hat{\Lambda} = +\hat{I}$ with structure [11, 13, 30]

$$\begin{aligned} \hat{O}(3.1): \quad \hat{\Lambda}(w) * x &= \left\{ \prod_k^* e_{\hat{\xi}}^{iX_k * w_k} \right\} T x \\ &= \left\{ \prod_k e^{iX_k T w_k} \right\} x \end{aligned} \quad (4.5a)$$

$$\begin{aligned} \hat{T}(3.1): \quad \left\{ e_{\hat{\xi}}^{iP\eta a} \right\} * x &= \left\{ e^{iP\hat{\eta} a} \right\} x \\ \left\{ e_{\hat{\xi}}^{iP\eta a} \right\} * p &\equiv 0. \end{aligned} \quad (4.5b)$$

where w_k and X_k are conventional [3] and T is given by (4.2). The isocommutation rules of $\hat{P}^0(3.1)$ are given by [loc cit]

$$[M_{\mu\nu}, M_{\alpha\beta}] = i(\hat{\eta}_{\nu\alpha} M_{\beta\mu} - \hat{\eta}_{\mu\alpha} M_{\beta\nu} - \hat{\eta}_{\nu\beta} M_{\alpha\mu} + \hat{\eta}_{\mu\beta} M_{\alpha\nu}) \quad (4.6a)$$

$$[M_{\mu\nu}, P_\alpha] = i(\hat{\eta}_{\mu\alpha} P_\nu - \hat{\eta}_{\nu\alpha} P_\mu) \quad [P_\mu, P_\nu] = 0 \quad (4.6b)$$

where the product is the isocommutator $[A, B] = ATB - BTA$. The isoCasimirs are then given by

$$\begin{aligned} \hat{C}^{(0)} &= \hat{I} & C^{(1)} &= P^{\hat{2}} = P * P = P_{\mu} \hat{\eta}^{\mu\nu} P_{\nu} \\ C^{(2)} &= \hat{W}^{\hat{2}} = \hat{W}_{\mu} \hat{\eta}^{\mu\nu} \hat{W}_{\nu} & \hat{W}_{\mu} &= \epsilon_{\mu\alpha\beta\rho} J^{\alpha\beta} * P^{\rho}. \end{aligned} \quad (4.7)$$

The general *isoPoincarè transformations* are given by [loc cit]

$$x' = \hat{A} * x \quad \text{isoLorentz transforms} \quad (4.8a)$$

$$x' = x + A(s, x, \dot{x}, \ddot{x}, \dots) \quad \text{isotranslations} \quad (4.8b)$$

$$x' = \hat{\pi}_r * x = (-x, x^4) \quad \text{isoinversions} \quad (4.8c)$$

$$x' = \hat{\pi}_t * x = (x, -x^4)$$

$$\begin{aligned} A_{\mu} &= a_{\mu} \left\{ g_{\mu\mu} + a^{\alpha} [g_{\mu\mu}, \hat{P}_{\alpha}] / 1! \right. \\ &\quad \left. + a^{\alpha} a^{\beta} [[g_{\mu\mu}, \hat{P}_{\alpha}], \hat{P}_{\beta}] / 2! + \dots \right\} \end{aligned} \quad (4.8d)$$

with the *general isoLorentz transformations* given by the isorotations (see [21] for brevity) and the *isoboosts* first constructed in [11]

$$x'^1 = x^1 \quad x'^2 = x^2 \quad (4.9a)$$

$$\begin{aligned} x'^3 &= x^3 \cosh \left[v(g_{33}g_{44})^{1/2} \right] \\ &\quad - x^4 g_{44} (g_{33}g_{44})^{-1/2} \sinh \left[v(g_{33}g_{44})^{1/2} \right] \\ &= \hat{\gamma} (x^3 - \beta x^4) \end{aligned} \quad (4.9b)$$

$$\begin{aligned} x'^4 &= -x^3 g_{33} (g_{33}g_{44})^{-1/2} \sinh \left[v(g_{33}g_{44})^{1/2} \right] \\ &\quad + x^4 \cosh \left[v(g_{33}g_{44})^{1/2} \right] = \hat{\gamma} (x^4 - \hat{\beta} x^3) \end{aligned} \quad (4.9c)$$

$$\hat{\beta} = v^k g_{kk} v^k / c_0 g_{44} c_0 \quad \hat{\gamma} = |1 - \hat{\beta}^2|^{-1/2}. \quad (4.9d)$$

The classification of all possible isosymmetries $\hat{O}(3.1)$ is then straightforward, as done in the original proposal [11]. In fact, in the above formulation without a define signature in the metric (class III), the *abstract isoLorentz symmetry* $\hat{O}(3.1)$ unifies all possible simple, six-dimensional Lie and Lie-isotopic algebras, i.e.: (1) all six-dimensional simple algebras of Cartan's classification $O(4)$, $O(3.1)$ and $O(2.2)$ (over a field of characteristic zero); (2) all their isoduals $O^d(4)$, $O^d(3.1)$ and $O^d(2.2)$; and (3) all infinitely possible isotopes for each of the preceding algebras. Similarly, the *abstract isoPoincarè algebra* $\mathcal{P}(3.1)$ unifies all possible ten-dimensional inhomogeneous algebras, isoalgebras and their isoduals.

The above structure, even though mathematically significant, is excessively broad for physical applications. From here on we shall restrict our analysis to the isosymmetries of class I with realization

$$\begin{aligned} \hat{\eta} &= T\eta = \text{diag.} (b_1^2, b_2^2, b_3^2, -b_4^2) \\ &\equiv \text{diag.} (n_1^{-2}, n_2^{-2}, n_3^{-2}, -n_4^{-2}) \quad b_{\mu}, n_{\mu} > 0 \end{aligned} \quad (4.10)$$

where the *bs* are called *characteristic functions* of the medium considered. The use of the quantity $T^d = -T$ then characterizes the isodual symmetry. The general functional dependence is needed for the study of a particle of an electromagnetic wave at

a given interior point of an inhomogeneous and anisotropic medium. When global-exterior conditions are studied (e.g., for the average speed of light throughout an inhomogeneous and anisotropic atmosphere), the characteristic functions can be effectively averaged into constants $b^\circ_\alpha = \text{const} = \text{Aver.}(b_\alpha)$, as is the case for most applications considered in this paper. In this case isotransforms (4.8) and (4.9) are called *restricted isoPoincarè and isoLorentz transformations*, respectively. Their primary implication is the regaining of locality and linearity, thus preserving inertial systems as in the conventional case, although the transformations remain noncanonical in $M(x, \eta, R)$.

It is easy to prove the local isomorphism $\hat{P}(3.1) \approx P(3.1)$ for all $T > 0$. This confirms a fundamental objective of section 1, the inapplicability of the Lorentz *transformations*, but the exact character of the Poincarè *symmetry*. The 'direct universality' of the isoPoincarè symmetry should be noted, i.e., its applicability for all infinitely possible isoseparations (4.3) (universality), directly in the x -frame of the experimenter (direct universality).

Despite their apparent simplicity, isotransformations (4.9) are highly nonlinear-nonlocal-noncanonical owing to the unrestricted functional dependence of the $g_{\mu\mu}$ -quantities. The simplicity of the final invariance should also be noted. In fact, the invariance of all infinitely possible isoseparations (3.3) is merely given by plotting the given $g_{\mu\mu}$ elements in equations (3.9).

The operator *relativistic isokinematics* on $\hat{M}(x, \hat{\eta}, \hat{R})$ [15, 30] is characterized by the linear momentum here presented for simplicity for the case $\partial b_\mu / \partial x^\nu = 0$ or for b°_μ -constants

$$\begin{aligned} p &= (p^\mu) = (\hat{m}u^\mu) = (m_0\hat{\gamma}\hat{c}v^k, m_0\hat{\gamma}\hat{c}) \\ \hat{m} &= m_0\hat{\gamma} \quad c = c_0b_4 \end{aligned} \quad (4.11)$$

with isoeigenvalue form (isoaxiom III of HM)

$$p_\mu * \hat{\psi} = -i \hat{I}_\mu^\nu \frac{\partial}{\partial x^\nu} \hat{\psi} = -i b_\mu^{-2} \frac{\partial}{\partial x^\mu} \hat{\psi} = -i \frac{\partial}{\partial x_\mu} \hat{\psi} \quad (4.12)$$

where the last identity is evidently due to the expressions $x_\mu = \hat{\eta}_{\mu\nu} x^\nu = b_\mu^2 x^\mu$. The fundamental isoinvariant is then given by from (4.7)

$$\begin{aligned} p^{\hat{2}} * | \hat{\psi} \rangle &= \hat{\eta}^{\mu\nu} p_\mu * p_\nu * | \hat{\psi} \rangle \\ &= (b_k^2 p_k * p_k - c^2 p_4 * p_4) * | \hat{\psi} \rangle = (-m_0^2 c^4) | \hat{\psi} \rangle. \end{aligned} \quad (4.13)$$

The *fundamental relativistic isocommutation rules* are then given by

$$\begin{aligned} [p_{a\mu}, \hat{x}_{a\nu}] * | \hat{\psi} \rangle &= -i \eta_{\mu\nu} * | \hat{\psi} \rangle \\ [x_{a\mu}, \hat{x}_{b\nu}] * | \hat{\psi} \rangle &= [p_{a\mu}, \hat{p}_{b\nu}] * | \hat{\psi} \rangle \equiv 0 \end{aligned} \quad (4.14)$$

namely, the isoeigenvalues do not exhibit b -terms, by coinciding with the corresponding conventional eigenvalues [15, 28, 30].

The operator *isoPoincarè algebra* $\hat{P}(3.1)$ can then be computed and it is given by [15, 30]

$$\begin{aligned} [J_{\mu\nu}, \hat{J}_{\alpha\beta}] * | \psi \rangle &= i (\eta_{\nu\alpha} J_{\beta\mu} - \eta_{\mu\alpha} J_{\beta\nu} \\ &\quad - \eta_{\nu\beta} J_{\alpha\mu} + \eta_{\mu\beta} J_{\alpha\nu}) * | \psi \rangle \end{aligned} \quad (4.15a)$$

$$\begin{aligned} [J_{\mu\nu}, \hat{P}_\alpha] * | \psi \rangle &= i (\eta_{\mu\alpha} P_\nu - \eta_{\nu\alpha} P_\mu) * | \psi \rangle \\ [J_\mu, \hat{P}_\nu] * | \psi \rangle &= 0 \end{aligned} \quad (4.15b)$$

namely, the structure constants of $\hat{P}(3.1)$ formally coincide with those of the conventional algebra, thus confirming not only the local isomorphisms $\hat{P}(3.1) \approx P(3.1)$, but also the identity at the abstract level of the conventional and isotopic symmetries and related relativities. The rest of the isoalgebras and isogroups can then be constructed via the preceding analysis for the matrix case. The isospinorial realization of the isoPoincarè symmetry is presented in section 8.

The first application of the isoPoincarè symmetry can be found in conventional, classical, exterior gravitation. To begin, the isoPoincarè symmetry $\hat{P}(3.1)$ provides the universal invariance of general relativity. In fact, the invariance of any gravitational (e.g., Schwartzschild's) line element is merely given by plotting the $g_{\mu\mu}$ elements in (4.9).

Another application is a geometric unification of the Minkowskian and Riemannian spaces. This result is achieved via the decomposition of the Riemannian metric $g(x) = T(x)\eta$, and the chain

$$\mathfrak{R}(x, g, R) \approx \hat{\mathfrak{R}}(x, g, \hat{R}) \equiv \hat{M}(x, \hat{\eta}, \hat{R}) \approx M(x, \eta, R) \tag{4.16}$$

where all isospaces are characterized by the gravitational isounit $\hat{I} = [T(x)]^{-1}$. Note that all Riemannian metrics admit the decomposition $g = T\eta$ with $T > 0$ (trivially, from their locally Minkowskian character).

The above geometric unification of Minkowskian and Riemannian spaces has been used by Lopez [31] to identify three geometric arguments supporting Logunov's [33] relativistic formulation of gravitation with a nowhere null source.

Further applications in gravitation under study are [15]: a novel treatment of singularities as singularities of the isounits $\hat{I} = [T(x)]^{-1}$; a novel operator form of gravitation given by its embedding in the unit of relativistic quantum theories; a novel 'iso-grand-unification'; and others, Applications of nongravitational character will be indicated below.

But perhaps the most remarkable aspect is the capability of the isoPoincarè symmetry to unify in one single abstract isosymmetry $\mathcal{P}(3.1)$ of class I: linear and nonlinear, local and nonlocal, Hamiltonian and nonhamiltonian, relativistic and gravitational, as well as exterior and interior systems, at classical, operator and statistical levels [13, 28].

5. Isotopies and isodualities of the special relativity

We shall now ignore gravitational profiles, and consider isotopic theories specifically built for interior relativistic problems of particles with $\partial b_\mu / \partial x_\nu = 0$ or $b^\circ_\alpha = \text{Aver.}(v_\alpha) = \text{const.} > 0$, $\alpha = 1, 2, 3, 4$.

The isotopies of the Poincarè symmetry $P(3.1) \rightarrow \hat{P}(3.1)$ imply corresponding, necessary liftings of the special relativity into a form called *isospecial relativity*, originally submitted in [11] and then studied in detail in [13, 15, 28, 30]. The objective is a form-invariant description of extended-deformable particles and electromagnetic waves propagating within inhomogeneous and anisotropic physical media represented by isospaces $\hat{M}(x, \hat{\eta}, \hat{R})$. The special relativity is admitted as a particular case in vacuum for which $\hat{I} \equiv I$.

The isospecial relativity is based on the isoPoincarè invariance on isospaces $\hat{M}(x, \hat{\eta}, \hat{R})$ of class I, with consequential isotopies of all basic postulates of the spe-

cial relativity. Those important for this paper are the following ones presented for $b_1 = b_2 = b_3 \neq b_4$, with $\hat{\beta}$ and $\hat{\gamma}$ given by (4.9d):

Isopostulate I. The maximal, causal, invariant speed is given by

$$V_{\max} = |dr/dt|_{\max} = c_0 b_4 / b_3. \quad (5.1)$$

Isopostulate II. The addition of speeds u and v is given by the isotopic law

$$v' = (u + v) / (1 + u_k b_k^2 v_k / c_0^2 b_4^2). \quad (5.2)$$

Isopostulate III. Time intervals and lengths follow the isodilation-isocontraction laws

$$\hat{\tau} = \hat{\gamma} \tau_0 \quad \hat{\Delta} L_0 = \hat{\gamma} \Delta L. \quad (5.3)$$

Isopostulate IV. Frequencies follow the isodoppler law [for aberration $\hat{a} = 90^\circ$]

$$\hat{\omega} = \omega \hat{\gamma}. \quad (5.4)$$

Isopostulate V. The energy equivalence of mass follows the isoequivalence principle

$$\hat{E} = mc^2 = mc_0^2 b_4^2 = mc_0^2 / n_4^2. \quad (5.5)$$

The above generalized postulates are implicit in the preceding formulations, e.g., in isoinvariant (4.3), or in isoLorentz transformations (4.9); they recover identically the conventional postulates in vacuum for which $b_\mu = 1$; and they coincide with the conventional postulate at the abstract, realization-free level, where we lose all distinctions between \hat{I} and I , \hat{x}^2 and x^2 , $\hat{\beta}^2$ and β^2 , $\hat{\tau}$ and τ , $\hat{\omega}$ and ω , \hat{E} and E , etc. Thus, criticism of the above isotopic postulates may result to be criticism on Einstein's postulates themselves.

A most visible departure from the conventional theories is the abandonment of the speed of light as the invariant speed in favour of quantity (5.1) which is intrinsic of the isoMinkowski geometry and represents the maximal causal speed as characterized by an effect following a cause due to particles, fields or other means. Note that in vacuum $V_{\max} \equiv c_0$ by therefore recovering as a particular case the speed of light as the maximal causal speed.

The best way to verify isopostulate I is in the simplest possible medium, the homogeneous and isotropic water, where the speed of light is no longer c_0 , but rather the familiar value $c = c_0 / n^0 < c_0$, where n^0 is the index of refraction. The insistence in keeping the speed of light as the invariant speed in water leads to a number of inconsistencies, such as: the violation of both the conventional and isotopic laws of addition of speeds, none of which yields the speed of light as the sums of two light speeds $u = v = c = c_0 / n^0$ for causal speed c_0 ; electrons can propagate in water at speeds bigger than the assumed invariant speed, as experimentally established by the Cherenkov light; and others. These inconsistencies are resolved by the isospecial relativity with isoinvariant given by the simple *scalar isotopy* $\hat{x}^2 = b^{02} x^2$ [11, 13, 30].

Even greater inconsistencies emerge if one insists in keeping the speed of light as the invariant speed for all media more complex than water, e.g., inhomogeneous and anisotropic atmospheres. To our best knowledge, a resolution of these inconsistencies requires the separation of the invariant speed from the speed of light, and the use of their identity only for the particular case in vacuum.

Since isopostulates I–V are quantitatively different than the conventional ones, they are suitable for experimental verifications. Intriguingly, all available experimental evidence appears to confirm the above isopostulates, not only for simple media such as water or atmospheres, but also for the more complex media, such as the hyperdense media inside hadrons (see section 9).

In summary, the isotopies identify four physically different but axiomatically equivalent formulations of the special relativity: (1) the *conventional special relativity* based on the $P(3.1)$ form-invariant description on $M(x, \eta, R)$ of point-like particles in vacuum; (2) the *isodual special relativity* based on the $P^d(3.1)$ -invariance on $M^d(x, \eta^d, R^d)$ for the point-like description of antiparticles in vacuum; (3) the *isospecial relativity* with $\hat{P}(3.1)$ form-isoinvariant description on $\hat{M}(x, \hat{\eta}, \hat{R})$ of extended-deformable particles in interior conditions; and (4) the *isodual isospecial relativity* with isodual $\hat{P}^d(3.1)$ -invariance on $\hat{M}^d(x, \hat{\eta}, \hat{R}^d)$ for the description of extended-deformable antiparticles in interior conditions.

6. IsoMinkowskian geometrization of physical media

As is well known, the Minkowski space $M(x, \eta, R)$ provides a geometrization of the homogeneous and isotropic vacuum. A fundamental aspect of the isospecial relativity is that the isoMinkowski space $\hat{M}(x, \hat{\eta}, \hat{R})$ provides a geometrization of interior, classical and operator, physical media, e.g., the geometrization of our inhomogeneous and anisotropic atmospheres, or of the medium in the interior of nuclei, hadrons and stars.

An intuitive understanding can be reached by noting that the characteristic functions $b_\mu = 1/n_\mu$ essentially extend the local index of refraction $1/n_4$ to all space-time components. Equivalently, by recalling that physical media are generally opaque to light, *the isotopy $M(x, \eta, R) \rightarrow \hat{M}(x, \hat{\eta}, \hat{R})$ essentially extends to all physical media the geometric structure of light in vacuum.*

This geometrization has permitted the classification of physical media into nine significant types [30], p 103, i.e., first, into the three primary classes $\hat{\gamma} = \gamma$, $\hat{\gamma} < \gamma$ and $\hat{\gamma} > \gamma$ from (4.9d) and then on the three subclasses for each of them $b^\circ = b^\circ_4$, $b^\circ > b^\circ_4$ and $b^\circ < b^\circ_4$, $b^\circ = \text{Aver.}(b^\circ_x)$.

This classification is of primary phenomenological relevance as researchers in the field know, because it implies automatic redefinition of the *intrinsic* characteristics of particles and electromagnetic waves, called *isorenormalizations*. They can be anticipated from the deviations of the isoCasimirs (4.7) from the conventional expressions, or from the differences between conventional and isorepresentations (see next section), and are finalized by isopostulates I–V, such as the isorenormalization of the rest energy (5.5) for a particle in interior conditions (only). The identification of the type of isoMinkowskian geometry characterized by a given medium is therefore a basic problem for practical applications.

In regard to electromagnetic waves, the isospecial relativity predicts no change in frequency (i.e., no loss of energy) for light propagating in media of type 1, 2, 3 with $\hat{\gamma} \equiv \gamma$, such as water (as experimentally established), plus two predictions from isopostulate IV suggested for tests [13, 28, 30]: an *isoDoppler redshift* for $\hat{\gamma} < \gamma$ (i.e., a loss of energy) for light propagating within media of type 4 such as inhomogeneous and anisotropic atmospheres of low density; and an *isoDoppler blueshift* for $\hat{\gamma} > \gamma$ for

light propagating in hyperdense media of type 9, such as those in the Bose–Einstein correlation fireball, and others.

In regard to particles, the isorenormalization of the intrinsic characteristics permits quantitative representations otherwise impossible, such as attractive interactions for the Cooper pair (e^-, e^-) in superconductivity (section 9), and imply the *novel* prediction of the apparent origin of cold fusion at the level of elementary particles, as an extension of that of the Cooper pair.

Note the *necessity* of the isotopies for the above results. Note also that the isoMinkowskian geometry is *isoflat* since it is the isotopy of a flat geometry (not so for the isotopies of Riemann [12, 13]). This implies the capability to reconstruct conventional and hyperbolic angles, even though the space has the most general possible curvature $\hat{\eta} = \hat{\eta}(x, \dot{x}, \ddot{x}, \dots)$. For instance, if θ_{1-2} is a conventional angle in 1–2 space, the corresponding *isoangle* is given by $\hat{\theta}_{1-2} = b_1 b_2 \theta_{1-2}$, and if v — is the ‘hyperbolic angle’ in 3–4 space, the corresponding *hyperbolic isoangle* is $\hat{v}_{1-2} = b_3 b_4 v_{1-2}$. Thus the isoMinkowskian geometry predicts the functional dependence of the isoLorentz transforms (4.9) in a way independent from, but in full agreement with the Lie-isotopic theory. For numerous other properties we refer to [13, 28]. A technical knowledge of the isoMinkowskian geometry and its isospecial functions is therefore essential for practical applications.

7. Isotopies and isodualities of SU(2) with applications to spin and isospin

The best way to begin an outline of the applications of the isoLorentz and isoPoincaré symmetries is via their most important component, the isospinorial $S\hat{U}(2)$ subalgebras studied in [15, 21, 22]. In fact, we have the following adjoint isorepresentation along the classification of section 2:

(1) *regular isoPauli matrices*

$$\begin{aligned} \hat{\sigma}_1 &= \Delta^{-1/2} \begin{pmatrix} 0 & g_{11} \\ g_{22} & 0 \end{pmatrix} & \hat{\sigma}_2 &= \Delta^{-1/2} \begin{pmatrix} 0 & -ig_{11} \\ +ig_{22} & 0 \end{pmatrix} \\ \hat{\sigma}_3 &= \Delta^{-1/2} \begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix} \end{aligned} \quad (7.1a)$$

$$\begin{aligned} T &= \text{diag.}(g_{11}, g_{22}) & \Delta &= \text{Det } Q = g_{11}g_{22} > 0 \\ [\hat{\sigma}_i, \hat{\sigma}_j]_{\hat{\xi}} &= i2\epsilon_{ijk}\hat{\sigma}_k \end{aligned} \quad (7.1b)$$

$$\hat{\sigma}_3^* | \hat{b} \rangle = \pm \Delta^{1/2} | \hat{b} \rangle \quad \hat{\sigma}_3^{\hat{2}*} | \hat{b} \rangle = 3\Delta | \hat{b} \rangle \quad (7.1c)$$

(2) *irregular isoPauli matrices*

$$\begin{aligned} \hat{\sigma}_1' &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 & \hat{\sigma}_2' &= \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2 \\ \hat{\sigma}_3' &= \begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix} = \Delta \hat{I} \sigma_3 \end{aligned} \quad (7.2a)$$

$$\begin{aligned} [\hat{\sigma}_1', \hat{\sigma}_2']_{\hat{\xi}} &= 2i\hat{\sigma}_3' & [\hat{\sigma}_2', \hat{\sigma}_3']_{\hat{\xi}} &= 2i\Delta\hat{\sigma}_1' \\ [\hat{\sigma}_3', \hat{\sigma}_1']_{\hat{\xi}} &= 2i\Delta\hat{\sigma}_2' \end{aligned} \quad (7.2b)$$

$$\hat{\sigma}_3'^* | \hat{b} \rangle = \pm \Delta | \hat{b} \rangle \quad \hat{\sigma}_3^{\hat{2}*} | \hat{b} \rangle = \Delta(\Delta + 2) | \hat{b} \rangle \quad (7.2c)$$

(3) standard isoPauli matrices

$$\begin{aligned} \hat{\sigma}_1 &= \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix} & \hat{\sigma}_2 &= \begin{pmatrix} 0 & -i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix} \\ & & \hat{\sigma}_3 &= \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix} \end{aligned} \quad (7.3a)$$

$$\begin{aligned} T &= \text{diag.}(\lambda, \lambda^{-1}) & \lambda &\neq 0 \\ \Delta &= \text{Det } Q = 1 & [\hat{\sigma}''_i, \hat{\sigma}''_j]_{\hat{\xi}} &= i\epsilon_{ijk}\hat{\sigma}''_k \end{aligned} \quad (7.3b)$$

$$\hat{\sigma}''_{3*} |\hat{b}\rangle = \pm |\hat{b}\rangle \quad \hat{\sigma}''_{2*} |\hat{b}\rangle = 3 |\hat{b}\rangle. \quad (7.3c)$$

As one can see isoPauli matrices (7.1) preserve the original structure constants, but exhibit new eigenvalues illustrating the isorenormalization of spin mentioned in section 6. Matrices (7.2) show different isorenormalizations because of the appearance of the structure functions in the isocommutation rules. Finally, matrices (7.3) preserve the original structure constants *and* eigenvalues of Pauli's matrices, yet they exhibit the presence of a 'hidden' parameter (actually a nonlinear–nonlocal function) λ in the very structure of the spin $\frac{1}{2}$.

Isorepresentations with mutated values of spin are used for particles in interior conditions under sufficiently intense nonlinear–nonlocal–noncanonical interactions (e.g., a neutron in the core of a neutron star). Particles under less extreme conditions do preserve their conventional spin, in which case isorepresentation (7.3) is applicable.

We now indicate some of the applications of the above isorepresentations. In regard to the spin, a clear application of the isotopes $S\hat{U}(2)$ is the proof that *Bell's inequality holds, specifically, for conventional quantum mechanics and it is inapplicable under isotopies*. The proof is transparent for the regular and irregular isoPauli matrices (because of the different eigenvalues). What is intriguing is that Bell's inequality is also inapplicable under the standard isoPauli matrices (7.3) (see [27] for details.)

Another QM property which is inapplicable under isotopies is von Neumann's theorem on the lack of existence of 'hidden variables' because based the uniqueness of the spectrum of eigenvalues of a Hermitean operator, which is lacking in HM owing to the infinite possibilities of the isotopic element T for each Hermitean operator. This has permitted the *isotopic realization* of the 'hidden variables' expressed precisely by the isotopic element T , i.e., the isoeigenvalue expressions $H* |\hat{\psi}\rangle = HT |\hat{\psi}\rangle = E_T |\hat{\psi}\rangle$, or its diagonal elements. In fact, standard isoPauli matrices (7.3) are an explicit realization of 'hidden variables' [15, 27].

These inapplicabilities are important because they permit the (otherwise prohibited) *isotopic completion* of QM, which has resulted to be considerably along the celebrated Einstein–Podolsky–Rosen argument. In fact, one can select a classical isospace such as to permit the identity between the classical and the operator, isotopic versions of Bell's inequality [27]. Similarly the isouncertainties of a particle in the interior of a star collapsing all the way into a gravitational singularity (with $\hat{I} \rightarrow \infty$) recover the classical determinism because, from the isoGaussian (2.10), we have

$$\begin{aligned} \lim_{\hat{I} \rightarrow \infty} \Delta x \Delta k &= \lim_{\hat{I} \rightarrow \infty} \langle \hat{I} \hat{S} \rangle \\ &= \lim_{T \rightarrow 0} \langle \hat{\psi} | TT^{-1}T | \hat{\psi} \rangle \equiv 0. \end{aligned} \quad (7.4)$$

We now illustrate the use of the $S\hat{U}(2)$ symmetry for the reconstruction of the *exact isospin symmetry under weak and ELM interactions*. The mechanism of the

reconstruction is so simple to appear trivial [22]. Consider the isonormalized isostates of (7.3)

$$\begin{aligned} |\hat{\psi}_p\rangle &= \begin{pmatrix} \lambda^{-1/2} \\ 0 \end{pmatrix} & |\hat{\psi}_n\rangle &= \begin{pmatrix} 0 \\ \lambda^{1/2} \end{pmatrix} \\ \langle \hat{\psi}_k | T | \hat{\psi}_k \rangle &= 1 & k &= p, n \end{aligned} \quad (7.5)$$

where $T = \text{diag.}(\lambda, \lambda^{-1})$. We now select such isospace to admit the same masses for the proton and the neutron. This is readily permitted by the 'hidden variable' λ when selected in such a way that $m_p \lambda^{-1} = m_n \lambda$, i.e., $\lambda^2 = m_p/m_n = 0.99862$. The mass operator is then defined by

$$\begin{aligned} \hat{M} &= \left\{ \frac{1}{2} \lambda (m_p + m_n) \hat{I} + \frac{1}{2} \lambda^{-1} (m_p - m_n) \hat{\sigma}_3 \right\} \hat{I} \\ &= \begin{pmatrix} m_p \lambda^{-1} & 0 \\ 0 & m_n \lambda \end{pmatrix} \end{aligned} \quad (7.7)$$

and manifestly represents equal masses $\hat{m} = m_p \lambda^{-1} = m_n \lambda$ in isospace.

The recovering of conventional masses in our physical space is readily achieved via the isoeigenvalue expression on an arbitrary isostate

$$\hat{M} * |\hat{\psi}\rangle = M \hat{I} Q |\psi\rangle = M |\psi\rangle = \begin{pmatrix} m_p & 0 \\ 0 & m_n \end{pmatrix} |\psi\rangle \quad (7.8)$$

or, equivalently, via the isoexpectation values $\langle \hat{\psi}_p | T \hat{M} T | \hat{\psi}_p \rangle = m_p$, $\langle \hat{\psi}_n | T \hat{M} T | \hat{\psi}_n \rangle = m_n$. Similarly, the charge operator can be defined by $Q = \frac{1}{2} e (\hat{I} + \hat{\sigma}_3)$ with charges on isospace $Q_p = e \lambda^{-1}$ and $Q_n = 0$. However, the charges in our physical space are the conventional ones, $\langle \hat{\psi}_p | T Q T | \hat{\psi}_p \rangle = e$, $\langle \hat{\psi}_n | T Q T | \hat{\psi}_n \rangle = 0$. See [22] for more details.

The *isodual isospin* then characterizes the antiparticle \bar{p} and \bar{n} .

8. Isotopies and isodualities of Dirac's equation

We are now sufficiently equipped to review the application of the isoPoincaré symmetry to the isotopies of Dirac's equation, called the *isoDirac equation* [15, 28, 33, 34]. The objective is to generalize the structure of the interactions admitted by the conventional Dirac's equation into their most general possible nonlinear-nonlocal-nonHamiltonian form.

The isolarization of 2nd order invariant (4.7) can be done by introducing the 12-dimensional isospace

$$\{ \hat{M}^{\text{orb.}}(x, \hat{g}, \hat{R}) \times \hat{S}^{\text{intr.}}(2) \} \times \{ \hat{M}^{\text{d,orb.}}(x, \hat{g}^d, \hat{R}^d) \times \hat{S}^{\text{d,intr.}}(2) \}$$

for the characterization of the orbital and intrinsic angular momentum for particles and antiparticles, respectively. The following expression in self-explanatory notation (see [33, 34] for details) then characterizes the *isogamma matrices* $\hat{\gamma}$

$$\begin{aligned} (\hat{\eta}^{\mu\nu} p_\mu *^{\text{orb.}} p_\nu + \hat{m}^2) *^{\text{orb.}} \hat{\psi}(x) \\ \equiv (\hat{\eta}^{\mu\nu} \hat{\gamma}_\mu *^{\text{tot.}} p_\nu + i \hat{m}) *^{\text{tot.}} (\hat{\eta}^{\alpha\beta} \hat{\gamma}_\alpha *^{\text{tot.}} p_\beta - i \hat{m}) *^{\text{tot.}} \hat{\psi}(x) \end{aligned} \quad (8.1a)$$

$$\{ \hat{\gamma}_\mu, \hat{\gamma}_\nu \}^{\text{tot.}} = \hat{\gamma}_\mu T^{\text{tot.}} \hat{\gamma}_\nu + \hat{\gamma}_\nu T^{\text{tot.}} \hat{\gamma}_\mu = 2 \hat{\eta}_{\mu\nu} \hat{I}^{\text{orb.}} \quad (8.1b)$$

$$\begin{aligned} \hat{\gamma}_\mu &= \tilde{\gamma}_\mu \hat{I}^{\text{orb.}} \\ \{\tilde{\gamma}_\mu, \hat{\gamma}_\nu\}^{\text{intr.}} &= \tilde{\gamma}_\mu T^{\text{intr.}} \tilde{\gamma}_\nu + \tilde{\gamma}_\nu T^{\text{intr.}} \tilde{\gamma}_\mu = 2\hat{\eta}_{\mu\nu}. \end{aligned} \tag{8.1c}$$

The above formulation is verified by Dirac's [37] generalization of his equation [33], but is excessively general for our needs here. We shall therefore assume the simpler realization

$$\begin{aligned} \hat{I}^{\text{orb.}} &\equiv \hat{I} = T^{-1} & T^{\text{orb.}} &\equiv T \\ \hat{I}^{\text{spin.}} &= I = \text{diag.}(1, 1) & \{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} &= 2\hat{\eta}_{\mu\nu} \hat{I} \end{aligned} \tag{8.2a}$$

$$\hat{\gamma}^k = b^k \hat{I} \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \quad \gamma^4 = ib^4 \hat{I} \begin{pmatrix} I_s & 0 \\ 0 & I_s^d \end{pmatrix} \tag{8.2b}$$

$$I_s = \text{diag.}(1, 1) \quad I_s^d = -I_s$$

where the γ - and σ -matrices are the conventional ones, and $\partial b_\mu / \partial x^\nu = 0$. One can see the emergence of the isodual isospaces $S^d(2)$ characterized by $I^d = -\text{diag.}(1, 1)$ beginning with the *conventional* Dirac's equation, which then persist under isotopies to $\hat{S}^d(2)$. The isogeometries permit the identification of the origin of the negative-energy solutions precisely in this negative-definite unit. Isoduality then characterizes antiparticles as in ordinary charge conjugation [24]. The desired *isoDirac equation* on $\hat{M}(x, \hat{\eta}, \hat{R})$ can then be written

$$\begin{aligned} (\hat{\gamma}_\mu * p^\mu + i\hat{m}) * \hat{\psi}(x) &= (\hat{\eta}^{\mu\nu} \hat{\gamma}_\mu T p_\nu + i\hat{m}) T \hat{\psi} = 0 \\ \hat{m} &= m \hat{I}. \end{aligned} \tag{8.3}$$

The extension to include electromagnetic potentials is trivial and will be ignored. Experts in isotopies however know that such an addition is not necessary to represent ELM interactions, because they can be equivalently represented with the Lie-isotopic tensor (see vol II of [9]), that is, with the characteristic b -functions [13, 28]. Contrary to its seemingly 'free' appearance, equation (8.3) represents a spinor under the most general known combination of linear and nonlinear, local and nonlocal, as well as potential and nonpotential interactions. Constant b^0 -quantities then represent their average. Note finally the lack of unitary equivalence of the Dirac and isoDirac equations, e.g., because of the lack of existence of a unitary transformation under which $U \hat{\gamma}_\mu U^\dagger = \gamma_\mu$, $\mu = 1, 2, 3, 4$.

The orbital and intrinsic angular momenta of particles with lowest admissible hadronic weight characterize the irregular isorepresentations

$$\hat{O}(3): \quad \hat{L}_k = \epsilon_{kij} r_i p_j \quad [\hat{L}_i, \hat{L}_j] = \epsilon_{ijk} b_k^{-2} \hat{L}_k \tag{8.4a}$$

$$\begin{aligned} \hat{L}^2 * \hat{\psi} &= (b_1^{-2} b_2^{-2} + b_2^{-2} b_3^{-2} + b_3^{-2} b_1^{-2}) \hat{\psi} \\ \hat{L}_3 * \hat{\psi} &= b_1^{-1} b_2^{-1} \hat{\psi} \end{aligned} \tag{8.4b}$$

$$S\hat{U}(2): \quad \hat{S}_k = \frac{1}{2} \epsilon_{kij} \hat{\gamma}_i * \hat{\gamma}_j \quad [\hat{S}_i, \hat{S}_j] = \epsilon_{ijk} b_k^{-2} \hat{S}_k \tag{8.4c}$$

$$\begin{aligned} \hat{S}^2 * \hat{\psi} &= (1/4)(b_1^2 b_2^2 + b_2^2 b_3^2 + b_3^2 b_1^2) \hat{\psi} \\ \hat{S}_3 * \hat{\psi} &= \frac{1}{2} b_1 b_2 \hat{\psi}. \end{aligned} \tag{8.4d}$$

which confirm the existence of nontrivial isorenormalizations.

It is easy to see the existence of the standard isorepresentations which preserve conventional eigenvalues of spin. In fact, the isoDirac equation characterizes the following standard realization of the *isospinorial Poincaré symmetry* $\mathcal{P}(3.1) = S\hat{L}(2.\hat{C}) \times \hat{T}(3.1)$ over the isofield \hat{C} with generators and isocommutators

$$J_{\mu\nu} = \left\{ \hat{S}'_{ij}, \hat{L}'_{k4} \right\} \quad (8.5a)$$

$$\hat{L}'_{k4} = \frac{1}{2} \hat{\gamma}_k * \hat{\gamma}_4 = \frac{1}{2} \{b_k b_4 \gamma_k \gamma_4\} \hat{I} \quad \hat{L}' \equiv L$$

$$\begin{aligned} \hat{S}'_{12} &= b_2^{-1} b_3^{-1} \hat{S}_{12} \\ \hat{S}'_{23} &= b_1^{-1} b_3^{-1} \hat{S}_{23} \quad \hat{S}'_{31} = b_1^{-1} b_2^{-1} \hat{S}_{31} \end{aligned} \quad (8.5b)$$

$$\begin{aligned} [J_{\mu\nu}, \hat{J}_{\alpha\beta}] * |\psi\rangle &= i(\eta_{\nu\alpha} J_{\beta\mu} - \eta_{\mu\alpha} J_{\beta\nu} \\ &\quad - \eta_{\nu\beta} J_{\alpha\mu} + \eta_{\mu\beta} J_{\alpha\nu}) * |\psi\rangle \end{aligned} \quad (8.5c)$$

$$\begin{aligned} [J_{\mu\nu}, \hat{P}_\alpha] * |\psi\rangle &= i(\eta_{\mu\alpha} P_\nu - \eta_{\nu\alpha} P_\mu) * |\psi\rangle \\ [J_\mu, \hat{P}_\nu] * |\psi\rangle &= 0 \end{aligned} \quad (8.5d)$$

which have conventional structure constants, thus coinciding with rules (5.15). The *standard isospinorial and isodual isoPoincarè group* can then be constructed via the rules of section 2.

A simple isotopy of the corresponding conventional derivation, yields the *magnetic and electric isodipole moments* (assumed for simplicity along the third axis)

$$\hat{\mu} = \frac{b_3}{b_4} \mu \quad \hat{m} = \frac{b_3}{b_4} m \quad (8.6)$$

first derived in [26], equations (4.20.16), p 803, and then isotopically reformulated in [15, 33, 34].

9. Experimental verifications

In the preceding sections we have outlined a number of applications of isotopic methods, such as the use of the isoPoincarè symmetry for the invariance of exterior gravitation, or the reconstruction of the exact isospin symmetry in isospace under weak and ELM interactions. In this final section we present a number of phenomenological applications and experimental verifications which, even though evidently preliminary, are nevertheless encouraging and sufficient to warrant additional studies.

1. A first verification is the use of the isoDirac equation for a quantitative representation of Rauch's interferometric measures on the 4π -spinorial symmetry of neutrons (see review [38] and references quoted therein), which do not yield the predicted angle of two spin flips, 720° , but the values $\theta = 715.87^\circ \pm 3.8^\circ$. Even though the deviation is smaller than the error, thus requiring experimental finalization, the measures are significant because the neutron beam of the experiment passes near the intense electric, magnetic and nuclear fields of Mu-metal nuclei placed in the electromagnet gap to reduce stray fields.

The expected physical origin of the measures is therefore a deformation of the charge distribution of the neutrons with consequential necessary (for Maxwell's electrodynamics) alteration of their intrinsic magnetic moments, under the preservation of the conventional spin $\frac{1}{2}$. These conditions are ideal for the isoDirac equation with expressions

$$\begin{aligned} \hat{I} &= \text{diag.} (b_1^{\circ-2}, b_2^{\circ-2}, b_3^{\circ-2}) \\ \hat{\psi}' &= \hat{R}(\theta) * \hat{\psi} = e^{i b_1^{\circ} b_2^{\circ} \gamma_1 \gamma_2 \theta_3 / 2} \hat{\psi} \end{aligned} \quad (9.1)$$

where the first characterizes the nonspherical charge distribution of the neutron, and the second expresses the covering isospinorial transformation. The use of: (a) the general rule for the isorotational symmetry $\hat{\theta}_3 = b^{\circ}_1 b^{\circ}_2 \theta_{3|\theta_3=716^\circ} = 720^\circ$ [13, 21]; (b) the value b°_4 from the geometrization of the $p-\bar{p}$ fireball in the Bose-Einstein correlation [30, 35]; and (c) the proportionality in first approximation $\hat{\mu}/\mu \approx 716^\circ/720^\circ$, where $\mu(\hat{\mu})$ is mutated (conventional) magnetic moment of the neutron, yield the numerical values of the characteristic constants of the isoDirac equation, as an average of the characteristic functions for the neutron in Rauch's experiment

$$b^{\circ}_1 = b^{\circ}_2 \approx 1.0028 \quad b^{\circ}_3 \approx 1.644 \quad b^{\circ}_4 = 1.653. \quad (9.2)$$

The mutated magnetic moments (along the third axis) is given by

$$\hat{\mu} = \mu b^{\circ}_3 / b^{\circ}_4 = -1902 < \mu \quad (9.3)$$

which is smaller than the conventional value thus confirming the 'angle-slow-down' occurred in all Rauch's measures (see [33] for more details). Note that data (9.2) characterize the neutron as an isoMinkowskian medium of type 9, a property verified by all phenomenological data known to date (see below).

2. Another application of the isoDirac equations for a *quantitative representation of the total magnetic moments for few-body nuclear structures*, a problem which has essentially remained unsolved for over half a century. It is essentially based on the isotopy of the conventional QM treatment, that is, on the direct representation of nucleons as extended nonspherical charge distributions which experience deformations under nuclear conditions, thus implying consequential alterations of their intrinsic magnetic moments. In turn, these deviations appear to be the reason for the inability of QM to reach a numerical representation of the total nuclear magnetic moments, despite relativistic and other corrections.

The isoDirac equation permits a direct representation of the actual nonspherical shape of nucleons, predicts their (generally small) deformation when members of a nuclear structure with consequential mutation of their intrinsic magnetic moments, and yields the following HM *model of total nuclear magnetic moments*

$$\begin{aligned} \mu_{\text{tot.}}^{\text{HM}} &= \sum_k \left(\hat{g}_k^{(L)} L_{k3} + \hat{g}_k^{(S)} S_{k3} \right) \\ \hat{g}_k^{(L)} &= 0.605 b^{\circ}_{k3} \hat{g}_k^{(L)} \quad \hat{g}_k^{(S)} = 0.605 b^{\circ}_{k3} \hat{g}_k^{(S)}. \end{aligned} \quad (9.4)$$

where we have the conventional values $g_p^{(s)} = 5.585$, $g_n^{(s)} = -3.816$, $g_p^{(L)} = 1$, $g_n^{(L)} = 0$, $b^{\circ}_4 = 1.653$ as for the neutron and b°_k must be determined from the experimental data. By assuming $L = 0$ for the ground state, by ignoring the contributions from $L = 2$ because they are very small, by recalling that $L = 1$ is unallowed by parity, and assuming that protons and neutrons experience the same deformation, we have the following *numerical representation of the total magnetic moment of the deuteron*

$$\mu_D^{\text{HM}} = 0.605 b^{\circ}_3 (g_p + g_n) \equiv \mu_D^{\text{Exp}} = 0.857 \quad b^{\circ}_3 = 1.611. \quad (9.5)$$

which simply shows that nucleons become oblate in the deuteron with their semiaxis $b^{\circ}_3^{-2}$. The representations of the magnetic moment of tritium, helium and other nuclei are studied in [36].

3. The isoMinkowskian geometrization of the interior of hadrons is confirmed by the phenomenological calculations [39] of deviations from the Minkowskian geometry