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Invited paper

Does antimatter emit a new light?

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Contemporary theories of antimatter have a number of insufficiencies which stimulated the recent construction of the new *isodual theory* based on a certain anti-isomorphic map of all (classical and quantum) formulations of matter called *isoduality*. In this note we show that the isodual theory predicts that antimatter emits a new light, called *isodual light*, which can be distinguished from the ordinary light emitted by matter via gravitational interactions (only). In particular, the isodual theory predicts that all stable antiparticles such as the isodual photon, the positron and the antiproton experience antigravity in the field of matter (defined as the reversal of the sign of the curvature tensor). The antihydrogen atom is therefore predicted to: experience antigravity in the field of Earth; emit the isodual photon; and have the same spectroscopy of the hydrogen atom, although subjected to an anti-isomorphic isodual map. In this note we also show that the isodual theory predicts that bound states of elementary particles and antiparticles (such as the positronium) experience ordinary gravitation in both fields of matter and antimatter, thus bypassing known objections against antigravity. A number of intriguing and fundamental, open theoretical and experimental problems of "the new physics of antimatter" are pointed out.

1. Introduction

Since the time of Dirac's prediction of antiparticles and their detection by Anderson (see [1] for historical accounts), the theory of antimatter has been essentially developed at the level of *second quantization.*

This occurrence has created an unbalance between the theories of matter and antimatter at the *classical* and *first quantization* levels, as well as a number of shortcomings, such as the inability for the classical theory of antimatter to have a quantized formulation which is the correct charge (or PTC) conjugate of that of matter.

In an attempt to initiate the scientific process toward the future resolution of the above problematic aspects, this author proposed in 1985 [2] a new anti-isomorphic image of conventional mathematics characterized by the map of the conventional unit

$$+1 \rightarrow 1^{d} = -1^{\dagger} = -1, \tag{1}$$

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called for certain technical reasons *isodual map*, or *isoduality*.

It should be noted that the change of the basic unit implies a simple, yet unique and non-trivial change of the totality of conventional mathematics, including numbers and angles, functions and transforms, vector and metric spaces, algebras and geometries, etc.

In 1991 this author [3] showed that the above isodual mathematics, since it is an anti-isomorphic image of the mathematics of matter, provides a novel *classical* representation of antimatter.

The proof that isoduality on a Hilbert space is equivalent to charge conjugation first appeared in paper [4] of 1994. A comprehensive operator treatment subsequently appeared in monographs [5].

The prediction that isoduality implies *antigravity* (defined as the reversal of the sign of the curvature tensor for massive antiparticles in the field of matter) was submitted in paper [6], which also included the proposal for its experimental verification via the use of a low energy (eV) positron beam in horizontal flight in a suitable vacuum tube. The latter experimental proposal was subsequently studied by Mills [7].

This note is devoted to a study of the *spectroscopy of antimatter* via the isodual characterization of the light emitted by the antihydrogen atom [8]. In particular, we show that *isoduality predicts that antimatter emits a new light here called isodual light* which can be solely differentiated from the conventional light via gravitational interactions.

In the events additional theoretical and experimental studies confirm the above hypothesis, isoduality would therefore permit the future experimental measures whether far away galaxies and quasars are made up of matter or of antimatter.

A more comprehensive analysis is presented in memoir [9], which also includes the study of isodual theories of antimatter at the more general *isotopic* and *isogravitational levels*. Reference [9] also shows that known objections against antigravity are *inapplicable* to (and not "violated" by) the isodual theory of antimatter, because the former are (tacitly) constructed on conventional mathematics, while the latter is formulated on a novel mathematics based on a new negative unit.

After reviewing the mathematical foundations and the main predictions of the isodual theory, in this note we identify a number of rather fundamental, open, theoretical and experimental problems of the emerging "new physics of antimatter".

2. Isodual mathematics

Our fundamental assumption on antimatter is that it is represented by the new *isodual mathematics* which is the anti-isomorphic image of conventional mathematics under map (1).

Since the latter mathematics is still vastly unknown (in both mathematical and physical circles), it appears recommendable to outline in this section the main notions in order to render understandable the physical analysis of the next section.

An *isodual field* $F^d = F^d(a^d, +^d, \times^d)$ [2,10,11] is a ring whose elements are the *isodual numbers*

$$a^d = a^\dagger \times 1^d = -a^\dagger, \tag{2}$$

(where a represents real numbers n, complex numbers c or quaternions q, † represents Hermitean conjugation, $1^d = -1$, and $\times$ represents the conventional multiplication) equipped with: the *isodual sum*

$$a_1^d +^d a_2^d = -(a_1^\dagger + a_2^\dagger), \tag{3}$$

with *isodual additive unit*

$$0^d \equiv 0, \quad a^d +^d 0^d = 0^d +^d a^d \equiv a^d \; \forall a^d \in F^d;$$

the *isodual multiplication*

$$a_1^d \times^d a_2^d = a_1^d \times (-1) \times a_2^d = -a_1^\dagger \times a_2^\dagger, \tag{4}$$

with *isodual multiplicative unit* (or *isodual unit* for short)

$$1^d = -1, \quad a^d \times^d 1^d = 1^d \times^d a^d \equiv a^d \; \forall a^d \in F^d;$$

and remaining *isodual operations*, such as the *isodual quotient*

$$a_3^d = a_1^d /^d a_2^d = -a_1^d / a_2^d = -a_1^\dagger / a_2^\dagger,$$

$$a_3^d \times^d a_2^d = a_1^d, \tag{5}$$

the *isodual square root*

$$(a^d)^{\frac{1}{2}d} = \sqrt{-a^d} = \sqrt{a^\dagger},$$

$$(a^d)^{\frac{1}{2}d} \times^d (a^d)^{\frac{1}{2}d} = a^d, \tag{6}$$

and others [5,10,11].

A property most important for this note is that the *norm of isodual fields, called isodual norm, is negative definite,*

$$|a^d|^d = |a^d| \times 1^d = -|a^d| = -(a \times a^\dagger)^{1/2}. \tag{7}$$

A quantity Q is called *isoselfdual* when it coincides with its isodual

$$Q \equiv Q^d = -Q^\dagger. \tag{8}$$

For instance, the imaginary quantity $\mathrm{i} = \sqrt{-1}$ is isoselfdual because $\mathrm{i}^d = -\mathrm{i}^\dagger = -\bar{\mathrm{i}} = -(-\mathrm{i}) \equiv \mathrm{i}$, where the upper symbol $^-$ denotes complex conjugation.

Note that for real numbers $n^d = -n$, and for complex number $c^d = (n_1 + \mathrm{i} \times n_2)^d = n_1^d +^d \mathrm{i}^d \times^d n_2^d = -n_1 + \mathrm{i} \times n_2 = -\bar{c}$. Note also that $|n^d|^d = -|n|$ and $|c^d|^d = -|\bar{c}| = -(n_1^2 + n_2^2)^{1/2}$.

We finally note that *isodual fields satisfy all axioms of a field* although in their isodual form. Thus, isodual fields $F^d(a^d, +^d, \times^d)$ are antiisomorphic to conventional fields $F(a, +, \times)$, as desired.

For further studies on isodual fields the interested reader may consult [5,10,11] (with the understanding that the *isodual number theory* has not yet been investigated by mathematicians until now).

An n-dimensional *isodual metric space* $S^d = S^d(x^d, g^d, R^d)$ [2,11] is a vector space with *isodual coordinates* $x^d = -x = \{x^1, x^2, \dots, x^n\}$ and *isodual metric* $g^d = -g$ (where g is an ordinary real and symmetric metric), defined over the isodual real field $R^d = R^d(n^d, +^d, \times^d)$.

By recalling that the interval of a metric space must be an element of the base field, the interval between two points $x_1^d, x_2^d \in S^d$ is given by

$$\begin{aligned}(x_1^d - x_2^d)^{2d} &= \left[(x_1^{id} - x_2^{id}) \times g_{ij}^d \times (x_1^{jd} - x_2^{jd})\right] \times 1^d \\ &= \left[(-x_1^i + x_2^i) \times (-g_{ij}) \times (-x_1^j + x_2^j)\right] \times (-1) \\ &\equiv (x_1 - x_2)^2. \end{aligned} \tag{9}$$

We reach in this way the fundamental property of isodual theories according to which the *interval of (real) metric spaces is isoselfdual* (i.e., invariant under isoduality).

As important particular cases we have [loc.cit.]:

1. The 3-dimensional *isodual Euclidean space*

$$E^d = E^d(r^d, \delta^d, R^d),$$
$$r^d = -r = -\{r^k\} = \{x^d, y^d, z^d\} = \{-x, -y, -z\}, \; k = 1,2,3,$$
$$\delta^d = -\delta = \mathrm{Diag}(-1,-1,-1),$$

with *isodual sphere*

$$r^{d\,2d} = (-xx - yy - zz) \times (-I) \equiv r^2 = (xx + yy + zz) \times (+I),$$

with isodual unit

$$I^d = \mathrm{Diag}(-1,-1,-1).$$

2. The $(3+1)$-dimensional *isodual Minkowski space*

$$M^d = M^d(x^d, r^d, R^d),$$
$$x^d = -x = -\{x^\mu\} = -\{x^k, x^4\} = -\{r, c, t\},$$

where c is the speed of light (in vacuum), and

$$\eta^d = -\eta = -\mathrm{Diag}(1,1,1,-1),$$

with *isodual light cone*

$$x^{d\,2d} = (x^{d\mu} \times \eta_{\mu\nu}^d \times x^{d\nu}) \times I^d \equiv x^2, \quad \mu,\nu = 1,2,3,4,$$

with isodual unit

$$I^d = \mathrm{Diag}(-1,-1,-1,-1).$$

3. The $(3+1)$-dimensional *isodual Riemannian space*

$$\mathcal{R}^{d} = \mathcal{R}^{d}(x^{d}, g^{d}, R^{d}),$$

$$x^{d} = -x \text{ and } g^{d} = -g(x) \text{ on } R^{d},$$

where g is a conventional Riemannian metric, with isodual interval

$$x^{d\,2d} = (x^{d\mu}\, g^{d}_{\mu\nu}\, x^{d\nu}) \times I^{d} \equiv x^{2} = (x^{\mu} g_{\mu\nu} x^{\nu}) \times (+I),$$

with isodual unit

$$I^{d} = \mathrm{Diag}(-1,-1,-1,-1).$$

The *isodual geometries* are the geometries of the isodual spaces. This includes the *isodual symplectic geometry* [5,11], which is the anti-isomorphic image of the conventional symplectic geometry on the *isodual cotangent bundle* $T^{d*}E^{d}(r^{d}, \delta^{d}, R^{d})$ with 6-dimensional isodual unit $I^{d}_{6} = I^{d}_{3} \times I^{d}_{3}$.

The *isodual differential calculus* [11] is characterized by

$$\begin{aligned} &\mathrm{d}^{d} x^{k} = -\mathrm{d}x^{k}, \quad \mathrm{d}^{d} x^{dk} = -\mathrm{d}(-x^{k}) = \mathrm{d}x^{k}, \\ &\partial^{d} f^{d} /^{d} \partial^{d} x^{dk} = -\partial f / \partial x^{k}. \end{aligned} \tag{10}$$

By recalling that p-forms must be elements of the base field, the *canonical one-form changes sign under isoduality,*

$$\theta^{d} = (p^{d}_{k} \times^{d} \mathrm{d}^{d} x^{dk}) \times I^{d}_{6} = -\theta, \tag{11}$$

while the *canonical symplectic two-form is isoselfdual,*

$$\begin{aligned} \omega^{d} &= (\mathrm{d}^{d} x^{dk} \wedge^{d} \mathrm{d}^{d} p^{d}_{k}) \times I^{d}_{6} \\ &\equiv (\mathrm{d}x^{k} \wedge \mathrm{d}p_{k}) \times (+I_{6}) = \omega. \end{aligned} \tag{12}$$

For further details one may consult [5,11].

The *isodual Lie theory* [5,12] is the anti-isomorphic image of the conventional Lie theory under isoduality 1. The *isodual enveloping association algebra* ξ^{rd} is characterized by the infinite-dimensional basis [loc.cit.]

$$\xi^{d} : I^{d},\ X^{d}_{k},\ X^{d}_{i} \times^{d} X^{d}_{j},\ i \leqslant j, \text{ etc.} \tag{13}$$

where $i, j, k = 1, 2, \ldots, n$, $X^{d} = -X = -X_{k}$, X_{k} is a conventional (ordered) basis of an n-dimensional Lie algebra $L \approx \xi^{-}$, and I^{d} is the n-dimensional isodual unit, $I^{d} = \mathrm{Diag}(-1, -1, \ldots, -1)$.

The attached antisymmetric algebra is the *isodual Lie algebra* $L^{d} \approx (\xi^{d})^{-}$ with basis $X^{d} = -X$ and *isodual commutators* [loc.cit.]

$$\begin{aligned} L^{d}: \quad [X^{d}_{i}, X^{d}_{j}]^{d} &= X^{d}_{i} \times^{d} X^{d}_{j} - X^{d}_{j} \times^{d} X^{d}_{i} \\ &= -(X_{i}) \times (-I) \times (-X_{j}) - (-X_{j}) \times (-I) \times (X_{i}) \\ &= C^{dk}_{ij} \times^{d} X^{d}_{k} = -[X_{i}, X_{j}] = -C^{k}_{ij} X_{k}. \end{aligned} \tag{14}$$

The *isodual exponentiation* is defined in terms of basis 13 [loc.cit.],

$$e^{\mathrm{d}\,X^{\mathrm{d}}} = I^{\mathrm{d}} + X^{\mathrm{d}}/^{\mathrm{d}}1!^{\mathrm{d}} + X^{\mathrm{d}} \times^{\mathrm{d}} X^{\mathrm{d}}/^{\mathrm{d}}2!^{\mathrm{d}} + \cdots = \\ = (-I)(1 + X/1! + X \times X/2! + \cdots) = -e^{-X^{\mathrm{d}}} = -e^{X}. \tag{15}$$

The (connected) *isodual Lie groups* G^{d} [loc. cit.] as characterized by the isodual Lie algebra L^{d} (under the conventional integrability conditions of L into G) are given by the isoexponential terms for Hermitean generators $X = X^{\dagger}$

$$G^{\mathrm{d}}: \quad U^{\mathrm{d}} = \prod_k^{\mathrm{d}} e^{\mathrm{d}\,\mathrm{i} \times^{\mathrm{d}} w_k^{\mathrm{d}} \times^{\mathrm{d}} X_k^{\mathrm{d}}} = -\prod_k e^{\mathrm{i} \times w_k \times X_k} = -U, \tag{16}$$

where $w_k^k = -w_k \in R^{\mathrm{d}}$ are the *isodual parameters* and we have used the isoselfduality of i.

It is evident that, for consistency, G^{d} characterizes the *isodual transforms* on $S^{\mathrm{d}}(x^{\mathrm{d}}, g^{\mathrm{d}}, R^{\mathrm{d}})$

$$x^{\mathrm{d}\prime} = U^{\mathrm{d}} \times^{\mathrm{d}} x^{\mathrm{d}} = -x', \tag{17}$$

and that the *isodual group laws* are given by

$$U^{\mathrm{d}}(w_1^{\mathrm{d}}) \times^{\mathrm{d}} U^{\mathrm{d}}(w_2^{\mathrm{d}}) = U^{\mathrm{d}}(w_1^{\mathrm{d}} +^{\mathrm{d}} w_2^{\mathrm{d}}), \\ U^{\mathrm{d}}(0^{\mathrm{d}}) = I^{\mathrm{d}} = -\mathrm{Diag}(1, 1, \ldots, 1). \tag{18}$$

For additional aspects of the isodual Lie theory one may consult [5,12].

An *isodual symmetry* is an invariance under an isodual group G^{d}. The fundamental isodual symmetries are:

- *isodual rotations* $0^{\mathrm{d}}(3)$;
- *isodual Euclidean symmetry* $\mathrm{E}^{\mathrm{d}}(3) = 0^{\mathrm{d}}(3) \times^{\mathrm{d}} \mathrm{T}^{\mathrm{d}}(3)$;
- *isodual Galilean symmetry* $\mathrm{G}^{\mathrm{d}}(3.1)$;
- *isodual Lorentz symmetry* $\mathrm{L}^{\mathrm{d}}(3.1)$;
- *isodual Poincaré symmetry* $\mathrm{P}^{\mathrm{d}}(3.1) = \mathrm{L}^{\mathrm{d}}(3.1) \times^{\mathrm{d}} \mathrm{T}^{\mathrm{d}}(3.1)$;
- *isodual spin symmetry* $\mathrm{SU}^{\mathrm{d}}(2)$;
- *isodual spinorial Poincaré symmetry* $\mathcal{P}^{\mathrm{d}} = \mathrm{SL}^{\mathrm{d}}(2.C^{\mathrm{d}}) \times^{\mathrm{d}} \mathrm{T}^{\mathrm{d}}(3.1)$;

and others [loc.cit.].

The *isodual Hilbert space* $\mathcal{H}^{\mathrm{d}}$ is characterized by the: *isodual states*

$$|\Psi\rangle^{\mathrm{d}} = -\langle\Psi| \quad (or\ \Psi^{\mathrm{d}} = -\Psi^{\dagger}); \tag{19}$$

isodual inner product

$$\langle\Phi|\Psi\rangle^{\mathrm{d}} = \langle\Phi|^{\mathrm{d}} \times (I^{\mathrm{d}-1}) \times |\Psi\rangle \times^{\mathrm{d}} I^{\mathrm{d}} \\ = \langle\Psi| \times I^{\mathrm{d}} \times |\Phi\rangle \times I^{\mathrm{d}} \in C^{\mathrm{d}}(c^{\mathrm{d}}, +, \times^{\mathrm{d}}); \tag{20}$$

and *isodual normalization*

$$\langle\Psi| \times (-I) \times |\Psi\rangle = -I. \tag{21}$$

The *isodual expectation values* of an operator Q^d are given by

$$\langle Q^d\rangle^d = \langle\Psi \mid \times^d Q^d \times^d \mid \Psi\rangle /^d \langle\Psi \mid \times(-I)\times \mid \Psi\rangle = -\langle Q\rangle, \tag{22}$$

where $\langle Q\rangle$ is the conventional expectation value on a conventional Hilbert space $\mathcal{H}$.

Similarly, the *isodual eigenvalue equations* for a Hermitean operator $H = H^\dagger$ are given by

$$H^d \times^d \mid \Psi\rangle^d = E^d \times^d \mid \Psi\rangle^d, \tag{23}$$

where $E^d = -E$. One can therefore see that the isodual eigenvalues E^d coincide with the isodual expectation values of H^d.

A property which is mathematically trivial, yet fundamental for the physical analysis of this note is that the *normalization on a Hilbert space is isoselfdual* [5],

$$\langle\Psi \mid \Psi\rangle^d = \langle\Psi \mid \times(-I)\times \mid \Psi\rangle \times (-I) \equiv \langle\Psi \mid \times(+I)\times \mid \Psi\rangle \times (+I) = \langle\Psi \mid \Psi\rangle. \tag{24}$$

The above property characterizes a new invariance which has remained undetected since Hilbert's conception. Note, however, that, as it is the case for the preceding novel invariance of the Minkowski line element, the discovery of new laws (9) and (24) required the prior identification of *new numbers*, the isodual numbers.

The theory of linear operators on a Hilbert space admits a simple, yet significant isoduality (see [5]) of which we can only mention for brevity the *isodual unitary law*

$$U^d \times^d U^{\dagger d} = U^{\dagger\dagger} \times^d U^d = I^d. \tag{25}$$

Functional analysis also admits a simple, yet significant isoduality which we cannot review for brevity [loc.cit.]. The non-initiated reader should be alerted that, to avoid insidious inconsistencies, the *totality* of conventional mathematical quantities, notions and operations must be subjected to isoduality, e.g., angles and related trigonometric functions must be isodual, conventional and special functions and transforms must be isodual, etc. [5].

We finally recall that the isodual mathematics of this section admits *three* sequential generalizations called *isotopic, genotopic* and *hyperstructural* which we cannot review here for brevity [9,11].

3. Isodual theory of antimatter

The central assumptions of this note are [3]:

1) *matter* is represented by conventional mathematics, including numbers, spaces, algebras, etc., based on the conventional positive unit $+1$; while
2) *antimatter* is represented by the isodual mathematics of the preceding section, including isodual numbers, isodual spaces, isodual algebras, etc., based on the isodual unit -1.

The above representations of matter and antimatter are then interconnected by the isodual map (1) which is bi-injective and anti-isomorphic, as desired.

In this way, isoduality permits, apparently for the first time, a representation of antimatter at all levels, beginning at *classical* level and then continuing at levels of *first and second quantization* in which it becomes equivalent to charge conjugation [4,9].

By recalling that the isodual norm is negative definite, eq. (7), an important consequence is that *all physical characteristics which are positive for matter become negative for antimatter.* The above assumptions are applied to the charge q whose change of sign in the transition from particles to antiparticles is re-interpreted as isoduality, $q \to q^{d} = -q$. Jointly, however, isoduality requires that the mass of antiparticles is negative, $m^{d} = -m$, their energy is negative, $E^{d} = -E$, etc. Finally, isoduality requires that antiparticles more backward in time, $t^{d} = -t$, as originally conceived [1].

One should note that the conventional positive values for particles $m > 0$, $E > 0$, $t > 0$, etc., are referred to corresponding positive units, while the negative values for antiparticles, $m^{d} < 0$, $E^{d} < 0$, $t^{d} < 0$, etc., are referred to negative units of mass, energy, time, etc. This implies the full equivalence of the two representations and removes the traditional objections against negative physical characteristics.

Moreover, isoduality removes the historical reason that forced Dirac to invent the "hole theory" [1], which subsequently restricted the study of antimatter at the level of second quantization. We are here referring to the fact that the negative energy solutions of Dirac's equation behave unphysically when (tacitly) referred to *positive units*, but they behave in a fully physical way when referred to *negative units* [4,9].

The isodual theory of antimatter *begins* at the primitive Newtonian level so as to achieve a complete equivalence of treatments with matter. The basic carrier space is the isodual space

$$S^{d}(t^{d}, r^{d}, v^{d}) = E^{d}(t^{d}, R^{d}) \times^{d} E^{d}(r^{d}, \delta^{d}, R^{d}) \times^{d} E^{d}(v^{d}, \delta^{d}, R^{d}),$$
$$v^{d} = d^{d}r^{d} /^{d} d^{d}t^{d} = -v, \tag{26}$$

with corresponding total 7-dimensional (dimensionless) unit

$$I^{d} = 1_t^{d} \times^{d} I_r^{d} \times^{d} I_v^{d},$$
$$1_t^{d} = -1, \quad I_r^{d} = -\mathrm{Diag}(1,1,1) = I_v^{d}. \tag{27}$$

The fundamental dynamical equations are the *isodual Newton's equations* first introduced by the author in [11].

$$m^{d} \times^{d} d^{d}v_k^{d} /^{d} d^{d}t^{d} = F_k^{d}(t^{d}, r^{d}, v^{d}), \quad k = x, y, x, z. \tag{28}$$

It is easy to see that the above theory represents correctly all Newtonian Coulomb interactions. In fact, the theory recovers the *repulsive* Coulomb force between two charges q_1 and q_2 of equal sign of matter, $F = k \times q_1 \times q_2 / r \times r > 0$; it recovers the *repulsive* force between the corresponding "anti-charges", $F^{d} = k^{d} \times^{d} q_1^{d} \times^{d} q_2^{d} /^{d} r^{d} \times^{d} r^{d} < 0$ (because now the force is referred to the unit -1); and recovers the *attractive* force between charges q and their conjugate q^{d} when computed in our space, $F = k \times q \times q^{d}/r \times r < 0$ referred to the unit $+1$, or in isodual space, $F^{d} = k^{d} \times^{d} q^{d} \times^{d} q/^{d} r^{d} \times^{d} r^{d} > 0$ referred to the unit -1.

Along similar lines, it is easy to see that the above theory recovers the conventional Newtonian gravitational *attraction* for matter–matter systems, $F^{d} = g \times m_1 \times m_2/r \times r > 0$; it predicts gravitational *attraction* for antimatter–antimatter systems, $F = g^{d} \times^{d} m_1^{d} \times^{d} m_2^{d}/^{d} r^{d} \times^{d} r^{d} < 0$; and it predicts gravitational *repulsion* (antigravity) for matter–antimatter systems, $F^{d} = g \times m_1 \times m_2^{d}/r \times r < 0$ on $S(t,r,v)$ or $F^{d} = g^{d} \times^{d} m_1^{d} \times^{d} m_2/^{d} r^{d} \times^{d} r^{d} > 0$ on $S^{d}(t^{d}, r^{d}, v^{d})$.

The above predictions are confirmed at all subsequent classical levels, including the representation of antimatter on isodual Riemannian spaces, which yields the characterization of antigravity for matter–antimatter systems via the reversal of the sign of the curvature tensor (see [5,9] for brevity). In different terms, even thogh antimatter–antimatter systems are attractive as the ordinary matter–matter ones, the gravitational fields of matter and of antimatter are different. Thus, gravity can tell whether it is made up of matter or antimatter.

The next level of study is that via the *isodual analytic mechanics* [11], which is characterized by the *isodual Lagrange equations* (here omitted for brevity), and the *isodual Hamilton equations* in the *isodual Hamiltonian* $H^{d}(t^{d}, r^{d}, p^{d}) = -H(t,r,p)$ [11]

$$\frac{d^{d} r^{kd}}{d^{d} t^{d}} d = \frac{\partial^{d} H^{d}}{\partial^{d} p_k^{d}} d, \qquad \frac{d^{d} p_k^{d}}{d^{d} t^{d}} d = -\frac{\partial^{d} H^{d}}{\partial^{d} x^{kd}} d, \tag{29}$$

which are defined on isodual space $S^{d}(t^{d}, r^{d}, p^{d})$ with isodual units (27), an $p_k^{d} = m^{d} \times^{d} v_k^{d}$.

Eq. (29) are derivable from the *isodual action*

$$A^{d} = \int^{d} {}_{t_1^{d}}^{t_2^{d}} (p_k^{d} \times d^{d} x^{kd} - H^{d} \times^{d} d^{d} t^{d}) = -A, \tag{30}$$

where $\int^{d} = -\int$ is the *isodual integral*, with *isodual Hamilton–Jacobi equations* [loc. cit.]

$$\begin{aligned} &\partial^{d} A^{d}/^{d} \partial^{d} t^{d} + H^{d} = 0, \\ &\partial^{d} A^{d}/^{d} \partial^{d} r^{dk} - p_k^{d} = 0. \end{aligned} \tag{31}$$

It is easy to see that the isodual analytic mechanics preserves all electromagnetic and gravitational predictions of the isodual Newtonian theory.

The operator formulation is characterized by a *new quantization for antimatter*, which is missing in current theories. It can be first expressed via the *naive isodual quantization* [11]

$$A^{d} \to \mathrm{i} \times^{d} \hbar^{d} \times^{d} \ln^{d} \Psi^{d}(t^{d}, r^{d}), \tag{32}$$

under which eqs. (31) are mapped into the *isodual Schrödinger equations* [11]

$$\begin{aligned} &\mathrm{i} \times^{d} \hbar^{d} \times^{d} \partial^{d} \Psi^{d}/^{d} \partial^{d} t^{d} = H^{d} \times^{d} \Psi^{d}, \\ &p_k^{d} \times^{d} \Psi^{d} = -\mathrm{i} \times^{d} \hbar^{d} \times^{d} \partial^{d} \Psi^{d}/^{d} \partial^{d} r^{dk}, \end{aligned} \tag{33}$$

with corresponding *isodual Heisenberg equation* [loc.cit.]

$$\mathrm{i} \times^{\mathrm{d}} \mathrm{d}^{\mathrm{d}} Q^{\mathrm{d}} /^{\mathrm{d}} \mathrm{d}^{\mathrm{d}} t^{\mathrm{d}} = Q^{\mathrm{d}} \times^{\mathrm{d}} H^{\mathrm{d}} - H^{\mathrm{d}} \times^{\mathrm{d}} Q^{\mathrm{d}}. \tag{34}$$

The above naive derivation is confirmed by the novel *isodual symplectic quantization* which is not reviewed here for brevity [loc.cit.].

Isodual techniques have therefore permitted the identification of a hitherto unknown image of quantum mechanics called by the author *isodual quantum mechanics*, whose structure is characterized by [5]:

1. Isodual fields of real numbers $R^{\mathrm{d}}(n^{\mathrm{d}}, +^{\mathrm{d}}, \times^{\mathrm{d}})$ and complex numbers $C^{\mathrm{d}}(c^{\mathrm{d}}, +^{\mathrm{d}}, \times^{\mathrm{d}})$.
2. Isodual carrier spaces, e.g., $E^{\mathrm{d}}(r^{\mathrm{d}}, \delta^{\mathrm{d}}, R^{\mathrm{d}})$.
3. Isodual Hilbert space $\mathcal{H}^{\mathrm{d}}$.
4. Isodual enveloping operator algebra ξ^{d}.
5. Isodual symmetries realized via isodual unitary operators in $\mathcal{H}^{\mathrm{d}}$, e.g., eq. (26).

The fundamental notion of the theory is evidently given by the *isodual Planck constant* $\hbar^{\mathrm{d}} = -\hbar$, although referred to a negative unit -1, thus being equivalent to $\hbar > 0$ when referred to its positive unit $+1$.

It is evident that the map from quantum mechanics to its isodual is bi–injective and anti-isomorphic, as desired and as occurring at all preceding levels.

Note in particular that the new Hilbert space invariance law (24) assures that all physical laws which hold for particles also hold for antiparticles, as confirmed by the equivalence between charge conjugation and isoduality.

It should be noted that charge conjugation is (bi-injective and) *homomorphic* because spaces are mapped into themselves. On the contrary, isoduality is (bi-injective and) *anti-isomorphic* because spaces are mapped into new ones, the isodual spaces, which are coexistent, yet physically distinct from conventional spaces. The latter occurrence will soon appear crucial for the main results of this note.

Intriguingly, *isodual quantum mechanics recovers the known electromagnetic and weak phemenology of antiparticles* [5,9], thus providing sufficient credibility for further studies. In fact, the isodual operator theory merely provides a *re-interpretation* of existing phenomenological knowledge, as the reader is encouraged to verify, e.g., for the quantum Coulomb interactions.

It should also be noted that the above results are reached for *antiparticles in first quantization*, because second quantization is done for exactly the same reasons used for *particles*, no more and no less, owing to the complete parallelism of the theories for matter and antimatter at all levels.

Our relativistic theory of antimatter also begins at the classical level and it is based on a new image of the special relativity called by this author the *isodual special relativity* [3,5,9]. The latter theory is based on the isodual Minkowski space $M^{\mathrm{d}}(x^{\mathrm{d}}, \eta^{\mathrm{d}}, R^{\mathrm{d}})$ with basic isodual unit of space and time (also in dimensional form) $I^{\mathrm{d}} = -\mathrm{Diag}(1, 1, 1, 1,)$.

The fundamental symmetry is the isodual Poincaré symmetry $P^d(3.1) = L^d(3.1) \times^d T^d(3.1)$, with *isodual Lorentz transforms*

$$\begin{cases} x'^{d1} = x^{d1}, \quad x'^{d2} = x^{d2}, \\ x'^{d3} = \gamma^d \times^d (x^{d3} - \beta^d \times^d x^{d4}), \\ x'^{d4} = \gamma^d \times^d (x^{d4} - \beta^d \times^d x^{d3}), \end{cases}$$
$$\beta^d = v^d /^d c^d = -\beta, \quad \beta^{d2d} = v^d \times^d v^d /^d c^d \times^d c^d = -\beta^2,$$
$$\gamma^d = 1/^d [(1-\beta^2)^d]^{\frac{1}{2}d} = -1/(1-\beta^2)^{1/2} = -\gamma, \tag{35}$$

where we have used properties (6).

It is instructive to verify that transforms (35) are the negative versions of the conventional transforms, thus confirming the isodual Lie theory [12].

The applicability of the isodual special relativity for the characterization of antimatter is established by the isoselfduality of the relativistic interval

$$\begin{aligned} [(x_1 - x_2)^d]^{2d} &= [(x_1^{d\mu} - x_2^{d\mu}) \times \eta^d_{\mu\nu} \times (x_1^{d\nu} - x_2^{d\nu})] \times I^d \\ &\equiv [(x_1^\mu - x_2^\mu) \times \eta_{\mu\nu}(x_1^\nu - x_2^\nu)] \times I \\ &= (x_1 - x_2)^2, \end{aligned} \tag{36}$$

which has remained unknown throughout this century because of the prior need of the isodual numbers.

It is an instructive exercise for the interested reader to see that the isodual special relativity recovers all known classical electromagnetic phenomenology for antiparticles [5,9].

The isodual theory of antimatter sees its best expression at the level of *isodual relativistic quantum mechanics* [5,9], which is given by a simple isoduality of the conventional theory here omitted for brevity.

We merely point out that *negative units and related isodual theory appear in the very structure of the conventional Dirac equation*

$$\gamma^\mu \times [p_\mu - e \times A_\mu(x)/c] + \mathrm{i} \times m \times \Psi(x) = 0,$$
$$\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma^4 = \mathrm{i} \times \begin{pmatrix} I_s & 0 \\ 0 & -I_s \end{pmatrix}. \tag{37}$$

In fact, the *isodual unit of spin* $I^d = -I_s = -\mathrm{Diag}(1,1)$ enters the very structure of γ^4, while the *isodual Pauli matrices* $\sigma^{kd} = -\sigma^k$ enter in the characterization of γ^k.

The above occurrence implies the emergence of a *novel interpretation* of the *conventional* Dirac equation based on the following total unit, total space and total symmetry

$$\begin{aligned} I_{\mathrm{Tot}} &= \{I_{\mathrm{orb}} \times I_{\mathrm{spin}}\} \times \{I^d_{\mathrm{orb}} \times^d I^d_{\mathrm{spin}}\}, \\ M_{\mathrm{Tot}} &= \{M(x,\eta,R) \times S_{\mathrm{spin}}\} \times \{M^d(x^d,\eta^d,R^d) \times^d S^d_{\mathrm{spin}}\}, \\ S_{\mathrm{Tot}} &= \{SL(2.C) \times T(3.1)\} \times \{SL^d(2.C^d) \times^d T^d(3.1)\}, \end{aligned} \tag{38}$$

where $I_{\mathrm{orb}} = \mathrm{Diag}(1,1,1,1)$ and $I_{\mathrm{spin}} = \mathrm{Diag}(1,1)$.

It should be indicated that the latter re-interpretation (which has also escaped attention thoughout this century) is necessary for consistency. In fact, *the conventional gamma matrices* (37) *are isoselfdual.* The conventional interpretation that the Poincaré symmetry $\mathcal{P}(3.1)$= SL(2.C)×T(3.1) is the symmetry of Dirac's equation then leads to inconsistencies because $\mathcal{P}(3.1)$ *is not* isoselfdual. Only the product $\mathcal{P}(3.1)\times\mathcal{P}^{\mathrm{d}}(3.1)$ is isoselfdual.

The above results permit the following novel re-interpretation of eqs. (37)

$$\begin{gathered}\tilde{\gamma}^{\mu}\times[p_{\mu}-e\times A(x)/c]+\mathrm{i}\times m\times\tilde{\Psi}(x)=0,\\ \tilde{\gamma}_k=\begin{pmatrix}0 & \sigma_k^{\mathrm{d}}\\ \sigma_k & 0\end{pmatrix},\quad \tilde{\gamma}^4=\mathrm{i}\begin{pmatrix}I_s^{\mathrm{d}} & 0\\ 0 & I_s\end{pmatrix},\\ \{\tilde{\gamma}_{\mu},\tilde{\gamma}_{\nu}\}=2\eta_{\mu\nu},\quad \tilde{\Psi}=-\tilde{\gamma}_4\times\Psi=\mathrm{i}\times\begin{pmatrix}\Phi\\ \Phi^{\mathrm{d}}\end{pmatrix},\end{gathered}\tag{39}$$

where $\Phi(x)$ is now two-dimensional.

Note that the above equations: remove the need of second quantization, eliminate the need of charge conjugation because the antiparticle is represented by σ^{d}, I^{d}, Φ^{d}, and restore the correct representation of spin 1/2 via the conventional *two-dimensional* regular representation of SU(2), rather than the current use of a *four-dimensional* γ-representation.

We are now equipped to address the main objective of this note, a study of *the anti-hydrogen atom and its spectroscopy.*

For this purpose we restrict our analysis to *massive particles* defined as irreducible unitary representations of the Poincaré symmetry $\mathcal{P}(3.1)$. This essentially restricts the analysis to the electron e and the proton p, both considered as *elementary*, due to certain ambiguities for composite hadrons indicated in Appendix A.

We then introduce the notion of *massive isodual particles* as irreducible unitary representations of the isodual Poincaré symmetry $\mathcal{P}^{\mathrm{d}}(3.1)$. This again restricts the analysis to the *isodual electron* e^{d} and *isodual proton* p^{d}.

At this point we should indicate the differences between "antiparticles" and "isodual particles". In first approximation, these two notions can be identified owing to the equivalence of charge conjugation and isoduality. However, at a deeper inspection, antiparticles and isodual particles result to be different on a number of grounds, such as:

1. Isoduality is broader than the PTC symmetry because, in addition to reversing space–time coordinates $x\to x^{\mathrm{d}}=-x$ (PT) and conjugating the charge $q\to q^{\mathrm{d}}=-q$ (C), it also reverses the sign of the background units.
2. Antiparticles are defined in our own space–time, while isodual particles are defined in a new space–time which is physically different than our own.
3. Antiparticles have *positive* mass, energy, (magnitude of) spin, etc., and *move forward in time,* while isodual particles have *negative* mass, energy, (magnitude of) spin, etc., and *move backward in time.*

Next, we consider the bound states of the above elementary particles which are given by the hydrogen atom $A = (\mathrm{p}, \mathrm{e})_{\mathrm{QM}}$, the isodual hydrogen atom $A^{\mathrm{d}} = (\mathrm{p}^{\mathrm{d}}, \mathrm{e}^{\mathrm{d}})_{\mathrm{IQM}}$, and the positronium $P = (\mathrm{e}, \mathrm{e}^{\mathrm{d}})_{\mathrm{QM}} = (\mathrm{e}^{\mathrm{d}}, \mathrm{e})_{\mathrm{IQM}}$, where QM (IQM) stands for Quantum Mechanics (Isodual Quantum Mechanics).

It is evident that the hydrogen atom is characterized by the familiar Schrödinger's equation with Coulomb spectrum E_n on the Hilbert space $\mathcal{H}$,

$$H \times \mid \Psi\rangle = E_n \times \mid \Psi\rangle. \tag{40}$$

The isodual hydrogen atom is then characterized by the isodual image on $\mathcal{H}^{\mathrm{d}}$,

$$H^{\mathrm{d}} \times^{\mathrm{d}} \mid \Psi\rangle^{\mathrm{d}} = E_n^{\mathrm{d}} \times^{\mathrm{d}} \mid \Psi^{\mathrm{d}}. \tag{41}$$

The isodual theory therefore predicts that *the isodual hydrogen atom has the same spectrum of the conventional atom, although with energy levels reversed in sign,* $E_n^{\mathrm{d}} = -E_n$.

The positronium is an isoselfdual state, because evidently invariant under the interchanges $\mathrm{e} \to \mathrm{e}^{\mathrm{d}}$, $\mathrm{e}^{\mathrm{d}} \to \mathrm{e}$. As such, it possesses a *positive* spectrum E_n in our space–time and a *negative* spectrum when studied in isodual space–time. In fact, the total state of the positronium is given by $|\mathrm{Pos}\rangle = |e\rangle \times |e\rangle^{\mathrm{d}}$ with Schrödinger's equation in our space–time ($\hbar = 1$)

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial t}|\mathrm{Pos}\rangle &= (p_k \times p^k/2m) \times |\mathrm{e}\rangle \times |\mathrm{e}\rangle^{\mathrm{d}} \\ &\quad + |\mathrm{e}\rangle \times (p_k \times p^k/2m)^{\mathrm{d}} \times^{\mathrm{d}} |\mathrm{e}\rangle^{\mathrm{d}} + V(r) \times |\mathrm{e}\rangle \times |\mathrm{e}\rangle^{\mathrm{d}} \\ &= E_n|\mathrm{Pos}\rangle, \quad E_n > 0, \end{aligned} \tag{42}$$

with a conjugate expression in isodual space–time.

As indicated earlier, the isodual theory recovers the available information on electromagnetic (and weak) interactions of antiparticles. No novelty is therefore expected along these lines in regard to the antihydrogen atom and the positronium.

However, the isodual theory has the following novel predictions for gravitational interactions [6,9]:

Prediction I. Massive stable isodual particles and their bound states (such as the isodual hydrogen atom) experience antigravity in the field of matter and ordinary gravity in the field of antimatter.

Prediction II. Bound states of massive stable particles and their isoduals (such as the positronium) experience ordinary gravity in both fields of matter and antimatter.

We now remain with the central open problem raised in this note: *Does antimatter emit a new light different than that emitted by ordinary matter?*

The answer provided by the isodual theory is in the affirmative. Recall that the *photon* γ emitted by the hydrogen atom has *positive* energy and time according to the familiar plane-waves characterization on $M(x, \eta, R)$ with unit $I = \mathrm{Diag}(1, 1, 1, 1)$

$$\Psi(t, r) = A \times \mathrm{e}^{\mathrm{i} \times (k \times r - E \times t)}. \tag{43}$$

appreciable electromagnetic origin, or it may imply the lack of physical validity of the mathematical notion of isoduality.

Finally, additional studies are needed on the classical and quantum isotopic representation of gravity and its isodual [9], because these studies contain all the preceding ones *plus* the inclusion of gravitation which is embedded in the unit for matter and in the isodual unit for antimatter.

Preliminary (unpublished) studies have indicated that the latter approach confirms the results of this note, while permitting further advances at the isooperator gravitational level, e.g., an axiomatically consistent formulation of the PTC theorem inclusive of gravitation.

We finally note that the possible lack of existence of antigravity for the isodual photon *will not* invalidate the isodual theory, because it will only imply the isoselfduality of the photon, that is, the presence in the photon of both retarded and advanced solutions, which would remain separated for massive particles. Thus, antigravity may exist for massive particles without necessarily existing for light.

Interested readers are encouraged to identify possible *theoretical* arguments against antigravity for light emitted by antimatter, with the clear understanding that the final scientific resolution one way or the other can only be the *experimental* one.

Appendix A. Gravitational problematic aspects of quark theories

The "new physics of antimatter" is expected to have an impact on all of elementary particle physics, because it focuses the attention on novel gravitational, rather than familiar electroweak aspects.

An illustration is given by a necessary reformulation of contemporary quark theories. In fact, gravitation is solely defined in our space–time, while quarks are solely defined in the mathematical, unitary, internal space, with no interconnection being possible due to the O'Rafeirtaigh theorem.

It *necessarily* follows that *all particles made up of quarks cannot have any gravitation at all*, which is grossly contrary to experimental evidence.

It should be indicated that O'Raifeartaigh's theorem has been superseded by graded Lie algebras and related supersymmetries, in which case a connection between space–time and internal symmetries is possible. However, the validity of such interconnection would require the prior establishment of the physical validity of supersymmetries and the existence of their predicted new particles. Irrespective of that, a correct formulation of the gravity of quarks within this latter setting is faced with serious technical problems and it has not been achieved until now, to our best knowledge.

The above conclusion is confirmed by the well known fact that *quarks cannot be characterized by irreducible representations of the Poincaré group*, that is, quark masses do not exist in our space–time, and are mere parameters in unitary spaces.

Even assuming that the above fundamental problem is somewhat resolved via hitherto unknown manipulations, additional equally fundamental problems exist in the construction of a quark theory of antimatter, because it does not yield in general an anti-isomorphic image of the phenomenology of matter.

When including the additional, well known problematic aspects of quark theories (e.g., the vexing problem of confinement which is not permitted by the uncertainty principle), a structural revision of contemporary quark theories becomes beyond *credible* doubts.

The only resolution of the current scientific impass known to this author is that advocated since 1981, *quarks cannot be elementary particles* [14], as apparently confirmed by recent experiments at Fermilab [15].

In fact, the compositeness of quarks would permit their construction as suitable bound states of physical massive particles existing in our space–time, in which case (only) there would be the regaining of the physical behavior under gravity.

The following aspects should however be clearly stated to separate science from fiction. First, the above new generation of quark theories requires the abandonment of the conventional Poincaré symmetry P(3.1) in favor of a nonlinear, nonlocal-integral and non-canonical generalization, e.g., the isoPoincaré-symmetry $\widehat{P}(3.1) \approx P(3.1)$ [16]. In fact, a consistent construction of composite quarks inside hadron requires the necessary alteration of the *intrinsic* characteristics of ordinary particles which is prohibited by P(3.1) but is rather natural for $\widehat{P}$ (3.1) [16] and related methodology [5].

The use of the q-, k- and quantum deformations should be excluded because they are afflicted by excessive problems of physical consistency (which are absent for isotopies), such as [5]:

1) Lack of invariance of the basic unit with consequential inapplicability to actual measurements;
2) Lack of preservation of Hermiticity in time with consequential lack of observables;
3) Lack of invariant special functions (because, e.g., the number q becomes an operator under the time evolution);
4) Lack of uniqueness and invariance of physical laws;
5) Loss of Eistein's axioms; etc.

Second, the real constituents of hadrons are expected to be the quark *constituents* and not the quarks themselves [14]. This new perspective removes altogether the need for confinement. As a matter of fact, the hadronic constituents are expected to be produced free and actually identified in the massive particles produced in the spontaneous decays with the lowest mode [16].

The latter particles become conceivable as constituents because of the novel renormalizations of their *intrinsic* characteristics which are permitted by internal non-Lagrangian and non-Hamiltonian effects.

Third, the primary physical meaning of unitary theories and related methodologies (Poincaré symmetry, SU(3) symmetry, relativistic quantum mechanics, etc.) is their historical one: having achieved the final classification of hadrons into families and the final understanding of the related exterior phenomenology.

In much of the way as it occurred for the atoms in the transition from the Mendeleev *classification* into families to the different problem of the *structure* of the individual atoms, the transition from the unitary classification of hadrons into families to the different problem of the structure of the individual hadrons, is expected to require a nonlinear, nonlocal-integral and nonpotential–non-Hamiltonian generalization of relativistic quantum mechanics, e.g., of the isotopic axion-preserving type of refs. [5,14,16].

We should not forget that hadrons are not ideal spheres with points in them, but are instead some of the densest objects measured in laboratory by mankind in which the constituent is in a state of total mutual penetration of the wavepackets. It is an easy prediction that, even though of clear preliminary physical value, the use for the latter conditions of theories which are linear, local-differential and Lagrangian–Hamiltonian will not resist the test of time.

The author would be grateful to colleagues who care to bring to his attention any credible alternative to the above lines [5,14], that is, a new theory with composite quarks which:

1) admits physical constituents unambiguously defined in our space–time;
2) represents without ambiguities the gravitational behavior of matter and antimatter; and
3) is based on the *exact* validity of the Poincaré-symmetry, quantum mechanics and all that.

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