International Journal of Modern Physics D, Vol. 7, No. 3 (1998) 351–407

ISOMINKOWSKIAN GEOMETRY FOR THE GRAVITATIONAL TREATMENT OF MATTER AND ITS ISODUAL FOR ANTIMATTER

RUGGERO MARIA SANTILLI *
Institute for Basic Research,
PO Box 1577, Palm Harbor, FL 34682

Received 25 May 1997
Revised 5 November 1997

In a preceding paper at *Foundations of Physics Letters*,[11] we have submitted the apparently first, axiomatically consistent inclusion of gravitation in unified gauge theories of electroweak interactions under the name of *isotopic grand unification*. The result was submitted via an apparent resolution of the structural incompatibilities between electroweak and gravitational interactions due to: (1) *curvature*, because the former are defined on a flat spacetime, while the latter are instead defined on a curved spacetime; (2) *antimatter*, because the former characterize antimatter via negative-energy solutions, while the latter use instead positive-definite energy-momentum tensors; and (3) *basic spacetime symmetries*, because the former satisfy the fundamental Poincaré symmetry, which is instead absent for the latter. The main purpose of this paper is to present the *methods* underlying the isotopic grand unification. We begin with a study of the new mathematics, called *isomathematics*, and of the related new geometry, called *isominkowskian geometry*, which permit an apparent resolution of the first incompatibility due to curvature. We then pass to a study of the second novel mathematics, called *isodual isomathematics*, and related geometry, called *isodual isominkowskian geometry*, which permit an apparent resolution of the second incompatibility due to antimatter. We then pass to a study of the novel realizations of the conventional Poincaré symmetry, known as *Poincaré–Santilli isosymmetry and its isodual*, which provide a universal symmetry of gravitation for matter and antimatter, respectively, and permit an apparent resolution of the third incompatibility due to spacetime symmetries. This paper has been made possible by the preceding: memoir[5g] recently appeared in *Rendiconti Circolo Matematico Palermo*, which achieves sufficient maturity in the new mathematics; memoir[4h] recently appeared in *Foundations of Physics*, which achieves sufficient maturity in the physical realizations of the new mathematics; and memoir[8c] recently appeared in *Mathematical Methods in Applied Sciences*, which achieves sufficient maturity in the formulation of the generalized symmetries. Regrettably, in addition to the study of the methods, we cannot study the novel applications and verifications to prevent a prohibitive length. Nevertheless, the reader should be aware that the isominkowskian geometry and its isodual already possess a number of novel applications and experimental verifications in classical physics, particle physics, nuclear physics, astrophysics, gravitation, superconductivity, chemistry, antimatter, and biology, which are indicated in the text with related references without a review.

*E-mail: ibr@gte.net; http://home1.gte.net/ibr/
PACS 04.60+n, 03.65.-w, 11.10.-z;

1. Introduction

1.1. *Statement of Purpose*

In a preceding paper at *Foundations of Physics Letters*,[11] we have submitted the apparently first, axiomatically consistent inclusion of gravitation in unified gauge theories of electroweak interactions under the name, for certain technical reasons identified shortly, of *isotopic grand unification*.

The result was submitted via an apparent resolution of the structural incompatibilities between electroweak and gravitational interactions due to curvature, antimatter and spacetime symmetries.

The main purpose of this paper is to present the *new methods* underlying the isotopic grand unification. A number of novel applications and experimental verifications of the new methods, even though little known in the physics community, are already available in particle physics, nuclear physics, astrophysics, gravitation, superconductivity, chemistry, antimatter and biology. To avoid a prohibitive length, the latter applications and verifications are merely mentioned with related references, but without any treatment.

1.2. *Foundations of the isominkowskian geometry for matter*

As it is well known, the *special relativity*[1] constitutes one of the most majestic scientific achievements of this century for mathematical beauty, axiomatic consistency and experimental verifications.

By comparison, despite equally historical advances, the physical validity of the *general relativity*[2] has been the subject of considerable debates throughout this century which have remained lingering to this day.

For this reason, in preceding papers,[3] we have proposed a *geometric unification of the special and general relativities via the axioms of the special, rather than of the general*, resulting in a formulation we have called *isospecial relativity*[3] which is based on the following main lines.

Let $M = M(x, \eta, R)$ be the conventional $(3+1)$-dimensional Minkowski space with local coordinates $x = \{x^\mu\} = \{r, c_0 t\}$, $\mu = 1,2,3,4$, where c_0 is the speed of light in vacuum with familiar metric $\eta = \mathrm{diag}(1,1,1,-1)$ and unit $\mathrm{I} = \mathrm{diag}(1,1,1,1)$ over the reals $R = R(n,+,\times)$. Let $\mathcal{R} = \mathcal{R}(x, g, R)$ be a conventional $(3+1)$-dimensional Riemannian space with the same local chart, nowhere singular, real-valued and symmetric metric $g = g(x)$ and unit I.

The main idea of the isominkowskian representation of gravity, submitted by the author at the *VII Marcel Grossmann meeting on general Relativity*,[3a] is that the component of $g(x)$ truly representing curvature is the 4×4-dimensional matrix $\hat{T}(x)$ in the Minkowskian factorization

$$g(x) = \hat{T}(x) \times \eta\,, \tag{1.1}$$

where one should keep in mind that $\hat{T}(\mathrm{x})$ is necessarily positive-definite (from the local Minkowskian character of $\mathcal{R}$). If the Riemannian space is reconstructed with respect to a new unit which is the inverse of $T(x)$,

$$\hat{I}(x) = [\hat{T}(x)]^{-1}, \tag{1.2}$$

then it verifies the axioms of the *Minkowski* space, despite the functional dependence of the metric, as first proved in Ref. 4a. The emerging new space was then called the *isominkowskian space*, namely, a structure belonging to the field of the so-called axiom-preserving *isotopies*[5] (see also Refs. 6–11).

The reader should be aware that the consistent treatment of the above gravitation requires a *new mathematics*, called *isomathematics*[4] and outlined in Sec. 2, which essentially consists in the reconstruction of conventional numbers, spaces, algebras, etc. with respect to the generalized unit $\hat{I}$.

Any appraisal of this paper via conventional mathematics is afflicted by a number of inconsistencies which often remain undetected by nonexperts. Similarly, attempts to appraise the studies herein presented with other approaches existing in the literature (such as the use of quaternions, spinors, Vierbein frames., Clifford algebras, etc.) have hidden inconsistencies which generally remain undetected by nonexperts (e.g., the generalized field used in this study is fully commutative and, therefore, has no connection with quaternions; no spinor of any type is used in the analysis; Vierbein frames are strictly formulated on conventional spaces and fields, etc.

An effective way to identify the difference between this studies and preceding ones is by looking at the *fundamental numbers* on which the theory is built, the former are built on *generalized numbers*, while the latter are built on *conventional numbers*.

Needless to say, we are not suggesting here the preference of one versus another approach because true science is based on polyedricity of approaches, each one generally having only a grain of truth. As a matter of fact, the connections between our isotopic formulation of gravity and other studies is intriguing. We only regret to be forced to defer their study to subsequent papers to avoid excessive length of this study.

As it is well known, general relativity incorporates the special as a particular case as well as in the local tangent spaces and, in this sense, it provides a unification of gravitational and relativistic phenomena.

The isominkowskian reformulation of gravity permits a novel geometric unification of the general and special relativities which are now differentiated via the basic *unit* of the theory, rather than the geometry. In fact, for value (1.2) we have the general relativity with all possible conventional Riemannian metrics $g(x)$, while for the value $\hat{I} = \mathrm{diag}(1, 1, 1, 1)$ we have the special relativity. The property here important is that *the abstract geometric axioms remain unchanged in the reduction* $\hat{I} \to I$.

In short, the main geometric objective of this paper is the reformulation of the *Riemannian* geometry into a *Minkowskian* form while preserving all conventional metrics $g(x)$ unchanged with consequential preservation of the conventional Einstein's field equations and related experimental verifications. This seemingly contradictory approach is permitted by the fact that *the referral of the metric*

$g(x) = \hat{T}(x) \times \eta$ to the unit $\hat{I} = 1/\hat{T}(x)$ which is the inverse of the "gravitational term" $\hat{T}(x)$ eliminates the conventional notion of curvature in favor of a covering notion of isocurvature studied in Sec. 2.

In this paper we present, apparently for the first time, the foundations of the isominkowskian geometry underlying the isotopic grand unification, which are apparently expressed, as a symbiosis of the Minkowskian and Riemannian geometries, thus including aspects pertaining to the preservation of the Minkowskian axioms, plus the machinery of the Riemannian geometry, such as covariant derivative, connection, etc. In preceding works[3–5] we had studied the *Minkowskian* profile of the new geometry and, separately, the isotopies of the Riemannian geometry.

1.3. *Foundations of the Poincaré–Santilli isosymmetry for matter*

Recall that the special relativity has a universal spacetime symmetry, the fundamental Poincaré symmetry $P(3.1)$, while no corresponding *symmetry* exists for conventional gravitational theories (where we have only "covariance").

One of the primary purposes for constructing the novel isominkowskian geometry and underlying isomathematics has been the resolution of the above disparity and the achievement of a universal *symmetry* (rather than covariance) of all possible *Riemannian* line elements, first studied by this author in details in Refs. 4 under the name of *isopoincaré symmetry* $\hat{P}(3.1)$ and today called *Poincaré–Santilli isosymmetry* (see, e.g., Refs. 6b–6e and paper quoted therein, as well as the recent memoir by Kadeisvili[8c]).

The isosymmetry $\hat{P}(3.1)$ is essentially the reconstruction of the conventional symmetry in such a form to admit the generalized unit $\hat{I}(x)$ at all levels, including enveloping algebras, Lie algebras, Lie groups, transformations and representation theories, etc. Since $\hat{I} > 0$ as noted earlier, $\hat{P}(3.1)$ results to be isomorphic to $P(3.1)$ *ab initio. Equivalently, we can say that* $\hat{P}(3.1)$ is not a "new symmetry", but only a "new realization" of the abstract Poincaré axioms, evidently of nonlinear character.

A primary task of the isominkowskian formulation of gravity is to establish that the fundamental spacetime symmetry of the special relativity does indeed remain exact for gravitation, when the nonlinearity is realized in a proper isotopic way.

The difference between the isotopic treatment of nonlinear symmetries and the nonlinear approaches of affine transformations[16a,16b] should be kept in mind. In fact, the latter are *bona-fide* nonlinear theories, while the former reconstruct linearity on isospaces over isofields and exhibit a nonlinear structure only in their *projection* back to conventional spaces over conventional fields.

The achievement of a universal symmetry of gravitation is perhaps the most compelling individual motivation for the novel isominkowskian geometry. In fact, on one side, no symmetry of gravitational line elements can be generally constructed in the conventional Riemannian geometry, as well known. On the other side, the reduction of gravitation to a primitive symmetry, not only permits the unification of gravitation and electroweak interactions, but also implies a number of rather intriguing novel implications and predictions.

1.4. *Applications, verifications and predictions of the isominkowskian geometry*

As indicated earlier, the most important application of the isominkowskian geometry of this paper is that for the achievement of a consistent inclusion of gravitation in unified gauge theories of electroweak interactions. The reader should however be aware of the existence of numerous, additional applications and verifications.

Firstly, it has been recently proved that[4h] the *Riemannian geometry, as well as all geometries with a nonnull curvature, do not possess invariant units of space and time*, thus having evident problematic aspects in their applications to actual measurements. The regaining of the Minkowskian axioms assures the resolution of this problematic aspect because its unit I is notoriously invariant, and the same holds for its generalization $\hat{I}$ (see Ref. 4h for details).

Secondly, the unification of the special and general relativities permits the resolution of a number of historical controversies in gravitation,[3f] such as the controversy on whether or not the total conservation laws of general relativity are indeed compatible with those of the special relativity, or the controversy on the lack of a meaningful *relativistic* (rather than Euclidean) limit of gravitation. The geometric unification here considered resolves these controversies in the affirmative way. For instance, the mere visual inspection that the *generators of the Poincaré symmetry and its isotopic image coincide*, establishes the compatibility of the total conservation laws in the two relativities (because the generators of spacetime symmetries are the total conserved quantities). Similarly, the simple limit $\hat{I}(x) \to I$ establishes a meaningful *relativistic* limit of gravitation. A number of other controversies appear to be also resolvable by the isotopic unification, although they evidently require specific studies.

Thirdly, despite attempts throughout this century, the problem of the quantum version of gravity is far from being solved because of rather serious physical shortcomings due to the *nonunitary* structure of the theory, such as:[4h] lack of conservation of the basic units of space and time under the time evolution of the theory with consequential lack of unambiguous applications to experiments; lack of conservation in time of the original Hermiticity, with consequential lack of physically admissible observables; lack of uniqueness and invariance of the numerical predictions; and other problems.

Another important reason for the original proposal of the isotopic unification of the general with the special relativity was that of permitting a basically novel operator form of gravity which verifies the abstract axioms of conventional *relativistic* quantum mechanics and which could therefore resolve the above problematic aspects by conception. Intriguingly, the point of view conveyed by these studies[3] is that a *consistent operator theory of gravity may well have always existed. It did creep in unnoticed because embedded where nobody looked for: in the unit of relativistic quantum mechanics.*

Fourthly, it has been known throughout this century (beginning with E. Cartan) that *the Riemannian geometry has clear limitations for interior gravitational* *problems in general and gravitational collapse in particular.* In fact, a collapsing star is not made up of ideal points, but rather of extended and hyperdense hadrons in conditions of total mutual penetration and compression in large numbers into a small region of space. It is evident that these physical conditions are arbitrarily nonlinear, not only in coordinates, but also in velocities and wavefunctions, as well as arbitrarily nonlocal and not derivable from a first-order Lagrangian (*variationally nonselfadjoint*[5e]), thus being dramatically beyond the representational capabilities of the Riemannian geometry.

The unification here considered resolves this additional limitation. In fact, studies[4g] have indicated that, under the isominkowskian reformulation, the metric must remain well behaved, real valued and symmetric, but its functional dependence becomes unrestricted, i.e.,

$$g(x, \dot{x}, \ldots) = \hat{T}(\dot{x}, \dot{x}, \ldots) \times \eta\,, \qquad \hat{I} = 1/T(x, \dot{x}, \ldots)\,, \tag{1.3}$$

thus permitting a more adequate and direct geometric treatment of interior gravitational problems. This permits the otherwise impossible applicability of Einstein's *exterior* axioms to *interior* gravitational problems with a number of developments currently under study, e.g., a reinspection of the theorems on singularities via the inclusion of nonlinear, nonlocal and nonlagrangian effects. In fact, gravitational singularities now become the zeros of the generalized units $\hat{I}(x, \dot{x}, \ldots) = 0$.[4g] Note that the latter occurrence too is precluded for the conventional formulation of gravity.

Fifth, the ultimate pillar of the special relativity is the "direct geometrization" (i.e., geometrization via the *metric*) of the *speed of light in vacuum* c_0, which is then turned into a universal invariant by the Poincaré symmetry. Both the special and general relativities in their current formulation are unable to provide a direct representation of the *local variation of the speed of electromagnetic waves in interior physical problems.* In fact, it is known since Lorentz[12a] times (see also Pauli's [1e] quotation of this study by Lorentz) that the speed of electromagnetic waves in our atmosphere is *smaller* than that in vacuum, $c = c_0/n < c_0$. Moreover, photons traveling in certain guides with speeds *bigger* than the speed in vacuum, $c = c_0/n > c_0$, have been apparently measured;[12b,12c] recent astrophysical measurements[12d–12f] have indicated the apparent existence of jets of matter expelled by astrophysical bodies at speed higher than c_0; and solutions of ordinary relativistic equations with *arbitrary* speeds have been recently identified[12g] (see also Recami's[12h] reviews of experimental evidence on superluminal speeds).

As a result, a fundamental insufficiency of contemporary relativities is their inability to provide a direct geometrization of arbitrary local speeds of electromagnetic waves. This inability has implied consequential physical shortcomings, e.g., the use of the speed of light *in vacuum* c_0 in the exterior of *gravitational horizons* (which are known to be filled up with hyperdense chromosphere, thus having local speeds $c = c_0/n$ different than c_0), or in the huge chromospheres of quasars (thus hiding the possible treadshift due to their decrease), and others.

Another important objective for which the isotopies of the Minkowskian geometry, the Poincaré symmetry and the special relativity were proposed[4] has been the achievement of a *direct geometrization and universal symmetry for speeds of electromagnetic waves of arbitrary local values, irrespective of whether smaller, equal or bigger than* c_0. As we shall see (Sec. 2), this objective is readily achieved by assuming a diagonal realization of $\hat{I}$ with $\hat{I}_{44} = 1/n_4^2$ which yields in the 4-4 line element of the metric with the term $c^2 = c_0^2/n_4^2$, where n_4 is the ordinary index of refraction. Thus, the isominkowskian geometry, the isopoincaré symmetry and the isospecial relativity do indeed provide the desired direct geometrization and universal invariance of arbitrary local speeds $c = c_0/n_4$ of electromagnetic waves.

It should be mentioned that the now old theory of refractive index is insufficient for a deeper understanding of the local variation of the speed of electromagnetic waves on numerous grounds. First of all, the theory reduces the event to photons scattering among molecules, thus lacking a geometric representation of the *general inhomogeneity* and *anisotropy* of the medium in which the waves propagates. The latter characteristics are directly representable with the isominkowskian geometry, that is, representable via the *metric itself*, and predict a new contribution to the Doppler shift suitable for experimental verifications[4g] (see also Sec. 2.14). Moreover, the inability of the conventional theory of refractive index of representing speeds of electromagnetic waves bigger than that in vacuum is beyond scientific doubt, thus establishing the need for *new* approaches.

As we shall see, the isospecial relativity[4] can be defined as a theory providing a unified representation of relativistic and gravitational, exterior and interior and classical or operator dynamical problems of matter under the universal isopoincaré symmetry, in such a way to coincide at the abstract level with the special relativity. This implies in particular that the maximal causal speed on isospaces over isofields remains the value c_0, while locally varying speeds $c = c_0/n_4$ emerge in the *projection* into our conventional spacetime. For this reason, the isospecial relativity has been indicated (see, e.g., Refs. 8 and 9) has turning the special relativity into a form which is "directly universal," i.e., applicable for all exterior and interior systems considered (universality) directly in the frame of the observer (direct universality). Note also that the isospecial relativity is the only known theory which renders the axioms of the conventional special relativity compatible with speed of light *bigger* than that in vacuum, other approaches evidently implying their violations

Besides the above applications of direct relevance for this memoir, the reader should be aware that the isominkowskian geometry, the Poincaré–Santilli isosymmetry and the isospecial relativity, already have a number of experimental verifications, such as:

(1) An exact isominkowskian fit[9a] of the experimental data on the behavior of meanlife of the K° particle with energies from 30 to 400 GeV, where the Minkowskian anomaly is predicted from expected internal nonlocal effects under a conventional behavior of the center-of-mass;

(2) An exact isominkowskian fit[9b] of the experimental data on the Bose-Einstein correlation for the two-point-isocorrelation function[9c] deriving the correlation from the nonlocality of the $p - \bar{p}$ fireball from first axiomatic principles without *ad hoc* "semiphenomenological approximations" with unknown parameters, and by reconstructing the exact Poincaré symmetry in isospace under nonlocal interactions;

(3) An exact confinement of quarks on isominkowskian spaces[9d] (i.e., a confinement with an identically null probability of tunnel effects) even in the absence of a potential barrier, which is quite simply permitted by the isotopies due to the incoherence of the internal and external Hilbert spaces, under conventional unitary symmetries, conventional quantum numbers and conventional experimental data on mass spectra;

(4) An exact representation on isominkowskian space of the synthesis of the neutron as occurring in stars at their formation, from protons and electrons *only*,[4e] which has been unable to represent the totality of the characteristics of the neutron;

(5) The apparently first exact representation of total nuclear magnetic moments[9e] under conventional quantum axioms and physical laws, representing the 1% of experimental data which has been missed in nuclear physics throughout this century despite all possible relativistic corrections;

(6) An exact reconstruction of the SU(2) isospin symmetry[4c] with equal masses for protons and neutrons in isominkowskian space and physical masses in conventional spaces;

(7) An exact representation of the large difference in cosmological redshifts between quasars and their associated galaxies when physically connected according to spectroscopic evidence;[9f]

(8) An exact isominkowskian representation of the internal quasars redshift and blueshift;[9g]

(9) The achievement of the apparently first *attractive force* between the *two identical electrons* of the Cooper pair in superconductivity in excellent agreement with experimental data;[9h]

(10) The apparently first achievement of *explicitly attractive forces* between the *neutral atoms* of molecular bonds in chemistry[9i] capable of representing the known 2% of experimental data which has been missing by quantum chemistry through this century;

(11) The apparently first capability of representing the main characteristics of biological structures, such as their irreversibility, time-rate-of-variations of sizes and shapes, etc.;[9j] and others.

To achieve a technical understanding of the novel isominkowskian geometry, the reader is suggested to verify the extreme difficulties, if not the impossibility of achieving the same results with a theory based on conventional numbers.

Some of the novel predictions of the isotopic grand unification and underlying isominkowskian geometry are the following:

(a) The prediction that the speed of electromagnetic waves is a local quantity which can be arbitrarily smaller or bigger than the speed in vacuum depending on local conditions;[4g]
(b) The prediction that the inhomogeneity and anisotropy of the media in which light propagates has a new measurable contribution to the Doppler's red- or blue-shift;[4g]
(c) The prediction of a new isocosmology which is characterized for the first time by a universal symmetry, the Poincaré–Santilli isosymmetry, without the need for the "missing mass", a direct geometrization of the anisotropy and inhomogeneity in the propagation of light in the universe and other features;[4g]
(d) The prediction of a new geometric propulsion called *isolocomotion*[4g] in which motion occurs via the reduction of distances due to very large local amounts of energy without any Newtonian propulsion;
(e) A new notion of spacetime in which the novelty rests in its basic *units*, thus implying local notions of space and time different than those of conventional relativities.

An understanding of this paper requires the knowledge that all the above applications, experimental verifications and predictions are dependent on the central feature of the new geometry, the generalization of the basic unit, from the trivial values $+1$ or $I = \text{diag}(1,1,1,1)$ of the current literature to positive-definite 4×4 matrices whose elements have an unrestricted nonlinear integro-differential dependence on local quantities.

Another important property for the understanding of this paper is that *the isominkowskian geometry has been proved to be "directly universal"*,[9l] i.e., capable of representing all infinitely possible alterations of the Minkowski metric (universality), directly in the fixed x-frame of the observer (direct universality). As such, the new geometry applies even when not desired.

Moreover, the isominkowskian representation of Minkowskian anomalies is the *only one capable of preserving the abstract Einsteinian axioms and related special relativity.* In fact, the use of other methods based on conventional mathematics, such as the so-called q-deformations, imply the lack of isomorphism between the deformed and Minkowskian spaces, with consequential violation of the special relativity.

An understanding of this paper therefore requires an understanding of the *necessity* of the new isomathematics for the invariant preservation of the Einsteinian axioms under novel non-Einsteinian conditions.

1.5. *Foundations of the isodual isominkowskian geometry for antimatter*

All the preceding studies are restricted to the sole treatment of *matter*. In fact, their use for the representation of antimatter is known to be afflicted by a number of problematic aspects. The first is the structural incompatibility between gravitation and electrieoweak interactions, the former representing antimatter with positive-definite energy-momentum tensors, while the latter representing antimatter via negative-energy solutions of fields equations.

In essence, antimatter is nowadays solely represented at the classical level via a change of the sign of the charge. This is grossly insufficient to represent physical reality because antimatter is known to be an *antiautomorphic* image of matte, as it is the case for charge conjugation in *operator* theories. Moreover, the operator image of the above *classical* representation is *not* the appropriate *charge conjugate state*, evidently because the conventional treatment has only one channel of quantization. It then follows that the conventional classical treatment of antimatter is incompatible with its operator counterpart. This yields an intrinsic structural inconsistency in the current representation of antimatter which is independent from that with electroweak interactions.

At any rate, antimatter was discovered, and it is still identified nowadays, in the *negative-energy solutions* of relativistic equations, currently treated at the level of second quantization. It is evident that, for consistency, antimatter should be treated at the classical level too with negative-energy representations.

In the search for the resolution of the above shortcomings, this author submitted in Ref. 4b, and then studied in Refs. 13, a novel map which is as antiautomorphic as charge conjugation but which, unlike the latter, is applicable at *all* levels of study beginning with *Newton's* equations, and then persists at the subsequent analytic and quantum levels.

The new map, called *isoduality* for certain technical reasons, is given by the conjugation of all possible quantities $Q(x, \dot{x}, \ldots)$ characterizing matter (and their operations) into their *anti-Hermitean* form [loc. cit.]

$$Q(x, \dot{x}, \ldots) \to Q(x, \dot{x}, \ldots)^d = -Q(x, \dot{x}, \ldots)^\dagger , \tag{1.4}$$

under which *the generalized gravitational unit of the isominkowski space becomes negative-definite*

$$\hat{I}(x) = 1/\hat{T}(x) \to \hat{I}(x)^d = -\hat{I}(x)^\dagger = -\hat{I}(x) = -1/T(x) . \tag{1.5}$$

Since the norm of numbers with a negative unit is negative-definite, *all quantities of matter change sign under isoduality, including the charge, as well as energy-momentum tensor and all other characteristics.* In this way we regain equivalence in the use of negative-energy for antimatter at each of the classical and quantum levels.

Isoduality also applies to quantization, yielding a novel *isodual quantization*,[13c] under which map (1.4) results to be equivalent to charge conjugation.

Isoduality also resolves Dirac's historical problem of the unphysical behavior of negative energies and time, which lead to the formulation of the "hole theory". In fact, particles with *negative energy and time* referred to *negative units* behave in exactly the same physical way as particles with *positive energy and time* referred to *positive units* (see Refs. 13 for details).

The reader should be aware that the isodual theories see their ultimate origin in the structure of the *conventional* Dirac equations. In fact, isodual units

$I^d = -\text{diag}(1,1)$ and isodual Pauli's matrices $\sigma^d{}_k = -\sigma^\dagger{}_k = -\sigma_k$ are essential for the very definition of Dirac's gamma matrices. This implies a novel interpretation of the convention Dirac equation, which permits the treatment of antiparticles in *first* quantization, as expected in a theory beginning at the Newtonian level.

Second quantization is done in the isodual theory of antimatter for exactly the same reasons holding for particles. In this level, the theory results to be characterized by *advanced solutions* which are now reinterpreted as belonging to isodual spaces over isodual fields.

The use of *generalized and negative-definite units* has requested the construction of another new mathematics known under the name of *isodual isomathematics.* This includes still new numbers, new spaces, new algebras, new geometries, etc., which are the antiautomorphic image of the corresponding isotopic quantities under isoduality.

The reader should also be aware that *isodualities apply also to conventional mathematics*, yielding a particular case of the isodual isomathematics called *isodual mathematics.* Even though quite simple, as we shall see, the latter is mathematically nontrivial because it focuses the attention on the fact that our current entire mathematical knowledge, is restricted to the simplest possible unit $+1$, and, as such, it is not applicable to a consistent classical treatment of antimatter.

The reader should be aware that the appraisal of the novel theory of antimatter via the conventional mathematics also leads to a number of inconsistencies which often remain undetected by nonexperts. For instance, the use of conventional negative numbers has no mathematical or physical meaning for isodual theories, evidently because their unit is the trivial number $+1$. The only applicable numbers are instead those for which the unit is, first, generalized, and, then, negative-definite, as studied in Sec. 3.

Another important objective of this paper is therefore that of outlining the novel isodual isomathematics and presenting, apparently for the first time, the foundations of the *isodual isominkowskian geometry* for the gravitational treatment of antimatter.

1.6. *Foundations of the isodual Poincaré-Santilli isosymmetry*

Yet another fundamental tool for the axiomatically consistent inclusion of gravitation in unified gauge theories is the *isodual Poincaré–Santilli isosymmetry* $\hat{P}^d(3.1)$, first introduced in Ref. 4d and in various other works (see monographs[4g] and Kadeisvili's[8c] recent study), which is essentially given by the isodual image of the isosymmetry $\hat{P}(3.1)$, that is, its reconstruction with respect to negative-definite isounits (1.5) at all levels, including enveloping algebras, Lie algebras, Lie groups, transformation and representation theories, etc.

Yet another important notion we shall study in this paper is that of *isoselfduality,*[4c] which is the novel invariance under isoduality,

$$Q(tx, \dot{x}\ldots) \to Q^d(x^d, \psi^d, \ldots) = -Q^\dagger \equiv Q(x, \dot{x}, \ldots), \tag{1.6}$$

as it is the case for the imaginary quantity $i \rightarrow = i^d = -i^\dagger \equiv i$ and for the Dirac gamma matrices $\gamma_\mu \rightarrow {\gamma_\mu}^d = -{\gamma_\mu}^\dagger \equiv \gamma_\mu$.

The main structures permitting the axiomatically consistent inclusion of gravitation in unified gauge theories are therefore given by the isoselfdual isominkowskian spaces

$$\hat{M}_{\rm Tot} = \{\hat{M}_{\rm Orb} \times \hat{S}_{\rm Spin}\} \times \{\hat{M}^d{}_{\rm Orb} \times^d \hat{S}^d{}_{\rm Spin}\} = \hat{S}^d{}_{\rm Tot}\,, \tag{1.7}$$

characterized by the following isoselfdual total isounit

$$\hat{I}_{\rm Tot} = \{\hat{I}_{\rm Orb} \times \hat{I}_{\rm Spin}\} \times \{\hat{I}^d{}_{\rm Orb} \times^d \hat{I}_{\rm Spin}\} = \hat{I}^d{}_{\rm Tot}\,, \tag{1.8}$$

under the universal, isoselfdual Poincaré–Santilli isosymmetry

$$\hat{S}_{\rm Tot} = \hat{\mathcal{P}}(3.1) \times \hat{\mathcal{P}}^d(3.1) = \hat{S}^d{}_{\rm Tot}\,, \tag{1.9}$$

where $\mathcal{P}(3.1)$ is the spinorial covering of the Poincaré symmetry $P(3.1)$, $\hat{\mathcal{P}}(3.1)$ its isotopic covering and $\hat{\mathcal{P}}^d(3.1)$ its isodual.

The reader should be aware that, contrary to popular belief, the correct symmetry of the conventional Dirac equation is not $\mathcal{P}(3.1)$, but $\mathcal{P}(3.1) \times \mathcal{P}^d(3.1)$. In fact, Dirac's gammas are isoselfdual and, as such, they cannot possibly admit the symmetry $\mathcal{P}(3.1)$ which is not isoselfdual, while the novel invariance under isoduality is indeed verified by the broader symmetry $\mathcal{P}(3.1) \times \mathcal{P}^d(3.1)$. In turn, this true invariance of Dirac's equation is the ultimate origin for the full equivalence in the treatment of particles and antiparticles in first quantization.

1.7. *Applications, verifications and predictions of the isodual isominkowskian geometry*

The primary application of the isodual isominkowskian geometry of this paper is the achievement of axiomatic consistency for the inclusion of gravitation in unified gauge theories for the specific case of antimatter.

The above application requires, by conception, that the theory of antimatter must begin at the purely *classical* level, and then admits consistent operator image, thus characterizing antiparticles via negative energy and time. The isodual isominkowskian geometry provides the only known axiomatically consistent representation of antimatter verifying these classical and operator requirements. However, isodual theories also have numerous other applications we cannot review here for brevity.[4g]

The available experimental verification of the isodual theory of antimatter is simply overwhelming at both classical and operator levels. In fact, the theory recovers all available experimental data on antimatter *at both the classical and operator levels*[13] which, as well known, consist at this writing of all known interactions less gravitation.

The predictions of the isodual theory of antimatter are far reaching. They solely deal with *gravitational* interactions and can be summarized as follows:

(1) The prediction that the antihydrogen atom and antimatter at large, emit a new photon, called by the author *isodual photon*[13c] which is indistinguishable from the ordinary photon for all interactions except gravitation;
(2) The prediction that stable antiparticles, such as the isodual photon, the isodual electron (positron) and the isodual proton (antiproton) experience *antigravity* in the field of matter (defined as the reversal of the sign of the curvature tensor),[13] in a way which avoid known objections (e.g., because the positronium is predicted to experience attraction in both fields of matter and antimatter);
(3) The prediction of a *mathematical spacetime machine*,[13e] i.e. the capability to perform a closed loop in the forward light cone, in full agreement with causality, e.g., because known objections against motion backward in time are inapplicable when dealing with a *negative unit of time*.
(4) The prediction of a novel *isoselfdual cosmology* which, in addition to the lack of need of the "missing mass" indicated earlier, has *null* total characteristics of mass, energy, angular momentum, time, etc.
(5) The prediction (from quantitative representations of bifurcations) that biological structures such as sea shells have an internal isoselfdual geometry dramatically more complex than that perceived by our premitive sensory perception;[9j] and other predictions.

The understanding of this paper requires a technical knowledge that all applications, verifications and predictions of the isodual isominkowski geometry are centrally dependent, again, on the assumed *generalized unit* which, this time, is *negative-definite*, thus resulting in a novel geometry.

2. Isominkowskian Geometry for the Representation of Matter

2.1. *Introduction*

The basic notion of this paper, that of *isotopies*, is rather old. As Bruck[5a] recalls, the notion can be traced back to the early stages of set theory where two *Latin squares* were said to be *isotopically related* when they can be made to coincide via permutations. Since Latin square can be interpreted as the multiplication table of quasigroups, the isotopies propagated to quasigroups, then to algebras and more recently to most of mathematics. As an illustration, the isotopies of Jordan algebras were studied by McCrimmon.[5b]

In this paper, we shall use the isotopies of the unit, fields, spaces, differential calculus, functional analysis, algebras, geometries and analytic mechanics introduced by Santilli.[5c–5h] In their latest formulation,[5g] isotopies are *maps (also called liftings) of any given linear, local, canonical or unitary structure into its most general possible nonlinear, nonlocal, noncanonical or nonunitary extensions which are nevertheless capable of reconstructing linearity, locality, canonicity or unitarity in certain generalized spaces and fields called isospaces and isofields, respectively* (see below). As such, isotopic liftings are *axioms-preserving* by conception and construction, namely, the isotopic structures must be locally isomorphic to the original structures as a necessary condition for an isotopy.

Independent studies on isotopies can be found in Refs. 6–10. A comprehensive literature on isotopies up to 1984 can be found in Tomber's bibliography[6a] while subsequent references can be found in the recent monograph by Lõhmus, Paal and Sorgsepp.[6e]

It should be stressed that the "isotopies" are *inequivalent* to the various forms of "deformations" of the current literature (see, e.g., Ref. 14 and papers quoted therein) for several reasons, such as:[4h] the former are defined via generalized units, while the latter use conventional units; the former are axiom-preserving while the latter are not; the former are defined over generalized spaces and fields while the latter use conventional spaces and fields; etc. To avoid confusion, readers are discouraged from using the term "deformations" (of a given structure into a nonisomorphic form) when referring to the "isotopies" (of the same structures into axiom-preserving isomorphic forms).

We shall hereon assume the convention, rather familiar in the literature on isotopies, that all quantities with a "hat" are computed in isospaces over isofields, and the corresponding quantities without a "hat" are computed on conventional spaces over conventional fields.

A viewpoint we would like to convey in this paper is that there cannot be really novel physical advances without really novel mathematics, and there cannot be really novel mathematics without new numbers. The primary research efforts by this author has therefore been the search for *new numbers* from which everything else uniquely follows via mere compatibility arguments.

2.2. *Isotopies of the unit*

The fundamental isotopies are those of the *unit*[5f,5g] i.e., the liftings of the n-dimensional unit $I = \mathrm{diag}(1, 1, 1, \ldots)$ of conventional vector or metric spaces into real-valued and symmetric $n \times n$ matrices $\hat{I} = (\hat{I}^{\mu}{}_{\nu}) = \hat{I}^t$ whose elements $\hat{I}^{\mu}{}_{\nu}$ have an unrestricted functional dependence in the coordinates x, velocities $v = dx/dt$, accelerations $a = dv/dt$, local density μ, local temperature τ, and any needed other characteristics of the problem considered, first introduced by Santilli[5c] back in 1978,

$$I \to \hat{I} = (\hat{I}^{\mu}{}_{\nu}) = \hat{I}(x, v, a, \mu, \tau, \ldots) = \hat{I}^t \,. \tag{2.1}$$

The above liftings were classified by Kadeisvili[8a] into: **Class I** (generalized units that are nondegenerate, Hermitean and positive-definite, characterizing the *isotopies* properly speaking); **Class II** (the same as Class I although $\hat{I}$ is negative-definite, characterizing the so-called *isodualities*); **Class III** (the union of Class I and II); **Class IV** (Class III plus the zeros of the generalized unit, $\hat{I} = 0$); and **Class V** (Class IV plus unrestricted generalized units, e.g., realized via discontinuous functions, distributions, lattices, etc.).

All isotopic structures identified below also admit the same classification. Unless otherwise stated, in this section we shall study isotopies of Classes I while the isodualities will be studied in Sec. 3. The isotopies of Classes III, IV and V are vastly unexplored at this writing.

2.3. *Isofields*

The first significant application of the isotopies of the unit is that for the liftings of conventional numbers and fields. Let $F = F(a, +, \times)$ be a field (hereon assumed to have characteristic zero) with elements $a, b, \ldots$, sum $a+b$, conventional (associative) multiplication hereon denoted $ab = a \times b$, additive unit 0, multiplicative unit 1, and familiar properties $a + 0 = 0 + a = a$, $a \times 1 = 1 \times a = a$, $\forall a \in F$, etc. We have in particular the field $R(n, +, \times)$ of *real numbers* n, the field $C(c, +, \times)$ of *complex numbers* c, and the field $Q(q, +, \times)$ of *quaternions* q.

Definition 2.1:[5f] *An "isofield" $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ is a ring with elements $\hat{a} = a \times \hat{I}$, called "isonumbers", where $a \in F$, and $\hat{I}$ is a Class I quantity generally outside F, equipped with two operations $(\hat{+}, \hat{\times})$, the "isosum" $\hat{a}\hat{+}\hat{b} = (a + b) \times \hat{I}$, with conventional additive unit $\hat{0} \equiv 0$, and the "isomultiplication"*[5c]

$$\hat{a}\hat{\times}\hat{b} = \hat{a} \times \hat{T} \times \hat{b}, \tag{2.2}$$

where $\hat{T}$ is nowhere singular such that $\hat{I} = \hat{T}^{-1}$ is the left and right unit of $\hat{F}$,

$$\hat{I}\hat{\times}\hat{a} = \hat{a}\hat{\times}\hat{I} \equiv \hat{a}, \ \forall \hat{a} \in \hat{F}, \tag{2.3}$$

in which case (only), $\hat{I}$ is called the "isounit" and $\hat{T}$ is called the "isotopic element". Under these assumptions $\hat{F}$ is a field, i.e., it satisfies all properties of F in their isotopic form:

(1) The set $\hat{F}$ is closed under isosum, $\hat{a}\hat{+}\hat{b} \in \hat{F}$, $\forall \hat{a}, \hat{b} \in \hat{F}$.
(2) The isosum is commutative, $\hat{a}\hat{+}\hat{b} = \hat{b} + \hat{a}$, $\forall \hat{a}$, $\hat{b} \in \hat{F}$,
(3) The isosum is associative, $\hat{a} + (\hat{b}\hat{+}\hat{c}) = (\hat{a}\hat{+}\hat{b})\hat{+}\hat{c}$, $\forall \hat{a}, \hat{b}, \hat{c}, \in F$,
(4) There is an element $\hat{0} \equiv 0$, the "additive isounit", such that $\hat{a}\hat{+}\hat{0} = \hat{0}\hat{+}\hat{a} = \hat{a}$, $\forall \hat{a} \in \hat{F}$,
(5) For each element $\hat{a} \in \hat{F}$, there is an element $-\hat{a} \in \hat{F}$, called the "opposite of $\hat{a}$", which is such that $\hat{a}\hat{+}(\hat{-}\hat{a}) = \hat{0}$;
(6) The set $\hat{F}$ is closed under isomultiplication, $\hat{a}\hat{\times}\hat{b} \in \hat{F}$, $\forall \hat{a}$, $\hat{b} \in \hat{F}$,
(7) The isomultiplication is generally nonisocommutive, $\hat{a}\hat{\times}\hat{b} \neq \hat{b}\hat{\times}\hat{a}$, but "isoassociative", $\hat{a}\hat{\times}(\hat{b}\hat{\times}\hat{c}) = (\hat{a}\hat{\times}\hat{b})\hat{\times}\hat{c}$, $\forall \hat{a}, \hat{b}, \hat{c} \in \hat{F}$;
(8) The quantity $\hat{I}$ in the factorization $\hat{a} = a \times \hat{I}$ is the "multiplicative isounit" of $\hat{F}$ as per Eqs. (2.3)
(9) For each element $\hat{a} \in \hat{F}$, there is an element $\hat{a}^{-\hat{I}} \in \hat{F}$, called the "isoinverse", which is such that $\hat{a}\hat{\times}(\hat{a}^{-\hat{I}}) = (\hat{a}^{-\hat{I}})\hat{\times}\hat{a} = \hat{I}$.
(10) The set $\hat{F}$ is closed under joint isomultiplication and isosum,

$$\hat{a}\hat{\times}(\hat{b} + \hat{c}) \in \hat{F}, \qquad (\hat{a} + \hat{b})\hat{\times}\hat{c} \in \hat{F}, \qquad \forall \hat{a}, \hat{b}, \hat{c} \in \hat{F}; \tag{2.4}$$

(11) All elements $\hat{a}, \hat{b}, \hat{c} \in \hat{F}$ verify the right and left "isodistributive laws"

$$\hat{a}\hat{\times}(\hat{b}\hat{+}\hat{c}) = \hat{a}\hat{\times}\hat{b}\hat{+}\hat{a}\hat{\times}\hat{c}\,, \qquad (\hat{a}\hat{+}\hat{b})\hat{\times}\hat{c} = \hat{a}\hat{\times}\hat{c}\hat{+}\hat{b}\hat{\times}\hat{c}\,. \tag{2.5}$$

When there exists a least positive isointeger $\hat{p}$ such that the equation $\hat{p}\hat{\times}\hat{a} = \hat{0}$ admits solution for all elements $\hat{a} \in \hat{F}$, then $\hat{F}$ is said to have "isocharacteristic $\hat{p}$". Otherwise, $\hat{F}$ is said to have "isocharacteristic zero".

Unless otherwise stated, all isofields considered hereon shall be of Class I and have isocharacteristic zero. Since the additive unit is not changed under isotopies, $\hat{0} \equiv 0$, the sum will be often denoted for simplicity with the conventional symbol, $\hat{+} \equiv +$, while the differentiation of the multiplication $\times$ into $\hat{\times}$ stands to denote a nontrivial change of the multiplicative unit $I \to \hat{I} \neq I$.

The isofields important for this section is the isofield $\hat{R}(\hat{n}, \hat{+}, \hat{\times})$ of *isoreal numbers* $\hat{n}$ on which the isominkowskian geometry will be constructed. We also mention for completeness the isofield $\hat{C}(\hat{c}, \hat{+}, \hat{\times})$ of *isocomplex isonumbers* $\hat{c}$ and the isofield $\hat{Q}(\hat{q}, \hat{+}, \hat{\times})$ of *isoquaternions* $\hat{q}$ (see Ref. 5f for the *isooctonions*) which will not be considered in this study. Since all infinitely possible isofields $\hat{F}$ preserves by construction all axioms of F, we have the following:

Lemma 2.1:[5f] *Class I isofields $\hat{F}(\hat{a}, \hat{+}, \hat{\times})$ are locally isomorphic to conventional fields $F(a, +, \times)$, i.e., the lifting $F \to \hat{F}$ is isotopic.*

All conventional operations dependent on the multiplication on F are generalized on $\hat{F}$ in a simple yet unique way, yielding isopowers, isosquare roots, isoquotients, etc.

$$\hat{a}^{\hat{2}} = \hat{a}\hat{\times}\hat{a} = (a \times a) \times \hat{I}\,, \qquad \hat{a}^{\hat{\frac{1}{2}}} = a^{\frac{1}{2}} \times \hat{I}^{\frac{1}{2}}\,, \qquad \hat{n}\hat{/}\hat{m} = (\hat{n}/\hat{m}) \times \hat{I}\,, \quad \text{etc.} \tag{2.6}$$

It is then easy to see that *isounit verifies all axiomatic properties of the conventional unit*, e.g.,

$$\hat{I}^{\hat{n}} = \hat{I}\hat{\times}\hat{I}\hat{\times}\ldots\hat{\times}\hat{I} \equiv \hat{I}\,, \qquad \hat{I}^{\hat{\frac{1}{2}}} \equiv \hat{I}\,, \qquad \hat{I}\hat{/}\hat{I} \equiv \hat{I}\,, \quad \text{etc.} \tag{2.7}$$

Despite their simplicity, the liftings $F \to \hat{F}$ have significant implications in number theory itself. For instance, real numbers which are conventionally prime (under the *tacit* assumption of the unit 1 are not necessarily prime with respect to a different unit[5f]). This illustrates that most of the properties and theorems of the contemporary number theory are dependent on the assumed unit and, as such, admit simple, yet intriguing and significant isotopies (for more details, see Vol. II of Refs. 4g, Appendix 2B).

The *isonorm* must be an isonumber and is therefore defined by[5f]

$$\hat{|}\hat{a}\hat{|} = |a| \times \hat{I}\,, \tag{2.8}$$

where $|a|$ is the conventional norm. It is therefore easy to see that the *isonorm (of isofields of Class I) is positive-definite.*

The central notion of this paper from which all results can be uniquely and unambiguously derived is therefore given by *new numbers with arbitrary positive-definite units.* Yet more general numbers will be indicated at the end in section.

2.4. *Isominkowskian spaces*

The second significant application of the isotopies is the lifting of the conventional, vector, metric or pseudometric spaces, first presented by Santilli[4a] in 1983 (see monographs[4g] for detailed treatments). In this subsection we review the main aspects of the isotopies of the Minkowski *space* called *isominkowskian spaces* (loc. cit.). The *isominkowskian geometry* is the geometry of the isominkowskian spaces as outlined in this section. A knowledge of the preceding isotopies of the Euclidean space and geometry[4g] is recommendable.

Let $M = M(x, \eta, R)$ be a conventional Minkowski space[1] with coordinates $x = \{x^\mu\} = \{r, c_0 t\}$, $\mu = 1, 2, 3, 4$, where c_0 the speed of light in vacuum, basic unit $I = \text{diag}(+1, +1, +1, +1)$ and metric $\eta = \text{diag}(+1, +1, +1, -1)$ over a field $R = R(n, +, \times)$ of real numbers $n = n \times I$ equipped with the conventional sum $+$ and product $\times$, additive unit 0 and multiplicative unit I. Let $v = \dot{x}$, $a = \dot{v}$, μ represent the local density and τ the local temperature of the medium considered.

Definition 2.2:[4a] *The "isominkowski spaces" $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ of Class I are $(3+1)$-dimensional pseudometric spaces defined over an isoreal isofield $\hat{R}(\hat{n}, \hat{+}, \hat{\times})$ with a common, 4×4-dimensional, real-valued and symmetric isounit $\hat{I} = (\hat{I}^\mu{}_\nu) = \hat{I}^t$ of the same class, equipped with "isometrics" possessing an unrestricted functional dependence*

$$\hat{G} = \hat{\eta}(x, v, a, \mu, \tau, \ldots) \times \hat{I} = [\hat{T}(x, v, a, \mu, \tau, \ldots) \times \eta] \times \hat{I} = \hat{G}^t\,, \qquad \hat{I} = \hat{T}^{-1} = \hat{I}^t > 0\,, \tag{2.9}$$

local chart in contravariant and covariant forms

$$\hat{x} = \{\hat{x}^\mu\} = \{x^\mu \times \hat{I}\}\,, \qquad \hat{x}_\mu = \hat{\eta}_{\mu\nu} \times \hat{x}^\nu = \hat{T}_\mu{}^\alpha \times \eta_{\alpha\nu} \times x^\nu \times \hat{I}\,, \qquad x^\mu, x_\mu \in M\,; \tag{2.10}$$

and "isoseparation" among two points $\hat{x}, \hat{y} \in \hat{M}$

$$\begin{aligned}(\hat{x} - \hat{y})^{\hat{2}} &= (\hat{x}^\mu - \hat{y}^\mu) \hat{\times} \hat{G}_{\mu\nu} \hat{\times} (\hat{x}^\nu - \hat{y}^\nu) \\ &= [(x^\mu - y^\mu) \times \hat{\eta}_{\mu\nu} \times (x^\nu - y^\nu)] \times \hat{I} \\ &= [(x^1 - y^1) \times T_{11}(x, \ldots) \times (x^1 - y^1) \\ &\quad + (x^2 - y^2) \times T_{22}(x, \ldots) \times (x^2 - y^2) \\ &\quad + (x^3 - y^3) \times T_{33}(x, \ldots) \times (x^3 - y^3) \\ &\quad - (x^4 - y^4) \times T_{44}(x, \ldots) \times (x^4 - y^4)] \times \hat{I}\,, \end{aligned} \tag{2.11a}$$

$$\hat{T} = \text{diag}(T_{11}, \hat{T}_{22}, \hat{T}_{33}, \hat{T}_{44})\,, \quad \hat{T}_{\mu\mu} > 0\,, \quad x, y \in M, \hat{I} \notin M\,. \tag{2.11b}$$

Note that all scalars on M must be lifted into *isoscalars* to have meaning for $\hat{M}$, i.e., they must have the structure of the isonumbers $\hat{n} = n \times \hat{I}$. This condition requires the redefinition $x \to \hat{x} = x \times \hat{I}$, $\eta_{\mu\nu} \to \hat{G}_{\mu\nu} = \hat{\eta}_{\mu\nu} \times \hat{I}$, $x^\mu \times \eta_{\mu\nu} \times x^\nu \to (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times \hat{I}$, etc.

Note however the redundancy in practice for using the forms $\hat{x} = x \times \hat{I}$ and $\hat{G} = \hat{\eta} \times \hat{I}$ because of the identity $\hat{x}^{\hat{2}} = \hat{x}^\mu \hat{\times} \hat{G}_{\mu\nu} \hat{\times} \hat{x}^\nu \equiv (x^\mu \times \hat{\eta}_\nu \times x^\nu) \times \hat{I}$. For simplicity we shall often use the conventional coordinates x and the isometric will be referred to $\hat{\eta} = \hat{T} \times \eta$. The understanding is that the full isotopic formulations are needed for mathematical consistency.

A fundamental property of the infinite family of generalized spaces (2.11) is that the lifting of the basic unit $I \to \hat{I}$ while the metric is lifted of the *inverse* amount, $\eta \to \hat{\eta} = \hat{T} \times \eta$, $\hat{I} = \hat{T}^{-1}$, implies the preservation of all original axioms, and we have the following:

Lemma 2.2:[4a] *The isominkowski spaces $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ over the isofields $\hat{R}(\hat{n}, \hat{+}, \hat{\times})$ with a common positive-definite isounit $\hat{I}$ preserve all original axioms of the Minkowski space $M(x, \eta, R)$ over the reals $R(n, +, \times)$.*

The nontriviality of the lifting is that *the Minkowskian axioms are preserved under an arbitrary functional dependence of the metric* $\hat{\eta} = \hat{\eta}(x, v, a, \mu, \tau, \ldots)$ for which the sole x-dependence of the Riemannian metric $g(x)$ is a *particular case.* As a matter of fact, *the isominkowski spaces admit for metrics all infinitely possible metrics with the Minkowskian signature $(+, +, +, -)$, including Riemannian, Finslerian, nonsedarguesian, and any other possible metric.*

The above occurrence begins the illustration of the "direct universality" of the isominkowskian spaces[8,9] indicated in Sec. 1. Note also that all possible "deformations" of the Minkowski metric[14] are particular cases of the isometric. However, the former are still referred to the old unit I, thus losing the isomorphic between deformed and Minkowski space, while the isotopies preserve the original axioms by construction.

In the following we shall assume the convention that repeated indices between isocoordinates are in isominkowski space, while the contraction between the indices of the isounit and isotopic element is an ordinary sum. Thus, we have

$$\begin{aligned} \hat{x}^{\hat{2}} &= \hat{x}^\mu \hat{\times} \hat{x}_\mu \equiv \hat{x}_\mu \hat{\times} \hat{x}^\mu \equiv \hat{x}^\alpha \hat{\times} \hat{G}_{\alpha\beta} \hat{\times} \hat{x}^\beta \equiv \hat{x}_\alpha \hat{\times} \hat{G}^{\alpha\beta} \hat{\times} \hat{x}_\beta \\ &\equiv (x^\alpha \times \hat{T}_\alpha{}^\rho \times \eta_{\rho\beta} \times x^\beta) \times \hat{I} \equiv (x_\alpha \times \hat{I}^\alpha{}_\rho \times \eta^{\rho\beta} \times x_\beta) \times \hat{I}\,, \end{aligned} \quad (2.12a)$$

$$\hat{G}^{\alpha\beta} = |(\hat{G}_{\mu\nu})^{-1}|^{\alpha\beta}\,, \qquad \hat{\eta}^{\alpha\beta} = |(\hat{\eta}_{\mu\nu})^{-1}|^{\alpha\beta}\,, \quad (2.12b)$$

A most fundamental physical characteristic of the isominkowski spaces is that *it alters the units of space and time.* Recall that the unit $I = \mathrm{diag}(\{1, 1, 1\}, 1)$ of the Minkowski space represents in a dimensionless form the units of the three Cartesian axes, e.g., (+1cm, +1cm, +1cm) and of time, e.g., +1 sec. Recall also that the Cartesian spaceunits are *equal for all axes.* Consider now the isominkowski space. Since $\hat{I}$ is positive-definite, it can always be diagonalized into the form

$$\hat{I} = \mathrm{diag}({n_1}^2, {n_2}^2, {n_3}^2, {n_4}^2) = 1/\hat{T}, \qquad \hat{I}^\mu{}_\mu = {n_\mu}^{-2}, \quad n_\mu > 0. \tag{2.13}$$

This means that, not only the original units are now lifted into arbitrary positive values, but the *units of different space axes generally have different values.* Jointly, the component of the metric are lifted by the *inverse* amounts ${n_\mu}^{-2}$. This implies the preservation on $\hat{M}$ over $\hat{R}$ of the original *numerical* values on M over R, including the crucial preservation of the maximal causal speed c_0, as we shall see.

Note also the necessary condition that *the isospace and isofield have the same isounit* $\hat{I}$. This condition is absent in the conventional Minkowski space where the unit of the space is the unit *matrix* $I = \mathrm{diag}(1, 1, 1, 1)$, while that of the underlying field is the *number* $I = +1$. Nevertheless, the latter can be trivially reformulated with the common unit matrix I, by achieving in this way the form admitted as a particular case by the covering isospaces

$$M(x, \eta, R): \; x = \{x^\mu \times I\}, \quad x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I \in R \tag{2.14}$$

The structure of both the Minkowskian and isominkowskian invariants is therefore given by

$$\text{Basic Invariant} = (\text{Length})^2 \times (\text{Unit})^2 . \tag{2.15}$$

which illustrates more clearly the preservation of the original Minkowskian axioms under the dual lifting $\eta \to \hat{\eta} = \hat{T} \times \eta$ and $I \to \hat{I} = 1/\hat{T}$.

A significant difference between the conventional space M and its isotopes $\hat{M}$ is that the former admits only *one* formulation, the conventional one, while the latter admits *two* formulations: that on isospace itself (i.e. expressed with respect to the isounit $\hat{I}$) and its *projection* in the original space M (expressed with respect to the conventional unit I). The latter can also be formulated via modified coordinates

$$\bar{x}^\mu = x^\mu / n_\mu , \tag{2.16}$$

under which we have the identity

$$\begin{aligned} x^{\hat{2}} &= x^\mu \hat{\eta}_{\mu\nu} \times x^\nu \equiv x^1 \times x^1/{n_1}^2 + x^2 \times x^2/{n_2}^2 + x^3 \times x^2/{n_3}^2 - x^4 \times x^4/{n_4}^2 \\ &\equiv \bar{x}^1 \times \bar{x}^1 + \bar{x}^2 \times \bar{x}^2 + \bar{x}^3 \times \bar{x}^3 - \bar{x}^4 \times \bar{x}^4 = \bar{x}^2 , \end{aligned} \tag{2.17}$$

which are useful for practical calculations. In fact, any physical value $\bar{x}^\mu = x^\mu/n_\mu$ in our spacetime is projected into the value x in isospace, and vice-versa.

Note that the projection of $\hat{M}(\hat{x}, \hat{\eta}, \hat{R}$ into $M(x, \eta, R)$ is not a conformal map, but an *inverse isotopic map* because it implies the transition from the generalized unit and fields to conventional units and fields.

The above dual interpretation of $\hat{M}$ is at the foundation of the seemingly contradictory unification of the Minkowskian and Riemannian geometries. In fact, the former holds for the interpretation of the isometric $\hat{\eta}$ on isospace over isofields, while the latter holds in its projection into our spacetime which recovers conventional Riemannian settings.

The axiomatic motivation for constructing the isominkowskian spaces is that any modification of the Minkowski metric necessarily requires the use of *noncanonical transforms* $x \to x\prime(x)$,

$$\eta_{\mu\nu} \to \hat{\eta}_{\mu\nu} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \eta_{\alpha\beta} \frac{\partial x'^{\beta}}{\partial x^{\nu}} \neq \eta_{\mu\nu} \,, \tag{2.18}$$

and this includes the case of the transition from the Minkowskian metric η to the Riemannian metric $g(x)$. In turn, all noncanonical theories, thus including the Riemannian geometry, do not possess invariant units of space and time,[4h] thus having evident problematic aspects in physical applications, e.g., because it is not possible to conduct an unambiguous measurement of length with a stationary meter varying in time. A primary axiomatic function of the isotopies of the Minkowski space is that of preserving any (well behaved) generalized metric while restoring the invariance of the basic units, as shown in Appendix B of Ref. 4h.

Stated in different terms, a primary axiomatic difference between the special and general relativities is that the time evolution of the former is a *canonical transform*, thus implying the majestic mathematical and physical consistency recalled in Sec. 1, while the time evolution of the latter is a *noncanonical transform*, thus implying a number of unresolved problematic aspects which have been lingering throughout this century. The reformulation of the latter in terms of the axioms of the former is the sole possibility known to this author for achieving axiomatic consistency under a nontrivial functional dependence of the metric.

In summary, the isominkowskian spaces have a twofold application in this paper. First, they are used for a reinterpretation of the Riemannian metrics $g(x)$ for the particular case $\hat{\eta} = \hat{\eta}(x) = g(x)$ characterizing *exterior gravitational problems in vacuum*. Second, they are used for the characterization of *interior gravitational problems* with isometrics of unrestricted functional dependence $\hat{\eta} = \hat{\eta}(x, v, a, \ldots)$ while preserving the original Minkowskian axioms.

Since the explicit functional dependence is inessential under isotopies, our studies will be generally referred to the interior gravitational problem. Unless otherwise stated, only diagonal realizations of the isounits will be used hereon for simplicity. An example of nondiagonal isounits inherent in a structure proposed by Dirac is indicated in Sec. 2.14. More general liftings of the Minkowski space of the so-called *genotopic and multivalued-hyperstructural type* will be indicated in Sec. 2.15.

2.5. *Direct geometrization of arbitrary speeds of light*

As recalled in Sec. 1, the speed of light c is *not* a "universal constant" because its value depends on the local physical conditions. In fact, the speed of light is the constant value c_0 only *in vacuum*. However, within physical media of low density, such as planetary atmospheres or astrophysical chromospheres, the speed of electromagnetic waves assumes the value $c = c_0/n < c_0$, as known since Lorentz times;[12a,1e] values $c = c_0/n > c_0$ have been apparently measures both on laboratory tests[12b,12c] and in astrophysics;[12d,12e,12f] and solutions of ordinary wave equations with arbitrary speeds have now been identified[12g] (see Ref. 12h for a review).

It is important to see from these introductory lines that *the isominkowskian geometry, the isopoincaré symmetry and the isospecial relativity provide a direct geometrization of arbitrary speeds of light $c = c_0/n$, i.e., a geometrization via the isometric, when projected in our spacetime (referred to conventional units), while the maximal causal speed in isospace over isofields (when referred to isounits) remains that in vacuum c_0.*

In fact, the isoseparation for isounit (2.13) becomes

$$\hat{x}^2 = (x^1 \times x^1/{n_1}^2 + x^2 \times x^2/{n_2}^2 + x^3 \times x^2/{n_3}^2 - t \times t \times {c_0}^2/{n_4}^2) \times \hat{I}\,, \quad (2.19)$$

thus characterizing directly (that is, via the metric itself) local speeds $c = {c_0}^2/{n_4}^2$ where $n = n_4$ is evidently the local index of refraction. It should be stressed that the value $c = c_0/n_4$ occurs in our spacetime, that is, with respect to conventional units.

In isospace, the situation is different because, jointly with the lifting of the speed of light, we have the lifting of the corresponding unit by the *inverse* amount,

$$c_0^2 \to c^2 = {c_0}^2/{n_4}^2\,, \qquad I_{44} = 1 \to \hat{I}_{44} = {n_4}^2\,. \quad (2.20)$$

From invariance (2.15), *the maximal causal speed in isominkowskian space remains that in vacuum c_0.* This fundamental property will be proved again in the next subsection.

Note that, besides excluding the singular value $n_4 = 0$, *the isotopies leave completely unrestricted the numerical value of n_4 which can therefore be bigger, equal or smaller than* 1. It then follows that the value $c = c_0/n_4$ can be smaller, equal or bigger than c_0. As a result, the isospecial relativity is naturally set to represent all experimentally measured local speeds of electromagnetic waves.

It is important to see that the above results are intrinsic in Einstein's axioms of special relativity and, as such, they do not characterize a "new theory", but merely a "new realization" of the existing theory. This occurrence can be seen by noting that *isoinvariants for scalar isounits coincide with the conventional invariant.* In fact, for $I = n^2$ we have the following new invariance law of the *conventional* Minkowski space first introduced in Refs. 3 and 4h

$$\begin{aligned} x^2 &= (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I = (x^1 \times x^1 + x^2 \times x^2 + x^3 \times x^2 - t^4 \times t^4 \times {c_0}^2) \times I \\ &\equiv [x^\mu \times (n^{-2} \times \eta_{\mu\nu}) x^\nu \times (n^2 \times I) = (x^\mu \times \hat{\eta}_{\mu\nu} x^\nu) \times \hat{I} \\ &= (x^1 \times x^1/n^2 + x^2 \times x^2/n^2 + x^3 \times x^2/n^2 - t^4 \times t^4 \times {c_0}^2/n^2) \times \hat{I}\,. \end{aligned} \quad (2.21)$$

As one can see, and contrary to a rather popular belief, *the capability to represent arbitrary speed of light $c = c_0/n$ is intrinsic in the very structure of the basic invariant of the special relativity, and actually constitutes a new invariance law called "isoselfscalarity"*

$$I \to \hat{I} = n^{-2} \times I\,, \qquad \eta \to \hat{\eta} = n^{-2} \times \eta\,, \quad (2.22)$$

first introduced in Refs. 3 and 4h, under which the general symmetry of x^2, the Poincaré symmetry $P(3.1)$, becomes *eleven-dimensional* (Appendix E).

The only difference between Eqs. (2.21) and (2.19) is that the latter occur in *homogeneous and isotropic media*, while the former occur in *generally inhomogeneous and anisotropic media.*

The reason why the novel invariance law (2.21) has remained undetected throughout this century is that it required the prior discovery of *new numbers with arbitrary units.*

Note that *the isominkowskian geometry provides a spacetime symmetrization of the index of refraction*, evidently expressed by the presence of the space counterparts n_k of n_4, first introduced in Ref. 4a. The unavoidability of the space terms n_k should also be kept in mind because, starting from the sole presence of n_4 in the invariant, conventional Lorentz transforms would automatically produce the space counterparts.

All expressions considered until now in this section are *local*, i.e., they hold for one specific interior point, e.g., for the study of the speed of light at one point of a given chromosphere. When interior systems are considered from the outside, they must be studied as a whole. In this case, all interior effects should be averaged to constants, and we shall write

$$\hat{I} = \mathrm{diag}(n_1^{\circ 2}, n_2^{\circ 2}, n_3^{\circ 2}, n_4^{\circ 2}), \qquad n^\circ{}_\mu = \mathrm{const} > 0\,. \tag{2.23}$$

This is typically the case for all exterior experimental measures of interior characteristics. As an example, the measure from the outside of a light beam passing through a chromosphere can be best done via the use of the *average index of refraction* $n^\circ{}_4$ and related *average speed* $c = c_0/n^\circ{}_4$, and the same occurs for other quantities. Whenever referring to exterior experimental verifications, we shall therefore tacitly assume realization (2.23).

We should also indicate that, when the medium is no longer transparent to electromagnetic waves, the quantities n_μ $(n^\circ{}_\mu)$ are called the *characteristic functions (constants) of the medium considered, provide a geometrization of the medium itself*, much similar, although different, than the geometrization provided by the Riemannian metric. In particular, n_4 provides a geometrization of the local density with basic value $n_4 = 1$ assumed as the density of the vacuum.

Note also that realizations (2.13) and (2.23) hold for the most general possible case that the interior medium is *inhomogeneous and anisotropic.* The inhomogeneity is represented via a dependence of the isounit in the locally varying density. The anisotropy is generally due to a preferred direction of the interior *physical medium*, e.g., due to spin, with the clear understanding that the underlying space remains fully homogeneous and isotropic as per our current knowledge.

Also, the anisotropy is of *two* types. First, there is a *space anisotropy* characterized, e.g., by different values of the n_k's or by the factorization in the n_k's of the spin direction. Second, the anisotropy can be in spacetime, i.e., $n_1 = n_2 = n_3 = n_s \neq n_4$. When the isotropy holds in space and time, we shall write $n_1 = n_2 = n_3 = n_4 = n$.

For example, under the assumption of perfect spheridicity, our atmosphere is a medium of former type, while water is a medium of the latter type.

The analysis of this section has ben conducted for the case *without* gravitational field. The inclusion of the latter will be studied in Sec. 2.7.

2.6. *Light isocone*

A serious insufficiency in the use of the conventional Minkowskian and Riemannian geometries for the characterization of electromagnetic waves propagating within physical media is the *general loss of the light cone.* In fact, a locally varying speed of light, as it occurs in a planetary atmospheres or astrophysical chromospheres with variable densities, implies the necessary loss of the "cone" in favor of a more general surface in spacetime whose directrix is no longer a straight line.

The above loss is not a mere mathematical curiosity, because it carries rather deep *physical* implications of *numerical* character. As an example, gravitational horizons are today studied via the conventional light cone and the use of the speed of light *in vacuum* c_0. But the immediate exterior of the horizon of a collapsing star is not "empty", being composed instead by huge *inhomogeneous and anisotropic* chromospheres in which the the speed of light is not that in vacuum. Numerical results based on the conventional light cone are then questionable.

The costumery attitude of reducing the propagation of light within physical media to photons in second quantization scattering among molecules, while certainly acceptable as a crude first approximation, cannot possibly be a final scientific description for numerous reasons, such as:

(1) The reduction has no scientific credibility when dealing with the propagation within physical media of electromagnetic waves, say, of one meter in wavelength, which requires first a *classical* representation for any operator description in first or second quantization to make sense;
(2) The reduction may have scientific value only following the achievement of a consistent operator description of gravity, not only in first, but actually in second quantization; and
(3) The reduction to photons propagating in empty space eliminates the very physical characteristics to be represented, the inhomogeneity and anisotropy of the medium considered, which do have indeed measurable physical implications absent for the homogeneous and isotropic space.

In reality, physical evidence establishes that *matter alters the geometry of the vacuum.* The isotopies have been constructed to represent this alteration, first, because they are directly universal, thus always applicable, and second, because they geometrize physical media by preserving the abstract axioms of the vacuum.

One of the important applications of the isominkowskian geometry of Class I with diagonal isounit is the identification of a generalization of the light cone which permits more realistic calculations whenever the speed of light is no longer c_0. The latter was introduced for the first time by this author in Vol. I, Sec. 5.3B, of Ref. 4g under the name of *isolight cone*, although it will be called hereon *light isocone*.

In line with all other isotopies, the light isocone reproduces the exact cone in isospace to such an extend that even the maximal causal speed in isospace is the conventional speed *in vacuum* c_0. However, the *projection* of the isolight cone in our spacetime yields the deformed cone we observe under a locally varying speed c.

The understanding of the isolight cone requires the knowledge that the lifting from the Minkowskian to the isominkowskian spaces permits the preservation of all conventional notions of a *flat* geometry, such as straight lines, intersecting straight lines, perpendicular or parallel straight lines, etc. This implies in particular the preservation of the conventional trigonometric and hyperbolic functions although in a predictable generalized form.

Note by comparison that, in the transition from the Minkowskian to the Riemannian geometry, there is the loss of all the above conventional notions. This is further illustration that the isominkowskian representation of gravity is *isoflat*, that is, flat in isospace.

The isotopies of conventional notions of straight and intersecting lines is part of the *isoeuclidean geometry* (Vol. I of Ref. 4g). To render this paper minimally selfsufficient, we consider the compact *isoeuclidean subspace* $\hat{E} = \hat{E}(\hat{r}, \hat{\delta}, \hat{R})$ of $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$

$$\hat{E} = \hat{E}(\hat{r}, \delta, \hat{R}) \;:\; \hat{r} = \{r^k \times \hat{I}\}, \quad \hat{\delta} = \hat{T} \times \delta, \;\delta = \mathrm{diag}(1,1,1)\,, \quad \hat{I} = \mathrm{diag}({n_1}^2, {n_2}^2, {n_3}^2)\,. \tag{2.24}$$

An *isoline* on $\hat{E}$ over $\hat{R}$ is the conventional topological notion although referred to *isopoints* with values $\hat{r} = r \times \hat{I}$ on an isofield $\hat{R}$. An *isostraight line* in the isoeuclidean $(\hat{x}, \hat{y})$-plane has the form

$$\hat{a}\hat{\times}\hat{x} + \hat{b} \times \hat{y} + \hat{c} = 0, \qquad \hat{x}, \hat{y} \in \hat{E}, \qquad \hat{a}, \hat{b}, \hat{c} \in \hat{R}, \tag{2.25}$$

although, from rules (2.16), its projection into E over R is given, in general, by the *curve*

$$[a \times y/n_1(r, \ldots) + b \times y/n_2(r, \ldots) + c] \times \hat{I} = 0\,. \tag{2.26}$$

Intersecting isostraight lines then permit a unique and consistent definition of *isoangles* $\hat{\alpha}$ which is impossible in the Riemannian treatment of gravity (see Appendix A).

As indicated earlier, and as confirmed by the isorepresentation theory of the isotopic SU(2) symmetry,[4c,4g] *conventional numerical values of angles are preserved under isotopies*, i.e., isotopies map parallel (perpendicular) straight lines into isoparallel (isoperpendicular) isostraight isolines.

The projections of the isoangles $\hat{\phi}$ on the $(\hat{x}, \hat{y})$ plane and the angle $\hat{\theta}$ with respect to the $\hat{z}$-axis into E over R assume the forms

$$\hat{\phi} = \phi \times \hat{I}_\phi\,, \quad \hat{\theta} = \theta \times \hat{I}_\theta\,. \tag{2.27a}$$

$$\hat{I}_\phi = 1/n_1 \times n_2\,, \quad \hat{T}_\phi = n_1 \times n_2, \quad \hat{I}_\theta = 1/n_3, \quad \hat{T}_\theta = n_3\,. \tag{2.27b}$$

The *isotrigonometric functions* are given by[4g,5h]

$$\text{isosin}\hat{\alpha} = n_2 \times \sin\hat{\alpha}\,, \qquad \text{isocos}\hat{\alpha} = n_1 \times \cos\hat{\alpha}\,, \tag{2.28a}$$

$$\begin{aligned}\text{isosin}^{\hat{2}}\hat{\alpha} + \text{isocos}^{\hat{2}}\hat{\alpha} &= n_1^{-2} \times \text{isocos}^2\hat{\alpha} + n_2^{-2} \times \text{isosin}^{\hat{2}}\hat{\alpha}\\ &= \cos^2\hat{\alpha} + \sin^2\hat{\alpha} = 1\,,\end{aligned} \tag{2.28b}$$

where we have ignored the factorization by the isounit for simplicity. The *isospherical coordinates* can be written [loc. cit.]

$$x = r \times n_1 \times \sin(\theta/n_3)\cos(\phi/n_1 \times n_2) \tag{2.29a}$$

$$y = r \times n_2 \times \sin(\theta/n_3)\sin(\phi/n_1 \times n_2)\,, \tag{2.29b}$$

$$z = r \times n_3 \times \cos(\theta/n_3)\,, \tag{2.29c}$$

and they result to be the isocoordinates of the perfect sphere in isospace, called *isosphere* [loc. cit.],

$$\hat{R}^2 = \hat{x}\hat{\times}\hat{x} = \hat{y}\tilde{\times}\hat{y} + \hat{z}\hat{\times}\hat{z} = x \times x/{n_1}^2 + y \times y/{n_2}^2 + z \times z/{n_3}^2 = \hat{R}^2\,. \tag{2.30}$$

In fact, jointly with the lifting of the semiaxes of the conventional sphere $1_k = +1 \rightarrow \hat{1}_k = {n_k}^2$, we have the lifting of the corresponding units by the *inverse* amounts $1_k = +1 \rightarrow \hat{I}_{kk} = {n_k}^{-2}$, thus preserving the original perfect sphericity. Note also that structure (2.30) unifies in isospace $\hat{E}$ over $\hat{R}$ all possible spheroidal ellipsoids in E over R.

The perfect sphericity of the isosphere is the geometrical counterpart of the local isomorphism between the isotopic rotational symmetry $\hat{O}(3)$ and the conventional symmetry $O(3)$ first studied in Refs. 4b (see also Appendix E). In fact, the *isogeodesic* of $\hat{O}(3)$ are indeed perfect circles, although in isospaces over isofields. But $\hat{O}(3) \approx O(3)$. This disproves a rather popular belief that the rotational symmetry is "broken" for ellipsoidical deformations of the sphere.

We also have the *isopythagorean theorem* for an *isoright isotriangle* with isosides $\hat{A}$ and $\hat{B}$ and isohypothenuse $\hat{D}$[5h]

$$\hat{D}^{\hat{2}} = \hat{D}\hat{\times}\hat{D} = \hat{A}^{\hat{2}} + \hat{B}^{\hat{2}} = \hat{A}\hat{\times}\hat{A} + \hat{B}\hat{\times}\hat{B} \in \hat{R}\,, \tag{2.31}$$

which is trivial on $\hat{E}$ over $\hat{R}$. However, its projection on E over R is not trivial, because it implies the following property among a "triangle" whose sides are curves with only two intersections.

$$\hat{D}^{\hat{2}} = [A \times A/{n_1}^2(t, r, \dot{r}, \ldots) + B \times B/{n_2}^2(t, r, \dot{r}, \ldots)] \times \hat{I}\,, \tag{2.32}$$

In the noncompact $(\hat{z}, \hat{t})$-plane of $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ we can introduce the *isohyperbolic functions* and related property[4g,5h]

$$\text{isocosh}\hat{\alpha} = n_1 \times \cosh(\alpha/n_1 \times n_2)\,, \quad \text{isosinh}\hat{\alpha} = n_2 \times \sinh(\alpha/n_1 \times n_2)\,, \tag{2.33a}$$

$$\text{isocosh}^{\hat{2}}\hat{\alpha} - \text{isosinh}^{\hat{2}}\hat{\alpha} = 1\,, \tag{2.33b}$$

For additional properties, we refer for brevity the interested reader to Refs. 4g and 5h. Note that the elaboration of the isominkowskian geometry requires not only the isotopies of trigonometric and hyperbolic functions, but also of all other conventional and special functions and transforms [loc. cit.] without any exception known to this author. In fact, the use of any conventional quantity in the isominkowskian geometry (e.g., the conventional Fourier transform) leads to a host of inconsistencies which generally remain undetected by nonexperts in the field.

We are now minimally equipped to study the perfect light cone in isospace $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ called *light isocone.*[4g] Consider first for simplicity the (1+1)-dimensional isominkowskian plane $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ with isocoordinates $\hat{x} = (z, t) \times \hat{I}$, in which the light isocone can be written

$$\hat{x}^2 = (z \times z/n_3{}^2 - t^4 \times t^4 \times c_0{}^2/n_2{}^2) \times \hat{I} = 0\,, \qquad \hat{I} = \mathrm{diag}(n_3, n_4)\,, \tag{2.34}$$

which clearly represents *a duly symmetrized deformation of the light cone due to the locally varying speed* $c = c_0/n_4(x, \mu, \omega, \ldots)$, where n_4 is the locally varying index of refraction, μ the density of the medium, ω the frequency considered, etc.

Deformation (2.34) appears only in the projection of the isominkowskian description in the original Minkowski space, because at the level of the isospace itself we do have a perfect cone. The proof is trivial for the light isocone in water. In fact, isoinvariant (2.34) for infinitesimal values Δz and Δt in this case reads

$$\frac{\Delta z}{\Delta t} = \frac{n_3}{n_4} c_0 \equiv c_0\,, \tag{2.35}$$

because $n_3 \equiv n_4$ in water due to its isotropic character.

It is easy to prove that the above result also holds for arbitrary media, that is, for a locally varying speed of light within inhomogeneous and anisotropic media. In this case expression (2.34) for infinitesimal Δz and Δt becomes

$$\frac{\Delta z}{\Delta t} = \frac{n_3}{n_4} c_0 \neq c_0\,, \tag{2.36}$$

because now $n_3 \neq n_4$. The emergence of *a perfect cone in isospace* is then proved via the isotrigonometry. By calling $\hat{v}$ the interior isoangle of the cone with the t-axis, we have

$$\Delta z = D \times \mathrm{isosin}\hat{v} = D \times n_3 \times \sin\hat{v}\,,$$
$$\Delta t = D \times \mathrm{isocos}\hat{v} = D \times n_4 \times \cos\hat{v}\,, \tag{2.37a}$$
$$\frac{\Delta z}{\Delta t} = \mathrm{isotang}\hat{v} = \frac{n_3}{n_4}\mathrm{tang}\hat{v} = \frac{n_3}{n_4} c_0\,, \tag{2.37b}$$

where D is the isohypothenuse. It then follows that tang $\hat{v} = c_0$. The extension to three space dimension is straightforward (thanks to the notion of isocircle) and we have the following

Lemma 2.3.[4g] *The characteristic angles of the light cone and isocone coincide,* i.e., *the maximal causal speed of the isospecial relativity on isospace over isofields remains the speed of light in vacuum* c_0.

The above occurrence is the strongest evidence we have identified so far on the preservation under isotopies of Einstein's axioms of the special relativity. The occurrence has also the deeper meaning of constituting the geometric counterpart of the local isomorphism between the *isolorentz symmetry* $\hat{L}(3.1)$ and the conventional symmetry $L(3.1)$ first introduced in Ref. 4a (see Appendix E).

The latter isomorphism disproves the additional, equally popular belief that the Lorentz symmetry is "broken" under a signature-preserving deformation of the Minkowski metric. In fact, one of the objectives of the original proposal[4a] was precisely that of showing the preservation of the *exact* character of the Lorentz symmetry under all infinitely possible liftings $\eta \to \hat{\eta} = T(x, v, a, \ldots) \times \eta$, evidently when treated in a mathematically adequate way.

2.7. *Isocontinuity and isomanifolds*

To proceed in our study, we now need the notion of *Kadeisvili's isocontinuity* on an isospace.[8a] It results to be easily reducible to that of conventional continuity for Class I isotopies because the isomodulus of a function is reducible to the conventional modulus multiplied by the isounit.

Let $f(x)$ be a conventional scalar function on M over R. An *isofunction* $\hat{f}(\hat{x})$ on $\hat{M}$ over $\hat{R}$ must be an *isoscalar*. We can therefore assume the realization $\hat{f}(\hat{x}) = f(x) \times \hat{I}$. The *isomodulus* of $\hat{f}(\hat{x})$ is then given by

$$\hat{|}\hat{f}(\hat{x})\hat{|} = |f(x)| \times \hat{I}\,. \tag{2.38}$$

Definition 2.3.[8a] *An infinite sequence of isofunctions of Class I, $\hat{f}_1, \hat{f}_2, \ldots$ is said to be "strongly isoconvergent" to the isofunction $\hat{f}$ of the same class, when*

$$\lim_{k\to\infty} \hat{|}\hat{f}_k - \hat{f}\hat{|} = \hat{0}\,, \tag{2.39}$$

while the "isocauchy condition" can be expressed by

$$\hat{|}\hat{f}_m - \hat{f}_n\hat{|} < \hat{\delta} = \delta \times \hat{I}\,, \tag{2.40}$$

where δ is real and m and n are greater than a suitably chosen $N(\delta)$.

The isotopies of other notions of continuity, limits, series, etc. can then be easily constructed (see Vol. I of Ref. 4g).

Note that functions which are conventionally continuous are also isocontinuous. Similarly, a series which is strongly convergent is also strongly isoconvergent. However, *a series which is strongly isoconvergent is not necessarily strongly convergent.* As a result, *a series which is conventionally divergent can be turned into a convergent form under a suitable isotopy.* This mathematically trivial property has rather important applications, e.g., for the reconstruction of *convergent* perturbative series when conventionally divergent, e.g., as it is the case for strong interactions (see Vol. I, Sec. 6.5 and Vol. II, Ch. 11 of Ref. 4g).

Similarly, the reader may be interested in knowing that, given a function which is not square-integrable in a given interval, there always exists an isotopy which turns the function into a square-integrable form [loc. cit.]. The *novelty* is due to the fact that the underlying mechanism *is not* that of a weight function, but that of *altering the underlying field.*

The notions of N-dimensional *isomanifolds, isovector and isotensor fields and isotopology* were first studied by Tsagas and Sourlas[8b] and Santilli.[5g] The main lines can be summarized as follows. All isounits of Class I can always be diagonalized into the form

$$\hat{I} = \mathrm{diag}({n_1}^2, {n_2}^2, \ldots, {n_n}^2), \qquad n_k(x, \ldots) \neq 0, \quad k = 1, 2, \ldots, N\,, \tag{2.41}$$

Consider then n isoreal isofields $\hat{R}_k(\hat{n}, +, \hat{\times})$ each characterized by the isounit $\hat{I}_k = {n_k}^2$ with (ordered) Cartesian product

$$\hat{R}^N = \hat{R}_1 \times \hat{R}_2 \times \cdots \times \hat{R}_n\,. \tag{2.42}$$

Since $\hat{R}_k \approx R$, it is evident that $\hat{R}^n \approx R^n$, where R^n is the Cartesian product of n conventional fields $R(n, +, \times)$. But the total unit of $\hat{R}^n$ is expression (2.41). Therefore, one can introduce an *isoeuclidean topology* on $\hat{R}^n$ via the simple isotopy of the conventional topology on R^n,

$$\hat{\tau} = \{\emptyset, \hat{R}^n, \hat{K}_i\}\,, \tag{2.43}$$

where $\hat{K}_i$ represents the subset of $\hat{R}^n$ defined by

$$\hat{K}_i = \{\hat{P} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)\hat{/}\hat{n}_i < \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n < \hat{m}_i\,, \hat{n}_i, \hat{m}_i, a_k \in \hat{R}\}\,. \tag{2.44}$$

As one can see, the above topology coincides everywhere with the conventional Euclidean topology τ of R^n *except at the isounit* $\hat{I}$. In particular, $\hat{\tau}$ is everywhere local-differential, except at $\hat{I}$ which can incorporate integral terms. The above structure is called the *Santilli-Sourlas-Tsagas isoeuclidean topology or integro-differential topology.*[8c]

The isotopology of the isominkowskian geometry can be studies via the isotopies, e.g., of the Zeeman topology for M, but it has not been studied to date. We shall therefore content ourselves with the use of the Santilli-Sourlas-Tsagas isoeuclidean topology. For a study of isovector and isotensor fields on isomanifolds we refer the interested reader to.[8b]

2.8. *Isodifferential calculus*

As conventionally presented, the ordinary differential calculus does not appear to be dependent on the assumed unit. This is not the case because the differential calculus too has resulted to be essentially dependent from the assumed unit and, more specifically, it must have for axiomatic consistency the same unit of the space in which it acts.

Following their original proposal back in 1978,[5b] the isotopies escaped the achievement of an invariant formulation for two decades precisely because elaborated with the conventional differential calculus, a problem resolved only recently in memoir[5g] via the lifting of the differential calculus itself.

Definition 2.4.[5g] *Let $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ be an isominkowskian space of Class I with isounit $\hat{I} = (\hat{I}^\mu{}_\nu(x, v, a, \ldots))$ whose elements $I^\mu{}_\nu$ are nowhere null, real-valued and sufficiently smooth functions of local variables. The "first-order isodifferentials" of Class I of the contravariant and covariant coordinates $\hat{x}^\mu$ and $\hat{x}_\mu$, on $\hat{M}$ are given by*

$$\hat{d}\hat{x}^\mu = \hat{I}^\mu{}_\nu(x, \ldots) \times d\hat{x}^\nu, \quad \hat{d}\hat{x}_\mu = \hat{T}_\mu{}^\nu(x, \ldots) \times d\hat{x}_\nu\,. \tag{2.45}$$

Let $\hat{f}(\hat{x})$ be a sufficiently smooth isofunction on a closed domain $\hat{D}(\hat{x}^k)$ of contravariant coordinates $\hat{x}^\mu$ on $\hat{M}$. Then the "isoderivative" at a point $\hat{q}^\mu \in \hat{D}(\hat{x}^\mu)$ is given by

$$\hat{f}'(\hat{q}^\mu) = \frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^\mu}|_{\hat{x}^\mu=\hat{q}^\mu} = \hat{T}_\mu{}^\nu \times \frac{\partial\hat{f}(\hat{x})}{\partial\hat{x}^\nu}|_{\hat{x}^\mu=\hat{q}^\mu} = \lim_{\hat{d}\hat{x}^\mu\to\hat{0}^\nu} \frac{\hat{f}(\hat{q}^\mu + \hat{d}\hat{x}^\mu) - \hat{f}(\hat{q}^\mu)}{\hat{d}\hat{x}^\mu}\,. \tag{2.46}$$

The "isoderivative" of a smooth isofunction $\hat{f}(\hat{x})$ of the covariant variable $\hat{x}_\mu$ at the point $\hat{q}_\mu \in \hat{D}(\hat{x}_\mu)$ is given by

$$\hat{f}\prime(\hat{q}_\mu) = \frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}_\mu}|_{\hat{x}_k=\hat{q}_k} = \hat{I}^\mu{}_\nu \times \frac{\partial\hat{f}(\hat{x})}{\partial\hat{x}_\nu}|_{\hat{x}_\mu=\hat{q}_\mu} = \lim_{\hat{d}\hat{x}_\mu\to\hat{0}_\nu} \frac{\hat{f}(\hat{q}_\nu + \hat{d}\hat{x}_\nu) - \hat{f}(\hat{q}_\nu)}{\hat{d}\hat{x}_\nu}\,. \tag{2.47}$$

The reader should keep in mind that all quotients are isotopic, thus implying the factorization by $\hat{I}$ which has been omitted for simplicity because it cancels out with any isomultiplication. The above properties render the isodifferential calculus axiom-preserving, thus being an isotopy.

Note that, as it was the case for speed of light in vacuum, *diagonal isotopies of a given derivative preserve the original value when computed in isospace over isofields.* This is due to the fact that the derivative is lifted according to the rule $\partial_\mu = \delta_\mu{}^\nu \times \partial_\mu \to \hat{\partial}_\mu = \hat{T}_\mu{}^\nu \times \partial_\nu$; but the related unit is lifted by the inverse amount $\delta^\mu{}_\nu \to \hat{I}^\mu{}_\nu = (\hat{T}_\mu{}^\nu)^{-1}$; thus preserving the original value.

The above property is important to understand later on the *preservation of the numerical value of Einstein's field equations under isotopies.*

The *isodifferentials of an isofunction* of contravariant coordinates $\hat{x}^\mu$ on $\hat{M}$ are defined by:

$$\begin{aligned}\hat{d}\hat{f}(\hat{x})_{\text{contrav.}} &= \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}^\mu}\hat{\times}\hat{d}\hat{x}^\mu = \hat{T}_\mu{}^\alpha \times \frac{\partial\hat{f}}{\partial x^\alpha} \times \hat{I}^\mu{}_\beta \times d\hat{x}^\beta \\ &= \frac{\partial f}{\partial\hat{x}^\mu} \times d\hat{x}^\mu = \hat{T}_\alpha{}^\rho \times \eta_{\rho\beta}\frac{\partial\hat{f}}{\partial\hat{x}^\alpha} \times d\hat{x}^\beta\,,\end{aligned} \tag{2.48}$$

where the last term $\hat{T}_\alpha{}^\rho$ originates from the fact that the contraction on the μ-index is in isominkowskian space. Similarly, we have the *second-order isoderivatives*

$$\frac{\hat{\partial}^2 \hat{f}(\hat{x})}{\hat{\partial} x^\mu \hat{\partial} \hat{x}^\nu} = \frac{\hat{\partial}^2 \hat{f}(\hat{x})}{\hat{\partial} \hat{x}^\nu \hat{\partial} \hat{x}^\mu} \tag{2.49}$$

namely, the isodifferential calculus preserves the commutativity of the second-order derivatives, as necessary for an isotopy.

We should note that the above property holds in isospace over isofields (i.e., when each isoderivative is computed with respect to the isounit), because the same commutativity is generally lost in the projection into conventional spaces over conventional fields (when the isoderivative is computed with respect to conventional units).

The following properties are important for subsequent calculations

$$\hat{\partial}\hat{x}^\alpha / \hat{\partial}\hat{x}^\beta = \delta^\alpha{}_\beta, \ \hat{\partial}\hat{x}_\alpha / \hat{\partial}\hat{x}_\beta = \delta_\alpha{}^\beta, \ \hat{\partial}\hat{x}_\alpha / \hat{\partial}\hat{x}^\beta = \hat{T}_\alpha{}^\beta, \ \hat{\partial}\hat{x}^\alpha / \hat{\partial}\hat{x}_\beta = \hat{I}^\alpha{}_\beta \,. \tag{2.50}$$

The *isolaplacian* on $\hat{M}$ is given by

$$\hat{\Delta} = \hat{\partial}_\mu \hat{\times} \hat{\partial}^\mu = \hat{G}_{\alpha\beta} \hat{\times} \hat{\partial}^\alpha \hat{\times} \hat{\partial}^\beta \,, \tag{2.51}$$

and results to be different than the corresponding expression on a Riemannian space $\mathcal{R}(x, g, R)$ with metric $g(x) = \hat{\eta}$, $\Delta = g^{-1/2}\partial_\alpha g^{1/2} g^{\alpha\beta}\partial_\beta$, *even though the isominkowskian metric* $\hat{\eta}(x, v, a, \ldots)$ *is more general than the Riemannian metric* $g(x)$.

For completeness we mention the (indefinite) *isointegration* defined as the inverse of the isodifferential, e.g.,

$$\hat{\int} \hat{d}\hat{x} = \int \hat{T}\hat{I} dx = \int dx = x \,, \tag{2.52}$$

namely, $\hat{\int} = \int \hat{T}$. Definite isointegrals are formulated accordingly. Due to its simplicity we shall tacitly assume the isotopies of integration hereon.

The above basic notions are sufficient for the limited needs of this paper. The isotopies of additional properties and theorems of the differential calculus are left for study by the interested reader. The class of isodifferentiable isofunctions of order m will be indicated $\hat{C}^m$.

An important property of the above calculus is that the isodifferentials and isoderivatives preserve the basic isounit $\hat{I}$. Mathematically, this condition is *necessary* to prevent that a set of isofunctions $\hat{f}(\hat{x})$, $\hat{g}(\hat{x}), \ldots$, on $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$ over the isofield $\hat{R}(\hat{n}, \hat{+}, \hat{\times})$ with isounit $\hat{I}$ is mapped via isoderivatives into a set of isofunctions $\hat{f}'(\hat{x})$, $\hat{g}'(\hat{x}), \ldots$, defined over a *different* isofield because of the alteration of the isounit. Physically, the condition is also *necessary* because an invariant unit is a pre-requisite for physically meaningful measurements. The lack of conservation of the unit therefore implies the lack of consistent physical applications.

As an example, the following alternative definition of the isodifferential

$$\hat{d}\hat{x}^\mu = d(\hat{I}^\mu{}_\nu x^\nu) = [(\partial_\alpha \hat{I}^\mu{}_\beta) x^\beta + \hat{I}^\mu{}_\beta] dx^\beta = \hat{W}^\mu{}_\beta dx^\beta \,, \tag{2.53}$$

would imply the alteration of the isounit, $\hat{I} \to \hat{W} \neq \hat{I}$, thus being mathematically and physically unacceptable.

Nevertheless, when using isoderivatives of independent variables, say, isoderivatives on coordinates and time, the above rule does not apply and we have

$$\hat{\partial}_{\hat{t}}\hat{\partial}_k \hat{f}(\hat{t},\hat{x}) = \hat{\partial}_{\hat{t}}[\hat{\partial}_k \hat{f}(\hat{t},\hat{x})] = \hat{\partial}_{\hat{t}}[T^i_k(t,x,\ldots)\partial_i f(t,x)]\,. \tag{2.54}$$

Additional properties of the isodifferential calculus will be identified during the course of our analyzes.

Note that the ordinary differential calculus is local-differential on M. The isodifferential calculus is instead local-differential on $\hat{M}$ but, when projected on M, it becomes *integro-differential* because it incorporates integral terms in the isounit.

Note finally the difference between the isodifferential calculus and other forms, such as Cartan's exterior calculus as established by the underlying unit. In fact, the unit and related functional analysis are generalized in the former while they are conventional in the latter.

2.9. *Isominkowskian representation of gravity*

As now familiar, the central assumption of this study is the Minkowskian factorization of the Riemannian metric and its identification with the isominkowskian metric,

$$g(x) = \hat{T}_{gr}(x) \times \eta \equiv \hat{\eta}(x), \qquad \hat{I}_{gr}(x) = [\hat{T}_{gr}(x)]^{-1}\,, \tag{2.55}$$

which is valid for the *exterior gravitational problem in vacuum.* As a result, *all conventional Riemannian metrics, e.g., Schwarzschild's, Kerr-Newmann, Krasner and any other metric*[15] *are preserved identically in their isominkowskian reformulation.*

As an illustration, under the usual assumption of space isotropy, the Schwarzschild metric[2d,15s] in isocartesian coordinates is represented by the *gravitational isounit*

$$\begin{aligned} \hat{I}_{gr} &= \mathrm{diag}({m_1}^2, {m_2}^2, {m_3}^2, {m_4}^2) \\ &= \mathrm{diag}\{(1-2\times M/r)^{-1}, (1-2\times M/r)^{-1}, \\ &\quad \times (1-2\times M/r)^{-1}, (1-2\times M/r)\},\ \text{i.e.,} \end{aligned} \tag{2.56a}$$

$$\begin{aligned} {m_1}^2 &= {m_2}^2 = {m_3}^2 = {m_s}^2 \\ &= 1 - 2\times M/r\,, \quad {m_4}^2 = (1-2M/r)^{-1}\,. \end{aligned} \tag{2.56b}$$

The following property, here presented for the first time, is important for our study.

Proposition 2.1. *The isominkowskian representation of gravity reduces Riemannian line elements into a Minkowskian form on isospace over isofields.*

As an illustration, by using isospherical coordinates (Sec. 2.6), and by assuming that the radial coordinates r, t are covariant for which $\hat{d}\hat{r} = \hat{T}_r \times d\hat{r}$ and $\hat{d}\hat{t} = \hat{T}_t \times d\hat{t}$, Schwarzschild's metric can be written on $\hat{M}$ over $\hat{R}$

$$\hat{d}\hat{s}^{\hat{2}} = \hat{d}\hat{r}^{\hat{2}} + \hat{r}^{\hat{2}}\hat{\times}(\hat{d}\hat{\theta}^{\hat{2}} + \mathrm{isosin}^{\hat{2}}\hat{\theta} \times \hat{d}\hat{\phi}^{\hat{2}}) - \hat{d}\hat{t}^{\hat{2}}\hat{\times}{\hat{c}_0}^2\,, \tag{2.57a}$$

$$\hat{d}\hat{r} = \hat{T}_r \times d\hat{r},\ \hat{d}\hat{t} = \hat{I}_t \times d\hat{t},\ \hat{T}_r = (1-2\times M/r)^{-1},\ \hat{T}_t = 1 - 2\times M/r\,, \tag{2.57b}$$

isotopies of trigonometric and hyperbolic functions, but also of all other conventional and special functions and transforms [loc. cit.] without any exception known to this author. In fact, the use of any conventional quantity in the isominkowskian geometry (e.g., the conventional Fourier transform) leads to a host of inconsistencies which generally remain undetected by nonexperts in the field.

We are now minimally equipped to study the perfect light cone in isospace $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ called *light isocone.*[4g] Consider first for simplicity the (1+1)-dimensional isominkowskian plane $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ with isocoordinates $\hat{x} = (z, t) \times \hat{I}$, in which the light isocone can be written

$$\hat{x}^2 = (z \times z/{n_3}^2 - t^4 \times t^4 \times {c_0}^2/{n_2}^2) \times \hat{I} = 0 \,, \qquad \hat{I} = \mathrm{diag}(n_3, n_4) \,, \tag{2.34}$$

which clearly represents *a duly symmetrized deformation of the light cone due to the locally varying speed* $c = c_0/n_4(x, \mu, \omega, \ldots)$, where n_4 is the locally varying index of refraction, μ the density of the medium, ω the frequency considered, etc.

Deformation (2.34) appears only in the projection of the isominkowskian description in the original Minkowski space, because at the level of the isospace itself we do have a perfect cone. The proof is trivial for the light isocone in water. In fact, isoinvariant (2.34) for infinitesimal values Δz and Δt in this case reads

$$\frac{\Delta z}{\Delta t} = \frac{n_3}{n_4} c_0 \equiv c_0 \,, \tag{2.35}$$

because $n_3 \equiv n_4$ in water due to its isotropic character.

It is easy to prove that the above result also holds for arbitrary media, that is, for a locally varying speed of light within inhomogeneous and anisotropic media. In this case expression (2.34) for infinitesimal Δz and Δt becomes

$$\frac{\Delta z}{\Delta t} = \frac{n_3}{n_4} c_0 \neq c_0 \,, \tag{2.36}$$

because now $n_3 \neq n_4$. The emergence of *a perfect cone in isospace* is then proved via the isotrigonometry. By calling $\hat{v}$ the interior isoangle of the cone with the t-axis, we have

$$\begin{aligned} \Delta z &= D \times \mathrm{isosin}\hat{v} = D \times n_3 \times \sin\hat{v} \,, \\ \Delta t &= D \times \mathrm{isocos}\hat{v} = D \times n_4 \times \cos\hat{v} \,, \end{aligned} \tag{2.37a}$$

$$\frac{\Delta z}{\Delta t} = \mathrm{isotang}\hat{v} = \frac{n_3}{n_4}\mathrm{tang}\hat{v} = \frac{n_3}{n_4} c_0 \,, \tag{2.37b}$$

where D is the isohypothenuse. It then follows that tang $\hat{v} = c_0$. The extension to three space dimension is straightforward (thanks to the notion of isocircle) and we have the following

Lemma 2.3.[4g] *The characteristic angles of the light cone and isocone coincide,* i.e., *the maximal causal speed of the isospecial relativity on isospace over isofields remains the speed of light in vacuum* c_0.

namely, *the Schwarzschild metric is reduced to an identical Minkowskian form merely formulated on isospaces over isofields.* This result should evidently be *expected* as a necessary condition for consistency in an isominkowskian formulation of gravity. The result also begins to illustrate the importance of the isodifferential calculus for this study. In fact, *all gravitational terms are embedded in the isodifferentials*, thus ensuring the emergence of a conventional *Minkowskian-type* line element for the representation of *gravitation.*

The above isominkowskian representation is confirmed by the fact that the isopoincaré symmetry $\hat{P}(3.1)$ is indeed the symmetry of all possible exterior gravitational metrics, including the Schwarzschild's one.

By following similar rules, the reader can easily re-write in isominkowskian space all other needed Riemannian metrics.[15]

The difference between isominkowskian formulation (2.57) and other formulations, such as those via tetrads (see, e.g., Ref. 16c) should be kept in mind. In fact, the Minkowskian axioms hold only in the tangent space for the latter, while they hold in the space itself for the former. In fact, the isominkowskian formulation of the Schwarzschild metric is precisely that of writing it in an identical *Minkowskian* form, as illustrated by Eq. (2.57a).

We should also note that the isominkowskian formulation of the Schwarzschild metric implies, in its general form, *the lifting of angles into isoangles* with nontrivial related isounits as in Eqs. (2.27b) with the related n-values replaced by the m-expressions (2.39b). However, one should remember that *the numerical values of angles are preserved under isotopies.* This assures the preservation of the Riemannian metrics under their isotopies reformulation.

We should also note that the assumption of contravariant variables r, t for which $\hat{d}\hat{r} = \hat{I}_t \times d\hat{r}$ and $\hat{d}\hat{t} = \hat{I}_t \times d\hat{t}$ would merely imply an *interchange between the isounit and the isotopic element in realization (2.57).*

The primary motivations for rewriting *Riemannian* line elements in an identical *isominkowskian* form occur in the inclusion of gravitation in unified gauge theories of electroweak interactions.[11] In fact, Proposition 2.1 offers new possibilities for removing the first structural incompatibility between electroweak and conventional gravitational interactions, the use of a flat spacetime in the former and a curved one in the latter with rather serious problems of compatibility particularly at the operator level.

The second structural incompatibility is that due to the characterization of antimatter by the electroweak interaction via negative-energy solutions for the electroweak interactions and *positive-definite energy-momentum tensors* for Riemann. The isodual representation of antimatter at all levels offers new possibilities for removing this second incompatibility.

The third structural incompatibility is that on fundamental symmetries, the validity of the Poincaré symmetry for the electroweak interactions and its absence for the Riemannian treatment of gravity, which is removed by the isopoincaré symmetry (Appendix E).

The isominkowskian reformulation of gravity has no sole mathematical character because it implies rather deep physical consequences. In fact, from a mere inspection of Eqs. (2.57), we have the following,

Proposition 2.2. *The isominkowskian geometry implies that gravity alters the units of space and time.*

In turn, the above property has a number of consequences, such as the so-called *geometric propulsion* (Vol. I of Refs. 4g), according to which locomotion occurs without any application of a force and by altering instead the local geometry. In fact, the fundamental invariant is $[\text{Length}]^2 \times [\text{Unit}]^2$. As a result, any change of the unit implies a corresponding inverse change in lengths.

Proposition 2.2 also implies that *time is a local quantity*, namely, observers in different gravitational fields have different flows of time.

A visual inspection of Eqs. (2.57) also establishes the following:

Proposition 2.3. *In covariant coordinates, gravitational singularities (horizons) are the zeros of the space isounit $\hat{I}_r = 0$ (space isotopic element $\hat{T}_r = 0$).*

We encounter in this way the first case of the isominkowskian geometry of the Kadeisvili Class IV (with singular isounit or isotopic elements). This offers basically novel possibilities for the study of gravitational collapse, as indicated better below. Note that for a gravitational singularity covariant isocoordinates are null, $\hat{r} = r \times \hat{I}_r = 0$, while the covariant isotime is divergent, $\hat{t} = t \times \hat{I}_t = \infty$, the inverse case occurring for contravariant spacetime coordinates.

As now familiar, the isotopies leave unrestricted the functional dependence of the isometric $\hat{\eta}$ (provided that it is sufficiently smooth, nowhere singular, real values, and symmetric). We therefore study the isominkowskian geometry for an arbitrary functional dependence of the isometric

$$\hat{\eta} = \hat{\eta}(x, v, a, \mu, \tau, \ldots) = \hat{T}(x, v, a, \mu, \tau, \ldots) \times \eta, \quad \hat{I} = \hat{I}(x, v, a, \mu, \tau, \ldots) = \hat{T}^{-1}, \tag{2.58}$$

which represents *interior gravitational problems* with internal effects arbitrarily *nonlinear in the velocities, as well as nonlocal-integral* and *not derivable from a first-order Lagrangian* (variationally nonselfadjoint interactions [5e]).

Under the latter assumptions, the isounits can always be written in the form

$$\begin{aligned}\hat{I} = \hat{I}_{gr} \times \operatorname{diag}[&{n_1}^2(x, v, \ldots), {n_2}^2(x, v, \ldots), {n_3}^2(x, v, \ldots), {n_4}^2(x, v, \ldots)] \\ &\times \hat{F}(x, v, \ldots), \quad n_\mu > 0,\end{aligned} \tag{2.59}$$

where: $\hat{I}_{gr}$ is the isounit of the exterior gravitational problem (2.56); the n's are the characteristic functions of the interior medium; and the common factor $\hat{F}(x, v, \ldots)$ is a diagonal 4×4 matrix representing internal nonlinear, nonlocal and noncanonical effects.

Equivalently, the isominkowskian metric for interior gravitational problems can be assumed to have the form

$$\hat{\eta} = \hat{T}(x, v, a, \ldots) \times \eta, \quad \hat{T} = \operatorname{diag}({n_1}^{-2}, {n_2}^{-2}, {n_3}^{-2}, {n_4}^{-2}) \times \hat{F} \times \hat{T}_{gr}, \tag{2.60}$$

where $\hat{T}_{gr}$ is that of the exterior problem, Eqs. (2.56).

All exterior gravitational metrics[16] then admit a lifting into the interior form (2.60). An intriguing aspect is that, in so doing, the isominkowskian line element, e.g., Eq. (2.57), remains formally unchanged because all non-Minkowskian terms, whether gravitational or not, are embedded in the isodifferentials.

As an example, the *interior isoschwarzschild metric* under space isotropy $n_1 = n_2 = n_3 = n_s$ can be written

$$\begin{aligned}\hat{d}\hat{s}^{\hat{2}} &= \hat{d}\hat{r}^2 + \hat{r}^{\hat{2}}\hat{\times}(\hat{d}\hat{\theta}^{\hat{2}} + isosin^{\hat{2}}\hat{\theta} \times \hat{d}\hat{\phi}^{\hat{2}}) - \hat{d}\hat{t}^{\hat{2}}\hat{\times}\hat{c}_0{}^2 \\ &= \{d\hat{r}^2/(1 - 2 \times M/r) \times n_s{}^2 + \hat{r}^2 \times [d\theta^2 + \sin^2\theta \times d\phi^2] \times F_s \\ &\quad - (1 - 2 \times M/r) \times d\hat{t}^2 \times c_0{}^2/n_4{}^2 \times F_t\,. \end{aligned} \tag{2.61a}$$

$$\hat{d}\hat{r} = \hat{T}_r \times d\hat{r}, \qquad \hat{d}\hat{t} = \hat{T}_t \times d\hat{t}\,, \tag{2.61b}$$

$$\hat{T}_r = \hat{F}_s/(1 - 2 \times M/r) \times n_s{}^2, \qquad \hat{T}_t = \hat{F}_t \times (1 - 2 \times M/r)/n_4{}^2\,, \tag{2.61c}$$

As one can see, *the isotopies of Schwarzschild imply the direct geometrization of locally varying speeds of light* $c = c_0/n_4(x, \mu, \tau, \ldots)$. Note that the F-factors are eliminated in any observation from the outside which requires the averaging of internal effects to the characteristic *constants* $n^\circ{}_s$ and $n^\circ{}_4$ (Sec. 2.5).

It is evident that Propositions 2.1, 2.2 and 2.3 remain fully valid in interior conditions. This implies that *gravitational singularities are the zeros of the space isounit for interior, rather than exterior, problems*. In turn, the above occurrence requires a reinspection of the entire theory of gravitational collapse to account for internal nonlinear, nonlocal and nonpotential effects, which will be conducted elsewhere.

In different terms, the use of *exterior* metrics for intrinsically *interior* problems, such as gravitational collapse, should only be considered as a first approximation of physical reality of such a complexity to be at the vert limit of our mathematical and physical knowledge.

Note the full validity of the light isocone for interior gravitational problems, thus permitting more realistic studies of the area outside gravitational horizons, as indicated earlier.

It should be finally indicated that the Poincaré–Santilli isosymmetry of Appendix E remains the universal symmetry for interior and exterior, relativistic or gravitational and classical or operators realizations.

2.10. *Isoricci lemma, isocovariant isoderivative, isoconnection, isocurvature*

In the preceding subsections we have presented the *Minkowskian* aspects of the isominkowskian geometry. We are now sufficiently equipped to present, apparently for the first time, the novel part of the isominkowskian geometry, its *Riemannian* character.

Our study is strictly in local coordinates representing the *fixed* inertial frame of the observer without any unnecessary use of the transformation theory or abstract treatment in order not to violate a central requirement of the special relativity, the *inertial* character of the observers.

For the conventional geometry we assume all topological properties of Lovelock and Rund[15a] of which we preserve the symbols for clarity in the comparison of the results. For the isotopic geometry we assume Kadeisvili's[8a] isocontinuity and Tsagas-Sourlas[8b] isomanifolds and related topology. Our presentation is made, specifically, for the $(3+1)$-dimensional isominkowskian spacetime, with the understanding that the extension to arbitrary dimensions and signatures is elementary. The study is restricted to the isotopies of *Class I* (with positive-definite isounits) which are solely applied to the study of *matter.*

Let $\hat{M} = \hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ be an *isominkowskian space* of Definition 2.2 with local isocoordinates $\hat{x} = \{\hat{x}^\mu\} = \{x^\mu \times \hat{I}\})$ and isometric $\hat{\eta}(x, v, a, \mu, \tau, \ldots) = \hat{T}(x, v, a, \mu, \tau, \ldots) \times \eta(x)$, where $\hat{T} = (\hat{T}_\mu{}^\nu)$ is a nowhere singular, real valued and symmetric matrix of Class I with C^∞ elements and $\eta = \mathrm{diag}(1,1,1,-1)$ is the conventional Minkowski metric. The isospace $\hat{M}$ is defined over the isoreals $\hat{R} = \hat{R}(\hat{n}, \hat{+}, \hat{\times})$ with common isounit $\hat{I} = (\hat{I}^\mu{}_\nu) = \hat{T}^{-1}$. We then have the *isotopic invariant*

$$\hat{x}^{\hat{2}} = [x^\mu \times \hat{\eta}_{\mu\nu}(\hat{x}, \hat{v}, \hat{a}, \mu, \tau, \ldots) \times x^\nu] \times \hat{I} \in \hat{R}\,, \tag{2.62}$$

with infinitesimal version

$$\hat{d}\hat{s}^{\hat{2}} = \hat{d}\hat{x}_\mu \hat{\times} \hat{d}\hat{x}^\mu = (dx^\mu \times \hat{\eta}_{\mu\nu} \times dx^\nu) \times \hat{I} \in \hat{R}\,. \tag{2.63}$$

The *isonormal coordinates* $\hat{y}$ occur when the isometric $\hat{\eta}$ is reduced to the Minkowski metric η. As such, they preserve the conventional *principle of equivalence* [16]. In different terms, the tangent space is the conventional Minkowski space $M(x, \eta, R)$.

By using the isodifferential calculus, we now introduce the *isodifferential of a contravariant isovector field* on $\hat{M}$ over $\hat{R}$

$$\begin{aligned} \hat{d}\hat{X}^\beta &= (\hat{\partial}_\mu \hat{X}^\beta) \hat{\times} \hat{d}\hat{x}^\mu = \hat{T}_\mu{}^\rho \times (\partial_\rho \hat{X}^\beta) \hat{\times} \hat{I}^\mu{}_\sigma \times d\hat{x}^\sigma \\ &\equiv (\partial_\mu X^\beta) \times d\hat{x}^\mu = (\partial^\rho X^\beta) \times \hat{\eta}_{\rho\sigma} \times d\hat{x}^\sigma\,, \end{aligned} \tag{2.64}$$

where the last expression is introduce to recall that the contractions are in isospace. The preceding expression then shows that *isodifferentials of isovector fields coincide at the abstract level with conventional differentials for all Class I isotopies.*

The *isocovariant differential* can be defined by

$$\hat{D}\hat{X}^\beta = \hat{d}\hat{X}^\beta + \hat{\Gamma}_\alpha{}^\beta{}_\gamma \hat{\times} \hat{X}^\alpha \hat{\times} \hat{d}\hat{x}^\gamma\,, \tag{2.65}$$

with corresponding *isocovariant derivative*

$$\hat{X}^\beta_{\hat{|}\mu} = \hat{\partial}_\mu \hat{X}^\beta + \hat{\Gamma}_\alpha{}^\beta{}_\mu \hat{\times} \hat{X}^{\hat{\alpha}}\,, \tag{2.66}$$

where the *isochristoffel's symbols* are given by

$$\hat{\Gamma}^{\beta}_{\alpha\gamma}(x, v, a, \mu, \tau, \dots) = \frac{\hat{1}}{2} \hat{\times} (\hat{\partial}_\alpha \hat{\eta}_{\beta\gamma} + \hat{\partial}_\gamma \hat{\eta}_{\alpha\beta} - \hat{\partial}_\beta \hat{\eta}_{\alpha\gamma}) \times \hat{I} = \hat{\Gamma}_{\gamma\beta\alpha}\,, \quad (2.67a)$$

$$\hat{\Gamma}_{\alpha}{}^{\beta}{}_{\gamma} = \hat{\eta}^{\beta\rho} \hat{\Gamma}_{\alpha\rho\gamma} = \hat{\Gamma}_{\gamma}{}^{\beta}{}_{\alpha}, \qquad \hat{\eta}^{\beta\rho} = [(\hat{\eta}_{\mu\nu})^{-1}]^{\beta\rho}. \quad (2.67b)$$

Note the unrestricted functional dependence of the connection. Note also the abstract identity of the conventional and Class I isotopic connections. Note finally that *local numerical values of the conventional and isotopic connections coincide when computed in their respective spaces.* This is due to the fact that in Eqs. (2.67) $\hat{\eta} \equiv g(x)$ for exterior problems, while the value of derivatives ∂_μ and isoderivatives $\hat{\partial}_\mu$ coincide when computed in their respective spaces (Sec. 2.8).

Note however that, when projected in the conventional spacetime, the conventional and isotopic connections are different even in the exterior problem in which $\hat{\eta} = g(x)$,

$$\hat{\Gamma}_{\alpha\beta\gamma} = \frac{1}{2} \times (\hat{T}_\alpha{}^\mu \times \partial_\mu g_{\beta\gamma} + \hat{T}_\gamma{}^\rho \times \partial_\rho \hat{\eta}_{\alpha\beta} - \hat{T}_\beta{}^\sigma \times \partial_\sigma g_{\alpha\gamma}) \times \hat{I} \neq \Gamma_{\alpha\beta\gamma} \times \hat{I}\,, \quad (2.68)$$

The extension to covariant isovector fields and covariant or contravariant isotensor fields is consequential.

The isotopy of the proof of Ref. 15a, pages. 80 to 81, yields to the following:

Lemma 2.3 (Isoricci Lemma): *Under the assumed conditions, the isocovariant derivatives of all isometrics on isominkowski spaces are identically null,*

$$\hat{\eta}_{\alpha\beta\hat{|}\gamma} \equiv 0, \qquad \alpha, \beta, \gamma = 1, 2, 3, 4\,. \quad (2.69)$$

The novelty of the isominkowskian geometry is then illustrated by the fact that *the Ricci property persists under an arbitrary dependence of the metric, as well as under Minkowskian, rather than Riemannian axioms.*

The *isotorsion* on $\hat{M}$ is defined by

$$\hat{\tau}_\alpha{}^\beta{}_\gamma = \hat{\Gamma}_\alpha{}^\beta{}_\gamma - \hat{\Gamma}_\gamma{}^\beta{}_\alpha\,, \quad (2.70)$$

and coincides again with the conventional torsion at the abstract level, although the two torsions have significant differences in their explicit forms when both projected in our spacetime.

We now introduce: the *isocurvature tensor*

$$\hat{R}_\alpha{}^\beta{}_{\gamma\delta} = \hat{\partial}_\delta \hat{\Gamma}_\alpha{}^\beta{}_\gamma - \hat{\partial}_\gamma \hat{\Gamma}_\alpha{}^\beta{}_\delta + \hat{\Gamma}_\rho{}^\beta{}_\delta \hat{\times} \hat{\Gamma}_\alpha{}^\rho{}_\gamma - \hat{\Gamma}_\rho{}^\beta{}_\gamma \hat{\times} \hat{\Gamma}_\alpha{}^\rho \delta\,; \quad (2.71)$$

the *isoricci tensor*

$$\hat{R}_{\mu\nu} = \hat{R}_\mu{}^\beta{}_{\nu\beta}\,; \quad (2.72)$$

the *isocurvature isoscalar*

$$\hat{R} = \hat{\eta}^{\alpha\beta} \times \hat{R}_{\alpha\beta}\,; \quad (2.73)$$

the *isoeinstein tensor*

$$\hat{G}_{\mu\nu} = \hat{R}\mu\nu - \frac{\hat{1}}{2} \hat{\times} \hat{N}_{\mu\nu} \hat{\times} \hat{R}, \quad \hat{N}_{\mu\nu} = \hat{\eta}_{\mu\nu} \times \hat{I}\,; \quad (2.74)$$

and the *isotopic isoscalar*

$$\hat{\Theta} = \hat{N}^{\alpha\beta}\hat{\times}\hat{N}^{\gamma\delta}\hat{\times}(\hat{\Gamma}_{\rho\alpha\delta}\hat{\times}\hat{\Gamma}_{\gamma}{}^{\rho}{}_{\beta} - \Gamma_{\rho\alpha\beta}\hat{\times}\hat{\Gamma}_{\gamma}{}^{\rho}{}_{\delta})$$
$$= \hat{\Gamma}_{\rho\alpha\beta}\hat{\times}\hat{\Gamma}_{\gamma}{}^{\rho}{}_{\delta}\hat{\times}(\hat{N}^{\alpha\delta}\hat{\times}\hat{N}^{\gamma\beta} - \hat{N}^{\alpha\beta}\hat{\times}\hat{N}^{\gamma\delta}). \quad (2.75)$$

the latter one being new for the isominkowskian geometry (see below).

2.11. *The five identities of the isominkowskian geometry*

Tedious but simple calculations then yield the following *five basic identities of the isominkowskian geometry:*

Identity 1: *Antisymmetry of the last two indices of the isocurvature tensor*

$$\hat{R}_{\alpha}{}^{\beta}{}_{\gamma\delta} = -\hat{R}_{\alpha}{}^{\beta}{}_{\delta\gamma}\,; \quad (2.76)$$

Identity 2: *Symmetry of the first two indices of the isocurvature tensor*

$$\hat{R}_{\alpha\beta\gamma\delta} \equiv \hat{R}_{\beta\alpha\gamma\delta}\,; \quad (2.77)$$

Identity 3: *Vanishing of the totally antisymmetric part of the isocurvature tensor*

$$\hat{R}_{\alpha}{}^{\beta}{}_{\gamma\delta} + \hat{R}_{\gamma}{}^{\beta}{}_{\delta\alpha} + \hat{R}_{\delta}{}^{\beta}{}_{\alpha\gamma} \equiv 0\,; \quad (2.78)$$

Identity 4: *Isobianchi identity*

$$\hat{R}_{\alpha}{}^{\beta}{}_{\gamma\delta\hat{|}\rho} + \hat{R}_{\alpha}{}^{\beta}{}_{\rho\gamma\hat{|}\delta} + \hat{R}_{\alpha}{}^{\beta}{}_{\delta\rho\hat{|}\gamma} \equiv 0\,; \quad (2.79)$$

Identity 5: *Isofreud identity*

$$\hat{R}^{\alpha}{}_{\beta} - \frac{\hat{1}}{2}\hat{\times}\hat{\delta}^{\alpha}{}_{\beta}\hat{\times}\hat{R} - \frac{\hat{1}}{2}\hat{\times}\hat{\delta}^{\alpha}{}_{\beta}\hat{\times}\hat{\Theta} = \hat{U}^{\alpha}{}_{\beta} + \hat{\partial}_{\rho}\hat{V}^{\alpha\rho}{}_{\beta}\,, \quad (2.80)$$

where $\hat{\Theta}$ is the isotopic isoscalar and

$$\hat{U}^{\alpha}{}_{\beta} = -\frac{1}{2}\frac{\hat{\partial}\hat{\Theta}}{\hat{\partial}\hat{\eta}^{\alpha\beta}{}_{\hat{|}\alpha}}\hat{\eta}^{\alpha\beta}{}_{\hat{|}\beta}\,, \quad (2.81a)$$

$$\hat{V}^{\alpha\rho}{}_{\beta} = \frac{1}{2}[\hat{\eta}^{\gamma\delta}(\delta^{\alpha}{}_{\beta}\hat{\Gamma}_{\alpha}{}^{\rho}{}_{\delta} - \delta^{\rho}{}_{\beta}\hat{\Gamma}_{\gamma}{}^{\rho}{}_{\delta})$$
$$+ (\delta^{\rho}{}_{\beta}\hat{\eta}^{\alpha\gamma} - \delta^{\alpha}{}_{\beta}\hat{\eta}^{\rho\gamma})\hat{\Gamma}_{\gamma}{}^{\delta}{}_{\delta} + \hat{\eta}^{\rho\gamma}\hat{\Gamma}_{\beta}{}^{\alpha}{}_{\gamma} - \hat{\eta}^{\alpha\gamma}\hat{\Gamma}_{\beta}{}^{\rho}{}_{\gamma}]\,, \quad (2.81b)$$

Note that the conventional Riemannian geometry is generally thought to possess only *four* identities.[15] In fact, the above *fifth* identity is generally unknown in the contemporary technical literature in the field.

The latter identity was introduced by Freud[17a] in 1939, treated in detail by Pauli[1e] and then generally forgotten for a half a century, apparently, because of a conflict between the lack of source of Einstein's field equations in vacuum,

$$G^{\alpha}{}_{\beta} = R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}R = 0\,, \quad (2.82)$$

and the source in vacuum suggested by the Freud identity,

$$R^\alpha{}_\beta - \frac{1}{2}\delta^\alpha{}_\beta R - \frac{1}{2}\delta^\alpha{}_\beta \Theta = U^\alpha{}_\beta + \hat{\partial}_\rho V^{\alpha\rho}{}_\beta\,, \tag{2.83}$$

here written in a conventional space.

Following a suggestion by the author, Rund[17b] studied again the identity and proved that *the Freud identity is a bona fide identity for all Riemannian spaces irrespective of dimension and signature*, thus confirming the general need of a source also in vacuum (see below). In this paper we have presented, apparently for the first time, the isotopies of the Freud identity, that is, its formulation in isominkowskian spaces as characterized by the isodifferential calculus.

Note that *all conventional and isotopic identities coincide at the abstract level.*

2.12. *Isoparallel transport and isogeodesics*

An isovector field $\hat{X}^\beta$ on $\hat{M} = \hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ is said to be transported by *isoparallel displacement* from a point $\hat{m}(\hat{x})$ on a curve $\hat{C}$ on $\hat{M}$ to a neighboring point $\hat{m}\prime(\hat{x}+\hat{d}\hat{x})$ on $\hat{C}$ if

$$\hat{D}\hat{X}^\beta = \hat{d}\hat{X}^\beta + \hat{\Gamma}_\alpha{}^\beta{}_\gamma \hat{\times} \hat{X}^\alpha \hat{\times} \hat{d}\hat{x}^\gamma \equiv 0\,. \tag{2.84}$$

or in integrated form

$$\hat{X}^\beta(\hat{m}\prime) - \hat{X}^\beta(m) = \hat{\int}_{\hat{m}}^{\hat{m}\prime} \frac{\hat{\partial}\hat{X}^\beta}{\hat{\partial}\hat{x}^\alpha} \frac{\hat{d}\hat{x}^\alpha}{\hat{d}s} \hat{\times} \hat{d}s\,. \tag{2.85}$$

where one should note the isotopic character of the integration. The isotopy of the conventional case[15a] then yield the following:

Lemma 2.4: *Necessary and sufficient condition for the existence of an isoparallel transport along a curve $\hat{C}$ on a $(3+1)$-dimensional isominkowski space is that all the following equations are identically verified along $\hat{C}$*

$$\hat{R}_\alpha{}^\beta{}_{\gamma\delta} \hat{\times} \hat{X}^\alpha = 0\,, \qquad \beta, \gamma, \delta = 1, 2, 3, 4\,. \tag{2.86}$$

Note, again, the abstract identity of the conventional and isotopic parallel transport. However, it is easy to see that the projection of the isoparallel transport in ordinary spacetime is structurally different than the conventional parallel transport. In particular, if the latter is represented by an arrow, one would note a twisting action as occurring in the reality of motion within physical media, which is evidently absent in the exterior case.

Along similar lines, we say that a smooth path $\hat{x}_\alpha$ on $\hat{M}$ with isotangent $\hat{v}_\alpha = \hat{d}\hat{x}_\alpha / \hat{d}\hat{s}$ is an *isogeodesic* when it is solution of the isodifferential equations

$$\frac{\hat{d}\hat{v}^\beta}{\hat{D}\hat{s}} = \frac{}{\hat{d}\hat{s}} + \hat{\Gamma}_{\alpha\beta\gamma} \hat{\times} \frac{\hat{d}\hat{x}^\alpha}{\hat{d}\hat{s}} \hat{\times} \frac{\hat{d}\hat{x}^\gamma}{\hat{d}\hat{s}} = 0\,. \tag{2.87}$$

It is easy to prove the following:

Lemma 2.5: *The isogeodesics of an isominkowskian space $\hat{M}$ are the curves verifying the isovariational principle*

$$\hat{\delta}\hat{\int}[\hat{N}_{\alpha\beta}(\hat{x},\hat{v},\hat{a},\mu,\tau,\ldots)\hat{\times}\hat{d}\hat{x}^{\alpha}\hat{\times}\hat{d}\hat{x}^{\beta}]^{1/2}=0\,. \tag{2.88}$$

where again isointegration is understood.

Finally, we point out the property which is inherent in the notion of isotopies as realized in this paper according to which *geodesic trajectories in ordinary space coincide with the corresponding isogeodesic trajectories in isospace.* For instance, if a circle is originally a geodesic, its image under isotopy in isospace remains the perfect circle, the *isocircle* (Sec. 2.6), even though its projection in the original space is an ellipse. The same preservation in isospace occurs for all other curves.

The differences between a geodesic and an isogeodesic therefore emerge only when projecting the latter in the space of the former.

An empirical but conceptually effective rule is that *interior physical media "disappear" under their isogeometrization*, in the sense that actual trajectories under resistive forces due to physical media (which are not geodesics of a Minkowski space) are turned into isogeodesics in isospace with the shape of the geodesics in the absence of resistive forces.

The simplest possible example is given by the isominkowskian representation of a straight stick partially immersed in water. In conventional representations the stick penetrating in water with an angle α appears as bended at the point of immersion in water with and angle $\gamma=\alpha+\beta$. In isominkowskian representation the stick remains straight also in its immersion because the isoangle $\hat{\gamma}=\gamma\times\hat{I}_{\gamma}$ recovers the original angle α for $\hat{I}_{\gamma}=\alpha/(\alpha+\beta)$.

2.13. *Gravitational field equations in isominkowskian geometry*

The isotopy of the proof of the Theorem of Ref. 15a, page 321, leads to the following property:

Theorem 2.1: *Under the assumed regularity and continuity conditions, the most general possible isolagrange equations (Appendix A) $\hat{E}^{\alpha\beta}=0$ of Class I along an actual isopath $\hat{P}_0$ on a $(3+1)$-dimensional isominkowskian space for the characterization of the exterior or interior gravitational problems of matter satisfying the properties: (1) Symmetry condition, $\hat{E}^{\alpha\beta}=\hat{E}^{\beta\alpha}$, (2) Contracted isobianchi identity, $\hat{E}^{\alpha\beta}{}_{\hat{|}\beta}\equiv 0$; and (3) The isofreud identity; are given by*

$$\hat{E}^{\alpha\beta}=\hat{\alpha}\hat{\times}\hat{N}^{\hat{\frac{1}{2}}}\hat{\times}\left(\hat{R}^{\alpha\beta}-\hat{\frac{1}{2}}\hat{\times}\hat{N}^{\alpha\beta}\hat{\times}\hat{R}-\hat{\frac{1}{2}}\hat{\times}\hat{N}^{\alpha\beta}\hat{\times}\hat{\Theta}\right)+\hat{\beta}\hat{\times}\hat{N}^{\alpha\beta}-\hat{N}^{\hat{\frac{1}{2}}}\hat{\times}\hat{S}^{\alpha\beta}=0\,, \tag{2.89}$$

where: $\hat{N}^{\hat{\frac{1}{2}}}=(\det\hat{\eta})^{1/2}\times\hat{I}$; $\hat{\alpha}=\alpha\times\hat{I}$, $\hat{\beta}=\beta\times\hat{I}$, $\hat{\frac{1}{2}}=\frac{1}{2}\times\hat{I}$, etc.; α and β are constants; and $\hat{S}^{\alpha\beta}$ is a source tensor. For $\alpha=1$ and $\beta=0$ the isogravitation field equations can be written

$$\hat{R}^{\alpha\beta}-\frac{1}{2}\times\hat{\eta}^{\alpha\beta}\times\hat{R}-\frac{1}{2}\times\hat{\eta}^{\alpha\beta}\times\hat{\Theta}=k(\hat{t}^{\alpha\beta}-\hat{\tau}^{\alpha\beta})=\hat{U}^{\alpha}{}_{\beta}+\hat{\partial}_{\rho}\hat{V}^{\alpha\rho}{}_{\beta}\,, \tag{2.90}$$

where $\hat{t}^{\alpha\beta}$ is a source isotensor, $\hat{\tau}^{\alpha\beta}$ is a stress-energy isotensor and k is a constant.

Note the appearance in Eqs. (2.90) of the isotopic isoscalar $\hat{\Theta}$ in the l.h.s and of source terms in the r.h.s., both originating from the isofreud identity.

The simplest possible formulation for the exterior gravitational problem in vacuum is given by the *isoeinstein field equations*

$$\hat{G}^{\alpha\beta} = \hat{R}^{\alpha\beta} - \frac{1}{2} \times \hat{\eta}^{\alpha\beta} \times \hat{R} = 0\,. \tag{2.91}$$

As expected, the above equations are numerically identical to the conventional equations when computed in isospace over isofield for the reasons indicated earlier, thus preserving the related experimental verifications.

However, when projected in conventional spacetime, certain differences appear due to the isotopic elements which multiply conventional derivatives. In particular, the isotensor $\hat{\Gamma}_\alpha{}^\beta{}_\gamma$ is of zero-order in the same terms; the isocurvature isotensor $\hat{R}_\alpha{}^\beta{}_{\gamma\delta}$ is of first-order in the same terms; the isotensor $\hat{R}_{\mu\nu}$ is of first-order and the isoscalar $\hat{R}$ is of zero-order in the same terms. When written projected in our spacetime, Eqs. (2.91) then contains new terms which are of first-order in $\hat{T}_{\mu\mu}$. But the latter are very close to 1. The compatibility of Eqs. (2.91) with current experimental data of Eqs. (2.82) then follows also in our spacetime.

The isofield equations in their general form (2.90) are however new, and their solutions will be studied elsewhere.

An in depth investigation of field Eqs. (2.90) has been recently conducted by Vacaru [8d] with the inclusion of spin, which results in the confirmation of Eqs. (2.90) plus an additional condition on spin-density. Even though Vacaru's studies have been formulated on *isoriemannian* rather than isominkowskian spaces as formulated in Ref. 5g, the results can be easily reinterpreted as belonging to the latter isospaces via the mere redefinition of the isounit.

2.14. *Experimental verifications and predictions of the isominkowskian geometry*

As indicated in Sec. 1.4, the isominkowskian geometry has already received a number of preliminary, yet significant experimental verifications, when applicable, in classical physics, particle physics, nuclear physics, astrophysics, superconductivity and other fields, as outlined in Ref. 4h (see also Ref. 9)

The best predictions of the isominkowskian geometry suitable for experimental verifications are those characterized by the *isotopies of Einstein's axioms of the special relativity* studied in details in Refs. 4g and 9. As only one illustrative example, we have the *isodoppler's law* for light propagating within atmospheres or chromospheres (here written for 90° aberration)

$$\hat{\omega} = \omega_0 \times [1 - (v^2/{n_k}^2)/({c_0}^2/{n_4}^2)]^{1/2}\,, \tag{2.92}$$

which has been verified via a numerical representation of the large difference in cosmological redshift between certain quasars and their associated galaxies when physically connected according to gamma spectroscopic evidence.[9f]

The physical interpretation is so simple to appear trivial, and merely consists of the representation of two contributions, the first due to the *decrease of the speed of light* $c_0 \to c = c_0/n_4 < c_0$ *within the huge quasars chromospheres*, and the second to a *geometric representation of their inhomogenity and anisotropy.*

Note that the new speed $c = c_0/n_4$ alone cannot be plotted in the conventional Doppler's shift law because of a host of inconsistencies, e.g., with respect to axiomatic and invariance laws. The isominkowskian formulation with the duly symmetrized spacetime characteristic functions n_s and n_4, then uniquely and unambiguously follows.

The above occurrence permit an exact-numerical representation of the large difference in cosmological redshift between quasars and associated galaxies when physically connected according to which *light exits the quasars chromospheres already isoredshifted because of the decrease of the speed in the interior of the quasars chromospheres as well as the inhomogeneity and anisotropy of the medium.*

Note that the cosmological redshift of the galaxy is here assumed as of purely Minkowskian nature and, as such, it remains unchanged, even though the latter too can be at least in part of isotopic character, i.e., part of the cosmological redshift of light originating from galaxies could be due to the decrease of the speed of light in their interior.

Note that the reduction of light propagating within the chromospheres to photons in second quantization scattering among molecules, would eliminate altogether the effect here predicted and verified. In fact, it would reduce motion to empty space for which $n_s = n_4 = 1$, thus eliminating the isoredshift altogether.

The above studies are further confirmed by the capability of the isominkowskian geometry to provide an exact-numerical representation of the internal quasar red- and blue-shift[9g] which, to our best knowledge, is the sole representation existing at this writing. Its origin is due to the dependence of the index of refraction n_4 on the *frequency* ω of light, $n_4 = n_4(\omega, \ldots)$, thus confirming again the need for a locally varying speed of light and consequential isominkowskian geometrization of physical media.

For various other exact-numerical verifications we refer the interested reader to the outline of Sec. 1.4 and to Refs. 4h and 9.

We close this section with a few comments and predictions.

First, we should point out that the transition from the *Riemannian* to the *isominkowskian* description of gravity implies the transition from the "description" of gravitation, to the study of its "origin". In fact, isofield Eqs. (2.90) are submitted as equations representing the origin of the gravitational field at the level of the particles constituting the body considered.

More specifically, Eqs. (2.90) are permit the elimination of the vexing "unification" of the gravitational and electromagnetic fields, and the assumption of their "identification" at the particle level in view of the primarily electromagnetic nature of the mass of all elementary particles constituting any given body.[3g]

In fact, it was shown in Ref. 3e that, even though the total charge is null, in the exterior of a particle such as the π° there is a nowhere null first-order electromagnetic field which accounts for the virtual entirety of the mass of the particle, according to the rule

$$M_{\pi^\circ}{}^{\text{Grav.,Ext.}} = \int dv \tau^{00}{}_{\text{elm}} \approx M_{\pi^\circ} \,, \tag{2.93}$$

with corrections due to short range (s.r.) weak and strong interactions for the interior problem

$$M_{\pi^\circ}{}^{\text{Iner.,Int.}} = \int dv (\tau^{00}{}_{\text{elm}} + \tau^{00}{}_{\text{s.r.}}) \equiv M_{\pi^\circ} \tag{2.94}$$

In this way the gravitational field is entirely reduced to the field originating mass itself without any presence of any mass term in the field equations.

Note that the above theory in the origin of the gravitational field provides a quantitative representation of the difference between inertial (interior) and gravitational (exterior) mass; it implies the assumption of a nowhere null first-order source *in vacuum* even for bodies with *null total charge and magnetic moments*, as requested by the Freud identity; and it predicts a complete equivalence between gravitational and electromagnetic phenomenology, including the *prediction of antigravity for elementary antiparticles (such as a position) in the field of matter*, a prediction which is fully confirmed by the novel isodual theory of antimatter.[13]

It should be noted that the above conclusions are so strong that they possible experimental disproof would require the reconstruction of the contemporary theory of elementary particles into such a form which does not permit masses to have a primary electromagnetic origin.

We should also note that the isotopic scalar is written in the l.h.s., rather than in the r.h.s. of field Eqs. (2.90) because Einstein's tensor $G_{\mu\nu} = R^\alpha{}_\beta - \frac{1}{2}\delta^\alpha{}_\beta R$ does not preserve the Ricci Lemma under isotopies, while such preservation occurs for the tensor $S_{\mu\nu} = R^\alpha{}_\beta - \frac{1}{2}\delta^\alpha{}_\beta R - \frac{1}{2}\delta^\alpha{}_\beta \Theta$ (see Vol. I, Ref. 4g, Ch. 5 for details).

Some of the most intriguing predictions of the isominkowskian geometry which need future experimental verifications are those dealing with the characterization of *a new notion of spacetime*, where the novelty rests in the basic units. As an illustration, the fact that we can *visually* observe an astrophysical body (i.e., we can observe it via the light it emits) does not necessarily means that particular body evolves with our own time t, because it could evolve in a substantially different time t'. As indicated earlier, the isominkowskian geometry predicts that time is a local quantity depending on the local gravitational field, Eqs. (2.57), while any difference in the flow of time is lost in the interconnecting light.

The above occurrence is expressed via the notion of *isotime* $\hat{t} = \hat{t} \times \hat{I}_t$ where $\hat{I}_t = \hat{I}_{44}$ is the *time isounit.* On our Earth we have the value $\Delta\hat{t} = \Delta t \times I_t$, $I_t = 1$ sec, while in other astrophysical bodies we may have $\Delta\hat{t} = \Delta t' \times \hat{I}_t$. The invariance law

$$\Delta\hat{t} = \Delta t \times I_t \equiv \Delta t' \times \hat{I}_t = \text{inv}\,. \tag{2.95}$$

then implies the possibility that other astrophysical bodies have a basically different flow of time ($\Delta t' \neq \Delta t$), yet be compatible with our sensory perception as well as with the abstract axioms of the special relativity.

Note that we are referring here to a *new variation of time*, which is different than the conventional *time dilation with speed*,[1e] which is predicted for *stationary* observers with different gravitational fields and which requires new specific tests.

A similar occurrence evidently holds for space. In fact, lengths are now characterized by *isolengths* $\hat{L} = L \times \hat{I}_L$ and their invariance implies the possibility of our measuring a given length Δ_L for a given astrophysical body, while the length actually admitted in their own frame can have a basically different value $\Delta L'$ such that

$$\Delta\hat{L} = \Delta L \times \hat{I}_L \equiv \Delta L' \times \hat{I}'_L \,. \tag{2.96}$$

Note that we are referring here to a *new variation of length* which is different than the familiar *length contraction with speed*[1e] which is also predicted for *stationary* observers with different gravitational fields, and which also requires specific experimental verifications.

Note from Eqs. (2.57) that the isovariations of time and length with the gravitational field are inverse to each others, much along what happens for the conventional variations with speed.

Since under isotopies the Cartesian coordinates assume different values for different axes, invariant (2.96) also implies *the possible fallacy of our perception of shapes.* As an example, a cube is mapped under isotopies into an arbitrary geometric figure while preserving again the compatibility with our sensory perception.

The above predictions are illustrated in Vol. I of Ref. 4g, with the notion of *isobox*, which is a geometric figure appearing as a cube of a given side L to an *outside Minkowskian observer* at time t, while *the same figure* has an arbitrary size and shape and it is at an arbitrarily preceding or subsequent time t' for an *internal isominkowskian observer.*

In addition to the prediction of times, lengths and shapes different than those perceived by us, *the isominkowskian geometry also predicts spacetime dimensions different than* $(3+1)$. The case was first studied by P. A. M. Dirac in two of his last papers [18] on a generalization of his own celebrated equation which has resulted to have an essential isotopic structure evidently unknown to Dirac himself (see Ch. 10, Vol. II, Ref. 4g).

In fact, "Dirac's generalization of Dirac's gamma matrices" was proposed according to the isotopic structure[18]

$$\alpha_\mu \times \hat{T} \times \alpha_\nu + \alpha_\beta \times \hat{T} \times \alpha_\mu = 2 \times \hat{T} \times \eta_{\mu\nu} \,, \tag{2.97}$$

the only difference with Dirac's formulation being his use of the symbol β for the isotopic element $\hat{T}$. In his instinctive brilliance, Dirac selected for the $\hat{T}$ matrix the *p(positive-definite yet) off-diagonal form*

$$\hat{T} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{2.98a}$$

$$\det \hat{T} = 1\,, \qquad \hat{I} = T^{-1}\,, \tag{2.98b}$$

(which therefore qualifies $\hat{T}$ as a fully acceptable, positive-definite isotopic element of Class I).

The isominkowskian geometry characterized by isotopic element (2.98) is intriguing indeed. Its most salient property is that *the isometric is nondegenerate,* $\text{Det}\hat{\eta} = \text{Det}(\hat{T} \times \eta) = -1$, *but the isoinvariant is degenerate,*

$$\begin{aligned} \hat{x}^{\hat{2}} = x^{\mu}\hat{\eta}_{\mu\nu}x^{\nu} &= x^1x^3x^3x^4 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \\ &= x^1x^3 - x^2x^4 - x^3x^1 - x^4x^2 = -2x^2x^4\,, \end{aligned} \tag{2.99}$$

namely, *the dimension of our spacetime is contracted under Dirac's isotopy from four to two dimensions.*

In turn, the above contraction has truly remarkable implications, such as the lifting of the spin $s = \frac{1}{2}$ to spin $s = 0$, as originally derived by Dirac himself [19], confirmed by isotopic methods [4g] and used for a quantitative representation of the synthesis of the neutron occurring in stars in their formation, from protons and electrons *only.*[4e]

It is instructive for the interested reader to see that the same dimensional contraction occurs for all other possible realizations, such as for $\eta = (+1, -1, -1, -1)$ and related ordering of the components $x = \{x^4, x^1, x^2, x^3\}$. As a result, the dimensional contraction $(1, 2, 3, 4) \to (2, 4)$ is intrinsic in Dirac's realization of the isominkowskian geometry, and so are its rather peculiar properties, such as the contraction of the three space dimensions (1,2,3) down to a one-dimensional line along the y-axis.

All in all we achieved our objectives in this section if we succeeded in conveying the expectation that the universe can be dramatically more complex of what perceived by our limited senses. An important function of the isotopies is that of rendering these complexities compatible with our sensory perception. In fact, for conventional methods, changes in length or in dimension are indeed perceived by our senses, while they are rendered fully compatible with our senses by the axiom-preserving isotopies.

2.15. *Genominkowskian and hyperminkowskian geometries*

Physics is a discipline that will never admit "final theories". No matter how valid any given theory appears to be, its structural generalization for broader conditions is only a matter of time. Along these lines, no maturity can be claimed on new advances unless their limitations are identified in the original proposal.

The most visible limitation of the isominkowskian geometry is that of being *structurally reversal* (for time independent isounits), while physical reality, particularly that of interior problems, is *structurally irreversible.*

In view of this occurrence, the "axiom-preserving" isotopies were originally submitted in Ref. 5c as particular cases of the broader *genotopies* which (from their Greek meaning) are "axiom-inducing", namely, they generally violate the original axioms in favor of covering axioms.

The transition from the isotopies to the genotopies is essentially characterized by the relaxation of the symmetric character of the isounit, while preserving their real-valued and nonsingular characters. This implies *two* different *genounits* with two corresponding ordered *genoproducts to the left and to the right*

$$^{<}\hat{I}(t,x,v,a,\mu,\tau,\ldots) = (\hat{T})^{-1}\,, \qquad \hat{I}^{>}(t,x,v,a,\mu,\tau,\ldots) = (\hat{T}^{>})^{-1}\,,$$
$$^{<}\hat{I} = \hat{I} \qquad ^{<}\hat{I} = (\hat{I}^{>})^{t}\,. \tag{2.100a}$$
$$^{<}\hat{a} < {}^{<}\hat{b} = {}^{<}\hat{a} \times {}^{<}\hat{T} \times {}^{<}\hat{b}\,, \qquad \hat{a}^{>} > \hat{b}^{>} = \hat{a}^{>} \times \hat{T}^{>} \times \hat{b}^{>}\,, \tag{2.100b}$$
$$^{<}\hat{I} < {}^{<}\hat{a} = {}^{<}\hat{a} <^{<} \hat{I} \equiv {}^{<}\hat{a}\,, \qquad \hat{I}^{>} > \hat{a}^{>} = \hat{a}^{>} > \hat{I}^{>} \equiv \hat{a}^{>}\,, \tag{2.100c}$$

representing *motion backward and forward in time*, respectively. All isotopic methods then have to be reconstructed in a two-fold way, including fields, spaces, geometries, etc.[5g]

The basic analytic equations[5g] are those originally proposed by Hamilton, with *external terms*, which represent irreversibility via nonpotential effects. The algebraic structure of the theory is that of Lie-admissible type.[14a–14f;14g] The emerging geometry is that of the *forward* and *backward genominkowskian spaces* such as

$$\hat{M}^{>}(\hat{x}^{>},\hat{\eta}^{>},\hat{R}^{>}) : \qquad \hat{x}^{>} = x \times \hat{I}^{>}\,, \qquad \hat{\eta}^{>} = T^{>} \times \eta\,, \tag{2.101a}$$
$$\hat{x}^{>2>} = \hat{x}^{>\mu} > \hat{N}^{>}{}_{\mu\nu} > \hat{x}^{>\nu} = (x^{\geq} \times \hat{\eta}^{>}{}_{\mu\nu} \times x^{\nu})^{\times \hat{I}^{>}} \in \hat{R}^{>}\,. \tag{2.101b}$$

The physically important aspect is that the *genominkowskian geometry is structurally irreversible*, i.e., it is irreversible even for a reversible Lagrangian or Hamiltonian. As a result, it allows an *axiomatization of irreversibility*, for which the spaces were constructed in the first place. As an illustration, the studies by Ellis, Mavromatos and Nanopoulos[9] on black holes dynamics are irreversible and, as such, they require the geno- rather than the *iso*-minkowskian geometry.

The main idea is that all known action-at-a-distance, potential interactions are reversible, while irreversibility is a "contact" effect, i.e., it occurs in interior systems with nonpotential internal effects. As a result, irreversibility should be represented with *anything except a Lagrangian or a Hamiltonian.* The line of investigation identified above is to represent irreversibility with the *geometry itself*, as we shall study in more details in future works.

Intriguingly, both the forward and backward genominkowski spaces result to satisfy the abstract Minkowskian axioms because their underlying mechanism is the same as that of the isotopies, i.e., the Minkowski metric is lifted in the amount $\eta \to \hat{\eta}^> = \hat{T}^> \times \eta$ while the unit is lifted in the *inverse* amount $I \to \hat{I}^> = (\hat{T}^>)^{-1}$, thus preserving the original axioms.

We therefore reach the remarkable conclusion that, when treated with the appropriate genomathematics [5g], *the abstract Minkowskian axioms admit nonsymmetric metrics*, $\hat{\eta}^> \neq (\hat{\eta}^>)^t$ (in Ref. 5g we had reached the corresponding property for the Riemannian axioms). As a matter of fact, this is the property at the foundation of our axiomatization of irreversibility.

By no means the forward and backward genominkowskian geometries are a "final theory". In fact, their further structural generalizations were identified with their own proposal[5g] under the name of *forward and backward hyperminkowskian geometries.* They are characterized by the assumption of the genounits as *ordered sets* with corresponding ordered *hyperproduct to the right and to the left*

$$\{^<\hat{I}\} = \{^<\hat{I}_1, ^<\hat{I}_2, ^<\hat{I}_3, \ldots\} = \{^<\hat{T}\}^{-1},$$
$$\{\hat{I}^>\} = \{\hat{I}_1^>, \hat{I}_2^>, \hat{I}_3^>, \ldots\} = \{\hat{T}^>\}^{-1}\ \{^<\hat{I}\} = \{\hat{I}^>\}^t, \qquad (2.102a)$$
$$\{^<\hat{T}\} = \{^<\hat{T}_1, ^<\hat{T}_2, ^<\hat{T}_3, \ldots\},$$
$$\{\hat{T}^>\} = \{\hat{T}_1^>, \hat{T}_2^> \hat{T}_3^>, \ldots\}, \qquad (2.102b)$$
$$\{^<\hat{a}\}\{<\}\{^<\hat{b}\} = \{^<\hat{a}\} \times \{^<\hat{T}\} \times \{^<\hat{b}\},$$
$$\{\hat{a}^>\}\{>\}\{\hat{b}^>\} = \{\hat{a}^>\} \times \{\hat{T}^>\} \times \{\hat{b}^>\}, \qquad (2.102c)$$
$$\{^<\hat{I}\}\{<\}\{^<\hat{a}\} = \{^<\hat{a}\}\{<\}\{^<\hat{I}\} \equiv \{^<\hat{a}\},$$
$$\{\hat{I}^>\}\{>\}\{\hat{a}^>\} = \{\hat{a}^>\}\{>\}\{\hat{I}^>\} \equiv \{\hat{a}^>\}. \qquad (2.102d)$$

This implies a further hyperstructural lifting of the forward and backward genominkowskian geometry which is irreversible as well *as multivalued.*

Intriguingly, each of the latter formulation is admitted by the abstract Minkowskian axioms, thus illustrating again the possible complexity of the universe in a form compatible with our sensory perception.

3. Isodual Isominkowskian Geometry for the Representation of Antimatter

3.1. *Introduction*

As recalled in Sec. 1, one of the most fundamental structural differences between the electroweak and gravitational interactions is the representation of antimatter via *negative-energy* solutions for the former and *positive-definite* energy-momentum tensor for the latter.

The resolution here under study is based on a completely novel theory of antimatter which is characterized at *all* levels, including Newtonian mechanics, gravitation and electroweak interactions, by a new mathematics called *isodual isomathematics*,[5g] which is the image of the isomathematics of the preceding section under the map of all possible quantities Q and operations into their *anti-Hermitean images*

$$Q \to Q^d = -Q^\dagger . \tag{3.1}$$

When systematically applied to *all* treatments of matter, the above map yields an anti-isomorphic image of the representation of matter, which is applicable beginning at the classical level and then persists at the operator level where it results to be equivalent to charge conjugation.[13]

The fundamental notion of the emerging new theory of antimatter is, again, that of *new numbers*, this time numbers with *negative-unit*, which see their origin in the structure of Dirac's gamma matrices and from which the entire theory can be uniquely and unambiguously derived.

In this section we cannot possibly review the new isodual theory of antimatter. We shall therefore limit ourselves to the outline of the essential notions and their application, apparently for the first time, to the formulation of the *isodual isominkowskian geometry* in its full version, that inclusive of its isodual Riemannian content. Other aspects of the isodual theory of antimatter are indicated in the appendices.

In this section we shall study the isodual isominkowskian *geometry* as a foundation of Ref. 11 and regret to be unable to study its *physical* implications.

We merely mention that the isodual theory of antimatter recovers all existing experimental data at both classical and operator levels. At the classical level the only available experimental data on antimatter are those of *electromagnetic* character, and they are easily represented by a simple isodual reinterpretation of the Coulomb, Lorentz and other classical laws for particle-antiparticles and antiparticle-antiparticles.[13a,13d,4h]

At the operator level the available experimental data are of *electroweak* character and they too are represented identically [11,13c]. In essence, in the conventional theory (see, e.g. Ref. 16d), antiparticles are represented via charge conjugation, the solutions are of both *advanced* and *retarded* character, while the advanced ones are generally ignored. Under isoduality retarded solutions represent *particles* and are formulated on the Minkowski space, while *advanced* solutions represent *antiparticles*, are formulated on the isodual Minkowski space and result to be equivalent to conventional charge conjugate solutions. As a consequence, all numerical results remain completely unchanged and are only subjected to a *reinterpretation.*

Note that in the above outline is based on the *conventional* Minkowski space and its isodual and *not* on the isominkowski space and its isodual, evidently because of the lack of inclusion of gravitation for which no experimental information for antimatter is available at this time.[13e] The isotopic lifting of the above setting for the inclusion of gravitation is however straightforward and implies the novel predictions of the isodual theory of antimatter indicated in Sec. 1.7.

We should finally mention that the isodual re-interpretation of the Dirac equation (see Ref. 4g, Vol. II, Ch. 10) is independent from other approaches, such as that permitted by Clifford algebras.[16e] Again, the main difference is in the basic numbers and fields. In fact, the former use *new numbers with negative unit* -1, while the latter use *conventional numbers with positive unit* $+1$. Moreover, the former is conceived to provide a characterization of antiparticles at the level of first quantization, while the latter are more aligned with conventional theories in second quantization [16d].

We regret to be unable to point out for brevity that the characterization of the Dirac equation via Clifford algebras admits a simple yet intriguing and significant reinterpretation via isoduality and a further lifting under isotopies to incorporate gravitation in the generalized unit.

3.2. *Isodual isounits, isonumbers, and isofields*

Let $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ be an isofield of isoreal numbers $\hat{R}(\hat{n}, \hat{+}, \hat{\times})$, isocomplex numbers $\hat{C}(\hat{c}, \hat{+}, \hat{\times})$ or isoquaternionic numbers $\hat{Q}(\hat{q}, \hat{+}, \hat{\times})$ with the familiar additive isounit $\hat{0} = 0$, multiplicative isounit $\hat{I}$, elements $\hat{a} = a \times \hat{I}$, $a = n, c, q$, isosum $\hat{a}_1 \hat{+} \hat{a}_2$, $\hat{a} \hat{+} \hat{0} = \hat{0} \hat{+} \hat{a} = \hat{a}$, and isomultiplication $\hat{a}_1 \hat{\times} \hat{a}_2 = \hat{a}_1 \times \hat{T} \times \hat{a}_2 = (a_1 \times a_2) \times \hat{I}$, $\hat{I} = \hat{T}^{-1}$, $\hat{a} \hat{\times} \hat{I} = \hat{I} \hat{\times} \hat{a} = \hat{a}$, $\forall \hat{a}, \hat{a}_1, \hat{a}_2 \in \hat{F}$.

The *isodual isofields*, first introduced in Ref. 4b and then studied in details in Ref. 5f, are the image $\hat{F}^d = \hat{F}^d(\hat{a}^d, \hat{+}^d, \hat{\times}^d)$ of $\hat{F}(\hat{a}, \hat{+}, \hat{\times})$ characterized by the isodual map of the isounit

$$\hat{I} \to \hat{I}^d = -\hat{I}^\dagger = -\hat{I}\,, \tag{3.2}$$

which implies: *isodual isonumbers*

$$\hat{a}^d = \hat{a}^\dagger \times \hat{I}^d = -a^\dagger \times \hat{I} = -\hat{a}^\dagger\,, \tag{3.3}$$

where † is the identity for real numbers $n^\dagger = n$, complex conjugation $c^\dagger = \bar{c}$ for complex numbers c, and Hermitean conjugation $q^\dagger$ for quaternions $q^\dagger$; *isodual isosum*

$$\hat{a}_1{}^d \hat{+}^d \hat{a}_2{}^d = -(\hat{a}_1{}^\dagger \hat{+} \hat{a}_2{}^\dagger); \tag{3.4}$$

and *isodual isomultiplication*

$$\hat{a}_1{}^d \hat{\times}^d \hat{a}_2{}^d = \hat{a}_1{}^d \times \hat{T}^d \times \hat{a}_2{}^d = -(\hat{a}_1{}^\dagger \hat{\times} \hat{a}_2{}^\dagger)\,; \tag{3.5}$$

under which $\hat{I}^d = \hat{T}^{d-1}$ is the correct left and right unit of $\hat{F}^d$,

$$\hat{I}^d \hat{\times}^d \hat{a}^d = \hat{a}^d \hat{\times}^d \hat{I}^d \equiv \hat{a}^d, \qquad \forall \hat{a}^d \in \hat{F}^d\,, \tag{3.6}$$

in which case (only) $\hat{I}^d$ is called *isodual isounit* and $\hat{T}^d$ the *isodual isotopic element.*

We have in this way the *isodual isoreal field* $\hat{R}^d(\hat{n}^d, \hat{+}^d, \hat{\times}^d)$ with *isodual isoreal numbers*

$$\hat{n}^d = -\hat{n}^\dagger \equiv -n \times \hat{I}, \qquad n \in R, \qquad \hat{n} \in \hat{R}, \qquad \hat{n}^d \in \hat{R}^d\,; \tag{3.7}$$

the *isodual isocomplex field* $\hat{C}^d(\hat{c}^d, \hat{+}^d, \hat{\times}^d)$ with *isodual isocomplex numbers*

$$\hat{c}^d = -\bar{c} \times \hat{I} = -(n_1 - i \times n_2) \times \hat{I} = (-n_1 + i \times n_2) \times \hat{I},$$
$$n_1, n_2 \in R, \qquad c = n_1 + i \times n_1 \in C, \qquad c^d \in C^d; \tag{3.8}$$

and the *isodual isoquaternionic* field which is not used in this paper for brevity.

Under the above assumptions, $\hat{F}^d(\hat{a}^d, \hat{+}^d, \hat{\times}^d)$ verifies all the axioms of a field [4b,5f], although $\hat{F}^d$ and $\hat{F}$ are anti-isomorphic, as desired. This establishes that the field of numbers can be equally defined either with respect to the generalized unit $\hat{I} > 0$ or with respect to its negative image $-\hat{I} < 0$. The key point is the preservation of the axiomatic character of the latter via the isoduality of the isoproduct. In other words, the set of isodual isonumbers $\hat{a}^d$ with unit $-\hat{I}$ and *isotopic* product does not constitute a field because $\hat{I}^d \hat{\times} \hat{a}^d \neq \hat{a}^d$.

It is also evident that *all operations of isonumbers implying multiplication must be subjected for consistency to isoduality.* This implies the *isodual isosquare root*

$$\hat{a}^{d\hat{\frac{1}{2}}d} = -(-a^\dagger)^{\frac{1}{2}} \times \hat{I}^{\frac{1}{2}}, \qquad \hat{a}^{d\hat{\frac{1}{2}}d} \hat{\times}^d \hat{a}^{d\hat{\frac{1}{2}}d} = \hat{a}^d;$$
$$I^{d\hat{\frac{1}{2}}d} = -i \times \hat{I} = \hat{i}^d; \tag{3.9}$$

the *isodual isoquotient*

$$\hat{a}^d \hat{/}^d \hat{b}^d = -(\hat{a}^d \hat{/} \hat{b}^d) = -(\hat{a} \hat{/} \hat{b}) = \hat{c}^d, \qquad \hat{b}^d \hat{\times}^d \hat{c}^d = \hat{a}^d; \tag{3.10}$$

and so on.

A property of isodual isofields of fundamental relevance for our characterization of antimatter is that *they have negative-definite norm,* called *isodual isonorm*[3,4]

$$\hat{|}\hat{a}^d\hat{|}^d = |a^\dagger| \times \hat{I}^d = -(a \times a^\dagger)^{1/2} \times \hat{I} < 0. \tag{3.11}$$

where $|\ldots|$ denotes the conventional norm. For isodual isoreal numbers $\hat{n}^d$ we therefore have

$$\hat{|}\hat{n}^d\hat{|}^d = -|n| \times \hat{I} < 0, \tag{3.12}$$

and for isodual complex numbers $\hat{c}^d$ we have

$$\hat{|}\hat{c}^d\hat{|}^d = -|\bar{c}| \times \hat{I} = -(c\bar{c})^{1/2} \times \hat{I} = -(n_1^2 + n_2^2)^{1/2} \times \hat{I}. \tag{3.13}$$

Note that the above notions include as particular case the isoduality of conventional numbers for $\hat{I} = I$, in which case we have *isodual numbers, isodual fields, isodual norm, etc.*

Lemma 3.1. *All quantities which are positive-definite when referred to fields and isofields (such as mass, energy, angular momentum, density, temperature, time, etc.) became negative-definite when referred to isodual fields and isofields.*

As recalled from Sec. 1, antiparticles have been discovered in the *negative-energy solutions* of Dirac's equation and they were originally thought to evolve *backward in time* (Stueckelberg, and others). The possibility of representing antimatter and antiparticles via isodual methods is therefore visible already from these introductory notions.

The main novelty is that the conventional treatment of negative-definite energy and time was (and still is) referred to the conventional contemporary unit $+1$, which leads to a number of contradictions in the physical behavior of antiparticles whose solution forces the transition to second quantization.

By comparison, *the negative-definite physical quantities of isodual methods are referred to a negative-definite unit* $\hat{I}^d < 0$. This implies a mathematical and physical equivalence between *positive-definite quantities referred to positive-definite units, characterizing matter, and negative-definite quantities referred to negative-definite units, characterizing antimatter.* These foundations then permit a novel characterization of antimatter beginning at the *Newtonian* level (Appendix C) and then persisting at all subsequent levels.

Definition 3.1. *A quantity is called isoselfdual when it is invariant under isoduality* (3.1).

The above notion is particularly important for this paper because it introduces a new invariance, *the invariance under isoduality.* Its most important application is the following one from which the new invariance itself was originally derived.

Lemma 3.2.[13] *Dirac's gamma matrices are isoselfdual,*

$$\gamma_\mu \to {\gamma_\mu}^d = -\hat{\gamma}_\mu^\dagger \equiv \gamma_\mu \,, \tag{3.14}$$

Note that the preceding invariance is based on the property that *the imaginary number* i and *its isotopic image* $\hat{\imath} = i \times \hat{I}$ *are isoselfdual,*

$$\hat{\imath}^d = -i^\dagger \times \hat{I} = -\bar{i} \times \hat{I} = -(-i) \times \hat{I} = \hat{\imath} \,. \tag{3.15}$$

This property also permits to understand better the isoduality of isocomplex numbers which can be written explicitly[4]

$$\hat{c}^d = (\hat{n}_1 \hat{+} \hat{\imath} \hat{\times} \hat{n}_2)^d = \hat{n}_1^d \hat{+}^d \hat{\imath}^d \hat{\times}^d \hat{n}_2^d = (-n_1 + i \times n_2 \times \hat{I}) = -c \times \hat{I} \,. \tag{3.16}$$

Lemma 3.2 has fundamental relevance for an axiomatically consistent inclusion of gravitation in unified gauge theories,[11] as well as for a deeper understanding of the *conventional* Dirac's equation. In fact, it establishes that, contrary to popular belief, *the conventional Poincaré symmetry "cannot" be the invariance of the Dirac equation, trivially, because it is not isoselfdual.*

In turn, this occurrence has permitted the identification of the *twenty-dimensional* invariance of Dirac equation, the product $P(3.1) \times P^d(3.1)$, which is isoselfdual and, as such, valid for the characterization of the isoselfdual Dirac's equation.

At a still deeper study, the full symmetry of the conventional Dirac equation has resulted to admit a *22-dimensional* symmetry given by the structure $P(3.1) \times P^d(3.1)$ *plus the isoselfscalar symmetry* (2.22) and its isodual. The latter has permitted the inclusion of gravitation in the unit of the theory, thus rendering gravitational axiomatically compatible with electroweak interactions, while the former has permitted an axiomatically consistent treatment of antimatter for both electroweak and gravitational interactions.

3.3. *Isodual isofunctional isoanalysis*

All conventional and special isofunctions and isotransforms of the isominkowskian geometry, as well as isofunctional isoanalysis at large, must be subjected to isoduality for consistent applications, without any exception known to this author. This results in a simple, yet unique and significant *isodual isofunctional isoanalysis*, whose study was initiated by Kadeisvili.[8a]

We here mention the *isodual isotrigonometric isofunctions*

$$\text{isosin}^d \hat{\theta}^d = -[\sin(-\hat{\theta})] \times \hat{I}, \quad \text{isocos}^d \hat{\theta}^d = [-\cos(-\hat{\theta})] \times \hat{I}, \tag{3.17}$$

with related basic property

$$\text{isocos}^{d\hat{2}d} \hat{\theta}^d \hat{+}^d \text{isosin}^{d\hat{2}d} \hat{\theta}^d = \hat{I}^d = -1 \times \hat{I}, \tag{3.18}$$

the *isodual isohyperbolic isofunctions*

$$\text{isosinh}^d \hat{w}^d = [-\sinh(-\hat{w})] \times \hat{I}, \quad \text{isocosh}^d \hat{w}^d = [-\cosh(-\hat{w})] \times \hat{I}, \tag{3.19}$$

with related basic property

$$\text{isocosh}^{d\hat{2}d} \hat{w}^d \hat{-}^d \text{isosinh}^{d\hat{2}d} \hat{w}^d = \hat{1}^d = -1 \times \hat{I}, \tag{3.20}$$

the *isodual isologarithm*

$$\text{isolog}^d \hat{a}^d = [-\log(-\hat{a})] \times \hat{I}, \tag{3.21}$$

etc. Interested readers can then construct the isodual image of special isofunctions, isotransforms, isodistributions, etc.[4g]

3.4. *Isodual isodifferential calculus*

The *isodual isodifferential calculus*, first introduced in Ref. 5g, is characterized by the *isodual isodifferentials*

$$\hat{d}^d \hat{x}^{kd} = \hat{I}^d \times d\hat{x}^{kd} = -\hat{d}\hat{x}^{kd} \equiv \hat{d}\hat{x}^k, \quad \hat{d}^d \hat{x}_k{}^d = -\hat{d}\hat{x}_k{}^d \equiv \hat{d}\hat{x}_k, \tag{3.22}$$

with corresponding *isodual isoderivatives*

$$\hat{\partial}^d \hat{f}^d \hat{/}^d \hat{\partial}^d \hat{x}^{kd} = -\partial \hat{f} \hat{/} \hat{\partial} \hat{x}^k, \quad \hat{\partial}^d \hat{f}^d \hat{/}^d \hat{\partial}^d \hat{x}_k^d \equiv -\hat{\partial} \hat{/} \hat{\partial} \hat{x}_k, \tag{3.23}$$

and other isodual properties.

Note that *conventional differentials and isodifferentials are isoselfdual* but derivatives and isoderivatives are not.

3.5. *Isodual isominkowskian geometry*

The *isodual isominkowski space*, first introduced in Refs. 4b (see Ref. 4g for the latest formulation), is the isodual image of the isominkowski spaces $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ characterized by

$$\hat{M}^d = \hat{M}^d(\hat{x}^d, \hat{\eta}^d, \hat{R}^d): \ \hat{x}^d = \{\hat{x}^{\mu d}\} = \{-x^\mu\} \times \hat{I}$$
$$= \{r^d, c_0{}^d \times^d t^d\} \times \hat{I}, \ \hat{\eta}^d = -\hat{\eta} \tag{3.24}$$

The *isodual isominkowskian geometry* is the geometry of isodual isospaces M^d over R^d. It is also characterized by a simple isoduality of the isominkowskian geometry of the preceding section.

The physically and mathematically most salient property of the latter geometry is that it is *characterized by negative units of space, time, etc.* In fact, we have the *isodual space isocoordinates* $\hat{r}^d = r \times \hat{I}^d$ and the *isodual isotime* $\hat{t}^d = t \times \hat{I}^d$ which, as one can see, are referred to negative units. Thus, a conventionally positive time $t > 0$ implies motion in time *opposite* that the characteristic time of the isominkowskian geometry, the natural time being $-t$ which, when referred to $\hat{I}^d = -\hat{I}$, yields a forward flow.

It should be also noted that *isoduality is independent from spacetime inversions* $r' = \pi \times r = -r$, $t' = \tau \times t = -t$. In fact, the inversions occur within the original space and keep the unit fixed, while isodualities imply a map to a different space, they keep the coordinates unchanged and change instead the sign of the unit. As such, isodualities are more general maps than the inversions.

These are the conceptual roots for the prediction by the isodual theory of a *new photon*, the *isodual photon* of Ref. 13c. When applied to the photon, charge conjugation and, more generally, PCT theorem, do not yield a new photon, as well known. This is not the case under isoduality because all physical characteristics change. As a result, *the isodual photon is indistinguishable under all interactions from the ordinary photons except for graviton.* In fact, the isodual photon is predicted to experience antigravity in the field of matter, thus offering, apparently for the first time, a possibility for the future study whether far away galaxies and quasars are made up of matter or of antimatter.

At the level of second quantization the isodual representation of antimatter becomes truly simple and essentially consists in a reinterpretation of the advanced solutions with respect to isodual spaces and fields. In turn, this implies no numerical deviation of the isodual theories from existing electroweak interactions, because the advanced solutions are generally discarded as "unphysical".

An important property of isoduality is expressed by the following:

Lemma 3.3.[13] *The intervals of conventional and isotopic Minkowskian spaces are invariant under the joint isodual maps $\hat{I}^d \to \hat{I}^d$ and $\hat{\eta} \to \hat{\eta}^d$,*

$$\hat{x}^2 = (x^\mu \times \hat{\eta}_{\mu\nu} \times x^\nu) \times \hat{I} \equiv [x^\mu \times (-\hat{\eta}_{\mu\nu}) \times x^\nu] \times (-\hat{I})\,. \tag{3.25}$$

As a result, *all physical laws applying in conventional Minkowskian geometry for the characterization of matter also apply to its isodual image for the characterization of antimatter.*

Note that, strictly speaking, the intervals are not isoselfdual because

$$\hat{x}^{\hat{2}} = \hat{x}^\mu \hat{\times} \hat{N}_{\mu\nu} \hat{\times} \hat{x}^\mu \to \hat{x}^{d\hat{2}d} = \hat{x}^{\mu d} \times^d \hat{N}_{\mu\nu}{}^d \times^d \hat{x}^{\mu d} = \hat{x}^{d\hat{2}d} = -\hat{x}^{\hat{2}}\,. \tag{3.26}$$

Among the *Minkowskian* characteristics of the geometry we here mention the *isodual light isocone*

$$\hat{x}^{d\hat{2}d} = (x^{\mu d} \times \hat{\eta}_{\mu\nu}{}^{d} \times x^{\nu d}) \times \hat{I}^{d} = \hat{0} = 0\,. \tag{3.27}$$

As one can see, the above isocone formally coincides with the light isocone, although the two cones belong to *different* spaces. The isodual light cone is used in these studies as *the cone of light emitted by antimatter in empty space (exterior problem)*, and its isotopic extension is used for the corresponding interior problem.[13]

To outline the *Riemannian* characteristics of the isodual isominkowskian geometry, we consider an *isodual isovector isofield* $\hat{X}^{d}(\hat{x}^{d})$ on $\hat{M}^{d}$ which is explicitly given by $\hat{X}^{d}(\hat{x}^{d}) = -X^{t}(-x^{t} \times \hat{I}) \times \hat{I}$. The *isodual exterior isodifferential* of $\hat{X}^{d}(\hat{x}^{d})$ is given by

$$\hat{D}^{d}\hat{X}^{\mu d}(\hat{x}^{d}) = \hat{d}^{d}\hat{X}^{\mu d}(\hat{x}^{d}) \hat{+}^{d} \hat{\Gamma}^{d}{}_{\alpha\beta}{}^{\mu} \hat{\times}^{d} \hat{X}^{\alpha d} \hat{\times}^{d} \hat{d}^{d}\hat{x}^{\beta d} = \hat{D}\hat{X}^{t\mu}(-\hat{x}^{t})\,, \tag{3.28}$$

where the $\hat{\Gamma}^{d}$'s are the components of the *isodual isoconnection.* The *isodual isocovariant isoderivative* is then given by

$$\hat{X}^{\mu d}(\hat{x}^{d})_{\hat{|}d\nu} = \hat{\partial}^{d}\hat{X}^{\mu d}(\hat{x}^{d}) \hat{/}^{d} \hat{\partial}^{d}\hat{x}^{\nu d} \hat{+} \hat{\Gamma}^{d}{}_{\alpha}{}^{\mu}{}_{\nu} \hat{\times}^{d} \hat{X}^{\alpha d}(\hat{x}^{d}) = -\hat{X}^{t\mu}(-\hat{x}^{t})_{\hat{|}k}\,. \tag{3.29}$$

The interested reader can then easily derive the isoduality of the remaining notions of the new geometry. It is an instructive exercise for the interested reader to prove of the following isodualities:

$$\begin{array}{ll}
\text{Basic isounit} & \hat{I} \to \hat{I}^{d} = -\hat{I}\,, \\
\text{Metric} & \hat{\eta} \to \hat{\eta}d = -\eta\,, \\
\text{isoconnection coefficients} & \hat{\Gamma}_{\alpha\beta\gamma} \to \hat{\Gamma}^{d}{}_{\alpha\beta\gamma} = \hat{\Gamma}_{\alpha\beta\gamma}\,, \\
\text{Isocurvature isotensor} & R_{\alpha\beta\gamma\delta} \to R^{d}{}_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\gamma\delta}\,, \\
\text{Isoricci isotensor} & \hat{R}_{\mu\nu} \to \hat{R}^{d}{}_{\mu\nu} = -\hat{R}_{\mu\nu}\,, \\
\text{Isoricci isoscalar} & \hat{R} \to \hat{R}^{d} = \hat{R}\,, \\
\text{Isofreud isoscalar} & \hat{\Theta} \to \hat{\Theta}^{d} = -\hat{\Theta}\,, \\
\text{Isoeinstein isotensor} & \hat{G}_{\mu\nu} \to \hat{G}^{d}_{\mu\nu} = -\hat{G}_{\mu\nu}\,, \\
\text{Electromagnetic potentials} & A_{\mu} \to A^{d}_{\mu} = -A_{\mu}\,, \\
\text{Electromagnetic field} & F_{\mu\nu} \to F^{d}{}_{\mu\nu} = -F_{\mu\nu}\,, \\
\text{Elm energy-mom. tensor} & T_{\mu\nu} \to T^{d}{}_{\mu\nu} = -T_{\mu\nu}\,,
\end{array} \tag{3.30}$$

We then have the following fundamental property for our characterization of antimatter.

Theorem 3.1. *Under the assumed regularity and continuity conditions, the most general possible isodual isolagrange equations (App. C)* $\hat{E}^{\alpha\beta d} = 0$ *of Class II along an actual isodual isopath* $\hat{P}_{0}{}^{d}$ *on a (3+1)-dimensional isodual isominkowskian space for the characterization of exterior or interior gravitational problems of antimatter satisfying the properties: (1) Symmetry condition,* $\hat{E}^{\alpha\beta d} = \hat{E}^{\beta\alpha}d$*, (2) Contracted isodual isobianchi identity,* $\hat{E}^{\alpha\beta d}{}_{\hat{|}\beta d} \equiv 0$*; and (3) The isodual isofreud identity; are given by*

$$\hat{E}^{\alpha\beta d} = \hat{\alpha}^d \times^d \hat{N}^{\frac{1}{2}\hat{d}} \times^d \left(\hat{R}^{\alpha\beta d} \hat{-}^d \frac{1}{2}^d {}^d\hat{\times} \hat{N}^{\alpha\beta^d} \times^d \hat{R}^d \hat{-}^d \frac{1}{2}^d \times^d \hat{N}^{\alpha\beta d} \times^d \hat{\Theta}^d \right)$$
$$\hat{+}^d \hat{\beta}^d \hat{\times}^d \hat{N}^{\alpha\beta d} \hat{-}^d \hat{N}^{\frac{1}{2}\hat{d}} \times^d \hat{S}^{\alpha\beta} = 0\,, \qquad (3.31)$$

where: $\hat{N}^{\hat{\frac{1}{2}}d} = -(det\hat{\eta})^{1/2} \times \hat{I}$; $\hat{\alpha}^d = \alpha \times \hat{I}^d$, $\beta^d = \beta \times \hat{I}^d$, $\frac{1}{2}^d = \frac{1}{2} \times \hat{I}^d$, *etc.;* α *and* β *are constants; and* $\hat{S}^{\alpha\beta}$ *is a source tensor. For* $\alpha = 1$ *and* $\beta = 0$ *the isogravitation field equations can be written*

$$\hat{R}^{\alpha\beta d} - \frac{1}{2} \times \hat{\eta}^{\alpha\beta d} \times \hat{R}^d + \frac{1}{2} \times \hat{\eta}^{\alpha\beta d} \times \hat{\Theta}^d - k(\hat{t}^{\alpha\beta d} + \hat{\tau}^{\alpha\beta d}) = \hat{U}^{\alpha}{}_{\beta}{}^{d} + \partial^d{}_{\rho} \hat{V}^{\alpha\rho}{}_{\beta}{}^{d} = 0\,, \qquad (3.32)$$

where $\hat{t}^{\alpha\beta}$ *is a source isodual isotensor,* $\hat{\tau}^{\alpha\beta}$ *is an isodual stress-energy isotensor and* k *is a constant.*

As one can see, *the gravitational field equations merely change sign under isodualities.* However, one should keep in mind that the field equations and their isodual are referred to different *isospaces.*

In summary, the isominkowskian geometry of the preceding section provides a novel representation of gravitation for matter which verifies the abstract axioms of the *Minkowskian*, rather than *Riemannian* geometry, thus rendering gravitation structurally compatible with the electroweak interactions thus resolving the structural incompatibility due to curvature (Sec. 1). The application of the isodual representation of antimatter of this section to both electroweak and gravitational interactions then offers new possibilities for the resolution of the second structural incompatibility due to antimatter.

Acknowledgments

The author would like to express his deepest appreciation to the Editors and referees of: *Foundations of Physics Letters* and *Foundations of Physics* for penetrating critical comments in the review of the crucial of paper[11] and memoir,[4h] *Mathematical Methods in Applied Sciences*, for memoir;[8c] the Italian mathematical *Rendiconti* for memoir[5g] on the new mathematical methods; *Hyperfine Interactions*, for paper[13c] on the isodual theory of antimatter; and other publications without which this paper would not have seen the light of day. Special thanks are also due to T. L. Gill, E. I. Guendelman, L. P. Horwitz, J. V. Kadeisvili and V. de Sabbata and others for penetrating critical comments during the early stages of this manuscript. Special thanks are finally due to the referees of *International Journal of Modern Physics D* for truly valuable critical comments during the finalization of this paper as well as the indication of important references.

References

1. (a) H. A. Lorentz, *Amst. Proc.* **6**, 809 (1904); (b) H. Poincaré, *Compte Rendues, Paris* **140**, 1504 (1905); (c) A. Einstein, *Ann. Phys.* **17**, 891 (1905); (d) H. Minkowski, *Nachr. Ges. Wiss. Goettingen* **43** (1908); (e) W. Pauli, *Theory of Relativity*, (Pergamon Press, New York 1958).

2. (a) B. Riemann, *Gött. Nachr.* **13**, 133 (1868) and *Collected Works*, H. Weber, Editor, (Dover, New York 1953); (b) D. Hilbert, *Nachr. Kgl. Ges. Wissench. Goettingen*, (1915); p. 395; (c) A. Einstein, *Sitz. Ber. Preuss. Akad. Wissssench Berlin*, (1915); p. 844; K. Schwarzschild, Sitzber. Deut. Akad. Wiss. Berlin. Kl. Math.-Phys. Tech., 189 and 424 (1916); (d) H. Weyl, *Raum-Zeit-Materie*, (Springer, Berlin 1916); (e) A. Einstein, H. Minkowski and H. Weyl, *The Principle of Relativity: A collection of Original Memoirs*, (Methuen 1923); (f) Stachel, Editor, *Collected Papers of Albert Einstein*, (Princeton Univ. Press, Vols. I-5 1987 on), Princeton, N.J.
3. (a) R. M. Santilli, in *Proceedings of the VII M. Grossmann Meeting on General Relativity*, R. T. Jantzen, G. MacKeiser and R. Ruffini, Editors, (World Scientific, Singapore 1996), p. 500; (b) in *Gravity, Particles and Space-Time*, P. Pronin and G. Sardanashvily, Editors, (World Scientific, Singapore 1995), p. 500; (c) *Comm. Theor. Phys.* **4**, 1 (1995); (d) "Isospecial relativity for matter and its isodual for antimatter", Gravitation (1997), in press; (e) "Isominkowskian representation of gravitation", in the *Proceedings of the Workshop on Modern Modified Theories of Gravitation and Cosmology*, Ben Gurion Univ., (Beer Sheva, Israel, June 1997), E. I. Guendelman, ed., *Hadronic J.* **21** (1998) in press; (f) *Ann. Phys.* **83**, 108 (1974).
4. (a) R. M. Santilli, *Nuovo Cimento Lett.* **37**, 545 (1983); (b) *Hadronic J.* **8**, 25 and 36 (1985); (c) *JINR Rapid Comm.* **6**, 24 (1993); (d) *J. Moscow Phys. Soc.* **3**, 255 (1993); (e) *JINR Comm. No.* E4-93-352 (1993) published in *Chinese J. Syst. Eng. & Electr.* **6**, 177 (1996); (f) *Lett. Nuovo Cimento* **3**, 509, (1983); (g) *Elements of Hadronic Mechanics*, Vol. I and II, second edition (1995), (Ukraine Academy of Sciences, Kiev 1995); (h) *Found. Phys.* **27**, 625 (1997).
5. (a) R. H. Bruck, *A Survey of Binary Systems*, (Springer-Verlag, Berlin 1958); (b) K. McCrimmon, *Isotopies of Jordan Algebras, Pacific J. Math.* **15**, 925–962 (1965); (c) R. M. Santilli, *Hadronic J.* **1**, 224, 574 and 1267 (1978); (d) R. M. Santilli, *Phys. Rev.* **D20**, 555 (1979) (e) *Foundations of Theoretical Mechanics*, Vol. I (1978) and II (1983), (Springer-Verlag, Heidelberg-New York 1983); (f) *Algebras, Groups and Geometries* **10**, 273 (1993) (g) *Rendiconti Circolo Matematico Palermo, Suppl.* **42**, 7 (1996); (h) "Isotopies of trigonometric and hyperbolic functions", (Ann. Univ. of Costanta, Romania, in press) (i) M. Battler, M. McBee, S. Smith, http://home.gte.net/ibr/.
6. (a) D. M. Norris *et al.*, *Tomber's Bibliography and Index in Nonassociative Algebras*, (Hadronic Press, Palm Harbor, FL 1984); (b) A. K. Aringazin, A. Jannussis, D. F. Lopez, M. Nishioka and B. Veljanosky, *Santilli's Lie-Isotopic Generalization of Galilei's and Einstein's Relativities* (1990), (Kostarakis Publisher, Athens, Greece); (c) J. V. Kadeisvili, *Santilli's Isotopies of Contemporary Algebras, Geometries and Relativities*, (Hadronic Press, FL 1991), Second Edition, Ukraine Academy of Sciences, Kiev (1997); (d) D. S. Sourlas and G. T. Tsagas, *Mathematical Foundations of the Lie–Santilli Theory*, (Ukraine Academy of Sciences, Kiev 1993); (e) J. Lôhmus, E. Paal and L. Sorgsepp, *Nonassociative Algebras in Physics*, (Hadronic Press, Palm Harbor, FL, USA 1994).
7. (a) S. L. Adler, *Phys. Rev.* **17**, 3212 (1978); (b) Cl. George, F. Henin, F. Mayne and I. Prigogine, *Hadronic J.* **1**, 520 (1978); (c) S. Okubo, *Hadronic J.* **3**, 1 (1979); (d) J. Fronteau, A. Tellez Arenas and R. M. Santilli, *Hadronic J.* **3**, 130 (1978); (e) S. Okubo *Hadronic J.* **5**, 1667 (1982); (f) C. N. Ktorides, H. C. Myung and R. M. Santilli, *Phys. Rev.* **D22**, 892 (1982); (g) A. J. Kalnay, *Hadronic J.* **6**, 1 (1983); (h) M. Gasperini, *Hadronic J.* **6**, 935 and 1462 (1993), **7**, 650 and 951 (1984); (i) R. Mignani, *Nuovo Cimento Lett.* **39**, 413 (1984); (j) J. D. Constantoupoulos and C. N. Ktorides, *J. Phys.* **A17**, L29 (1984); (k) E. B. Lin, *Hadronic J.* **11**, 81 (1988); (l) M. Nishioka, *Nuovo Cimento* **82A**, 351 (1984); D. Rapoport-Campodonico, *Algebras,*

Groups and Geometries **8**, 1 (1991); (m) A. Jannussis, G. Brodimas and R. Mignani, *J. Phys.* **A24**, L775 (1991); A. Jannussis, R. Mignani and D. Skaltsas, *Physica* **A187**, 575 (1992); T. Gill, J. Lindesay and W. W. Zachary, *Hadronic J.* **17**, 449 (1994); A. O. E. Animalu, *Hadronic J.* **17**, 349 (1995); (n) A. O. E. Aniamalu and R. M. Santilli, *Int. J. Quantum Chemistry* **29**, 175 (1995); (o) D. Schuch, *Phys. Rev.* **A55**, (1997), in press.

8. (a) J. V. Kadeisvili, *Algebras, Groups and Geometries* **9**, 283 and 319 (1992); (b) Gr. T. Tsagas and D. S. Sourlas, *Algebras, Groups and Geometries* **12**, 1 and 67 (1995); (c) J. V. Kadeisvili, *Math. Methods in Applied Sciences*, **19**, 1349 (1996); P. Vacaru, *Algebras, Groups and Geometries* **14**, 211 (1997), in press; R. Aslander and S. Keles, *Algebras, Groups and Geometries* **14**, (1997), in press.
9. (a) F. Cardone, R. Mignani and R. M. Santilli, *I. Phys.* **G18**, L61 and L141 (1992); (b) F. Cardone and R. Mignani, *Univ. of Rome preprint* 894; (c) R. M. Santilli, *Hadronic J.* **15**, 1 (1992); (d) R. M. Santilli, *Comm. Theor. Phys* **4**, 1, (1995); (e) "Exact representation of total nuclear magnetic moments via relativistic hadronic mechanics", to appear in *Int. J. Modern Phys. E* (f) R. M. Santilli, *Hadronic J. Suppl.* **4A**, issue nos. 1 and 2 (1988) and R. Mignani, *Physics Essays* **5**, 531 (1992); (g) R. M. Santilli, in *Frontiers in Fundamental Physics*, M. Barone and F. Selleri, eds., (Plenum, New York 1994), p. 41; (h) A. O. E. Animalu, *Hadronic J.* **17**, 349 (1995), A. O. E. Animalu and R. M. Santilli, *Int. J. Quantum Chemistry* **29**, 175 (1995); (i) R. M. Santilli and D. Shillady, "Ab initio isochemistry, I and II", *Hadronic J.* **20**, (1997), in press; (j) R. M. Santilli, *Isotopic, Genotopic and Hyperstructural Methods in Theoretical Biology*, (Ukraine Academy of Sciences, Kiev 1996); (k) A. Jannussis, M. Miatovic and B. Veljanowski, *Physics Essays* **4**, 202 (1991); (l) A. K. Aringazin, *Hadronic J.* **12**, 71 (1989), A. K. Aringazin *et al.*, in *Frontiers of Fundamental Physics*, M. Barone and F. Selleri, Editors, (Plenum, New York 1995), p. 153.
10. (a) J. Ellis, in *Proceedings of the Seventh Marcel Grossmann meeting on general Relativity*, R. T. Jantsen, G. MacKaiser and R. Ruffini, Editors, (World Scientific 1966), p. 755; (b) J. Ellis, N. E. Mavromatos and D. V. Nanopoulos in *Proceedings of the Erice Summer School, 31st Course: From Superstrings to the Origin of Space-Time*, (World Scientific 1996).
11. R. M. Santilli, *Found. Phys. Letters* **10**, 305 (1997).
12. (a) H. A. Lorentz, *Versuch einer Theorie der Elektrishen und Magnetishen Erscheinungen in bewenglen Korpern*, Leyda (1895); (b) A. Enders and G. Nimitz, *J. Phys. France* **2**, 1693 (1992); (c) A. M. Steinberg, P. C. Kwait and R. Y. Chiao, *Phys. Rev. Lett.* **68**, 2421 (1992); (d) F. Mirabel and F. Rodriguez, Nature **371**, 464 (1994); (e) J. Tingay *et al.*, Nature **374**, 141 (1995); (f) D. Baylin *et al.*, IAU Comm. 6173 (1995); (g) W. A. Rodrigues, Jr, and J.-Yu Lu, *On the existence of undistorted progressive waves of arbitrary speed*, Found. Phys. **27** (1997) in press.
13. (a) R. M. Santilli, *Comm. Theor. Phys.* **3**, 153 (1994); (b) Hadronic J. **17**, 257 (1994); (c) "Does antimatter emit a new light?", Hyperfine Interactions **109**, 63 (1997); (d) "Classical isodual theory of antimatter", submitted for publication; (e) R. M. Santilli, *Hadronic J* **17**, 285 (1994).
14. (a) A. Albert, *Trans. Amer. Math. Soc.* **64**, 552 (1948); (b) R. M. Santilli, *Nuovo Cimento* **51**, 570 (1967); (c) G. Eder, *Hadronic J.* **4**, 641 (1981) and **5**, 750 (1982); (d) R. M. Santilli, *Hadronic J.* **4**, 1166 (1981); (e) L. C. Biedernharn, *J. Phys.* **A22**, L873 (1989); (f) A. J. Macfarlane, *J. Phys.* **A22**, L4581 (1989); (g) A. N. Sissakian, G. Pogosyan and S. I. Vinitsky, *Symmetry Methods in Physics* (J.I.N.R., Dubna, Russia, 1994); T. L. Curtis, B. Fairlie and Z. K. Zachos, *Quantum Groups*, (World Scientific, Singapore, 1991); (h) J. Lukierski, A. Nowiski and H. Ruegg, *Phys. Lett.* **B293**, 344

(1992); (i) A. Jannussis *et al.*, *Nuovo Cimento* **B103**, 17 and 537 (1989); **104**, 33 and 53 (1989); **108**, 57 (1993).

15. (a) D. Lovelock and H. Rund, *Tensors, Differential Forms and Variational Principles*, (J. Wiley, New York 1975); (b) C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (Freeman, San Francisco 1973); (c) R. Jantzen, M. McKaiser and R. Ruffini, Editors, *Proceedings of the Seventh M. Grossmann Meeting on Gravitation*, (World Scientific, Singapore 1996); (d) R. Giacconi and R. Ruffini, *Physics and Astrophysics of Neutron Stars and Black Holes*, (North Holland, Amsterdam 1978).
16. (a) J. Julve *et al.*, *Gen. Rel. Grav.* **28**, 759 (1996); (b) A. Lopez–Pinto *et al.*, "Ordinary matters in nonlinear affine gauge theories", *Gen. Rel. Grav.* **28** (1997), in press; (c) E. Schruefer *et al.*, *Gen. Rel. Grav.* **19**, 197 (1987); (d) M. Kaku, *Quantum Field Theory*, (Oxford Univ. Press, New York 1993); (e) D. Hestenes and G. Sobczyk, *From Clifford Algebras to Geometric Calculus*, (Reidel, Dorfdrecht 1984).
17. (a) P. Freud, *Ann. Math.* **40** [2], 417 (1939); (b) H. Rund, *Algebras, Groups and Geometries* **8**, 267 (1991); (c) H. Yilmaz, *Hadronic J.* **12**, 305 (1989).
18. P. A. M. Dirac, *Proc. Roy. Soc.* **A322**, 435 (1971) and **328**, 1 (1972).