Nonlocal Isotopic Representation of the Cooper Pair in Superconductivity

A. O. E. ANIMALU AND R. M. SANTILLI

Istituto per la Ricerca di Base, Castello Principe Pignatelli, I-86075 Monteroduni (IS), Molise, Italy; E-mail: ibrrms@pinet.aip.org

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ABSTRACT

We study a model of the Cooper pair in superconductivity whose basic mechanism is due to nonlinear–nonlocal and non-Hamiltonian interactions; we show the capability of the model of representing available data while possessing intriguing predictive capacities, and we propose its experimental verification. © 1995 John Wiley & Sons, Inc.

1. Introduction

There are few doubts that the currently available theory of electron pairing in superconductivity (Cooper pair) provides a satisfactory representation of its structure, by achieving a representation of the *attractive* interaction among the two *identical* electrons of the pair via the mediation of cuprite ions in agreement with available data (see, e.g., [1, 2]).

Despite this success, studies of alternative representations of the Cooper pair are not expected to halt, and, in fact, they have continued for various reasons. The first is that the conventional model has exhausted all predictive capacities. This sets the challenge of identifying a new model which also represents available data while permitting novel predictions for possible basic advances in the field.

Also, the mechanism providing attraction in the conventional model, that based on the electron–phonon interaction [1, 2], has not been fully established experimentally in other fields, e.g., in particle physics, and additional tests are warranted prior to a final assessment. Moreover, despite its effectiveness, the contemporary notion of valence, or molecular bonding at large, is also far from having reached a final stage. In fact, it may well be that the same mechanism at the basis of the Cooper pair in superconductivity is also responsible for molecular bonding, evidently after due adjustments due to different physical conditions.

The mechanism responsible for Pauli's exclusion principle is also far from a complete understanding. In fact, quantum mechanics is simply unable to represent *interactions* among two electrons in the same orbit capable of excluding the same quantum numbers. Again, it may well be that the mechanism at the foundation of the Cooper

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pair is also responsible for Pauli's exclusion principle. After all, the electron couple themselves in the Cooper pairs in singlet states, precisely in agreement with Pauli's principle.

2. The New Model of the Cooper Pair

This article surveys and develops a new model of electron pairing as a conceivable *alternative* to the conventional model [1, 2]. The mechanism underlying the new model was submitted by Santilli [3] in attempting a new structure model of the π° as a novel bound state of an electron e^- and a positron e^+ in the singlet state, $(e^+_\uparrow, e^-_\downarrow)$. Animalu [4] showed that the same mechanism implies an attraction also for equal charges and applied the model for a representation of the Cooper pair $(e^-_\uparrow, e^-_\downarrow)$ (see also the most recent studies [5]).

The expectations are that the new model recovers all available experimental data, while exhibiting a more promising predictive capacity for lower T_c values, as well as possibilities of being applied to similar systems, such as molecular bondings or Pauli's exclusion principle. Needless to say, in this article, we can study only some of these open issues. The remaining open aspects have to be studied in subsequent articles.

The conceptual foundations of the Animalu–Santilli model of the Cooper pair are the following: Despite historical achievements, quantum mechanics is not expected to be the final theory of the microcosm because it is linear, local–differential, and potential–Hamiltonian. A central issue of contemporary physics is the identification of a generalization of quantum mechanics which is effective for the study of interactions which are arbitrarily nonlinear (in the wavefunctions ψ *and* their derivatives $\partial\psi$), nonlocal-integral (admitting of essential integral terms in the equations of motion), and nonpotential–non-Hamiltonian (admitting of effects which violate the necessary and sufficient conditions for the existence of a potential, the *conditions of variational self-adjointness* [6]). Needless to say, quantum mechanics can provide a good *approximation* under nonlinear, nonlocal, and non-Hamiltonian interactions by merely assuming that they are null. The central issue is, however, their quantitative study.

The above studies are directly relevant for superconductivity because a more adequate theory of the Cooper pair is expected to be precisely nonlinear, nonlocal, and non-Hamiltonian. In fact, the fundamental assumption of the Animalu–Santilli model is the presence of internal nonlinear–nonlocal–non-Hamiltonian effects due to overlappings of the wavepackets of the electrons among themselves and with those of the Cu^{2+} ions. As such, the new model is conceptually, mathematically, and physically beyond realistic capabilities of exact treatment via conventional quantum mechanics. As is well known, the latter mechanics can represent only interactions mediated by particle exchanges (i.e., of potential type) among a finite number of isolated points (linear and local character). By comparison, interactions caused by wave-overlappings are of "contact" type (i.e., having a "zero range") for which the notion of potential and related particles exchange have no mathematical or physical meaning. Moreover, wave-overlappings require contributions necessarily of the nonlocal-integral type because they cannot be *exactly* reduced to a finite set of isolated points. Finally, contact interactions, whether in classical or in operator mechanics, are notoriously nonlinear in the velocities, wave functions, and their derivatives.

Again, we could ignore nonlinear–nonlocal–non-Hamiltonian effects expected in the interior of the Cooper pair, thus remaining within the context of the current quantum mechanical model [1, 2]. The scope of this article is, however, the initiation of their quantitative study in a way suitable for experimental resolutions.

3. Hadronic Mechanics

A generalization of quantum mechanics under the name of *hadronic mechanics* was proposed by Santilli [3] when he was at Harvard University under DOE support for the representation of linear and nonlinear, local and nonlocal, and potential as well as nonpotential interactions. The new mechanics was subsequently studied by a number of mathematicians, theoreticians, and experimentalists and recently reached sufficient maturity for applications and verifications (see monographs [7] for a comprehensive presentation). A knowledge of at least the rudiments of the new mechanics is essential for an understanding of the model of the Cooper pair studied in this note.

Hadronic mechanics is constructed via the *isotopies* of the various aspects of quantum mechanics, whose most salient characteristic is that of being *axiom-preserving*. In particular, the isotopies map any linear–local–canonical structure into the most general possible nonlinear–nonlocal–noncanonical forms in such a way to recover linearity–locality–canonicity in certain generalized spaces and fields. Unlike other generalizations (see below), the isotopies therefore assure a correct axiomatic structure under generalized interactions because they preserve established quantum axioms and only realize them in a more general form.

The main postulate of the new mechanics is the representation of conventional linear–local–potential interactions via the familiar Hamiltonian H and the representation of the nonlinear–nonlocal–nonpotential interactions via a generalization of the *unit* of quantum mechanics:

$$\hbar = 1 \to \hat{\hbar} = \hbar \hat{I}(x, p, \psi, \partial\psi, \dots), \qquad (3.1)$$

where $\hat{I}$ is a matrix having the same dimension of the underlying (Euclidean, Minkowskian, or Riemannian) space, being nonsingular, Hermitian, and positive-definite, and admitting of the value I as a particular case, in which case quantum mechanics is a particular case of hadronic mechanics. The description of systems via hadronic mechanics therefore requires the knowledge of *two* quantities, H and $\hat{I}$. All quantum mechanical methods are then subjected to a map, called *isotopic lifting*, such to admit $\hat{I}$ as the left and right units.

The fundamental quantities of hadronic mechanics are characterized by the lifting of the field R of real numbers $n, m, \dots$, with conventional multiplication nm and multiplicative unit I, $In = nI \equiv N$, $\forall n \in R$, into the so-called *isofields* $\hat{R}$ [8] whose elements are the *isonumbers* $\hat{n} = n\hat{I}$, $\hat{m} = m\hat{I}$, with isomultiplication $\hat{n} \times \hat{m} = \hat{n}T\hat{m}$, where T is a fixed and invertible quantity (outside the original set R). Under the assumption $\hat{I} = T^{-1}$, $\hat{I}$ is the correct left and right units of $\hat{R}$, $\hat{I} \times \hat{n} = \hat{n} \times \hat{I} = \hat{I}$, $\forall \hat{n} \in \hat{R}$, in which case (only) $\hat{I}$ is called the *isounit* and T is called the *isotopic element*. Note that $\hat{R} \simeq R$ by construction. This illustrates the isotopic character of the lifting $R \to \hat{R}$. The lifting of the complex field $C \to \hat{C}$ is then consequential [7, 8].

The next basic method of hadronic mechanics is the lifting of the conventional differential calculus into a form called *isodifferential calculus* [7] which is based on the isotopes $dx^k \to \hat{dx}^k = T_i^k\, dx^i$, $\partial_k = \partial/\partial x^k \to \hat{\partial}_k = \hat{I}_k^i\, \partial_i = \hat{I}_k^i\, \partial/x^i$ and on the corresponding isotopic lifting of the operations on them, e.g.,

$$\Delta = \sum_k \partial_k^2 = \sum_k \partial_k\, \partial_k \to \hat{\Delta} = \sum_k \hat{\partial}_k^2 = \hat{\partial}_i T^{ij} \hat{\partial}_j = \sum_k \hat{I}_k^i\, \partial_i\, \partial_i,$$

the latter being necessary to map a ring of function over $\hat{R}$ with isounit $\hat{I}$ into another ring *over the same isofiled with the same isounit* $\hat{I}$ (e.g., the definition

$$\hat{dx}^k = d(T_i^k x^i) = x^i\, dT_i^k + T_i^k\, dx^i$$

is incorrect because it would alter the basic unit of the theory, thus preventing consistent measurements). As the reader can predict, the isodifferential calculus implies a step-by-step generalization of classical Hamiltonian mechanics, as well as of quantization procedures which we cannot possibly review here [7]. It is nevertheless important to know that the nonlinear–nonlocal–non-Hamiltonian interactions studied in this article for the Cooper pair have their primitive origin at the Newtonian level.

The next important structure of the new mechanics is the lifting of the enveloping associative algebra ξ of quantum mechanical operators $A, B, \dots$ with the simplest possible associative product AB over R and unit I, $IA = AI = A$, $\forall A \in \xi$, into the *enveloping isoassociative algebra* $\hat{\xi}$ with the same original operators $A, B, \dots$ (because, mathematically, the basis of a vector space is unchanged under isotopes [7, Vol. I] and, physically, the operators $A, B, \dots$ represent physical quantities independent from their interactions [7, Vol. II]), but now equipped with the *isoassociative product* $A \times B = ATB$ over the isofield $\hat{R}$ for which $\hat{I} = T^{-1}$ is the correct left and right isounit,

$$\hat{\xi}: A \times B = ATB, \qquad \hat{I} \times A = A \times \hat{I} \equiv A, \qquad \forall A \in \hat{\xi}, \qquad \hat{I} = T^{-1}. \qquad (3.2)$$

The lifting $\xi \to \hat{\xi}$ implies a step-by-step generalization of all aspects of Lie's theory into the Lie–Santilli theory [9–12] with isoalgebra $\hat{\xi}^-$ characterized by the generalized product

$$[A, B]_{\hat{\xi}} = A \times B - B \times A = ATB - BTA, \qquad (3.3)$$

which still verifies the Lie axioms (as a necessary condition for an isotopy); *isoexponentiation*

$$\hat{e}^{iA\times\hat{w}} = \hat{e}^{iAw} = \hat{I} + (iAw)/1! + (iAw) \times (iAw)/2! + \cdots = (e^{iATw})\hat{I}, \qquad w \in R, \quad \hat{w} \in \hat{R}, \quad (3.4)$$

where e denotes the conventional exponentiation, and $\hat{e}$ its isotopic image; the *isogroup* and related *isotopic group laws*

$$x' = \hat{g}(\hat{w}) \times x = \hat{g}Tx = \hat{e}^{iA\times\hat{w}} \times x = e^{iATw}x,$$
$$\hat{g}(\hat{w}) \times \hat{g}(\hat{w}') = \hat{g}(\hat{w} + \hat{w}'),$$
$$\hat{g}(\hat{0}) = \hat{g}(\hat{w}) \times \hat{g}(-\hat{w}) = \hat{I}; \qquad (3.5)$$

the *isorepresentation theory;* and others [7, 9]. The nontriviality of the new theory is illustrated by *the appearance of a nonlinear–nonlocal–noncanonical operator T in the exponent of the group structure,* thus implying that conventional linear–local–canonical space–time symmetries (rotation, Lorentz, Poincaré symmetries) are lifted into a nonlinear–nonlocal–noncanonical form as *necessary* to characterize nonlinear–nonlocal–noncanonical systems.

Conventional Hilbert spaces $\mathscr{H}$ with states $|\psi\rangle, |\phi\rangle, \ldots$, and inner product $\langle\psi \mid \phi\rangle$ over C are lifted into the *isohilbert space* $\hat{\mathscr{H}}$ [7] with *isostates* $|\hat{\psi}\rangle, |\hat{\phi}\rangle, \ldots$, and *isoinner product*

$$\hat{\mathscr{H}}: \langle\hat{\psi}\,\hat{|}\,\hat{\phi}\rangle = \langle\hat{\psi}|T|\hat{\phi}\rangle\hat{I} \in \hat{C}. \qquad (3.6)$$

All operations on $\mathscr{H}$ are then submitted to a simple, unique, and unambiguous lifting into isooperations on $\hat{\mathscr{H}}$. For instance, unitary operators $UU^\dagger = U^\dagger U = I$ are lifted into *isounitary operators* $\hat{U} \times \hat{U}^\dagger = \hat{U}^\dagger \times \hat{U} = \hat{I}$ with realization $\hat{U} = \hat{e}^{iA\times\hat{w}}$, where A is isohermitian, $A = A^\dagger \equiv A^\dagger$.

The fundamental dynamical equations of hadronic mechanics are given by the *isoschrödinger representation*

$$i\,\hat{\partial}_t\hat{\psi} = i\hat{I}_t\,\partial_t\hat{\psi}(t,r) = H \times \hat{\psi} = H(t,r,p)T\left(\hat{\psi}, \partial\hat{\psi}, \partial\,\partial\hat{\psi}, \ldots\right)\hat{\psi}(t,r) = \hat{E} \times \hat{\psi} = E\hat{\psi}(t,r), \qquad (3.7a)$$

$$p_k \times \hat{\psi} = p_k T\left(\hat{\psi}, \partial\hat{\psi}, \partial\,\partial\hat{\psi}, \ldots\right)\hat{\psi}(t,r) = -i\,\hat{\partial}_k\hat{\psi} = -i\hat{I}^i_k\left(\hat{\psi}, \partial\hat{\psi}, \partial\,\partial\hat{\psi}, \ldots\right)\partial_i\hat{\psi}(t,r), \qquad (3.7b)$$

$$r^i \times p_j - p_j \times r^i = i\hat{I}^i_j,$$
$$r^i \times r^j - r^j \times r^i = p_i \times p_j - p_j \times p_i = 0, \qquad (3.7c)$$

where $\hat{I}_t$ is the *time isounit,* with a corresponding *isoheisenberg form* and relativistic extensions here omitted for brevity [7, Vol. II].

All remaining mathematical methods of quantum mechanics must be subjected to an isotopic lifting with respect to the same isounit $\hat{I}$ for their consistent application to specific problems. This implies the isotopies of trigonometry, special functions, distributions, transforms, etc., which we cannot possibly review here [7]. We limit ourselves here to a few additional comments essential for an understanding of our model of the Cooper pair:

A most important property of the lifting $\mathscr{H} \to \hat{\mathscr{H}}$ is that *operators $A \in \xi$ over C which are Hermitian on $\mathscr{H}$ remain Hermitian when reinterpreted as elements of the isoenvelope $\hat{\xi}$ over the isofield $\hat{C}$.* Thus, *physical quantities such as linear momentum, angular momentum, energy, etc., which are observable for quantum mechanics (read, under linear–local–potential interactions only) remain observable for hadronic mechanics (read, under additional nonlinear–nonlocal–nonpotential interactions).*

Another important property of isohilbert spaces is given by the *isoexpectation values*

$$\langle\hat{A}\rangle = \langle\psi|TAT|\psi\rangle/\langle\hat{\psi}|T|\hat{\psi}\rangle, \qquad (3.8)$$

which are on R (rather than $\hat{R}$) where the isotopic action is *necessary* for consistency [7, Vol. II]. As a result, the isoexpectation value of the isounit of hadronic mechanics $\hat{I} = T^{-1}$ recovers the conventional value:

$$\langle\hat{I}\rangle = \langle\hat{\psi}|TT^{-1}T|\hat{\psi}\rangle/\langle\hat{\psi}|T|\hat{\psi}\rangle \equiv I. \qquad (3.9)$$

This implies that *nonlinear–nonlocal–non-Hamiltonian effects are internal and cannot be detected in the center-of-mass trajectory in which only the total energy is measurable.* In fact, we know that the total sum of internal nonpotential forces must be identically null as a condition for the conservation of the total energy both at the classical [6, Vol. II] and operator [7, Vol. II] levels. At any rate, only the total energy is measurable in the center-of-mass system, with the evident impossibility of any measure on internal nonpotential effects. Thus, hadronic mechanics can be applied solely to the interior structure of the Cooper pair, while quantum mechanics is and remains exactly valid for the center-of-mass trajectories.

Note in Eq. (3.7a) that the *isoeigenvalues* are generalized, $\hat{E} = E\hat{I} \in \hat{R}$, but the measurable numbers of the theory are the conventional ones because of the cancellation $\hat{E} \times \hat{\psi} = E\hat{I}\hat{T}\hat{\psi} = E\hat{\psi}$. Also, all isoeigenvalues of isohermitian operators are real [7] as in the conventional case. However, *the same Hermitian operators have different eigenvalues in quantum and hadronic mechanics*, i.e., if $H\psi = E_0\psi$ in quantum mechanics, $H \times \hat{\psi} = H\hat{T}\hat{\psi} = E\hat{\psi}$ in hadronic mechanics with $E \neq E_0$. The alteration of the numerical value of the eigenvalue for the same Hermitian Hamiltonian, $E_0 \to E$, is called *isorenormalization* and plays a crucial role in our model of the Cooper pair.

Note also that the isoeigenvalue equation $H \times \hat{\psi} = H\hat{T}\hat{\psi} = E_T\hat{\psi}$ constitutes an *explicit operator realization of the theory of "hidden variables."* We can therefore say that *our model of the Cooper pair is based on an explicit operator realization of the theory of hidden variable and a corresponding "completion" of quantum mechanics much along the known Einstein–Podolsky–Rosen argument.*

At the abstract level, all hadronic and quantum structures coincide (for a positive-definite isounit), $I \equiv \hat{I}$, $\hat{R} \equiv R$, $\hat{\xi} \equiv \xi$, $\hat{\mathscr{H}} \equiv \mathscr{H}$, etc. This assures the axiomatic consistency of hadronic mechanics (see [7, Vol. II] for the explicit forms of the nonrelativistic and relativistic axioms). Hadronic mechanics is, therefore, a mere nonlinear–nonlocal–non-Hamiltonian *realization* of the abstract quantum axioms. In fact, Planck's quantum is recovered in full under isoexpectation values [Eq. (3.9)].

Despite these abstract mathematical similarities, *quantum and hadronic mechanics are physically inequivalent*. This is established by the fact that *all aspects of hadronic mechanics can be constructed via a nonunitary transformation of the corresponding quantum aspects as suggested since the original proposal* [3]. Consider a nonunitary transform and put $UU\dagger = U^\dagger U = \hat{I} \neq I$. It is then easy to see that the quantum unit is lifted into the hadronic isounit, $\hat{I} = UIU^\dagger$; the associative product in ξ is lifted into the isoassociative form in $\hat{\xi}$,

$$AB \to UABU^\dagger = UAU^\dagger(UU^\dagger)^{-1}UBU^\dagger = \tilde{A}T\tilde{B},$$

where T has the correct form needed for $\hat{I}$ to be the isounit, $T = (UU^\dagger)^{-1} = \hat{I}^{-1}$, and it is correctly Hermitian, $T^\dagger = [(UU^\dagger)^{-1}]^\dagger = T$; the Lie product is lifted into the Lie–Santilli product, $AB - BA \to \tilde{A}T\tilde{B} - \tilde{B}T\tilde{A}$; conventional eigenvalue equations are lifted into their isotopic form

$$H\psi \to E_0\psi \to UH\psi = UHU^\dagger(UU^\dagger)^{-1}U\hat{\psi} = \bar{H}T\hat{\psi}$$
$$= UE\psi = UEU^\dagger(UU^\dagger)U\psi = E\hat{I}\hat{T}\hat{\psi} = E\hat{\psi};$$

the canonical commutation rules are lifted into their fundamental isotopic form,

$$rp - pr = iI \to U(rp - pr)U^\dagger = \bar{r}T\bar{p} - \bar{p}T\hat{r} = i\hat{I};$$

linear–local–canonical models of quantum mechanics are lifted into nonlinear–nonlocal–noncanonical forms which do verify linearity, locality, and canonicity in isospaces over isofields; etc. (see [7, Vol. II] for all details). This illustrates the necessity for hadronic mechanics of lifting the *totality* of the mathematical methods of quantum mechanics.

In fact, the approaching of hadronic models via old quantum mechanical thinking is particularly insidious because it leads to a host of inconsistencies which generally remain undetected by researchers not familiar with the new mechanics (e.g., the conventional magnitude of the total angular momentum $J^2 = \Sigma_k J_k J_k$ *violates linearity and it is not conserved for hadronic mechanics*, $[\hat{J^2}, J_k] \neq 0$, and must be replaced with the isotopic form $\hat{J^2} = \Sigma_k J_k \times J_k$, which does indeed verify isolinearity and is indeed conserved, $[\hat{J^2}, J_k] \equiv 0$, $k = 1, 2, 3$).

Even though hadronic mechanics is not expected (or intended) to be unique, other currently available generalizations of quantum mechanics are afflicted by rather serious problems of *physical* consistency [13, 14]. For instance, the known q- and k-deformations alter quantum products, thus implying a nonunitary time evolution, while preserving the original unit, the original Hilbert spaces, and the original fields. As shown by Lopez [13], these theories are afflicted by the following physical drawbacks: They do not allow measurements because their basic unit is not preserved under the time evolution of the theory, $I' = UIU^\dagger = \hat{I} \neq I$; they do not have observables at all times because operators which are originally Hermitian on $\mathscr{H}$, $(\langle\phi|H^\dagger|\psi\rangle \equiv \langle\phi|(H|\psi\rangle)$, do not remain Hermitian on $\mathscr{H}'$ under a nonunitary time evolution, $(\langle\hat{\phi}|T\bar{I}^\dagger|\hat{\psi}\rangle \neq \langle\hat{\phi}|(i\bar{H}T|\hat{\psi}\rangle)$, $\bar{H}^\dagger = T^{-1}\bar{H}T \neq \bar{H}$ (this property is now known as *Lopez's lemma* [13]); the q- and k-special functions are not applicable at all times (because the q- and *k-numbers* are transformed in *operators* under nonunitary time evolutions), thus preventing meaningful data elaborations, such as partial wave

analysis of the scattering amplitude and other drawbacks.

The above problematic aspects are all resolved by hadronic mechanics. As a necessary condition for a nontrivial departure from quantum mechanics, the time evolution of hadronic mechanics is indeed nonunitary, but of the isounitary type $\hat{U} = \hat{e}^{i\hat{H}t} = (e^{iHT})\hat{I}$, $\hat{U} \times \hat{U}^\dagger = \hat{U}^\dagger \times \hat{U} = \hat{I}$ under which the basic isounit $\hat{I}$ is invariant at all times $\hat{I}' = \hat{U} \times \hat{I} \times \hat{U}^\dagger \equiv \hat{I}$ (also, $\hat{I}$ isocommutes with all possible operators

$$[\hat{I}, A] = \hat{I} \times A - A \times \hat{I} = A - A \equiv 0,$$

thus being conserved despite its nonlinear–nonlocal–noncanonical functional dependence); the isoassociative and isolie products are invariant,

$$\hat{U} \times (A \times B - B \times A) \times \hat{U}^\dagger = A' \times B' - B' \times A';$$

the isoeigenvalues are preserved

$$\hat{U} \times H \times \hat{\psi} = \hat{U} \times H \times \hat{U}^\dagger \times \hat{U} \times \hat{\psi} = H' \times \hat{\psi}$$
$$= \hat{U} \times E \times \psi = E\hat{U} \times \hat{I} \times \hat{U}^\dagger$$
$$\times \hat{U} \times \hat{\psi} \equiv E\hat{\psi};$$

the T-isotopic special functions, distributions, and transforms are applicable at all times *because the numerical value of T, let alone its operator structure, is preserved at all times;* etc. (see [7, Vol. II] for comprehensive studies of these basic invariance properties). It should also be noted that a conventionally nonunitary operator U can always be identically decomposed in the form $U = \hat{U}T^{1/2}$, with isounitary reformulation $UU^\dagger = \hat{U} \times \hat{U}^\dagger$.

Conventional nonlinear extensions of quantum mechanics (i.e., those based on the eigenvalue equation $H(t, r, p, \psi)\psi = E\psi$) have additional problematic aspects also of *physical* character. In fact, they generally represent open-nonconservative systems with consequential nonunitary time evolutions. But they are defined on a conventional Hilbert space. Thus, they are afflicted by the same drawbacks of q- and k-deformations. Even when $H(t, r, p, \psi)$ is Hermitian, there is the loss of the superposition principle which prevents quantitative studies of composite systems (such as the Cooper pair) with the consistency on axiomatic grounds necessary to warrant experimental verifications (see [7, Vol. II] for details). Hadronic mechanics resolved the latter problematic aspects, too, by representing all nonlinear terms in the isounit with achievement of isolinearity. In particular, hadronic mechanics permits the *identical* isotopic reformulation of conventional nonlinear models

$$H(t, r, p, \psi)\psi \equiv H_0(t, r, p)T(\psi, \ldots)\psi$$
$$= H_0^*\psi = E\psi$$

with a consequential, simple, unique, and effective form of the superposition principle (see [7, Vol. II] for details). This implies an axiomatically consistent form for quantitative studies of composite systems.

Numerous, additional problematic aspects of other formulations have been studied in [14] and are not reviewed here for brevity. These problematic aspects illustrate the reason for the selection of hadronic mechanics for this study.

4. Nonrelativistic Formulation

Once the rudiments of hadronic mechanics are known, the understanding of the Animalu–Santilli model of the Cooper pair $(e_\uparrow^-, e_\downarrow^-)_{HM}$ is straightforward [3, 4]. Consider an electron with charge $-e$, spin up, and wave function $\psi_\uparrow$ in the field of another electron with the same charge, spin down, and wave function $\psi_\downarrow$ considered as *external*. Its Schrödinger equation is given by the familiar expression

$$H_{\text{Coul.}}\psi(t, r) = \left\{\frac{1}{2m} p_k p^k + \frac{e^2}{r}\right\}\psi_\uparrow(t, r)$$
$$= E_0\psi_\uparrow(t, r),$$
$$p_k\psi_\uparrow(t, r) = -i\,\partial_k\psi_\uparrow(t, r), \qquad (4.1)$$

where m is the electron rest mass. The above equation and related wave function $\psi_\uparrow(t, r)$ represent *repulsion*, as is well known. We are interested in the physical reality in which there is *attraction* represented by a new wave function, here denoted $\hat{\psi}_\uparrow(t, r)$.

By recalling that quantum mechanical Coulomb interactions are invariant under unitary transforms, the map $\psi_\uparrow \to \hat{\psi}_\uparrow$ is representable by a transform $\hat{\psi} = U\psi$ which is *necessarily nonunitary*, $UU^\dagger = U^\dagger U = \hat{I} \neq I$, where $\hat{I}$ has to be determined (see below). This activates ab initio the applicability of hadronic mechanics. The first step of the proposed model is therefore that of trans-

forming system (4.1) in $\psi_\uparrow$ into a new system in $\hat{\psi}_\uparrow = U\psi_\uparrow$ where U is nonunitary:

$$\begin{aligned} UH_{\text{Coul.}}U^\dagger(UU^\dagger)^{-1}U\psi_\uparrow(t,r) &= \tilde{H}_{\text{Coul.}}T\hat{\psi}_\uparrow(t,r) \\ &= \left\{\frac{1}{2m}\,\tilde{p}_k T\tilde{p}^k + \frac{e^2}{r}\,\hat{I}\right\}T\hat{\psi}_\uparrow(t,r) \\ &= E\hat{\psi}_\uparrow(t,r), \end{aligned}$$

$$\tilde{p}_k T\hat{\psi}_\uparrow(t,r) = -i\hat{I}_k^i\,\partial_i\hat{\psi}_\uparrow(t,r). \tag{4.2}$$

However, system (4.2) is incomplete because it misses the interaction with the Cu^{z+} ion represented by the familiar term $-ze^2/r$ [1, 2]. The latter are not transformed (i.e., they are conventionally quantum mechanical) and therefore they should be merely added to the transformed Eqs. (4.2). The formal equations of the proposed model $(e_\uparrow^-, e_\downarrow^-)_{HM}$ are therefore given by

$$\begin{aligned} &\left\{\frac{1}{2m}\,\tilde{p}_k T\tilde{p}^k + \frac{e^2}{r}\,\hat{I} - z\,\frac{e^2}{r}\right\}T\hat{\psi}_\uparrow(t,r) \\ &\quad = \frac{1}{2m}\,\tilde{p}_k T\tilde{p}^k T\hat{\psi}_\uparrow + \frac{e^2}{r}\,\hat{\psi}_\uparrow - z\,\frac{e^2}{r}\,T\hat{\psi}_\uparrow(t,r) \\ &\quad = E\hat{\psi}_\uparrow(t,r) \end{aligned}$$

$$\tilde{p}_k T\hat{\psi}_\uparrow(t,r) = -i\hat{I}_k^i\,\partial_i\hat{\psi}_\uparrow(t,r). \tag{4.3}$$

To achieve a form of the model confrontable with experimental data, we need an explicit expression of the isounit $\hat{I}$. Among various possibilities, we select here the simplest possible isounit for the problem at hand, first identified in [4] and today called *Animalu's isounit* [7], which we write in this article in the form

$$\begin{aligned} \hat{I} &= e^{-\langle\hat{\psi}_\uparrow|\hat{\psi}_\downarrow\rangle\psi_\uparrow/\hat{\psi}_\uparrow} \approx 1 - \langle\hat{\psi}_\uparrow|\hat{\psi}_\downarrow\rangle\psi_\uparrow/\hat{\psi}_\uparrow + \cdots \\ T &= e^{+\langle\hat{\psi}_\uparrow|\hat{\psi}_\downarrow\rangle\psi_\uparrow/\hat{\psi}_\uparrow} \approx 1 + \langle\hat{\psi}_\uparrow|\hat{\psi}_\downarrow\rangle\psi_\uparrow/\hat{\psi}_\uparrow + \cdots, \end{aligned} \tag{4.4}$$

under which Eqs. (4.3) can be written

$$\begin{aligned} &\left\{\frac{1}{2m}\,\tilde{p}_k T\tilde{p}^k T\hat{\psi}_\uparrow - (z-1)\frac{e^2}{r}\,\hat{\psi}_\uparrow\right. \\ &\left. - z\,\frac{e^2}{r}\,\langle\hat{\psi}_\uparrow|\hat{\psi}_\downarrow\rangle\left(\psi_\uparrow/\hat{\psi}_\uparrow\right)\hat{\psi}_\uparrow(t,r)\right\} = E\hat{\psi}_\uparrow. \end{aligned} \tag{4.5}$$

Now, it is well known from quantum mechanics that the radial part of $\psi_\uparrow$ in the ground state ($L = 0$) behaves as $\psi_\uparrow(r) \approx Ae^{-r/R}$, where A is (approximately) constant and R is the coherence length of the pair. The radial solution for $\hat{\psi}_\uparrow$ also in the ground state is known from Eqs. (5.1.21) [3, p. 837] to behave as $\hat{\psi}_\uparrow(r) \approx B(1 - e^{-r/R})/r$, where B is also approximately a constant. The last term in the l.h.s. of Eq. (4.5) therefore behaves like a Hulten potential $V_0e^{-r/R}/(1 - e^{-r/R})$, $V_0 = e^2\langle\hat{\psi}_\uparrow|\hat{\psi}_\downarrow\rangle$. After substituting the expression for the isomomentum, the radial isoschrödinger equation can be written

$$\begin{aligned} &\left\{-\frac{\hat{I}}{2\hat{m}}\,r^2\frac{d}{dr}\,r^2\frac{d}{dr} - (z-1)\frac{e^2}{r}\right. \\ &\left. - V_0\,\frac{e^{-r/R}}{1-e^{-r/R}}\right\}\hat{\psi}_\uparrow(r) = E\hat{\psi}_\uparrow(r), \end{aligned} \tag{4.6}$$

where $\hat{m}$ is the isorenormalized mass (see next section). The solution of the above equation is known from [3, Sect. 5.1]. The Hulten potential behaves at small distances like the Coulomb potential,

$$V_{\text{Hulten}} = V_0e^{-r/R}/(1 - e^{-r/R}) \approx V_0R/r.$$

At distances smaller than the coherent length of the pair, Eq. (4.6) can therefore be effectively reduced to the form

$$\begin{aligned} &\left\{-\frac{1}{2\hat{m}}\,r^2\frac{d}{dr}\,r^2\frac{d}{dr} - V\,\frac{e^{-r/R}}{1-e^{-r/R}}\right\} \\ &\qquad \times\hat{\psi}_\uparrow(r) = E\hat{\psi}_\uparrow(r), \end{aligned} \tag{4.7}$$

where $V = V_0R + (z-1)e^2$ with general solution, boundary condition, and related spectrum [3, pp. 837–838]

$$\begin{aligned} \hat{\psi}_\uparrow(r) &= {}_2F_1(2\alpha + 1 + n, 1 - \alpha, 2\alpha + 1, e^{-r/R}) \\ &\quad \times e^{-\alpha r/R}(1 - e^{-r/R})/r, \end{aligned} \tag{4.8a}$$

$$\alpha = (\beta^2 - n^2)/2n > 0,$$

$$\beta^2 = \hat{m}VR^2/\hbar^2 > n^2, \tag{4.8b}$$

$$E = -\frac{\hbar^2}{4\hat{m}R^2}\left(\frac{\hat{m}VR^2}{\hbar^2}\,\frac{1}{n} - n\right)^2,$$

$$n = 1, 2, 3, \ldots, \tag{4.8c}$$

where we have reinstated $\hbar$ for clarity.

Santilli [3] identified the numerical solution of Eqs. (4.8) for the hadronic model $\pi^\circ = (e^+_\uparrow, e^-_\downarrow)_{HM}$ (in which there is evidently no contribution from the cuprite ions to the constant V), by introducing the parameters $k_1 = \hbar/2\hat{m}Rc_0$ and $k_2 = \hat{m}VR^2/\hbar$, where c_0 is the speed of light in vacuum, for which $V = 2k_1k_2^2\hbar c_0/R$ and the total energy of the state $(e^+_\uparrow, e^+_\downarrow)_{IIM}$ becomes in the ground state (which occurs for $n = 1$ for the Hulten potential)

$$E_{tot,\pi^\circ} = 2k_1\left[1 - (k_2 - 1)^2/4\right]\hbar c_0/R$$
$$= 2k_1(1 - \epsilon^2)\hbar c_0/R. \qquad (4.9)$$

The use of the total energy of the π° (135 MeV), its charge radius ($R \approx 10^{-13}$ cm), and its meanlife ($\tau \approx 10^{-16}$ s), then yields the values [3, Eqs. (5.1.33), p. 840]

$$k_1 = 0.34, \quad \epsilon = 4.27 \times 10^{-2},$$
$$k_2 = 1 + 8.54 \times 10^{-2} > 1. \qquad (4.10)$$

Animalu [4, 5] identified the solution of Eqs. (4.8) for the Cooper pair by introducing the parameters $k_1 = \epsilon_F R/\hbar c_0$ and $k_2 = KR/\epsilon_F$, where ϵ_F is the iso-Fermi energy of the electron (that for hadronic mechanics).

The total energy of the Cooper pair in the ground state is then given by

$$E_{tot,\text{Cooper pair}} = 2k_1\left[1 - (k_2 - 1)^2/4\right]\hbar c_0/R$$
$$\approx k_2 T_c/\theta_D, \qquad (4.11)$$

where θ_D is the Debye temperature.

Several numerical examples were considered in [5]. The use of experimental data for aluminum ($\theta_D = 428$ K, $\epsilon_F = 11.6$ eV, $T_c = 1.18$ K) yields the values

$$k_1 = 94, \qquad k_2 = 1.6 \times 10^{-3} < 1. \qquad (4.12)$$

For the case of $YBa_2Cu_3O_{6-\chi}$, the model yields [5]

$$k_1 = 1.3z^{-1/2} \times 10^{-4}, \qquad k_2 = 1.0z^{1/2} > 1, \qquad (4.13)$$

where the effective valence $z = 2(7 - \chi)/3$ varies from a minimum of $z = 4.66$ for $YBa_2Cu_3O_{6.96}$ ($T_c = 91$ K) to a maximum of $z = 4.33$ for $YBa_2Cu_3O_{6.5}$ ($T_c = 20$ K). The general expression predicted by hadronic mechanics for $YBa_2Cu_3O_{6-\chi}$ is given by [5, Eq. (5.15), p. 373]

$$T_c = 367.3 \times z \times e^{-13.6/z}, \qquad (4.14)$$

and it is in remarkable agreement with experimental data (see Figures 1–3).

A few comments are now in order: The above model of the Cooper pair is indeed nonlinear, nonlocal, and nonpotential. In fact, the nonlinearity in $\hat{\psi}_\uparrow$ is expressed by the presence of such a quantity in Eqs. (4.5). The nonlocality is expressed by the term $\langle \hat{\psi}_\uparrow \mid \hat{\psi}_\downarrow \rangle$ representing the overlapping of the wavepackets of the electrons, and the nonpotentiality is expressed by the presence of interactions, those characterized by the isounit (4.4a), which are outside the representational capabilities of the Hamiltonian H. This illustrates the *necessity* of using hadronic mechanics or other similar non-Hamiltonian theories (provided that they are physically consistent), because of the strictly linear–local—potential character of quantum mechanics. Note that whenever the wave-overlapping is no longer appreciable, i.e., for $\langle \hat{\psi}_\uparrow \mid \hat{\psi}_\downarrow \rangle = 0$, $\hat{I} \equiv I$, and quantum mechanics is recovered identi-

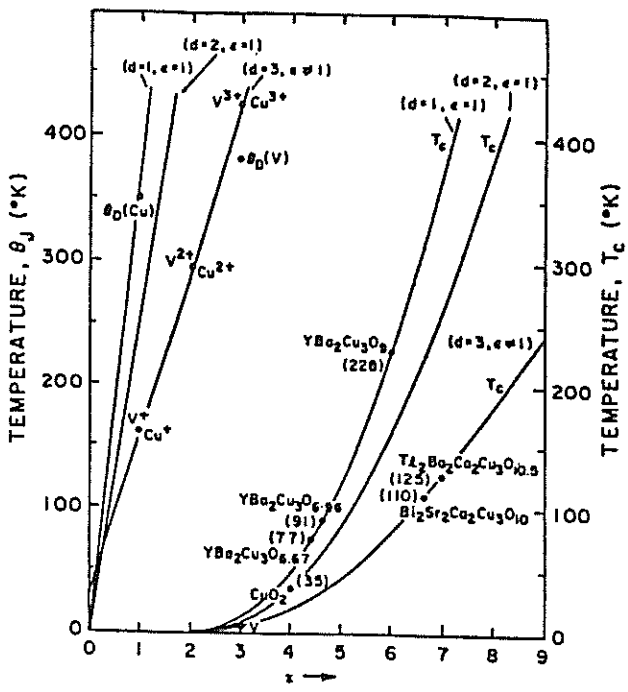

FIGURE 1. A reproduction of Figure 10 of [4] illustrating the remarkable agreement between the predicted dependence of T_c from the effective valence z of ions (continuous curve) and the experimental values on the "jellium temperature" for various compounds (solid dots).

Table 1. $YBa_2Cu_{3-x}Mn_xO_y$
(After N.L. Saini *et al.*, Int. J. Mod. Phys. B6, 3515 (1992)

x	y	z	T_c (theory)	T_c (expt.)
0.00	6.92	4.613	88.9	91
0.03	6.88	4.541	83.5	86.6
0.09	6.87	4.447	76.7	79.0
0.15	6.91	4.387	72.6	75.0
0.21	6.92	4.312	67.6	72.0
0.30	6.95	4.212	61.3	67.0

Note: T_c (theory) = $367.3z\exp(-13.6/z)$, where the effect of replacing Cu_3 by $Cu_{3-x}Mn_x$ is obtained by replacing 3 by $(3-x)+2x=3+x$, which lowers the effective valence (z) on Cu^{2+} ions to $z=2y/(3+x)$.

Table 2. $GdBa_2(Cu_{1-x}Ni_x)_3O_{7-\delta}$
(After, Chin Lin *et al.*, Phys. Rev. B42, 2554 (1990))

x	$y=7-\delta$	z	T_c (theory)	T_c (expt.)
0.000	6.96	4.640	91.0	91
0.025	6.96	4.527	82.4	79
0.050	6.96	4.419	74.8	71
0.075	6.96	4.316	67.9	65

Note: T_c (theory) = $367.3z\exp(-13.6/z)$, $z=2y/3(1+x)$ as discussed in Table 1.

Table 3. $GdBa_2(Cu_{1-x}Zn_x)_3O_{7-\delta}$
(After, Chin Lin *et al.*, Phys. Rev. B42, 2554 (1990))

x	$y=7-\delta$	z	T_c (theory)	T_c (expt.)
0.000	6.96	4.640	91.0	91
0.025	6.96	4.309	67.4	54
0.050	6.96	4.009	49.0	37
0.075	6.96	3.737	36.1	35

FIGURE 2. A reproduction of the tables of [5, p. 379], illustrating the agreement between the predictions of the model with experimental data from other profiles.

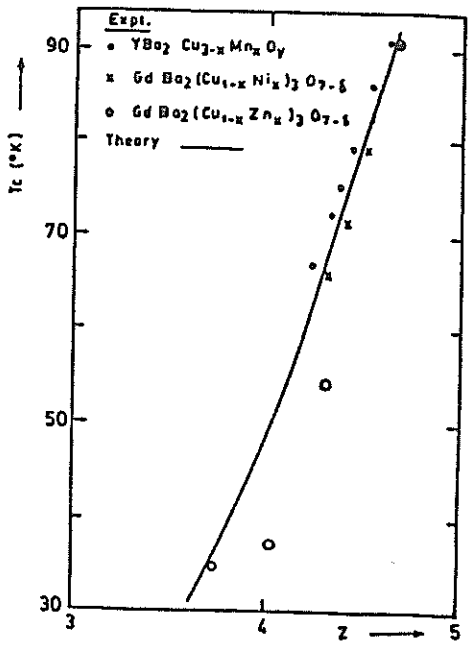

FIGURE 3. A reproduction of Fig. 5 of [5, p. 380], showing the agreement between the predictions of Eq. (4.14) for the doped 1 : 2 : 3 cuprates and the experimental data.

cally as a particular case, although without attraction.

The mechanism of the creation of the *attraction* among the *identical* electrons of the pair via the intermediate action of cuprate ions is a general law of hadronic mechanics according to which *nonlinear–nonlocal–non-Hamiltonian interactions due to wave-overlappings at short distances are always attractive in singlet couplings and such to absorb Coulomb interactions, resulting in total attractive interactions irrespective of whether the Coulomb contribution is attractive (as in the model of* [3] *for the* π°*) or repulsive (as in the model of* [4] *for the Cooper pair).* As noted earlier, the Hulten potential is known to behave as the Coulomb one at small distances and therefore absorbs the latter within the coherent length of the Cooper pair. But the Hulten interaction is stronger than is the Coulomb one within the same coherent length. This results in the overall attraction. The similarities between the model for the π° and that for the Cooper pair are remarkable. The applicability of the same model for other aspects should then be expected, such as for a deeper understanding of the valence, and will be studied elsewhere.

Another main feature of the model is characterized also by a general law of hadronic mechanics, that *bound state of particles due to wave-overlappings at short distances in singlet states suppress the atomic spectrum of energy down to only one possible level.* The Hulten potential is known to admit a *finite* number of energy levels. Santilli's [3] solution for the π° shows the suppression of the energy spectrum of the positronium down to only one energy level, 135 MeV of the π° for $k_2 > 1$. Animalu's [4, 5] solutions for the Cooper pair also reduce the same finite spectrum down to only one admissible level, that of the Cooper pair. Excited states are indeed admitted, but they imply large distances R for which nonlinear–nonlocal–non-Hamiltonian interactions are ignorable. This implies that all excited states are fully quantum mechanical, i.e., they *do not* represent the π° (the Cooper pair), but the ordinary positronium (repulsive Coulomb interactions among the two identical electrons). Alternatively, we can say that, in addition to the conventional, quantum mechanical, Coulomb interactions among two electrons, there is *only one* additional system of hadronic type with only one energy level per each couple of particles, one for $\pi^\circ = (e^+_\uparrow, e^-_\downarrow)_{HM}$ and the other for the Cooper pair $= (e^-_\uparrow, e^-_\downarrow)_{HM}$.

The case of possible triplet couplings also follows a general law of hadronic mechanics. While singlets and triplets are equally admitted in quantum mechanics (read, coupling of particles under their pointlike approximation), this is no longer the case for hadronic mechanics (read, couplings of particles when represented as being extended and at mutual distances smaller than their wavepackets/wavelengths). In fact, *all triplet couplings of particles under nonlinear–nonlocal–non-Hamiltonian interactions are highly unstable, the only stable states being the singlets.* This law was first derived in [3] via the "gear model," i.e., the illustration via gears which experience a known highly repulsive force in triplet couplings, while they can be coupled in a stable way in singlets. The possibility of applying the model to a deeper understanding of Pauli's exclusion principle is then consequential and will be studied in a subsequent article.

The connection between the proposed model and the conventional theory of the Cooper pair [1, 2] is intriguing, as studied in [5]. The constant in the Hulten potential can be written $V_0 = \hbar\omega$, where ω is precisely the (average) phonon frequency. Expression (4.11) can then be rewritten

$$E_{tot} = 2\epsilon_F - E \approx 2k_1k_2\hbar c_0/R(e^{1/NV} - 1),$$

where NV is the (dimensionless) electron–phonon coupling constant. The main results of our model can therefore be reformulated in terms of the electron–phonon interactions, as expected because the latter too represents available data [1, 2]. The resolution of which of the two models is preferable is therefore deferred to specific tests (see next section) as well as to the model with the more effective predictive capacity.

5. Relativistic Formulation and Experimental Verification

The proposed model of the Cooper pair see its true formulation at the relativistic level because it provides a *geometrization* of the Cooper pair, better possibilities for novel predictions, and the best possible experimental tests. These profiles require a technical knowledge of relativistic hadronic mechanics which we cannot possibly outline here (see [7, Vol. II]). We must therefore limit ourselves to an indication of the main lines. The first study of the Cooper pair via relativistic hadronic mechanics was conducted by Animalu [5].

The central mechanism of the Animalu–Santilli model of the Cooper pair is a *new type of renormalization of the intrinsic characteristics of particles caused by nonlinear–nonlocal–non-Hamiltonian interactions, called isorenormalization.* In essence, all interactions are known to imply renormalizations of the characteristics of particles. The renormalizations due to particle exchanges are known to be insufficient (when used alone without the cuprite action) to achieve an interpretation of the Cooper pair. But these interactions are strictly of potential-Lagrangian type. The point here conveyed is that the renormalization due to nonpotential interactions (isorenormalizations) do indeed permit a quantitative representation of the Cooper pair.

The notion of isorenormalization requires a knowledge of the generalization of the Minkowski space which is applicable under nonlinear–nonlocal–non-Hamiltonian interactions and a knowledge of its universal symmetry.

Let $M(x, \eta, R)$ be the conventional Minkowski space, where $x = (r, x^4)$, $x^4 = c_0 t$, and $\eta = \text{diag.}(1, 1, 1, -1)$ is the Minkowski metric over the real R. It is evident that M is inapplicable for a relativistic treatment of the proposed model because of its strict local–differential character, while the interactions here studied are nonlocal–integral. Moreover, the Minkowski space can geometrize only the homogeneity and isotropy of empty space, while the interior region of the Cooper pair is manifestly inhomogeneous and anisotropic.

Many different deformations–generalizations of the Minkowski space can be considered. That selected by these authors is the so-called *isominkowskian space* first introduced by Santilli in [15] and then studied in detail in various articles (see the most recent study [16] and the general presentation in [7, Vol. II, Chap. 8]) which can be written for diagonal isounits:

$$\hat{M}(x, \hat{\eta}, \hat{R}): \hat{\eta} = T\eta, \quad \hat{I} = T^{-1}, \qquad (5.1a)$$

$$\hat{x}^2 = (x^1T_{11}x^1 + x^2T_{22}x^2 + x^3T_{33}x^3 - x^4T_{44}x^4)\hat{I} \in \hat{R}. \qquad (5.1b)$$

The main reasons for the selection of isospace $\hat{M}$ over other possibilities is its "isotopic" character, i.e., the capability of preserving the original axioms of M. In fact, $\hat{M}$ is constructed by deform-

ing the Minkowski metric $\eta \to \hat{\eta} = T\eta$, while jointly deforming the unit by the amount inverse of the preceding deformation, $I \to \hat{I} = T^{-1}$. Geometrically, this implies no axiomatic change and, thus, $\hat{M}(x, \hat{\eta}, \hat{R} \approx M(x, \eta, R)$. In particular, this implies the preservation of all axioms of the special relativity at the abstract level, although realized in a nonlinear and nonlocal way [15, 16]. By comparison, other approaches deform the Minkowski metric, but keep the old unit I, resulting in spaces which are no longer isomorphic to M, thus implying the abandonment of the Einsteinian axioms in favor of yet unknown axioms.

By using the realization

$$\hat{I} = \hat{I}_0 \times \hat{I}_1, \quad \hat{I}_0 = \text{diag.}(n_1^2, n_2^2, n_3^2, n_4^2) \quad (5.2)$$

and $\hat{I}_1$ representing the factorized nonlinear–nonlocal terms, the isoseparation in $\hat{M}$ can be written

$$\hat{x}^2 = (x^1 n_1^{-2} x^1 + x^2 n_2^{-2} x^2 + x^3 n_3^2 x^3 - t c_0^2 / n_4^2 t) \hat{I}_0.$$

We see in this way that *nonlinear–nonlocal–nonpotential interactions treated via the isominkowskian geometry can represent a locally varying speed of light* $c = c_0/n_4$ *due to propagation within a physical medium, where* n_4 is the familiar index of refraction. The space components n_k emerge from the space–time symmetrization (or use of the applicable symmetry, see below) and permit a direct geometrization (i.e., a geometrization via the metric itself) of inhomogeneous and anisotropic media inside the Cooper pair. As an example, the Minkowski space provides a geometrization of light propagating in empty space, while the isominkowskian space provides a geometrization of light propagating within inhomogeneous and anisotropic media, such as our atmosphere.

The first, and perhaps most basic, isorenormalization can now be seen. In fact, the lifting $M \to \hat{M}$ implies the corresponding lifting of the mass–energy equivalence $E_0 = mc_0^2$ into the *isoprinciple of mass–energy equivalence* $\hat{E}_0 = \hat{m}c^2 = mc_0^2/n_4^2$ [7, 15, 16]. The total energy of the Cooper pair is then predicted to be given by the expression

$$\hat{E}_{tot} = 2mc_0^2/n_4^2 - E, \quad (5.3)$$

where m is the conventional mass of the electron and $\hat{m} = m_0/n_4^2$ is the *isorenormalized mass*.

The n_μ's are called the *characteristic quantities of the medium considered*. They are locally varying when one studies the behavior at one internal point (e.g., the local speed of light at one point of our atmosphere). When studying the system as a whole, the characteristic quantities must be averaged to constant (this is the case when studying the average speed of light throughout our atmosphere which requires an average of the local index of refraction). When considering the Cooper pair from the outside, the characteristic quantities must therefore be averaged to constants. This allows the reduction of the isorenormalized mass $\hat{m}$ to a constant.

Explicit calculations done by Animalu [5] for the Cu^{z+} ion under a space–isotropy, $n_1 = n_2 = n_3 = n$, but space–time anisotropy, $n \neq n_4$, have identified the following solution:

$$n^{-2} = 1 - K/4, \quad n_4^{-2} = 1 + 3K/4, \quad k^{1/2} = \langle \hat{\psi}_\uparrow \mid \hat{\psi}_\downarrow \rangle, \quad (5.4)$$

which recover the conventional Minkowski space for null wave-overlappings, $K = 0$.

This result permits the identification of the following consequences of the model: First, we have the *isonormalization of the rest energy of the electron of the Cooper pair* $E = mc_0^2 \to \hat{E} = mc^2 = mc_0^2(1 + 3K/4)$, namely, our model predicts that *the rest energy of the electron in the Cooper pair is bigger than that predicted by relativistic quantum mechanics*. The isorenormalization of the charge will be indicated below. The models considered in this note are therefore written $\pi^\circ = (\hat{e}_\uparrow^+, \hat{e}_\downarrow^-)_{HM}$ [3] and Cooper pair $= (\hat{e}_\uparrow^-, \hat{e}_\downarrow^-)_{HM}$ [4, 5], with the symbol $\hat{e}^\pm$ to emphasize the fact that the "electrons" are in reality "isoelectrons," i.e., particles with physical characteristics different from those of quantum mechanics.

As a numerical example, for $YBa_2Cu_3O_{6.5}$, Animalu [4, 5] found

$$n = 2.081, \qquad n_4 = 0.550. \quad (5.5)$$

The isorenormalized rest energy of the electron is then $\hat{E}_0 = 0.511/n_4^2 = 1.824$ MeV. Note that the isorenormalization varies from case to case and depends on the compound at hand (i.e., it depends on the nonlinear–nonlocal–nonpotential interactions which vary from case to case). As such, hadronic mechanics does not appear to introduce any new constant and actually turns quantum mechanical constants in vacuum into local variables

in interior conditions, e.g., $\hbar = \text{const} \to \hat{\hbar} = \hat{\hbar}(x, p, \hat{\psi}, \dots)$, $c_0 = \text{cost.} \to c = c_0/n_4(x, \dots)$, etc.

The above relativistic treatment also permits the identification of the applicable geometry. In fact, it emerges that the interior region of the Cooper pair can be geometrized via the *isominkowskian geometry of Type 9* [7, Vol. II], i.e., the most general possible geometry in which $n_4 < 1$ and $n = (b_1 + n_2 + n_3)/3 > n_4$ as in values (5.5). This is not a mere formality because it possesses rather intriguing predictive capacities to be studied elsewhere.

Note that the emerging geometry has the most general possible isometric $\hat{\eta}$ with a nonlinear and nonlocal dependence on the wavefunction, $\hat{\eta} = T\eta = \hat{\eta}(x, \dot{x}, \hat{\psi}, \partial\hat{\psi}, \dots)$. However, such geometry *is not* curved because $\hat{M} \approx M$, but *isoflat*, verifying the axioms of flatness in isospace (for detailed geometrical studies, see [7, Vols. I and II]). The isominkowskian geometrization of the different type 5 (with $n < n_4$, $n_4 > 1$, see [7, Vol. II, p. 386]) permits the representation of ordinary conductors. This offers a geometric representation of the transition from conductors to superconductors. The open issue studied elsewhere is the identification of the predictions of the new geometry in superconductivity.

Finally, the isominkowskian geometrization permits the identification of specific experiments for the test of the proposed model. In fact, *the isominkowskian geometry of Type 9 predicts that light exits the system "isoblueshifted," i.e., with a frequency bigger than that predicted by relativistic quantum mechanics due to the absorption of energy from the medium itself.* Explicitly, the conventional Doppler shift is given by the familiar rule for null angle of aberration

$$\omega = \omega_0(1 - v/c_0)/(1 - v^2/c_0^2)^{1/2} \approx \omega_0\left(1 - \beta + \tfrac{1}{2}\beta^2 + \cdots\right), \quad \beta = v/c_0. \quad (5.6)$$

But c_0 is the speed of light *in vacuum*. As such, the above law is inapplicable within physical media.

The issue is the identification of the generalization which is appropriate for physical media. In this respect, note that the replacement of c_0 with c in Eq. (5.6) would be contradicted by experimental evidence. In fact, light propagating within homogeneous and isotropic media such as water experience a decrease of speed $c < c_0$, but no frequency shift, contrary to the prediction of redshift by replacing c_0 with $c < c_0$ in Eq. (5.6).

The resolution of this and other inconsistencies is provided by the isominkowskian geometry which predicts (uniquely and unambiguously) the following *isodoppler law* first identified by Santilli [7, 14, 16], here also expressed for null angle of aberration:

$$\omega = \omega_0(1 - vn_4/c_0 n)/(1 - v^2 n_4^2/c_0^2 n^2)^{1/2} \approx \omega_0\left(1 - \beta(n_4/n) + \tfrac{1}{2}\beta^2(n^2/n^2) + \cdots\right). \quad (5.7)$$

For homogeneous and isotropic media such as water, $n \equiv n_4 > 1$, and the isominkowskian geometry correctly represents the lack of frequency shift in water. However, for $n \neq n_4$, the isominkowskian geometry predicts an *isoredshift* (i.e., loss of energy to the medium) for $n > n_4$ and *isoblueshift* (i.e., the acquisition of energy from the medium) for $n < n_4$. The isominkowskian medium for the Cooper pair is of Type 9, thus implying an isoblueshift.

The Animalu–Santilli model can therefore be tested by verifying whether or not light emitted from the Cooper pair is indeed isoblueshifted. This can be done in a variety of ways, e.g., via the scattering of positrons on a compound including the pair, their annihilation with one of the electrons of the pair, and the measure of the frequency of the emitted gammas. Similar independent experiments should be conceived and conducted to test the electron–phonon interactions in a way independent from superconductivity.

In closing, we would like to indicate that all aspects of the model here submitted can be derived from the universal symmetry of the isominkowskian geometry, the isotopies $\hat{P}(3.1)$ of the Poincaré symmetry $P(3.1)$, also called *isopoincaré symmetry*, first proposed by Santilli in [15] and then studied in various articles (see, the latest study [16] or the comprehensive presentation [7, Vol. II]). In essence, $\hat{P}(3.1)$ is the conventional symmetry although constructed with respect to the most general possible isounit $\hat{I}$ yielding the universal symmetry of isoseparation (5.2).

According to our model, the electrons $\hat{e}^-$ in the Cooper pair are not conventional particles because they are not characterized by the Poincaré symmetry on M over R. Instead, they are *isoparticles*, i.e., particles characterized by the isopoincaré symmetry $\hat{P}(3.1)$ on $\hat{M}$ over $\hat{R}$. The reader not familiar with isotopies should be aware that the transition from particles to isoparticles is more general than that from particles to quasi-particles [1, 2]. This can be seen from the fact indicated earlier that the lifting $I \to \hat{I}$ necessarily requires *nonunitary* transforms. Particles and isoparticles are therefore related by nonunitary transforms.

It is a truism to say that all aspects of the nonlinear–nonlocal–nonpotential model of the Cooper pair are derivable from the isopoincaré symmetry $\hat{P}(3.1)$. We regret the inability to study in detail this property at this time. We merely indicate the *isorenormalization of the charge* $e^{\pm} \to \hat{e}^{\pm}$ with the possible realization

$$e^- \to \hat{e}^- = e^- \left(1 - K\langle \hat{\psi}_\uparrow \mid \hat{\psi}_\downarrow \rangle r^2 \psi_\uparrow / \hat{\psi}_\uparrow\right)$$
$$\approx e^- \left(1 - \langle \hat{\psi}_\uparrow \mid \hat{\psi}_\downarrow \rangle r e^{-r/R}/(1 - e^{-r/R})\right). \quad (5.8)$$

This turns the Coulomb interactions into the combination of Coulomb and Hulten interactions,

$$e^- e^-/r \to \hat{e}^- e^-/r$$
$$= e^- e^-/r - V_0 e^{-r/R}/(1 - e^{-r/R}), \quad (5.9)$$

which results in attraction irrespective of the Coulomb contribution. The reduction of the proposed model of the Cooper pair to the primitive universal symmetry $\hat{P}(3.1)$ of $\hat{M}$ is then consequential.

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