Isorepresentations of the Lie-Isotopic $SU(2)$ Algebra with Applications to Nuclear Physics and to Local Realism

RUGGERO MARIA SANTILLI
The Institute for Basic Research, PO Box 1577, Palm Harbor, FL 34682, U.S.A.

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Abstract. In this note, we study the nonlinear-nonlocal-noncanonical, axiom-preserving isotopies/Q-operator deformations $S\widehat{U}_Q(2)$ of the $SU(2)$ spin-isospin symmetry. We prove the local isomorphism $S\widehat{U}_Q(2) \approx SU(2)$, construct and classify the isorepresentations of $S\widehat{U}_Q(2)$, identify the emerging generalizations of Pauli matrices, and show their lack of unitary equivalence to the conventional representations. The theory is applied for the reconstruction of the exact $SU(2)$-isospin symmetry in nuclear physics with equal p and n masses in isospaces. We also prove that Bell's inequality and the von Neumann theorem are inapplicable under isotopies, thus permitting the isotopic completion/Q-operator deformation of quantum mechanics studied in this note which is considerably along the celebrated argument by Einstein, Podolsky and Rosen.

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1. Statement of the Problem

According to current knowledge (see, e.g., [1, 4]), the $SU(2)$ spin or isospin symmetry can solely characterize the familiar eigenvalues $j(j+1)$ and m, $j = 0, \frac{1}{2}, 1, \ldots$, $m = j, j-1, \ldots, -j$.

In this note, we show that the isotopic generalization of $SU(2)$, herein denoted $S\widehat{U}_Q(2)$, while being locally isomorphic to $SU(2)$, can characterize more general eigenvalues of the type

$$\widehat{J^2} \Rightarrow f(\Delta)j[f(\Delta)j+1], \qquad J_3 \Rightarrow f(\Delta)m, \tag{1.1}$$

where j and m have conventional values and $f(\Delta)$ is a real valued, positive-definite function of the determinant of the background metric $\Delta = \operatorname{Det} g = \operatorname{Det} Q\delta$ such that $f(1) = 1$.

For the two-dimensional case, the condition $\det g = 1$ for $g = \operatorname{diag}(g_{11}, g_{22})$ is realized by $g_{11} = g_{22}^{-1} = \lambda$. This implies the preservation of the conventional value $\frac{1}{2}$ of the spin, but the appearance of a nontrivial generalization of Pauli's matrices, herein called *iso-Pauli matrices*, with an explicit realization of the 'hidden variable' λ in the structure of the spin $\frac{1}{2}$ itself.

As a first application, we construct the isotopies of the conventional isospin (see, e.g., [2, 6]), and show that the iso-Pauli matrices permit the reconstruction of an *exact* $S\hat{U}(2)$-isospin symmetry *under electromagnetic and weak interactions* because protons and neutrons acquire equal masses in the underlying isospace.

As a second application, we show that Bell's inequality and the von Neumann theorem (see, e.g., review [7]) are inapplicable under isotopies, thus permitting the isotopic completion of quantum mechanics studied in this note, which is considerably much along the lines of the celebrated Einstein–Podolski–Rosen (EPR) argument.

It should be noted that, at the International Workshop on Symmetry Methods in Physics held at the JINR in July 1993, Lopez [9] showed that the so-called *q-deformations* (see, e.g., [17, 45]) can be put in an axiomatic form precisely via the isotopic Q-operator deformations studied in this note.

One isotopy of Pauli matrices was first presented by this author at the Third International Wigner Symposium (held at Oxford University in September 1993, [13]). In this note, we present, apparently for the first time, a systematic study and classification of the fundamental (adjoint) isorepresentation of the Lie-isotopic $S\hat{U}_Q(2)$ algebra, their applications to the reconstruction of the exact isospin symmetry as well as to the limitation of Bell's inequality and von Neumann's theorem. Additional applications to nuclear magnetic moments, particle physics and other fields will be presented elsewhere.

2. Isotopies of $SU(2)$ Symmetry

The understanding of this note requires a knowledge of the nonlinear-nonlocal-noncanonical, axiom-preserving isotopies of the theory of numbers [11] and of Lie's theory as reviewed in the article [8] in this issue and studied in detail in the monographs [16, 18].

The fundamental notion is the *isotopy of the unit* of the theory considered [4, 6–14], in this case, the unit $I = \mathrm{diag}(1,1)$ of $SU(2)$, into a two-dimensional matrix $\hat{I}$ whose elements have the most general possible dependence on complex coordinates z, $\bar{z}$ of the underlying carrier space of $SU(2)$, their derivative with respect to time of arbitrary order, the wave functions ψ, $\psi^\dagger$ and their derivatives also of arbitrary order, and any needed additional quantity, subject to the condition of preserving the original axioms of I (smoothness, boundedness, nonsingularity, Hermiticity and positive-definiteness, as a necessary condition for isotopy),

$$I = \mathrm{diag}(1,1) > 0 \Rightarrow \hat{I} = \hat{I}(t, z, \bar{z}, \dot{z}, \bar{z}\psi, \psi\dagger, \partial\psi, \partial\psi\dagger, \ldots) > 0. \tag{2.1}$$

The isotopy of the unit then demands, for consistency, a corresponding, compatible lifting of all associative products AB among generic QM quantities A, B, into the isoproduct

$$AB \Rightarrow A * B := AQB, \quad Q \text{ fixed}, \tag{2.2}$$

where the isotopic character of the lifting is established by the preservation of associativity by the isoproduct, $A * (B * C) = (A * B) * C$.

The assumption $\hat{I} = Q^{-1}$ then implies that $\hat{I}$ is the correct left and right unit of the theory, $\hat{I} * A = A * \hat{I} \equiv A$, in which case Q is called the *isotopic element*, and $\hat{I}$ is called the *isounit*. Note the invariant appearance of q-deformations in their Q-operator form at the very foundation of the theory, provided that they are reformulated with respect to the new unit $\hat{I} = q^{-1}$ ([10, 11]).

The isotopies of the unit $I \Rightarrow \hat{I}$ and of the product $AB \Rightarrow A * B$ then imply the necessary lifting of *all* mathematical structures of quantum mechanics (QM) into those of a covering discipline called *hadronic mechanics* (HM) [16]. Here we mention the lifting of the field of complex numbers $C(c, +, \times)$, with elements c, ordinary sum $+$ and multiplication $c \times c' = cc'$, into the infinitely possible isotopes $\hat{C}_Q(\hat{c}, +, *)$, with *isocomplex numbers* $\hat{c} = C\hat{I}$, conventional sum $+$ and isomultiplication $\hat{c}_1 * \hat{c}_2 = \hat{c}_1 Q \hat{c}_2 = (c_1 c_2)\hat{I}$. Note for future use that, for an arbitrary quantity A, $\hat{c} * A = c\hat{I}QA \equiv cA$.

The isotopies of the unit, multiplication and fields then demand, for mathematical consistency, corresponding compatible isotopies of the basic carrier space, the two-dimensional complex Euclidean space $E(z, \bar{z}, \delta, C)$ with familiar metric $\delta = \mathrm{diag}(1,1)$ into the complex two-dimensional *iso-Euclidean spaces* introduced in [15, 16]

$$\hat{E}_Q(z, \bar{z}, \hat{\delta}, \hat{C})\colon\ z = (z_1, z_2), \quad \hat{\delta} = Q\delta \equiv g = \mathrm{diag}(g_{11}, g_{22}) = g\dagger > 0, \tag{2.3a}$$

$$z \dagger_i g_{ij}(t, z, \bar{z}, \ldots) z_j = \bar{z}_1 g_{11} z_1 + \bar{z}_2 g_{22} z_2, \tag{2.3b}$$

where the assumed diagonalization of Q is always possible (although not necessary) from its positive-definiteness.

The isotopic character (as well as novelty) of the generalization is established by the fact that, under the *joint* lifting of the metric $\delta \Rightarrow \hat{\delta} = Q\delta = g$ and of the field $C \Rightarrow \hat{C}_Q$, $\hat{I} = Q^{-1}$, all infinitely possible isospaces $\hat{E}_Q(z, \bar{z}, \hat{\delta}, \hat{C})$ are locally isomorphic to the original space $E(z, \bar{z}, \delta, C)$ under the sole condition of positive-definiteness of the isounit $\hat{I}$ [15]. In turn, this evidently sets the foundation for the local isomorphism of the corresponding symmetries.

Note that separation (2.3) is the most general possible nonlinear, nonlocal and noncanonical generalization of the original separation $z \dagger z$ under the sole condition of remaining positive-definite, i.e., of preserving the topology $\mathrm{sig}\,\delta = \mathrm{sig}\,\hat{\delta} = (+,+)$. The symmetries of invariant (2.3) are then expected to be nonlinear, nonlocal and noncanonical, as desired.

The preceding isotopies imply, for consistency, the isotopies of Hilbert spaces $\mathcal{H}$: $\langle\psi|\phi\rangle \in C$ into the so-called *iso-Hilbert space* $\hat{\mathcal{H}}_Q$ with *isoproduct* and *isonormalization*

$$\hat{\mathcal{H}}_Q\colon\ \langle\hat{\psi}\hat{|}\hat{\phi}\rangle = \langle\hat{\psi}|Q|\hat{\phi}\rangle\hat{I} \in \hat{C}_Q; \quad \langle\hat{\psi}\hat{|}\hat{\psi}\rangle = \hat{I}. \tag{2.4}$$

Then, operators which are Hermitian (observable) for QM remain Hermitian (observable) for HM [16].

The liftings of the Hilbert space require corresponding isotopies of all operations in $\mathcal{H}$ [13, 14]. We here mention isounitarity $\widehat{U} * \widehat{U}^\dagger = \widehat{U}^\dagger * \widehat{U} = \widehat{I}$; isoeigenvalue equations $H * |\widehat{\psi}\rangle = HQ|\widehat{\psi}\rangle = \widehat{E} * |\widehat{\psi}\rangle \equiv E|\widehat{\psi}\rangle$; isoexpectation values $\widehat{\langle A \rangle} = \langle\widehat{\psi}|QAQ|\widehat{\psi}\rangle / \langle\widehat{\psi}|Q|\widehat{\psi}\rangle$; etc.

The lifting of the unit, base field and carrier space then require, for mathematical consistency, the lifting of the entire structure of Lie's theory first submitted in [10]. We are here referring to the isotopies of enveloping associative algebras ξ, Lie algebras L, Lie groups G, representation theory, etc., today called Lie–Santilli theory. Here we mention the *isoassociative enveloping operator algebras* $\widehat{\xi}_Q$ with isoproduct (2.2), $A * B \equiv AQB$; the *Lie-isotopic algebras* $\widehat{L}_Q$ with isoproduct

$$[A, B]_{\widehat{\xi}_Q} = [A, \widehat{\ } B] = A * B - B * A \equiv AQB - BQA; \tag{2.5}$$

the (connected) *Lie-isotopic groups* $\widehat{G}_Q$ of *isolinear isounitary transforms* on $\widehat{E}_Q(z, \overline{z}, \widehat{\delta}, \widehat{C})$

$$z' = \widehat{U}(w) * z = \widehat{U}(w)Qz = \widehat{U}(w)Q(z, \overline{z}, \dot{z}, \dot{\overline{z}}, \widehat{\psi}, \widehat{\psi}\dagger, \ldots)z, \tag{2.6a}$$

$$\widehat{U}(w) = e^{iX*\widehat{w}}_{|\widehat{\xi}_Q} = \widehat{I} + (iXw)/1! + (iXw) * (iXw)/2I + \cdots$$
$$= \{e^{iXQw}\}\widehat{I}, \tag{2.6b}$$

$$\widehat{U}(w) * \widehat{U}(w') = \widehat{U}(w') * \widehat{U}(w) = \widehat{U}(w + w'),$$
$$\widehat{U}(w) * \widehat{U}(-w) = \widehat{U}(0) = \widehat{I}, \tag{2.6c}$$

where the reformulation in terms of the conventional exponentiation has been done for simplicity of calculations.

The *isounitary* $\widehat{U}_Q(2)$ *symmetry* is the most general possible, nonlinear, nonlocal and noncanonical, simple, Lie-isotopic invariance group of separation (2.3b) with realization in terms of isounitary operators on $\widehat{\mathcal{H}}_Q$

$$\widehat{U} * \widehat{U}^\dagger = \widehat{U}^\dagger * \widehat{U} = \widehat{I} = Q^{-1}, \tag{2.7}$$

verifying isotopic laws (2.6). $\widehat{U}(2)$ can be decomposed into the *connected, special isounitary symmetry* $S\widehat{U}_Q(2)$ for

$$\det(\widehat{U}Q) = +1, \tag{2.8}$$

plus a discrete part which is similar to that for $\widehat{O}(3)$ [10] and is here ignored for brevity.

The connected $S\widehat{U}_Q(2)$ components admit the realization in terms of new generators $\widehat{J}_k$ and the same parameters $\theta_k \in R(n, +, \times)$ of $SU(2)$ although re-expressed in the isofield $\widehat{R}(\widehat{n}, +, *)$

$$\widehat{U} = \prod_k e_{\widehat{\xi}}^{i\widehat{J}_k * \widehat{\theta}_k} = \left\{\prod_k e^{i\widehat{J}_k Q\theta_k}\right\}\widehat{I}, \tag{2.9}$$

under the conditions (necessary for isounitarity)

$$\mathrm{tr}(\widehat{J}_k Q) \equiv 0, \quad k = 1, 2, 3. \tag{2.10}$$

The *isorepresentations of the isotopic algebras* $S\widehat{U}_Q(2)$ can be studied by imposing that the isocommutation rules have the same structure constants of $SU(2)$, i.e., for the rules

$$[\widehat{J}_i, \widehat{\ } \widehat{J}_j] = \widehat{J}_i Q \widehat{J}_j - \widehat{J}_j Q \widehat{J}_i = i\varepsilon_{ijk}\widehat{J}_k. \tag{2.11}$$

with iso-Casimir

$$\widehat{J}^2 = \sum_k \widehat{J}_k * \widehat{J}_k. \tag{2.12}$$

The maximal isocommuting set is then given by $\widehat{J}^2$ and $\widehat{J}_3$ as in the conventional case. These assumptions ensure the local isomorphism $S\widehat{U}(2) \approx SU(2)$ by construction.

Let $|\widehat{b}^d_k\rangle$ be the d-dimensional isobasis of $S\widehat{U}_Q(2)$ with iso-orthogonality conditions

$$\langle\widehat{b}^d_i| * |\widehat{b}^d_j\rangle := \langle\widehat{b}^d_i|Q|\widehat{b}^d_j\rangle = \delta_{ij}, \quad i, j = 1, 2, \ldots, n. \tag{2.13}$$

By putting as in the conventional case $\widehat{J}_\pm = \widehat{J}_1 \pm \widehat{J}_2$, and by repeating the same procedure as the familiar one [1], we have

$$\widehat{J}_3 * |\widehat{b}^d_k\rangle = b^d_k|\widehat{b}^d_k\rangle, \qquad \widehat{J}^2 * |\widehat{b}^d_k\rangle = b^d_1(b^d_1 - 1)|\widehat{b}^d_1\rangle, \tag{2.14a}$$

$$d = 1, 2, \ldots, \quad k = 1, 2, \ldots, d,$$
$$b^d_1 \equiv -b^d_d, \qquad b^d_1(b^d_1 - 1) \equiv b^d_d(b^d_d + 1). \tag{2.14b}$$

A consequence is that *the dimensions of the isorepresentations of* $S\widehat{U}_Q(2)$ *remain the conventional ones*, i.e., they can be characterized by the familiar expression $n = 2j + 1$, $j = 0, \frac{1}{2}, 1, \ldots$ as expected from the isomorphism $S\widehat{U}_Q(2) \approx SU(2)$.

However, *the explicit form of the matrix representations are different than the conventional ones*, as expressed by the rules

$$(\widehat{J}_1)_{ij} = \tfrac{1}{2}i\langle\widehat{b}^d_i| * (\widehat{J}_- - \widehat{J}_+) * |\widehat{b}^d_j\rangle, \tag{2.15a}$$
$$(\widehat{J}_2)_{ij} = \tfrac{1}{2}i\langle\widehat{b}^d_i| * (\widehat{J}_- + \widehat{J}_+) * |\widehat{b}^d_j\rangle, \tag{2.15b}$$
$$(\widehat{J}_3)_{ij} = \langle\widehat{b}^d_i| * \widehat{J}_3 * |\widehat{b}^d_j\rangle, \tag{2.15c}$$

under condition (2.10).

The isorepresentations of the desired dimension can then be constructed accordingly. In the next section we shall compute the two-dimensional isorepresentations, while those of higher dimensions will be studied in a subsequent paper.

A new image of the conventional $SU(2)$ symmetry is characterized by our isotopic methods via the antiautomorphic map $I = \mathrm{diag}(1, 1) \Rightarrow I^d = -I$ called *isoduality* ([12, 16]), which provides a novel and intriguing characterization of antiparticles. The corresponding *isodual isosymmetry* $\widehat{SU}^d_Q(2)$ will be studied in a separate work.

In summary, our isotopic methods permit the identification of four physically relevant isotopies and isodualities of $SU(2)$ which, for the case of isospin, are given by the broken conventional $SU(2)$ for the usual treatment of $p-n$; the exact isotopic $\widehat{SU}_Q(2)$ for the characterization of $p-n$ (see next section); the broken isodual $SU^d(2)$ symmetry for the characterization of the antiparticles $\overline{p}-\overline{n}$ in isodual spaces; and the exact, isodual, isotopic $\widehat{SU}^d_Q(2)$ for the characterization of antiparticles $\overline{p}-\overline{n}$ in isodual isospace.

The reader may be interested in knowing that, when the positive- (or negative-) definiteness of the isotopic element Q is relaxed, the isotopes $\widehat{SU}(2)$ unifies all three-dimensional simple Lie groups of Cartan classification over a complex field (of characteristic zero). In fact, we have the compact isotopes $\widehat{SU}_Q(2) \approx SU(2)$ for $g_{11} > 0$, $g_{22} > 0$, and the noncompact isotopes $\widehat{SU}_Q(2) \approx SU(1,1)$ for $g_{11} > 0$ and $g_{22} < 0$ (see [8] for the corresponding unification of orthogonal groups over the reals). In this note we consider only positive-definite isotopic elements Q.

3. Isotopies of Pauli Matrices

Recall that the conventional Pauli matrices σ_k (see, e.g., [2, 6]) verify the rules $\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k$, $i, j, k = 1,2,3$. In this section we identify and classify the generalizations of these familiar matrices implied by the isoalgebra $\widehat{SU}_Q(2)$.

To have a guiding principle, we recall that ([8, 15]), in general, *Lie-isotopic algebras are the image of Lie algebras under nonunitary transformations*. In fact, under a transformation $UU\dagger = \hat{I} \neq I$, a Lie commutator among generic matrices A, B, acquires the Lie-isotopic form

$$U(AB - BA)U\dagger = A'QB' - B'QA', \\ A' = UAU\dagger, \qquad B' = UBU\dagger, \qquad Q = (UU\dagger)^{-1} = Q\dagger. \tag{3.1}$$

We therefore expect a first class of fundamental (adjoint) isorepresentations, here called *regular adjoint isorepresentations*, which are characterized by the maps $J_k = \frac{1}{2}\sigma_k \to \hat{J}_k = UJ_kU\dagger$, $UU\dagger\hat{I} \neq I$ with isotopic contributions that are factorizable in the spectra, $\pm\frac{1}{2} \to +\frac{1}{2}f(\Delta)$, $3/4 \to (3/4)f^2(\Delta)$, where $\Delta = \det Q$ and $f(\Delta)$ is a smooth nowhere-null function such that $f(1) = 1$.

An example is readily constructed via Equations (2.15) resulting in the following generalization of Pauli's matrices here called *regular iso-Pauli matrices*

$$\hat{\sigma}_1 = \Delta^{-\frac{1}{2}}\begin{pmatrix} 0 & g_{11} \\ g_{22} & 0 \end{pmatrix}, \qquad \hat{\sigma}_2 = \Delta^{-\frac{1}{2}}\begin{pmatrix} 0 & -ig_{11} \\ +ig_{22} & 0 \end{pmatrix}, \\ \hat{\sigma}_3 = \Delta^{-\frac{1}{2}}\begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix}, \tag{3.2}$$

where $\Delta = \det Q = g_{11}g_{22} > 0$. The above isorepresentation verifies the isotopic rules $\hat{\sigma}_iQ\hat{\sigma}_j = i\Delta^{\frac{1}{2}}\varepsilon_{ijk}\hat{\sigma}_k$ and, consequently, the following isocommutator rules and generalized isoeigenvalues for $f(\Delta) = \Delta^{\frac{1}{2}}$

$$[\hat{\sigma}_1 \hat{,} \hat{\sigma}_j] = \hat{\sigma}_iQ\sigma\partial_j - \hat{\sigma}_jQ\hat{\sigma}_i = 2i\Delta^{\frac{1}{2}}_{ijk}\hat{\sigma}_k, \tag{3.3a}$$

$$\hat{\sigma}_3 * |\hat{b}^2_i\rangle = \pm\Delta^{\frac{1}{2}}|\hat{b}^2_i\rangle, \tag{3.3b}$$

$$\hat{\sigma}^{\hat{2}} * |\hat{b}^2_i\rangle = 3\Delta|\hat{b}^2_i\rangle, \qquad i = 1,2, \tag{3.3c}$$

which confirm the 'regular' character of the generalization here considered (that is, the factorizability of the isotopic contribution in the spectrum of eigenvalues). The isonormalized isobasis is then given by a trivial extension of the conventional basis $|\hat{b}\rangle = Q^{-\frac{1}{2}}|b\rangle$.

Recall that Pauli's matrices are essentially unique in the sense that their transformations under unitary equivalence do not yield significant changes in their structure, as well known ([1, 4]). The situation is different for the iso-Pauli matrices, because isorepresentations are based on various degrees of freedom which are absent in the conventional $SU(2)$ theory, such as: (1) infinitely possible isotopic elements Q; (2) formulation of the isoalgebra in terms of structure functions [7, 9]; (3) use of an isotopic element for the iso-Hilbert space different than that of the isoalgebra [13, 14]; and others.

In fact, we can identified a second class of isorepresentations, here called *irregular adjoint isorepresentations*, in which the isotopic contributions is no longer factorizable in the entirety of the spectra of eigenvalues. A first example is given by the following *irregular iso-Pauli matrices*

$$\hat{\sigma}'_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \qquad \hat{\sigma}'_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2, \\ \hat{\sigma}'_3 = \begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix} = \Delta\hat{I}\sigma_3, \tag{3.4}$$

which verify the isocommutation rules

$$[\hat{\sigma}'_1 \hat{,} \hat{\sigma}'_2] = 2i\hat{\sigma}'_3, \quad [\hat{\sigma}'_2 \hat{,} \hat{\sigma}'_3] = 2i\Delta\hat{\sigma}'_1, \quad [\hat{\sigma}'_1 \hat{,} \hat{\sigma}'_1] = 2i\Delta\hat{\sigma}'_2, \tag{3.5}$$

without evidently altering the local isomorphism $\widehat{SU}_Q(2) \approx SU(2)$. The new isoeigenvalue equations are given by

$$\hat{\sigma}'_3 * |\hat{b}^2_i\rangle = \pm\Delta|\hat{b}^2_i\rangle, \quad \hat{\sigma}'^{\hat{2}} * |\hat{b}^2_i\rangle = \Delta(\Delta+2)|\hat{b}^2_i\rangle, \tag{3.6}$$

which confirm the 'irregular' character consideration (that is, the lack of factorizability of the isotopic contributions in the entirety of the spectrum of eigenvalues). Isorepresentation (3.4) also provide an illustration of Equations (1.1) with the nontrivial lifting of the spin $s = \frac{1}{2} \to \hat{s} = \frac{1}{2}\Delta$.

The 'degrees of freedom' of isorepresentations are then illustrated via the following second example of irregular Pauli matrices

$$\widehat{\sigma}_1'' = \begin{pmatrix} 0 & g_{22}^{-\frac{1}{2}} \\ g_{11}^{-\frac{1}{2}} & 0 \end{pmatrix}, \qquad \widehat{\sigma}_2'' = \begin{pmatrix} 0 & -ig_{22}^{-\frac{1}{2}} \\ ig_{11}^{-\frac{1}{2}} & 0 \end{pmatrix},$$
$$\widehat{\sigma}_3'' = \begin{pmatrix} g_{11}^{-1} & 0 \\ 0 & -g_{22}^{-1} \end{pmatrix}, \tag{3.7}$$

with isocommutation rules and isoeigenvalues for $\widehat{J}_k = \frac{1}{2}\widehat{\sigma}_k''$

$$[\widehat{J}_1, \widehat{\ } \widehat{J}_2''] = i\Delta \widehat{J}_3'', \qquad [\widehat{J}_2'', \widehat{\ } \widehat{J}_3] = i\widehat{J}_1'', \qquad [\widehat{J}_3'', \widehat{\ } \widehat{J}_1''] = i\widehat{J}_2'', \tag{3.8a}$$
$$\widehat{J}_3'' * |\widehat{b}_i^2\rangle = \pm\frac{1}{2}|\widehat{b}_i^2\rangle, \qquad \widehat{J}''^{\hat{2}} * |\widehat{b}_i^2\rangle = \frac{1}{2}\left(\frac{1}{2} + \Delta\right)|\widehat{b}_i^2\rangle, \tag{3.8b}$$

where, as one can see, the eigenvalue of the third component is conventional, but that of the magnitude is generalized with a nonfactorizable isotopic contribution.

Intriguingly, the isorepresentations generally occurring in physical applications are the irregular ones ([15, 16]) because the generators represent physical quantities and, as such, are not changed under isotopies [7–9]. Their embedding in an isotopic algebra then generally implies the appearance of the structure functions and irregular isorepresentations.

By no mean do the above two classes exhaust all possible, physically significant isorepresentations. We therefore introduce a third class of isorepresentations without any claim of completeness (in fact, we do not study here for brevity the isorepresentations with different isotopic elements for the isoenvelopes and iso-Hilbert space which characterize yet more general isorepresentations).

We here define as *standard adjoint isorepresentations* those occurring when the spectra of eigenvalues are conventional, but the representations are nontrivially generalized, i.e., remain nonunitarily equivalent to the conventional representations. In fact, regular iso-Pauli matrices (3.2) admit the conventional eigenvalues 1/2 and 3/4 for $\Delta = 1$. This condition can be verified by putting $g_{11} = g_{22}^{-1} = \lambda$. We discover in this way the existence of the *standard iso-Pauli matrices* first presented in [13]

$$\widehat{\sigma}_1 = \begin{pmatrix} 0 & g_{22}^{-1} \\ g_{11}^{-1} & 0 \end{pmatrix}, \qquad \widehat{\sigma}_2 = \begin{pmatrix} 0 & -ig_{22}^{-1} \\ ig_{11}^{-1} & 0 \end{pmatrix},$$
$$\widehat{\sigma}_3 = \begin{pmatrix} g_{11}^{-1} & 0 \\ 0 & -g_{22}^{-1} \end{pmatrix}, \tag{3.9}$$

which admit all conventional structure constants and eigenvalues for $\widehat{J}_k = \frac{1}{2}\widehat{\sigma}_k$,

$$[\widehat{J}_i, \widehat{\ } \widehat{J}_j] = i\varepsilon_{ijk}\widehat{J}_k, \qquad \widehat{J}_3 * |\widehat{b}\rangle = \pm\tfrac{1}{2}|\widehat{b}\rangle, \qquad \widehat{J}^{\hat{2}} * |\widehat{b}\rangle = \tfrac{3}{4}|\widehat{b}\rangle \tag{3.10}$$

yet exhibit the 'hidden functions' g_{kk} in their structure.

Needless to say, isorepresentation (3.9) remains standard under the physically significant condition

$$\det Q = g_{11}g_{22} = 1,$$

which is realized for

$$g_{11} = g_{22} = \lambda \neq 0,$$

where is a sufficiently smooth, real-valued and nowhere-null function of the local variables. In this case, isorepresentation (3.9) assumes the form used in physical applications (see the next sections)

$$\widehat{\sigma}_1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \qquad \widehat{\sigma}_2 = \begin{pmatrix} 0 & -i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix},$$
$$\widehat{\sigma}_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}. \tag{3.11}$$

Similarly, irregular isorepresentations also become standard under condition (3.9) and realization (3.10). We therefore have the following additional standard iso-Pauli matrices

$$\widehat{\sigma}_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \qquad \widehat{\sigma}_2' = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2,$$
$$\widehat{\sigma}_3' = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}, \tag{3.12a}$$
$$\widehat{\sigma}_1'' = \begin{pmatrix} 0 & \lambda^{\frac{1}{2}} \\ \lambda^{-\frac{1}{2}} & 0 \end{pmatrix}, \qquad \widehat{\sigma}_2'' = \begin{pmatrix} 0 & -i\lambda^{\frac{1}{2}} \\ i\lambda^{-\frac{1}{2}} & 0 \end{pmatrix},$$
$$\widehat{\sigma}_3'' = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}. \tag{3.12b}$$

Iso-Pauli matrices with generalized eigenvalues are useful for interior structural problems, i.e., the description of a neutron in the core of a neutron star or, along the same lines, for a hadron constituent. As such, the applications of the general case of the $\widehat{SU}_Q(2)$ isosymmetry is studied elsewhere [16].

When studying conventional particles, e.g., those of nuclear physics, the physically relevant subclass of $\widehat{SU}_Q(2)$ is the special one with conventional eigenvalues, which is studied in the next sections. The image $\widehat{\sigma}_k^d$ under isoduality, called *isodual Pauli matrices*, will be studied elsewhere.

4. Application to Isospin

As is well known (see, e.g., [2, 6]), the conventional $SU(2)$-isospin symmetry is broken by electromagnetic and weak interactions. One of the first applications of our isotopic/Q-operator-deformation theory is to show that the $SU(2)$-isospin symmetry can be reconstructed as exact at the isotopic level, namely, there exists a realization of the underlying isospace $\widehat{E}_Q(z, \overline{z}, \widehat{\delta}, \widehat{C})$ in which protons and neutrons have the same mass, although the conventional values of mass are recovered under isoexpectation values.

The main idea is that the $SU(2)$-isospin symmetry is broken when realized via the simplest conceivable Lie product $AB - BA$. However, when the same symmetry is realized via a lesser trivial product, such as our Lie-isotopic product $AQB - BQA$ [7], it can be proved to be exact even under electromagnetic and weak interactions. In this case, the elements of the Q-matrix are constants and acquires the meaning of average of these interactions.

The reader should be aware that this is an isolated occurrence, because it represents a rather general capabilities of the Lie-isotopic theory. In fact, it is referred to as the *isotopic reconstruction of exact spacetime and internal symmetries when conventionally broken*. For example, the rotational symmetry has been reconstructed as exact for all infinitely possible ellipsoidical deformations of the sphere; the Lorentz symmetry has been reconstructed as exact at the isotopic level for all possible signature preserving deformations $\widehat{\eta} = Q\eta$ of the Minkowski metric, etc. [15].

The reconstruction of the exact $S\widehat{U}_Q(2)$-isospin symmetry is so simple to appear trivial. Consider a 2-component isostate

$$\widehat{\psi}(x) = \begin{pmatrix} \widehat{\psi}_p(x) \\ \widehat{\psi}_n(x) \end{pmatrix}, \tag{4.1}$$

where $\widehat{\psi}_p(x)$ and $\widehat{\psi}_n(x)$ are solutions of the isodirac equation [19] which transforms isocovariantly under a standard isorepresentation of $\widehat{P}_Q(3.1) \times S\widehat{U}_{Q'}(2)$. In this note we study only the $S\widehat{U}_Q(2)$ part without any iso-Minkowskian coordinates, thus restricting our attention to the isonormalized isostates

$$|\widehat{\psi}_p\rangle = \begin{pmatrix} \lambda^{-\frac{1}{2}} \\ 0 \end{pmatrix}, \qquad |\widehat{\psi}_n\rangle = \begin{pmatrix} 0 \\ \lambda^{\frac{1}{2}} \end{pmatrix},$$
$$\langle\widehat{\psi}_k|Q|\widehat{\psi}_k\rangle = 1, \; k = p, n, \tag{4.2}$$

where $Q = \mathrm{diag}(\lambda, \lambda^{-1})$, $\widehat{I} = Q^{-1} = \mathrm{diag}(\lambda^{-1}, \lambda)$.

We then introduce the $S\widehat{U}_Q(2)$-isospin with isorepresentation (3.11) admitting conventional eigenvalues $\pm 1/2$ and $3/4$, defined over the isospace $\widehat{E}_Q(z, \overline{z}, \widehat{\delta}, \widehat{C})$, $\widehat{\delta} = Q\delta$.

We now select such an isospace to admit the same masses for the proton and the neutron. This is readily permitted by the 'hidden variable' λ when selected in such a way that

$$m_p\lambda^{-1} = m_n\lambda, \quad \text{i.e.,} \quad \lambda^2 = m_p/m_n = 0.99862. \tag{4.3}$$

The mass operator can then be defined by

$$\widehat{M} = \left\{\frac{1}{2}\lambda(m_p + m_n)\widehat{I} + \frac{1}{2}\lambda^{-1}(m_p - m_n)\widehat{\sigma}_3\right\}\widehat{I}$$
$$= \begin{pmatrix} m_p\lambda^{-1} & 0 \\ 0 & m_n\lambda \end{pmatrix}, \tag{4.4}$$

and manifestly represents equal masses $\widehat{m} = m_p\lambda^{-1} = m_n\lambda$ in isospace.

The recovering of conventional masses in our physical space is readily achieved via the isoeigenvalue expression on an arbitrary isostate

$$\widehat{M} * |\widehat{\psi}\rangle = M\widehat{I}Q|\psi\rangle = M|\widehat{\psi}\rangle = \begin{pmatrix} m_p & 0 \\ 0 & m_n \end{pmatrix}|\widehat{\psi}\rangle \tag{4.5}$$

or, equivalently, via the isoexpectation values

$$\langle\widehat{\psi}_p|Q\widehat{M}Q|\widehat{\psi}_p\rangle = m_p, \qquad \langle\widehat{\psi}_n|Q\widehat{M}Q|\widehat{\psi}_n\rangle = m_n. \tag{4.6}$$

Similarly, the charge operator can be defined by

$$q = \frac{1}{2}e(\widehat{I} + \widehat{\sigma}_3) = \begin{pmatrix} e\lambda^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.7}$$

Thus, the $S\widehat{U}_Q(2)$ charges on isospace are $q_p = e\lambda^{-1}$ and $q_n = 0$. However, the charges in our physical space are the conventional ones

$$\langle\widehat{\psi}_p|QqQ|\psi_p\rangle = e, \qquad \langle\widehat{\psi}_n|QqQ|\widehat{\psi}_n\rangle = 0. \tag{4.8}$$

The *isodual* $S\widehat{U}^Q_Q(2)$*-isospin* characterizing the antiparticle $\overline{p}$ and $\overline{n}$ will be studied elsewhere. The entire theory of isospin and its application can then be lifted in an isotopic form which remains exact under all interactions. This is not a mere mathematical curiosity, because it carries a corresponding isotopy of the nuclear force, e.g., via $S\widehat{U}_Q(2)$-isotopic exchange mechanism, essentially representing the old legacy of a (generally small) nonlocal component in the nuclear structure. These dynamical implications are studied elsewhere.

5. Applications to Local Realism

The $S\widehat{U}_Q(2)$ theory studied above is based on a structural generalization of QM of nonlinear-nonlocal-non-Hamiltonian, although axiom-preserving type. However, in the so-called literature of *local realism* (see, e.g., [7]) there exist certain arguments, most notably Bell's inequality and von Neumann's theorem, *prohibiting* a generalization of quantum mechanics.

This note would therefore not be completed without an inspection of these issue and the proof that both Bell's inequality and von Neumann's theorem are *inapplicable* (and not 'violated') under isotopies. This then sets the foundations for the isotopic completion of QM studied in this note. The study also serves as an application of the $S\widehat{U}_Q(2)$ symmetry to spin.

The lack of applicability of Bell's inequality and von Neumann's theorem under regular and irregular isotopies is transparent from the alteration of the spectra of eigenvalues and, as such, deserves no additional comment.

In the following we show that the above inapplicability persists not only for standard isorepresentations (3.9) but also for the particular case of $\det Q = 1$, isorepresentations (3.11).

Consider two *standard isoparticles* with spin $\frac{1}{2}$, i.e., particles characterized by standard iso-Pauli matrices (3.9). Even though their spin is the same, there is no necessary reason to restrict their isotopic degrees of freedom λ to be the same outside isospin treatments (e.g., because their density may be different). We can therefore assume

$$\text{Particle 1: } Q = \operatorname{diag}(\lambda, \lambda^{-1}), \quad \Delta = \det Q = 1, \ \text{spin } \tfrac{1}{2}, \tag{5.1a}$$

$$\text{Particle 2: } Q' = \operatorname{diag}(\lambda', \lambda'^{-1}), \quad \Delta' = \det Q' = 1, \ \text{spin}' \ \tfrac{1}{2} \tag{5.1b}$$

Next, consider the composite system of the two isoparticles 1 and 2 which is characterized by the isounit

$$\widehat{I}_{\text{tot}} = \widehat{I}_1 \times \widehat{I}_2 = Q_{\text{Tot}}^{-1} = (Q \times Q')^{-1}. \tag{5.2}$$

To properly recompute the isotopies of Bell's inequality (see, e.g., [13] for the conventional case), it is necessary to identify the isonormalized basis $|\widehat{S}_{1-2}\rangle$, that is, the basis of the total spin of the particles 1 and 2 normalized to $\widehat{I}_{\text{tot}}$,

$$\langle \widehat{S}_{1-2} \widehat{|} \widehat{S}_{1-2}\rangle = \langle \widehat{S}_{1-2}|Q_{\text{tot}}|\widehat{S}_{1-2}\rangle \widehat{I}_{\text{tot}} = \widehat{I}_{\text{tot}}. \tag{5.3}$$

A simple isotopy of the conventional case (see, e.g., [3], Sect. 17.9) then leads to the *isobasis for the singlet state*

$$|\widehat{S}_{1-2}\rangle = \frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} \lambda^{-\frac{1}{2}} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \lambda'^{\frac{1}{2}} \end{pmatrix} - \begin{pmatrix} 0 \\ \lambda'^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \lambda'^{-\frac{1}{2}} \\ 0 \end{pmatrix} \right\}. \tag{5.4}$$

It is a tedious but instructive exercise for the interested reader to verify isonormalization condition (5.3) by constructing the adjoint of basis (5.4), by sandwiching the quantity $T_{\text{tot}} = Q \times Q'$, by contracting only quantities of the same particle, and then multiplying the scalar results of the two different particles.

Next, recall that the conventional scalar product $\boldsymbol{\sigma} \cdot \mathbf{a}$, where a is a three-vector, has no mathematical or physical meaning in isospace $\widehat{E}_Q(z, \bar{z}, \widehat{\delta}, \mathbb{R})$ and must be replaced by the isoscalar product for isorepresentation (3.11)

$$\widehat{\sigma} Q \mathbf{a} = \begin{pmatrix} a_z & (a_x - i a_y) \\ (a_x + i a_y) & -a_z \end{pmatrix}. \tag{5.5}$$

The tedious but straightforward repetition of the conventional procedure [7] under isotopy then leads to the expression

$$\begin{aligned} &\langle \widehat{S}_{1-2}|(Q \times Q')\{(\widehat{\sigma} * \mathbf{a}) \times (\widehat{\sigma}' * \mathbf{b})\}(Q \times Q)|\widehat{S}_{1-2}\rangle \\ &\quad = -a_x b_x - a_y b_y - \tfrac{1}{2}(\lambda\lambda'^{-1} + \lambda^{-1}\lambda') a_z b_z. \end{aligned} \tag{5.6}$$

Consider now unit vectors a, b, a', b' along the z-axis. Then the Bell's inequality under the conventional $SU(2)$ symmetry [7]

$$D_{\text{Bell}} = \operatorname{Max}|P(a,b) - P(a,b')| + |P(a',b) + P(a',b')| \leqslant 2, \tag{5.7a}$$

$$P(a,b) = \langle S_{1-2}|(\sigma_1 \cdot a) \times (\sigma_2 \cdot b)|S_{1-2}\rangle = -a \cdot b \tag{5.7b}$$

admits the following isotopic image under the covering $S\widehat{U}_Q(2)$ symmetry

$$D_{\text{Bell}} \leqslant D_{\text{Max}}^{\text{HM}} = \tfrac{1}{2}(\lambda\lambda'^{-1} + \lambda^{-1}\lambda') D_{\text{Bell}}. \tag{5.8}$$

But, the factor $\frac{1}{2}(\lambda\lambda'^{-1} + \lambda^{-1}\lambda')$ can be easily proved to admit values bigger than one. This establishes the statement of Section 1, to the effect that Bell's inequality is not universally valid, but holds, specifically, for the conventional, linear, local and canonical realization of the $SU(2)$ symmetry. The proof for arbitrary orientations of the unit vectors follows the conventional one [3] and it is here omitted for brevity.

Similarly, von Neumann theorem [7] is inapplicable under isotopies because based on the uniqueness of the spectrum of eigenvalues of Hermitian operators. In fact, isotopic theories establish that the *same* Hermitian operator H admits *an infinite variety of different spectra of eigenvalues*, trivially, because of the infinitely possible isotopic elements Q, $H * |\widehat{\psi}\rangle = HQ|\widehat{\psi}\rangle = E_Q|\widehat{\psi}\rangle$ [13].

Similar obstacles to the completion of QM into a covering theory are removed under isotopies as shown elsewhere [16]. We here merely mention the reason why HM is indeed a completion of QM much along the EPR argument [3]. Recall that

$$D_{\text{Max}}^{\text{Class.}} = \operatorname{Max}|a \cdot b - a \cdot b'| + |a' \cdot b + a' \cdot b'| = 2\sqrt{2} > 2. \tag{5.9}$$

and that $D_{\text{Bell}} < D_{\text{Max}}^{\text{Class.}}$, thus preventing the completion of quantum mechanics.

However, under isotopic liftings, one can assume a classical iso-Euclidean space $\widehat{E}(r, \widehat{\delta}, \widehat{\mathbb{R}})$ (representing motion of extended objects within physical media [11]) with isotopic scalar product

$$a * b = a^t Q b = a_x g_{11} b_x + a_y g_{22} b_y + a_z g_{33} b_z. \tag{5.10}$$

Then, there *always exists a realization of* $\widehat{E}(r, \widehat{\delta}, \mathbb{R})$ *under which we have the identity of the maximal operator and classical values*, $\widehat{D}_{\text{Max}}^{\text{HM}} \equiv \widehat{D}_{\text{Max}}^{\text{Classical}}$, as it is the case for the orientation of the unnit vectors as above, and values

$$g_{11} = g_{22} = 1, \qquad g_{33} = \frac{1}{2}(\lambda\lambda'^{-1} + \lambda^{-1}\lambda') = \sqrt{2}. \tag{5.11}$$

A number of additional, intriguing completions of QM are provided by HM along the EPR argument, such as the recovering of classical determinism for a particle in the interior of a gravitational singularity and others [16].

In closing, it is hoped that systematic studies on the isorepresentations of Lie-isotopic algebras, such as the isotopic $\hat{O}(3)$, $\hat{O}(3.1)$, $S\hat{L}(2.\hat{\mathbb{C}})$, $\hat{P}(3.1)$, $S\hat{U}(3)$, etc., are conducted by interested colleagues because of their capabilities of novel applications, that is, results beyond the capacity of the conventional Lie theory.

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