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Ruggero Maria Santilli

# Foundations of Theoretical Mechanics I

## The Inverse Problem in Newtonian Mechanics

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## The Inverse Problem in Newtonian Mechanics

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*Questo volume è dedicato a mia moglie*

*Carla*

*con infinito amore*

# Contents

# Preface

The objective of this monograph is to present some methodological foundations of theoretical mechanics that are recommendable to graduate students prior to, or jointly with, the study of more advanced topics such as statistical mechanics, thermodynamics, and elementary particle physics.

A program of this nature is inevitably centered on the methodological foundations for Newtonian systems, with particular reference to the central equations of our theories, that is, Lagrange's and Hamilton's equations. This program, realized through a study of the analytic representations in terms of Lagrange's and Hamilton's equations of generally nonconservative Newtonian systems (namely, systems with Newtonian forces not necessarily derivable from a potential function), falls within the context of the so-called Inverse Problem, and consists of three major aspects:

1. The study of the necessary and sufficient conditions for the existence of a Lagrangian or Hamiltonian representation of given equations of motion with arbitrary forces;
2. The identification of the methods for the construction of a Lagrangian or Hamiltonian from the given equations of motion; and
3. The analysis of the significance of the underlying methodology for other aspects of Newtonian Mechanics, e.g., transformation theory, symmetries, and first integrals for nonconservative Newtonian systems.

This first volume is devoted to the foundations of the Inverse Problem, with particular reference to aspects 1 and 2. The second volume deals with some generalizations and applications of the Inverse Problem, with

particular reference to aspect 3, and the problem of the construction of equivalent forms of the equations of motion that satisfy the conditions for the existence of a Lagrangian or Hamiltonian representation.

I had several motivations for undertaking this task. The first motivation came to me as a teacher. Indeed, the decision to study the analytic representations of systems with arbitrary Newtonian forces grew out of my uneasiness in teaching a graduate course in classical mechanics in the conventional manner. Typically, an articulated body of interrelated methodological formulations (i.e., analytic, variational, algebraic, geometrical, etc.) is presented; but in the final analysis, in view of the lack of knowledge of the methods for computing a Lagrangian for systems with more general Newtonian forces, these formulations are nowadays applicable only to systems with forces derivable from a potential function (basically, conservative systems). My uneasiness was ultimately due to the fact that, strictly speaking, conservative systems do not exist in our Newtonian environment. As a result, the Lagrangian representation of conservative Newtonian systems is, in general, only a crude approximation of physical reality.

A few remarks are sufficient to illustrate this point. For instance, the entire conventional theory of the Lagrangian representation in the space of the generalized coordinates of conservative Newtonian systems subject to holonomic constraints, is based on the often tacit assumption that the constraints are frictionless. But in practice, holonomic constraints are realized by mechanical means, e.g., hinges, rods, etc. Therefore, the presence of frictional forces is inevitable whenever holonomic constraints occur and, in turn, a Lagrangian representation that does not reflect this dissipative nature can only be considered a first approximation of the systems considered.

Owing to the fundamental nature of the knowledge of a Lagrangian or Hamiltonian, the above limitation of the conventional approach to Newtonian Mechanics is present at virtually all levels of the theory. For instance, the theory of canonical transformations for the one-dimensional harmonic oscillator is well known. But the extension of this theory to the more realistic case of the damped oscillator is not treated in currently available textbooks, again because of the lack of methods for constructing a Hamiltonian when damping forces are present. Similarly, the Hamilton–Jacobi theory of the frictionless spinning top is well known, but its extension to the system which actually occurs in our environment, namely, the spinning top with damping torque, is unknown at this time to the best of my knowledge. Therefore, the analysis presented in this monograph, the analytic representations of nonconservative Newtonian systems, grew out of my attempts to more closely represent Newtonian reality.

Other motivations for undertaking this task came to me as a theoretical physicist. As we all know, the significance of Newtonian Mechanics goes beyond the pragmatic aspect of merely studying Newtonian systems, because its methodological foundations apply, apart from technical rather than conceptual modifications, to several branches of physics, such as quantum mechanics and elementary particle physics. As soon as I became aware of

new methodological prospectives within the context of purely Newtonian systems, I became intrigued by their possible significance for other branches of physics.

Predictably, it will take a considerable amount of time and effort by more than one researcher to ascertain the possible significance of the Inverse Problem for non-Newtonian frameworks. Nevertheless, to stimulate research along these lines, a few remarks are presented in the Introduction.

Owing to the lack of recent accounts of the Inverse Problem in both the mathematical and the physical literature, one of the most time-consuming parts of my program has been the identification of the prior state of the art. Indeed, it was only after a laborious library search, which I conducted over a three-year period by moving backward in time to the beginning of the past century, that I came to realize that the methodological foundations of the Inverse Problem were fully established in the mathematical literature by the first part of this century within the context of the calculus of variations. This was the result of the contributions of several authors, such as Jacobi (1837), Helmholtz (1887), Darboux (1891), Mayer (1896), Hirsh (1898), Bohem (1900), Könisberger (1901), Hamel (1903), Kurshak (1906), and others. The most comprehensive account of which I am aware is the thesis of D. R. Davis in 1926 at the Department of Mathematics of the University of Chicago, under the supervision of G. A. Bliss, subsequently expanded and published in three articles in 1928, 1929, and 1931 (see References). Since that time, regrettably, the problem remained largely ignored in both the mathematical and physical literature, with only a few exceptions known to me, which I shall indicate in the Introduction.

In this volume I present the results of my search in specialized mathematical and physical literature, and of my efforts on aspects such as the use of the Converse of the Poincaré Lemma for the proof of the central theorem on the necessary and sufficient conditions for the existence of a Lagrangian, the methods for the construction of a Lagrangian from the given equations of motion, the independent treatment of the Inverse Problem for phase space formulations without the prior knowledge of a Lagrangian, and the algebraic or geometrical significance of the necessary and sufficient conditions for the existence of a Hamiltonian. However, owing to the vast accumulation of literature in classical mechanics, calculus of variations, and other disciplines over the centuries, I make no claim to originality.

I make no claim to mathematical rigor, either. I concentrated my efforts primarily on presenting and illustrating the basic concepts in as simple a manner as possible. In essence, by specific intent, this volume should be readable by first- or second-year graduate students without major difficulties. In writing this monograph, I have also attempted to render it self-sufficient—extensive reference study is needed only for certain complementary aspects, such as for certain problems of the theory of differential equations or for certain geometrical interpretations, and a sound knowledge of undergraduate mechanics is the only prerequisite.

I have also made an effort to adopt the most widely used notations and symbols. When necessary, new notations are identified by footnotes.

Equations are referred to by notation of the form (1.2.3a), where 1, 2, and 3a indicate the chapter, the section, and the equations therein, respectively.

The references are listed at the end of the volume in chronological and then alphabetical order. My list of textbooks must be considered as purely representative, though incompletely so, of contributions in theoretical mechanics and related disciplines. However, for specialized topics not treated in currently available textbooks, I have listed all the relevant references of which I am aware.

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Almost needless to say, I am solely responsible for the content of this volume, including several modifications implemented in the final version of the manuscript.

# Volume Organization

In the Introduction, I formulate the Inverse Problem of the calculus of variations, point out its reduction to a Newtonian context, and indicate all the relevant references on such problem of which I am aware.

In Chapter 1, I outline the rudiments of three disciplines, ordinary differential equations, calculus of differential forms, and calculus of variations, which are prerequisites for the methodology of the Inverse Problem.

In Chapter 2, I introduce the central mathematical tool of the analysis, the so-called variational approach to self-adjointness, and I specialize it to the most important forms of Newtonian systems.

In Chapter 3, I work out the central objectives of this monograph, which consist of the necessary and sufficient conditions for the existence of a Lagrangian or, independently, a Hamiltonian, the methods for their computation from given equations of motion, and an analysis of those Newtonian forces that are admissible by a Lagrangian or Hamiltonian representation.

In Appendix A, I review those concepts of Newtonian Mechanics that are useful for the analysis of the main text, to avoid excessive reference to the existing literature.

The presentation is organized into a main text, a series of charts, a set of examples, and problems. In the main text, I treat the essential concepts and formulations of the approach. In the charts, I present those complementary aspects which, even though not essential for the basic lines of the approach, are valuable for a deeper insight, and I touch on topics of more advanced nature for subsequent study by the interested reader. The examples are intended to illustrate the basic concepts introduced in the text only. The

problems are intended to test the student's understanding of the given methodology and then his capability to work out specific applications.

The generalization of methodology of the Inverse Problem for the construction of a Lagrangian or Hamiltonian representation of systems of ordinary differential equations which, as given, violate the integrability conditions, is treated in Santilli (1979).

# Use Suggestions

This book can be used as a textbook for a one-term graduate course on the Inverse Problem or on Nonconservative Newtonian Mechanics.

For the use of this book as a reference book for a section of a regular graduate course in classical mechanics devoted to the Inverse Problem, the instructor is recommended to work out a summary presentation of Chapters 2 and 3.

For graduate students in physics, I recommend first reading the main text, verifying the illustrative examples, and working out the problems. Subsequent study should then incorporate the charts and quoted references. A prior inspection of the Appendix A might be recommendable.

For graduate students and instructors in mathematics, this book can be complemented by currently available treatises on the calculus of variations, optimal control theory, differential geometry, and other topics to formulate the Inverse Problem in these disciplines.

# Introduction

At present, there are several well-established and interrelated methodological formulations for describing Newtonian systems with (local) forces derivable from a potential, i.e.,[1]

$$m_k \ddot{\mathbf{r}}_k - \mathbf{f}_k(t, \mathbf{r}, \dot{\mathbf{r}}) = 0, \qquad k = 1, 2, \ldots, N, \tag{I.1a}$$

$$\mathbf{f}_k = -\frac{\partial U}{\partial \mathbf{r}^k} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\mathbf{r}}^k}. \tag{I.1b}$$

For instance, by simply emphasizing the most significant aspects under consideration, we can distinguish among[2]

1. *analytic formulations*, e.g., Lagrange's and Hamilton's equations, Hamilton–Jacobi theory, etc.[3];
2. *variational formulations*, e.g., variational problems, variational principles, etc.[4];
3. *algebraic formulations*, e.g., infinitesimal and finite canonical transformations, Lie algebras and Lie groups, symmetries and conservation laws, etc.[5];

[1] For a review of basic concepts, see the Appendix.

[2] For each of the listed topics there exists a vast bibliography. We quote below only a few representative references.

[3] Whittaker (1904), Goldstein (1950), Pars (1965).

[4] Lanczos (1949), Rund (1966).

[5] Saletan and Cromer (1971), Sudarshan and Mukunda (1974).

4. *geometric formulations*, e.g., symplectic geometry, canonical structure, etc.[6];
5. *statistical formulations*, e.g., Liouville's theorem, equilibrium and nonequilibrium statistical mechanics, etc.[7];
6. *thermodynamic formulations*, e.g., irreversible processes, entropy, etc.[8];
7. *many-body formulations*, e.g., stability of orbits, quadrature problems, etc.[9];

and so on.

In practice, a particular formulation is selected according to actual needs. For instance, when dealing with a small number of particles, analytic formulations may be used instead of, say, the statistical formulations. However, when dealing with a large number of particles, the opposite selection may be preferable.

One reason for constructing such a variety of formulations is that a sufficient depth in studying a given system is reached only when a sufficient number of aspects are taken into consideration. Physical reality is polyhedric, to say the least, in relation to our capability to represent it. Therefore, the level of our knowledge depends on how many aspects are considered and how deeply each of them is analyzed. This does not imply, however, that theoretical formulations are compartmentalized. Actually, all the above-mentioned formulations are so deeply interrelated that they form a single articulated body of methodological tools. As a set, they could be called by a single name, e.g., "the methodological formulations for Newtonian systems with (local) forces derivable from a potential."

This interrelationship is due to the fact that the various formulations are centrally dependent, in either a direct or an indirect way, on the fundamental analytic equations of the theory, namely, the conventional *Lagrange's equations*:

$$\frac{d}{dt}\frac{\partial L_{\text{tot}}}{\partial \dot{\mathbf{r}}^k} - \frac{\partial L_{\text{tot}}}{\partial \mathbf{r}^k} = 0, \tag{I.2a}$$

$$L_{\text{tot}} = T(\dot{\mathbf{r}}) - U(t, \mathbf{r}, \dot{\mathbf{r}}); \tag{I.2b}$$

and *Hamilton's equations*:

$$\dot{\mathbf{r}}^k = \frac{\partial H_{\text{tot}}}{\partial \mathbf{p}_k}, \qquad \dot{\mathbf{p}}_k = -\frac{\partial H_{\text{tot}}}{\partial \mathbf{r}^k}, \tag{I.3a}$$

$$H_{\text{tot}} = T(\mathbf{p}) + U(t, \mathbf{r}, \mathbf{p}); \tag{I.3b}$$

[6] Jost (1964), Abraham and Marsden (1967), Guillemin and Sternberg (1977).
[7] Gibbs (1948), Katz (1967).
[8] Sommerfeld (1956), Tisza (1966).
[9] Wintner (1941), Khilmi (1961), Hagihara (1970).

with their interconnecting *Legendre transform*:

$$\mathbf{p}_k = \frac{\partial L_{\text{tot}}}{\partial \dot{\mathbf{r}}^k}, \tag{I.4a}$$

$$H_{\text{tot}} = \mathbf{p}_k \cdot \dot{\mathbf{r}}^k - L_{\text{tot}}. \tag{I.4b}$$

The assumption of Equations (I.3) implies a unique characterization of the time evolution law in phase space:

$$\dot{A}(\mathbf{r}, \mathbf{p}) = [A, H_{\text{tot}}]_{\text{cl}}, \tag{I.5}$$

where[10]

$$[A, B]_{\text{cl}} = \frac{\partial A}{\partial \mathbf{r}^k} \cdot \frac{\partial B}{\partial \mathbf{p}_k} - \frac{\partial A}{\partial \mathbf{p}_k} \cdot \frac{\partial B}{\partial \mathbf{r}^k} \tag{I.6}$$

are the *Poisson brackets*. Then, methodological tools such as canonical transformations, Liouville's theorem, etc. can be characterized in a sequential manner.

An aspect of central methodological significance is that the algebraic structure which underlies the above-mentioned formulations is a *Lie algebra*. This is due to the fact that Poisson brackets (I.6) satisfy the Lie algebra identities

$$[A, B]_{\text{cl}} + [B, A]_{\text{cl}} = 0 \tag{I.7a}$$

and

$$[[A, B]_{\text{cl}}, C]_{\text{cl}} + [[B, C]_{\text{cl}}, A]_{\text{cl}} + [[C, A]_{\text{cl}}, B]_{\text{cl}} = 0. \tag{I.7b}$$

As a matter of fact, the methodological significance of Lie algebras is so prominent that the theory considered here can well be called the "Lie algebra approach to Newtonian systems."

Methodological formulations 1–7 have a significance which goes beyond the description of Newtonian systems (I.1). Indeed, they are the foundations of virtually the entire current theoretical knowledge for the representation of non-Newtonian systems, such as quantum mechanical and quantum field theoretical systems.

For instance, in the transition from Newtonian to quantum mechanical systems[11] we have, in essence, the transition from functions in phase space $A, B, \ldots$ obeying the time evolution law (I.5) to a *Hilbert space* of Hermitian operators $A, B, \ldots$ satisfying the *Heisenberg law*:

$$\dot{A}(\mathbf{r}, \mathbf{p}) = \frac{1}{i\hbar}[A, H_{\text{tot}}]_{\text{qm}} \tag{I.8a}$$

$$H_{\text{tot}} = H_{\text{free}} + H_{\text{int}} = -\hbar^2 \sum_{k=1}^{n} \frac{\Delta_k}{2m_k} + U. \tag{I.8b}$$

[10] The familiar convention on the sum of the repeated (Latin and Greek) indices, unless otherwise stated, will tacitly used throughout our analysis.

[11] For the problem of quantization see, for instance, van Hove (1951), Prosser (1964), Kostant (1970), and Souriau (1970).

Poisson bracket Equation (I.6) is then replaced by the algebraic product,

$$[A, B]_{\mathrm{qm}} = AB - BA, \tag{I.9}$$

with the interconnecting limiting procedure (for polynomial functions)

$$\lim_{\hbar \to 0} \frac{1}{i\hbar} [A, B]_{\mathrm{qm}} = [A, B]_{\mathrm{cl}}. \tag{I.10}$$

The relevant aspect here is that the algebras characterized by brackets (I.6) and (I.9) coincide as abstract algebras. Thus, the theory considered here can also be called the "Lie algebra approach to quantum mechanical systems."

In the transition from Newtonian to classical continuous systems, the methodology essentially demands a generalized form of the Lagrangian, i.e., the *Lagrangian density*[12]:

$$\mathscr{L}_{\mathrm{tot}} = \mathscr{L}_{\mathrm{free}}(\varphi^k, \partial_\mu \varphi^k) + \mathscr{L}_{\mathrm{int}}(\varphi^k, \partial_\mu \varphi^k) \tag{I.11}$$

$$k = 1, 2, \ldots, N, \qquad \partial_\mu \varphi^k = \frac{\partial \varphi^k}{\partial x^\mu}, \qquad \mu = 0, 1, 2, 3, \ldots \qquad x^0 = ct, \qquad \mathbf{x} = \mathbf{r},$$

where $\mathscr{L}_{\mathrm{free}}$ represents the free systems and $\mathscr{L}_{\mathrm{int}}$ is an additive term which couples the fields; the independent variables are now both the time and space coordinates, and the fields $\varphi^k(t, \mathbf{r})$ take the place of the coordinates $\mathbf{r}^k(t)$. The basic analytic equations of the theory are now *Lagrange's equations for continuous systems*:

$$d_\mu \frac{\partial \mathscr{L}_{\mathrm{tot}}}{\partial \partial_\mu \varphi^k} - \frac{\partial \mathscr{L}_{\mathrm{tot}}}{\partial \varphi^k} = 0; \tag{I.12}$$

or *Hamilton's equations for continuous systems*:

$$\frac{\partial \varphi^k}{\partial t} = \frac{\delta \mathscr{H}_{\mathrm{tot}}}{\delta \pi_k} = \frac{\partial \theta^{00}}{\partial \pi_k} - \nabla \cdot \frac{\partial \theta^{00}}{\partial (\nabla \pi_r)}, \tag{I.13a}$$

$$\frac{\partial \pi_k}{\partial t} = -\frac{\delta \mathscr{H}_{\mathrm{tot}}}{\delta \varphi^k} = -\frac{\partial \theta^{00}}{\partial \varphi^k} + \nabla \cdot \frac{\partial \theta^{00}}{\partial (\nabla \varphi^k)}; \tag{I.13b}$$

where the canonical momentum density $\pi_k$, the Hamiltonian density $\mathscr{H}_{\mathrm{tot}}$, the energy-momentum density $\theta^{00}$, and the functional derivative $\delta$ are defined by

$$\pi_k = \frac{\partial \mathscr{L}_{\mathrm{tot}}}{\partial (\partial \varphi^k / \partial t)}, \tag{I.14a}$$

$$\mathscr{H}_{\mathrm{tot}} = \pi_k \frac{\partial \varphi^k}{\partial t} - \mathscr{L}_{\mathrm{tot}} = \mathscr{H}_{\mathrm{free}} + \mathscr{H}_{\mathrm{int}}, \tag{I.14b}$$

[12] See, for instance, Goldstein (1950) and Roman (1969).

$$\theta^{\mu\nu} = \frac{\partial \mathscr{L}_{\text{tot}}}{\partial(\partial_\mu \varphi^k)} \partial^\nu \pi_k - g^{\mu\nu} \mathscr{L}_{\text{tot}}, \tag{I.14c}$$

$$\frac{\delta \mathscr{H}_{\text{tot}}}{\delta \pi_k} = \frac{\partial \mathscr{H}_{\text{tot}}}{\partial \pi_k}, \tag{I.14d}$$

and

$$\frac{\partial \mathscr{H}_{\text{tot}}}{\delta \varphi^k} = \frac{\partial \mathscr{H}_{\text{tot}}}{\partial \varphi^k} - \frac{d}{d\mathbf{r}} \cdot \frac{\partial \mathscr{H}_{\text{tot}}}{\partial(\partial \varphi^k / \partial \mathbf{r})}. \tag{I.14e}$$

Here, the relevant aspect is that the time evolution law is again law (I.5), only written in the form

$$\dot{A} = [A, H_{\text{tot}}] = \int_v d^3r \left( \frac{\delta A}{\delta \varphi^k} \frac{\delta H_{\text{tot}}}{\delta \pi_k} - \frac{\delta A}{\delta \pi_k} \frac{\delta H_{\text{tot}}}{\delta \varphi^k} \right) \tag{I.15a}$$

$$A = \int_v d^3r \, \mathscr{A}, \qquad H_{\text{tot}} = \int_v d^3r \, \mathscr{H}_{\text{tot}}. \tag{I.15b}$$

Therefore, the underlying algebraic structure is still a Lie algebra.

In the transition from classical to quantum field theory,[13] there is essentially a transition from the space of functions $\varphi^k(t, \mathbf{r})$ to a space of operator-valued distributions. But, again, the algebraic structure of the theory remains a Lie algebra.

A fundamental physical property common to all systems represented by structures (I.3b), (I.8b), and (I.14b), is that the acting Newtonian, quantum mechanical, or quantum field theoretical forces (or couplings) are (local and) derivable from a potential. A central contention of this monograph is that these systems do not exhaust the physical reality. As a result, the study of the methodological formulations for more general systems is needed.

For instance, the Newtonian systems of our everyday experience *are not* of type (I.1) because their forces are generally not derivable from a potential. This is the case, for instance, of the motion of particles with drag forces—damped and forced oscillators, spinning tops with drag torques, etc. Indeed, a more general (local) Newtonian system is characterized by a collection of forces some of which are derivable from a potential and some not, and we write

$$m_k \ddot{\mathbf{r}}_k - \mathbf{f}_k(t, \mathbf{r}, \dot{\mathbf{r}}) - \mathscr{F}_k(t, \mathbf{r}, \dot{\mathbf{r}}) = 0, \tag{I.16}$$

where the $\mathbf{f}_k(\mathscr{F}_k)$ forces are the collection of forces derivable (not derivable) from a potential, i.e.,

$$\mathbf{f}_k(t, \mathbf{r}, \dot{\mathbf{r}}) = -\frac{\partial U}{\partial \mathbf{r}^k} + \frac{d}{dt} \frac{\partial U}{\partial \dot{\mathbf{r}}^k}, \tag{I.17a}$$

$$\mathscr{F}_k(t, \mathbf{r}, \dot{\mathbf{r}}) \neq -\frac{\partial U}{\partial \mathbf{r}^k} + \frac{d}{dt} \frac{\partial U}{\partial \dot{\mathbf{r}}^k}. \tag{I.17b}$$

[13] See, for instance, Streater and Wightman (1964).

At a quantum mechanical level, the force responsible for the atomic structure, the Coulomb force, is known to be derivable from a potential.[14] Thus, Hamiltonian (I.8b) produces a fully satisfactory representation of the atomic phenomenology. The situation is somewhat different at the nuclear level. The current representations of the nuclear forces as derivable from a potential are known to produce an excellent agreement with the experimental data. Neverthless, the nature of the nuclear forces is a problem which is still open to a considerable extent at this time, while the study of nonconservative nuclear processes has lately been increased. The situation is still more different at the level of the structure of the hadrons, i.e., the strongly interacting particles such as mesons, nucleons, etc., where the need of forces more general than the atomic and nuclear forces is conceivable, and its study is, in any case, recommendable.

At a quantum field theoretical level, structures of the Lagrangian densities of type (I.11) are known to produce a physically effective representation of the electromagnetic interactions. The same structure, once implemented within the context of the so-called gauge theories, has also produced a physically effective unification of the weak and electromagnetic interactions. Nevertheless, the problem of whether the same structure can also produce a physically effective representation of the strong interactions is still open today.

These remarks are intended to indicate that the study of the methodological formulations for systems with forces not necessarily derivable from a potential, besides its direct physical significance in Newtonian mechanics, might prove to be significant for other branches of physics, too. In any case, a study of this nature first demands the identification of the necessary methodology within the arena of our best intuition—Newtonian Mechanics. In turn, as indicated earlier, such methodology is centrally dependent on the assumed analytic equations.

At this point, an alternative of major methodological implications occurs:

1. *Use of the equations originally conceived by Lagrange and Hamilton,* i.e., those with external terms.

$$\frac{d}{dt}\frac{\partial L_{\text{tot}}}{\partial \dot{\mathbf{r}}^k} - \frac{\partial L_{\text{tot}}}{\partial \mathbf{r}^k} = \mathscr{F}_k \tag{I.18a}$$

$$\dot{\mathbf{r}}^k = \frac{\partial H_{\text{tot}}}{\partial \mathbf{p}_k}, \qquad \dot{\mathbf{p}}_k = -\frac{\partial H_{\text{tot}}}{\partial \mathbf{r}^k} + \tilde{\mathscr{F}}_k \tag{I.18b}$$

$$\mathbf{p}_k = \frac{\partial L_{\text{tot}}}{\partial \dot{\mathbf{r}}^k} \tag{I.18c}$$

$$H_{\text{tot}} = \mathbf{p}_k \cdot \dot{\mathbf{r}}^k - L_{\text{tot}}$$
$$\tilde{\mathscr{F}}_k = \tilde{\mathscr{F}}_k(t, \mathbf{r}, \mathbf{p}) = \mathscr{F}_k(t, \mathbf{r}, \dot{\mathbf{r}}). \tag{I.18d}$$

[14] Notice that Equations (I.1) contain, as a particular case, the more general Lorentz force. In this respect, see, for instance, Goldstein (1950).

In essence, Lagrange[15,16] and Hamilton[17] were fully aware that the Newtonian forces are generally not derivable from a potential. Therefore, to avoid an excessive approximation of the physical reality, they formulated their equations with external terms. Oddly, only since the beginning of this century have Lagrange's and Hamilton's equations been "truncated" with the removal of the external terms, by acquiring the form of Equations (I.2) and (I.3), which is primarily used in the current physical literature. Notice that for Equations (I.18b), the Hamiltonian can characterize the *total energy*, i.e., the sum of the kinetic and potential energies of all forces admitting a potential function[18]

$$H_{\text{tot}} = T + U, \tag{I.19}$$

while all forces that do not admit a potential function are represented by the external terms. The methodological implications of Equations (I.18) are nontrivial. To illustrate this aspect, it is sufficient to note that the time evolution law for Equations (I.18b) is now given by

$$\dot{A}(\mathbf{r}, p) = (A, H)_{\text{cl}} = \frac{\partial A}{\partial \mathbf{r}^k} \cdot \frac{\partial H_{\text{tot}}}{\partial \mathbf{p}_k} - \frac{\partial A}{\partial \mathbf{p}_k} \cdot \frac{\partial H_{\text{tot}}}{\partial \mathbf{r}^k} + \frac{\partial A}{\partial \mathbf{p}_k} \cdot \tilde{\mathscr{F}}_k \tag{I.20}$$

while brackets $(A, H)$ *violate* the Lie algebra laws, Equations (I.7). By taking into account the central role of Lie algebras in physics, as recalled earlier, this is an indication of deep methodological implications in the transition from systems with forces derivable from a potential to systems with forces not derivable from a potential. This approach to systems (I.16) is explored in Santilli (1978, Vols. I, II, and III). Here, let us only indicate that, when properly written, Equations (I.18b) and (I.20) characterize an algebraic covering of Lie algebras called *Lie-admissible algebras.*[19] Predictably, this broader algebraic character has implications at several levels of the theory, e.g., the underlying geometry is no longer of symplectic type but rather of a broader type which has been called *symplectic-admissible geometry* (Santilli, 1978, Vol. II).

2. *Use of Lagrange's and Hamilton's equations without external terms*, i.e.,

$$\frac{d}{dt} \frac{\partial L_{\text{tot}}^{\text{gen}}}{\partial \dot{\mathbf{r}}^k} - \frac{\partial L_{\text{tot}}^{\text{gen}}}{\partial \mathbf{r}^k} = 0, \tag{I.21a}$$

$$\dot{\mathbf{r}}^k = \frac{\partial H_{\text{tot}}^{\text{gen}}}{\partial \mathbf{P}_k}, \qquad \dot{\mathbf{P}}_k = -\frac{\partial H_{\text{tot}}^{\text{gen}}}{\partial \mathbf{r}^k}, \tag{I.21b}$$

$$\mathbf{P}_k = \frac{\partial L_{\text{tot}}^{\text{gen}}}{\partial \dot{\mathbf{r}}^k}, \tag{I.21c}$$

and

$$H_{\text{tot}}^{\text{gen}} = \mathbf{P}_k \cdot \dot{\mathbf{r}}^k - L_{\text{tot}}^{\text{gen}}. \tag{I.21d}$$

[15] Lagrange (1788).

[16] For historical notes see, for instance, Whittaker, (1904) and Pars (1965).

[17] Hamilton (1834). For historical notes see also Whittaker (1904) and Pars (1965).

[18] Notice that here we are referring to the notion of total energy for a *nonconservative* system, i.e., a system which does not conserve such energy by assumption. For a treatment of the case of damped-oscillator systems see, for instance, Symon (1960).

[19] Santilli (1968, 1969, 1970, 1978).

In essence, systems with forces derivable from a potential can be effectively treated with only one type of analytic equation, i.e., Equations (I.2) or (I.3). The case of systems with forces not derivable from a potential is different. In this latter instance, two types of equations are admissible—those with and without external terms. However, when Equations (I.21) are used for systems (I.16), the Lagrangian or Hamiltonian structure is not longer of type (I.2b) or (I.3b), respectively. Instead, these functions must possess generalized structures which, as we shall see in Section 3.7, can be written in the form[20]

$$L_{\text{tot}}^{\text{gen}} = \sum_{k=1}^{N} \sum_{a=x,y,z} L_{\text{int,I}}^{(ka)}(t, \mathbf{r}, \dot{\mathbf{r}}) L_{\text{free}}^{(ka)}(\dot{r}^{ka}) + L_{\text{int,II}}(t, \mathbf{r}, \dot{\mathbf{r}}),$$

$$L_{\text{free}}^{(ka)} = \tfrac{1}{2} m_k (\dot{r}_{ka})^2, \tag{I.22}$$

with a corresponding form for the Hamiltonian. The aspect of this second alternative, which is relevant here, is that time evolution law (I.5) is insensitive to the explicit functional dependence of the Hamiltonian. As a result, the analytic brackets of the approach are still Poisson brackets (I.6) and the underlying algebraic structure is still a Lie algebra. In conclusion, when the conventional Lagrange's and Hamilton's equations can be used for the representation of nonconservative systems (I.16), the methodological profile is basically that for systems with forces derivable from a potential, in the sense that the analytic equations, the time evolution law, the underlying algebraic and geometrical structure, etc. remain formally unchanged. Nevertheless, the nonconservative nature of the represented systems is reflected

[20] Notice that generalized Lagrangian (I.22): (a) demands the use of $3N + 1$ interaction terms, $3N$ multiplicative and one additive to the terms representing the free motion; (b) is a covering of conventional structure (I.2b) in the sense that it is a nontrivial generalization capable of reproducing structure (I.2b) at the limit when the multiplicative interaction terms reduce to unity (i.e., all the acting forces are derivable from a potential); and (c) does not necessarily possess the dimension of the energy. (I.22) will be studied further in Chapter 3. At this stage, the following example may be useful to illustrate the concept under consideration. One of the simplest systems with forces not derivable from a potential is the particle under a drag force which is linearly dependent on the velocity. Assume that the motion is in one dimension. The equation of motion is then given by

$$m\ddot{r} + \gamma\dot{r} = 0, \qquad \dot{r} \neq 0.$$

As we shall see in Example 3.1, the above system admits a representation in terms of Lagrange's equations (I.21a) and (at least) the following two nontrivially different Lagrangians.

$$L_{\text{tot}}^{\text{gen}} = L_{\text{int,I}} L_{\text{free}} + L_{\text{int,II}}$$
$$L_{\text{int,I}} = e^{\gamma t/m}, \qquad L_{\text{free}} = \tfrac{1}{2} m\dot{r}^2, \qquad L_{\text{int,II}} = 0$$
$$L_{\text{tot}}^{*\text{gen}} = L_{\text{int,I}}^{*} L_{\text{free}} + L_{\text{int,II}}^{*}$$

$$L_{\text{int,I}}^{*} = \frac{2}{\dot{r}} \ln \frac{\dot{r}}{c}, \qquad L_{\text{free}} = \tfrac{1}{2} m\dot{r}^2, \qquad L_{\text{int,II}} = -\gamma c r, \qquad c = \text{const.}$$

Since the acting force $f = -\gamma\dot{r}$ is not derivable from a potential, both Lagrangians contain an essential term which multiplies the term for the free motion. In particular, Lagrangian $L_{\text{tot}}^{\text{gen}}$ has the dimension of energy, while Lagrangian $L_{\text{tot}}^{*\text{gen}}$ has not. Notice also the differences involved to recover the case $f = 0$. For a study of the differences between these two Lagrangians from the viewpoint of the transformation theory, see Santilli (1979).

in a number of aspects. For instance, a necessary condition for Equations (I.21b) to represent nonconservative system (I.16) is that the generalized Hamiltonian *does not* represent the total physical energy [Equation (I.19)], because otherwise it would imply the absence of forces not derivable from a potential. On equivalent grounds, one can see that, under the same conditions, the canonical momentum $\mathbf{P}$ of prescriptions (I.21c) does not represent the physical linear momentum $\mathbf{p} = m\dot{\mathbf{r}}$, or that the canonical angular momentum $\mathbf{M}_{can} = \mathbf{r} \times \mathbf{P}$ does not represent the physical angular momentum $\mathbf{M} = \mathbf{r} \times \mathbf{p}$, etc.

To summarize, a dual methodological profile appears to be conceivable for systems with forces not derivable from a potential, as induced by the use of analytic equations with external terms (alternative 1 above) or without external terms (alternative 2 above).[21] In the former case, the underlying methodology is generalized but the fundamental quantities of the approach, such as the Hamiltonian $H_{tot}$, the momentum $\mathbf{p}$, and the angular momentum $\mathbf{M} = \mathbf{r} \times \mathbf{p}$, possess a direct physical significance. In the latter case, the opposite situation occurs, namely, the methodology is the conventional one, while the indicated fundamental quantities of the approach lose their direct physical significance. It is hoped that a judicious interplay between these two complementary approaches to the same systems will be effective on methodological as well as physical grounds. On the former grounds, certain aspects which are difficult to treat within the context of one approach could be more manageable within the context of the other approach, and vice versa. On the latter grounds, the two complementary approaches could be useful for the identification of the physical significance of the algorithms at hand, which is one of the most insidious aspects of the study of nonconservative systems.[22]

[21] See chart A.9 (of the Appendix) for a dual formulation of Liouville's Theorem.

[22] One of the best examples to illustrate this aspect is given by the problem of the quantization of forces not derivable from a potential which, despite initial efforts, is still an open question today. As is well known, in order to comply with the correspondence principle, the familiar quantization rule

$$\text{Linear momentum} \to \frac{\hbar}{i}\nabla$$

must be applied to the canonical momentum [Equation (I.21c)]. The comparison of the prediction of the theory with the experimental data is then based on expectation values of the quantum mechanical operator $(\hbar/i)\nabla$. Within the context of systems with forces derivable from a potential, the approach is fully consistent on both mathematical and physical grounds. The corresponding situation for the case of systems with forces not derivable from a potential appears to be consistent on mathematical grounds. Nevertheless, its physical consistency demands a specific study. Indeed, we are now dealing with the expectation values of the operator $(\hbar/i)\nabla$, whose Newtonian limit under the correspondence principle, Equations (I.21c), *is not* representative of the physical linear momentum, i.e., it is a mathematical quantity of the type

$$\mathbf{P} = \gamma \csc^{-1} \beta\dot{\mathbf{r}}^2, \qquad \gamma, \beta = \text{const.}$$

Clearly, until the problem of the identification of the quantum mechanical representative of the *physical* linear momentum $\mathbf{p} = m\dot{\mathbf{r}}$ is not resolved, the physical consistency of the theory in general, and the comparison of its prediction with the experimental data in particular, are in question. Notice that the same problem also occurs for other dichotomies, e.g., the canonical Hamiltonian-vs-total physical energy and the canonical angular momentum-vs-physical angular momentum. For a study of these quantum mechanical aspects of nonconservative systems, see Santilli (1978, particularly Volume III).

The reader should be aware that a rather sizable methodological gap exists between systems (I.1) and (I.16), and that several conceptual, methodological, and physical aspects which are fully established for systems (I.1) need a reinspection for systems (I.16). To illustrate this situation, it is sufficient to indicate that systems (I.16) are nonconservative by assumption and, thus, they generally violate all Galilean conservation laws (for the energy, linear momentum, angular momentum, and uniform motion of the center of mass). Besides, the forces not derivable from a potential do not transform form-invariantly, in general, under the Galilean transformations. As a result, the problem of the applicable relativity demands a specific study [see Santilli (1978), particularly Vol. II].

This monograph is devoted to the following aspects of alternative 2.

(a) The necessary and sufficient conditions for the existence of a representation of systems (I.16) in terms of conventional analytic Equations (I.21)
(b) The methods for the construction of a Lagrangian or Hamiltonian from given equations of motion when their existence is ensured by the integrability conditions
(c) The identification of the most general (local) Newtonian forces that are admissible by Equations (I.21).

Other methodological aspects are studied in Santilli (1979).

The analysis will be conducted within the context of the so-called *Inverse Problem of the calculus of variations*, which, for the case of single integral path functionals, can be formulated as follows.

*Given the totality of solutions* $y(x) = \{y^1(x), \ldots, y^n(x)\}$ *of a system of* $n$ *ordinary differential equations of order* $r$,[23]

$$F_k(x, y^{(0)}, y^{(1)}, \ldots, y^{(r)}) = 0, \tag{I.23}$$

$$y^{(i)} = \frac{d^i y}{dx^i}, \qquad i = 1, 2, \ldots, r, \qquad k = 1, 2, \ldots, n,$$

*determine whether there exists a functional*

$$A(y) = \int_{x_1}^{x_2} dx L(x, y^{(0)}, \ldots, y^{(r-1)}) \tag{I.24}$$

*which admits such solutions as extremals.*

This problem is based on the study of the conditions under which there exists a function $L(x, y^{(0)}, y^{(1)}, \ldots, y^{(r-1)})$ such that Euler's equations[24] of

[23] That is, the $nr$-parameter family of solutions of Equations (I.23).
[24] Euler (1736 and 1765). For historical notes see, for instance, Dugas (1950).

functional (I.24) coincide with system (I.23), i.e.,

$$\sum_{i=0}^{r-1}(-1)^i \frac{d^i}{dx^i}\frac{\partial L}{\partial y^{k(i)}} = F_k. \tag{I.25}$$

To reformulate this problem within a Newtonian context, suppose that the order $r$ of Equations (I.23) is two, the independent variable $x$ is the time $t$, and the $n$ dependent variables $y^k(x)$ are the generalized coordinates $q^k(t)$ (see Appendix) or, in the absence of holonomic constraints, the $n = 3N$ Cartesian coordinates $\{q^k(t)\} = \{\mathbf{r}^i(t)\}$ in a given ordering. System (I.23) can then be interpreted as a system of Newton's equations of motion in configuration space, i.e.,

$$F_k(t, q, \dot{q}, \ddot{q}) = 0,$$

$$\dot{q} = \frac{dq}{dt}, \qquad \ddot{q} = \frac{d^2q}{dt^2}, \qquad q = \{q^k(t)\}, \qquad k = 1, 2, \ldots, n. \tag{I.26}$$

The Inverse Problem then consists of studying the conditions under which there exists a Lagrangian $L(t, q, \dot{q})$ such that Lagrange's equations[25] in $L$ coincide with system (I.23), i.e.,

$$\sum_{i=0}^{1}(-1)^i \frac{d^i}{dt^i}\frac{\partial L}{\partial q^{k(i)}} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} + \frac{\partial L}{\partial q^k}$$

$$= F_k(t, q, \dot{q}, \ddot{q}) \qquad k = 1, 2, \ldots, n. \tag{I.27}$$

On physical grounds, the primary significance of the Inverse Problem rests on the fact that the acting forces of Newtonian system (I.26) need not necessarily be derivable from a potential. Therefore, the Inverse Problem allows one to study the Lagrangian representations of systems with arbitrary (i.e., generally nonconservative but local) Newtonian forces.

This problem constitutes the central objective of this monograph. It should be indicated in this respect that our analysis is restricted primarily to the problem of the existence of a Lagrangian or, independently, a Hamiltonian, and that the extremal aspect is ignored. For further study, the interested reader may consult some of the readily available textbooks on the calculus of variations.

In the rest of this Introduction we shall outline of the prior state of the art. It should be stressed here that the following quotations must not be interpreted as historical notes, which are left to interested historians, but simply as a report on my findings.

[25] Throughout the course of our analysis we shall use the term "Lagrange's equations" rather than "Euler–Lagrange equations," as often (although not universally) used, whenever the function $L$ is the Lagrangian of a Newtonian system.

Apparently, the case $n = 1, r = 2$ was first solved by Darboux (1891) with conventional techniques, because it consists of one partial differential equation, i.e.,

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q} = -\frac{\partial^2 L}{\partial \dot{q}^2}\ddot{q} - \frac{\partial^2 L}{\partial \dot{q}\,\partial q}\dot{q} - \frac{\partial^2 L}{\partial \dot{q}\,\partial t} + \frac{\partial L}{\partial q} = F(t, q, \dot{q}, \ddot{q}) \quad \text{(I.28)}$$

in one unknown, i.e., the function $L$. Under certain continuity and regularity conditions, the theory of partial differential equations guarantees the existence of a solution. Similar conclusions can be reached for the case $n = 1$ and $r > 2$.

The case $n > 1, r = 2$ is not trivial. This is due to the fact that in this case the problem consists of $n$ partial differential equations, i.e.,

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} + \frac{\partial L}{\partial q^k} = -\sum_{i=1}^{n}\left(\frac{\partial^2 L}{\partial \dot{q}^k\,\partial \dot{q}^i}\ddot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k\,\partial q^i}\dot{q}^i\right) - \frac{\partial^2 L}{\partial \dot{q}^k\,\partial t} + \frac{\partial L}{\partial q^k}$$
$$= F_k(t, q, \dot{q}, \ddot{q}), \qquad k = 1, 2, \ldots, n, \quad \text{(I.29)}$$

in only one unknown, i.e., again the function $L$. As a result, the system is overdetermined and a solution in this case does not necessarily exist despite continuity and regularity assumptions. A similar situation occurs for $n > 1$ and $r > 2$.

The necessary and sufficient conditions for the existence of a solution $L$ of system (I.29) were apparently formulated for the first time by Helmholtz (1887)[26] on quite remarkable intuitional grounds. In essence, Helmholtz's starting point was the property of the self-adjointness of Lagrange's equations, i.e., their system of variational forms coincides with the adjoint system (see Chapter 2 and following). This is a property which goes back to Jacobi (1837).[27] Without providing a rigorous proof, Helmholtz indicated that the necessary and sufficient condition for the existence of a solution $L$ of system (I.28) is that the system $F_k = 0$ be self-adjoint.

The problem was subsequently studied by several authors, including Mayer (1896),[28] Hirsch (1897 and 1898),[29] Bohem (1900),[29] Könisberger (1901),[30] Hamel (1903), Kurshak (1906), Davis (1928, 1929, and 1931),[31]

[26] Helmholtz did not consider an explicit dependence of the equations of motion on time. Subsequent studies indicated that his findings were insensitive to such a dependence.

[27] The equations of variations of Lagrange's equations or, equivalently, of Euler's equations of a variational problem, are called *Jacobi's equations* in the current literature of the calculus of variations. We shall use the same terminology for our Newtonian analysis.

[28] This author apparently attempted the first proof of sufficency. The proof of necessity is self-evident from the self-adjointness of Lagrange's equations.

[29] These authors apparently conducted the first studies for $n > 1$ and $r > 2$.

[30] This is the first detailed and comprehensive account on the problem.

[31] These references contain the best treatment of sufficiency known to me, and one of the first studies of the indirect representations, namely, the representations of equivalent systems rather than the original systems as given. These papers are an elabroation of Davis's thesis at the Department of Mathematics of the University of Chicago in 1926 under the supervision of G. A. Bliss. Oddly, there is no direct quotation in Davis's papers of the previous crucial results by Helmholtz and Mayer.

de Donder (1935),[32] and Rapoport (1938).[33] These studies provided major contributions toward the proof that Helmholtz's condition, namely, the condition of variational self-adjointness for the system $F_k = 0$, was indeed both necessary and sufficient not only for the case $n > 1$ and $r = 2$, but also for the general case of arbitrary (but finite) dimensionality and order of the system. Neverthless, as we shall see, a number of technical aspects still remained open.

A somewhat different but rather laborious approach was subsequently provided by Douglas (1941).[34] The remaining and most recent studies on the Inverse Problem known to me are those by Dedecker (1949 and 1950),[35] Havas (1957),[36] Klein (1962),[37] Vainberg (1964),[38] Edelen (1969),[39] Tonti

[32] See page 204 of de Donder. Rather unpredictably, this is one of the very few treatises in the calculus of variations with a treatment of the Inverse Problem that I have been able to identify, despite a laborious search.

[33] This author apparently confronts, for the first time in a direct way, the problem of constructing a Lagrangian once its existence is ensured by the conditions of self-adjointness.

[34] Rather than using the conditions of self-adjointness, this author uses the so-called Riquier theory of partial differential equations for the case of a system with $n = r = 2$, by reducing it to a completely integrable system.

[35] This author conducted a detailed analysis of a prolongation method previously introduced by Bateman (1931).

[36] In this reference, the case of indirect representations is studied to a considerable extent. It should be indicated here that this study has a central significance on practical grounds, because Newton's equations of motion are generally non-self-adjoint. The problem of the existence of their Lagrangian representation is then reduced to the problem of finding equivalent self-adjoint forms. The article quoted here also constitutes one of the very few accounts on the Inverse Problem in the physical literature known to me.

[37] The memoir by this author which is quoted here constitutes, to the best of my knowledge, the first comprehensive attempt at a geometrical interpretation of the integrability conditions for the existence of a Lagrangian, as identified by Helmholtz (1887). The context is that of metric differential geometry, with particular reference to certain applications of the theory of generalized Finsler manifolds to analytic mechanics via the use of the calculus of differential forms. The significance of this memoir for the Inverse Problem is that it reduces the integrability conditions for the existence of a Lagrangian to primitive geometrical concepts.

[38] The monograph by this author which is quoted here constitutes, to the best of my knowledge, the first operational attempt at the integrability conditions for the existence of an action functional within the context of modern functional analysis. The generally nonlinear nature of the considered operators is essentially rendered treatable with the conventional theory of linear operators on function spaces via the use of the Frechet derivative. A significance of this monograph for the Inverse Problem is that it provides a basis for the study of the relationship between the variational and operational approaches to self-adjointness.

[39] The monograph by this author which is quoted here deals with the case of continuous systems. Even though, in this reference, there is no explicit use of the conditions of self-adjointness, the author reaches the rather remarkable result that a Lagrangian density for the representation of linear integro-differential systems of second-order partial differential equations always exists. In a subsequent monograph (Edelen, 1977), this author provides a detailed study of nonconservative nonholonomic systems in terms of the calculus of differential forms.

(1968),[40] Horndeski (1974),[41] Atherton and Homsey (1975),[42] Allcock (1975),[43] and Santilli (1977a, b, and c).[44]

The Inverse Problem can be studied today with a variety of modern and sophisticated mathematical tools which (as emerges from the footnotes to this Introduction) include the use of functional analysis, prolongation theory, and differential geometry, to cite only a few.

However, the emphasis of the analysis of this monograph is on physical content as well as simplicity of presentation, rather than mathematical completeness. As a consequence, I have selected what apparently is the simplest but most effective mathematical approach to the integrability conditions for the existence of a Lagrangian—the variational approach to self-adjointness, with an economical use of its prerequisites, e.g., the existence theory of differential equations, the calculus of differential forms, and the calculus of variations. Other approaches quoted in this Introduction will be outlined, for the reader's convenience, in the charts of Chapter 3 [as well as those of Santilli (1979)].

[40] Vainberg's study on the integrability conditions of the Inverse Problem (1964) was considerably abstract, to the point of remaining either unknown or inaccessible to the broad audience of applied mathematicians. Tonti's merit is that of having recognized the significance of Vainberg's studies by developing a reformulation of the operational approach to the Inverse Problem of considerable practical applicability. A significance of the memoir quoted here is that the integrability conditions for the existence of an action functional, as derived within the context of the operational approach, coincide with those obtained via a variational approach, i.e., they are in both cases the conditions identified by Helmholtz (1887). This renders the two approaches equivalent.

[41] To the best of my knowledge, this author initiated the use in the Inverse Problem of the cohomology theory and cochain complexes. A point which is significant for our analysis is that the emerging integrability conditions for the existence of a Lagrangian again coincide with those obtained with the variational approach to self-adjointness [Lovelock and Anderson (1976) private communication)]. This indicates that the same integrability conditions can be expressed in a variety of different, but equivalent, mathematical languages. For further studies, see also Horndeski (1975).

[42] These authors made significant contributions within the context of the operational approach to the Inverse Problem. The paper quoted here also contains a summary of previous contributions along the same lines of study.

[43] This author considers the problem of the existence of an action functional within an algebraic-geometric setting consisting of the reduction of a Pfaffian linear differential form on a manifold to a locally Hamiltonian form via the use of certain properties of the Lagrange brackets. This approach, which is equivalent to the variational approach to self-adjointness for vector fields on manifold, is particularly significant, e.g., for the extension of the Inverse Problem to the case of nonintegrable subsidiary constraints.

[44] In these papers I studied the Inverse Problem in classical relativistic field theories and initiated the study of the application of this problem to transformation theory. These papers are based on a variational approach to self-adjointness complemented by the use of the calculus of differential forms in general and the Converse of the Poincaré Lemma in particular, on account of the known effectiveness of these latter mathematical tools in studying integrability conditions. The Newtonian analysis of the Inverse Problem presented in this monograph closely follows the field theoretical analysis presented in these papers.

CHAPTER 1

# Elemental Mathematics

## 1.1 Existence Theory for Implicit Functions, Solutions, and Derivatives in the Parameters

In this section, we shall study some aspects of the theory of ordinary differential equations, which will later play a central role in several aspects of our analysis.

The first objective of this section is to review the existence theorems for implicit functions. Later, these theorems will be useful for the study of topics such as the Legendre transform, the construction of equivalent forms of the equations of motion, and others.

The second objective of this section is to recall the existence theorems for solutions with or without initial conditions. These theorems are useful to ascertain whether a given system of ordinary differential equations is consistent and, thus, whether or not it represents a physically admissible motion.

As a third objective of this section, we shall outline the theorems for the embedding of a solution into a parametric family of solutions and the theorems of the existence and continuity of the derivatives of such solutions in the parameters. Later on, this third aspect will turn out to be useful for the characterization of the necessary and sufficient conditions for the existence of a Lagrangian, namely, the variational forms, their adjoint system, and, finally, the conditions of variational self-adjointness.

Consider a system of $n$ second-order ordinary differential equations in the generalized coordinates $q^k$,[1] i.e.,

$$F_i(t, q, \dot{q}, \ddot{q}) = F_i(t, q^i, \ldots, q^n, \dot{q}^i, \ldots, \dot{q}^n, \ddot{q}^i, \ldots, \ddot{q}^n) = 0, \quad i = 1, 2, \ldots, n, \tag{1.1.1}$$

where

$$\dot{q}^k = \frac{dq^k}{dt} \quad \text{and} \quad \ddot{q}^k = \frac{d^2q^k}{dt^2} \tag{1.1.2}$$

are the first- and second-order time derivatives, respectively.

In general, we shall use $\mathscr{C}^m$ to denote the class of functions possessing continuous partial derivatives of order $0, 1, 2, \ldots, m$ on a region $R$ of their variables. In particular, we shall say that the (real-valued) function $F_i$ of Equations (1.1.1) are of class $\mathscr{C}^m$ in a region $R^{3n+1}$ of the (real-valued) elements $(t, q, \dot{q}, \ddot{q})$ when they possess continuous partial derivatives up to and including the order $m$ everywhere in $R^{3n+1}$, and we shall write

$$F_i \in \mathscr{C}^m(R^{3n+1}). \tag{1.1.3}$$

Notice that this property, when it holds, implies that partial derivatives up to and including the order $m$ exist, are continuous, and "commute," e.g.,

$$\frac{\partial^{m_3} F_i}{(\partial q^j)^{m_1}(\partial \dot{q}^k)^{m_2}} = \frac{\partial^{m_3} F_i}{(\partial \dot{q}^k)^{m_2}(\partial q^j)^{m_1}}, \tag{1.1.4}$$

$$m_1 + m_2 = m_3 = 0, 1, 2, \ldots, m.$$

When the functions $F_i$ are of class $\mathscr{C}^0$ in $R^{3n+1}$, they are continuous at all points of (but their derivatives are not necessarily continuous in) $R^{3n+1}$.

A *point* $P_0$ of $R^{3n+1}$ is a set of $3n + 1$ values $(t_0, q_0, \dot{q}_0, \ddot{q}_0)$. A *neighborhood* $(P_0)_\varepsilon$ of $P_0$ is the totality of points $P = (t, q, \dot{q}, \ddot{q})$ satisfying the inequalities

$$|t - t_0| \leq \varepsilon, \quad |q - q_0| \leq \varepsilon, \quad |\dot{q} - \dot{q}_0| \leq \varepsilon, \quad |\ddot{q} - \ddot{q}_0| \leq \varepsilon. \tag{1.1.5}$$

A neighborhood of a given value $t_0$ of $t$ will be denoted by $(t_0)_\varepsilon$. The same notation will also be used for other neighborhoods.

In general, the *region* of definition $R$ for the functions we shall consider in this volume will be a connected set. Unless otherwise stated, such a set will be tacitly assumed to be open, e.g., $t_1 < t < t_2$. Our "minimal" region will be a point *and* its neighborhood.

In this chapter, we shall first consider the *local* aspect of the existence theory (for implicit functions, solutions, and derivatives with respect to parameters) by restricting the values of $t$ to those lying in a neighborhood $(t_0)_\varepsilon$ of a given value $t_0$. Then we shall touch on the problem of the global existence theory,

[1] This chapter is formulated for the reader's convenience specifically in terms of generalized coordinates. The results, however, trivially apply for differential equations $F_k(x, y, y', y'') = 0$ in $3n + 1$ arbitrary variables $x$ and $y^k(x)$, $y'^k(x)$, $y''^k(x)$, $y' = dy/dx$, $y'' = dy'/dx$, $k = 1, 2, \ldots, m$. For a review of the concept of generalized coordinates, see the Appendix.

that is, when $t$ can take all the values of an interval $t_1 \leq t \leq t_2$. By writing $t \in (t_1, t_2)$ we mean the totality of values of $t$ on the interval $(t_1, t_2)$, *including* the end times.

The matrix

$$\mathscr{H} = \left(\frac{\partial F_i}{\partial \ddot{q}^j}\right) = \begin{pmatrix} \dfrac{\partial F_1}{\partial \ddot{q}^1} & \cdots & \dfrac{\partial F_1}{\partial \ddot{q}^n} \\ \vdots & & \\ \dfrac{\partial F_n}{\partial \ddot{q}^1} & \cdots & \dfrac{\partial F_n}{\partial \ddot{q}^n} \end{pmatrix} \tag{1.1.6}$$

is termed the *functional matrix*[2] of Equation (1.1.1). Since Equations (1.1.1) are not necessarily linear in the accelerations, the elements $\partial F_i/\partial \ddot{q}^j$ can, in general, be functions of $(t, q, \dot{q}, \ddot{q})$, and we shall write, symbolically,

$$\mathscr{H} = \mathscr{H}(R^{3n+1}). \tag{1.1.7}$$

The determinant

$$|\mathscr{H}| = |\mathscr{H}|(R^{3n+1}) \tag{1.1.8}$$

is called the *functional determinant* of system (1.1.1) and it is, in general, also a function of the $3n + 1$ elements $(t, q, \dot{q}. \ddot{q})$.

**Definition 1.1.1.** System (1.1.1) is called *regular* (*degenerate*)[3] when its functional determinant (1.1.8) is everywhere non-null (null) in a region $R^{3n+1}$ of points $(t, q, \dot{q}, \ddot{q})$, with the possible exception of a (finite) number of isolated points.

In nontechnical terms, we can say that two possibilities exist for determinant (1.1.8). Either it is identically null as a function, in which case the system is degenerate, or not, in which case the system is regular. The fact that determinant (1.1.8) is not null as a function, however, does not exclude the possible existence of isolated zeros, i.e., the solution of the *equation* $|\mathscr{H}| = 0$. This yields functional determinants which are null at their zeros, but not in the neighborhood of the same zeros. Thus, according to Definition 1.1.1, the systems are regular in this case. Alternatively, we can say that for a system to be degenerate, its functional determinant must be null at a point of its variables *and* in its neighborhood. For illustrations, see Examples 1.3 and 2.6.

[2] This terminology has been derived from the calculus of variations. See, for instance, Bliss (1946). Matrix (1.2.6) is also called the *Jacobian matrix* or, sometimes, the *Hessian matrix* when related to Lagrange's equations.

[3] It should be mentioned that the above terminology of *regular* and *degenerate* systems does not appear to be universally adopted and several different terms exist, such as *standard* and *nonstandard*; and *regular* and *singular*. We believe that the term "singular" is inappropriate for our context because the singularities of a system of differential equations, e.g., of the type $w' = A(z)w$, are customarily associated with the singularities of $A(z)$ and not with the properties of the functional matrix of the system [see, for instance, Coddington and Levinson (1955, Section 4.2)]. Levi-Civita and Amaldi (1927, Vol. II, part 2) called a system *normal* when $|\mathscr{H}| \neq 0$. Within the context of the calculus of variations for single integrals, such a condition is equivalent to the *Legendre condition* [see, for instance, G. A. Bliss (1946, page 23)]. The case of an infinite number of nonisolated zeros is not considered for simplicity.

A subcase of degeneracy is significant for our analysis. System (1.1.1) will be called *totally degenerate* when each and every element of its functional matrix is identically null. In this case, (1.1.1) reduces to a system of $n$ first-order ordinary differential equations.

In this book, we shall study the inverse problem for the case of systems (1.1.1), which are *regular*. The extension of the results to totally degenerate system is straightforward and will be indicated in the charts of Chapters 2 and 3. However, the extension of the methodology to the case of degenerate systems is considerably more delicate and will not be considered. For a better identification of the arena of applicability (and nonapplicability) of the analysis of this book, see the end of Section 2.1.

**Theorem 1.1.1** (Local Existence of Implicit Functions).[4] *Suppose the following conditions hold.*

(1) *The functions* $F_i(t, q, \dot{q}, \ddot{q})$ *are of class* $\mathscr{C}^m$, $m \geq 1$, *in a neighborhood* $(P_0)_\varepsilon$ *of a point* $P_0 = (t_0, q_0, \dot{q}_0, \ddot{q}_0)$,[5] *i.e.,*

$$F_i \in \mathscr{C}^m[(P_0)_\varepsilon], \qquad m \geq 1. \tag{1.1.9}$$

(2) *The functional determinant* $|\mathscr{H}|$ *is nonzero at* $P_0$, *i.e.,*

$$|\mathscr{H}|(P_0) \neq 0. \tag{1.1.10}$$

(3) *The point* $P_0$ *satisfies all Equations* (1.1.1), *i.e.,*

$$F_i(t_0, q_0, \dot{q}_0, \ddot{q}_0) = 0, \qquad i = 1, 2, \ldots, n. \tag{1.1.11}$$

*Then there exists a unique system*

$$\ddot{q}^i = f^i(t, q, \dot{q}) \tag{1.1.12}$$

*such that*

(*a*) *the functions* $f^i$ *are single-valued and continuous in a neighborhood* $(N_0)_\varepsilon$ *of* $N_0 = (t_0, q_0, \dot{q}_0)$, *i.e.,*

$$f^i \in \mathscr{C}^0[(N_0)_\varepsilon], \tag{1.1.13}$$

(*b*) *the values* $(t, q, \dot{q}, f) \in (P_0)_\varepsilon$ *satisfy Equations* (1.1.1), *i.e.,*

$$F_i(t, q, \dot{q}, f) = 0, \qquad (t, q, \dot{q}, f) \in (P_0)_\varepsilon, \tag{1.1.14}$$

(*c*) *there exists a constant* $\delta$ *such that for each element* $(t, q, \dot{q}) \in (N_0)_\varepsilon$, *the set* $(t, q, \ddot{q}, f)$ *is the only solution of Equation* (1.1.1) *satisfying the inequalities*

$$f^i - \delta < \ddot{q}^i < f^i + \delta, \tag{1.1.15}$$

[4] See, for instance, Bliss (1946, Appendix A), Rektorys (1969), Loomis and Sternberg (1968). For an alternative formulation, see Chart 1.1.

[5] It should be stressed that the theorem holds under the weaker assumption that the functions $F_i$ are continuous and possess continuous partial derivatives with respect to all $q$, $\dot{q}$ and $\ddot{q}$ in $(P_0)_\varepsilon$, but not necessarily with respect to $t \in (t_0)_\varepsilon$ (Bliss, 1946). Other authors (e.g., Rektorys, 1969) prefer continuity assumptions of the type of expression (1.1.9). We have selected the latter type of assumptions because the occurrence of the former in Newtonian mechanics is rather problematic and we shall not encounter it in this book.

(*d*) *at* $P_0$, *the identities*

$$\ddot{q}_0^i = f^i(t_0, q_0, \dot{q}_0) \tag{1.1.16}$$

*hold, and*

(*e*) *in a sufficiently small neighborhood* $(N_0)_\varepsilon$ *of* $(t_0, q_0, \dot{q}_0)$, *the functions* $f^i(t, q, \dot{q})$ *have continuous partial derivatives of as many orders as are possessed by the functions* $F_i$ *in* $(P_0)_\varepsilon$, *i.e.*,

$$f^i \in \mathscr{C}^m[(N_0)_\varepsilon]. \tag{1.1.17}$$

The functions $f^i(t, q, \dot{q})$ are customarily termed *implicit functions.* System (1.1.12) will be termed the *kinematical form in configuration space.*[6]

Theorem 1.1.1 essentially indicates that if a solution $P_0 = (t_0, q_0, \dot{q}_0, \ddot{q}_0)$ of Equations (1.1.1) is known and conditions (1) and (2) above are verified, then $P$ can be considered as part of a larger, continuous set of solutions.

In turn, it is possible to prove that when the conditions of Theorem 1.1.1 hold at each and every point of a region $R^{3n+1}$, then the solutions $P$ can be considered as part of a larger set defined in the interior[7] of $R^{3n+1}$.

In the following, if $T$ is a set of values of time $t$, the notation $T_\delta$ indicates the set of points $t$ characterized by $|t - t_0| \leq \delta$ with at least one value $t_0 \in T$, and similarly for the case of more variables.

**Theorem 1.1.2** (Global Existence of Implicit Functions).[8] *Let* $R^{3n+1}$ *be a set of points* $(t, q(t), \dot{q}(t), \ddot{q}(t))$ *defined by a set of functions* $q^k(t)$, $\dot{q}^k(t)$, *and* $\ddot{q}^k(t)$, *which are single-valued and continuous in a bounded and closed region* $T$ *of* $t$ *space. Suppose that*

(1) *the functions* $F_i(t, q, \dot{q}, \ddot{q})$ *are of class* $\mathscr{C}^m$, $m \geq 1$, *in a neighborhood* $(R^{3n+1})_\varepsilon$ *of* $R^{3n+1}$, *i.e.*,

$$F_i \in \mathscr{C}^m[(R^{3n+1})_\varepsilon], \tag{1.1.18}$$

(2) *the functional determinant* $|\mathscr{H}|$ *is non-null everywhere in* $R^{3n+1}$, *i.e.*,

$$|\mathscr{H}|(R^{3n+1}) \neq 0, \tag{1.1.19}$$

(3) *Equations* (1.1.1) *are identically verified in* $R^{3n+1}$, *i.e.*,

$$F_i(R^{3n+1}) = 0. \tag{1.1.20}$$

[6] We must mention the fact that Equations (1.1.12) are customarily referred to as the *canonical form* of system (1.1.1) [see for instance, Rectorys (1969, page 817)]. This terminology, however, could be misleading in our context due to the fact that in mechanics the term "canonical" relates to phase space formulations, while Equations (1.1.12) belong, by assumption, to configuration space formulations. The term *kinematical form* for Equations (1.1.12) has been suggested to me by A. Shimony. The term *normal form* will be used for systems of *first-order* differential equations in the form $dy^k/dx = f^k(x, y)$ [see, for instance, Rektorys, (1969, page 818)].

[7] This restriction avoids certain delicate aspects related to frontier points which are not essential for our analysis.

[8] See, for instance, Bliss (1946).

*Then there exists a bounded and closed region* $B^{2n+1} \subset R^{3n+1}$ *with points* $(t, q, \dot{q})$ *and neighborhoods*

$$(B^{2n+1})_\delta = \{t, q, \dot{q} \mid |t - t_0| \leq \delta, |q - q_0| \leq \delta, |\dot{q} - \dot{q}_0| \leq \delta\}, \quad (1.1.21)$$

$$(t_0, q_0, \dot{q}_0) \in B_{2n+1},$$

*and a unique system*

$$\ddot{q}^i = f^i(t, q, \dot{q}), \quad (1.1.22)$$

*such that the following holds.*

($a'$) *The functions* $f^i$ *are single-valued and continuous in* $(B^{2n+1})_\delta$, *i.e.,*

$$f^i \in \mathscr{C}^0[(B^{2n+1})_\delta]. \quad (1.1.23)$$

($b'$) *The points* $(t, q, \dot{q}, f)$ *which they define in* $R^{3n+1}$ *satisfy Equations* (1.1.1), *i.e.,*

$$F_i(t, q, \dot{q}, f) = 0, \qquad (t, q, \dot{q}, f) \in (R^{3n+1})_\varepsilon. \quad (1.1.24)$$

($c'$) *There exists a constant* $\varepsilon$ *such that for each* $(t, q, \dot{q}) \in (B^{2n+1})_\delta$ *the set* $[t, q, \dot{q}, f(t, q, \dot{q})]$ *is the only solution of Equations* (1.1.1) *satisfying the inequalities*

$$f^i - \varepsilon \leq \ddot{q}^i \leq f^i + \varepsilon. \quad (1.1.25)$$

($d'$) *At all points of* $B^{2n+1}$, $\ddot{q}^i$ *coincide with* $f^i(t, q, \dot{q})$, *i.e.,*

$$\ddot{q}^i = f^i(t, q, \dot{q}), \qquad (t, q, \dot{q}) \in B^{2n+1}. \quad (1.1.26)$$

($e'$) *In a sufficiently small neighborhood* $(B^{2n+1})_\delta$, *the functions* $f^i$ *have continuous partial derivatives of as many orders as are possessed by the function* $F_i$ *in* $R^{3n+1}$, *i.e.,*

$$f^i \in \mathscr{C}^m[(B^{2n+1})_\delta]. \quad (1.1.27)$$

In practice, region (1.1.21) can often be reformulated in terms of two values of time, $t_1$ and $t_2$, in the interior of $R^{3n+1}$. Then the implicit functions $f^i(t, q, \dot{q})$, when they exist, can be defined for all values $t, q(t), \dot{q}(t), t \in (t_1, t_2)$ in the interior of $R^{3n+1}$.

On practical grounds, when a given system (1.1.1) is assigned, it is often advisable to first check whether the conditions of Theorem 1.1.1 or 1.1.2 are verified. Indeed, when this is the case, the system can be transformed into the *equivalent* form of Equations (1.1.12), which is particularly significant in view of its *uniqueness* guaranteed by the same Theorem 1.1.1.[9]

From a Newtonian viewpoint, the implicit functions are often proportional to the acting forces. When this is the case, the anticipated restrictions on the implicit functions for the existence of a Lagrangian are restrictions on the acting forces (see Section 3.7).

[9] This uniqueness of Equations (1.1.12) should be compared with the nonuniqueness of Equations (1.1.1) in the sense that there may exist different functions $F_i'$ such that the totality of solutions of $F_i = 0$ and $F_i' = 0$ (when definable) coincide.

The existence, uniqueness, and continuity theorems for solutions[10] are generally formulated for systems of first-order ordinary differential equations.

Customarily, one introduces $2n$ variables, say $a^\mu$, $\mu = 1, 2, \ldots, 2n$, defined by

$$a^\mu = \begin{cases} q^\mu & \mu = 1, 2, \ldots, n \qquad (1.1.28a) \\ y^{\mu-n} & \mu = n+1, n+2, \ldots, 2n, \qquad (1.1.28b) \end{cases}$$

and $2n$ functions, say $\Xi^\mu(t, a^\alpha)$, defined by

$$\Xi^\mu = \begin{cases} y^\mu & \mu = 1, 2, \ldots, n, \\ f^{\mu-n} & \mu = n+1, n+2, \ldots, 2n. \end{cases} \qquad (1.1.29)$$

Then the system of $2n$ first-order equations in the *normal form*[11]

$$\dot{a}^\mu - \Xi^\mu(t, a^\alpha) = 0 \qquad \mu = 1, 2, \ldots, 2n, \qquad (1.1.30)$$

is equivalent to system (1.1.12) and, thus, to system (1.1.1). We shall now study the question of the existence, uniqueness, and continuity of solutions for systems of type (1.1.30).

We must mention, for completeness, that prescriptions more general than Equation (1.1.28) exist, are equally admissible, and are often advisable for our context.

Suppose, for instance, that $n$ new variables $y'_k$ and a set of prescriptions

$$G_i(t, q, \dot{q}, y') = 0 \qquad (1.1.31)$$

are assigned in such a way as to have a one-to-one mapping of points $(t, q, \dot{q})$ of the region $R^{2n+1}$ into points $(t, q, y')$ of an "image" region $\tilde{R}^{2n+1}$. This implies that the functions $G_i$ of Equation (1.1.31) satisfy Theorem 1.1.1, so that the normal forms

$$\dot{q}^i = \tilde{g}^i(t, q, y) \qquad (1.1.32)$$

exist and are unique, single-valued, and of class $\mathscr{C}^1$. The substitution of Equation (1.1.32) into Equation (1.1.1) with the identifications

$$a^\mu = \begin{cases} q^\mu \\ y'^\mu \end{cases} \qquad (1.1.33)$$

then yields a system of $2n$ first-order equations in the variables $a^\mu$, which can be reduced to a form of type (1.1.30) [but, in general, with functions $\Xi^\mu$ different from those of Equations (1.1.29)].

We must stress the fact that, in the above approach, the variables $(q, y)$ are not necessarily canonically conjugate in the sense that Equations (1.1.30) are not necessarily derivable from Hamilton's equations.

[10] See, for instance, Bliss (1946, Appendix A), Brauer and Nohel (1969, Chapter 3), Akhiezer (1962, Section I.9), and Rektorys (1969, Chapter 17). For a more advanced account see, for instance, Coddington and Levinston (1955) and Friedrichs (1965). Alternative versions are also given in the charts at the end of this chapter.

[11] See footnote 6 on page 19 for this terminology.

However, alternatives (1.1.28) and (1.1.33) are patterned along the canonical approach. Indeed, when system (1.1.1) can be represented in terms of Lagrange's equations with a Lagrangian $L(t, q, \dot{q})$, then prescriptions (1.1.31) become

$$G_i = p_i - \frac{\partial L}{\partial \dot{q}^i} = 0 \qquad y_i = p_i \tag{1.1.34}$$

and, in this case, the variables $(a^\mu) = (q, p)$ are indeed canonically conjugate.

In particular, when a Lagrangian for Equations (1.1.12) exists, then prescriptions (1.1.34) may reduce to the form (1.1.28b), i.e.,

$$G_i = p_i - \dot{q}_i = 0, \tag{1.1.35}$$

and in this case the variables $(a^\mu) = (q, p)$ are again canonically conjugate.

Therefore, our approach consists of constructing the not necessarily canonical system[12] of $2n$ first-order ordinary differential Equations (1.1.30), which is equivalent to system (1.1.1). However this system is constructed in such a way that when a Lagrangian (or a Hamiltonian) exists, the variables $a^\mu$ can be embodied in a canonical structure[13] without any formal modification of the procedure.

We shall term Equations (1.1.30) the *normal form* of system (1.1.1). We shall term Equations (1.1.30) the *normal form in phase space* only when the existence of a canonical structure is either established or assumed.

By a *solution* (*integral*) of Equations (1.1.30), we mean a set of functions $a^\mu(t)$ that are (at least) of class $\mathscr{C}^1$ on (at least) a neighborhood $(t_0)_\varepsilon$ of $t_0$ and which, when substituted together with their derivatives in Equations (1.1.30), all such equations are identically satisfied in $(t_0)_\varepsilon$.

By the *general solution* (*general integral*) of Equations (1.1.30), we mean the system of functions

$$a^\mu = a^\mu(t; c) \tag{1.1.36}$$

depending on $2n$ constants $c^\mu$, which are continuous together with (at least) their (time) derivatives $\dot{a}^\mu(t; c)$ on (at least) a neighborhood $(t_0)_\varepsilon$ of $t_0$, and which satisfy Equations (1.1.30) identically for all values of $t \in (t_0)_\varepsilon$ and for all values of $c^\mu$ in neighborhoods $(c_0^\mu)_\varepsilon$ of given values $c_0^\mu$.

Let the modulus of $a$ be defined by the *Euclidean norm*

$$\operatorname{mod} a = \left[ \sum_{\mu=0}^{2n} (a^\mu)^2 \right]^{1/2}. \tag{1.1.37}$$

Without proof, we quote the following theorem.

**Theorem 1.1.3** (Local Existence of a Solution Through Initial Conditions). *Suppose the following conditions are satisfied.*

[12] That is, a system of $2n$ first-order equations which is not necessarily representable in terms of Hamilton's equations.

[13] A geometrical definition of canonical (or symplectic) structure will be given later on in Chart 2.3.

(1) *The functions* $\Xi^\mu(t, a)$ *are single-valued and continuous in a neighborhood* $I_\varepsilon$ *of point* $I = (t_0; c_0)$.

(2) *There exists a constant* $k$ *such that for every pair of points* $(t, a)$, $(t, a') \in I_\varepsilon$ *the* Lipschitz condition

$$|\Xi^\mu(t, a) - \Xi^\mu(t, a')| < \frac{k}{(2n)^{1/2}} \operatorname{mod}(a - a') \tag{1.1.38}$$

*is verified. Then there exists in* $I_\varepsilon$ *one and only one set of* $2n$ *functions* $a^\mu(t)$, *which is the solution of Equations* (1.1.30) *in the interior of* $I_\varepsilon$ *and which satisfies the initial conditions*

$$a^\mu(t_0) = c_0^\mu. \tag{1.1.39}$$

The relationship between a solution $a^\mu(t)$ and the initial conditions (1.1.39) is made clearer by the following theorem.

**Theorem 1.1.4** (Local Embedding of a Solution into a $2n$-Parameter Family of Solutions). *If the functions* $\Xi^\mu(t, a)$ *satisfy conditions* (1) *and* (2) *of Theorem* 1.1.3, *and if* $a^\mu(t)$ *is a solution of Equations* (1.1.30) *in* $I_\varepsilon$ *satisfying the initial conditions* (1.1.39), *then at a value* $t \in I_\varepsilon$ *there exists one and only one solution*

$$a^\mu = a^\mu(t; c) \tag{1.1.40}$$

*of Equations* (1.1.30) *which passes through every point* $(t; c) \in I_\varepsilon$.

*The functions* $a^\mu(t; c)$ *and* $\dot{a}^\mu(t; c)$ *are continuous and satisfy Equations* (1.1.30) *identically for all* $(t; c) \in I_\varepsilon$.

In essence, the above theorem establishes that when a solution is known and conditions (1) and (2) of Theorem 1.1.3 are satisfied, then such a solution can always be *embedded* in a family of solutions characterized by neighboring initial points.

The reformulations of Theorems 1.1.3 and 1.1.4 in configuration space is straightforward.

For instance, when prescriptions (1.1.28) are used, the solution $a^\mu(t; c)$ can be explicitly written

$$a^\mu(t; c) = \begin{cases} q^\mu(t; u, v) & \text{(1.1.41a)} \\ \dot{q}^{\mu-n}(t; u, v) & \text{(1.1.41b)} \end{cases}$$

with initial conditions

$$q^k(t_0; u, v) = u_0^k, \qquad \dot{q}^k(t_0; u, v) = v_0^k, \tag{1.1.42}$$

where $c^\mu$ and $c_0^\mu$ have been separated into the values $(u, v)$ and $(u_0, v_0)$, respectively.

When the more general prescriptions of type (1.1.33) are used, one can arrive at equivalent results, usually through simple algebraic manipulations.

The family (1.1.41a) of solutions depending on the $2n$ parameters $(u, v)$ will be referred to as an $\infty^{2n}$ *family of possible paths in configuration space*, and will be denoted by

$$\tau^{2n}_{(t,q)} = \{q \mid q = q(t; u, v),\ t \in (t_0)_\varepsilon,\ (u,\ v) \in (u_0, v_0)_\varepsilon\}. \tag{1.1.43}$$

The family (1.1.40) is then also an $\infty^{2n}$ family of possible paths and will be denoted by

$$\tilde{\tau}^{2n}_{(t,a)} = \{a^\mu \mid a^\mu = a^\mu(t; c),\ t \in (t_0)_\varepsilon,\ c^\mu \in (c^\mu_0)_\varepsilon\}. \tag{1.1.44}$$

When the family $\tau^{2n}_{(t,q)}$ is restricted to satisfy all $2n$ initial conditions, we shall call it *the actual path.*

The existence and uniqueness Theorem 1.1.3 is of *local* character, that is, it deals with solutions in a neighborhood $I_\varepsilon$ of the point $I = (t_0; c_0)$.

We shall now briefly consider the question of the global existence of a solution, e.g., when joining two distinct points.

Consider a region $\tilde{R}^{2n+1}$ of points $(t, a)$ such that all points of the *closed box*

$$\tilde{B}^{2u+1} = \{(t, a) \mid |t - t_0| \leq K^0,\ |a^\mu - a^\mu_0| \leq K^\mu\} \tag{1.1.45}$$

for $(t_0, c_0) \in \tilde{R}^{2n+1}$ and for suitably chosen constants $K^0$ and $K^\mu$ lies entirely in the interior of $\tilde{R}^{2n+1}$.

Examples of *open boxes* are the "entire space" $-\infty < t,\ a^\mu < +\infty$, the "half space" $0 < t < +\infty,\ 0 < a^\mu < +\infty$, or the "infinite strips" $-\infty < t < +\infty,\ -|K| < a^\mu < +|K|$, where $K$ is a number. Examples of closed boxes can be constructed in a similar way.

All considered paths $a^\mu(t)$ for, say, $t \in (t_1, t_2)$, are such that all admissible elements $(t, a)$ are in the interior of $\tilde{R}^{2n+1}$.

The following existence theorem holds.

**Theorem 1.1.5** (Global Existence of a Solution Through Initial Conditions). *Let*

(1) $\Xi^\mu$ *and* $\partial\Xi^\mu/\partial a^\nu$ *be continuous in the closed box* $\tilde{B}^{2n+1}$, *i.e., Equation* (1.1.45), *for some positive constants* $K^0$ *and* $K^\mu$,

(2) *the bounds*

$$|\Xi^\mu(t, a)| \leq M, \qquad \left|\frac{\partial\Xi^\mu(t, a)}{\partial a^\nu}\right| \leq M \tag{1.1.46}$$

*be everywhere verified in* $\tilde{B}^{2n+1}$, *and*

(3) $\sigma$ *be the smaller of the numbers* $K^0$ *and* $K^\mu/M$. *Then the sequence of successive approximations*

$$\begin{aligned} a^\mu_0(t) &= c^\mu_0, \\ a^\mu_1(t) &= c^\mu_0 + \int_{t_0}^{t} dt\ \Xi^\mu[t, a^\mu_0(t)], \\ &\vdots \\ a^\mu_n(t) &= c^\mu_0 + \int_{t_0}^{t} dt\ \Xi^\mu[t, a^\mu_{n-1}(t)] \end{aligned} \tag{1.1.47}$$

*converges (uniformly) in the interval* $|t - t_0| < \sigma$ *to a unique solution*[14] $a^\mu(t)$ *of Equations* (1.1.30), *which satisfies the intial conditions*

$$a^\mu(t_0) = c_0^\mu. \tag{1.1.48}$$

Notice that the assumptions of continuity of $\Xi^\mu$ and $\partial\Xi^\mu/\partial a^\nu$ in $\tilde{B}^{2n+1}$ assure the fulfillment of the Lipschitz conditions (1.1.38) (however, the inverse statement is not generally true; namely, a function which satisfies the Lipschitz condition need not be of class $\mathscr{C}^1$).

In essence, assumptions (1), (2), and (3) of Theorem 1.1.5 guarantee that the *initial value problem*

$$\begin{gathered} \dot{a}^\mu = \Xi^\mu(t, a), \qquad \mu = 1, 2, \ldots, 2n \\ a^\mu(t_0) = c_0^\mu \end{gathered} \tag{1.1.49}$$

is equivalent to that of finding continuous functions $a^\mu(t)$ defined in some interval containing $t_0$ that are the solution of the *integral equations of Volterra type*,

$$a^\mu(t) = c_0^\mu + \int_{t_0}^{t} dt\, \Xi^\mu(t, a). \tag{1.1.50}$$

Then, such assumptions guarantee the uniform convergence of sequence (1.1.47).

The uniqueness of the solution can be proved under weaker continuity conditions than those of Theorem 1.1.5. But such refined results are not needed for our analysis. Besides, a single set of conditions which guarantees both the existence and the uniqueness of the solutions of Equations (1.1.30) is advantageous from a practical viewpoint.

Embedding Theorem 1.1.4 can also be formulated at large:

**Theorem 1.1.6** (Global Embedding of a Solution into a $2n$-Parameter Family of Solutions). *Suppose that the initial value problem* (1.1.49) *admits a solution* $a^\mu(t)$ *for all* $t$ *in an interval* $(t_1, t_2)$ *containing* $t_0$. *If in some neighborhood* $(\tilde{R}^{2n+1})_\varepsilon$ *of* $(t, a(t))$ *for* $t \in (t_1, t_2)$ *all functions* $\Xi^\mu$ *are of class* $\mathscr{C}^1$, *then* $a^\mu(t)$ *can be embedded in an* $\infty^{2n}$ *family* $\tilde{\tau}^{2n}_{(t,a)}$ *of solutions* $a^\mu(t; c)$ *for all values of* $c^\mu$ *in a neighborhood* $(c_0^\mu)_\varepsilon$ *of* $c_0^\mu$, *only one element of which satisfies the initial conditions.*

It should be noted that the above theorem is often formulated by demanding that the functions $\Xi^\mu$ are continuous and satisfy the Lipschitz condition (1.1.38) for some *Lipschitz constant* $K$.

The formulation of Theorem 1.1.6 is based on the fact that in our analysis of Newtonian systems we shall not encounter functions $\Xi^\mu$ which satisfy the Lipschitz condition but are not of class $\mathscr{C}^1$, or which are of class $\mathscr{C}^1$ in $a^\alpha$ but not in $t$.

[14] Here a "solution" is a set of functions $a^\mu(t)$ such that $a^\mu(t)$ satisfies Equations (1.1.30) identically in the considered box for all $t$ with $|t - t_0| < \sigma$.

To properly deal with the equations of variations of system (1.1.1) (to be introduced in Chapter 2) we finally need the existence theorem of the partial derivatives with respect to parameters of a solution of Equations (1.1.49).

**Theorem 1.1.7** (Global Existence of the Derivatives with Respect to Parameters).[15] *Suppose that the initial value problem* (1.1.49) *admits an* $\infty^{2n}$-*parameter family of solutions* $a^\mu(t; c)$ *for all* $t$ *in an interval* $(t_1, t_2)$ *containing* $t_0$ *and all* $c^\mu$ *in a neighborhood* $(c_0^\mu)_\varepsilon$ *of* $c_0^\mu$. *If all functions* $\Xi^\mu$ *are of class* $\mathscr{C}^m$, $m > 1$, *in a neighborhood* $(\tilde{R}^{2n+1})_\varepsilon$ *of the solution* $(t, a(t))$, $t \in (t_1, t_2)$, then $a^\mu(t; c)$ *possess continuous partial derivatives up to and including the order* $m$ *with respect to all parameters* $c^\mu$ *in* $(c_0^\mu)_\varepsilon$ *at all times* $t \in (t_1, t_2)$.

In the following, for convenience of notation, we shall refer to all theorems quoted in this section as the *existence theorems* or *existence theory*.

It should be mentioned that our analysis of Newtonian systems can essentially be conducted on the basis of Theorem 1.1.2 on the global existence, uniqueness, and continuity of the implicit functions, Theorem 1.1.6 on the global existence, uniqueness, and continuity of a solution through initial conditions, and Theorem 1.1.7 on the global existence and continuity of the derivatives with respect to the parameters.

For the reader's convenience, we give in Charts 1.1, 1.2, and 1.3 a simplified version of the above theorems and outline their application to Newtonian systems.

## 1.2 Calculus of Differential Forms, Poincaré Lemma, and Its Converse

The problem of the existence of a Lagrangian can be reduced, as we shall see in Chapter 3, to the study of the integrability conditions for a certain system of partial differential equations.

One of the most effective mathematical tools for the study of the integrability conditions is the *calculus of differential forms* in general and the so-called *Converse of the Poincaré Lemma* in particular.

In this section, we review some basic aspects of the calculus of differential forms specialized, for the reader's convenience, to the case where the local coordinates are the generalized coordinates $q^i$. It should be indicated from the outset that all the formulations considered in this section also apply to the case where the local coordinates are the generalized velocities $\dot{q}^i$ or, for that matter, any set of (independent) variables.

Our review closely follows the presentation by Lovelock and Rund (1975). Due to the elementary nature of our analysis, the interested reader is urged to study this reference (or some other reference[16]) for an in-depth treatment of the subject.

[15] See, for instance, Akhiezer (1962, Section I-8).
[16] See, for instance, Flanders (1963).

Let $M^n$ be a differentiable manifold[17] with local coordinates $q^i$, $i = 1, 2, \ldots, n$. A generic tensor on $M^n$ with $r$ contravariant and $s$ covariant indices is customarily written $T^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ and termed a tensor of type $(r, s)$. In particular, a scalar is a (0, 0)-tensor, a covariant vector is a (0, 1)-tensor, a contravariant vector is a (1, 0)-tensor, etc.

The contraction of an infinitesimal displacement $dq^i$, i.e., a (1, 0)-tensor, with a (0, 1)-tensor $A_i(q)$,

$$A^{(1)} = A_i(q) dq^i, \tag{1.2.1}$$

is termed a 1-*form* (or *Pfaffian forms*).[18] The addition of 1-forms is carried out according to the conventional rule, e.g.,

$$A^{(1)} + B^{(1)} = A_i\, dq^i + B_i\, dq^i = (A_i + B_i)\, dq^i. \tag{1.2.2}$$

The multiplication of 1-forms, however, demands a new operation, called the *exterior product* and often denoted by the symbol $\wedge$, which preserves the distributive law of ordinary multiplication but obeys the anticommutative rather than the commutative law according to the rule:

$$\begin{aligned} A^{(1)} \wedge B^{(1)} &= (A_i\, dq^i) \wedge (B_j\, dq^j) \\ &= A_i B_j\, dq^i \wedge dq^j \\ &= -A_i B_j\, dq^j \wedge dq^i \\ &= \tfrac{1}{2}(A_i B_j - B_i A_j)\, dq^i \wedge dq^j. \end{aligned} \tag{1.2.3}$$

The structure emerging from the above product is termed a 2-*form*.

Repeated use of the exterior product then induces the so-called *p-forms* ($p \leq n$), which are scalars characterized by the contraction of the antisymmetric $dq^{i_1} \wedge \cdots \wedge dq^{i_p}$, $(p, 0)$-tensor with a $(0, p)$-tensor $A_{i_1 \cdots i_p}(q)$, i.e.,

$$A^{(p)} = A_{i_1 \cdots i_p}(q)\, dq^{i_1} \wedge \cdots \wedge dq^{i_p}. \tag{1.2.4}$$

The ordinary concept of a derivative is now generalized to that of the *exterior derivative* of a $p$-form, which is defined by the scalar $(p + 1)$-form

$$dA^{(p)} = \frac{\partial A_{i_1 \cdots i_p}}{\partial q^k}\, dq^k \wedge dq^{i_1} \wedge \cdots \wedge dq^{i_p}. \tag{1.2.5}$$

For the case of a 1-form, we have

$$\begin{aligned} dA^{(1)} &= \frac{\partial A_{i_1}}{\partial q^{i_2}}\, dq^{i_2} \wedge dq^{i_1} \\ &= -\frac{\partial A_{i_2}}{\partial q^{i_1}}\, dq^{i_2} \wedge dq^{i_1} \\ &= \frac{1}{2}\left(\frac{\partial A_{i_2}}{\partial q^{i_1}} - \frac{\partial A_{i_1}}{\partial q^{i_2}}\right) dq^{i_1} \wedge dq^{i_2}, \end{aligned} \tag{1.2.6}$$

[17] For differentiable manifolds, see Chart 2.1.

[18] It should be indicated that a 1-form or, more generally, a $p$-form, need not necessarily be a scalar. For references on Pfaffians, see von Weber (1900), Goursat (1922), and Cartan (1922 and 1937).

where the last equality results from the antisymmetric nature of the (2.0)-tensor $dq^i \wedge dq^k$. Equation (1.2.5) can thus be written, more generally,

$$dA^{(p)} = (-1)^p \frac{\partial A_{i_1 \cdots i_p}}{\partial q^{p+1}} dq^{i_1} \wedge \cdots \wedge dq^{i_p} \wedge dq^{i_{p+1}}. \tag{1.2.7}$$

The algebraic manipulation of $p$-forms and their exterior derivatives is considerably simplified by the use of the so-called *generalized Kronecker delta.* This is a $(p, p)$-tensor defined by the determinant

$$\delta^{i_1 \cdots i_p}_{j_1 \cdots j_p} = \begin{vmatrix} \delta^{i_1}_{j_1} \cdots \delta^{i_1}_{j_p} \\ \vdots \\ \delta^{i_p}_{j_1} \cdots \delta^{i_p}_{j_p} \end{vmatrix}, \qquad p \leq n. \tag{1.2.8}$$

For $p = 1$, we recover from the above definition the ordinary Kronecker delta $\delta^i_j$. However, for $p = 2$ we have

$$\delta^{i_1 i_2}_{j_1 j_2} = \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} - \delta^{i_1}_{j_2} \delta^{i_2}_{j_1}. \tag{1.2.9}$$

For an arbitrary $p(\leq n)$, $\delta^{i_1 \cdots i_p}_{j_1 \cdots j_p}$ is the sum of $p!$ terms, each of which is the product of $p$ ordinary deltas.

If any two contravariant (covariant) indices are identical, then the generalized Kronecker delta is null from a known property of determinants. For $p > n$, at least two indices must coincide and, therefore,

$$\delta^{i_1 \cdots i_p}_{j_1 \cdots j_p} \equiv 0, \qquad p > n. \tag{1.2.10}$$

When the contravariant and covariant indices are pairwise equal, the *normalization rule*

$$\delta^{i_1 \cdots i_p}_{i_1 \cdots i_p} = \frac{n!}{(n-p)!} \tag{1.2.11}$$

can be proved.[19] The determinant of a $p \times p$ matrix $(m^i_j)$ can be written[19]

$$|m^i_j| = \frac{1}{p!} \delta^{i_1 \cdots i_p}_{j_1 \cdots i_p} m^{j_1}_{i_1} \cdots m^{j_p}_{i_p}. \tag{1.2.12}$$

Finally, the identities

$$\delta^{i_1 \cdots i_p}_{j_1 \cdots j_p} dq^{j_1} \wedge \cdots \wedge dq^{j_p} = p!\, dq^{i_1} \wedge \cdots \wedge dq^{i_p} \tag{1.2.13}$$

also hold.[19]

By using Equations (1.2.9), we can now write a 2-form as follows.

$$\begin{aligned} A^{(2)} &= \tfrac{1}{2}(A_{i_1 i_2} - A_{i_2 i_1})\, dq^{i_1} \wedge dq^{i_2} \\ &= \frac{1}{2!} \delta^{i_1 i_2}_{j_1 j_2} A_{i_1 i_2}\, dq^{i_1} \wedge dq^{j_2}. \end{aligned} \tag{1.2.14}$$

[19] See Problem 1.4.

More generally, a $p$-form (1.2.4) can be written in terms of the generalized Kronecker delta,

$$A^{(p)} = \frac{1}{p!}\,\delta^{i_1\cdots i_p}_{j_1\cdots j_p} A_{i_1\cdots i_p}\, dq^{i_1} \wedge \cdots \wedge dq^{i_p}. \tag{1.2.15}$$

Similarly, for the exterior derivative (1.2.6) we have

$$\begin{aligned} dA^{(1)} &= \frac{1}{2}\left(\frac{\partial A_{i_2}}{\partial q^{i_1}} - \frac{\partial A_{i_1}}{\partial q^{i_2}}\right) dq^{i_1} \wedge dq^{i_2} \\ &= \frac{1}{2!}\,\delta^{i_1 i_2}_{j_1 j_2}\frac{\partial A_{i_1}}{\partial q^{i_2}}\, dq^{j_1} \wedge dq^{j_2}, \end{aligned} \tag{1.2.16}$$

and, more generally, for the exterior derivative (1.2.7) we can write

$$dA^{(p)} = \frac{(-1)^p}{(p+1)!}\,\delta^{i_1\cdots i_{p+1}}_{j_1\cdots j_{p+1}}\frac{\partial A_{i_1\cdots i_p}}{\partial q^{i_{p+1}}}\, dq^{j_1} \wedge \cdots \wedge dq^{j_{p+1}}. \tag{1.2.17}$$

It then follows that the necessary and sufficient conditions for a $p$-form (1.2.15) or for its exterior derivative (1.2.17) to be null are

$$\delta^{i_1\cdots i_p}_{j_1\cdots j_p} A_{i_1\cdots i_p} = 0, \tag{1.2.18a}$$

$$\delta^{i_1\cdots i_{p+1}}_{j_1\cdots j_{p+1}}\frac{\partial A_{i_1\cdots i_p}}{\partial q^{i_{p+1}}} = 0, \tag{1.2.18b}$$

$$j_1, \ldots, j_{p+1} = 1, 2, \ldots, n,$$

respectively.

The above property can easily be seen for the case of a 2-form (1.2.14). Indeed, we can write

$$\begin{aligned} A^{(2)} &= \tfrac{1}{2}(A_{i_1 i_2} - A_{i_2 i_1})\, dq^{i_1} \wedge dq^{i_2} \\ &= \frac{1}{2!}\,\delta^{i_1 i_2}_{j_1 j_2} A_{i_1 i_2}\, dq^{i_1} \wedge dq^{j_2} \\ &= \sum_{i_1 < i_2} (A_{i_1 i_2} - A_{i_2 i_1})\, dq^{i_1} \wedge dq^{i_2} = 0. \end{aligned} \tag{1.2.19}$$

The conditions

$$\delta^{i_1 i_2}_{j_1 j_2} A_{i_1 i_2} = A_{j_1 j_2} - A_{j_2 j_1} = 0, \qquad j_1, j_2 = 1, 2, \ldots, n, \tag{1.2.20}$$

follow from the linear independence of the elements $dq^{i_1} \wedge dq^{i_2}$ for $i_1 < i_2$. Equations (1.2.20) then recover the known property that a necessary and sufficient condition for the contraction of a tensor $A_{i_1 i_2}$ with an antisymmetric tensor, say $B^{i_1 i_2}$, to be identically null is that $A_{i_1 i_2}$ be symmetric in its indices. Conditions (1.2.18) are then a generalization of the above familiar case to higher orders.

We are now equipped to formulate the Poincaré Lemma. For this purpose, we recall that a $p$-form (1.2.15) is termed *exact* if there exists a $(p-1)$-form, say $B^{(p-1)}$, called a *primitive form*, such that

$$A^{(p)} = dB^{(p-1)}. \tag{1.2.21}$$

Also, a $p$-form (1.2.15) is termed *closed* whenever its exterior derivative is identically null, i.e.,

$$dA^{(p)} \equiv 0. \tag{1.2.22}$$

We then have the following.

**Lemma 1.2.1** (Poincaré Lemma). *Every p-form* $A^{(p)} = A_{i_1 \cdots i_p} dp^{i_1} \wedge \cdots \wedge dq^{i_p}$ *on an n-dimensional differentiable manifold* $M^n$ *with local coordinates* $q^i (i = 1, 2, \ldots, n)$, *which is exact and of (at least) class* $\mathscr{C}^1$ *in a region* $R^n$ *of points* $q^i$, *is closed in* $R^n$.

PROOF. From the assumption that $A^{(p)}$ is exact, it follows that

$$\begin{aligned} dA^{(p)} &= d(dB^{(p-1)}) \\ &= \frac{\partial^2 B_{i_1 \cdots i_{p-1}}}{\partial q^{k_1} \partial q^{k_2}} dq^{k_1} \wedge dq^{k_2} \wedge dq^{i_1} \wedge \cdots \wedge dq^{i_{p-1}} = 0, \end{aligned} \tag{1.2.23}$$

where the last equality results from the symmetry of the tensor $\partial B_{i_1 \cdots i_{p-1}} / \partial q^{k_1} \partial q^{k_2}$ in the $k_1$ and $k_2$ indices. *Q.E.D.*

A simple illustration of the above lemma is given by the case $p = 1$, for which

$$\begin{aligned} dA^{(1)} &= d(dA^{(0)}) = d(d\phi) \\ &= d\left(\frac{\partial \phi}{\partial q^{i_1}} dq^{i_1}\right) = \frac{\partial^2 \phi}{\partial q^{i_1} \partial q^{i_2}} dq^{i_1} \wedge dq^{i_2} = 0 \end{aligned} \tag{1.2.24}$$

This is equivalent to the well-known property that (under the minimal continuity conditions indicated above) the curl of the gradient of a scalar is identically null. Equations (1.2.22) then express the generalization of this property to higher orders $p = 2, 3, 4, \ldots$.

As indicated at the beginning of this section, what is particularly significant for the study of the integrability conditions is the Converse of the Poincaré Lemma, rather than the Lemma per se. In order to formulate and prove the Converse, we need a more adequate characterization of the region of definition of the $p$-forms.

A region[20] $R^{*n}$ on $M^n$ is termed *star-shaped* when, jointly with a given open and connected set of points $q^i (i = 1, 2, \ldots, n)$, all points $q'^i = \tau q^i, 0 \leq \tau \leq 1$, are also contained in $R^{*n}$. Notice that such a region contains the local origin $q^i = 0$.

[20] For the definition of a region, see page 16.

**Lemma 1.2.2** (Converse of the Poincaré Lemma). *Every p-form* $A^{(p)} = A_{i_1 \cdots i_p}\, dq^{i_1} \wedge \cdots \wedge dq^{i_p}$ *on an n-dimensional differentiable manifold* $M^n$ *with local coordinates* $q^i (i = 1, 2, \ldots, n)$ *which is closed, well defined, and of (at least) class* $\mathscr{C}^1$ *in a star-shaped region* $R^{*n}$ *on* $M^n$, *is exact on* $R^{*n}$.

PROOF. Introduce the following operation on $p$-forms

$$\mathscr{D}A^{(p)} = \sum_{r=1}^{p} (-1)^{r-1} \left[ \int_0^1 d\tau\, \tau^{p-1} A_{i_1 \cdots i_p}(\tau q) \right] q^{i_r}\, dq^{i_1} \wedge \cdots \wedge dq^{i_{r-1}} \wedge dq^{i_{r+1}} \wedge \cdots \wedge dq^{i_p}$$

$$= \frac{1}{(p-1)!} \left[ \int_0^1 d\tau\, \tau^{p-1} A_{i_1 \cdots i_p}(\tau q) \right] q^{j_1} \delta^{i_1 \cdots i_p}_{j_1 \cdots j_p}\, dq^{j_2} \wedge \cdots \wedge dq^{j_p}. \tag{1.2.25}$$

Its exterior derivative reads[21]

$$d(\mathscr{D}A^{(p)}) = \sum_{r=1}^{p} (-1)^{r-1} \left[ \int_0^1 d\tau\, \tau^p \frac{\partial A_{i_1 \cdots i_p}(\tau q)}{\partial q^k} \right] q^{i_r}$$
$$dq^k \wedge dq^{i_1} \wedge \cdots \wedge dq^{i_{r-1}} \wedge dq^{i_{r+1}} \wedge \cdots \wedge dq^{i_p}$$
$$+ p \left[ \int_0^1 d\tau\, \tau^{p-1} A_{i_1 \cdots i_p}(\tau q) \right] dq^{i_1} \wedge \cdots \wedge dq^{i_p}. \tag{1.2.26}$$

But we can also write

$$\mathscr{D}(dA^{(p)}) = -\sum_{r=1}^{p} (-1)^{r-1} \left[ \int_0^1 d\tau\, \tau^p \frac{\partial A_{i_1 \cdots i_p}(\tau q)}{\partial q^k} \right] q^{i_r}$$
$$dq^k \wedge dq^{i_1} \wedge \cdots \wedge dq^{i_{r-1}} \wedge dq^{i_{r+1}} \wedge \cdots \wedge dq^{i_p}$$
$$+ \left[ \int_0^1 d\tau\, \tau^p \frac{\partial A_{i_1 \cdots i_p}(\tau q)}{\partial q^k} \right] q^k\, dq^{i_1} \wedge \cdots \wedge dq^{i_p}. \tag{1.2.27}$$

Therefore, under the assumptions that the $p$-form is well defined and of (at least) class $\mathscr{C}^1$ on $R^{*n}$, the following identity holds.

$$d(\mathscr{D}A^{(p)}) + \mathscr{D}(dA^{(p)}) = \left\{ \int_0^1 d\tau \frac{d}{d\tau} [\tau^p A_{i_1 \cdots i_p}(\tau q)] \right\} dq^{i_1} \wedge \cdots \wedge dq^{i_p}$$
$$= A_{i_1 \cdots i_p}\, dq^{i_1} \wedge \cdots \wedge dq^{i_p}$$
$$= A^{(p)}. \tag{1.2.28}$$

Under the additional assumption that the $p$-form is closed, we have

$$d(\mathscr{D}A^{(p)}) = A^{(p)}. \tag{1.2.29}$$

This establishes the existence of a primitive form

$$B^{(p-1)} = \mathscr{D}A^{(p)} \tag{1.2.30}$$

such that

$$A^{(p)} = dB^{(p-1)} \tag{1.2.31}$$

and completes the proof of the lemma. *Q.E.D.*

[21] See Problem 1.8.

A few comments are now in order. First of all, it should be stressed that on a comparative basis with respect to the Poincaré Lemma, its converse demands a new condition, namely, that the *p-forms are well behaved on a star-shaped region rather than an ordinary region.* This condition is needed to ensure the existence of the integrals of Equations (1.2.25)–(1.2.28). This is clearly a restriction on the types of $p$-forms for which Lemma 1.2.2 applies. The problem of the removal of such a restriction would go outside the scope of this monograph. Therefore, we shall content ourselves with differential forms that obey such a requirement.

Under the condition that the $A^{(p)}$ form is of (at least) class $\mathscr{C}^1$ in $R^{*n}$, the *integrability condition* for the existence of a primitive form $B^{(p-1)}$ is that the $A^{(p)}$ form be closed, i.e., that each and every one of the conditions

$$\delta^{i_1 \cdots i_{p+1}}_{j_1 \cdots j_{p+1}} \frac{\partial A_{i_1 \cdots i_p}}{\partial q^{i_{p+1}}} = 0, \qquad j_1, \ldots, j_{p+1} = 1, 2, \ldots, n \tag{1.2.32}$$

are everywhere identically satisfied in $R^{*n}$.

To summarize, the conditions for the existence of a primitive form (1.2.30) are that

1. The $A^{(p)}$ form is of (at least) class $\mathscr{C}^1$ in a star-shaped region $R^{*n}$, namely, that the tensor $A_{i_r \cdots i_p}(q)$ satisfies this continuity requirement;
2. The $A^{(p)}$ form is well behaved in $R^{*n}$ and, thus, the integral of Equation (1.2.25) exists; and
3. The $A^{(p)}$ form is closed, namely, that each and every one of the integrability conditions (1.2.32) are identically satisfied in $R^{*n}$.

The Converse of the Poincaré Lemma is, therefore, centered on the identification of the conditions under which a primitive form exists. As a consequence, such a lemma will be crucial for the study of the conditions under which a Lagrangian exists, as we shall see in Chapter 3. For initial illustrative applications, see the examples at the end of this chapter.

The significance of Lemma 1.2.2, however, goes beyond the identification of the integrability conditions. Indeed, it also provides a *solution* for the primitive form. More specifically, given a $p$-form satisfying conditions 1, 2, and 3 above, rule (1.2.30) provides a solution for the primitive $B^{(p-1)}$ form. As we shall see in Chapter 3, this property will play a crucial role in the methodology of computing a Lagrangian.

It should be stressed that such a solution is not unique. Indeed, given a solution (1.2.30), one can construct an infinite family of forms according to the rule

$$B'^{(p-1)} = B^{(p-1)} + dC^{(p-2)}, \tag{1.2.33}$$

all of which satisfy Equation (1.2.31) identically in view of property (1.2.23). This fact is significant for the problem of the "degrees of freedom" of a Lagrangian, namely, the construction of equivalent Lagrangians [see Santilli (1979)].

## 1.3 Calculus of Variations, Action Functional, and Admissible Variations

The calculus of variations originated in the eighteenth century with the problem of determining the maxima or minima of definite integrals of known functions. Subsequently, the significance of this problem was identified for several branches of mathematics, physics, and engineering, and the calculus of variations was developed up to the present degree of sophistication, which includes disciplines such as functional analysis, differential geometry, and algebraic topology.

In this section, we shall review the rudiments of the branch of the calculus of variations which, as Caratheodory puts it,[22] is the "servant of mechanics." This review appears to be advisable on the following grounds. (a) The subject of this monograph, the Inverse Problem, was originally developed within the context of the calculus of variations, as indicated in the Introduction. A review of the rudiments of this discipline is, therefore, useful to provide a proper methodological perspective. (b) The approach we have selected for the study of the integrability conditions for the existence of a Lagrangian or Hamiltonian (to be introduced in the next chapter) is based on variational techniques. Therefore, it is useful to identify the admissible variations within their proper methodological context, the calculus of variations. (c) Later, we shall be involved with variational principles [see the charts of Chapter 3 and of Santilli (1979)]. Therefore, it is appropriate to recall that the variational principles customarily used in analytic mechanics are, actually, a particularization of the more general methodological context of the variation problems.

As indicated in the Introduction, we are primarily interested in the methodology which underlies the variational problems, rather than these problems per se. More specifically, in this section we shall review certain aspects related to paths, path functionals, and their variations, which are of direct significance for our subsequent analysis, while the extremal aspect of the methodology will be referred to the several excellent treatises on the subject.[23]

For unity of notation throughout this volume we shall use the symbol $t$ to denote the independent variable and the symbols $q^k(t)$, $k = 1, 2, \ldots, n$, to denote the dependent variables, although the symbols more commonly used in the calculus of variations are, instead, $x$ and $y^k(x)$, respectively.

A *path* (or *path segment*) is the set of values

$$E = \{q^k(t) | t \in (t_1, t_2), k = 1, 2, \ldots, n\} \tag{1.3.1}$$

for given functions $q^k$. The values $t_1$ and $t_2$ are termed the *end points*, with $t_1(t_2)$ being the *initial (final) point*. The values $q^k(t_s)$, $s = 1, 2$, are called *end values*, with $q^k(t_1)(q^k(t_2))$ being the *initial (final) value*. A path $E$ is of class $\mathscr{C}^m$ in the (closed) interval $(t_1, t_2)$, when each and every function $q^k(t)$, $k = 1$,

[22] Caratheodory (1935, preface).

[23] See, for instance, Caratheodory (1935), Bliss (1946), Gelfand and Fomin (1963), Rund (1966), and Hestenes (1966).

$2, \ldots, n$, is of class $\mathscr{C}^m$ in it. A similar definition applies for the case of an (open) region $R_t$.

The set of all possible paths forms a rather large space $S(E)$, called *function space*. A subset of $S(E)$ constituted by paths $E^1, E^2, \ldots$ of class $\mathscr{C}^0$ in $(t_1, t_2)$, when equipped with the nonnegative number

$$\|E^k\|^0 = \{\text{Max. value}|E^k(t)|, t \in (t_1, t_2), k = 1, 2, \ldots\}, \tag{1.3.2}$$

called the *norm*, constitutes a *normed function space of class* $\mathscr{C}^0$, $\mathscr{N}^0(E)$. When paths $E^1, E^2, \ldots$ of order $\mathscr{C}^m, m > 0$, are considered, the *norm of order m*,

$$\|E^k\|^m = \left\{ \sum_{i=0}^{m} \text{Max. value} \left| \frac{d^i E^k}{dt^i} \right|, t \in (t_1, t_2) \right\}, \tag{1.3.3}$$

applies with the properties

$$\|E^k\|^m = 0 \quad \text{if and only if } E^k = 0, \tag{1.3.4a}$$

$$\|cE^k\|^m = |c| \|E^k\|^m, \qquad c = \text{const.}, \tag{1.3.4b}$$

$$\|E^i + E^j\|^m \leq \|E^i\|^m + \|E^j\|^m, \qquad i, j, k = 1, 2, \ldots, \tag{1.3.4c}$$

yielding a *normed function space of class* $\mathscr{C}^m$, $\mathscr{N}^m(E)$.

The norm allows the characterization of the *distance of order m* between two paths $E^1$ and $E^2$, given by

$$D^m(E^1, E^2) = \|E^1 - E^2\|^m. \tag{1.3.5}$$

In turn, the concept of distance allows the characterization of the *neighborhood of order m* of a given path $E$, which is the space of all paths $E' = E + \delta E$ whose distance of order $m$ from $E$ is less than $\varepsilon > 0$, i.e.,

$$\mathscr{E}^m(E')_E = \{E' | E' = E + \delta E, \|E' - E\|^m = \|\delta E\|^m < \varepsilon\}. \tag{1.3.6}$$

The path $E'$ so characterized is the *varied path*. From these definitions, we see that the continuity properties of a path $E$ and those of its varied path $E' = E + \delta E$ can be different, e.g., $E \in \mathscr{C}^2$ and $E' \in \mathscr{C}^0$. Within the context of the calculus of variations, the neighborhood of order zero of a given path is customarily considered. This is essential to study certain aspects of the extremal problems, such as the so-called *Weierstrass necessary condition*. Within the context of the Inverse Problem and, more specifically, the variational approach to self-adjointness, the minimal continuity conditions for paths and their variations can be unified for simplicity but without loss of generality, and assumed to be that of class $\mathscr{C}^2$ for configuration space formulations (and of class $\mathscr{C}^1$ for phase space formulations). This implies that we shall be dealing with paths $E^1, E^2, \ldots$ which are of at least class $\mathscr{C}^2$ and their neighborhoods also of class $\mathscr{C}^2$ (i.e., $E'^1, E'^2, \ldots \in \mathscr{C}^2$). This also implies that the variations $\delta E^1, \delta E^2, \ldots$ have the same continuity properties of $E^1, E^2, \ldots$. From a Newtonian profile, these continuity assumptions imply the study of trajectories $\{q^k(t)\}$ and their variations $\{\delta q^k(t)\}$, which possess continuous derivatives up to and including that of order two, i.e., not only the velocities, but also the accelerations and their variations are continuous.

A *path functional* is a correspondence which assigns a (real) number $A(E)$ to a given (real) path $E$. Several types of path functionals for a given path $E$ can be conceived. The correspondence which is of central relevance for the calculus of variations, as well as for the Inverse Problem, is given by the quantity

$$A(E) = \int_{t_1}^{t_2} dt\, L(t, q, \dot{q})|_E = \int_{t_1}^{t_2} dt\, L(t, E, \dot{E}). \tag{1.3.7}$$

When the integrand function $L(t, q, \dot{q})$ is a Lagrangian of a Newtonian system, path functional (1.3.7) is generally (although not universally) called the *action functional*, or *action* for short.

The so called *simplest variational problem* essentially consists of finding the path $E_0$ satisfying the end conditions

$$E_0(t_1) = u_0 = \{u_0^k\}, \qquad E_0(t_2) = v_0 = \{v_0^k\}, \tag{1.3.8}$$

along which the functional $A(E)$ affords an *extremum* (i.e., either a maximum or a minimum).[24] Such a path $E_0$ is then called the *extremal path* of problem (1.3.7).

Notice that the continuity of a given function $L(t, q, \dot{q})$ can be different than that of the path $E$ along which it is computed. Throughout our analysis, we shall only consider functions $L(t, q, \dot{q})$ that are of at least class $\mathscr{C}^4$ in a region $R^{2n+1}$ of their variables. Within a Newtonian context, these functions will be called *admissible Lagrangians*. A path $E$ will be called an *admissible path* and its varied path $E' = E + \delta E$ an *admissible varied path* (or its variation $\delta E$, *admissible variation*) when $E$ and $E'$ (or $\delta E$) are of at least class $\mathscr{C}^2$.

Path functional (1.3.7) is called *regular* or *degenerate* (also *regular* or *singular*, and *standard* or *nonstandard*) in a region $R^{2n+1}$ when the integrand $L$ is regular or degenerate in it in the sense of Definition 1.1.1. Throughout our analysis, we shall only consider regular path functionals.

Path functional (1.3.7) is said to possess an *absolute maximum* or an *absolute minimum* along $E_0$ when the inequalities

$$A(E) - A(E_0) \leq 0 \tag{1.3.9a}$$

and

$$A(E) - A(E_0) \geq 0 \tag{1.3.9b}$$

hold, respectively, for all paths $E$ satisfying certain continuity properties (usually $E \in C^0$). The fundamental quantity for the study of the variational problems is, therefore, $A(E) - A(E_0)$.

Of particular significance is the study of the *relative maximum* or *minimum*, which occur when inequalities (1.3.9a) and (1.3.9b), respectively, hold for all paths $E$ in the neighborhood of $E_0$ of order zero, $\mathscr{E}^0(E)_E$. If the case $E \in \mathscr{E}^1(E)_{E_0}$ is considered, inequalities (1.3.9a) and (1.3.9b) characterize the *weak relative maximum* and *minimum*, respectively.

[24] The problem we refer to here is also called the *variational problem with fixed end points.*

An absolute extremum is also a relative extremum and a weak relative extremum, but the inverse property does not necessarily hold. This is due to the fact that if $A(E)$ has an extremum along $E_0$ for functions $E$ of class $\mathscr{C}^0$, the same property will be satisfied with paths of class $\mathscr{C}^m, m \geq 1$, but the inverse property does not necessarily hold.

The comparison of the above continuity conditions for the extremal problem and the corresponding sufficient conditions for the Inverse Problem (or the variational principles) is instructive.

The *total variation* of path functional (1.3.7) is given by

$$\Delta A(E, \delta E) = A(E + \delta E) - A(E), \qquad \|\delta E\|^m < \varepsilon \tag{1.3.10}$$

and it is a functional of both $E$ and $\delta E$. Notice that $\Delta E(E, \delta E)$ is not a linear functional of $\delta E$.

A path functional $A(E)$ is said to be *differentiable* when $\Delta A$ admits the decomposition

$$\Delta A(E, \delta E) = \delta^1 A(E, \delta E) + \alpha \|\delta E\|^0, \tag{1.3.11}$$

$$\lim_{\|\delta E\|^0 \to 0} \alpha = 0$$

where $\delta^1 A$, called the *first-order variation* of $A$, is a linear functional of $\delta E$.

Without proof, we quote the following theorem.

**Theorem 1.3.1** (A First Necessary Condition for an Extremum). *A necessary condition for a differentiable path functional $A(E)$ to have a relative extremum at $E = E_0$ is that its first-order variation $\delta^1 A$ vanishes for $E = E_0$ and for all variations $\delta E_0$ of at least class $\mathscr{C}^0$, i.e.,*

$$\delta^1 A(E_0, \delta E_0) = 0, \qquad \|\delta E_0\|^0 < \varepsilon. \tag{1.3.12}$$

As we shall see, the above theorem characterizes a methodological context of variational principles.

A path functional is said to be *twice differentiable* if $\Delta A$ admits the decomposition

$$\Delta A(E, \delta E) = \delta^1 A(E, \delta E) + \delta^2 (A, \delta E) + \beta(\|\delta E\|^0)^2, \tag{1.3.13}$$

$$\lim_{\|\delta E\|^0 \to 0} \beta = 0$$

where $\delta^2 A$, called the *second-order variation* of $A$, is a quadratic functional of $\delta E$.

Also without proof we quote the following theorem.

**Theorem 1.3.2** (A second Necessary Condition for an Extremum). *A necessary condition for a twice differentiable functional $A(E)$ to have a relative extremum at $E = E_0$ is that one of the inequalities*

$$\delta^2 A(E_0, \delta E_0) \geq 0, \tag{1.3.14a}$$

$$\delta^2 A(E_0, \delta E_0) \leq 0 \tag{1.3.14b}$$

*holds for all variations $\delta E_0$ of at least class $\mathscr{C}^0$.*

This second fundamental theorem of the calculus of variations has no counterpart in contemporary analytic mechanics. This is essentially due to the fact that this discipline considers the customary variational principles only, which are first-order principles from the viewpoint of the calculus of variations. Therefore, higher-order variations of the action are customarily ignored.

As we shall see during the course of our analysis, this analtyic context is modified by the methodology of the Inverse Problem because the use of the second-order variation of the action functional will be necessary for the study of the conditions under which a Lagrangian exists or not. In turn, this implies an implementation of the customary variational principles into second- (as well as higher-) order forms.

In summmary, of fundamental importance for the Inverse Problem is the study of both the first- and the second-order variation of path (or action) functionals, even within the context where the extremal aspect of the problem is ignored. The rest of this section is primarily devoted to the identification of these variations.

First, we shall identify the *first- and second-order variations of the path (action) functional with fixed end points*, i.e., variations for which

$$\delta E(t_s) = \{\delta q^k(t_s)\} = 0, \qquad s = 1, 2. \tag{1.3.15}$$

If we assume the following explicit form of the variations,

$$\begin{aligned} \delta q^k(t) &= \varepsilon\eta(t), \\ \delta \dot{q}^k(t) &= \varepsilon\dot{\eta}(t) = \frac{d}{dt}\,\delta q^k(t), \end{aligned} \tag{1.3.16}$$

often called *weak variations*, Equations (1.3.13) can be explicitly written[25]

$$\begin{aligned} \Delta A(E, \delta E) &= \varepsilon \left| \frac{d}{d\varepsilon} A(E + \delta E) \right|_{\varepsilon=0} + \frac{\varepsilon^2}{2!} \left| \frac{d^2}{d\varepsilon^2} A(E + \delta E) \right|_{\varepsilon=0} + \beta(\|\delta E\|^0)^2 \\ &\approx \int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial q^k}\,\delta q^k + \frac{\partial L}{\partial \dot{q}^k}\,\delta \dot{q}^k \right)_E \\ &\quad + \frac{1}{2} \int_{t_1}^{t_2} dt \left( \frac{\partial^2 L}{\partial q^i\,\partial q^j}\,\delta q^i \delta q^j + 2\,\frac{\partial^2 L}{\partial q^i\,\partial \dot{q}^j}\,\delta q^i \delta \dot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i\,\partial \dot{q}^j}\,\delta \dot{q}^i \delta \dot{q}^j \right)_E . \end{aligned} \tag{1.3.17}$$

The first- and second-order variations under consideration at this time are then given, respectively, by

$$\delta^1 A(E, \delta E) = \int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial q^k}\,\delta q^k + \frac{\partial L}{\partial \dot{q}^k}\,\delta \dot{q}^k \right)_E , \tag{1.3.18a}$$

$$\delta^2 A(E, \delta E) = \frac{1}{2} \int_{t_1}^{t_2} dt \left( \frac{\partial^2 L}{\partial q^i\,\partial q^j}\,\delta q^i \delta q^j + 2\,\frac{\partial^2 L}{\partial q^i\,\partial \dot{q}^j}\,\delta q^i \delta \dot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i\,\partial \dot{q}^j}\,\delta \dot{q}^i \delta \dot{q}^j \right)_E \tag{1.3.18b}$$

Without proof, we quote the following lemma.

[25] See, for instance, Bliss (1946).

**Lemma 1.3.1** (A Fundamental Lemma of the Calculus of Variations). *If the functions* $\beta_k(t)$, $k = 1, 2, \ldots, n$, *are of class* $\mathscr{C}^0$ *in* $(t_1, t_2)$, *and if*

$$\int_{t_1}^{t_2} dt\, \beta_k(t)\eta^k(t) = 0 \tag{1.3.19}$$

*for all functions* $\eta^k(t)$ *of at least class* $\mathscr{C}^0$ *in the same interval, which are identically null at end points,*

$$\eta^k(t_s) = 0, \qquad s = 1, 2, \tag{1.3.20a}$$

*then*

$$\beta_k(t) = 0, \qquad k = 1, 2, \ldots, n \tag{1.3.20b}$$

*for all values* $t \in (t_1, t_2)$.

By integrating by parts Equation (1.3.18a) and using Equations (1.3.15), Theorem 1.3.1 and Lemma 1.3.1 imply the following theorem.

**Theorem 1.3.3** (Euler's Necessary Condition). *A necessary condition for a path* $E_0$ *of at least class* $\mathscr{C}^1$ *to be an extremal of functional* (1.3.7) *is that the* Euler equations

$$L_k(E_0) = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k}\right)_{E_0} = 0, \qquad k = 1, 2, \ldots, n, \tag{1.3.21}$$

*be identically verified along* $E_0$.

As indicated in the Introduction, Equations (1.3.21) will be referred to as *Lagrange's equations* whenever the function $L$ is a Lagrangian of a Newtonian system, as customarily used in the recent literature of analytic mechanics. A more adequate term would, however, be *Euler–Lagrange equations.*

For later use in Chapter 3, let us recall that if the path $E_0$ is of class $\mathscr{C}^0$, the *integral form of Euler's* (*Lagrange's*) *equations*

$$\left(\frac{\partial L}{\partial \dot{q}^k} - \int_{t_1}^{t} dt \frac{\partial L}{\partial q^k}\right)_{E_0} = c_k = \text{const.} \tag{1.3.22}$$

must be used to treat possible *corners* of $E_0$, i.e., points with discontinuous first-order derivatives. Clearly, the total derivative with respect to $t$ of Equations (1.3.22) reproduces Equations (1.3.21) identically. Thus, Equations (1.3.21) and (1.3.22) are equivalent for paths of at least class $\mathscr{C}^1$.

The fundamental variational principle of analytic mechanics, *Hamilton's principle,*

$$\delta^1 A(E_0, \delta E_0) = -\int_{t_1}^{t_2} dt\, L_k(E_0)\delta q_0^k = 0, \tag{1.3.23}$$

follows from Theorem 1.3.3.

Of particular importance for the extremal problem $A(E)$ is the study of the so called *accessory extremal problem.* This consists of the study of the second order variation (1.3.18b) as a variational problem per se. Theorem 1.3.3, applied to this problem, yields

$$\delta^1(\delta^1 A(E_0, \delta E_0)) = -\int_{t_1}^{t_2} dt\, J_k(\delta E_0)\delta(\delta q_0^k) = 0 \tag{1.3.24a}$$

and

$$J = \frac{1}{2}\left(\frac{\partial^2 L}{\partial q^i\, \partial q^j}\,\delta q^i \delta q^j + 2\frac{\partial^2 L}{\partial q^i\, \partial \dot{q}^j}\,\delta q^i \delta \dot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}\,\delta \dot{q}^i \delta \dot{q}^j\right). \tag{1.3.24b}$$

The underlying equations,

$$J_k(\delta E_0) = \left(\frac{d}{dt}\frac{\partial J}{\partial(\delta \dot{q}^k)} - \frac{\partial J}{\partial(\delta q^k)}\right)_{\delta E_0} = 0, \tag{1.3.25}$$

are called *Jacobi's equations* in the literature of the calculus of variations and their solutions are referred to as the *accessory extremals.* The study of accessory problem (1.3.18b) is essential for the characterization of the behavior of the second-order variations within the context of Theorem 1.3.2 and results in the so-called *Jacobi's necessary condition.* A review of this condition (which is centered on the theory of the so-called *conjugate points*) would bring us considerably outside the scope of this monograph; therefore, we urge the interested reader to study the literature on the calculus of variations.

For our needs, it is sufficient here to recall that, when the variations are restricted to satisfy Jacobi's equations, the varied path of an extremal path is also an extremal. This can be seen with the Taylor expansion

$$\begin{aligned} L_k(E_0 + \delta E_0) &= \left(\frac{d}{dt}\frac{\partial L(t, q + \delta q, \dot{q} + \delta \dot{q})}{\partial \dot{q}^k} - \frac{\partial L(t, q + \delta q, \dot{q} + \delta \dot{q})}{\partial q^k}\right)_{E_0 + \delta E_0} \\ &= \left(\frac{d}{dt}\frac{\partial L(t, q, \dot{q})}{\partial \dot{q}^k} - \frac{\partial L(t, q, \dot{q})}{\partial q^k}\right)_{E_0} \\ &\quad + \left(\frac{d}{dt}\frac{\partial J(t, \delta q, \delta \dot{q})}{\partial(\delta \dot{q}^k)} - \frac{\partial J(t, \delta q, \delta \dot{q})}{\partial(\delta q^k)}\right)_{\delta E_0} + \cdots \\ &= L_k(E_0) + J_k(\delta E_0) + \cdots \\ &= J_k(\delta E_0) + \cdots = 0, \end{aligned} \tag{1.3.26}$$

where we have used Euler's condition (1.3.21). This point also illustrates the deep relationship between Euler's equations and their associated Jacobi's equations. The latter are uniquely characterized by the former. Also, while the latter are always linear, the former are generally nonlinear. Finally, the regularity or degeneracy properties of the latter and those of the former coincide, owing to the value of the functional determinant

$$|\mathscr{H}|(R) = \left|\frac{\partial^2 J_i}{\partial(\delta \ddot{q}^j)}\right|(R) = \left|\frac{\partial^2 L_i}{\partial \ddot{q}^j}\right|(R) = \left|\frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}\right|(R). \tag{1.3.27}$$

Regrettably, Jacobi's equations are customarily ignored in the current literature of analytic mechanics. As we shall see in Chapter 3, these equations will play a fundamental role in the identification of the necessary and sufficient conditions for the existence of a Lagrangian. This is a reason why the Inverse Problem also demands the use of second-order variations. It is in this sense that, even though the extremal aspect can be ignored within the context of the Inverse Problem, the methodology of the calculus of variations plays a fundamental role in it.

When the independent variable $t$ represents the time, the variations considered until now are often called *contemporaneous variations*, because they occur at a fixed value of time. The *total contemporaneous variation of a path* can then be expanded:

$$\Delta E = E'(t) - E(t) = \{q'^k(t) - q^k(t)\} = \left\{ \sum_{s=1}^{m} \delta^s q^k(t) + O_q^{m+1} \right\}, \tag{1.3.28a}$$

$$\delta^s q^k(t) = \frac{1}{s}\, \delta^1(\delta^{s-1} q^k(t)), \tag{1.3.28b}$$

by characterizing in this way the *contemporaneous variations of order* $s = 1, 2, 3, \ldots,$ *of* a path, i.e., $\delta^s q^k(t)$. The *total contemporaneous variation of a path functional* can be subject to a corresponding expansion,

$$\Delta A = A(E') - A(E) = \sum_{s=1}^{m} \delta^s A + O_A^{m+1}, \tag{1.3.29a}$$

$$\delta^s A = \frac{1}{s}\, \delta^1(\delta^{s-1} A), \tag{1.3.29b}$$

by characterizing the *contemporaneous variations of order s of* $A(E)$, $\delta^s A$. For the case $s = 1$, we recover Equation (1.3.18a). However, for the case $s = 2$, we have the expression

$$\begin{aligned} \delta^2 A(E, \delta E) &= \frac{1}{2}\, \delta^1(\delta^1 A) = \frac{1}{2}\, \delta^1 \int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial q^k}\, \delta^1 q^k + \frac{\partial L}{\partial \dot{q}^k}\, \delta^1 \dot{q}^k \right)_E \\ &= \frac{1}{2} \int_{t_1}^{t_2} dt \left( \frac{\partial^2 L}{\partial q^i\, \partial \dot{q}^j}\, \delta q^i \delta q^j + 2\, \frac{\partial^2 L}{\partial q^i\, \partial \dot{q}^j}\, \delta q^i \delta \dot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}\, \delta \dot{q}^i \delta \dot{q}^j \right)_E \\ &\quad + \int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial q^k}\, \delta^2 q^k + \frac{\partial L}{\partial \dot{q}^k}\, \delta^2 \dot{q}^k \right)_E, \end{aligned} \tag{1.3.30}$$

which exhibits the presence of the two additional terms in the second-order variations of the path, which we had ignored in Equation (1.3.18b) because they are not essential for the accessory extremal problem.[26] The computation

[26] By using an integration by parts and the fixed end point conditions, the last integral of Equation (1.3.30) becomes

$$-\int_{t_1}^{t_2} dt\, L_k(E) \delta^2 q^k$$

and, as such, it is identically null along the extremal path $E_0$.

of higher-order variations can then be done by using iterative formula (1.3.29b).

The above variations are a particular case of the more general *noncontemporaneous variations*, i.e., those also involving a variation of time. The *total noncontemporaneous variation of a path* can then be written and expanded as follows.

$$\hat{\Delta}E = E'(t') - E(t) = \{q'^k(t') - q^k(t)\} = \left\{\sum_{s=1}^{m} \delta^s q^k(t) + \hat{O}_q^{m+1}\right\}, \tag{1.3.31a}$$

$$\hat{\delta}^s q^k(t) = \frac{1}{s}\hat{\delta}^1(\hat{\delta}^{s-1}q^k(t)) \tag{1.3.31b}$$

$$\Delta t = t' - t = \sum_{s=1}^{m} \hat{\delta}^s t + \hat{O}_t^{m+1}, \tag{1.3.31c}$$

$$\hat{\delta}^s t = \frac{1}{s}\hat{\delta}^1(\hat{\delta}^{s-1}t). \tag{1.3.31d}$$

The *noncontemporaneous variations of order s of a path* can then be computed with the iterative formulae

$$\hat{\delta}^1 q^k = q'^k(t + \hat{\delta}^1 t) - q^k(t) = \delta^1 q^k + \dot{q}^k \hat{\delta}^1 t, \tag{1.3.32a}$$

$$\begin{aligned}\hat{\delta}^2 q^k &= \tfrac{1}{2}\hat{\delta}^1(\hat{\delta}^1 q^k) = \tfrac{1}{2}\hat{\delta}^1(\delta^1 q^k + \dot{q}^k \hat{\delta}^1 t) \\ &= \delta^2 q^k + \tfrac{1}{2}\delta^1 \dot{q}^k \hat{\delta}^1 t + \tfrac{1}{2}\delta^1 \dot{q}^k \hat{\delta}^1 t + \tfrac{1}{2}\ddot{q}^k (\hat{\delta}^1 t)^2 + \tfrac{1}{2}\dot{q}^k \frac{d}{dt}(\hat{\delta}^1 t)\hat{\delta}^1 t \\ &= \delta^2 q^k + \hat{\delta}^1 \dot{q}^k \hat{\delta}^1 t - \ddot{q}^k \tfrac{1}{2}(\hat{\delta}^1 t)^2 + \tfrac{1}{2}\dot{q}^k \frac{d}{dt}(\hat{\delta}^1 t)\hat{\delta}^1 t.\end{aligned} \tag{1.3.32b}$$

This also yields the relationship between the contemporaneous and noncontemporaneous variations. Notice that the operations of variation and derivative commute for the former, i.e., $\delta^1 \dot{q}^k = (d/dt)\delta^1 q^k$, while they do not for the latter, i.e., $\hat{\delta}^1 \dot{q}^k \neq (d/dt)\hat{\delta}^1 q^k$.

On similar grounds, the *total noncontemporaneous variation of a path functional* can be written and expanded as follows.

$$\begin{aligned}\hat{\Delta}A(E, \hat{\delta}E) &= \int_{t_1'}^{t_2'} dt'\, L(t', q', \dot{q}') - \int_{t_1}^{t_2} dt\, L(t, q, \dot{q}) \\ &= \sum_{s=1}^{m} \hat{\delta}^s A + \hat{O}_A^{m+1},\end{aligned} \tag{1.3.33a}$$

$$\hat{\delta}^s A = \frac{1}{s}\hat{\delta}^1(\hat{\delta}^{s-1}A). \tag{1.3.33b}$$

By using the reduction

$$dt' = dt\frac{dt'}{dt} \approx dt\left[1 + \frac{d}{dt}(\hat{\delta}^1 t)\right] \tag{1.3.34}$$

the *first-order noncontemporaneous variation of* $A(E)$ is given by

$$\begin{aligned}
\hat{\delta}^1 A &= \int_{t_1}^{t_2} dt \left[\hat{\delta}^1 L + L\frac{d}{dt}(\hat{\delta}^1 t)\right]_E \\
&= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial t}\hat{\delta}^1 t + \frac{\partial L}{\partial q^k}\hat{\delta}^1 q^k + \frac{\partial L}{\partial \dot{q}^k}\hat{\delta}^1 \dot{q}^k + L\frac{d}{dt}(\hat{\delta}^1 t)\right]_E \\
&= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q^k}\delta^1 q^k + \frac{\partial L}{\partial \dot{q}^k}\delta^1 \dot{q}^k + \frac{d}{dt}(L\hat{\delta}^1 t)\right]_E \\
&= \int_{t_1}^{t_2} dt \left[\delta^1 L + \frac{d}{dt}(L\hat{\delta}^1 t)\right]_E,
\end{aligned} \tag{1.3.35}$$

where we have used Equations (1.3.32a).

The *second-order noncontemporaneous variation of* $A(E)$ is then given by

$$\begin{aligned}
\hat{\delta}^2 A &= \tfrac{1}{2}\hat{\delta}^1(\hat{\delta}^1 A) \\
&= \tfrac{1}{2}\hat{\delta}^1 \int_{t_1}^{t_2} dt \left[\delta^1 L + \frac{d}{dt}(L\hat{\delta}^1 t)\right]_E \\
&= \frac{1}{2}\int_{t_1}^{t_2} dt \left\{\delta^1\left[\delta^1 L + \frac{d}{dt}(L\hat{\delta}^1 t)\right] + \frac{d}{dt}\left[\left(\delta^1 L + \frac{d}{dt}(L\hat{\delta}^1 t)\right)\hat{\delta}^1 t\right]\right\}_E \\
&= \int_{t_1}^{t_2} dt \left\{\delta^2 L + \frac{d}{dt}\left[\delta^1 L\hat{\delta}^1 t + \frac{1}{2}\frac{d}{dt}(L\hat{\delta}^1 t)\hat{\delta}^1 t\right]\right\}_E.
\end{aligned} \tag{1.3.36}$$

Higher-order variations can then be computed with iterative formula (1.3.33b).

For these more general variations, the concept of neighborhood according to Equation (1.3.6) is insufficient. This is due to the need for using a concept of distance between two paths $E$ and $E'$ of different end values. An extended definition of distance of order $m$, often used in the literature of the calculus of variations, is given by[27]

$$\begin{aligned}
\hat{D}^m(E^1, E^2) = \|E^1 - E^2\|^m &+ |E^1(t_1') - E^2(t_1')| \\
&+ |E^1(t_2') - E^2(t_2')| + |t_1' - t_1| + |t_2' - t_2|.
\end{aligned} \tag{1.3.37}$$

This yields the extended definition of a neighborhood of order $m$:

$$\hat{\mathscr{E}}(E')_E = \{E' | E' = E + \hat{\delta}E, \hat{D}^m(E', E) < \varepsilon\}. \tag{1.3.38}$$

When considering noncontemporaneous variations, we shall always assume an extended neighborhood (1.3.38) of at least order two.

[27] Gelfand and Fomin (1963).

When the independent variable $t$ does not necessarily represent time, the more general variations considered here are referred to as *variations with variable end points*, and they characterize the so-called *variational problem with variable end points*. In this case, by integrating by parts, the *first-order variations with variable end points of* $A(E)$ can be written

$$\hat{\delta}^1 A(E, \hat{\delta}E) = -\int_{t_1}^{t_2} dt\, L_k(E)\delta q^k + \left| \frac{\partial L}{\partial \dot{q}^k}\,\hat{\delta}^1 q^k - \left(\frac{\partial L}{\partial \dot{q}^k}\,\dot{q}^k - L\right)\hat{\delta}^1 t \right|_{t_1}^{t_2}(E), \tag{1.3.39}$$

where we have used Equations (1.3.32a). The quantity

$$(\mathrm{EPC})^1(E) = \left| \frac{\partial L}{\partial \dot{q}^k}\,\hat{\delta}^1 q^k - \left(\frac{\partial L}{\partial \dot{q}^k}\,\dot{q}^k - L\right)\hat{\delta}^1 t \right|_{t_1}^{t_2}(E) \tag{1.3.40}$$

is called the *first-order end-points contribution.*

When Equation (1.3.39) is computed along an extremal (and thus a possible) path, we have the so-called *Weiss's principle*[28] (also called *Holder's principle*[29]) of analytic mechanics,

$$\hat{\delta}^1 A(E_0, \hat{\delta}E_0) = \hat{\delta}^1 \int_{t_1}^{t_2} dt\, L(t, q, \dot{q})|_{E_0} = (\mathrm{EPC})^1(E_0), \tag{1.3.41}$$

which is clearly a generalization of Hamilton's principle (1.3.23). This broader principle is particularly significant in analytic mechanics for a number of methodological aspects, such as the derivation of the Hamilton–Jacobi equation, the computation of conserved quantities, etc., as can be seen in Santilli (1979).

The inclusion of the Inverse Problem will inevitably demand the use of higher-order noncontemporaneous variations. The *second-order variation with variable end points* of $A(E)$ can be written

$$\hat{\delta}^2 A(E, \hat{\delta}E) = -\int_{t_1}^{t_2} dt\, L_k(E)\delta^2 q^k - \int_{t_1}^{t_2} dt\, J_k(\delta E)\delta^1 q^k + (\mathrm{EPC})^2(E), \tag{1.3.42}$$

where the quantity

$$\begin{aligned}(\mathrm{EPC})^2(E) = \Bigg| & \frac{\partial L}{\partial \dot{q}^k}\,\hat{\delta}^2 q^k - \left(\frac{\partial L}{\partial \dot{q}^k}\,\dot{q}^k - L\right)\hat{\delta}^2 t - \frac{1}{2}\left(\frac{\partial L}{\partial q^k}\,\dot{q}^k - \frac{\partial L}{\partial t}\right)(\hat{\delta}^1 t)^2 \\ & + \frac{\partial L}{\partial q^k}\,\hat{\delta}^1 q^k \hat{\delta}^1 t + \frac{1}{2}\,\frac{\partial^2 L}{\partial q^i\,\partial \dot{q}^j}\,(\hat{\delta}^1 q^i - \dot{q}^i \hat{\delta}^1 t)(\hat{\delta}^1 q^j - \dot{q}^j \hat{\delta}^1 t) \\ & + \frac{1}{2}\,\frac{\partial^2 L}{\partial \dot{q}^i\,\partial \dot{q}^j}\,(\hat{\delta}^1 q^i - \dot{q}^i \hat{\delta}^1 t)(\hat{\delta}^1 \dot{q}^j - \ddot{q}^j \hat{\delta}^1 t)\Bigg|_{t_1}^{t_2}(E)\end{aligned} \tag{1.3.43}$$

is called *second-order end-points contribution.* Higher-order variations and end-point contributions can then be computed with an iterative procedure.

[28] See, for instance, Sudarshan and Mukunda (1974).
[29] See, for instance, Pars (1965).

The variations $\delta^s q^k(\hat{\delta}^s q^k)$ will be referred to as the *abstract contemporaneous (noncontemporaneous) variations of order s of a path.* They can have an arbitrary explicit and/or implicit dependence on the independent variable, the path, and its first-order derivative, i.e.,

$$\delta^s q^k = \delta^s q^k(t, q, \dot{q}), \qquad \hat{\delta}^s q^k = \hat{\delta}^s q^k(t, q, \dot{q}), \tag{1.3.44a}$$

$$D^m(E + \delta E, E) < \varepsilon, \qquad \hat{D}^m(E + \hat{\delta} E, E) < \varepsilon, \tag{1.3.44b}$$

which is admissible by the assumed continuity conditions. When variations (1.3.44) are computed along a given path, they reduce to the explicit dependence on the independent variable only, i.e.,

$$\begin{aligned} &\delta^s q^k(t, q, \dot{q})|_E \approx \delta^s q^k(t), \\ &\hat{\delta}^s q^k(t, q, \dot{q})|_E \approx \hat{\delta}^s q^k(t). \end{aligned} \tag{1.3.45}$$

When an explicit functional dependence in Equations (1.3.44) is assumed, we have a *realization of an abstract (admissible) variation.* For instance, weak variations (1.3.16) are a simple realization of abstract variations (1.3.44). However, they are not the only admissible form, and any other functional dependence which characterizes varied paths in the neighborhood (of the considered order) of the given path is equally admissible.

The use of the broader functional dependence of the variations according to Equations (1.3.44) will play a crucial role for the broadening of the direct representational capability of Hamilton's principle [see Santilli (1979)].

**Chart 1.1** A Theorem on the Existence, Uniqueness, and Continuity of the Implicit Functions for Newtonian Systems[30]

This simplified version of Theorem 1.1.1 is often useful for practical applications in Newtonian Mechanics.

**Theorem.** *Given a system of ordinary second-order differential equations*

$$F_i(t, q, \dot{q}, \ddot{q}) = 0, \qquad i = 1, 2, \ldots, n, \tag{1}$$

*let*:

(1) *the point* $P_0 = (t_0, q_0, \dot{q}_0, \ddot{q}_0)$ *satisfy all the equations of system* (1);
(2) *the functions* $F_i$ *be of class* $\mathscr{C}^m$, $m \geq 1$, *in the neighborhood of* $P_0$; *and*
(3) *the functional determinant* $|\mathscr{H}| = |\partial F_i/\partial \ddot{q}^j|$ *be different from* 0 *at* $P_0$.

[30] See Rektorys (1969).

*Nota Bene*: The continuity conditions in the accelerations $\ddot{q}^i$ can be ignored for system (3) due to their linearity in such variables. The functional determinant for Newtonian systems in configuration space is the determinant of the factor terms $A_{ki}$. Unlike the case for system (1), such a determinant is always independent of the accelerations. Notice the vital role of the regularity condition $|A_{ij}| \neq 0$ for the very existence of the implicit functions. Indeed, when such a condition is violated, the inverse $(A_{ij})^{-1}$ does not exist and the set of *all* implicit functions $f^i$ does not exist, either.

*Then in a neighborhood of* $(t_0, q_0, \dot{q}_0)$ *there exists a unique system of functions* $f^i(t, q, \dot{q})$, *termed the* implicit functions, *of class* $\mathscr{C}^m$ *such that*

$$\ddot{q}^i = f^i(t, q, \dot{q}) \tag{2}$$

*for all* $(t, q, \dot{q})$ *in the neighborhood of* $(t_0, q_0, \dot{q}_0)$.

The application of the above theorem (or, similarly, of Theorems 1.1.1 and 1.1.2) to Newtonian systems is straightforward. Consider such systems in their fundamental form in configuration space (A.7.5) (of the Appendix), i.e.,

$$F_i = A_{ij}(t, q, \dot{q})\ddot{q}_j + B_i(t, q, \dot{q}). \tag{3}$$

Let: (a) the point $P = (t_0, q_0, \dot{q}_0, \ddot{q}_0)$ satisfy such equations, (b) the functions $A_{ij}$ and $B_i$ be of class $\mathscr{C}^m$, $m \geq 1$, and (c) the functional determinant

$$|\mathscr{H}| = \left|\frac{\partial F_i}{\partial \ddot{q}^j}\right| = |A_{ij}| \tag{4}$$

be different from zero at $(t_0, q_0, \dot{q}_0)$. Then the implicit functions $f^i$ exist, are unique and of class $\mathscr{C}^m$ in a neighborhood of $(t_0, q_0, \dot{q}_0)$, and are given, trivially, by

$$f^i = -A^{ij}B_j, \qquad (A^{ij}) = (A_{ij})^{-1}. \tag{5}$$

**Chart 1.2** A Theorem on the Existence, Uniqueness, and Continuity of a Solution of a Newtonian Initial Value Problem[31]

The following simplified version of Theorem 1.1.5 is particularly useful for Newtonian systems.

**Theorem.** *If all the functions* $\Xi^\mu(t, a)$ *are of (at least) class* $\mathscr{C}^1$ *in a region* $\tilde{R}^{2n+1}$ *of points* $(t, a^\alpha)$, *then the initial value problem*

$$\begin{aligned} a^\mu &= \Xi^\mu(t, a), \\ a^\mu(t_0) &= c_0^\mu, \qquad \mu = 1, 2, \ldots, 2n, \end{aligned} \tag{1}$$

*admits a unique solution* $a^\mu(t)$ *in any interval of time* $(t_1, t_2)$ *containing* $t_0$ *for which all points* $(t, a(t))$ *lie in the interior of* $\tilde{R}^{2n+1}$ *and such a solution is continuous in* $(t_1, t_2)$.

If the region $\tilde{R}^{2n+1}$ is the entire space $-\infty < t, a^\mu < +\infty$, then a solution exists provided its norm remains finite. This remark is useful to determine whether a Newtonian system admits a solution for all values of time in the interval $(-\infty, +\infty)$. For configuration space formulations of

[31] Akhiezer (1962).

*Nota Bene*: The theorem of this chart also holds under the weaker continuity conditions that all $\Xi^\mu$ and $\partial\Xi^\mu/\partial a^\nu$ are continuous. However, the case when a discontinuity of the $\Xi^\mu$ functions in their time dependence occurs, is rather problematic within a Newtonian context and, as such, we shall ignore it. Notice also that to avoid certain delicate aspects related to frontier points, we have assumed that the interval $(t_1, t_2)$ and all elements $(t, a(t))$ are in the *interior* of $\tilde{R}^{2n+1}$.

Newtonian systems (Appendix A), the requirement of the above theorem is that, for instance, the implicit functions $f^i(t, q, \dot{q})$, are of at least class $\mathscr{C}^1$ in a region $R^{2n+1} = \{t, q, \dot{q}\}$. Then the solution $q^k(t)$ of the initial value problem,

$$\begin{aligned} \ddot{q}^i &= f^i(t, q, \dot{q}), \\ q^k(t_0) &= u_0^k, \qquad \dot{q}^k(t_0) = v_0^k, \qquad i, k = 1, 2, \ldots, n, \end{aligned} \tag{2}$$

exists, is unique, and is of class $\mathscr{C}^1$ in any interval $(t_1, t_2)$ containing $t_0$ such that $(t, q(t), \dot{q}(t))$ lies in the interior of $R^{2n+1}$. But along such solutions, $\ddot{q}^k = f^k$ (from the theorem on implicit functions) and the $f$'s are continuous functions of time in $(t_1, t_2)$. Thus, $q^k(t)$ possesses continuous second-order derivatives in $(t_1, t_2)$ (see also the last statement of Theorem 1.1.4), namely, not only the generalized coordinates and velocities but also the generalized accelerations are continuous in $(t_1, t_2)$.

**Chart 1.3** A Theorem on the Existence, Uniqueness, and Continuity of the Derivatives with respect to the Parameters of Solutions of Newtonian Systems[32]

For the reader's convenience, we shall again give a simplified version of certain aspects of the "existence theory," this time of Theorems 1.1.4 and 1.1.7.

**Theorem.** *When the conditions of the theorem of Chart* 1.2 *are met, and the* $\Xi^\mu$ *functions are single valued, then through every point of the neighborhood* $I_\varepsilon = (t_0, c_0)_\varepsilon$ *there passes one and only one system of functions of time and* $2n$ *parameters* $c^\mu$, $a^\mu = a^\mu(t; c)$, *which, together with* $a^\mu = \partial a^\mu/\partial t$ *satisfy system* (1) *of Chart* 1.2 *identically, are continuous functions of* $t \in (t_1, t_2)$ *and possess continuous first-order partial derivatives with respect to all* $c^\mu \in (c_0^\mu)$.

Again, the application of this theorem to Newtonian systems is straightforward. Suppose that the initial value problem

$$\begin{aligned} \ddot{q}^i &= f^i(t, q, \dot{q}), \\ q^i(t_0) &= u_0^i, \qquad \dot{q}^i(t_0) = v_0^i, \qquad i = 1, 2, \ldots, n \end{aligned} \tag{1}$$

satisfies the conditions of Chart 1.2 and, in addition, the implicit functions $f^i$ are single-valued. Then the solution $q^i(t)$ exists, is unique, and can be uniquely imbedded into a $2n$-parameter family of solutions $q^i(t; u, v)$, only one element of which satisfies system (1). Furthermore, such solutions possess continuous partial derivatives up to and including the second-order with respect to all values $t \in (t_1, t_2)$, $u^i \in (u_0^i)_\varepsilon$ and $v^i \in (v_0)_\varepsilon^i$.

[32] Akhiezer (1962).

*Nota Bene*: The appearance of the additional condition of single-valuedness should be indicated here but not overemphasized. In essence, such a condition can be related to the uniqueness of the imbedding of a solution into a $2n$-parameter family of solutions. Such uniqueness, however, will not be crucial for our analysis. What will turn out to be of primary importance, particularly for the conditions of self-adjointness (Section 2.1), is the continuity of the derivatives of the solutions with respect to the parameters, namely, that the variations (2.1.2) and (2.1.3) be continuous. Such variations, however, are never unique in the sense that for system (1) there always exists a family of admissible variations.

**Chart 1.4** A Relationship between Local and Global Solutions for Conservative Systems[33]

Consider a one-dimensional system of one particle of mass $m$ and coordinate $x(t)$ moving on the half-line $(0, \infty)$ under the action of a conservative force with potential $\mathscr{V}(x(t))$. The Hamiltonian is $H = \frac{1}{2}m\dot{x}^2 + \mathscr{V}(x)$ and the equations of motion are

$$\begin{aligned} \ddot{x} &= v, \\ v &= -\frac{1}{m}\frac{\partial \mathscr{V}}{\partial x}. \end{aligned} \tag{1}$$

Suppose that $\partial\mathscr{V}/\partial x \in (\mathscr{C}^0, \text{Lips})$ uniformly on every compact subset of $(0, \infty)$. Then system (1) admits a unique local solution $x^0(t) > 0$ in the neighborhood of each value $t_0 > 0$.

It is possible to prove that the only case where the local solution $x^0(t)$ does not extend to a global solution is when the particle runs into zero or off to infinity in a finite time. If none of these two possibilities occurs, the motion represented by Equations (1) in $(0, \infty)$ is called *complete*. We can, therefore, say that under the assumption that the motion is complete, a local solution of Equations (1) always extends to a global solution in $(0, \infty)$.

Without proof, we quote the following theorem.

**Theorem.** *Suppose that* $\partial\mathscr{V}/\partial x \in (\mathscr{C}^0, \text{Lips})$ *uniformly on each compact subset of* $(0, \infty)$. *Then*:
*the motion is not complete at* 0 *if* $\mathscr{V}$ *is bounded above in* $0_\varepsilon$;
*the motion is not complete at* $\infty$ *if* $\mathscr{V}$ *is bounded above for* $x \geqslant 1$ *and*

$$\int_1^\infty \frac{dx}{\sqrt{N - \mathscr{V}(x)}} < \infty$$

*for some* $N > \text{Sup } \mathscr{V}(x)$, $x \geq 1$.

*Significance*: The above properties are significant both within the framework of classical mechanics as well as in conducting a comparative study of the corresponding case in quantum mechanics. In the latter case, the Hamiltonian is the (symmetric) operator $-(1/2m)\, d^2/dx^2 - \mathscr{V}(x)$. Then, it is possible to show that the classical and quantum mechanical motions are *not* equivalent when the potential $\mathscr{V}(x)$ is complete at $\infty$ both classically and quantum mechanically and $\partial\mathscr{V}/\partial x$ is "too large" compared to $\mathscr{V}$.

**Chart 1.5** Hilbert Space Approach to Newtonian Mechanics[34]

It is generally assumed that *Hilbert spaces* appear only within a quantum mechanical context. However, recent studies indicate that Newtonian systems can also be studied within the framework of such spaces. In this chart, we touch on certain background questions only. The Hilbert space approach to the Inverse Problem will be considered in Charts 3.16 and 3.17.

[33] Reed and Simon (1975, Vol. II, Appendix to Chapter X.1).

[34] Reed and Simon (1975, Vol. II, Section X-14).

Hamilton's equations characterize a generally *nonlinear* system of $2n$ first-order ordinary differential equations in the *finite-dimensional* phase space of the $a = \{q, p\}$ variables. In order to introduce a Hilbert space, such equations must be turned into an equivalent *linear* system of equations in an *infinite-dimensional* space. This can be acomplished by considering first a map $m(a_0, t) = a(t)$ from the $(2n + 1)$-dimensional $\tilde{R}^{2n+1}$ space of initial data $a_0$ and time $t$ to the $2n$-dimensional space $R^{2n}$ of the solutions $a(t)$ of Hamilton' equations satisfying such initial data, and then introducing complex valued functions $f$ on $\tilde{R}^{2n}$ through the action of an operator $U_t$ according to the expression

$$(U_t f)(a) = f(m(a, t)). \tag{1}$$

Then we can write

$$\left.\frac{d(U_t f)}{dt}\right|_{t=0} = \{f, H\};$$

$$\int \prod_{\mu=1}^{2n} da^\mu \, h\{f, g\} = \int \prod_{\mu=1}^{2n} da^\mu \{h, f\} g;$$

$$f, g, h \in C^\infty(\tilde{R}^{2n}), \tag{2}$$

where $\{f, g\}$ is the Poisson brackets and the last property can be proved by using an integration by parts. The *Liouville form* and *Liouville operator* are defined, respectively, by

$$I(f, g) = \int \prod_{\mu=1}^{2n} da^\mu \{f, g\} H$$

and

$$Lf = \{f, H\}. \tag{2}$$

It is then possible to prove that (a) the Liouville form is *skew-symmetric*, i.e., $I(f, g) = -I(g, f)$, (b) if $H \in \mathscr{C}^1$ and $f, g$ belong to the domain of $L$, then $(f, Lg) = I(f, g)$, and (c) $-iL$ is a *symmetric* operator. If the theorems for the existence, continuity, and uniqueness of a *global* solution of Hamilton's equations hold, then it is also possible to prove the *Liouville Theorem*, namely, that $U_t$ is a *unitary operator*. In this case, $U_t$ characterizes a one-dimensional unitary group whose infinitesimal generator is $-iL$, and $-iL$ is *essentially self-adjoint* on $C^\infty(R^{2n})$.

However, Hamilton's equations do not generally possess global solutions (e.g., when there are forces due to collisions), in which case it is not possible to extend $-iL$ to a self-adjoint operator. As a result, self-adjointness properties can be more easily established in quantum mechanics than in Newtonian Mechanics (in their operational sense). This confirms a predictable difference in the treatment of these two disciplines within the context of Hilbert spaces. Indeed, even for simple conservative systems, the classical Hamiltonian $H = T(q) + V(q)$ can be unbounded from below or above, while the corresponding quantum mechanical Hamiltonian can be bounded on account of the *Uncertainty Principle*. Besides, possible singularities of the potential $V$ (e.g., the Coulomb potential) are made worse in the Newtonian case because they enter into the definition of $L$, with the overall consequence that they can be better handled in a quantum mechanical context. Despite (or, if you like, because of) these and other technical difficulties, the Hilbert space approach to Newtonian Mechanics remains intriguing and potentially effective.

EXAMPLES[35]

*Example 1.1*

The equation $F(\dot{q}, \ddot{q}) = \ddot{q} - \dot{q}^{2/3} = 0$ is regular (Definition 1.1.1) in any region $R^2$ of points $(\dot{q}, \ddot{q})$ because its functional determinant $|\partial F/\partial \ddot{q}| = 1$ is a numerical constant. The function $F(\dot{q}, \ddot{q})$ is continuous in $R^2$ (of class $\mathscr{C}^0$), but it does not possess continuous derivatives nor does it satisfy the Lipschitz conditions in $q$ for any region $R^2$ that includes the value $\dot{q} = 0$. Therefore, its solutions exist but are not unique. Indeed, both integrals

$$\dot{q} = 0 \quad \text{and} \quad \dot{q} = \tfrac{1}{27}(t - 2)^3 \tag{1}$$

are solutions of the equation satisfying the initial conditions

$$\dot{q}(t_0) = 0, \qquad t_0 = 2. \tag{2}$$

The solution $\dot{q} = 0$ is then called a *singular integral* (Rektorys, 1969, page 737) because the uniqueness condition is broken.

*Example 1.2*

The equation

$$F(\dot{q}, \ddot{q}) = \ddot{q} - q^2 = 0 \tag{1}$$

is regular and satisfies the existence and uniqueness theorem. Indeed, its integral is given by

$$\dot{q} = \frac{1}{c - t}, \qquad c = \text{const.} \tag{2}$$

and it is unique.

*Example 1.3*

The system

$$\begin{aligned} &\ddot{q}_1 + \ddot{q}_2^{1/2} + \alpha^2 q_2 - \alpha q_1^{1/2} = 0, \\ &\ddot{q}_1 - \ddot{q}_2^{1/2} + \alpha^2 q_2 + \alpha q_1^{1/2} = 0, \\ &\alpha \neq 0, \qquad \ddot{q}_2 \neq 0 \end{aligned} \tag{1}$$

satisfies the existence and uniqueness theorems for implicit functions and, therefore, can be reduced to the kinematic form

$$\begin{aligned} \ddot{q}_1 &= f_1 = -\alpha q_2, \\ \ddot{q}^2 &= f_2 = +\alpha q_1, \end{aligned} \tag{2}$$

The system also satisfies Theorem 1.1.5. Its solutions can be written

$$\begin{aligned} q_1 &= (c_1 \cos \tfrac{1}{2}\alpha t + c_2 \sin \tfrac{1}{2}\alpha t)e^{(1/2)\alpha t} + (c_3 \cos \tfrac{1}{2}\alpha t + c_4 \sin \tfrac{1}{2}\alpha t)e^{-(1/2)\alpha t} \\ q_2 &= (c_1 \sin \tfrac{1}{2}\alpha t - c_2 \cos \tfrac{1}{2}\alpha t)e^{(1/2)\alpha t} + (-c_3 \sin \tfrac{1}{2}\alpha t + c_4 \cos \tfrac{1}{2}\alpha t)e^{-(1/2)\alpha t}. \end{aligned}$$

Theorem 1.1.7 is satisfied, too, for an interval of time $(t_1, t_2)$, say, in the interior of $(0, \pi/2\alpha)$. Then $q_i, \dot{q}_i$ and $\ddot{q}_i, i = 1, 2$ are continuous and possess continuous derivatives with respect to the four constants of integration.

[35] For references on differential equations with extensive applications see, for instance, Rektorys (1969) or Brauer and Nohel (1969).

*Example 1.4*

Consider the 1-form $A^{(1)} = A_i(q)\, dq^i$ and suppose that it satisfies all conditions (1). (2) and (3) given at the end of Section 1.2 (page 32) for the applicability of the Converse of the Poincaré Lemma. Integrability conditions (1.2.32), in particular, read

$$\delta^{i_1 i_2}_{j_1 j_2} \frac{\partial A_{i_1}}{\partial q^{i_2}} = \frac{\partial A_{j_1}}{\partial q^{j_2}} - \frac{\partial A_{j_2}}{\partial q^{i_1}} = 0. \tag{1}$$

Then Lemma 1.2.2 ensures the existence of a 0-form, i.e., a scalar function $\phi(q)$, such that its exterior derivative coincide with $A^{(1)}$. By using Equation (1.2.25), such a scalar is given by

$$\phi = \left[\int_0^1 d\tau\, A_k(\tau q)\right] q^k. \tag{2}$$

The above statements are a reformulation, in the language of the calculus of differential forms, of the known property according to which a necessary and sufficient condition for a vector to be the gradient of a scalar is that its curl vanishes. Besides a more rigorous formulation of this property, the use of the calculus of differential forms also provides a solution for the scalar function. It should also be indicated that, in the ultimate analysis, this is a solution of the system of *partial* differential equations in the unknown $\phi$,

$$A_i - \frac{\partial \phi}{\partial q^i} = 0, \qquad i = 1, 2, \ldots, n, \tag{3}$$

which is *overdetermined* (because the number of equations exceeds the number of unknowns). One of the most significant applications of this case to Newtonian Mechanics is that when $A_i$ represents an acting force $F_i(q)$. Then, the above integrability conditions are the necessary and sufficient conditions for such force to be conservative, i.e., Equations (A.4.7) (see Appendix). The extension of the case to a differentiable manifold with local coordinates $\dot{q}^k$ (rather than $q^k$) yields the necessary and sufficient conditions for the existence of a power function, i.e., Equations (A.5.11). Notice that the method also provides a *solution* for the primitive form.

*Example 1.5*

Consider the 2-form $A^{(2)} = A_{i_1 i_2}\, dq^{i_1} \wedge dq^{i_2}$ with antisymmetric (2, 0)-tensor $A_{i_1 i_2} (= -A_{i_2 i_1})$, and suppose that it satisfies all the conditions of Lemma 1.2.2. Then the integrability conditions read

$$\delta^{i_1 i_2 i_3}_{j_1 j_2 j_3} \frac{\partial A_{i_1 i_2}}{\partial q^{i_3}} = 2\left(\frac{\partial A_{j_1 j_2}}{\partial q^{j_3}} + \frac{\partial A_{j_2 j_3}}{\partial q^{j_1}} + \frac{\partial A_{j_3 j_1}}{\partial q^{i_2}}\right) = 0. \tag{1}$$

They are the familiar necessary and sufficient conditions for an antisymmetric tensor to be the curl of a vector, say $B_i(q)$. Indeed, Lemma 1.2.2 guarantees the existence of the 1-form $B^{(1)} = B_i\, dq^i$ such that $dB^{(1)} = A^{(2)}$. In this case, the underlying system of partial differential equations in the unknown $B_i$ is

$$\tfrac{1}{2}\delta^{i_1 i_2}_{j_1 j_2}\left(A_{i_1 i_2} - \frac{\partial B_{i_2}}{\partial q^{i_1}}\right) = A_{i_1 i_2} - \frac{1}{2}\left(\frac{\partial B_{i_2}}{\partial q^{i_1}} - \frac{\partial B_{i_1}}{\partial q^{i_2}}\right) = 0 \tag{2}$$

and its solution, again by using Equation (1.2.25), is given by

$$B_i = 2\left[\int_0^1 d\tau\, \tau A_{i_k}(\tau q)\right] q^k. \tag{3}$$

Notice that this solution is not unique because of the "degree of freedom"

$$B^{(1)} \to B'^{(1)} = B^{(1)} + dB^{(0)}, \qquad B^{(0)} = \phi(q), \tag{4}$$

for which $dB'^{(1)} \equiv dB^{(1)}$. Notice also that the formulation of the problem within the context of the calculus of differential forms is more restrictive than that of the ordinary approach, because of the condition that the tensor $A_{i_1 i_2}(q)$ be well behaved in a star-shaped region $R^{n*}$ of points $q$. In turn, this guarantees the existence of the integral of its solution. One of the most significant applications of this case to classical mechanics is that when the tensor $A_{k_1 k_2}$ represents the electromagnetic field $F_{\mu\nu}$ in the variables $(q^k) = (t, r)$, $(c = 1)$. The method provides not only the necessary and sufficient conditions for $F_{\mu\nu}$ to be the curl of the 4-potential $(A_\mu) = (\varphi, \mathbf{A})$, but also a *solution* for $A_\mu$.

*Example 1.6*

Consider again a 2-form $A^{(2)} = A_{i_1 i_2}\, dq^{i_1} \wedge dq^{i_2}$ which satisfies all the conditions of Lemma 1.2.2, but suppose now that the tensor $A_{i_1 i_2}$ is symmetric, i.e., $A_{i_1 i_2} = A_{i_2 i_1}$. Our problem is that of identifying the necessary and sufficient conditions for a symmetric tensor $A_{i_1 i_2}$ to be derivable from a vector $B_i$ according to

$$A_{i_1 i_2} = \frac{1}{2}\left(\frac{\partial B_{i_1}}{\partial q^{i_2}} + \frac{\partial B_{i_2}}{\partial q^{i_1}}\right). \tag{1}$$

This problem can be solved by using Lemma 1.2.2 twice. The 1-form

$$\Omega_{i_1 i_2} = \left(\frac{\partial A_{i_1 i_3}}{\partial q^{i_2}} - \frac{\partial A_{i_2 i_3}}{\partial q^{i_1}}\right) dq^{i_3} = -\Omega_{i_2 i_1} \tag{2}$$

is closed if and only if

$$d\Omega_{i_1 i_2} = \left(\frac{\partial^2 A_{i_1 i_3}}{\partial q^{i_4}\, \partial q^{i_3}} - \frac{\partial^2 A_{i_2 i_3}}{\partial q^{i_4}\, \partial q^{i_2}}\right) dq^{i_4} \wedge dq^{i_3} = 0. \tag{3}$$

When the above conditions hold, there exists a 0-form $\Gamma_{i_1 i_2} = -\Gamma_{i_2 i_1}$[36] for which

$$\Omega_{i_1 i_2} = d\Gamma_{i_1 i_2} = \frac{\partial \Gamma_{i_1 i_2}}{\partial q^{i_3}}\, dq^{i_3}. \tag{4}$$

From the first use of Lemma 1.2.2, we then have the identities

$$\frac{\partial A_{i_1 i_3}}{\partial q^{i_2}} - \frac{\partial A_{i_2 i_3}}{\partial q^{i_1}} = \frac{\partial \Gamma_{i_1 i_2}}{\partial q^{i_3}}. \tag{5}$$

Now introduce the 1-form

$$\omega_{i_1} = A_{i_1 i_2}\, dq^{i_2} - \Gamma_{i_1 i_2}\, dq^{i_2}. \tag{6}$$

[36] According to footnote 18 of page 27, this is a case in which a tensor, without any contraction with the differentials of the variables, is considered as a zero-form.

This form is closed, i.e., $d\omega_{i_1} = 0$, in view of

$$\left(\frac{\partial A_{i_1 i_3}}{\partial q^{i_2}} - \frac{\partial A_{i_2 i_3}}{\partial q^{i_2}}\right) dq^{i_2} \wedge dq^{i_3} = \frac{\partial \Gamma_{i_1 i_2}}{\partial q^{i_3}} dq^{i_2} \wedge dq^{i_3}. \tag{7}$$

From the second use of Lemma 1.2.2, then, a 0-form $B_i$ exists such that

$$\omega_{i_1} = \frac{\partial B_{i_1}}{\partial q^{i_2}} dq^{i_2}, \tag{8}$$

i.e.,

$$A_{i_1 i_2} = \frac{\partial B_{i_1}}{\partial q^{i_2}} + \Gamma_{i_1 i_2}. \tag{9}$$

By interchanging the $i_1$ and $i_2$ indices and summing up, the desired relation holds in view of the antisymmetry of the $\Gamma_{i_1 i_2}$ terms. The integrability conditions are then, from $d\Omega_{i_1 i_2} = 0$,

$$\frac{\partial^2 A_{i_1 i_2}}{\partial q^{i_4} \partial q^{i_2}} - \frac{\partial^2 A_{i_1 i_4}}{\partial q^{i_3} \partial q^{i_2}} - \frac{\partial^2 A_{i_2 i_3}}{\partial q^{i_4} \partial q^{i_1}} + \frac{\partial^2 A_{i_2 i_4}}{\partial q^{i_3} \partial q^{i_1}} = 0. \tag{10}$$

A significant application of this case is that when the symmetric tensor $A_{k_1 k_2}$ represents the *strain tensor* of the Theory of Elasticity.

## Problems

1.1 Consider the second-order ordinary differential equation

$$t\ddot{q} \ln t - \dot{q} - tq \ln {}^3 t = 0.$$

Identify an interval of time for which the theorem of Chart 1.1 (for the existence of the implicit functions) holds. Prove that in such an interval the theorem of Chart 1.2 (for the existence of a solution) also holds and identify such a solution.

1.2 Consider the following variation of the system of Example 1.3.

$$\begin{aligned} \ddot{q}_1 + \ddot{q}_2 + \alpha^2 q_2 - \alpha q_1^{1/2} &= 0, \\ \ddot{q}_2 + \ddot{q}_2 + \alpha^2 q_2 + \alpha q_1^{1/2} &= 0, \\ \alpha \neq 0, \qquad \ddot{q}_2 &\neq 0. \end{aligned}$$

Prove that for such a system both implicit functions $f^1$ and $f^2$ do not exist.

1.3 Compute a solution at large of the system

$$\begin{aligned} \ddot{q}_1 - \dot{q}_1 + \dot{q}_2 &= 0, \\ \ddot{q}_1 + \ddot{q}_2 - q_1 &= 0. \end{aligned}$$

1.4 Prove properties (1.2.11), (1.2.12), and (1.2.13).

1.5 Prove that the generalized Kronecker delta (1.2.8) can be written in terms of the contravariant and covariant *Levi-Civita tensors* as follows

$$\delta^{i_1 \cdots i_p}_{j_1 \cdots j_p} = \varepsilon_{j_1 \cdots j_p} \varepsilon^{i_1 \cdots i_p}.$$

1.6 Prove that the exterior product of a $p_1$-form with a $p_2$-form satisfies the rule

$$A^{(p_1)} \wedge A^{(p_2)} = (-1)^{p_1 p_2} A^{(p_2)} \wedge A^{(p_1)}$$

1.7 Prove that the exterior derivative of the exterior product of problem 1.6 satisfies the rule

$$d(A^{(p_1)} \wedge A^{(p_2)}) = (dA^{(p_1)}) \wedge A^{(p_2)} + (-1)^{p_1} A^{(p_1)} \wedge (dA^{(p_2)})$$

1.8 Prove the identities

$$\frac{\partial}{\partial q^k} \int_0^1 d\tau\, \tau^{p-1} A_{i_1 \cdots i_p}(\tau q) = \int_0^1 d\tau\, \tau^p \frac{\partial A_{i_1 \cdots i_p}(\tau q)}{\partial q^k},$$

which have been tacitly used for Equations (1.2.26).

1.9 By using the Poincaré Lemma, prove that the exterior derivative of the elements $dq^{i_1} \wedge \cdots \wedge dq^{i_p}$ is identically null.

1.10 Prove Equations (1.3.39) and (1.3.42).

CHAPTER 2

# Variational Approach to Self-Adjointness

## 2.1 Equations of Motion, Admissible Paths, Variational Forms, Adjoint Systems, and Conditions of Self-Adjointness

In this chapter we introduce a methodological tool of central relevance for our analysis, the conditions of variational self-adjointness, which will later result to be necessary and sufficient conditions for the existence of a Lagrangian or Hamiltonian (Chapter 3).

This section is devoted to a presentation of the basic ideas as close as possible to their original derivation,[1] i.e., for systems of second-order ordinary differential equations that are generally nonlinear in the second-order derivatives and, as such, generally non-Newtonian (see Appendix A). The specialization of these ideas to the various Newtonian forms of differential equations will be worked out in the subsequent sections. The comparison of the variational approach to self-adjointness and that for operators acting on linear spaces will be considered in Section 2.8 and Chart 3.16. The algebraic significance of the variational approach to self-adjointness will be worked out in Section 2.9, while its geometrical significance will be indicated in the Charts at the end of this chapter and in Chart 3.18.

Throughout this section we consider a system of $n$ second-order ordinary differential equations

$$F_i(q) = F_i(t, q, \dot{q}, \ddot{q}) = 0, \qquad i = 1, 2, \ldots, n, \tag{2.1.1}$$

[1] See, for instance, Frobenius (1878), Helmholtz (1887), Mayer (1896), Kurshak (1906), and Davis (1929).

which is generally nonlinear in all the variables $q^k$, $\dot{q}^k$, and $\ddot{q}^k$, and we assume that it satisfies the global existence theorems of Section 1.1 in such a way that the $\infty^{2n}$-family of solutions $\tau^{2n}_{(t,q)}$ exists, is unique, is of (at least) class $\mathscr{C}^2$ in $(t_1, t_2)$, and possesses continuous derivatives with respect to the $2n$ parameters in the neighborhood of given values.

Our objective is the identification of the conditions of variational self-adjointness without the necessary knowledge of the solutions. This will allow us later (Chapter 3) to compute a Lagrangian, when it exists, also without the necessary knowledge of the solutions. By recalling that the systems considered are generally nonlinear, this is clearly a central requirement for the practical effectiveness of the methodology of the Inverse Problem.

This objective is achieved by

1. Considering the *functions* $F_i$ rather than the equations $F_i = 0$;
2. Computing these functions along a one-parameter path $\mathscr{P} = \{q^k(t; w)\}$ $t \in (t_1, t_2)$, $w \in 0_\varepsilon$, $k = 1, 2, \ldots, n$, which is of (at least) class $\mathscr{C}^2$ in $t$ and of class $\mathscr{C}^1$ in $w$ but is not necessarily a solution of Equations (2.1.1);
3. Applying the variational approach to self-adjointness to the functions $F_i$ along $\mathscr{P}$, i.e., $F(\mathscr{P})$.

More generally and in line with the assumptions of Section 1.3, we shall call the *family of admissible paths* that characterized by *all* one-parameter functions $q^k(t; w)$ satisfying the indicated continuity conditions. The function $F_i$ can then be computed along any element of this family.

To implement the variational approach to self-adjointness we begin with the construction of the *variations* of the admissible paths, which can be defined by[2]

$$\eta^k(t) = \left.\frac{\partial q^k}{\partial w}\right|_{w=0}, \qquad k = 1, 2, \ldots, n. \tag{2.1.2}$$

From the viewpoint of the calculus of variations (Section 1.3), quantities (2.1.2) are, in essence, the finite part of the contemporaneous first-order variations of $q^k$. Indeed, we can write $\delta^1 q^k = \eta^k(t)w$, $w \in 0_\varepsilon$.

It is readily seen that, by construction, the variations $\eta^k(t)$, considered as functions of time, have the same continuity properties of $q^k(t; w)$, namely, they are of (at least) class $\mathscr{C}^2$ in $(t_1, t_2)$. This implies that the derivatives

$$\dot{\eta}^k(t) = \left.\frac{\partial \dot{q}^k}{\partial w}\right|_{w=0}, \ddot{\eta}^k(t) = \left.\frac{\partial \ddot{q}^k}{\partial w}\right|_{w=0} \tag{2.1.3}$$

exist and are continuous in $(t_1, t_2)$.

[2] The analysis can be equivalently carried out for $w$ in the neighborhood of any (finite) value $w^0$.

Our second step is that of constructing the so-called system of (first-order) *variational forms* of Equations (2.1.1). This can be done by computing the functions $F_i$ along a one-parameter admissible path, by differentiating with respect to $w$ and letting $w = 0$. In this way, we reach the system

$$M_i(\eta) = \frac{dF_i}{dw}\bigg|_{w=0} = \frac{\partial F_i}{\partial q^k}\bigg|_{w=0}\eta^k + \frac{\partial F_i}{\partial \dot{q}^k}\bigg|_{w=0}\dot{\eta}^k + \frac{\partial F_i}{\partial \ddot{q}^k}\bigg|_{w=0}\ddot{\eta}^k, \quad (2.1.4)$$

which, from the viewpoint of the calculus of variations, can be considered as the finite part of the contemporaneous first-order variations of $F_i$, i.e., $\delta^1 F_i = M_i(\eta)w$, $w \in 0_\varepsilon$.

Notice again the distinction between the variational forms $M_i(\eta)$ and the associated equations $M_i(\eta) = 0$, called *equations of variations*. These latter equations essentially restrict the class of variations along which the forms $M_i$ are computed.[3] This restriction, even though not excluded, is not necessary for the variational approach to self-adjointness and, therefore, we shall not assume it.

Since the original functions $F_i$ are known and the path $\mathscr{P}$ along which they are computed is also known, all coefficients of the variations in Equations (2.1.4) are known, and we shall write

$$M_i(\eta) = a_{ik}(t)\eta^k + b_{ik}(t)\dot{\eta}^k + c_{ik}(t)\ddot{\eta}^k, \quad (2.1.5)$$

where (*at* $w = 0$),

$$a_{ik} = \frac{\partial F_i}{\partial q^k}, \qquad b_{ik} = \frac{\partial F_i}{\partial \dot{q}^k}, \qquad c_{ik} = \frac{\partial F_i}{\partial \ddot{q}^k}. \quad (2.1.6)$$

Notice that, given a system of functions [Equations (2.1.1)] not necessarily linear in $q^k$, $\dot{q}^k$, and $\ddot{q}^k$, their variational forms (equations) are *always* linear in $\eta^k$, $\dot{\eta}^k$ and $\ddot{\eta}^k$.

The variations $\eta^k(t)$, as defined by Equations (2.1.2), are not unique. We shall call the *family of admissible variations* that family characterized by the variations (2.1.2) of all possible admissible paths. This means, in practice, that *all* functions $\eta^k(t)$ of at least class $\mathscr{C}^2$ in $(t_1, t_2)$ are admissible. Two or more elements of this family can be constructed by considering two or more admissible paths, say $q^k(t; w)$, $\tilde{q}^k(t; w)$, etc. Equations (2.1.2) then yield the variations

$$\eta^k = \frac{\partial q^k}{\partial w}\bigg|_{w=0}, \qquad \tilde{\eta}^k = \frac{\partial \tilde{q}^k}{\partial w}\bigg|_{w=0}, \ldots, \quad (2.1.7)$$

which are admissible because of class $\mathscr{C}^2$ in $(t_1, t_2)$.

[3] It is significant to note that, when the original system (2.1.1) is consistent, so is the system of equations of variations (see Problem 2.11).

We shall call the *family of admissible ordinary variational forms* that family characterized by the computation of the forms $M_i$ along all admissible variations, i.e.,

$$\{M_i(\alpha)\} = \{a_{ik}\alpha^k + b_{ik}\dot{\alpha}^k + c_{ik}\ddot{\alpha}^k, \qquad \alpha = \eta, \tilde{\eta}, \ldots \in \mathscr{C}^2(t_1, t_2)\}, \quad (2.1.8)$$

where the term "ordinary" is introduced to stress the fact that the procedure is here referred, specifically, to ordinary differential equations.[4]

Our next step can be characterized by the following definition.

**Definition 2.1.1.** A system of (ordinary) variational forms $\tilde{M}_i(\tilde{\eta})$ is termed the *adjoint system*[5] of forms $M_i(\eta)$ defined by Equations (2.1.5) when there exists a function $Q(\eta, \tilde{\eta})$[6] such that the *Lagrange identity*[7]

$$\tilde{\eta}^i M_i(\eta) - \eta^i \tilde{M}_i(\tilde{\eta}) = \frac{d}{dt} Q(\eta, \tilde{\eta}) \quad (2.1.9)$$

holds for all admissible variations.

To identify a possible structure of the adjoint system $\tilde{M}_i(\tilde{\eta})$ and of the functions $Q(\eta, \tilde{\eta})$, consider the relations

$$\begin{aligned}
\tilde{\eta}^i M_i(\eta) &= \tilde{\eta}^i a_{ij}\eta^j + \tilde{\eta}^i b_{ij}\dot{\eta}^j + \tilde{\eta}^i c_{ij}\ddot{\eta}^j \\
&= \tilde{\eta}^i a_{ij}\eta^j + \frac{d}{dt}(\tilde{\eta}^i b_{ij}\eta^j) + \frac{d}{dt}\left[(\tilde{\eta}^i c_{ij})\dot{\eta}^j - \eta^j \frac{d}{dt}(\tilde{\eta}^i c_{ij})\right] \\
&\quad - \eta^j \frac{d}{dt}(\tilde{\eta}^i b_{ij}) + \eta^i \frac{d^2}{dt^2}(\tilde{\eta}^i c_{ij}) \qquad (2.1.10) \\
&= \eta^j\left[\tilde{\eta}^i a_{ij} - \frac{d}{dt}(\tilde{\eta}^i b_{ij}) + \frac{d^2}{dt^2}(\tilde{\eta}^i c_{ij})\right] \\
&\quad + \frac{d}{dt}\left[\tilde{\eta}^i b_{ij}\eta^j + \tilde{\eta}^i c_{ij}\dot{\eta}^j - \eta^j \frac{d}{dt}(\tilde{\eta}^i c_{ij})\right].
\end{aligned}$$

[4] For the extension to partial differential equations, see Santilli (1977a).

[5] The term *adjoint* was, apparently, proposed by Fuchs (1873).

[6] $Q(\eta, \tilde{\eta})$ is sometimes called the *bilinear concomitant*.

[7] Condition (2.1.9) was, apparently, introduced by Lagrange. See Bocher (1917, page 23) and Ince (1927, page 124), Notice that this condition can also be turned into an integral form. This yields the *Green identity*:

$$\int_{t_1}^{t_2} dt[\tilde{\eta}^i M_i - \eta^i \tilde{M}_i] = |Q|_{t_1}^{t_2}.$$

Our analysis will be based on the Lagrange rather than the Green identity due to certain uniqueness problems related to the latter.

Thus, $\tilde{M}_i(\tilde{\eta})$ and $Q(\eta, \tilde{\eta})$ can be given, respectively, by

$$\tilde{M}_i(\tilde{\eta}) = \tilde{\eta}^k a_{ki} - \frac{d}{dt}(\tilde{\eta}^k b_{ki}) + \frac{d^2}{dt^2}(\tilde{\eta}^k c_{ki}) \tag{2.1.11a}$$

and

$$Q(\eta, \tilde{\eta}) = \tilde{\eta}^i b_{ij}\eta^j + \tilde{\eta}^i c_{ij}\eta^j - \eta^j \frac{d}{dt}(\tilde{\eta}^i c_{ij}). \tag{2.1.11b}$$

**Theorem 2.1.1** (Uniqueness of the Adjoint System). *Every system of (ordinary) variational forms $M_i$ which is a continuous function of time in $(t_1, t_2)$ possesses one and only one*[8] *adjoint system $\tilde{M}_i$.*

The idea of the proof is the following. Suppose that there are two adjoints $\tilde{M}_i(\tilde{\eta})$ and $\tilde{M}'_i(\tilde{\eta})$ for each given form $M_i(\eta)$. This implies the existence of two functions $Q(\eta, \tilde{\eta})$ and $Q'(\eta, \tilde{\eta})$, such that

$$\tilde{\eta}^i M_i - \eta^i \tilde{M}_i = \dot{Q}, \; \tilde{\eta}^i M_i - \eta^i \tilde{M}'_i = \dot{Q}' \tag{2.1.12}$$

Then we can write

$$\eta^i(\tilde{M}_i - \tilde{M}'_i) = \frac{d}{dt}(Q - Q') \tag{2.1.13}$$

But the integral

$$\int_{t_1}^{t_2} dt\, \eta^i(\tilde{M}_i - \tilde{M}'_i) = |Q - Q^1|_{t_1}^{t_2} \tag{2.1.14}$$

must be independent of the path in $(\eta, \tilde{\eta})$-space. This can be so, in view of the continuity conditions, if and only if $\tilde{M}_i(\tilde{\eta}) = \tilde{M}'_i(\tilde{\eta})$.

Notice that the above argument excludes also the trivial degrees of freedom $\tilde{M}'_i = \tilde{M}_i + c_{i'} c_i = \text{constant}$.

Theorem 2.1.1 can also be studied under weaker continuity conditions of the forms $M_i(\eta)$, but we shall not indulge in analyzing this aspect at this time.[9]

Notice that the actual functional dependence of $M_i$ is in the elements $(t, \eta, \dot{\eta}, \ddot{\eta})$ and, thus, the notation $M_i(\eta)$ must be considered symbolic. The function $Q$ is also, in general, a function of the elements $(t; \eta, \tilde{\eta}; \dot{\eta}, \dot{\tilde{\eta}})$.

Clearly, under the assumed continuity and regularity conditions, the concept of adjointness is reciprocal and involutive. It is reciprocal in the sense

[8] The uniqueness is referred here to the functional structure of the forms $\tilde{M}_i$ and not to the variations $\eta_i$ along which they are computed.

[9] However, we must indicate that, as we shall see more clearly later, when the variations $\eta^k$ are of class $\mathscr{C}^m$, $m < 2$, in the considered region of time there exist considerable difficulties for the conditions of self-adjointness. This is a reason for our restriction to equations of motion which possess solutions of at least class $\mathscr{C}^2$.

that if $\tilde{M}_k(\tilde{\eta})$ is the adjoint of $M_k(\eta)$, then $M_k(\eta)$ is the adjoint of $\tilde{M}_k(\tilde{\eta})$. It is involutive in the sense that if applied twice it reproduces the original form identically.

We are now equipped to introduce a concept of central significance for our analysis.

**Definition 2.1.2.** A system of (ordinary) variational forms $M_i(\eta)$ is termed *self-adjoint* when it coincides with its adjoint system $\tilde{M}_i(\eta)$ for all admissible variations, i.e.,

$$M_i(\eta) = \tilde{M}_i(\eta),$$
$$i = 1, 2, \ldots, n, \qquad \eta \in \mathscr{C}^2. \tag{2.1.15}$$

The conditions of self-adjointness can be derived by imposing the identity between forms (2.1.5) and (2.1.11a), i.e.,

$$a_{ik}\eta^k + b_{ik}\dot{\eta}^k + c_{ik}\ddot{\eta}^k = \eta^k a_{ki} - \frac{d}{dt}(\eta^k b_{ki}) + \frac{d^2}{dt^2}(\eta^k c_{ki}), \tag{2.1.16}$$

which yields

$$c_{ik} = c_{ki}, \tag{2.1.17a}$$
$$b_{ik} + b_{ki} = 2\dot{c}_{ki}, \tag{2.1.17b}$$
$$a_{ik} - a_{ki} = \ddot{c}_{ki} - \dot{b}_{ki}. \tag{2.1.17c}$$

We shall call Equations (2.1.17) the *conditions of self-adjointness of the variational forms* (2.1.5).

A system of ordinary differential Equations (2.1.1) is called *self-adjoint* when its variational forms are self-adjoint. By substituting definitions (2.1.6) into Equations (2.1.17), we obtain the relations

$$\frac{\partial F_i}{\partial \ddot{q}^k} = \frac{\partial F_k}{\partial \ddot{q}^i}, \tag{2.1.18a}$$

$$\frac{\partial F_i}{\partial \dot{q}^k} + \frac{\partial F_k}{\partial \dot{q}^i} = 2\frac{d}{dt}\frac{\partial F_i}{\partial \ddot{q}^k} = \frac{d}{dt}\left(\frac{\partial F_i}{\partial \ddot{q}^k} + \frac{\partial F_k}{\partial \ddot{q}^i}\right), \tag{2.1.18b}$$

$$\frac{\partial F_i}{\partial q^k} - \frac{\partial F_k}{\partial q^i} = \frac{d}{dt}\left[\frac{d}{dt}\left(\frac{\partial F_k}{\partial \ddot{q}^i}\right) - \frac{\partial F_k}{\partial \dot{q}^i}\right]$$
$$= \frac{1}{2}\frac{d}{dt}\left(\frac{\partial F_i}{\partial \dot{q}^k} - \frac{\partial F_k}{\partial \dot{q}^i}\right), \tag{2.1.18c}$$

which must be satisfied everywhere in the considered region $R^{4n+1}$ of points $(t, q, \dot{q}, \ddot{q}, \dddot{q})$. We shall call Equations (2.1.18) the *conditions of self-adjointness for systems of ordinary second-order differential equations.* To my best knowledge, such conditions were first identified by Helmholtz (1887).

Theorem 2.1.1 and Definition 2.1.2 imply the following theorem.

**Theorem 2.1.2** (Self-Adjointness of Systems of Ordinary Second-Order Differential Equations). *A necessary and sufficient condition for a (regular) system of second-order ordinary differential equations* (2.1.1) *to be self-adjoint in a region* $R^{4n+1}$ *of points* $(t, q, \dot{q}, \ddot{q})$ *is that all conditions* (2.1.18) *are everywhere satisfied in* $R^{4n+1}$.

When at least *one* of the conditions (2.1.18) is violated, we shall call the system *non-self-adjoint.*

In practice, given a (regular) system (2.1.1), one first sees whether the functions $F_i$ are of at least class $\mathscr{C}^2$ in a region $R^{3n+1}$ of points $(t, q, \dot{q}, \ddot{q})$.[10] When this is the case, for the self-adjointness of the system it is sufficient that all Equations (2.1.18) are identities among functions. Notice that Equations (2.1.18) imply, in general, third-order time derivatives of $q^k$. Thus, even though the continuity condition $F_k \in \mathscr{C}^2$ is sufficient, when Equations (2.1.18) are computed along a path (rather than considered as identities among functions), such a path is assumed to be of at least class $\mathscr{C}^3$. For later use (see next section), notice also that, if the functions $F_k$ are linear in the $\ddot{q}$'s, then no third-order derivative of $q^k$ appears in Equations (2.1.18), and their computation along a path of class $\mathscr{C}^2$ is sufficient.

A most important property is that, according to Theorem 2.1.2, the self-adjointness or non-self-adjointness of system (2.1.1) can be ascertained without any knowledge of the solutions. Indeed, it is sufficient to ascertain whether conditions (2.1.18) are satisfied as identities among functions (without necessarily considering their computation along given paths), as we shall illustrate with the examples at the end of this and the next chapter.

This is a rather remarkable occurrence. In more explicit but nontechnical terms, we can say that the machinery of the variational approach to self-adjointness, after producing the central conditions (2.1.18), can be ignored in practical applications. We are referring here to: (1) the family of admissible paths, (2) the family of admissible variations, (3) the systems of variational forms (and equations), (4) the systems of adjoint forms (and equations), and (5) the self-adjoint or non-self-adjoint systems of variational forms (or equations). All these tools play a crucial role in the derivation of conditions (2.1.18). Nevertheless, the net result is a set of conditions on the original functions $F_i$ that can be directly tested without any need of variational techniques and without any knowledge of the solutions of the system considered. As indicated earlier, the latter occurrence will allow us to ascertain whether a Lagrangian exists and, in case of affirmative answer, to compute it, without any knowledge of the solutions of the system considered.

For completeness, however, it should be recalled that the terms *self-adjoint or non-self-adjoint systems of ordinary differential equations* refer to *systems of ordinary differential equations whose systems of variational forms are self-adjoint or non-self-adjoint.*

[10] This is due to the fact that conditions (2.1.18) involve second-order partial derivatives.

We are now in a position to identify the arena of applicability of the methods considered in this book. This can be done by considering the following five different classifications of Newtonian systems. The emerging *restrictions* on the applicability of the methods under consideration then trivially extend to ordinary differential equations of non-Newtonian interpretation (e.g., those of the Optimal Control Theory).

## Classification I: Holonomic or Nonholonomic Systems

This classification arises from the nature of the acting constraints (see Appendix). Its significance for our analysis rests on the fact that the methodology for the problem of the existence of a Lagrangian demands a considerable amount of technical implementation in the transition from holonomic to nonholonomic systems. The analysis of this volume is restricted to holonomic systems that can be written in configuration space according to *fundamental form* (A.7.5):

$$F_k(q) = A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0,$$
$$k = 1, 2, \ldots, n. \tag{2.1.19}$$

If the constraints are nonholonomic, the reduction to the configuration space of the generalized coordinates cannot be performed. We are dealing, then, with a system of second-order ordinary differential equations (i.e., Newton's equations of motion) subject to a system of subsidiary constraints. The methods of this book are expected to be extendable to this latter class of systems. Nevertheless, such an extension will not be considered at this time.

## Classification II: Local or Nonlocal Systems

In principle, a Newtonian system can be *nonlocal*, in which case it demands the use of integro-differential equations. These systems are excluded by the analysis of this volume, which is restricted to *local* ordinary differential equations, i.e., equations of type (2.1.1). It should be indicated in this respect that this latter class is sufficient for our needs because it includes conservative, dissipative, and dynamic systems according to their conventional interpretation as reviewed in the Appendix. Notice that the problem of the existence of a Lagrangian or Hamiltonian is trivial for (holonomic) conservative systems but is not trivial for arbitrary local systems. From now on, we shall tacitly assume that fundamental form (2.1.19) represents a holonomic system with arbitrary (but local) Newtonian forces. When the holonomic constraints are absent, we shall tacitly assume that the variables $q^k, k = 1, 2, \ldots, n = 3N$, represent the Cartesian coordinates $r^{ia}, i = 1, 2, \ldots, N, a = x, y, z$ (in a given ordering) of the considered systems of $N$ particles in a three-dimensional Euclidean space.

## Classification III: Systems of Class $\mathscr{C}^m$, $m \geq 2$ or of Class $\mathscr{C}^m$, $m < 2$ in Their Region of Definition

This classification arises from the integrability conditions for the existence of a Lagrangian, which, as we shall see in Chapter 3, are the conditions for self-adjointness (2.1.18) and, as such, exhibit the presence of second-order partial derivatives. From now on, we shall restrict our analysis to systems (2.1.19) which satisfy the continuity conditions

$$A_{ki}, B_k \in \mathscr{C}^m(R^{2n+1}), \qquad m \geq 2. \tag{2.1.20}$$

The above assumption, together with the condition of regularity,[11] also guarantees[12] the consistency of the system considered, (i.e., the existence of a physically acceptable motion). When a Lagrangian for the representation of systems (2.1.19) exists, conditions (2.1.20) correspond to the assumption that such a Lagrangian is of at least class $\mathscr{C}^4$ in $R^{2n+1}$, an assumption rather familiar in the calculus of variations.[13] It should be indicated that the problem of the existence of analytic representations for systems (2.1.19) can also be considered with the minimal continuity conditions,

$$A_{ki}, B_k \in \mathscr{C}^m(R^{2n+1}), \qquad m \geq 1, \tag{2.1.21}$$

by means of their reduction to normal forms (Section 2.4) and the identification of a Hamiltonian (Section 3.12), rather than a Lagrangian. The Hamiltonian would then be, when it exists, of class $\mathscr{C}^3$, a minimal continuity assumption which is also familiar in the canonical formulation of the calculus of variations.[13]

Notice that the assumed continuity conditions exclude impulsive motions.

## Classification IV: Regular or Degenerative Systems

This classification arises from the nature of the functional determinant (Section 1.1). Its significance rests on the fact that the methodology for the existence of a Lagrangian is highly sensitive to the regularity or degeneracy of the functional determinant. The analysis of this volume is restricted to systems (2.1.19), which satisfy the regularity condition

$$|\mathscr{H}|(R) = \left|\frac{\partial F_i}{\partial \ddot{q}^j}\right|(R) = |A_{ij}|(R^{2n+1}) \neq 0 \tag{2.1.22}$$

[11] This condition is requested for the existence of the implicit functions which, in turn, allow the reduction of Equations (2.1.19) to the normal form (Section 1.1). If the condition of regularity is violated, continuity assumptions (2.1.20) are not even sufficient to guarantee the existence of a solution.

[12] Assumptions (2.1.20) are actually redundant for this purpose, with their weaker form (2.1.21) being more familiar (see Chart 1.2).

[13] See, for instance, Bliss (1946, page 7).

at every point $(t, q, \dot{q})$ of the considered region of definition. It should be indicated here that, on practical grounds, it is sufficient to demand that the functional determinant is non-null as a function of $(t, q, \dot{q})$. However, this does not exclude possible zeros of such a function along the path considered. To avoid occurrences of this type we shall, from now on, tacitly assume that the region of definition of systems (2.1.19) is selected in such a way that its functional determinant is everywhere non-null in it. For an illustration of this point, see Example 2.6.

### Classification V: Self-Adjoint or Non-Self-Adjoint Systems

This classification arises from the use of conditions (2.1.18) and it will be shown to be crucial for the problem of the existence of a Lagrangian. The analysis of this volume is restricted to self-adjoint systems. [The problem of the existence of analytic representations for nonselfadjoint systems is treated in Santilli (1979)].

## 2.2 Conditions of Self-Adjointness for Fundamental and Kinematic Forms of Newtonian Systems

Under the restrictions of the preceding section, the central objective of this volume is the study of the necessary and sufficient conditions for the existence of a Lagrangian representation of (local) Newtonian systems in the *fundamental form*:

$$F_k = A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0, \tag{2.2.1a}$$

$$A_{ki}, B_k \in \mathscr{C}^2(R^{2n+1}), \tag{2.2.1b}$$

$$|A_{ij}|(R^{2n+1}) \neq 0; \tag{2.2.1c}$$

or in the equivalent *kinematic form*,

$$\ddot{q}^k - f^k(t, q, \dot{q}) = 0, \tag{2.2.2a}$$

$$f^k \in \mathscr{C}^2(R^{2n+1}), \tag{2.2.2b}$$

where, from the theorem on implicit functions,

$$f^k = -A^{ki}B_i, \qquad (A^{ij}) = (A_{ij})^{-1}. \tag{2.2.3}$$

This objective demands the specialization of the conditions of self-adjointness (2.1.18) to Equations (2.2.1a) and (2.2.2a). Let us begin with the former case. Assume that for Equations (2.1.1) *all* conditions (2.1.18) hold. Then the system is necessarily linear in the accelerations $\ddot{q}^i$. This is due to the fact that, since the left-hand side of Equations (2.1.18b) is independent of $\ddot{q}^i$, all terms $\partial F_i/\partial \ddot{q}^i$ must be independent of the accelerations. Therefore, we

reach the important conclusion that a necessary (but not sufficient) condition for a system (2.1.1) to be self-adjoint is that it is of the Newtonian type (2.2.1a).

Conditions (2.1.18a) now read simply

$$A_{ij} = A_{ji}. \tag{2.2.4}$$

Since the coefficients of the $\ddot{q}^i$ terms of Equation (2.1.18c) must vanish, we have

$$\frac{\partial A_{ik}}{\partial \dot{q}^j} = \frac{\partial A_{jk}}{\partial \dot{q}^i}. \tag{2.2.5}$$

The above conditions, together with Equations (2.2.4), imply that the expressions $\partial A_{ik}/\partial \dot{q}^j$ remain unchanged under all permutations of the indices $i, j, k$.

From Equation (2.1.18b), by using Equation (2.2.1a) and properties (2.2.5), we obtain

$$\begin{aligned}\frac{\partial B_i}{\partial \dot{q}^j} + \frac{\partial B_j}{\partial \dot{q}^i} &= 2\frac{d}{dt}A_{ij} - \left(\frac{\partial A_{ik}}{\partial \dot{q}^j} + \frac{\partial A_{jk}}{\partial \dot{q}^i}\right)\ddot{q}^k \\ &= 2\left[\frac{\partial A_{ij}}{\partial t} + \left(\frac{\partial A_{ij}}{\partial q^k}\dot{q}^k + \frac{\partial A_{ij}}{\partial \dot{q}^k}\ddot{q}^k\right)\right] - \left(\frac{\partial A_{ik}}{\partial \dot{q}^j} + \frac{\partial A_{jk}}{\partial \dot{q}^i}\right)\ddot{q}^k \\ &= 2\left\{\frac{\partial}{\partial t} + \dot{q}^k\frac{\partial}{\partial q^k}\right\}A_{ij}.\end{aligned} \tag{2.2.6}$$

Equations (2.1.18c), by using Equations (2.2.5), can be written

$$\frac{\partial B_i}{\partial q^j} - \frac{\partial B_j}{\partial q^i} = \left(\frac{\partial A_{jk}}{\partial q^i} - \frac{\partial A_{ik}}{\partial q^j}\right)\ddot{q}^k + \frac{1}{2}\frac{d}{dt}\left(\frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i}\right), \tag{2.2.7}$$

which can hold identically if and only if the following separate sets of identities hold.

$$\frac{\partial A_{jk}}{\partial q^i} - \frac{\partial A_{ik}}{\partial q^j} = \frac{1}{2}\frac{\partial}{\partial \dot{q}^k}\left(\frac{\partial B_j}{\partial \dot{q}^i} - \frac{\partial B_i}{\partial \dot{q}^j}\right) \tag{2.2.8a}$$

$$\frac{\partial B_i}{\partial q^j} - \frac{\partial B_j}{\partial q^i} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{q}^k\frac{\partial}{\partial q^k}\right\}\left(\frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i}\right). \tag{2.2.8b}$$

Equations (2.2.8a) are not independent, since they can be obtained from Equations (2.2.6) and (2.2.5).[14] Equations (2.2.8b), however, constitute an independent set of conditions.

[14] See Problem 2.7.

In this way, we obtain the following sets of independent conditions.

$$A_{ij} = A_{ji}, \tag{2.2.9a}$$

$$\frac{\partial A_{ik}}{\partial \dot{q}^j} = \frac{\partial A_{jk}}{\partial \dot{q}^i}, \tag{2.2.9b}$$

$$\frac{\partial B_i}{\partial \dot{q}^j} + \frac{\partial B_j}{\partial \dot{q}^i} = 2\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\} A_{ij}, \tag{2.2.9c}$$

$$\frac{\partial B_i}{\partial q^j} - \frac{\partial B_j}{\partial q^i} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\}\left(\frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i}\right), \tag{2.2.9d}$$

$$i, j, k = 1, 2, \ldots, n,$$

which must hold everywhere on $R^{2n+1}$ for system (2.2.1) to be self-adjoint. We shall call Equations (2.2.9) the *conditions of self-adjointness for the fundamental form.*[15]

The above situation can be summarized by the following theorem.

**Theorem 2.2.1** (Self-Adjointness of the Fundamental Form). *A necessary and sufficient condition for a holonomic Newtonian system,*

$$A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0, \qquad k = 1, 2, \ldots, n, \tag{2.2.10}$$

*satisfying the continuity and regularity conditions*

$$A_{ki}, B_k \in \mathscr{C}^m(R^{2n+1}), \qquad m \geq 2, \tag{2.2.11a}$$

$$|A_{ki}|(R^{2n+1}) \neq 0, \tag{2.2.11b}$$

*in a region $R^{2n+1}$ of points $(t, q, \dot{q})$ to be self-adjoint in $R^{2n+1}$, is that all conditions* (2.2.9) *are satisfied everywhere in $R^{2n+1}$.*

We must stress again that, strictly speaking, when conditions (2.2.9) hold, system (2.2.10) has self-adjoint variational forms. When at least one of conditions (2.2.9) is violated, we shall call the system *non-self-adjoint.*

Essentially, continuity conditions (2.2.11a) guarantee the existence and continuity of all derivatives appearing in Equations (2.2.9) everywhere in $R^{2n+1}$.[16] Regularity condition (2.2.11b) plays a crucial role in Theorem 2.2.1. Indeed, the study of the corresponding theorem for degenerate systems is considerably more delicate.[17]

[15] Conditions (2.2.9) have apparently been derived for the first time by Mayer (1896) and then worked out in more details by Davis (1928 and 1929).

[16] Notice that, strictly speaking, the conditions $A_{ki} \in \mathscr{C}^1$ and $B_k \in \mathscr{C}^2$ are sufficient. Nevertheless this occurrence, as compared with the unified condition (2.2.11a), has little practical significance in Newtonian mechanics.

[17] Basically, the difficulties are due to the fact that, when the condition of regularity is relaxed, the continuity conditions (2.2.11a) alone *are not* sufficient to guarantee the existence of a solution. This, in turn, affects the question of the existence of the adjoint system as well as its uniqueness, both of which are needed to properly define the conditions of self-adjointness.

Another aspect which we must stress is that conditions for self-adjointness (2.2.2) or (2.2.9) *do not* imply the linearity of the system either in $q^k$ or $\dot{q}^k$.

When the $\infty^{2n}$ family of possible paths $q^k(t; u, v)$ for $t \in (t_2, t_2)$ and $(u, v) \in (u_0, v_0)_\varepsilon$ is known, and conditions (2.2.9) are verified, we can alternatively say that system (2.2.1) is everywhere self-adjoint along all possible paths $q(t; u, v)$.

For an illustration of Theorem 2.2.1, see the examples at the end of this chapter as well as those in Chapter 3. As a trivial (but significant) example, consider conservative systems of the type[18]

$$m_i \ddot{\mathbf{r}}_i - \mathscr{F}_i(r) = m_i \ddot{\mathbf{r}}_i + \frac{\partial \mathscr{V}}{\partial \mathbf{r}_i} = 0, \tag{2.2.12}$$

where

$$\frac{\partial^2 V}{\partial \mathbf{r}_i \, \partial \mathbf{r}_i} \neq 0, \qquad \mathscr{V} \in \mathscr{C}^m(R^{3N}), \qquad m \geq 2, \qquad i = 1, 2, \ldots, N. \tag{2.2.13}$$

Then conditions (2.2.9) reduce to the single set of conditions

$$\frac{\partial^2 \mathscr{V}}{\partial \mathbf{r}^i \, \partial \mathbf{r}^j} - \frac{\partial^2 \mathscr{V}}{\partial \mathbf{r}^j \, \partial \mathbf{r}^i} = 0, \tag{2.2.14}$$

which are implicit in continuity properties (2.2.13). Thus, when the acting forces are derivable from a potential $\mathscr{V}(\mathbf{r})$ of (at least) class $\mathscr{C}^2$, the systems in the form (2.2.12) are self-adjoint.

We now study the sets of all independent conditions of self-adjointness for the kinematical form (2.2.2a). Conditions (2.1.18a) are always identically verified for systems in this form. Conditions (2.2.2b) become

$$\frac{\partial f_i}{\partial \dot{q}^j} + \frac{\partial f_j}{\partial \dot{q}^i} = 0 \tag{2.2.15}$$

and constitute a first independent set. Conditions (2.2.2c) can be written

$$\frac{\partial f_i}{\partial q^j} - \frac{\partial f_j}{\partial q^i} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\}\left(\frac{\partial f_i}{\partial \dot{q}^j} - \frac{\partial f_j}{\partial \dot{q}^i}\right) + \frac{1}{2}\left(\frac{\partial^2 f_i}{\partial \dot{q}^j \, \partial \dot{q}^k} - \frac{\partial^2 f_j}{\partial \dot{q}^i \, \partial \dot{q}^k}\right)\ddot{q}^k \tag{2.2.16}$$

and can hold identically if and only if the following distinct sets of conditions hold.

$$\frac{\partial^2 f_i}{\partial \dot{q}^j \, \partial \dot{q}^k} - \frac{\partial^2 f_j}{\partial \dot{q}^i \, \partial \dot{q}^k} = 0, \tag{2.2.17a}$$

$$\frac{\partial f_i}{\partial q^j} - \frac{\partial f_j}{\partial q^i} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\}\left(\frac{\partial f_i}{\partial \dot{q}^j} - \frac{\partial f_j}{\partial \dot{q}^i}\right). \tag{2.2.17b}$$

[18] We assume here no summation on repeated indices.

In this way, we obtain the following sets of all independent conditions

$$\frac{\partial f_i}{\partial \dot{q}^j} + \frac{\partial f_j}{\partial \dot{q}^i} = 0, \tag{2.2.18a}$$

$$\frac{\partial^2 f_i}{\partial \dot{q}^j \partial \dot{q}^k} - \frac{\partial^2 f_j}{\partial \dot{q}^i \partial \dot{q}^k} = 0, \tag{2.2.18b}$$

$$\frac{\partial f_i}{\partial q^j} - \frac{\partial f_j}{\partial q^i} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\}\left(\frac{\partial f_i}{\partial \dot{q}^j} - \frac{\partial f_j}{\partial \dot{q}^i}\right), \tag{2.2.18c}$$

which must be verified everywhere in $R^{2n+1}$ for system (2.2.2a) to be self-adjoint. We shall call Equations (2.2.18) the *conditions of self-adjointness for the kinematical form.*

From Equations (2.2.18a) and (2.2.18b), we see that a necessary (but not sufficient) condition for system (2.2.2a) to be self-adjoint is that it is linear in the velocities,[19] i.e., of the type

$$\ddot{q}_i - \rho_{ij}(t, q)\dot{q}^j - \sigma_i(t, q) = 0. \tag{2.2.19}$$

Then equations (2.2.18a) for system (2.2.19) become

$$\rho_{ij} + \rho_{ji} = 0, \tag{2.2.20}$$

and Equations (2.2.18c), in view of Equations (2.2.20), can be explicitly written

$$\left(\frac{\partial \rho_{ik}}{\partial q^j} + \frac{\partial \rho_{kj}}{\partial q^i} + \frac{\partial \rho_{ji}}{\partial q^k}\right)\dot{q}^k + \left(\frac{\partial \sigma_i}{\partial q^j} - \frac{\partial \sigma_j}{\partial q^i} - \frac{\partial \rho_{ij}}{\partial t}\right) = 0. \tag{2.2.21}$$

Clearly, Equation (2.2.21) can hold everywhere in $R^{2n+1}$ if and only if each term within the parentheses individually vanishes. We obtain, in this way, the following theorem.

**Theorem 2.2.2** (Self-Adjointness of the Kinematical Form). *A necessary and sufficient condition for a Newtonian system in the kinematical form*

$$\ddot{q}_i - f_i(t, q, \dot{q}) = 0, \qquad i = 1, 2, \ldots, n, \tag{2.2.22a}$$

$$f_i \in \mathscr{C}^m(R^{2n+1}), \qquad m \geq 1 \tag{2.2.22b}$$

*to be self-adjoint in a region $R^{2n+1}$ of points $(t, q, \dot{q})$ is that the system is linear in the velocities, i.e., of the type*[20]

$$\ddot{q}_i - \rho_{ij}(t, q)\dot{q}^j - \sigma_i(t, q) = 0, \tag{2.2.23a}$$

$$\rho_{ij}, \sigma_i \in \mathscr{C}^m(R^{n+1}), \qquad m \geq 1, \tag{2.2.23b}$$

[19] Problem 2.5.

[20] We assume the minimal continuity conditions (2.2.23b) in view of the appearance of only first-order partial derivatives in conditions (2.2.24).

*and all the conditions*

$$\rho_{ij} + \rho_{ji} = 0, \tag{2.2.24a}$$

$$\frac{\partial \rho_{ij}}{\partial q^k} + \frac{\partial \rho_{jk}}{\partial q^i} + \frac{\partial \rho_{ki}}{\partial q^j} = 0, \tag{2.2.24b}$$

$$\frac{\partial \rho_{ij}}{\partial t} = \frac{\partial \sigma_i}{\partial q^j} - \frac{\partial \sigma_j}{\partial q^i}, \tag{2.2.24c}$$

*are satisfied everywhere in the subregion* $R^{n+1} \in R^{2n+1}$ *of points* $(t, q)$.

When at least one of the conditions of the above theorem is not verified, we shall say that the kinematical form is *non-self-adjoint.*

When the implicit functions represent the acting forces, the conditions of self-adjointness for the kinematical form are ultimately restrictions on the acting forces. Notice that conditions (2.2.24) *do not* imply linearity in the coordinates $q^k$.

The analysis of this section relates to an arbitrary number $n$ of dimensions. For $n = 1$, we have the following corollaries.

**Corollary 2.2.1a.** *A necessary and sufficient condition for a holonomic one-dimensional Newtonian system in the fundamental form*

$$\begin{gathered} A(t, q, \dot{q})\ddot{q} + B(t, q, \dot{q}) = 0, \\ A, B \in \mathscr{C}^m(R^3), \qquad m \geq 1, \qquad A(R^3) \neq 0, \end{gathered} \tag{2.2.25}$$

*to be self-adjoint in a region* $R^3$ *of points* $(t, q, \dot{q})$ *is that the condition*

$$\frac{\partial B}{\partial \dot{q}} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial q}\dot{q} \tag{2.2.26}$$

*holds everywhere in* $R^3$.

**Corollary 2.2.2a.** *A necessary and sufficient condition for a one-dimensional holonomic Newtonian system in the kinematical form,*

$$\begin{gathered} \ddot{q} - f(t, q, \dot{q}) = 0, \\ f \in \mathscr{C}^m(R^3), \qquad m \geq 1, \end{gathered} \tag{2.2.27}$$

*to be self-adjoint in a region* $R^3$ *of points* $(t, q, \dot{q})$ *is that the implicit function is independent of the velocity* $\dot{q}$.

By comparing conditions of self-adjointness (2.2.9) for the fundamental form (2.2.1a) and conditions (2.2.24) for the kinematical form (2.2.2a), an aspect of considerable methodological significance emerges. Let us first recall that a (regular holonomic) Newtonian system in the fundamental form can be *equivalently* written in its kinematical form. The aspect in which we are interested at this point is that, despite the above equivalence, if the system in the fundamental form is self-adjoint, the same system in its equivalent kinematical form is not necessarily self-adjoint, and vice versa.

Let us illustrate this property with a simple example. Consider the conservative system (2.2.12) under the assumption that the potential function is not linear in the coordinates. Its kinematic form is[21]

$$\ddot{\mathbf{r}}_i + \frac{1}{m_i}\frac{\partial \mathscr{V}}{\partial \mathbf{r}_i} = 0, \tag{2.2.28a}$$

$$\frac{\partial^2 \mathscr{V}}{\partial \mathbf{r}_i \, \partial \mathbf{r}_j} \neq 0, \tag{2.2.28b}$$

i.e., it is of type (2.2.23), where all $\rho$'s are identically null. Then conditions (2.2.24) for Equations (2.2.28) reduce to

$$\frac{1}{m_i}\frac{\partial^2 \mathscr{V}}{\partial \mathbf{r}_i \, \partial \mathbf{r}_j} - \frac{1}{m_j}\frac{\partial^2 \mathscr{V}}{\partial \mathbf{r}_j \, \partial \mathbf{r}_i} = 0, \tag{2.2.29}$$

and system (2.2.28a) is self-adjoint if and only if all masses are equal.[22]

Therefore, if we start with a conservative system (2.2.12) of $N$ particles, all with different masses ($m_i \neq m_j$), which is self-adjoint when written in the fundamental form (2.2.12), after performing the transition to its equivalent kinematic form (2.2.28) the system is no longer self-adjoint. In this case, we can say that the simple operation of "division by the masses" applied to system (2.2.12) is insufficient to break its self-adjointness.

More generally, we have the following property.

**Lemma 2.2.1** (Independence of the Solutions from Self-Adjointness Properties of the Equations). *Systems of ordinary differential equations that are equivalent to a self-adjoint system are not necessarily self-adjoint.*

This property will turn out to be crucial for the construction of an analytic representation of non-self-adjoint systems [see Santilli (1979)].

The variational approach to self-adjointness presented in Section 2.1 and in this section extends to systems of ordinary differential equations of arbitrary order. A unified approach to the conditions of self-adjointness for systems of ordinary differential equations of arbitrary (finite) order is presented in Chart 3.10. The following theorem for the first-order case is significant for our analysis.

**Theorem 2.2.3**[23] (Self-Adjointness of First-Order Differential Equations in Configuration Space). *A necessary and sufficient condition for a system of ordinary first-order differential equations in configuration space,*

$$F_k(t, q, \dot{q}) = 0, \qquad k = 1, 2, \ldots, n, \tag{2.2.30}$$

[21] We assume here no summation for repeated Latin indices.

[22] See also Example 3.5.

[23] See Mayer (1896) and Havas (1973, Appendix B). For the field theoretical case, see, for instance, Santilli (1978, Vol. I).

*which is of at least class $\mathscr{C}^2$ in a region $R^{2n+1}$ of points $(t, q, \dot{q})$, to be self-adjoint is that the system is linear in the velocities, i.e., of the type*

$$F_k = X_{ki}(t, q)\dot{q}^i + Y_k(t, q) = 0, \tag{2.2.31}$$

*and all the conditions of self-adjointness,*

$$X_{ki} + X_{ik} = 0, \tag{2.2.32a}$$

$$\frac{\partial X_{ij}}{\partial q^k} + \frac{\partial X_{jk}}{\partial q^i} + \frac{\partial X_{ki}}{\partial q^j} = 0, \tag{2.2.32b}$$

$$\frac{\partial X_{ij}}{\partial t} = \frac{\partial Y_j}{\partial q^i} - \frac{\partial Y_i}{\partial q^j}, \tag{2.2.32c}$$

*are identically verified in the subregion $R^{n+1} \in R^{2n+1}$ of points $(t, q)$.*

PROOF. Conditions (2.2.18) for system (2.2.30), after simple manipulations, become

$$\frac{\partial^2 F_i}{\partial \dot{q}^k \partial \dot{q}^j} - \frac{\partial^2 F_k}{\partial \dot{q}^i \partial \dot{q}^j} = 0, \tag{2.2.33a}$$

$$\frac{\partial F_i}{\partial \dot{q}^k} + \frac{\partial F_k}{\partial \dot{q}^i} = 0, \tag{2.2.33b}$$

$$\frac{\partial F_i}{\partial q^j} - \frac{\partial F_j}{\partial q^i} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\}\left(\frac{\partial F_i}{\partial \dot{q}^j} - \frac{\partial F_j}{\partial \dot{q}^i}\right). \tag{2.2.33c}$$

The combined use of Equations (2.2.33a) and (2.2.33b) demands that the system be linear in the velocities (Problem 2.5). Conditions (2.2.32) then follows by specializing Equations (2.2.33) to system (2.2.31). *Q.E.D*

Let us recall that system (2.2.30), in the terminology of Section 1.1, is *totally degenerate*. The above theorem indicates that the variational approach to self-adjointness for regular second-order systems can be trivially extended to the case when these systems are totally degenerate. The reader should, however, be alerted that the extension of the analysis to the case of "bona fide" *degenerate* systems of second-order differential equations is not trivial, owing to the presence of subsidiary constraints. See, in this respect, footnote 17 of page 65. In any case, this extension demands specific investigations that will not be considered in this book.

The important property that the Lorentz force is variationally self-adjoint is presented in Example 2.7.

## 2.3 Reformulation of the Conditions of Self-Adjointness within the Context of the Calculus of Differential Forms

As indicated earlier, our study of the integrability conditions for the existence of a Lagrangian will be based on the calculus of differential forms in general, and the Converse of the Poincaré Lemma in particular. This requires a reformulation of the conditions of self-adjointness within such a context.

The Lagrangian representations of Newtonian systems in their fundamental forms (2.2.1a), i.e., the identifications

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^{k_1}} - \frac{\partial L}{\partial q^{k_1}} = \frac{\partial^2 L}{\partial \dot{q}^{k_1}\,\partial \dot{q}^{k_2}}\ddot{q}^{k_2} + \frac{\partial^2 L}{\partial \dot{q}^{k_1}\,\partial q^{k_2}}\dot{q}^{k_2} + \frac{\partial^2 L}{\partial \dot{q}^{k_1}\,\partial t} - \frac{\partial L}{\partial q^{k_1}}$$
$$= A_{k_1 k_2}\ddot{q}^{k_2} + B_{k_1} \tag{2.3.1}$$
$$k_1 = 1, 2, \ldots, n$$

demand the validity of the following separate equations

$$\frac{\partial^2 L}{\partial \dot{q}^{k_1}\,\partial \dot{q}^{k_2}} = A_{k_1 k_2}, \tag{2.3.2a}$$

$$\frac{\partial^2 L}{\partial \dot{q}^{k_1}\,\partial q^{k_2}}\dot{q}^{k_2} + \frac{\partial^2 L}{\partial \dot{q}^{k_1}\,\partial t} - \frac{\partial L}{\partial q^{k_1}} = B_{k_1}. \tag{2.3.2b}$$

Suppose a particular solution, say $K(t, q, \dot{q})$, of Equations (2.3.2a) exists. From the continuity and regularity conditions of Equations (2.2.1), it follows that such a solution must be of (at least) class $\mathscr{C}^4$ and nonlinear[24] in the velocities. The most general solution $L$ of Equations (2.3.2a) can then be written

$$L(t, q, \dot{q}) = K(t, q, \dot{q}) + D_k(t, q)\dot{q}^k + C(t, q). \tag{2.3.3}$$

Indeed, if $K$ is a particular solution of Equations (2.3.2a), then so is function (2.3.3), because of the appearance in such equations of the *second-order* partial derivatives in the velocities.

We now substitute structure (2.3.3) in Equations (2.3.2) and obtain the equations

$$\frac{\partial^2 K}{\partial \dot{q}^{k_1}\,\partial \dot{q}^{k_2}} = A_{k_1 k_2}, \tag{2.3.4a}$$

$$\left(\frac{\partial D_{k_1}}{\partial t} - \frac{\partial C}{\partial q^{k_1}}\right) + \left(\frac{\partial D_{k_1}}{\partial q^{k_2}} - \frac{\partial D_{k_2}}{\partial q^{k_1}}\right)\dot{q}^{k_2} = B_{k_1} + \frac{\partial K}{\partial q^{k_1}} - \frac{\partial^2 K}{\partial \dot{q}^{k_1}\,\partial t} - \frac{\partial^2 K}{\partial \dot{q}^{k_1}\,\partial q^{k_2}}\dot{q}^{k_2}, \tag{2.3.4b}$$

where we have written all terms involving the $K$ function on the right-hand side because they can be assumed to be known from the solution of Equations (2.3.2a). But the left-hand side of Equations (2.3.4b) is linear in the velocities. By differentiating with respect to $\dot{q}^{k_2}$, we obtain the equations

$$\frac{\partial D_{k_1}}{\partial q^{k_2}} - \frac{\partial D_{k_2}}{\partial q^{k_1}} = \frac{\partial B_{k_1}}{\partial \dot{q}^{k_2}} - \left\{\frac{\partial}{\partial t} + \dot{q}^{k_3}\frac{\partial}{\partial q^{k_3}}\right\}\frac{\partial^2 K}{\partial \dot{q}^{k_1}\,\partial \dot{q}^{k_2}} + \frac{\partial^2 K}{\partial q^{k_1}\,\partial \dot{q}^{k_2}} - \frac{\partial^2 K}{\partial \dot{q}^{k_1}\,\partial q^{k_2}} \tag{2.3.5}$$

[24] The regularity condition implies that at least some of the second-order derivatives of the function $K$ in the velocities must be non-null and this, in turn, can occur if and only if the function is at least quadratic (and, thus, nonlinear) in the velocities.

which constitute a second independent set of equations for identifications (2.3.1) to hold, jointly, with Equations (2.3.4a).

By assuming that a solution $D_k$ of these equations exists, we can substitute Equations (2.3.5) into (2.3.4b), yielding the equations

$$\frac{\partial C}{\partial q^{k_1}} = \frac{\partial D_{k_1}}{\partial t} - B_{k_1} - \frac{\partial K}{\partial q^{k_1}} + \frac{\partial^2 K}{\partial \dot{q}^{k_1}\,\partial t} + \frac{\partial^2 K}{\partial q^{k_1}\,\partial \dot{q}^{k_2}}\,\dot{q}^{k_2} + \frac{\partial B_{k_1}}{\partial \dot{q}^{k_2}}\,\dot{q}^{k_2} - \left\{\frac{\partial}{\partial t} + \dot{q}^k\,\frac{\partial}{\partial q^k}\right\}\frac{\partial^2 K}{\partial \dot{q}^{k_1}\,\partial \dot{q}^{k_2}}\,\dot{q}^{k_2}, \quad (2.3.6)$$

which constitute the third (and last) independent set of conditions for identifications (2.3.1) to hold, jointly with Equations (2.3.4a) and (2.3.5).

Suppose now that *all* conditions of self-adjointness (2.2.9) are verified for system (2.2.1a). Then, by using Equations (2.3.4a) and (2.2.9c), the system of equations (2.3.4a), (2.3.5), and (2.3.6) can be written

$$\frac{\partial^2 K}{\partial \dot{q}^{k_1}\,\partial \dot{q}^{k_2}} = A_{k_1 k_2}, \quad (2.3.7a)$$

$$\frac{\partial D_{k_1}}{\partial q^{k_2}} - \frac{\partial D_{k_1}}{\partial q^{k_2}} = \frac{1}{2}\left(\frac{\partial B_{k_1}}{\partial \dot{q}^{k_2}} - \frac{\partial B_{k_2}}{\partial \dot{q}^{k_1}}\right) + \left(\frac{\partial^2 K}{\partial q^{k_1}\,\partial \dot{q}^{k_2}} - \frac{\partial^2 K}{\partial \dot{q}^{k_1}\,\partial q^{k_2}}\right) \equiv Z_{k_1 k_2} \quad (2.3.7b)$$

$$\frac{\partial C}{\partial q^{k_1}} = \frac{\partial D_{k_1}}{\partial t} - B_{k_1} - \frac{\partial K}{\partial q^{k_1}} + \frac{\partial^2 K}{\partial \dot{q}^{k_1}\,\partial t} + \left[\frac{\partial^2 K}{\partial q^{k_1}\,\partial \dot{q}^{k_2}} + \frac{1}{2}\left(\frac{\partial B_{k_1}}{\partial \dot{q}^{k_2}} - \frac{\partial B_{k_2}}{\partial \dot{q}^{k_2}}\right)\right]\dot{q}^{k_2} \equiv W_{k_1} \quad (2.3.7c)$$

where the right-hand side of each of these equations at this point is assumed to be known.

Equations (2.3.7a) constitute a generally overdetermined system of second-order differential equations in only one unknown, the function $K$. Since the partial derivatives are in the velocities only, we can assume that the $t$ and $q^k$ variables are fixed. The underlying differential form is then defined on â (differentiable) manifold with local coordinates $\dot{q}^k$ and can be written (see Section 1.2)

$$A^{(2)} = A_{k_1 k_2}\, d\dot{q}^{k_1} \wedge d\dot{q}^{k_2} = \frac{1}{2!}\,\delta^{i_1 i_2}_{k_1 k_2} A_{i_1 i_2}\, d\dot{q}^{k_1} \wedge d\dot{q}^{k_2}. \quad (2.3.8)$$

The conditions of self-adjointness (2.2.9a) then imply that this 2-form is identically null. Indeed, Equations (2.2.9a) can be rewritten in terms of the generalized Kronecker delta (1.2.9),

$$\delta^{i_1 i_2}_{k_1 k_2} A_{i_1 i_2} = 0, \qquad k_1, k_2 = 1, 2, \ldots, n. \quad (2.3.9)$$

We consider now the exterior derivatives of a (not necessarily null) form (2.3.8). From Equations (1.2.5) and (1.2.17), we can write

$$dA^{(2)} = \frac{\partial A_{k_1k_2}}{\partial \dot{q}^{k_3}}\, d\dot{q}^{k_3} \wedge d\dot{q}^{k_1} \wedge d\dot{q}^{k_2}$$

$$= \frac{1}{3!}\, \delta^{i_1i_2i_3}_{k_1k_2k_3} \frac{\partial A_{i_1i_2}}{\partial \dot{q}^{i_3}}\, d\dot{q}^{k_1} \wedge d\dot{q}^{k_2} \wedge d\dot{q}^{k_3}. \tag{2.3.10}$$

Conditions of self-adjointness (2.2.9b) then imply that this exterior derivative is identically null, as the reader can verify by a simple inspection, because they imply the identities

$$\delta^{i_1i_2i_3}_{k_1k_2k_3} \frac{\partial A_{i_1i_2}}{\partial \dot{q}^{i_3}} = 0, \qquad k_1, k_2, k_3 = 1, 2, \ldots, n. \tag{2.3.11}$$

From now on, whenever working within the context of the calculus of differential forms, we shall use conditions of self-adjointness (2.3.9) and (2.3.11) rather than (2.2.9a) and (2.2.9b).

Next, we consider Equations (2.3.7b), which also characterize a generally overdetermined system of, in this case, first-order partial differential equations. Since the partial derivatives appearing in these equations are in the coordinates, the underlying differential form is defined on a manifold with (local) coordinates $q^k$, and can be written

$$Z^{(2)} = Z_{k_1k_2}\, dq^{k_1} \wedge dq^{k_2}. \tag{2.3.12}$$

The closure condition in this case is

$$dZ^{(2)} = \frac{1}{3!}\, \delta^{i_1i_2i_3}_{k_1k_2k_3} \frac{\partial Z_{i_1i_2}}{\partial q^{i_3}}\, dq^{k_1} \wedge dq^{k_2} \wedge dq^{k_3} = 0 \tag{2.3.13}$$

and it identically holds if and only if

$$\delta^{i_1i_2i_3}_{k_1k_2k_3} \frac{\partial Z_{i_1i_2}}{\partial q^{i_3}} = 0, \qquad k_1, k_2, k_3 = 1, 2, \ldots, n. \tag{2.3.14}$$

But the $Z$-terms are antisymmetric in their indices. Therefore, conditions (2.3.14) reduce to (see Example 1.5)

$$\frac{\partial Z_{k_1k_2}}{\partial q^{k_3}} + \frac{\partial Z_{k_2k_3}}{\partial q^{k_1}} + \frac{\partial Z_{k_3k_1}}{\partial q^{k_2}} = 0. \tag{2.3.15}$$

By substituting the values of the $Z$ terms from Equations (2.3.7b), the above equations take the explicit form

$$\frac{1}{2}\left[\frac{\partial}{\partial q^{k_1}}\left(\frac{\partial B_{k_2}}{\partial \dot{q}^{k_3}} - \frac{\partial B_{k_3}}{\partial \dot{q}^{k_2}}\right) + \frac{\partial}{\partial q^{k_2}}\left(\frac{\partial B_{k_3}}{\partial \dot{q}^{k_1}} - \frac{\partial B_{k_1}}{\partial \dot{q}^{k_3}}\right) + \frac{\partial}{\partial q^{k_3}}\left(\frac{\partial B_{k_1}}{\partial \dot{q}^{k_2}} - \frac{\partial B_{k_2}}{\partial \dot{q}^{k_1}}\right)\right] = 0, \tag{2.3.16}$$

where we have used the continuity properties of the $K$ function. By again using the generalized Kronecker delta, the above equations can be written

$$\delta^{i_1 i_2 i_3}_{k_1 k_2 k_3} \frac{\partial^2 B_{i_1}}{\partial q^{i_2} \partial \dot{q}^{i_3}} = 0, \qquad k_1, k_2, k_3 = 1, 2, \ldots, n. \tag{2.3.17}$$

Consider, now, three sets of conditions of self-adjointness (2.2.9d) in the pair of indices $(k_2, k_3)$, $(k_3, k_1)$, and $(k_1, k_2)$. By differentiating them with respect to $\dot{q}^{k_1}$, $\dot{q}^{k_2}$, and $\dot{q}^{k_3}$, respectively, and adding up, we obtain the expressions

$$\delta^{i_1 i_2 i_3}_{k_1 k_2 k_3} \frac{\partial^2 B_{i_1}}{\partial q^{i_2} \partial \dot{q}^{i_3}} = \frac{1}{3} \left\{ \frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k} \right\} \delta^{i_1 i_2 i_3}_{k_1 k_2 k_3} \frac{\partial^2 B_{i_1}}{\partial \dot{q}^{i_2} \partial \dot{q}^{i_3}} \tag{2.3.18}$$

From the commutativity of the second-order derivatives in the velocities and the antisymmetry properties of the generalized Kronecker delta, it follows that Equations (2.3.18) are identically null. Therefore, under the conditions of self-adjointness, integrability conditions (2.3.17) hold and the 2-form (2.3.12) is closed.

We now consider Equation (2.3.7c), which again characterizes a generally overdetermined system of first-order partial differential equations in the unknown function $C$. Since the partial derivatives are again in the $q^k$ variables, the underlying differential form is defined in a manifold in such (local) coordinates and can be written

$$W^{(1)} = W_k \, dq^k. \tag{2.3.19}$$

The related closure conditions read

$$dW^{(1)} = \frac{1}{2!} \delta^{i_1 i_2}_{k_1 k_2} \frac{\partial W_{i_1}}{\partial q^{i_2}} \, dq^{k_1} \wedge dq^{k_2} = 0 \tag{2.3.20}$$

and can identically hold if and only if

$$\delta^{i_1 i_2}_{k_1 k_2} \frac{\partial W_{i_1}}{\partial q^{i_2}} = 0, \qquad k_1, k_2 = 1, 2, \ldots, n. \tag{2.3.21}$$

By substituting the explicit form of the $W_i$ terms from Equations (2.3.7c), we obtain the relations

$$\begin{aligned} \delta^{i_1 i_2}_{k_1 k_2} \frac{\partial W_{i_1}}{\partial q^{i_2}} = {} & \left( \frac{\partial B_{k_2}}{\partial q^{k_1}} - \frac{\partial B_{k_1}}{\partial q^{k_1}} \right) - \frac{1}{2} \frac{\partial}{\partial t} \left( \frac{\partial B_{k_2}}{\partial \dot{q}^{k_1}} - \frac{\partial B_{k_1}}{\partial \dot{q}^{k_2}} \right) \\ & + \frac{1}{2} \left[ \frac{\partial}{\partial q^{k_2}} \left( \frac{\partial B_{k_1}}{\partial \dot{q}^{k_3}} - \frac{\partial B_{k_3}}{\partial \dot{q}^{k_1}} \right) + \frac{\partial}{\partial q^{k_1}} \left( \frac{\partial B_{k_3}}{\partial \dot{q}^{k_2}} - \frac{\partial B_{k_2}}{\partial \dot{q}^{k_3}} \right) \right] \dot{q}^{k_3} = 0 \end{aligned} \tag{2.3.22}$$

which, from Equations (2.2.9d), can be written

$$\delta^{i_1 i_2}_{k_1 k_2} \frac{\partial W_{i_1}}{\partial q^{i_2}} = \frac{1}{2} \delta^{i_1 i_2 i_3}_{k_1 k_2 k_3} \frac{\partial^2 B_{i_1}}{\partial q^{i_2} \partial \dot{q}^{i_3}} \dot{q}^{k_3} = 0 \tag{2.3.23}$$

and they identically hold whenever Equations (2.3.17) hold. Therefore, under the conditions of self-adjointness, 1-form (2.3.19) is closed. Notice that the closure conditions for Equations (2.3.7b) and (2.3.7c) are equivalent.

Finally, for consistency, the right-hand sides of Equations (2.3.7b) and (2.3.7c) must be independent of the velocity. By differentiating these equations with respect to $\dot{q}^{k_3}$ and after some simple algebra, we obtain the respective consistency conditions

$$\frac{1}{2}\frac{\partial}{\partial \dot{q}^{k_3}}\left(\frac{\partial B_{k_1}}{\partial \dot{q}^{k_2}} - \frac{\partial B_{k_2}}{\partial \dot{q}^{k_1}}\right) + \left(\frac{\partial A_{k_2 k_3}}{\partial q^{k_1}} - \frac{\partial A_{k_1 k_2}}{\partial q^{k_2}}\right) = 0 \qquad (2.3.24a)$$

and

$$\left[\frac{1}{2}\frac{\partial}{\partial \dot{q}^{k_3}}\left(\frac{\partial B_{k_1}}{\partial \dot{q}^{k_1}} - \frac{\partial B_{k_2}}{\partial \dot{q}^{k_1}}\right) + \left(\frac{\partial A_{k_1 k_3}}{\partial q^{k_1}} - \frac{\partial A_{k_1 k_3}}{\partial q^{k_2}}\right)\right]\dot{q}^{k_3} = 0, \qquad (2.3.24b)$$

which are clearly equivalent among themselves. But, as indicated in Section 2.2, Equation (2.3.24a) can be derived from Equations (2.2.9b) and (2.2.9c). Therefore, under the conditions of self-adjointness, consistency conditions (2.3.24) identically hold.

We now summarize the contents of this section:

1. The most general structure of the Lagrangian for the representation of Newtonian systems in their fundamental form (2.2.1) is given by structure (2.3.3).
2. The underlying system of independent partial differential equations for the existence of such a Lagrangian is given by Equations (2.3.7).
3. The reinterpretation of the conditions of self-adjointness within the context of the calculus of differential forms leads to the following closure and consistency conditions.

$$\delta^{i_1 i_2}_{k_1 k_2} A_{i_1 i_2} = 0, \qquad (2.3.25a)$$

$$\delta^{i_1 i_2 i_3}_{k_1 k_2 k_3} \frac{\partial A_{i_1 i_2}}{\partial \dot{q}^{i_3}} = 0, \qquad (2.3.25b)$$

$$\delta^{i_1 i_2 i_3}_{k_1 k_2 k_3} \frac{\partial^2 B_{i_1}}{\partial q^{i_2} \partial \dot{q}^{i_3}} = 0, \qquad (2.3.25c)$$

$$\frac{1}{2}\frac{\partial}{\partial \dot{q}^{k_3}}\left(\frac{\partial B_{k_1}}{\partial \dot{q}^{k_2}} - \frac{\partial B_{k_2}}{\partial \dot{q}^{k_1}}\right) + \left(\frac{\partial A_{k_2 k_3}}{\partial q^{k_1}} - \frac{\partial A_{k_1 k_3}}{\partial q^{k_2}}\right) = 0 \qquad (2.3.25d)$$

It should be stressed that *all* conditions of self-adjointness (2.2.9) enter into the reformulation (2.3.25). Also, when conditions (2.2.9) hold, Equations (2.3.25) are automatically verified and there is no need to reinspect them.

Finally, it should be stressed that Equations (2.2.9) and (2.3.25) *are not* equivalent. What we have proved is simply that the former equations always imply the latter. This is sufficient for our needs (see Chapter 3).

The interpretation of the conditions of self-adjointness (2.2.24) for kinematic form (2.2.23) within the context of the calculus of differential forms is left to the interested reader (see Problem 2.8).

## 2.4 The Problem of Phase Space Formulations

As is well known, the transition from configuration space to phase space of a Newtonian system (2.2.1) requires the knowledge of the Lagrangian $L(t, q, \dot{q})$ and is based on the prescriptions for the *canonical* generalized momenta $p_k$,

$$G_k(t, q, \dot{q}, p) = p_k - \frac{\partial L}{\partial \dot{q}^k} = 0, \qquad k = 1, 2, \ldots, n. \tag{2.4.1}$$

However, at this stage of our analysis we do not know yet whether a Lagrangian capable of representing system (2.2.1) exists and, thus, we are not yet in a position to introduce canonically conjugate variables $(q, p)$.

In order to treat this aspect, we shall first transform Newtonian systems of either one of the forms (2.2.1) or (2.2.2) into equivalent, not necessarily canonical systems of $2n$ first-order equations. This is done in this section. We shall then express these systems in a more adequate tensor notation (Section 2.5).

Let us introduce a set of prescriptions for the characterization of new variables $y_k$, which are linear in the velocities, i.e., of the type

$$\tilde{G}_k(t, q, \dot{q}, y) = \alpha_{ki}(t, q, y)\dot{q}^i + \beta_k(t, q, y) = 0, \tag{2.4.2}$$

where $\alpha_{ij}$ and $\beta_i$ represent known functions.

In order to avoid cases rather delicate to handle, we assume that prescriptions (2.4.2) are selected in such a way as to produce a one-to-one mapping of points $(t, q, \dot{q})$ of the region $R^{2n+1}$ onto points $(t, q, y)$ of an "image" region $\tilde{R}^{2n+1}$. We shall fulfill this requirement by assuming that the functions $\alpha_{ij}(t, q, y)$ and $\beta_k(t, q, y)$ are single-valued, of class $\mathscr{C}^2$ in $\tilde{R}^{2n+1}$,[25] and are such that the determinants

$$\left|\frac{\partial \tilde{G}_i}{\partial \dot{q}^j}\right| = |\alpha_{ij}| \quad \text{and} \quad \left|\frac{\partial \tilde{G}_i}{\partial y^j}\right| \tag{2.4.3}$$

are non-null everywhere in their respective regions of definition.

These assumptions imply that the normal forms

$$y_k = g_k(t, q, \dot{q}) \tag{2.4.4a}$$

and

$$\dot{q}_k = \tilde{g}_k(t, q, y) \tag{2.4.4b}$$

exist and are unique, single-valued, and at least of class $\mathscr{C}^2$ in $R^{2n+1}$ and $\tilde{R}^{2n+1}$, respectively.

[25] This assumption will later be consistent with class $\mathscr{C}^3$ Hamiltonians.

We should stress the fact that the $2n$ variables $(q, y)$ are not necessarily canonically conjugate. Nevertheless, the above approach is useful for the reduction of the system of $n$ second-order equations (2.2.1) to an equivalent system of $2n$ first-order equations in the variables $(q, y)$.

Indeed, under the above assumptions, the second-order derivatives

$$\ddot{q}^k = \frac{d}{dt}\tilde{g}^k = \frac{\partial \tilde{g}^k}{\partial q^i}\dot{q}^i + \frac{\partial \tilde{g}^k}{\partial y_i}\dot{y}_i + \frac{\partial \tilde{g}^k}{\partial t} \tag{2.4.5}$$

exist (and are continuous). Therefore, they can be substituted into Equations (2.2.1a), yielding the following equation[26] in the $y$.

$$\begin{aligned}\tilde{F}_i(t, q, y, \dot{y}) &= A_{ij}(t, q, \tilde{g})\left(\frac{\partial \tilde{g}^j}{\partial q^k}\tilde{g}^k + \frac{\partial \tilde{g}^j}{\partial y_k}\dot{y}_k + \frac{\partial \tilde{g}^k}{\partial t}\right) + B_i(t, q, \tilde{g}) \\ &\equiv \alpha_i'^j(t, q, y)\dot{y}_j + \beta_i'(t, q, y).\end{aligned} \tag{2.4.6}$$

The system of $2n$ first-order equations linear in $\dot{q}^k$ and $\dot{y}_k$ so constructed, i.e.,

$$\tilde{G}_i = \alpha_{ij}(t, q, y)\dot{q}^j + \beta_i(t, q, y) = 0, \tag{2.4.7a}$$

$$\tilde{F}_i = \alpha_i'^j(t, q, y)\dot{y}_j + \beta_i'(t, q, y) = 0, \tag{2.4.7b}$$

is equivalent to system (2.2.1). Indeed, the procedure is everywhere invertible.

Kinematic form (2.2.2) can be reduced accordingly.

In should be mentioned that when a canonical structure exists, prescriptions (2.4.2) do not coincide in form with the conventional prescriptions (2.4.1) (with $y_k \equiv p_k$). However, when the functions $\alpha_{ij}$ reduce to the Kronecker $\delta_{ij}$, then the resulting form, i.e.,

$$\tilde{G}_i = \dot{q}_i - \tilde{g}_i(t, q, y) = 0 \tag{2.4.8}$$

must coincide with the normal form in $\dot{q}_i$ of Equations (2.4.1) for such a canonical structure to exist.

More generally, in view of the assumed regularity conditions, prescriptions (2.4.2) can always be written in the "factorized" form

$$\tilde{G}_i = \alpha_{ij}(t, q, y)[\dot{q}^j - \tilde{g}^j(t, q, y)] = 0, \tag{2.4.9a}$$

$$\tilde{g}^j = -\alpha^{jk}\beta_k, \qquad (\alpha^{ij}) = (\alpha_{ij})^{-1}. \tag{2.4.9b}$$

Thus, when a canonical structure exists in the space of $(q, y)$ variables, in view of the regularity and continuity of the $\alpha_{ij}$ functions, Equation (2.4.9a) is equivalent to the normal form in $\dot{q}^i$ of the conventional prescriptions (2.4.1).

The reasons for selecting prescriptions (2.4.2) or (2.4.9) linear in $\dot{q}_i$ rather than an equivalent form of type (2.4.1) (i.e., linear in $y_i$) are related to certain properties of the conditions of self-adjointness, and will be discussed later on.

[26] We want here to obtain equations which contain time derivatives of $y_k$ variables *only*. Therefore, we substitute the $\dot{q}^k$ terms with the $\tilde{g}^k$ functions of Equations (2.4.4b).

## 2.5 General and Normal Forms of the Equations of Motion

In this section we implement a second step which will be crucial for the necessary and sufficient condition for Equations (2.4.7) to admit a Hamiltonian representation. This step essentially consists of the identification within Equations (2.4.7) of two forms of first-order equations termed general and normal, each one expressible in either contravariant or covariant form. The problem of the analytic representation of the normal forms in terms of Hamilton's equations will be studied in Chapter 3. [The significance of the general tensorial forms for the problem of equivalent canonical formulations are studied in Santilli (1979)].

The reader should be aware of the fact that the proper handling of these tensorial forms, or of any tensorial quantity in general, inevitably demands a geometrical analysis, with particular reference to transformation theory, as an essential tool for the same characterization of the tensors considered.

This program will be implemented in sequential steps. In this section we shall simply identify such tensor forms on somewhat empirical grounds without reference to their geometric significance or their transformation properties. In the charts at the end of this chapter, we shall point out the geometric interpretation of such tensor forms for the primary objective of identifying the differentiation between the contravariant and covariant versions of the same forms. The program is completed in Santilli (1979) when studying the phase space transformation theory.

The ultimate significance of the tensorial forms, however, will be transparent only after the introduction of Hamilton's equations (Chapter 3). Indeed, one of the most effective ways of expressing the canonical equations is precisely in terms of contravariant and covariant normal forms. In turn, such formulations will play a crucial role in the problem of identifying the necessary and sufficient condition for the existence of a Hamiltonian capable of representing Equations (2.4.7).

Introduce the vector $a^\mu$ with $2n$ components

$$a^\mu = \begin{cases} q^\mu, & \mu = 1, 2, \ldots, n, \qquad (2.5.1a) \\ y_{\mu-n}, & \mu = n+1, n+2, \ldots, 2n \qquad (2.5.1b) \end{cases}$$

and the matrices

$$(C_{\mu\nu}(t, a^\sigma)) = \begin{pmatrix} (\alpha_{ij}(t, a^\sigma)) & 0_{n\times n} \\ 0_{n\times n} & (\alpha'^{ij}_i(t, a^\sigma)) \end{pmatrix}, \qquad (2.5.2a)$$

$$(D_\mu(t, a^\sigma)) = \begin{pmatrix} \beta_i(t, a^\sigma) \\ \beta'_i(t, a^\sigma) \end{pmatrix}, \qquad (2.5.2b)$$

where the functions $\alpha_{ij}$ and $\beta_i$ are the functions appearing in prescriptions (2.4.2) and the functions $\alpha'_{ij}$ and $\beta'_i$ are defined by Equations (2.4.6). Then

system (2.4.7) can be written in the form

$$C_{\mu\nu}(t, a^\sigma)\dot{a}^\nu + D^\mu(t, a^\sigma) = 0, \tag{2.5.3a}$$

$$C_{\mu\nu}, D_\mu \in \mathscr{C}^m(\tilde{R}^{2n+1}), \qquad m \geq 2, \tag{2.5.3b}$$

$$|C_{\mu\nu}|(\tilde{R}^{2n+1}) \neq 0, \tag{2.5.3c}$$

$$\mu, \nu = 1, 2, \ldots, 2n,$$

which we shall call the *general first-order form of a Newtonian system,* or *general form* for short.

Owing to the linearity of the derivatives as well as regularity condition (2.5.3c), the normal form of system (2.5.3) can be identified easily.

It is convenient, for reasons to be illustrated later, to denote the matrix elements of the inverse matrix $(C)^{-1}$ with upper indices, i.e.,

$$(C_{\mu\nu})^{-1} \equiv (C^{\mu\nu}). \tag{2.5.4}$$

Then, in view of the identities

$$C_{\mu\alpha}C^{\alpha\nu} = C^{\nu\alpha}C_{\alpha\mu} = \delta^\nu_\mu, \tag{2.5.5}$$

system (2.5.3) can be written in the equivalent[27] form

$$\dot{a}^\mu - \Xi^\mu(t, a^\sigma) = 0, \mu = 1, 2, \ldots, 2n, \tag{2.5.6a}$$

$$\Xi^\mu \in \mathscr{C}^m(\tilde{R}^{2n+1}), \qquad m \geq 2, \tag{2.5.6b}$$

$$\Xi^\mu = -C^{\mu\nu}D_\nu, \tag{2.5.6c}$$

which we call the *normal form.*[28]

Notice that system (2.5.6) is of the type considered for the existence theorems of Section 1.1.

Clearly, an inspection of systems (2.5.3) and (2.5.6) indicates the need for an interpretation of the significance of the upper and lower indices as well as the identification of a tensor of rank 2 suitable for raising and lowering such indices.

The geometrical framework is undoubtedly the best arena for analyzing these problems. However, as indicated at the beginning of this section, we shall not consider the geometric aspect at this point, but we refer the interested reader to Chart 2.5. Essentially, we shall now proceed to a characterization of the above tensor forms suitable for the identification of their conditions of self-adjointness.

[27] In the sense that systems (2.5.3) and (2.5.6) characterize the same family of possible paths.
[28] See footnote 6 of Chapter 1 for this terminology.

For reasons to be justified later (primarily from the conditions of self-adjointness), we assume the elements of the matrix[29]

$$(\omega_{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ +1_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{2.5.7}$$

as the tensor needed for lowering the index of the $a^\mu$ vector.

By inspection, we note the following.

1. The elements $\omega_{\mu\nu}$ are numerical constants given explicitly by

$$\omega_{\mu\nu} = \begin{cases} 0 & \text{for } \mu, \nu \leq n, \quad \mu, \nu \geq n, \\ +1 & \text{for } \quad \nu \leq n, \quad \mu = \nu + n, \\ -1 & \text{for } \quad \mu \leq n, \quad \nu = \mu + n, \end{cases} \tag{2.5.8}$$

   and, thus, are independent of the paths in $\tilde{R}^{2n+1}$.
2. The matrix $(\omega_{\mu\nu})$ is antisymmetric:

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{2.5.9}$$

3. The inverse matrix $(\omega^{\mu\nu}) \equiv (\omega_{\mu\nu})^{-1}$ exists [because $(\omega_{\mu\nu})$ is regular] and is given explicitly by

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & +1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix}. \tag{2.5.10}$$

   Thus, the properties

$$\omega_{\mu\alpha}\omega^{\alpha\nu} = \omega^{\nu\alpha}\omega_{\alpha\mu} = \delta_\mu^\nu \tag{2.5.11}$$

   always hold.[30]
4. The transition from the $a^\mu = (q, y)$ coordinates to a new set of coordinates $a_\mu$ defined by

$$a_\mu \equiv \omega_{\mu\nu}a^\nu = (-y, q) \tag{2.5.12}$$

   preserves the equivalence with configuration space formulations, in the sense that the equations with lower indices that can be constructed with the tensor $\omega_{\mu\nu}$ from Equations (2.5.6) are also equivalent to the corresponding equations in configuration space.
5. The tensor $\omega_{\mu\nu}$ (or $\omega^{\mu\nu}$) is independent of the Lagrangian or the Hamiltonian and, thus, can be introduced at this stage of our analysis.

[29] The initiated reader has eventually identified the introduction of a *symplectic structure* in the space of the coordinate $a = (q, y)$ through the inverse of matrix (2.5.7) (for more details, see Chart 2.3).

[30] In the language of matrix theory we can say that the matrix $(\omega^{\mu\nu})$ is *unimodular*, *antisymmetric*, and *orthogonal*, i.e.,

$$|\omega^{\mu\nu}| = 1, (\omega)^T + (\omega) = 0, (\omega)^T(\omega) = (\omega)(\omega)^T = 1,$$

where $T$ denotes the transpose.

We now introduce the quantities

$$\Xi_\mu = \omega_{\mu\nu}\Xi^\nu. \tag{2.5.13}$$

By contracting equations (2.5.6) with $\omega_{\mu\nu}$ and by using definition (2.5.12), we obtain the system

$$\dot{a}_\mu - \Xi_\mu(t, a^\sigma) = 0, \tag{2.5.14}$$

which constitutes another form for representing Newton's equations (2.2.1).

On similar grounds, one can construct a fourth form of the type

$$C^{\mu\nu}(t, a^\sigma)\dot{a}_\nu + D^\mu(t, a^\sigma) = 0. \tag{2.5.15}$$

It should be mentioned at this point that the tensor for raising or lowering the indices of the normal form does not necessarily apply to the general form because, in this case, a suitably generalized tensor with a possible path dependence is, in principle, admissible. Therefore, the form $(\omega_{\mu\nu})$ is not unique.[31]

By construction, all tensor forms (2.5.3), (2.5.6), (2.5.14), and (2.5.15) are equivalent among themselves as far as the characterization of the solutions in $q^k$ is concerned. However, they have different algebraic (or geometric) properties which will be indicated later on.

By anticipating some of these properties, we shall label all equations with upper (lower) indices *contravariant* (*covariant*). Explicitly, we shall call Equations (2.5.3) and (2.5.15) *the covariant and contravariant general forms*, respectively, and equations (2.5.6) and (2.5.14) *the contravariant and covariant normal forms*, respectively.[32]

Notice that the above forms can be related in a "crosswise" way, in the sense that the theorem on implicit functions applied to the contravariant (covariant) general form gives rise to the covariant (contravariant) normal form. Alternatively, we can write the factorizations

$$\begin{aligned} C_{\mu\nu}\dot{a}^\nu + D_\mu &= C_{\mu\nu}(\dot{a}^\nu - \Xi^\nu) = 0 \\ C^{\mu\nu}\dot{a}_\nu + D^\mu &= C^{\mu\nu}(\dot{a}_\nu - \Xi_\nu) = 0 \end{aligned} \tag{2.5.16}$$

in which the "crosswise" relationship is transparent.

Again, we must stress the point that the forms considered in this section are not necessarily embodied in a canonical structure owing to the independence of prescriptions (2.4.2) from the existence of a Lagrangian.

[31] See Section 2.7, the comments after Theorem 2.7.3.

[32] This terminology is introduced from the geometrical significance of the upper and lower indices as indicated in Chart A.13. In particular, Equations (2.5.6) characterize *contravariant vector fields* and, similarly, Equations (2.5.14) characterize *covariant vector fields* (Chart 2.2).

## 2.6 Variational Forms of General and Normal Systems

Consider a covariant system of the type

$$\Gamma_\mu(t, a^\alpha, \dot{a}^\alpha) = 0, \qquad \mu = 1, 2, \ldots, 2n, \tag{2.6.1a}$$

$$\Gamma_\mu \in \mathscr{C}^m(\tilde{R}^{4n+1}), \qquad m \geq 2, \tag{2.6.1b}$$

$$\left|\frac{\partial \Gamma_\mu}{\partial \dot{a}^\nu}\right|(\tilde{R}^{4n+1}) \neq 0, \tag{2.6.1c}$$

which is not necessarily linear in $\dot{a}^\mu$. Let $a^\mu(t; w)$ denote a one-parameter family $\tilde{\tau}^1_{(t,a)}$ of admissible paths which is of at least class $\mathscr{C}^2$ for all $t \in (t_1, t_2)$ and possesses continuous first-order derivatives with respect to $w$ in a neighborhood $0_\varepsilon$.

Then the variations (see Section 2.1) of $\tilde{\tau}^1_{(t,a)}$,

$$\eta^\mu = \left.\frac{\partial a^\mu}{\partial w}\right|_{w=0}, \tag{2.6.2}$$

exist and are continuous together with their derivatives,

$$\dot{\eta}^\mu = \left.\frac{\partial \dot{a}^\mu}{\partial w}\right|_{w=0}, \tag{2.6.3}$$

for all $t \in (t_1, t_2)$.

The (first-order) *variational forms* of system (2.6.1) can be written[33]

$$M_\mu(\eta) = c_{\mu\nu}(t)\dot{\eta}^\nu + d_{\mu\nu}(t)\eta^\nu, \tag{2.6.4}$$

where

$$c_{\mu\nu} = \frac{\partial \Gamma_\mu}{\partial \dot{a}^\nu}, \qquad d_{\mu\nu} = \frac{\partial \Gamma_\mu}{\partial a^\nu}, \tag{2.6.5}$$

and they are of covariant type.

In view of the identity

$$\begin{aligned}\tilde{\eta}^\mu M_\mu(\eta) &= \tilde{\eta}^\mu c_{\mu\nu}\dot{\eta}^\nu + \tilde{\eta}^\mu d_{\mu\nu}\eta^\nu \\ &= \left[\tilde{\eta}^\mu d_{\mu\nu} - \frac{d}{dt}(\tilde{\eta}^\mu c_{\mu\nu})\right]\eta^\nu + \frac{d}{dt}(\tilde{\eta}^\mu c_{\mu\nu}\eta^\nu),\end{aligned} \tag{2.6.6}$$

which hold for all admissible variations (Section 2.1), the *adjoint system* of Equations (2.6.4) is

$$\tilde{M}_\mu(\tilde{\eta}) = \tilde{\eta}^\nu d_{\nu\mu} - \frac{d}{dt}(\tilde{\eta}^\nu c_{\nu\mu}). \tag{2.6.7}$$

Indeed, by introducing the scalar

$$Q(\eta, \tilde{\eta}) = \tilde{\eta}^\mu c_{\mu\nu}\eta^\nu, \tag{2.6.8}$$

[33] Notice the need for the proper characterization of the equations of variation of considering a "covariant" ("contravariant") system (2.6.1) in the "contravariant" ("covariant") variables $a$.

the identity

$$\tilde{\eta}^{\mu}M_{\mu}(\eta) - \eta^{\mu}\tilde{M}_{\mu}(\tilde{\eta}) = \frac{d}{dt}Q(\eta, \tilde{\eta}) \tag{2.6.9}$$

holds for all admissible variations.

When system (2.6.1) is the covariant general form

$$\Gamma_{\mu} = C_{\mu\nu}(t, a^{\sigma})\dot{a}^{\nu} + D_{\mu}(t, a^{\sigma}) = 0, \tag{2.6.10}$$

the quantities (2.6.5) are given by

$$c_{\mu\nu} = C_{\mu\nu}, \qquad d_{\mu\nu} = \frac{\partial C_{\mu\alpha}}{\partial a^{\nu}}\dot{a}^{\alpha} + \frac{\partial D_{\mu}}{\partial a^{\nu}} \tag{2.6.11}$$

and Equations (2.6.4), (2.6.7), and (2.6.9) are correspondingly defined.

When system (2.6.1) is the covariant normal form

$$\Gamma_{\mu} = \omega_{\mu\nu}\dot{a}^{\nu} - \Xi_{\mu}(t, a^{\alpha}) = 0, \tag{2.6.12}$$

quantities (2.6.5) become

$$c_{\mu\nu} = \omega_{\mu\nu}, \qquad d_{\mu\nu} = -\frac{\partial \Xi_{\mu}}{\partial a^{\nu}} \tag{2.6.13}$$

and system (2.6.4) with its adjoint system (2.6.7) are correspondingly defined.

Notice that both systems (2.6.10) and (2.6.12) are "covariant" systems with a functional dependence on the "contravariant" vector $a^{\mu}$. These systems originate in a natural way from the reduction of Equation (2.2.1) to a first-order form and, as such, they constitute the most direct framework for the study of the conditions of self-adjointness.

The condition of self-adjointness can, however, be equivalently studied for the contravariant forms when considered as functions of the covariant vector $a_{\mu}$.

Indeed, for a contravariant system of the type[34]

$$\Gamma^{\mu}(t, a_{\sigma}, \dot{a}_{\sigma}) = 0, \qquad \mu = 1, 2, \ldots, 2n, \tag{2.6.14a}$$

$$\Gamma^{\mu} \in \mathscr{C}^{m}(\tilde{R}_{4n+1}), \qquad m \geq 2, \tag{2.6.14b}$$

$$\left|\frac{\partial \Gamma^{\mu}}{\partial a_{\nu}}\right|(\tilde{R}_{4n+1}) \neq 0, \tag{2.6.14c}$$

one can also introduce the variations

$$\eta_{\mu} = \left.\frac{\partial a_{\mu}}{\partial w}\right|_{w=0}, \qquad \dot{\eta}_{\mu} = \left.\frac{\partial \dot{a}_{\mu}}{\partial w}\right|_{w=0}. \tag{2.6.15}$$

[34] From now on, symbols of the type $\tilde{R}^{4n+1}$, and $\tilde{R}_{4n+1}$ will denote regions in the contravariant and covariant variables, respectively.

Then the contravariant variational forms are given by

$$M^{\mu}(\eta) = c^{\mu\nu}\dot{\eta}_{\nu} + d^{\mu\nu}\eta_{\nu} \tag{2.6.16}$$

where

$$c^{\mu\nu} = \frac{\partial \Gamma^{\mu}}{\partial a_{\nu}}, \qquad d^{\mu\nu} = \frac{\partial \Gamma^{\mu}}{\partial a_{\nu}} \tag{2.6.17}$$

and the adjoint system

$$\tilde{M}^{\mu}(\tilde{\eta}) = \tilde{\eta}_{\nu}\, d^{\nu\mu} - \frac{d}{dt}(\tilde{\eta}_{\nu} c^{\nu\mu}) \tag{2.6.18}$$

satisfies, for all admissible variations, the identity

$$\tilde{\eta}_{\mu} M^{\mu}(\eta) - \eta_{\mu}\tilde{M}^{\mu}(\tilde{\eta}) = \frac{d}{dt} Q'(\eta, \tilde{\eta}), \tag{2.6.19}$$

where

$$Q'(\eta, \tilde{\eta}) = \tilde{\eta}_{\mu} c^{\mu\nu} \eta_{\nu}. \tag{2.6.20}$$

Consider now the contravariant normal form

$$\dot{a}^{\mu} - \Xi^{\mu}(t, a^{\sigma}) = 0. \tag{2.6.21}$$

In order to obtain an expression suitable for the construction of the equations of variations, we must re-express its dependence in terms of the covariant vector $a_{\mu}$. This can be done by using the quantities

$$a^{\mu} = \omega^{\mu\nu} a_{\nu}, \tag{2.6.22}$$

for which

$$\dot{a}^{\mu} - \Xi^{\mu}(t, \omega^{\sigma\rho} a_{\rho}) = \omega^{\mu\nu}\dot{a}_{\nu} - \Xi'^{\mu}(t, a_{\rho}) = 0, \tag{2.6.23}$$

where the prime emphasizes the fact that the functions $\Xi^{\mu}(t, a^{\sigma})$ are now regarded as new functions on $a_{\rho}$.

Quantities (2.6.17) for systems (2.6.23) become

$$c^{\mu\nu} = \omega^{\mu\nu}, \; d^{\mu\nu} = -\frac{\partial \Xi'^{\mu}}{\partial a_{\nu}} \tag{2.6.24}$$

Then system (2.6.16) and the adjoint system (2.6.18) are defined accordingly.

For the contravariant general form, notice that in the transition from the matrix $(C_{\mu\nu}(t, a^{\sigma}))$ to its inverse $(C^{\mu\nu}(t, a^{\sigma}))$, the functional dependence of its elements remains on the contravariant vectors $a^{\sigma}$.

In order to be in a form suitable for the study of the conditions of self-adjointness, the contravariation general form (2.5.15) must be rewritten as a function of the covariant vector $a_{\rho}$, i.e.,

$$C^{\mu\nu}(t, a^{\sigma'})\dot{a}_{\nu} + D^{\mu}(t, a^{\sigma}) = C'^{\mu\nu}(t, a_{\rho})\dot{a}_{\nu} + D'^{\mu}(t, a_{\rho}) = 0. \tag{2.6.25}$$

For such a system, definitions (2.6.17) become

$$c^{\mu\nu} = C'^{\mu\nu}, \qquad d^{\mu\nu} = \frac{\partial C'^{\mu\alpha}}{\partial a_\nu}\dot{a}_\alpha + \frac{\partial D'^{\mu}}{\partial a_\nu}. \tag{2.6.26}$$

System (2.6.16) and adjoint system (2.6.18) then follow accordingly.

## 2.7 Conditions of Self-Adjointness for General and Normal Systems

We now study the conditions under which the covariant system

$$\Gamma_\mu(t, a^\sigma, \dot{a}^\sigma) = 0, \qquad \mu = 1, 2, \ldots, 2n \tag{2.7.1a}$$

$$\Gamma_\mu \in \mathscr{C}^m(\tilde{R}^{4n+1}), \qquad m \geq 2 \tag{2.7.1b}$$

$$\left|\frac{\partial \Gamma'_\mu}{\partial a^\nu}\right|(\tilde{R}^{4n+1}) \neq 0 \tag{2.7.1c}$$

is self-adjoint in a region $\tilde{R}^{4n+1}$ of points $(t, a^\sigma, \dot{a}^\sigma)$. This is the case, according to Definition 2.1.2, when the system of variational forms (2.6.4) coincides with adjoint system (2.6.7) for all admissible variations, i.e.,

$$M_\mu(\eta) \equiv \tilde{M}_\mu(\eta), \qquad \eta \in \mathscr{C}^2 \tag{2.7.2}$$

The above conditions explicitly read

$$c_{\mu\nu}\dot{\eta}^\nu + d_{\mu\nu}\eta^\nu \equiv \tilde{\eta}^\nu d_{\mu\nu} - \frac{d}{dt}(\eta^\nu c_{\nu\mu}), \tag{2.7.3}$$

and they can hold identically for all admissible variations if and only if the properties

$$c_{\mu\nu} + c_{\nu\mu} = 0, \tag{2.7.4a}$$

$$d_{\mu\nu} - d_{\nu\mu} = \dot{c}_{\mu\nu}, \tag{2.7.4b}$$

$$\mu, \nu = 1, 2, \ldots, 2n$$

are satisfied everywhere in the interval $(t_1, t_2)$. We shall call Equations (2.7.4) the *conditions of self-adjointness for a covariant first-order system of variational forms.*

By substituting definitions (2.6.5) into Equations (2.7.4) for $c_{\mu\nu}$ and $d_{\mu\nu}$, we obtain the identities

$$\frac{\partial \Gamma_\mu}{\partial \dot{a}^\nu} + \frac{\partial \Gamma_\nu}{\partial \dot{a}^\mu} = 0, \tag{2.7.5a}$$

$$\frac{\partial \Gamma_\mu}{\partial a^\nu} - \frac{\partial \Gamma_\nu}{\partial a^\mu} = \frac{d}{dt}\frac{\partial \Gamma_\mu}{\partial a^\nu}, \tag{2.7.5b}$$

$$\mu, \nu = 1, 2, \ldots, 2n,$$

which must be satisfied everywhere in $\tilde{R}^{4n+1}$.

**Theorem 2.7.1** (Self-Adjointness of a Covariant First-Order Ordinary System in Tensor Notation). *A necessary and sufficient condition for a covariant system* (2.7.1) *to be self-adjoint in a region* $\tilde{R}^{6n+1}$ *of points* $(t, a^\alpha, \dot{a}^\alpha, \ddot{a}^\alpha)$ *is that all conditions* (2.7.5) *are satisfied everywhere in* $\tilde{R}^{6n+1}$.

It should be recalled that the conditions of self-adjointness (2.7.5) do not demand the knowledge of a solution for their applicability to systems (2.7.1). This point will be tacitly implied from here on.

Notice the significance of continuity conditions (2.7.1b) for the right-hand side of Equations (2.7.5b) to be well defined. Conditions (2.7.5b) explicitly read

$$\frac{\partial \Gamma_\mu}{\partial a^\nu} - \frac{\partial \Gamma_\nu}{\partial a^\mu} = \left\{\frac{\partial}{\partial t} + \dot{a}^\alpha \frac{\partial}{\partial a^\alpha}\right\} \frac{\partial \Gamma_\mu}{\partial \dot{a}^\nu} + \frac{\partial^2 \Gamma_\mu}{\partial \dot{a}^\nu \, \partial \dot{a}^\alpha} \ddot{a}^\alpha. \tag{2.7.6}$$

Since the left-hand sides of these equations are independent of $\ddot{a}^\alpha$, we see that a necessary (but not sufficient) condition for system (2.7.1) to be self-adjoint is that it is linear in the first-order derivatives, i.e., it is of the type[35]

$$\Gamma_\mu = C_{\mu\nu}(t, a^\sigma)\dot{a}^\nu + D_\mu(t, a^\sigma) = 0, \tag{2.7.7a}$$

$$C_{\mu\nu}, D_\mu \in \mathscr{C}^m(\tilde{R}^{2n+1}), \qquad m \geq 2, \tag{2.7.7b}$$

$$|C_{\mu\nu}|(\tilde{R}^{2n+1}) \neq 0. \tag{2.7.7c}$$

We shall now derive the sets of all independent conditions of self-adjointness, specifically, for this form.

Conditions (2.7.5a) for system (2.7.7) read

$$C_{\mu\nu} + C_{\nu\mu} = 0, \tag{2.7.8}$$

namely, the matrix $(C_{\mu\nu})$ must be antisymmetric. Let us now recall that an $m \times m$ matrix which is antisymmetric is (is not) necessarily singular when the dimension $m$ is odd (even). Therefore, property (2.7.8) is consistent with the regularity condition (2.7.7c) because its dimension is always even.

Conditions (2.7.7b) for system (2.7.7) become

$$\left(\frac{\partial C_{\mu\alpha}}{\partial a^\nu} - \frac{\partial C_{\nu\alpha}}{\partial a^\mu}\right)\dot{a}^\alpha + \frac{\partial D_\mu}{\partial a^\nu} - \frac{\partial D_\nu}{\partial a^\mu} = \frac{\partial C_{\mu\nu}}{\partial t} + \frac{\partial C_{\mu\nu}}{\partial a^\alpha}\dot{a}^\alpha \tag{2.7.9}$$

and they can hold identically if and only if the conditions

$$\frac{\partial C_{\mu\nu}}{\partial a^\alpha} + \frac{\partial C_{\nu\alpha}}{\partial a^\mu} + \frac{\partial C_{\alpha\mu}}{\partial a^\nu} = 0 \tag{2.7.10a}$$

and

$$\frac{\partial C_{\mu\nu}}{\partial t} = \frac{\partial D_\mu}{\partial a^\nu} - \frac{\partial D_\nu}{\partial a^\mu} \tag{2.7.10b}$$

[35] This is the reason for selecting prescriptions (2.4.3), which are linear in the first-order derivatives.

are satisfied everywhere in $\tilde{R}^{2n+1}$, where for conditions (2.7.10a) we have used properties (2.7.8).

In this way, we obtain the following theorem.

**Theorem 2.7.2** (Self-Adjointness of the Covariant General Form). *A necessary and sufficient condition for the covariant system*

$$C_{\mu\nu}(t, a^\sigma)\dot{a}^\nu + D_\mu(t, a^\sigma) = 0, \qquad \mu = 1, 2, \ldots, 2n, \tag{2.7.11a}$$

$$C_{\mu\nu}, D_\mu \in \mathscr{C}^m(\tilde{R}^{2n+1}), \qquad m \geq 1, \tag{2.7.11b}$$

$$|C_{\mu\nu}|(\tilde{R}^{2n+1}) \neq 0 \tag{2.7.11c}$$

*to be self-adjoint in a region $\tilde{R}^{2n+1}$ of points $(t, a^\sigma)$ is that all the conditions*

$$C_{\mu\nu} + C_{\nu\mu} = 0, \tag{2.7.12a}$$

$$\frac{\partial C_{\mu\nu}}{\partial a^\rho} + \frac{\partial C_{\nu\rho}}{\partial a^\mu} + \frac{\partial C_{\rho\mu}}{\partial a^\nu} = 0, \tag{2.7.12b}$$

$$\frac{\partial C_{\mu\nu}}{\partial t} = \frac{\partial D_\mu}{\partial a^\nu} - \frac{\partial D_\nu}{\partial a^\mu}, \tag{2.7.12c}$$

$$\mu, \nu, \rho = 1, 2, \ldots, 2n,$$

*are satisfied everywhere in $\tilde{R}^{2n+1}$.*

When at least one of the conditions (2.7.12) is violated, we shall call the system (2.7.11) *non-self-adjoint.*

Notice that conditions (2.7.5) or (2.7.12) *do not* imply linearity in $a^\mu$ and that continuity conditions (2.7.11b) with $m \geq 1$ (rather than $m \geq 2$) are now sufficient for Equations (2.7.12).

Again, strictly speaking, conditions (2.7.12) are the necessary and sufficient conditions for system (2.7.11) to possess a self-adjoint system of variational forms. When the general solution $a^\mu(t; c)$ of system (2.7.11) is known and conditions (2.7.12) hold, we can say that the system is self-adjoint everywhere along all possible paths. For an illustration of Theorem 2.7.2, see the examples at the end of this chapter as well as those of Chapter 3.

We now introduce the $\omega$ structure defined by Equation (2.5.7) and transform system (2.7.11) into the covariant normal form

$$\omega_{\mu\nu}\dot{a}^\nu - \Xi_\mu(t, a^\sigma) = 0, \tag{2.7.13a}$$

$$\Xi_\mu \in \mathscr{C}^m(\tilde{R}^{2n+1}), \qquad m \geq 1, \tag{2.7.13b}$$

Our problem is to study the conditions of self-adjointness, specifically, for system (2.7.13). This can be done by using either conditions (2.7.5) or conditions (2.7.12). In the latter case, expressions (2.7.13) can be interpreted as a subcase of Equations (2.7.11), with the substitutions

$$C_{\mu\nu} \to \omega_{\mu\nu}, \tag{2.7.14}$$

$$D_\mu \to -\Xi_\mu.$$

Conditions (2.7.12a) then become

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0, \tag{2.7.15}$$

and they are always identically satisfied because the matrix $(\omega_{\mu\nu})$ is antisymmetric by assumption. Conditions (2.7.12b) identically hold because

$$\frac{\partial \omega_{\mu\nu}}{\partial a^{\sigma}} \equiv 0, \tag{2.7.16}$$

also by assumption. Finally, conditions (2.7.12c) become

$$\frac{\partial \Xi_{\mu}}{\partial a^{\nu}} - \frac{\partial \Xi_{\nu}}{\partial a^{\mu}} = 0 \tag{2.7.17}$$

in view of substitution (2.7.14) and the assumptions

$$\frac{\partial \omega_{\mu\nu}}{\partial t} = 0. \tag{2.7.18}$$

Thus, we have the following theorem.

**Theorem 2.7.3** (Self-Adjointness of the Covariant Normal Form). *A necessary and sufficient condition for the covariant system*

$$\omega_{\mu\nu}\dot{a}^{\nu} - \Xi_{\mu}(t, a^{\sigma}) = 0, \qquad \mu = 1, 2, \ldots, 2n, \tag{2.7.19a}$$

$$\Xi_{\mu} \in \mathscr{C}^{m}(\tilde{R}^{2n+1}), \qquad m \geq 1 \tag{2.7.19b}$$

*to be self-adjoint in a region $\tilde{R}^{2n+1}$ of points $(t, a^{\sigma})$ is that all the conditions*

$$\frac{\partial \Xi_{\mu}}{\partial a^{\nu}} - \frac{\partial \Xi_{\nu}}{\partial a^{\mu}} = 0, \qquad \mu, \nu = 1, 2, \ldots, 2n \tag{2.7.20}$$

*are satisfied everywhere in $\tilde{R}^{2n+1}$.*

We are now in a position to comment about the assumption of Section 2.6 for the $\omega_{\mu\nu}$ tensor as the lowering tensor of the contravariant normal form.

In essence, the matrix $(\omega_{\mu\nu})$ is selected (independently from any geometrical consideration) to comply with the conditions of self-adjointness of the covariant normal form, or, more specifically, the matrix $(\omega_{\mu\nu})$ is identified as a solution with constant elements of conditions of self-adjointness (2.7.12a) and (2.7.12b).[36] Since these two sets of conditions also admit solutions with an explicit dependence on time and path, the above situation illustrates the comment after equations (2.5.15) related to the nonuniqueness of the $\omega$ form.[37]

[36] The geometrical significance of the form $\omega$ has been indicated in footnote 29 of this chapter. Here we would like to stress the fact that the selection of $\omega$ has been done to comply with the conditions of self-adjointness and, thus, quite independently from a symplectic structure. Not surprisingly, the symplectic approach and our differential approach converge to the same nondegenerate antisymmetric form $\omega$. See Charts 2.1 to 2.5 for more details.

[37] This is an aspect of typical geometrical significance. In this respect, see also Charts 2.1 to 2.5.

On similar grounds, one can study the conditions of self-adjointness for the contravariant forms when considered as functions of the covariant vector $a_\mu$. This is left as an exercise for the interested reader.

A problem of particular *algebraic* significance, as we shall see in Section 2.9, is the following problem.

*Given a self-adjoint covariant general form in $a^\mu$, what are the differential properties of the corresponding contravariant general form when considered as a function of $a^\mu$, too?*

In turn, this problem is centered on the properties of matrix $(C^{\mu\nu}(t, a^\sigma))$ when its inverse $(C_{\mu\nu}(t, a^\sigma))$ is a regular $2n \times 2n$ matrix satisfying the identities (2.7.12a) and (2.7.12b).

As is known from matrix theory, the inverse of a (regular) antisymmetric matrix is also antisymmetric. Thus, conditions (2.7.12a) imply that

$$C^{\mu\nu} + C^{\nu\mu} = 0. \tag{2.7:21}$$

To identify the properties that correspond to Equations (2.7.12b), consider the relation

$$C_{\mu\alpha}C^{\alpha\nu} = C^{\nu\alpha}C_{\alpha\mu} = \delta^\nu_\mu; \tag{2.7.22}$$

then the identity

$$\frac{\partial C_{\mu\alpha}}{\partial a^\rho} C^{\alpha\nu} + C_{\mu\alpha}\frac{\partial C^{\alpha\nu}}{\partial a^\rho} = 0 \tag{2.7.23}$$

can be written

$$C^{\tau\rho}\frac{\partial C^{\sigma\nu}}{\partial a^\rho} = C^{\tau\rho}C^{\alpha\sigma}C^{\nu\mu}\frac{\partial C_{\mu\alpha}}{\partial a^\rho}. \tag{2.7.24}$$

By permuting the indices and summing up, we can write[38]

$$C^{\nu\rho}\frac{\partial C^{\sigma\nu}}{\partial a^\rho} + C^{\sigma\rho}\frac{\partial C^{\nu\tau}}{\partial a^\rho} + C^{\nu\rho}\frac{\partial C^{\tau\sigma}}{\partial a^\rho} = C^{\tau\rho}C^{\sigma\alpha}C^{\nu\mu}\left(\frac{\partial C_{\mu\alpha}}{\partial a^\rho} + \frac{\partial C_{\alpha\rho}}{\partial a^\mu} + \frac{\partial C_{\rho\mu}}{\partial a^\alpha}\right) \equiv 0, \tag{2.7.25}$$

which is identically null in view of conditions (2.7.12b).

In this way, we obtain the following theorem.

**Theorem 2.7.4** (A Connection Between Covariant and Contravariant Self-Adjoint Structures). *A necessary and sufficient condition for a matrix $(C^{\mu\nu}(t, a^\sigma))$ to be the inverse of a $2n \times 2n$ matrix $(C_{\mu\nu}(t, a^\sigma))$ satisfying the continuity and regularity conditions*

$$C_{\mu\nu} \in \mathscr{C}^m(\tilde{R}^{2n+1}), \qquad m \geq 1, \qquad |C_{\mu\nu}|(\tilde{R}^{2n+1}) \neq 0, \tag{2.7.26}$$

[38] W. Pauli (1953).

*and the identities*

$$C_{\mu\nu} + C_{\nu\mu} = 0, \tag{2.7.27a}$$

$$\frac{\partial C_{\mu\nu}}{\partial a^{\rho}} + \frac{\partial C_{\nu\rho}}{\partial a^{\mu}} + \frac{\partial C_{\rho\mu}}{\partial a^{\nu}} = 0 \tag{2.7.27b}$$

*in a region* $\tilde{R}^{2n+1}$ *of points* $(t, a^{\sigma})$, *is that all the conditions*

$$C^{\mu\nu} + C^{\nu\mu} = 0, \tag{2.7.28a}$$

$$C^{\tau\rho}\frac{\partial C^{\mu\nu}}{\partial a^{\rho}} + C^{\mu\rho}\frac{\partial C^{\nu\tau}}{\partial a^{\rho}} + C^{\nu\rho}\frac{\partial C^{\tau\mu}}{\partial a^{\rho}} = 0 \tag{2.7.28b}$$

*are satisfied everywhere in* $\tilde{R}^{2n+1}$. *Conversely, the necessary and sufficient conditions for a matrix* $(C_{\mu\nu}(t, a^{\sigma}))$ *to be the inverse of a* $2n \times 2n$ *matrix* $(C^{\mu\nu}(t, a^{\alpha}))$, *which satisfies equations* (2.7.28) *in a region* $\tilde{R}^{2n+1}$ *in which it is of at least class* $\mathscr{C}^1$ *and regular, is that all conditions* (2.7.27) *are satisfied everywhere in* $\tilde{R}^{2n+1}$.

In essence, the above theorem indicates that conditions (2.7.27) and (2.7.28) are equivalent in the sense that, under the assumed continuity and regularity properties, each set of conditions uniquely implies the other, and vice versa.

But we can have, as a particular case, $C_{\mu\nu} = \omega_{\mu\nu}$. The theorem then illustrates a relationship between the contravariant and covariant forms $\omega^{\mu\nu}$ and $\omega_{\mu\nu}$. We can also say, more generally, that the forms $(C_{\mu\nu})$ and $(\omega_{\mu\nu})$ are solutions of Equations (2.7.27) satisfying conditions (2.7.26), while their inverses $(C^{\mu\nu})$ and $(\omega^{\mu\nu})$ are solutions of Equations (2.7.28), satisfying corresponding continuity and regularity properties.

## 2.8 Connection with Self-Adjointness of Linear Operators

The concept of self-adjointness which is nowadays familiar in theoretical physics is that for *linear operators* acting on vector spaces. Such a concept, as we shall now see, is closely related to the variational approach to self-adjointness considered in this chapter.

For simplicity (but without loss of generality), consider a finite-dimensional, real linear space $S$ over the field $F$ of real numbers. Let $S$ be of dimension $n$ with elements $u_i$, $w_j$, etc. Consider an operator $A_{ij}$ which transforms an $n$ vector $u$ into an $n$ vector $w$ according to

$$W_i = A_{ij}u_j \tag{2.8.1}$$

or, in symbolic notation,

$$w = Au = A(u). \tag{2.8.2}$$

Define the inner product of $S$:

$$(w', w) = (w', A(u)) = w'_i A_{ij} u_j. \tag{2.8.3}$$

Consider another operator, $\tilde{A}_{ij}$, which transform a vector $v$ into a vector $z$, i.e.,

$$z_i = \tilde{A}_{ij} v_j \tag{2.8.4}$$

or

$$z = \tilde{A}v = \tilde{A}(v) \tag{2.8.5}$$

The operator $\tilde{A}$ is called the *adjoint operator* of $A$ when

$$(v, A(u)) - (u, \tilde{A}(v)) = v_i A_{ij} u_j - u_i \tilde{A}_{ij} v_j = 0 \tag{2.8.6}$$

Clearly, the definition of an *adjoint system* for a system of second-order differential equations (Section 2.1),

$$\tilde{\eta}^i M_i(\eta) - \eta^i \tilde{M}(\tilde{\eta}) = \dot{Q}(\eta, \tilde{\eta}), \tag{2.8.7}$$

can be considered as a generalization of condition (2.8.6). An operator in $S$ is called *self-adjoint* (or *Hermitian*) when it coincides with its adjoint.[39] For the space under consideration, the adjoint operator $\tilde{A}$ is simply the transpose of $A$, i.e.,

$$\tilde{A} = A^T. \tag{2.8.8}$$

Thus, an operator $A$ in $S$ is self-adjoint when it coincides with its transpose, i.e.,

$$A_{ij} = A_{ji}. \tag{2.8.9}$$

Conditions of self-adjointness (2.2.9) can clearly be considered as a generalization of conditions (2.8.9). Indeed, within the framework of our variational approach to the self-adjointness of Newton's equation (2.2.1a), at the limit when the functions $B_k$ are identically null and all $A_{ij}$ are independent of $t$, $q^k$, and $\dot{q}^k$, the quantity $Q$ becomes identically null, definitions (2.8.6) and (2.8.7) formally coincide, and conditions (2.2.9) reduce to conditions (2.8.9).

Notice that the above correspondence holds, as mentioned before, for systems of "second-order" differential equations.

It is of some interest, then, to note that when considering the case of "first-order" differential equations, instead of the symmetry property (2.8.9) we have the antisymmetry property $C_{\mu\nu} = -C_{\nu\mu}$ or $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Thus, in this case, the conditions of self-adjointness can be considered as a generalization of the concept of *skew-Hermiticity*.

For additional informations on the relationship between the operational and variational approach to self-adjointness, see Charts 3.16 and 3.17.

[39] For a recent account, see Reed and Simon (1975).

## 2.9 Algebraic Significance of the Conditions of Self-Adjointness

An *algebra* $U$ is a vector space of elements, say, $A, B, C, \ldots$ (e.g., operators) over a field $F$ of elements $N_1, N_2, N_3, \ldots$ (e.g., complex numbers) equipped with a *product* $AB$ satisfying the *right and left distributive laws* (also called *axioms* or *identities*)

$$A(B + C) = AB + AC, \qquad (A + B)C = AC + BC, \tag{2.9.1}$$

and the *scalar law*

$$N(AB) = (NA)B = A(NB), \tag{2.9.2}$$

for all $A,B,C \in U$ and $N \in F$. In essence, laws (2.9.1) and (2.9.2) ensure that the product $AB$ is bilinear.

A major classification of algebras is that into *associative* and *nonassociative* algebras, depending on whether the product $AB$ verifies or does not verify, respectively, the *associative law*

$$(AB)C = A(BC) \tag{2.9.3}$$

for all elements $A,B,C \in U$.

As recalled in the Introduction, some algebras of particular relevance in physics are the *Lie algebras*. These are the nonassociative algebras $L$ which verify the laws

$$AB + BA = 0, \tag{2.9.4a}$$

$$(AB)C + (BC)A + (CA)B = 0, \tag{2.9.4b}$$

called the *anticommutative* and *Jacobi law*, respectively, for all elements $A,B,C \in L$. It is here understood that the product $AB$, in order to properly characterize a Lie algebra, must first obey the distributive and scalar laws (that is, it must first constitute an algebra as commonly understood) and then verify laws (2.9.4). In this respect, it should be noted that Lie algebras verify the particular form of the scalar law

$$AN = NA = 0 \tag{2.9.5}$$

for all $A \in L$ and $N \in F$. Lie algebras also verify, in the realizations used in physics, the *differential laws*

$$(A \circ B)C = (AC) \circ B + A \circ (BC) \tag{2.9.6a}$$

$$A(B \circ C) = (AB) \circ C + B \circ (AC), \tag{2.9.6b}$$

where $A \circ B$ is the associative product.

It should be stressed that in all the above identities the product $AB$ is intended as the *abstract product* of the algebra $L$.

Among all possible *realizations* of the algebra $L$ and of its abstract product $AB$, we are now interested in those significant for Newtonian systems. Suppose that the quantities $A$, $B$, etc. are functions of the $2n$-component contravariant vector $a^\mu$, $\mu = 1, 2, \ldots, 2n$ and, possibly, of time. Suppose also that

such functions, $A$, $B$, etc., are of class $\mathscr{C}^m$, $m \geq 2$ in a region $\tilde{R}^{2n+1}$ of points $(t, a^\alpha)$.

In a set $\zeta$ of such functions $A$, $B$, etc., introduce the composition law

$$[A, B]^* = C^{\mu\nu}(t, a^\alpha) \frac{\partial A}{\partial a^\mu} \frac{\partial B}{\partial a^\nu}, \tag{2.9.7a}$$

$$C^{\mu\nu} \in \mathscr{C}^m(\tilde{R}^{2n+1}), \qquad m \geq 1, \tag{2.9.7b}$$

$$|C^{\mu\nu}|(\tilde{R}^{2n+1}) \neq 0. \tag{2.9.7c}$$

Clearly, the "product" $[A, B]^*$ satisfies bilinearity laws (2.9.1) and (2.9.2), i.e.,[40]

$$[N_1 A + N_2 B, C]^* = N_1[A, C]^* + N_2[B, C]^*, \tag{2.9.8a}$$

$$[A, N_1 B + N_2 C]^* = N_1[A, B]^* + N_2[A, C]^*, \tag{2.9.8b}$$

rules (2.9.6), i.e.,[40]

$$[AB, C]^* = [A, C]^* B + A[B, C]^*, \tag{2.9.9a}$$

$$[A, BC]^* = [A, B]^* C + B[A, C]^*, \tag{2.9.9b}$$

and laws (2.9.5), i.e.,

$$[A, N]^* = [N, A]^* = 0. \tag{2.9.10}$$

If, in addition, the set $\zeta$ and the matrix $(C^{\mu\nu})$ are selected in such a way that the closure law

$$[A, B]^* = C, \qquad A, B, C \in \zeta, \tag{2.9.11}$$

also holds for all (ordered) pairs of elements $A$ and $B \in \zeta$, then we have a (finite-dimensional or infinite-dimensional) *closed algebra.*

In order for such an algebra $\zeta$ to be a Lie algebra, the product $[A, B]^*$ must obey Equations (2.9.4). Antisymmetry law (2.9.4a) implies that

$$C^{\mu\nu} + C^{\nu\mu} = 0. \tag{2.9.12}$$

In order to obtain the (necessary and sufficient) conditions to satisfy the Jacobi law (2.9.4b), notice that such a law must hold for arbitrary elements $A$, $B$, and $C$. Therefore, one can select for those elements the components $a^\tau$, $a^\mu$, and $a^\nu$ of the vector $a$ resulting in the equations

$$[a^\tau, [a^\mu, a^\nu]^*]^* + [a^\mu, [a^\nu, a^\tau]^*]^* + [a^\nu, [a^\tau, a^\mu]^*]^*$$
$$= C^{\tau\rho} \frac{\partial C^{\mu\nu}}{\partial a^\rho} + C^{\mu\rho} \frac{\partial C^{\nu\tau}}{\partial a^\rho} + C^{\nu\rho} \frac{\partial C^{\tau\mu}}{\partial a^\rho} = 0 \tag{2.9.13}$$

Clearly, Equations (2.9.12) and (2.9.13) express the Lie algebra identities (2.9.4) for the "product" $[A, B]^*$.

[40] Here we revert to the conventional notations whereby $NA$ and $AB$ are the ordinary (associative) products.

When axioms (2.9.12) and (2.9.13) are satisfied, $[A, B]^*$ is called the *generalized Poisson brackets*[41] and, in view of the antisymmetry law (2.9.12), it can also be written

$$[A, B]^* = \sum_{\substack{\mu,\nu=1 \\ \mu<\nu}}^{2n} C^{\mu\nu}\left(\frac{\partial A}{\partial a^\mu}\frac{\partial B}{\partial a^\nu} - \frac{\partial A}{\partial a^\nu}\frac{\partial B}{\partial a^\mu}\right) \tag{2.9.14}$$

Clearly, the simplest solution of Equations (2.9.12) and (2.9.13) is given by the now familiar matrix $(\omega^{\mu\nu})$. The related product,

$$[A, B] = \omega^{\mu\nu}\frac{\partial A}{\partial a^\mu}\frac{\partial B}{\partial a^\nu}, \tag{2.9.15}$$

is then the conventional *Poisson brackets* in tensor notation. To see this, suppose that the vector $a^\mu$ represents the canonically conjugate generalized positions $q^k$ and momenta $p_k$, i.e., $a^\mu \equiv (q, p)$. Then

$$\begin{aligned}[A, B] = \omega^{\mu\nu}\frac{\partial A}{\partial a^\mu}\frac{\partial B}{\partial a^\nu} &= \sum_{\mu,\nu=1}^{2n}\omega^{\mu\nu}\left(\frac{\partial A}{\partial a^\mu}\frac{\partial B}{\partial a^\nu} - \frac{\partial A}{\partial a^\nu}\frac{\partial B}{\partial a^\mu}\right) \\ &= \frac{\partial A}{\partial q^k}\frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial q^k}.\end{aligned} \tag{2.9.16}$$

For later use, notice the important property

$$(\omega^{\mu\nu}) = ([a^\mu, a^\nu]) = \begin{pmatrix}([q^i, q^j]) & ([q^i, p_j]) \\ [(p^i, q_j]) & ([p_i, p_j])\end{pmatrix} = \begin{pmatrix} 0_{n\times n} & +1_{n\times n} \\ -1_{n\times n} & 0_{n\times n}\end{pmatrix} \tag{2.9.17}$$

namely, the matrix $(\omega^{\mu\nu})$ represents the *fundamental Poisson's brackets.*

By inspection, we see that Equations (2.9.12) and (2.9.13) coincide with conditions (2.7.27) which, in turn, are equivalent to conditions of self-adjointness (2.7.28) (Theorem 2.7.4). Thus, conditions of self-adjointness (2.7.28) are the necessary and sufficient conditions for generalized brackets (2.9.7) to satisfy Lie algebra identities (2.9.4). This property can also be seen on a more direct basis by noting that

$$\begin{aligned}&[a^\gamma, [a^\alpha, a^\beta]^*]^* + [a^\alpha, [a^\beta, a^\gamma]^*]^* + [a^\beta, [a^\gamma, a^\alpha]^*] \\ &\qquad = C^{\tau\gamma}C^{\rho\alpha}C^{\sigma\beta}\left(\frac{\partial C_{\rho\sigma}}{\partial a^\tau} + \frac{\partial C_{\sigma\tau}}{\partial a^\rho} + \frac{\partial C_{\tau\rho}}{\partial a^\sigma}\right) = 0\end{aligned} \tag{2.9.18}$$

We can thus state the following theorem.

**Theorem 2.9.1** (Indirect Algebraic Significance of the Conditions of Self-Adjointness). *Necessary and sufficient condition for the brackets*

$$[A, B]^* = C^{\mu\nu}(t, a^\alpha)\frac{\partial A}{\partial a^\mu}\frac{\partial B}{\partial a^\nu} \tag{2.9.19}$$

[41] The asterisk in the notation $[A, B]^*$ stands for the generalized nature of the brackets. For brackets $[A, B]^*$ see, for instance, Pauli (1953).

*characterized by the contravariant factor tensor $C^{\mu\nu}$ of the (regular holonomic) Newtonian system,*

$$C^{\mu\nu}(t, a^\alpha)[\dot{a}_\nu - \Xi_\nu(t, a^\alpha)] = 0, \tag{2.9.20a}$$

$$C^{\mu\nu}, \Xi_\nu \in \mathscr{C}^m(\tilde{R}^{2n+1}), \qquad m \geq 1, \qquad |C^{\mu\nu}|(\tilde{R}^{2n+1}) \neq 0, \tag{2.9.20b}$$

$$\mu, \nu = 1, 2, \ldots, 2n,$$

*to satisfy the laws of the generalized Poisson brackets,*

$$[A, B]^* + [B, A]^* = 0, \tag{2.9.21a}$$

$$[A, [B, C]^*]^* + [B, [C, A]^*]^* + [C, [A, B]^*]^* = 0, \tag{2.9.21b}$$

*in a region $\tilde{R}^{2n+1}$ of points $(t, a^\alpha)$ is that the covariant version of system* (2.9.20) *satisfies all the conditions of self-adjointness* (2.7.27) *everywhere in $\tilde{R}^{2n+1}$.*

Other brackets that are particularly significant in classical mechanics are the *Lagrange brackets,*

$$\{A, B\} = \omega_{\mu\nu} \frac{\partial a^\mu}{\partial A} \frac{\partial a^\nu}{\partial B} = -\frac{\partial q^k}{\partial A} \frac{\partial p_k}{\partial B} + \frac{\partial p_k}{\partial A} \frac{\partial q^k}{\partial B}, \tag{2.9.22}$$

or the *generalized Lagrange brackets,*

$$\{A, B\}^* = C_{\mu\nu}(t, a^\sigma) \frac{\partial a^\mu}{\partial A} \frac{\partial a^\nu}{\partial B}, \tag{2.9.23}$$

where $(C_{\mu\nu})$ is the inverse of the matrix $(C^{\mu\nu})$ of the generalized Poisson brackets.

For brackets (2.9.23), Equations (2.9.1) are replaced by

$$\{A, B\}^* + \{B, A\}^* = 0, \tag{2.9.24a}$$

$$\frac{\partial}{\partial A}\{B, C\}^* + \frac{\partial}{\partial B}\{C, A\}^* + \frac{\partial}{\partial C}\{A, B\}^* = 0 \tag{2.9.24b}$$

and they can be written

$$C_{\mu\nu} + C_{\nu\mu} = 0, \tag{2.9.25a}$$

$$\frac{\partial C_{\mu\nu}}{\partial a^\rho} + \frac{\partial C_{\nu\rho}}{\partial a^\mu} + \frac{\partial C_{\rho\mu}}{\partial a^\nu} = 0. \tag{2.9.25b}$$

Clearly, the above axioms coincide with conditions of self-adjointness (2.7.28), and the following theorem holds.

**Theorem 2.9.2** (Direct Algebraic Significance of the Conditions of Self-Adjointness). *Necessary and sufficient condition for the brackets*

$$\{A, B\}^* = C_{\mu\nu}(t, a^\sigma) \frac{\partial a^\mu}{\partial A} \frac{\partial a^\nu}{\partial B}, \tag{2.9.26}$$

*characterized by the covariant factor tensor $C_{\mu\nu}$ of the (regular holonomic) Newtonian system*

$$C_{\mu\nu}(t, a^\sigma)[\dot{a}^\nu - \Xi^\nu(t, a^\sigma)] = 0, \tag{2.9.27a}$$

$$C_{\mu\nu}, \Xi^\nu \in \mathscr{C}^m(\tilde{R}^{2n+1}), \qquad m \geq 1, \tag{2.9.27b}$$

$$|C_{\mu\nu}|(\tilde{R}^{2n+1}) \neq 0 \tag{2.9.27c}$$

*to satisfy the laws of the generalized Lagrange brackets*

$$\{A, B\}^* + \{B, A\}^* = 0, \tag{2.9.28a}$$

$$\frac{\partial}{\partial A}\{B, C\}^* + \frac{\partial}{\partial B}\{C, A\}^* + \frac{\partial}{\partial C}\{A, B\}^* = 0 \tag{2.9.28b}$$

*in a region $\tilde{R}^{2n+1}$ of points $(t, a^\sigma)$, is that the system satisfies conditions of self-adjointness* (2.7.27) *everywhere in $\tilde{R}^{2n+1}$.*

Notice that the identity

$$[A_i, A_k]^*\{A_k, A_j\}^* = \delta_{ij}, \tag{2.9.29}$$
$$i, j = 1, 2, \ldots, n,$$

hold. In this sense, each of the brackets $[A, B]^*$ and $\{A, B\}^*$ can be considered as the "inverse" of the other.

To make a crude summary of the contents of this section, we can say that, under suitable technical implementations, *the conditions for self-adjointness guarantee the existence of a Lie algebra structure.*

**Chart 2.1** Hausdorff, Second-Countable, $\infty$-Differentiable Manifolds[42]

A geometric approach to Newtonian systems can be formulated by representing the equations of motion in their first-order forms (Sections 2.4 and 2.5), by interpreting these forms as vector fields on suitably selected manifolds, and then by using the so-called symplectic geometry. In this chart, we shall outline certain basic concepts of point-set topology and identify the needed notion of manifold. In Chart 2.2, we shall outline the interpretation of Newtonian systems as vector fields on manifold. The symplectic geometry will be outlined in Chart 2.3. The concept of contact manifold will be indicated in Chart 2.4. Finally, in Chart 2.5 we shall point out the geometrical significance of the conditions of variational self-adjointness for Newtonian systems in their first-order forms. The geometrical interpretation of the integrability conditions for the existence of a Hamiltonian are also treated in Santilli (1979). The interested reader is urged to consult the quoted literature for all technical details.

A *topological space* $M$ is a set, together with a collection of subsets $O$ called *open sets*, such that $M \in O$, if $O_1, O_2 \in O$, then $O_1 \bigcap O_2 \in O$ and the

[42] See, for instance, Abraham and Marsden (1977).

union of any collection $O_i$ of open sets is open. The *relative topology* on a subset $M_1 \in M$ is given by $O_M = \{O_1 \bigcap M_1 \mid O_1 \in O\}$. A *basis* of the topology is a collection $D$ of open sets such that every open set of $M$ is a union of elements of $D$. This topology is called *first-countable* if and only if for each element $m \in M$ there is a countable collection $\{N_i(m)\}$ of neighborhoods of $m$, such that for any neighborhood $N(m)$ of $m$ there is a set $M_1$ such that $N_M(m) \subset N(m)$. The topology is called *second-countable* if and only if it has a countable basis.

A topological space $M$ is called *Hausdorff* if and only if each two distinct points have disjoint neighborhoods. Alternatively, a first-countable space is Hausdorff if and only if all sequences have at most one limit point.

A *local chart* $(M_1, \varphi)$ is a bijection $\varphi$ from a subset $M_1 \in M$ to an open subset $O$ of a (finite-dimensional, real) vector space $M$. $M_1$ is then called the *domain* of $M$. An *atlas* on $M$ is a family of charts $A = \{M_i, \varphi_i\}$ such that $M = \bigcup M_i$. Two atlases are *equivalent* if and only if their union is an atlas. A *differentiable structure* $\Omega$ on $M$ is an equivalence class of atlases on $M$.

A *differentiable manifold* can be conceived as a topological space $M$ equipped with a differentiable structure $\Omega$, and we shall write $M(\Omega)$. Throughout our analysis, we shall consider only $\infty$-differentiable manifolds.

Consider a map $f: M \to M'$, where $M$ and $M'$ are two (differentiable) manifolds. Let $(V, \Psi)$ be a chart of $M'$ with $f(m) \in V$ for $m \in M$ and $(U, \Psi)$ be a chart of $M$ with $m \in U$ and $f(U) \in V$. The *local representatives* (or *local coordinates*) of $f$ can be introduced as $q = f_{\ell\psi} = \Psi \cdot f \cdot \ell^{-1}$.

The notion of manifold, which is often used in the study of Newtonian systems, is that of a *Hausdorff, second-countable, $\infty$-differentiable manifold* $M$. An example of this type of manifold, which is relevant here, is given by the *configuration space* of a Newtonian system (see, Appendix A) with (local) coordinates $q^k$, when equipped with the indicated topology and restricted to satisfy the indicated differentiability properties.

### Chart 2.2 Newtonian Systems as Vector Fields on Manifolds[43]

Let $M$ be a (Hausdorff, second-countable, $\infty$-differentiable) manifold realized in terms of the configuration space of a Newtonian system with (local) coordinates $q$. This manifold is insufficient to characterize the system because, for instance, nonequivalent trajectories may pass through each point of $M$. In this chart, we shall indicate the additional notions needed to achieve a characterization of a Newtonian system.

Let $V_1$ and $V_2$ be (finite-dimensional, real) vector spaces. An open subset of $V_1$ will be denoted with $O$. A (local) *vector bundle* is the Cartesian product $O \times V_2$ with $O$ being the *base space*. Let the points of $O$ be denoted with $q$. A *fiber* over $q \in 0$ is the product $\{q\} \times V_2$. In essence, $O \times V_2$ is an open subset of $V_1 \times V_2$ and, as such, it is (locally) a manifold. Thus, a vector bundle is, at least locally, a manifold with a vector space attached to each of its points. For the case of Newtonian systems, therefore, the notion of vector bundle allows complementing the generalized coordinates $q$ with, say, the generalized velocities $\dot{q}$.

A (local) *vector bundle isomorphism* is a $\mathscr{C}^\infty$ map $\varphi: O \times V_2 \to O' \times V'_2$. A (local) *bundle chart* is a pair $(O, \varphi)$ where $\mathscr{C}$ is the bijection $\varphi: O \to O' \times V'_2$ for an open subset 0 of a set S. A *vector bundle atlas* on $S$ is a family

[43] See, for instance, Abraham and Marsden (1967), Herman (1973), and Caratù et al. (1976).

$\{O_i, \varphi_i\}$ of local bundle charts which covers $S$ and whose overlap is a local vector bundle isomorphism. A *vector bundle structure* on $S$ is an equivalence class of vector bundle atlases. A vector bundle can now be reinterpreted as a pair $(S, B)$, where $S$ is a set and $B$ is a vector bundle structure.

A $\mathscr{C}^m$ *section* of a map $\pi: B_1 \to B_2$ of a vector bundle is a class $\mathscr{C}^m$ map $\rho: B_2 \to B_1$ such that for each $b \in B_2$, $\rho(\pi(b)) = b$. The set of all $\mathscr{C}^m$ sections of a map $\pi$ will be denoted by $\Gamma^m(\pi)$. It should be indicated here that the $\mathscr{C}^m$ sections form a linear space, though this is not necessarily the case for other $\mathscr{C}^m$ maps.

A *curve* at a (local) point $q_0$ of a manifold $M$ is a map $q: I \to M$ of (at least) class $\mathscr{C}^1$ from an open interval $I$ of the field of real numbers $\mathbb{R}$ such that for $t_0 \in I$, $q(t_0) = q_0$. The *tangent space* of $M$ at $q_0$ is the set of all equivalence classes of curves at $q_0$, $T_{q_0} M$. The *tangent bundle* $TM$ of $M$ is the union of all tangent spaces of $M$, $TM = \bigcup T_q M$ for all $q \in M$. The *tangent bundle projection* is the map $\tau_M: TM \to M$.

As is known, given a vector space $V$, one can form new vector spaces by means of tensors $T^r_s$ of contravariant index $r$ and covariant index $s$ (Chart A.13). The procedure can be extended to tensors $T^r_s$ on manifolds. This leads to a generalization of the notion of tangent space $TM$. The vector bundle of tensors $T^r_s$ on $M$ is the *tangent bundle of contravariant index r and covariant index s*, $T^r_s(M)$. In particular, the ordinary tangent bundle $TM$ is given by $T^1_0(M)$. The quantity $T^0_1(M)$ is called the *cotangent bundle* and is denoted with $T^*M$.

A *tensor field* on a manifold $M$ is a $\mathscr{C}^\infty$ section of $T^r_s(M)$. A *contravariant vector field* is an element of $\Gamma^\infty(T^1_0(M))$. A *covariant vector field* is instead an element of $\Gamma^\infty(T^0_1(M))$. All operations of tensors apply to tensor fields fiberwise.

Consider a Newtonian system in the kinematical form in configuration space (2.2.2), i.e.,

$$\ddot{q}^k - f^k(t, q, \dot{q}) = 0, \qquad f^k \in \mathscr{C}^\infty, \qquad k = 1, 2, \ldots, n. \tag{1}$$

The first possibility of reinterpreting this system as a vector field on a manifold $M$ with local coordinates $q$ is through the use of the tangent bundle $TM$ of $M$ via the association, e.g., of the velocities $\dot{q}$ at each point $q$ and the interpretation of the functions $\Xi$ of the normal forms (2.1.30), i.e.,

$$\dot{a} - \Xi(a) = 0, \tag{2}$$

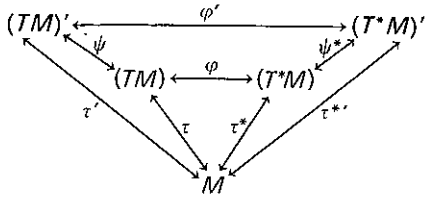
$$a = (q, y), \qquad y = \dot{q}, \qquad \Xi = (y, f) \in \mathscr{C}^\infty,$$

as elements of $\Gamma^\infty(T^1_0(M))$. The underlying mechanism is called the *lifting* of $M$ to $TM$.

In Section 2.4 we have stressed that the transition from Equation (1) to Equation (2) is not unique because it depends on the assumed prescriptions for the characterization of the new variables $y$. It then follows that the lifting from $M$ to $TM$ is not unique and many different liftings can be conceived for the interpretation of a Newtonian system as a vector field on a manifold. Notice that the lifting from $M$ to $TM$ is of contravariant character, e.g., in the sense that it associates the contravariant quantities $\dot{q}^k$ to $q^k$, $k = 1, 2, \ldots, n$.

A second important lifting is that to the cotangent bundle $T^0_1(M) = T^*M$. Suppose that a Lagrangian for the representation of system (1) is known. The conventional canonical prescriptions $p_k = \partial L / \partial \dot{q}^k$ then allow the association of new quantities, the covariant canonical momenta $p_k$, to $q^k$. The construction of the generally different normal form for Equations (1) in terms of $p_k$ then leads to the lifting to $T^*M$, i.e., the interpretation of

Newtonian systems as vector fields in $T^*M$. If different Lagrangians for the representation of the same systems in the same generalized coordinates are known [see Santilli (1979)], this leads to different liftings to $T^*M$. Therefore, the lifting to $T^*M$ is also not unique and other alternatives are possible. This situation can be schematically represented with a commutative diagram of the type




where $\varphi$ and $\varphi'$ are (fibre preserving) diffeomorphisms. Notice that the diagram implies the existence of mapping from functions $\Xi$ on $TM$ to functions $\Xi^*$ on $T^*M$.

## Chart 2.3 Symplectic Manifolds

The geometrical, coordinate-free generalization of the conventional phase space of Analytic Mechanics for *autonomous systems* (i.e., systems that do not depend explicitly on time) is characterized by the so-called symplectic manifolds.

Consider a contravariant tensor field of rank 2, $\Omega^{(2)}$ on the cotangent bundle $T^*M$ of a (Hausdorff, second-countable, $\infty$-differentiable) manifold $M$. When restricted to exact differentials, it induces the structure in local coordinates

$$\Omega^{(2)}(df, dg) = (f, g) = \frac{\partial f}{\partial a^\mu}\ \Omega^{\mu\nu}(a)\ \frac{\partial g}{\partial a^\nu} \tag{1}$$

satisfying the properties

$$\begin{gathered}(f, g_1 + g_2) = (f, g_1) + (f, g_2),\\ (f, c) = 0 \qquad (f, g_1 g_2) = (f, g_1)g_2 + g_1(f, g_2),\\ c = \text{const.}\end{gathered} \tag{2}$$

$\Omega^{(2)}$ is nowhere degenerate if it is nondegenerate at all points of $T^*M$. This is possible if and only if the dimension of $T^*M$ is even. By writing $\Omega^{(2)}(f, g) = \lambda_f(g)$ a covariant tensor field of rank 2, $\Omega_{(2)}$, can be uniquely associated to $\Omega^{(2)}$ (under the assumption of nowhere degeneracy) through the equations $\Omega^{(2)}(f, g) = \Omega_{(2)}(\lambda_f, \lambda_g)$ which, when restricted to exact differentials, can be written in local coordinates

$$\Omega_{(2)}(df, dg) = \langle f, g\rangle = \frac{\partial a^\mu}{\partial f}\ \Omega_{\mu\nu}(a)\ \frac{\partial a^\nu}{\partial g}. \tag{3}$$

An even-dimensional manifold $M$, equipped with a covariant nowhere degenerate tensor field of rank two, $\Omega_{(2)}$, is called a *symplectic manifold*,

and we shall write $M(\Omega)$, when structure (1) satisfies the additional properties

$$J(f, g) = (f, g) - (g, f) = 0, \tag{4a}$$

$$J(f, g, h) = (f, (g, h)) + (g, (h, f)) + (h, (f, g)) = 0. \tag{4b}$$

$\Omega_{(2)}$ ($\Omega^{(2)}$) is then called a *symplectic structure* (*cosymplectic structure*) Conditions (4) essentially ensure that brackets (1) satisfy the Lie algebra identities (Section 2.9), i.e., $(f, g) = [f, g]^*$ are the *generalized Poisson brackets.* Then $\langle f, g\rangle = \{f, g\}^*$ are the *generalized Lagrange brackets.* It should be indicated that this connection with conventional notions of Analytic Mechanics occurs through the introduction of local coordinates, while the full geometrical treatment is coordinate-free.

**Theorem 1** (Darboux). *Suppose that $\Omega_{(2)}$ is a nondegenerate 2-form on a $2n$-dimensional manifold $M$. Then $\Omega_{(2)}$, when restricted to exact differentials, satisfies Equations* (4) *if and only if there is a chart $(U, \varphi)$ at each $m \in M$ such that $\varphi(m) = 0$ and, with $\varphi(t) = x^1(t), \ldots, x^n(t), y_1(t), \ldots, y_n(t)$, we have*

$$(U, \varphi): \Omega_{(2)} \to \omega_{(2)} = dx^k \wedge dy_k \tag{5}$$

*or, in matrix form,*

$$(U, \mathscr{C}): \Omega_{(2)} \to \omega_{(2)} = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ +1_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{6}$$

The product $\wedge$ of Equation (5) is the exterior product of Section 1.2.

The form $\omega_{(2)}$ is sometimes called the *fundamental symplectic form.* The analytic counterpart of Darboux's theorem is offered by the following property

**Theorem 2** (Pauli). *To every point $m$ of a $2n$-dimensional cosymplectic manifold $M(\Omega)$ there exist local, canonical coordinates $a = (q, p)$ such that*

$$\Omega^{\mu\nu}(a) \to \Omega'^{\mu\nu} = \frac{\partial a'^{\mu}}{\partial a^{\rho}} \Omega^{\rho\sigma} \frac{\partial a'^{\nu}}{\partial a^{\sigma}} = \omega^{\mu\nu}. \tag{7}$$

In essence, Pauli's theorem provides the possibility of reinterpreting Darboux's theorem in terms of the conventional transformation theory of analytic mechanics. The connection of Theorem 1 or 2 with the conventional canonical formulations is that under a Darboux chart or under a Pauli transformation, brackets (1) become the *conventional Poisson brackets* and then brackets (3) become the *conventional Lagrange brackets* (Section 2.9). It then follows that vector fields (Chart 2.2) on a symplectic manifold can be *locally Hamiltonian.* This is the case when, given a vector field $\Xi$ on $M(\Omega)$ and a point $m \in M$, there exists a neighborhood $N(m)$ of $m$ and a function $H$ on $N(m)$ such that $L_{\Xi}(f) = [f, H]^*$ for all $f$ on $N(m)$, where $L_{\Xi}$ is the so-called *Lie derivative.* If, in particular, we have $L_{\Xi}(f) = [f, H]$ for all $f$ on $N(m)$, then the vector field is called *globally Hamiltonian* (or *Hamiltonian* for short). In this case,

$$\Xi_{(1)} = \omega_{\mu\nu}\Xi^{\nu}\, da^{\mu} = \Xi_{\mu}\, da^{\mu} = dH = \frac{\partial H}{\partial a^{\mu}}\, da^{\mu}. \tag{8}$$

Alternatively, the connection between symplectic geometry and canonical formulations can be seen from the fact that Hamilton's equations characterize a symplectic manifold $M(\omega)$. For the explicit form of Hamilton's equations with the fundamental symplectic structure $\omega$, see Section 3.9. For the necessary and sufficient conditions for a vector field to be globally Hamiltonian, see Section 3.12.[44]

**Chart 2.4** Contact manifolds[45]

We now consider the case of *nonautonomous systems* (i.e., systems with an explicit dependence on time). In this case, the base manifold is generalized into a $(2n + 1)$ dimensional manifold $\mathbb{R} \times M$, where $\mathbb{R}$ is representative of the time variable. The odd dimensionality of the base manifold then demands a suitable generalization of the notion of symplectic manifold.

A *contact manifold* $\hat{M}(\hat{\Omega})$ is a $(2n + 1)$-dimensional (Hausdorff, second-countable, $\infty$-differentiable) manifold $\mathbb{R} \times TM$ equipped with a covariant two-form $\hat{\Omega}_{(2)}$ of maximal rank (i.e., $2n$) which, when restricted to TM is symplectic, i.e., $\hat{\Omega}_{(2)}(df, dg)|_{TM} = \Omega_{(2)}(df, dg) = \langle f, g \rangle$ is symplectic in the sense of Chart 2.3. $\hat{\Omega}_{(2)}$ is then called a *contact structure* and, in local coordinates, it induces the more general form of the generalized Lagrange brackets

$$\hat{\Omega}_{(2)}(df, dg) = \{f, g\}^* = \frac{\partial a^\mu}{\partial f} \Omega_{\mu\nu}(t, a) \frac{\partial a^\nu}{\partial g} \tag{1}$$

Despite the presence of an explicit time dependence, the $\hat{\Omega}_{(2)}$ structure can still be reduced to a fundamental structure $\omega_{(2)}$ by means of a Darboux chart

$$(\mathbb{R} \times U, t \times \varphi): \Omega_{(2)} \to \omega_{(2)} = dx^k \wedge dy_k \tag{2}$$

Alternatively, there exist the more general Pauli transformations $t \to t'$ and $a \to a'$ under which

$$\hat{\Omega}_{(2)}|_{(t,a)} \to \omega_{(2)}|_{(t', a')} \tag{3}$$

This allows the connection with conventional canonical formulation.

[44] We have here mostly followed Jost (1964). It should be indicated that this author calls a symplectic structure $\Omega^{(2)}$ a nowhere degenerate contravariant two-structure which is only skew-symmetric, i.e., it does not necessarily satisfy Jacobi identity (4b). When condition (4b) is included, Jost calls $\Omega^{(2)}$ a *canonical structure*. We have preferred the name of symplectic structure over that of canonical structure because it appears to be more generally adopted. Also, we have called symplectic structure the covariant two-form $\Omega_{(2)}$, rather than the contravariant form $\Omega^{(2)}$ as in Jost, because it is more widely accepted. See, for instance, Sternberg (1964) and Abraham and Marsden (1977). The formulation of the Darboux theorem has been derived from the latter authors. Theorem 2 is that presented by Jost (1964) with reference to Pauli (1953). The notion of symplectic structure will be more properly presented in Chart 2.5 via the concept of closure. The purpose of this chart (which was, perhaps, Jost's objective) is restricted to the indication of the deep interelation between symplectic geometry and Lie algebras. As a final note, the reader should be aware that a symplectic structure (or manifold) is often called *canonical* or *Hamiltonian structure* (or manifold) in the literature of differential geometry.

[45] Abraham and Marsden (1967).

A given vector field $\Xi$ can be globally Hamiltonian at a point $(t, m) \in \mathbb{R} \times T^*M$ when there exist a neighborhood $N(t, m)$ of $(t, m)$ and a function $H$ on $N(t, m)$ such that $L_\Xi(f) = [f, H] + \partial f/\partial t$ for all functions $f$ on $N(t, m)$. This recovers the more general time evolution law $\dot{f} = [f, H] + \partial f/\partial t$. Alternatively, Hamilton's equations for nonautonomous systems characterize a contact manifold $M(\omega)$. In essence, mappings (2) or (3) allow the representation of Hamilton's equations in terms of the fundamental structure $\omega$ irrespective of whether the Hamiltonian is explicitly dependent on time or not (Section 3.9).

**Chart 2.5** Geometrical Significance of the Conditions of Self-Adjointness

We are now equipped to study the geometrical significance of the conditions of variational self-adjointness

$$C_{\mu\nu} + C_{\nu\mu} = 0, \tag{1a}$$

$$\frac{\partial C_{\mu\nu}}{\partial a^\rho} + \frac{\partial C_{\nu\rho}}{\partial a^\mu} + \frac{\partial C_{\rho\mu}}{\partial a^\nu} = 0, \tag{1b}$$

$$\frac{\partial C_{\mu\nu}}{\partial t} = \frac{\partial D_\mu}{\partial a^\nu} - \frac{\partial D_\nu}{\partial a^\mu}, \tag{1c}$$

$$\mu, \nu, \rho = 1, 2, \ldots, 2n$$

for first-order systems

$$C_{\mu\nu}(t, a)\dot{a}^\nu + D_\mu(t, a) = 0$$

$$C_{\mu\nu}, D_\mu \in \mathscr{C}^\infty(\tilde{R}^{2n+1}), \qquad |C_{\mu\nu}|(\tilde{R}^{2n+1}) \neq 0. \tag{2}$$

Consider first the autonomous case for which $C_{\mu\nu} = C_{\mu\nu}(a)$ and $D_\mu = D_\mu(a)$. One of the fundamental properties of a symplectic structure $\Omega_{(2)}$ is that, when written as the 2-form $\Omega_{(2)} = \Omega_{\mu\nu}\, da^\mu \wedge da^\nu$, is closed (in the sense of section 1.2), i.e., $d\Omega_{(2)} = 0$. As a matter of fact, one can equivalently define a *symplectic structure* as a nondegenerate, closed 2-form on a $2n$-dimensional manifold M (Abraham and Marsden (1967, page 95)]. The geometrical significance of conditions of self-adjointness (1a) and (1b) is then straightforward: under the indicated continuity and regularity conditions, they are the necessary and sufficient conditions for the tensor $C_{\mu\nu}$ to characterize a symplectic structure or, equivalently, for the vector field (2) to be embodied in a symplectic manifold. This is a consequence of the fact that Equations (1a) and (1b) are the necessary and sufficient conditions for the form $C_{\mu\nu}\, da^\mu \wedge da^\nu$ to be closed (Example 1.5).

In the transition to nonautonomous systems, the closure property of a contact structure persists, although it is now referred to as a $(2n + 1)$-dimensional space $\mathbb{R} \times M$ with local coordinates $a^i$, $i = 0, 1, 2, \ldots, 2n$, $a^0 = t$. As a matter of fact, a *contact structure* can be equivalently defined as a closed 2-form of maximal rank on a $(2n + 1)$-dimensional manifold [Abraham and J. E. Marsden (1967, page 132)]. In this case, the full set of conditions (1) must be considered, and their geometrical significance is also straightforward.

Introduce the tensor

$$\Omega_{ij} = \begin{array}{l} \Omega_{\mu\nu} = C_{\mu\nu}, \quad , \nu = 1, 2, \ldots, 2n, \\ \Omega_{0\mu} = -D_\mu. = -\Omega_{\mu 0}, \Omega_{00} = 0 \end{array} \tag{3}$$

Then conditions (1) are necessary and sufficient conditions for the 2-form $\hat{\Omega}_{(2)} = \Omega_{ij}\, da^i \wedge da^j$, $i, j = 0, 1, 2, \ldots, 2n$ to be closed. Thus, the conditions of variational self-adjointness for vector fields (2) are the necessary and sufficient conditions for the tensor $C_{\mu\nu}$ to characterize a contact structure or, equivalently, for vector fields (2) to be embodied in a contact manifold.

The connection between the geometrical and algebraic significance of the conditions of self-adjointness (1) is also significant. The necessary and sufficient conditions for the closure property of a symplectic form can also be proved by using Equations (4) of Chart 2.3 [Jost (1964)]. But these equations are the Lie algebra identities for the brackets $\hat{\Omega}^{(2)}(df, dg) = [f, g]^*$. On the other side, conditions (1a) and (1b) are equivalent to the Lie algebra identities (Section 2.9). Thus, the conditions of self-adjointness (1) can be interpreted as the analytic counterpart of the necessary and sufficient conditions for either the brackets $\hat{\Omega}^{(2)}(df, dg) = [f, g]^*$ to characterize a Lie algebra or for the 2-form $\hat{\Omega}_{(2)} = \Omega_{ij}\, da^i \wedge da^j$ to be a contact (symplectic) structure for autonomous (nonautonomous) systems.

As we shall see in Section 3.12, conditions (1), when restricted to covariant normal forms, are the necessary and sufficient conditions for the existence of a Hamiltonian. We can, therefore, say that the integrability conditions for the existence of Hamilton's equations, as expected, are also the integrability conditions for the (classical realizations of the) Lie algebra identities and for the symplectic structure.

In conclusion, the Inverse Problem for Hamilton's equations results to be an effective arena for the study of the deep relationships that exist between canonical formulations, Lie algebras, and differential geometry.

## EXAMPLES

*Example 2.1*

The kinematical form for the one-dimensional harmonic oscillator,

$$\ddot{x} + \omega_0^2 x = 0, \qquad \omega_0^2 = \frac{k}{m}$$

is self-adjoint,[46] since it satisfies Corollary 2.2.2a. However, if for $x \neq 0$ the same equation is written in the form

$$\frac{\ddot{x}}{x} + \omega_0^2 = 0,$$

then it is non-self-adjoint.

*Example 2.2*

The kinematical form for a particle under a drag force

$$\ddot{x} + \gamma\dot{x} = 0$$

[46] When the region $R^{2n+1}$ in which a system is self-adjoint (non-self-adjoint) can be arbitrarily selected for all finite values of $(t, q, \dot{q})$ we shall ignore it and simply call the system self-adjoint (non-self-adjoint).

is not self-adjoint. However, if the same equation is written in the form

$$\frac{\ddot{x}}{\dot{x}} + \gamma = 0, \qquad \dot{x} \neq 0,$$

then it is self-adjoint.

*Example 2.3*

The equation for the one-dimensional damped oscillator,

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

is non-self-adjoint.

*Example 2.4*

The fundamental form

$$m_1\ddot{x}_1 + m_c\ddot{x}_2 + k_1x_1 + k_cx_2 = 0$$
$$m_c\ddot{x}_q + m_2\ddot{x}_2 + k_cx_1 + k_2x_2 = 0$$

can represent a system of two linear, coupled, and undamped oscillators with ocupling constants $m_c$ and $k_c$. For $m_c = k_c = 0$, the two oscillators are decoupled (i.e., they do not "interact" between themselves even though each individual particle "interacts" with its own elastic force field).

Since the system is in more than one dimension, its regularity must be checked prior to studying its self-adjointness.

The functional determinant in this case is

$$\begin{vmatrix} m_1 & m_c \\ m_c & m_2 \end{vmatrix} = m_1m_2 - m_c^2.$$

Therefore, the (necessary and sufficient) condition for the system to be regular is that

$$m_1m_2 \neq m_c^2.$$

By inspection, we then see that conditions (2.2.9) are satisfied and the system is self-adjoint.

*Example 2.5*

Consider the *Whittaker equations* (Problem 2.9)

$$\ddot{q}_1 = q_1,$$
$$\ddot{q}_2 = \dot{q}_1.$$

Assume for prescriptions (2.4.2) the functions

$$G_i = y_i - \dot{q}_i = 0, \qquad i = 1, 2.$$

Then the equations can be written in the equivalent first-order form

$$\dot{q}_1 = y_1,$$
$$\dot{q}_2 = y_2,$$
$$\dot{y}_1 = q_1,$$
$$\dot{y}_2 = y_1.$$

By introducing tensor notation (2.6.23), we can write

$$\begin{pmatrix} q_1 \\ q_2 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \\ a^4 \end{pmatrix} = \begin{pmatrix} 0_{2\times 2} & 1_{2\times 2} \\ -1_{2\times 2} & 0_{2\times 2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} a_3 \\ a_4 \\ -a_1 \\ -a_2 \end{pmatrix}.$$

Therefore, contravariant normal form (2.6.23) holds with the following functions

$$\begin{pmatrix} \Xi^1 \\ \Xi^2 \\ \Xi^3 \\ \Xi^4 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ q_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -a_1 \\ -a_2 \\ a_3 \\ -a_1 \end{pmatrix}.$$

By inspection, we see that the conditions of self-adjointness, i.e., the contravariant version of Equations (2.7.20), are violated and such a normal form is non-self-adjoint. However, this does not preclude the possible existence of other prescriptions (2.4.2), for which the corresponding normal form is self-adjoint.

*Example 2.6*

The system

$$t\ddot{q}_1 + \ddot{q}_2 + f_1(t, q, \dot{q}) = 0$$
$$\ddot{q}_1 + \ddot{q}_2 + f_2(t, q, \dot{q}) = 0$$

is regular because its functional determinant

$$|\mathscr{H}| = \begin{vmatrix} t & 1 \\ 1 & 1 \end{vmatrix} = t - 1$$

is non-null as a function. However, at the value $t = 1$, such a determinant is null. According to the assumption of Section 1.1, we then consider the system in any interval of time that does not contain the value $t = 1$. Notice that the existence of the zero $t = 1$ of the functional determinant is in line with Definition 1.1.1 of regularity. Indeed, according to such a definition, for a system to be degenerate it must be null at least at a point *and* in its neighborhood, which is not the case for the system considered.

*Example 2.7*

In this example, we shall indicate the important property according to which *the Lorentz force is variationally self-adjoint*, and identify in more detail the objective of this monograph.

Consider for simplicity, but without loss of generality, the case of one particle of mass $m$ and charge $e$ moving in a three-dimensional Euclidean space with Cartesian coordinates $r^i$, $i = x, y, z$, under the action of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ in Gaussian units. We also assume, for simplicity, that the speed of light $c = 1$. The *Lorentz force* is then expressed by the familiar equations of motion

$$\begin{aligned} m\ddot{\mathbf{r}} &= \mathbf{F}^L \\ &= e(\dot{\mathbf{r}} \times \mathbf{B} + \mathbf{E}) \\ &= e\dot{\mathbf{r}} \times \nabla \times \mathbf{A} - e\left(\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}\right), \end{aligned}$$

where $\mathbf{A}$ and $\varphi$ are the electromagnetic potentials. The components of $\mathbf{F}^L$ can be written

$$F_i^L = e\delta_{ij}^{mn} \frac{\partial A_m}{\partial r^n} \dot{r}^j - e\left(\frac{\partial \varphi}{\partial r^i} - \frac{\partial A_i}{\partial t}\right)$$

$$i = x, y, z$$

where $\delta_{ij}^{mn}$ is the generalized Kronecker delta (Section 1.2).

It is easy to see that the Lorentz force satisfies all the conditions of Theorem 2.2.2 and, therefore, it is variationally self-adjoint. To see it, we first note that $\mathbf{F}^L$ is *linear in the velocity* and, as such, it satisfies the first part of Theorem 2.2.2. Secondly, by using the notation of Equations (2.2.23a), we can write

$$\rho_{ij} = e\delta_{ij}^{mn} \frac{\partial A_m}{\partial r^n}, \qquad \sigma_i = -e\left(\frac{\partial \varphi}{\partial r^i} - \frac{\partial A_i}{\partial t}\right)$$

It is a matter of simple algebra to see that the above realization of the $\rho_{ij}$ and $\sigma_i$ terms satisfies all the conditions of self-adjointness (2.2.24), and this concludes our proof.[47] It is significant here to indicate that the property of variational self-adjointness of the Lorentz force persists in the transition to relativistic, field theoretical, and gravitational generalizations. The proof of this property, of course, demands the generalization of the methods of the Inverse Problem to Minkoswki space, field theory, and Riemannian manifolds. As a result, we can say, more generally, that the forces or couplings of the electromagnetic interactions (satisfying the needed minimal continuity conditions) are variationally self-adjoint. Still more generally, we can say that the couplings of the recently unified gauge theories (of Abelian or non-Abelian type)

[47] By taking into account the geometrical significance of the conditions of self-adjointness (Chart (2.5), the property indicated essentially implies, on geometrical grounds, that the terms $\rho_{ij}$ of the Lorentz force characterize a symplectic structure under even-dimensionality and nowhere-degeneracy conditions. Indeed, conditions of self-adjointness (2.2.24a) and (2.2.24b) are the necessary and sufficient conditions for the (nowhere degenerate) two-form

$$\rho_{(2)} = e\delta_{ij}^{mn} \frac{\partial A_m}{\partial r^n} dr^i \wedge dz^j, \qquad i, j = x, y,$$

to be closed. Similarly, the terms $\rho_{ij}$ and $\sigma_i$ of the Lorentz force characterize a contact structure, in which case the full set of conditions (2.2.24) is used (see Chart 2.5 for details).

of weak and electromagnetic interactions are all variationally selfadjoint. Indeed, all these couplings or forces are linear in the derivative terms and satisfy the conditions of variational self-adjointness in the appropriate form. For a detailed study we refer the reader to Santilli (1978 I, II, and III). As we shall see in Chapter 3, the variational self-adjointness of the Lorentz force essentially implies that this force possesses the structure of the most general (acceleration independent) Newtonian forces derivable from a potential. Indeed, for the Lorentz force, we have the familiar expressions

$$F_i^L = -\frac{\partial U}{\partial r^i} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}^i}$$

$$U = -e\varphi + e\mathbf{A}\cdot\dot{\mathbf{r}} = U(t, \mathbf{r}, \dot{\mathbf{r}}),$$

yielding the most general functional dependence of a potential function in Newtonian mechanics.

It is significant at this point to indicate that this monograph and the forthcoming second part (Santilli 1979) are devoted to the study of Newtonian forces which are analytically *more general* than the Lorentz force, that is, *nonderivable from a potential.* Indeed, this is the dominant analytic character of the Newtonian forces, in general, as recalled in Appendix A. Besides this Newtonian profile, the analysis could be of some value also for other physical aspects. For instance, an initial study of the problem whether the strong interactions can be interpreted in terms of forces analytically more general than the Lorentz force (variationally non-self-adjoint strong forces) has been conducted by Santilli (1978 I, II, and III; for a review, see the quoted articles in the Hadronic Journal). Additional studies by a number of authors on this intriguing physical problem are currently in progress.

## Problems

2.1 Prove the following table of conditions of self-adjointness for one-dimensional systems

| Equation | Condition of self-adjointness |
|---|---|
| $A(t, q, \dot{q})\ddot{q} + B(t, q, \dot{q}) = 0$ | $\frac{\partial B}{\partial \dot{q}} = \frac{\partial A}{\partial t} + \dot{q}\frac{\partial A}{\partial q}$ |
| $A(t, q)\ddot{q} + B(t, q) = 0$ | $\frac{d}{dt}A = 0$ |
| $a(t)\ddot{q} + b(t)q + c(t)q + d(t) = 0$ | $\frac{d}{dt}a(t) = b(t)$ |
| $\ddot{q} + f(t, q, \dot{q}) = 0$ | $\frac{\partial f}{\partial \dot{q}} = 0$ |

2.2 Prove that the *Bessel equations*, when they are written in the form

$$t\ddot{q} + \dot{q} + \frac{t^2 + n^2}{t} q = 0, \qquad t \neq 0,$$

are non-self-adjoint and that, when they are written in the equivalent form

$$t^2\ddot{q} + t\dot{q} + (t^2 - n^2)q = 0,$$

they become self-adjoint.

2.3 Consider this variation of the system of Example 2.4:

$$m_1\ddot{x}_1 + m_c\ddot{x}_2 + k_1x_1 + k_cx_2 = 0$$
$$m_c'\ddot{x}_1 + m_2\ddot{x}_2 + k_c'x_1 + k_2x_2 = 0$$
$$m_1m_2 \neq m_cm_c', \qquad m_c \neq m_c', \qquad k_c \neq k_c'.$$

Prove that it is non-self-adjoint.

2.4 Select an interval of time in which the system

$$\ddot{q}_1(\sin \omega t + \tfrac{1}{4}) + \ddot{q}_2 \cos \omega t + f_1(t, q, \dot{q}) = 0,$$
$$\ddot{q}_1 \cos \omega t + \ddot{q}_2 + f_2(t, q, \dot{q}) = 0$$

is everywhere regular.

2.5 Prove that Equations (2.2.18a) and (2.2.18b) imply the linearity of the related functions in the velocities $\dot{q}^i$.

2.6 Consider the harmonic oscillator $\ddot{q} + q = 0$. Assume for prescriptions (2.4.2) the functions

$$G = y + c_1\dot{q} + c_2 = 0,$$
$$G' = qy + c_1\dot{q} + c_2 = 0,$$

with $c_1, c_2 =$ constants. Construct the corresponding contravariant normal form (2.6.23) as functions of the covariant vector $a_\mu$, and prove that for prescription $G(G')$ such a form is self-adjoint (non-self-adjoint).

2.7 Prove that Equations (2.2.8a) can be obtained from Equations (2.2.6) and (2.2.5).

2.8 By following the analysis of Section 2.3, reinterpret the conditions of self-adjointness (2.2.24) of the kinematical form (2.2.23) within the context of the calculus of differential forms.

2.9 Determine whether the system of equations (sometimes called *Whittaker equations*)

$$\ddot{q}_1 = q_1,$$
$$\ddot{q}_2 = \dot{q}_1$$

is self-adjoint or non-self-adjoint

2.10 The *Mathieu equations*

$$\frac{d^2\theta}{dz^2} + (a - 2b \cos 2z) = 0,$$

where

$$z = \tfrac{1}{2}\omega t, \qquad a = \frac{4ml_g}{I\omega^2}, \qquad b = \frac{2\alpha_0 ml}{I\omega^2},$$

can represent an inverted pendulum with an applied force proportional to $\cos \omega t$, a moment of inertia $I$, and a downward acceleration of the applied force $\alpha_0 \cos \omega t$. Determine whether the system is self-adjoint or non-self-adjoint.

2.11 Prove that, when a system $F_i(t, q, \dot{q}, \ddot{q}) = 0$, satisfies the (local or global) existence theorems for solutions, so does its system of equations of variations, $M_i(\eta) = 0$.

CHAPTER 3

# The Fundamental Analytic Theorems of the Inverse Problem

## 3.1 Statement of the Problem

We consider now the *conventional analytic equations*[1] *in configuration space*, i.e., *Lagrange's equations*:

$$L_k(q) = \frac{d}{dt}\frac{\partial L(t, q, \dot{q})}{\partial \dot{q}^k} - \frac{\partial L(t, q, \dot{q})}{\partial q^k} = 0, \qquad (3.1.1)$$

$$k = 1, 2, \ldots, n.$$

Our problem is to identify the necessary and sufficient conditions for regular holonomic Newtonian systems in their fundamental form (2.2.1) to admit a representation in terms of Equations (3.1.1).

This problem can be studied first by searching for the conditions under which a Lagrangian $L$ exists that satisfies each of the identifications

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} = A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}), \qquad (3.1.2)$$

$$k = 1, 2, \ldots, n.$$

But Equations (3.1.1), after expanding the total time derivative, explicitly read

$$\frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^i}\ddot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k \partial q^i}\dot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k \partial t} - \frac{\partial L}{\partial q^k} = 0. \qquad (3.1.3)$$

[1] We shall use the term "conventional" when referring to Equation (3.1.1) to avoid possible confusion when we encounter other types of analytic equations in Santilli (1979).

From the linearity in the accelerations $\ddot{q}^k$ of both the equations of motion and the analytic equations, it then follows that identifications (3.1.2) demand the validity of each of the equations (Section 2.3)

$$\frac{\partial^2 L}{\partial \dot{q}^k \, \partial \dot{q}^i} = A_{ki}, \tag{3.1.4a}$$

$$\frac{\partial^2 L}{\partial \dot{q}^k \, \partial q^i} \dot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k \, \partial t} - \frac{\partial L}{\partial q^k} = B_k. \tag{3.1.4b}$$

By inspection, we see that this is a system of $n^2 + n$ partial differential equations in only one unknown, the Lagrangian $L$, and as such it is overdetermined. From the current literature on the subject[2], we know that, in view of the overdetermined nature of the system, a solution $L$ does not necessarily exist irrespective of any assumption concerning the regularity and continuity of the system.[3]

Our analysis of this problem will proceed as follows. In Chapters 1 and 2 we have studied certain methodological aspects related to the equations of motion, i.e., the right-hand side of identifications (3.1.2). The first sections of this chapter are devoted to the extension of this study to the left-hand side of identifications (3.1.2), i.e., to Lagrange's equations. In particular, we shall first study (Section 3.2) the basic continuity, regularity, and consistency conditions for Equations (3.1.1). Secondly, we shall analyze (Section 3.3) Lagrange's equations from the variational approach to self-adjointness as introduced in Chapter 2. In this way, we will complete our analysis of certain basic properties of each separate member of identifications (3.1.2). In order to combine those results, we shall then give an appropriate definition of the concept of analytic representation in configuration space (Section 3.4). The problem of the necessary and sufficient conditions for the existence of a Lagrangian will then be treated in Section 3.5. Section 3.6 will be devoted to a method for the construction of a Lagrangian. A reinterpretation of the results for ascertaining the most general form of Newtonian forces admissible by a Lagrangian representation will be given in Section 3.7.

[2] See, for instance, Goldschmidt (1967), Spencer (1969), and Gasqui (1975).

[3] It is significant here to point out the dual nature of Lagrange's equations depending on whether one considers the *Direct Problem* or the *Inverse Problem* of Newtonian Mechanics. The Direct Problem is essentially the conventional approach whereby one first assigns a Lagrangian and then computes the equations of motion through Lagrange's equations. Within the context of this problem, Lagrange's equations are *ordinary second-order differential equations* in the $q$'s. The Inverse Problem, on the contrary, as by now familiar, consists of assigning the equations of motion and then computing a Lagrangian through identifications (3.1.4). Within the context of this latter problem Lagrange's equations are *partial second-order differential equations* in the unknown $L$. On methodological grounds, the existence theory for ordinary differential equations reviewed in Section 3.1 is, therefore, sufficient to ascertain the consistency of Lagrange's equations in the context of the Direct Problem. As we shall see in this chapter, the theory of differential forms reviewed in Section 1.2 (which is essentially centered on the study of partial differential equations) is sufficient to study the integrability conditions of the fundamental identifications (3.1.2) of the Inverse Problem. These remarks illustrate the need for both methodologies, i.e., those for ordinary *and* partial differential equations, although from different profiles.

The remaining sections of this chapter are devoted to the independent Inverse Problem of phase space formulations. After reviewing the Legendre transform (Section 3.8), we shall study the equivalence of Lagrange's and Hamilton's equations (Section 3.9) and establish the self-adjointness of Hamilton's equations (Section 3.10). The concept of an analytic representation in phase space will be treated in Section 3.11, and the study of the necessary and sufficient conditions for the existence of a Hamiltonian, treated in a manner *independent* from that of a Lagrangian, will be presented in Section 3.12.

## 3.2 The Conventional Lagrange's Equations

Let us recall that Lagrange's equations are linear in the accelerations $\ddot{q}^k$ (but not necessarily linear in the $q^k$ and $\dot{q}^k$ variables) and, as such, they are of "Newtonian type" (in the sense of the Appendix, Section A.7). Their continuity properties can thus be studied, as was the case for Newtonian systems, in a region $R^{2n+1}$ of points $(t, q, \dot{q})$.

By inspection we see that for the second-order partial derivatives of Equations (3.1.3) to exist and be continuous, the Lagrangian $L(t, q, \dot{q})$ must be at least of class $\mathscr{C}^2$ in a region $R^{2n+1}$ of its variables. However, such an assumption corresponds to Newtonian system (A.7.5) whose functions $A_{ki}$ and $B_k$ are only of class $\mathscr{C}^0$ in $R^{2n+1}$, and, as such, it is insufficient to guarantee the uniqueness of the solution.

For consistency with the continuity assumptions of Section 2.1, we shall assume from now on, unless otherwise specified, that all Lagrangians are of at least class $\mathscr{C}^4$ in a region of their variables, i.e.,

$$L \in \mathscr{C}^4(R^{2n+1}). \tag{3.2.1}$$

The above assumption implies that all partial derivatives, up to and including fourth-order, exist, are continuous, and "commute," i.e.,

$$\frac{\partial^m L}{(\partial q^i)^{m_1}(\partial \dot{q}^j)^{m_2}} = \frac{\partial^m L}{(\partial \dot{q}^j)^{m_2}(\partial q^i)^{m_1}} \tag{3.2.2}$$

$$m_1 + m_2 = m = 0, 1, 2, 3, 4.$$

The functional determinant of Equations (3.1.1), also called the *Hessian determinant*, is given by

$$|\mathscr{H}|(R^{2n+1}) = \left|\frac{\partial^2 L}{\partial \dot{q}^i \, \partial \dot{q}^j}\right|(R^{2n+1}) \tag{3.2.3}$$

and it is also defined in a region $R^{2n+1}$ of points $(t, q, \dot{q})$ in a way similar to the corresponding Newtonian determinant (Chart 1.1).

In line with Definition 1.1.1, we then have the following definition.

**Definition 3.2.1.** The Lagrangian $L(t, q, \dot{q})$ or Lagrange's equations (3.1.1) are called *regular* (*degenerate*)[4] in a region $R^{2n+1}$ of points $(t, q, \dot{q})$ when the functional determinant (3.2.3) is non-null (null) in it, with the possible exception of a (finite) number of isolated zeros.

Again, let us stress that, according to the above definition, determinant (3.2.3) must be null at a point $(t, q, \dot{q})$ *and* at least in its neighborhood $(t, q, \dot{q})$ for a Lagrangian (or Lagrange's equations) to be degenerate (Section 1.1). In practice, a simple inspection as to whether the Hessian is non-null or null as a function is sufficient to ascertain the regularity or degeneracy of a Lagrangian. If the Hessian is non-null as a function, this does not prohibit the possible existence of its zeros. Such zeros, when they occur, render the value of the Hessian null only at a (finite) number of points and *not* in their neighborhoods. Such a Lagrangian is then still regular according to the above definition.[5] More generally, we can say that when Hessian (3.2.3) admits a (finite) number of isolated zeros, the Lagrangian is still regular. From here on we shall tacitly assume that the region of definition of a Lagrangian has been selected in such a way as to avoid possible zeros of the Hessian (see Problem 2.4).

Identifications (3.1.2) demand, through identities (3.1.4a), that

$$\left|\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right|(R^{2n+1}) = |A_{ij}|(R^{2n+1}). \tag{3.2.4}$$

This clearly implies the following theorem.

**Theorem 3.2.1** (A Necessary Condition for the Existence of a Lagrangian). *A necessary condition for regular* (*degenerate*) *Newtonian systems* (A.7.5) *to admit an analytic representation* (3.1.2) *in a region* $R^{2n+1}$ *of points* $(t, q, \dot{q})$, *is that the Lagrangian be regular* (*degenerate*) *in it.*

It should be emphasized here that the condition of the above theorem is necessary but *not* sufficient.

A *path* (or *path segment*)

$$E = \{q | q^k(t), t \in (t_1, t_2), k = 1, 2, \ldots, n\} \tag{3.2.5}$$

will be called a *possible path* when it is a solution of Equations (3.1.1). Such a path is often called *regular* or *degenerate* (also *nonsingular* or *singular*) depending on whether the Lagrangian is regular or degenerate along it, that is, Hessian determinant (3.2.3) is null or non-null along $(t, E, \dot{E})$.

[4] See footnote 3 of Chapter 1 for comments on this terminology.

[5] Let us recall that, according to our definition (Section 1.1), a "region" is an open and connected set. Thus, the minimal region of definition of a Lagrangian is constituted by a point $(t, q, \dot{q})$ *and* its neighborhood.

We shall call the *actual path*, and denote it with $E_0$ the solution of *Lagrange's initial value problem*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} = 0, \qquad L \in \mathscr{C}^4(R^{2n+1}), \tag{3.2.6a}$$

$$q^k(t_0) = u_0^k, \qquad \dot{q}^k(t_0) = v_0^k. \tag{3.2.6b}$$

In essence, with the terms "possible path" and "actual path" we intend to indicate the fact that the former, since it is a solution of Lagrange's equations without initial conditions, depends on $2n$ arbitrary constants and, as such, it does not necessarily represent the actual trajectory of the system. The latter, however, is the solution of Lagrange's initial value problem (3.2.6); it does not depend on arbitrary constants and, as such, does represent the actual trajectory of the system in the space of the $q$'s.

We shall call the $\infty^{2n}$ *family of possible paths* the family

$$\tau^{2n}_{(t,q)} = \{E \mid E = \{q^k(t; u, v),\, t \in (t_1, t_2),\, u^k \in (u_0^k)_\varepsilon,\, v^k \in (v_0^k)_\varepsilon\} \tag{3.2.7}$$

induced by the general solution of Equations (3.1.3) for all admissible values of the parameters.

Given a Lagrangian $L(t, q, \dot{q})$, after computing the indicated partial derivatives, Equations (3.1.3) become *ordinary* differential equations. Therefore, under the assumption that the Lagrangian is of (at least) class $\mathscr{C}^4$ and regular, all the theorems of the existence theory of Section 1.1 apply. We then say that the assumed continuity and regularity conditions guarantee the *consistency* of Lagrange's equations, namely, the representation of a physically admissible motion.

Suppose that all admissible paths (including actual and possible paths) are of (at least) class $\mathscr{C}^1$ in a given interval of time. This corresponds to the exclusion of an impulsive motion[6] or any type of discontinuous force. Then, for class $\mathscr{C}^4$ regular Lagrangians, Equations (3.1.1) are equivalent to the so-called *integral form of Lagrange's equations* (1.3.22), i.e.,

$$\frac{\partial L}{\partial \dot{q}^k} - \int_{t_1}^{t} dt \frac{\partial L}{\partial q^k} = c_k = \text{const.}, \tag{3.2.8}$$

everywhere in the interval considered. Indeed, under the continuity properties $L \in \mathscr{C}^4(R^{2n+1})$ and $E = \{q_k(t; u, v)\} \in \mathscr{C}^1(t_1, t_2)$ with $t_1$ and $t_2$ in the interior of $R^{2n+1}$, the continuity of $L$ along the possible path $E$ is ensured.[7] Then, the

[6] See Chart A.3 (Appendix).

[7] It is essential in this respect to differentiate between the continuity of $L$ as a function of the variables $(t, q, \dot{q})$ and the continuity of $L$ when computed along a given path $E$, because the former does not imply the latter. For instance, the Lagrangian $L = \dot{q}^3 - q^3$ is of class $\mathscr{C}^\infty$ in $(q, \dot{q})$, but when such a Lagrangian is computed along the path $E = q = \|t\|, t \in (-t_0, +t_0), t_0 \neq 0$, then $L(E, \dot{E})$ is discontinuous at $t = 0$. Our restriction that all paths are of at least class $\mathscr{C}^1$ eliminates occurrences of this type.

total time derivative of Equations (3.2.8) exists and is continuous everywhere in $(t_1, t_2)$, yielding

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^k} - \int_{t_1}^{t} dt \frac{\partial L}{\partial q^k} - c_k\right) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k}. \tag{3.2.9}$$

A similar situation exists for the inverse transition through an integration.

Despite the assumptions $E \in \mathscr{C}^1$ and $L \in \mathscr{C}^4$, we are still left with the problem of the continuity properties of the accelerations $\ddot{q}^k$. This problem is solved by the so-called *Hilbert Differentiability Theorem*, which indicates the deep relationship existing between the continuity of (possible or actual) paths and the regularity of the Lagrangian.

**Theorem 3.2.2** (Hilbert Differentiability Theorem).[8] *Suppose that the path $E$ is of class $\mathscr{C}^1$ in a neighborhood $(t_0)_\varepsilon$ of a point $t_0$ and satisfies Lagrange's equations* (3.1.1) *or* (3.2.8) *for a given $L(t, q, \dot{q})$. Then, in the neighborhood of every point $(t_0, E(t_0), \dot{E}(t_0))$, in which the Lagrangian is of class $\mathscr{C}^{m+2}$, $m > 1$, and regular, the path $E$ is of class $\mathscr{C}^m$ in $(t_0)_\varepsilon$. If the assumptions hold everywhere in an interval $(t_1, t_2)$, so do the conclusions.*

The proof can be outlined as follows. Equations (3.2.8) can be interpreted as a system of first-order differential equations, i.e.,

$$F_k(t, q, \dot{q}) = \frac{\partial L}{\partial \dot{q}^k} - \int_{t_1}^{t} dt \frac{\partial L}{\partial q^k} - c_k = 0. \tag{3.2.10}$$

In the neighborhood of a point $P_0 = (t_0, q(t_0), \dot{q}(t_0))$, the Implicit Function Theorem 3.1.1 holds for Equations (3.2.10) with respect to $\dot{q}^k$ if and only if its functional determinant

$$\left|\frac{\partial F_i}{\partial \dot{q}^j}\right| = \left|\frac{\partial^2 L}{\partial \dot{q}^i \, \partial \dot{q}^j}\right| \tag{3.2.11}$$

is different from zero at $P_0$. But the Lagrangian $L$ is regular by assumption in $(P_0)_\varepsilon$. Thus, Theorem 1.1.1 holds and from property (e) of the same theorem it follows that $\dot{q}^k$ has the same continuity properties of $F_k$, namely, $\dot{q}^k \in \mathscr{C}^{m-1}(t_0)_\varepsilon$ and, thus, $q^k \in \mathscr{C}^m(t_0)_\varepsilon$.

It should be stressed that the regularity condition plays a crucial role in the above theorem. Indeed, if such a condition is removed, the conclusions of the Hilbert Theorem 3.2.2 do not necessarily hold.

Without proof, we also quote the following reformulation of Theorem 1.1.6 for Lagrange's equations.

**Theorem 3.2.3** (Embedding Theorem for Solutions of Lagrange's Initial value Problem).[9] *Every regular actual path $E_0 = \{q \mid q^k = q^k(t; u_0, v_0), t \in (t_1, t_2), k = 1, 2, \ldots, n\}$ of Lagrange's initial value problem* (3.2.6) *can be*

[8] See, for instance, Bliss (1946), Akhiezer (1962).
[9] See, for instance, Bliss (1946), Sec. I.7.

*embedded in a $2n$-parameter family of possible paths* $E = \{q^k | q^k = q^k(t; u, v), t \in (t_1, t_2), u^k \in (u_0^k)_\varepsilon, v^k \in (v_0^k)_\varepsilon, k = 1, 2, \ldots, 3n\}$ *whose functions* $q^k$ *possess continuous partial derivatives of at least the second order for all values* $t \in (t_1, t_2)$, $u^k \in (u_0^k)_\varepsilon$, $v^k \in (v_0^k)_\varepsilon$, *and the determinant*

$$D(t; u, v) = \begin{vmatrix} \dfrac{\partial q^i}{\partial u^j} & \dfrac{\partial q^i}{\partial v^j} \\ \dfrac{\partial \dot{q}^i}{\partial u^j} & \dfrac{\partial \dot{q}^i}{\partial v^j} \end{vmatrix} \tag{3.2.12}$$

*is everywhere non-null along* $E$.

Unless otherwise stated, all Lagrangians considered from here on will be regular and of at least class $\mathscr{C}^4$ in their region of definition.

## 3.3 Self-Adjointness of the Conventional Lagrange's Equations

Consider a class $\mathscr{C}^2$, regular, holonomic Newtonian system (2.2.1). The problem of the existence of a Lagrangian for its representation (3.1.2) centrally depends on the variational characteristics of both members of these identifications.

In Chapter 2, we established that systems (2.2.1) can be either self-adjoint or non-self-adjoint. In this section, we shall investigate the corresponding properties of the conventional Lagrange's equations.

Let us begin by constructing the *equations of variations* and *variational forms* of Lagrange's equations. Consider a one-parameter family of possible paths

$$\tau^1_{(t,q)} = \{E | E = q(t; w); t \in (t_1, t_2), w \in 0_\varepsilon\}, \tag{3.3.1}$$

which are solutions of the equations

$$L_k(E) = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k}\right)(E) = 0, \tag{3.3.2a}$$

$$L \in \mathscr{C}^4(E), \tag{3.3.2b}$$

$$\left|\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right|(E) \neq 0. \tag{3.3.2c}$$

From the existence theorem of Section 1.1 and Theorem 3.2.2, we know that the paths $q^k(t; w)$ are of class $\mathscr{C}^2$ in $(t_1, t_2)$. Theorems 1.1.4 and 1.1.7 then apply,

and $q^k(t; w)$, $\dot{q}^k(t; w)$, and $\ddot{q}^k(t; w)$ possess continuous derivatives with respect to all $w \in 0_\varepsilon$.[10] Thus, the *variations*

$$\eta^k = \left.\frac{\partial q^k}{\partial w}\right|_{w=0}, \qquad \dot{\eta}^k = \left.\frac{\partial \dot{q}^k}{\partial w}\right|_{w=0}, \qquad \ddot{\eta}^k = \left.\frac{\partial \ddot{q}^k}{\partial w}\right|_{w=0} \tag{3.3.3}$$

exist and are continuous in $(t_1, t_2)$.

Equation (3.3.2) can be more explicitly written

$$L_k(E) = \left\{\frac{d}{dt}\frac{\partial}{\partial \dot{q}^k} - \frac{\partial}{\partial q^k}\right\} L(t, q(t; w), \dot{q}(t; w)), \tag{3.3.4}$$

where now both $L$ and $q$ are known. By differentiating with respect to $w$ and by putting $w = 0$, we obtain the system

$$\begin{aligned} J_k(\eta) &= \left.\frac{dL_k}{dw}\right|_{w=0} \\ &= \left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{q}^k\, \partial \dot{q}^i}\,\dot{\eta}^i + \frac{\partial^2 L}{\partial \dot{q}^k\, \partial q^i}\,\eta^i\right) - \left(\frac{\partial^2 L}{\partial q^k\, \partial \dot{q}^i}\,\dot{\eta}^i + \frac{\partial^2 L}{\partial q^k\, \partial q^i}\,\eta^i\right)\right]_{w=0} \end{aligned} \tag{3.3.5}$$

which, as is the case for all equations of variations (see Section 2.1), is *linear* in $\eta^i$, $\dot{\eta}^i$, and $\ddot{\eta}^i$.

By introducing the function

$$J = J(t, \eta, \dot{\eta}) = \frac{1}{2}\left(\frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}\,\dot{\eta}^i\dot{\eta}^j + 2\frac{\partial^2 L}{\partial \dot{q}^i\, \partial q^j}\,\dot{\eta}^i\eta^j + \frac{\partial^2 L}{\partial q^i\, \partial q^j}\,\eta^i\eta^j\right), \tag{3.3.6}$$

Equations (3.3.5) can be written in the form (1.3.25), i.e.,

$$J_k(\eta) = \frac{d}{dt}\frac{\partial J}{\partial \dot{\eta}^k} - \frac{\partial J}{\partial \eta^k} = 0, \tag{3.3.7a}$$

$$J \in \mathscr{C}^2(R^{2n+1}_{(t,\,\eta,\,\dot{\eta})}),$$

$$\left|\frac{\partial^2 J}{\partial \dot{\eta}^i\, \partial \dot{\eta}^j}\right|(R^{2n+1}_{(t,\,\eta,\,\dot{\eta})}) = \left[\left|\frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}\right|(R^{2n+1}_{(t,\,E,\,E)})\right]_{w=0} \neq 0. \tag{3.3.7b}$$

These are the *equations of variations of Lagrange's equations*, also called *Jacobi's equations* (see Section 1.3). When the forms $J_k(\eta)$ are computed along a variation $\eta$ which is not necessarily a solution of Equations (3.3.7), we shall call them *variational forms of Lagrange's equations* in line with the assumption of Section 2.1.

Equations (3.3.7) are similar in structure to Lagrange's equations. Nevertheless, it must be recalled in this respect that the former are always linear in

[10] The analysis of this section can be equivalently carried out for $w$ in the neighborhood of any (finite) value $w_0$.

$\eta^i$, $\dot{\eta}^i$, and $\ddot{\eta}^i$ while the latter are linear only in $\ddot{q}^i$ but not necessarily in $q^i$ and $\dot{q}^i$ as well.

We are now equipped to prove a theorem of major significance for our analysis.

**Theorem 3.3.1** (Self-Adjointness of the Conventional Lagrange's Equations).[11] *Under the assumptions that the Lagrangians $L(t, q, \dot{q})$ are of (at least) class $\mathscr{C}^4$ and regular in a region $R^{2n+1}$ of points $(t, q, \dot{q})$, the conventional Lagrange's equations are always self-adjoint in $R^{2n+1}$.*

PROOF. The variational forms

$$\begin{aligned} J_i(\eta) = &\left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial q^k}\right) - \frac{\partial^2 L}{\partial q^i \partial q^k}\right]\eta^k \\ &+ \left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^k}\right) + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^k} - \frac{\partial^2 L}{\partial q^i \partial \dot{q}^k}\right]\dot{\eta}^k + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^k}\ddot{\eta}^k, \end{aligned} \tag{3.3.8}$$

under the assumed continuity and regularity conditions, always coincide with the adjoint system (see Section 2.1)

$$\begin{aligned} \tilde{J}_i(\eta) = &\eta^k\left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{q}^k \partial q^i}\right) - \frac{\partial^2 L}{\partial q^k \partial q^i}\right] \\ &- \frac{d}{dt}\left\{\eta^k\left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^i}\right) + \frac{\partial^2 L}{\partial \dot{q}^k \partial q^i} - \frac{\partial^2 L}{\partial q^k \partial \dot{q}^i}\right]\right\} + \frac{d^2}{dt^2}\left(\eta^k \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^i}\right) \end{aligned} \tag{3.3.9}$$

everywhere in $R^{2n+1}$. *Q.E.D.*

In essence, Theorem 3.3.1 states that, under the indicated continuity and regularity assumptions, Lagrange's equations are self-adjoint for "all" possible Lagrangians.

Notice that the continuity property $L \in \mathscr{C}^3(R^{2n+1})$ is sufficient to establish the existence and continuity of Jacobi's forms (3.3.8). However, such a continuity property is insufficient to establish the self-adjointness of Lagrange's equations because adjoint system (3.3.9) demands the use of partial derivatives up to the fourth order. The emerging minimal continuity condition $L \in \mathscr{C}^4(R^{2n+1})$ then coincides with the minimal continuity conditions $A_{ki}$, $B_k \in \mathscr{C}^2(R^{2n+1})$ to study the self-adjointness of Newtonian systems in their

[11] As indicated in the Introduction, this property goes back to Jacobi (1837). For a subsequent proof see, for instance, Davis (1929). For an extension to relativistic field theories see Santilli (1977a, Theorem 7.1). Notice that Theorem 3.3.1 deals with the variational self-adjointness of Lagrange's *equations*. The self-adjointness of the Lagrange *operator*

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}^k} - \frac{\partial}{\partial q^k}$$

is a well-known property, but it demands a different treatment. Therefore, the approach to self-adjointness which is followed in this monograph is *variational* rather than operational in nature. For the latter approach, see Vainberg (1964). For the equivalence of the operational and variational approach, see Charts 3.16 and 3.17.

fundamental form (Sections 2.2 and 2.3). Indeed, when the Lagrangian is of class $\mathscr{C}^4$, Lagrange's equations are of class $\mathscr{C}^2$.

Theorem 3.3.1 can also be proved by showing that Equations (3.1.3) satisfy all the conditions of self-adjointness (see Problem 3.1).

It should be stressed that Jacobi's equations (3.3.7) are intimately linked to Lagrange's equations in the sense that they demand the prior knowledge of the latter. The combination of Lagrange's equation and the related Jacobi's equations puts the conventional analytic framework in a different light. The need for the joint use of these equations to properly characterize an analytic representation in configuration space will be indicated in the next section. Its significance for the problem of the existence of a Lagrangian will be pointed out in Section 3.5. However, the joint use of Lagrange's equations and the related Jacobi's equations might have a significance that goes beyond the problem of the existence of a Lagrangian. This is due to the fact that while Lagrange's equations are *generally nonlinear*, and therefore their general solution is usually unknown,[12] the related Jacobi's equations are *always linear* and, as such, their general solution can be computed with conventional techniques. To the best of my knowledge, the possible significance of the joint use of these equations for nonlinear systems has not been investigated until now.

As a final remark, we would like to stress that the knowledge of a solution of Lagrange's equations *is not* necessary for establishing the self-adjointness of the same equations, as can also be seen, for instance, by proving Theorem 3.3.1 along the lines of Problem 3.1. This is the reason why we have used, in the proof of Theorem 3.3.1, the variational *forms* $J_k(\eta)$ and their adjoint $\tilde{J}_k(\eta)$ rather than the corresponding variational *equations* $J_k(\eta) = 0$ and $\tilde{J}_k(\eta) = 0$. This point has a crucial relevance for our analysis owing to the generally nonlinear nature of the equations of motion considered.

## 3.4 The Concept of Analytic Representation in Configuration Space

As previously stated, a central objective of this monograph is to consider a given system of Newton's equations of motion and then study the conditions under which a Lagrangian capable of "representing" such a system exists. In order to achieve this objective it is essential to clarify the concept of an "analytic representation in configuration space," namely, the representation of the system in terms of Lagrange's equations.

In principle, we can say that a system of Newton's equations of motion admits a representation in terms of Lagrange's equations for a given Lagrangian when the solutions of those two systems coincide. Predictably, this

[12] We are referring here to exact solutions. Approximate solutions of nonlinear equations can be computed, e.g., with numerical methods.

approach encounters severe practical difficulties because the equations of motion are generally nonlinear.

In order to overcome these difficulties, we introduce the following definition.

**Definition 3.4.1.**[13] A class $\mathscr{C}^2$, regular, holonomic, system of Newton's equations of motion admits an *analytic representation in configuration space in terms of the conventional Lagrange's equations* in a region $R^{2n+1}$ of the variables $(t, q, \dot{q})$ when there exist $n^2$ functions $h_k^i(t, q, \dot{q})$ which are of (at least) class $\mathscr{C}^2$ and whose matrix $(h_k^i)$ is regular in $R^{2n+1}$, such that the conventional Lagrange's equations coincide with the equations of motion up to the equivalence transformation induced by such a matrix $(h_k^i)$, i.e.,[14]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} = h_k^i(A_{ij}\ddot{q}^j + B_i), \qquad k = 1, 2, \ldots, n, \tag{3.4.1a}$$

$$h_k^i \in \mathscr{C}^2(R^{2n+1}), \tag{3.4.1b}$$

$$|h_k^i|(R^{2n+1}) \neq 0, \tag{3.4.1c}$$

or, equivalently, when the equations of motion coincide with the conventional Lagrange's equations up to the equivalence transformation induced by the inverse matrix $(h_k^{-1\,i}) \equiv (h_k^i)^{-1}$, i.e.,

$$h_k^{-1\,i}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right) = A_{kj}\ddot{q}^j + B_k, \tag{3.4.2a}$$

$$(h_k^{-1\,i}) = (h_k^i)^{-1}. \tag{3.4.2b}$$

A few comments are in order here. First of all, let us note that the right-hand side of definition (3.4.1a) is indeed an equivalence transformation of the equations of motion, precisely in view of the assumption of regularity of the matrix $(h_k^i)$. Assumptions (3.4.1b) are introduced to preserve the minimal continuity condition of the equations of motion in their fundamental form. The functional dependence $h_k^i = h_k^i(t, q, \dot{q})$ is the *maximal* functional dependence of these functions, which is admissible within the context of Definition 3.4.1. Indeed, any additional functional dependence of these functions on the accelerations would not preserve the Newtonian character of the equations of motion, namely, their linearity in the accelerations (Appendix). Finally, the equivalence of definitions (3.4.1) and (3.4.2) is also self-evident from the assumed regularity of the matrix $(h_k^i)$. From now on, we shall refer to this matrix as the *matrix of the factor functions.*

The first significant implication of Definition 3.4.1 is that, given a system of Newton's equations of motion, the knowledge of only one function, i.e., the

[13] Santilli (1977a).

[14] Notice that these identities *are not* set equal to zero because they must hold along any admissible path.

Lagrangian $L$, is generally *insufficient* to characterize an analytic representation because $n^2$ additional functions, i.e., the factor functions $h_k^i$, are generally needed. This fact, which will have an impact at several levels of our analysis, will be illustrated later in this section.

To better elaborate the significance of the factor matrices, we now introduce the following definition.

**Definition 3.4.2.**[15] The analytic representation of Definition 3.4.1 is called *direct* (*indirect*) when the matrix $(h_k^i)$ of the factor functions is (is not) the unit matrix $(\delta_k^i)$.

We therefore have a direct analytic representation when the equations of motion are represented as given, without any equivalence transformation. But, as will be evident later, the Lagrangian for this type of representation exists only under special circumstances. As a consequence, one remains in the general case with the study of the analytic representations of equivalent systems, rather than the original system as given. This indicates the significance of the concept of indirect analytic representations, particularly when non-conservative forces occur. However, it should be pointed out that this type of analytic representations is also significant for the case when all acting forces are conservative.[16]

It is useful here to illustrate the concepts of direct and indirect analytic representations with a simple example. Consider the self-adjoint system of two uncoupled harmonic oscillators with the equations of motion

$$\begin{aligned} \ddot{q}_1 + \omega^2 q_1 &= 0, \\ \ddot{q}_2 + \omega^2 q_2 &= 0. \end{aligned} \tag{3.4.3}$$

A well-known Lagrangian for the representation of this system is given by

$$L = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{\omega^2}{2}(q_1^2 + q_2^2). \tag{3.4.4}$$

By computing Lagrange's equations with this Lagrangian, we obtain the identity

$$\begin{pmatrix} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}^1} - \dfrac{\partial L}{\partial q^1} \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}^2} - \dfrac{\partial L}{\partial q^2} \end{pmatrix}_{\mathrm{SA}} = \begin{pmatrix} \ddot{q}_1 + \omega^2 q \\ \ddot{q}_2 + \omega^2 q \end{pmatrix}_{\mathrm{SA}} = 0, \tag{3.4.5}$$

[15] Santilli (1977a).

[16] This is because conservative Newtonian systems can also be given in non-self-adjoint forms [see in this respect Equations (2.2.28) or Example 3.5], in which case a Lagrangian for their *direct* analytic representation *does not* exist, as follows from the Fundamental Analytic Theorem of the next section.

where the symbol SA stands for self-adjointness. Therefore, we have in this case, according to our terminology, a direct analytic representation of the equations of motion (3.4.3).

But an equally acceptable Lagrangian for the representation of system (3.4.3) is given by[17]

$$L^* = \dot{q}_1\dot{q}_2 - \omega^2 q_1 q_2. \tag{3.4.6}$$

By again computing Lagrange's equations with this new Lagrangian, we now have

$$\begin{pmatrix} \dfrac{d}{dt}\dfrac{\partial L^*}{\partial \dot{q}^1} - \dfrac{\partial L^*}{\partial q^1} \\ \dfrac{d}{dt}\dfrac{\partial L^*}{\partial \dot{q}^2} - \dfrac{\partial L^*}{\partial q^2} \end{pmatrix}_{SA} = \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 + \omega^2 q_1 \\ \ddot{q}_2 + \omega^2 q_2 \end{pmatrix}_{SA} \right]_{SA} = 0. \tag{3.4.7}$$

This is precisely an indirect analytic representation of system (3.4.3), where the factor function $h_k^i$, in this case, characterizes a permutation of the indices. Indeed, a simple inspection of Lagrangian (3.4.6) indicates that Lagrange's equations, say, in the $(q_1, q_2)$ variables, reproduce the equations of motion in the $(q_2, q_1)$ variables, and vice versa. This did not happen for Lagrangian (3.4.4). Another inspection also indicates that both Lagrangians (3.4.4) and

[17] A brief digression to the field theoretical case is significant here. Under the transitions

$$q_1(t) \to \varphi(x),\ q_2(t) \to \bar{\varphi}(x),\ \omega^2 \to m^2 (\hbar = c = 1), \qquad \frac{d}{dt} \to \frac{d}{dx^\mu}$$

Equations (3.4.3) become those of the complex scalar field, i.e.,

$$(\Box + m^2)\varphi = 0,$$
$$(\Box + m^2)\bar{\varphi} = 0.$$

The Lagrangian density customarily used in field theory for the representation of these equations is precisely one of type (3.4.6), i.e.,

$$\mathscr{L}^* = \bar{\varphi}_\mu; \varphi^\mu; - m^2\bar{\varphi}\varphi, \qquad \varphi^\mu; = \frac{\partial \varphi}{\partial x^\mu}$$

$$\mu = 0, 1, 2, 3,$$

while the Lagrangian corresponding to structure (3.4.4), i.e.,

$$\mathscr{L} = \tfrac{1}{2}(\varphi_\mu; \varphi^\mu; - m^2\varphi^2) + \tfrac{1}{2}(\varphi_\mu; \varphi^\mu; - m^2\varphi^2),$$

even though it is fully acceptable on grounds of its real-valuedness and capability of reproducing the considered field equations, is usually ignored. Oddly, this position in field theory is opposite to the corresponding position in Newtonian Mechanics, where Lagrangian (3.4.4) is generally assumed, while its equivalent form (3.4.6) is customarily ignored. In actuality, this multiplicity of the functional structure of the Lagrangian has rather deep implications within the context of both Newtonian Mechanics [see Santilli (1979)] and field theory [see Santilli (1978, Vol. I)].

(3.4.6) are regular, as they must be from Theorem 3.2.1, in view of the values of the corresponding Hessians

$$\left|\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$
$$\left|\frac{\partial^2 L^*}{\partial \dot{q}^i \partial \dot{q}^j}\right| = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1. \tag{3.4.8}$$

Strictly speaking, Lagrangian (3.4.4) alone is capable of representing the considered system in the form (3.4.3), while representation (3.4.7) demands the knowledge of Lagrangian (3.4.6) *and* four elements of the factor functions $h_k^i$. Notice that in this simple case the question of whether the representation is direct or indirect is, in the final analysis, immaterial because it is merely related to the assumed order of the equations of motion. For instance, Equations (3.4.3) can be assigned in their reverse order. In this case, representation (3.4.7) becomes direct, while representation (3.4.5) is indirect. What will be significant for our study of the transformation theory [see Santilli (1979)] is the possibility that the same system admits *both* a direct and an indirect analytic representation.

The combined use of Definitions 3.4.1 and 3.4.2 is still insufficient to characterize properly, according to our needs, the behavior of the equations of variations. To fulfill this last requirement, we introduce the following definition.

**Definition 3.4.3.**[18] The analytic representations of Definitions 3.4.1 and 3.4.2 are termed *ordered* (*nonordered*) when the left-hand and right-hand sides of Equations (3.4.1) or (3.4.2) coincide (do not coincide), member by member, for all values of the index $k = 1, 2, 3, \ldots, n$ in a given ordering.

In essence, the concept of *ordered direct analytic representations* implies that Lagrange's equations and the equations of motion not only coincide as systems, but also coincide member by member for a given ordering of the index $k$. In this case, we shall write

$$L_k(q) \equiv A_{ki}\ddot{q}^i + B_k, \qquad k = 1, 2, \ldots, n. \tag{3.4.9}$$

On the contrary, the concept of *nonordered direct analytic representations* implies that the left-hand and right-hand sides of the above equations only coincide as systems, with no reference to their respective orderings. In this case, we shall write

$$\{L_k(q)\} = \{A_{ki}\ddot{q}^i + B_k\}. \tag{3.4.10}$$

[18] Santilli (1977a).

Clearly, the only possible differences between the left-hand and the right-hand sides of the above identifications are permutations of the indices. For instance, Equations (3.4.5) characterize an ordered direct analytic representation of the equations of motion (3.4.3) in the ordering $q = (q_1, q_2)$, while Equations (3.4.7) characterize an ordered indirect analytic representation of the same system in the same ordering. If the condition of ordering is removed, then *both* Lagrangians (3.4.4) and (3.4.6) characterize nonordered *direct* analytic representations of the equations of motion (3.4.3).

From the viewpoint of our objective of studying the necessary and sufficient conditions for the existence of a Lagrangian, the condition of ordering plays a crucial role. Indeed, the term "ordered direct analytic representation" implies that each of the following three sets of identities for class $\mathscr{C}^2$ and regular Newtonian systems hold in the assumed ordering:

| Lagrange's equations | | Equations of motion | |
|---|---|---|---|
| $L_k(q)$ | $\equiv$ | $F_k(q)$, | (3.4.11a) |
| Jacobi's equations | | Equations of variations of the equations of motion | |
| $J_k(\eta)$ | $\equiv$ | $M_k(\eta)$, | (3.4.11b) |
| Adjoint system of Jacobi's equations | | Adjoint system of the equations of variations | |
| $\tilde{J}_k(\tilde{\eta})$ | $\equiv$ | $\tilde{M}_k(\tilde{\eta})$. | (3.4.11c) |

This is a consequence of the uniqueness of the variational forms and their adjoint system as identified in Section 2.1.

Similarly, the terms "ordered indirect analytic representation" implies that each of the following three sets of identities, again for class $\mathscr{C}^2$ and regular systems, also hold in the assumed ordering:

| Lagrange's equations | | Equivalent equations of motion | |
|---|---|---|---|
| $L_k(q)$ | $\equiv$ | $F_k^{eq}(q) = h_k^i F_i(q)$, | (3.4.12a) |
| Jacobi's equations | | Equations of variation of the equivalent system | |
| $J_k(\eta)$ | $\equiv$ | $M_k^{eq}(\eta)$, | (3.4.12b) |
| Adjoint system of Jacobi's equations | | Adjoint system of the equations of variations | |
| $\tilde{J}_k(\tilde{\eta})$ | $\equiv$ | $\tilde{M}_k^{eq}(\tilde{\eta})$. | (3.4.12c) |

It is essential to illustrate the above remarks with an example. Consider the simple generalization of system (3.4.3),

$$\begin{aligned}\begin{pmatrix}\ddot{q}_1 + b\dot{q}_1 + \omega^2 q_1\\ \ddot{q}_1 - b\dot{q}_2 + \omega^2 q_2\end{pmatrix} &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\ddot{q}_1\\ \ddot{q}_2\end{pmatrix} + \begin{pmatrix}b & 0\\ 0 & -b\end{pmatrix}\begin{pmatrix}\dot{q}_1\\ \dot{q}_2\end{pmatrix} + \begin{pmatrix}\omega^2 & 0\\ 0 & \omega^2\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix}\\ &= (c_{ki}\ddot{q}^i) + (b_{ki}\dot{q}^i) + (a_{ki}q^i)\\ &= 0,\end{aligned} \tag{3.4.13}$$

which exhibits the presence of a *nonconservative* force $F = (-b\dot{q}_1, + b\dot{q}_2)$.[19] This is a two-dimensional linear system of ordinary second-order differential equations with constant coefficients. The conditions for self-adjointness (2.1.17) now become

$$c_{ki} = c_{ik}, \tag{3.4.14a}$$

$$b_{ki} = -b_{ik}, \tag{3.4.14b}$$

$$a_{ki} = a_{ik}. \tag{3.4.14c}$$

Since Equations (3.4.14b) are violated, system (3.4.13) is *non-self-adjoint*. By anticipating the Fundamental Analytic Theorem of the next section, one might, therefore, conclude that a Lagrangian for the representation of such a system does not exist. This conclusion, however, is erroneous. Indeed, an analytic representation of system (3.4.13) is known in the physics literature and is given by the *Morse–Feshbach Lagrangian*[20]

$$L^* = \dot{q}_1\dot{q}_2 + \frac{b}{2}(q_1\dot{q}_2 - \dot{q}_1 q_2) - \omega^2 q_1 q_2. \tag{3.4.15}$$

Our objective here is that of identifying the behavior of the analytic representation characterized by such a Lagrangian from the viewpoint of the equations of variations and their adjoint system.

By computing Lagrange's equations with Lagrangian (3.4.15), we can write

$$\begin{pmatrix}\dfrac{d}{dt}\dfrac{\partial L^*}{\partial \dot{q}^1} - \dfrac{\partial L^*}{\partial q^1}\\[2ex] \dfrac{d}{dt}\dfrac{\partial L^*}{\partial \dot{q}^2} - \dfrac{\partial L^*}{\partial q^2}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\ddot{q}_1 + b\dot{q}_1 + \omega^2 q_1\\ \ddot{q}_2 - b\dot{q}_2 + \omega^2 q_2\end{pmatrix} = 0. \tag{3.4.16}$$

[19] Notice that this force *is not* dissipative, because while the component $-b\dot{q}_1$ is passive, the component $+b\dot{q}_2$ is active.

[20] Morse and Feshbach (1953).

Therefore, we have an ordered indirect analytic representation of the equations of motion.

A striking apparent discrepancy, however, emerges between the left-hand and right-hand sides of the above identifications. This is because Lagrange's equations, from Theorem 3.3.1, are expected to be self-adjoint, while equations of motion (3.4.13), as indicated earlier, are non-self-adjoint. This indicates the need to inspect the variational behavior of this analytic representation in order to ensure that a structure of type (3.4.12) actually occurs.

Let us first verify that Theorem 3.3.1 is indeed satisfied by Lagrangian (3.4.15). Jacobi's equations for such a Lagrangian are

$$J_k^*(\eta) = -\frac{\partial^2 L^*}{\partial q^k\,\partial q^i}\,\eta^i + \left(\frac{\partial^2 L^*}{\partial \dot{q}^k\,\partial q^i} - \frac{\partial^2 L^*}{\partial q^k\,\partial \dot{q}^i}\right)\dot{\eta}^i + \frac{\partial^2 L^*}{\partial \dot{q}^k\,\partial \dot{q}^i}\,\ddot{\eta}^i = 0. \quad (3.4.17a)$$

The adjoint system of Jacobi's equations is, from Equations (3.3.9),

$$\tilde{J}_k^*(\eta) = -\eta^i\frac{\partial^2 L^*}{\partial q^i\,\partial q^k} + \dot{\eta}^i\left(\frac{\partial^2 L^*}{\partial q^i\,\partial \dot{q}^k} - \frac{\partial^2 L^*}{\partial \dot{q}^i\,\partial q^k}\right) + \ddot{\eta}^i\frac{\partial^2 L^*}{\partial \dot{q}^i\,\partial \dot{q}^k} = 0. \quad (3.4.17b)$$

But Lagrangian (3.4.15) is trivially of class $\mathscr{C}^\infty$. Therefore, systems (3.4.17a) and (3.4.17b) coincide, i.e.,

$$J_k^*(\eta) \equiv \tilde{J}_k^*(\eta), \qquad \eta \in \mathscr{C}^2(R_t) \quad (3.4.18)$$

and Lagrange's equations for Lagrangian (3.4.15), i.e., the left-hand side of representation (3.4.16), is self-adjoint, as expected.

To complete our analysis, we must now inspect the right-hand side of the same representation. This is easily achieved by rewriting such a system in the form

$$\begin{aligned}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\ddot{q}_1 + b\dot{q}_1 + w^2 q_1\\ \ddot{q}_2 - b\dot{q}_2 + w^2 q_2\end{pmatrix} &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\ddot{q}_1\\ \ddot{q}_2\end{pmatrix}\\ &\quad + \begin{pmatrix}0 & -b\\ b & 0\end{pmatrix}\begin{pmatrix}\dot{q}_1\\ \dot{q}_2\end{pmatrix} + \begin{pmatrix}0 & \omega^2\\ \omega^2 & 0\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix}\\ &= (c'_{ki}\ddot{q}^i) + (b'_{ki}\dot{q}^i) + (a'_{ki}q^i)\\ &= 0. \end{aligned} \quad (3.4.19)$$

A simple inspection then indicates that such an equivalent form of the equations of motion does indeed satisfy all conditions (3.4.14) and, therefore, it is self-adjoint. All Equations (3.4.12) for this representation then hold.

*The variational significance of the analytic representation characterized by the Morse–Feshbach Lagrangian* (3.5.15) *is therefore that of transforming the equations of motion from the original non-self-adjoint form* (3.4.13) *to the equivalent self-adjoint form* (3.4.19) *through a simple permutation of the indices,*

which we symbolically write[21]

$$\left(\begin{matrix}\dfrac{d}{dt}\dfrac{\partial L^*}{\partial \dot{q}^1} - \dfrac{\partial L^*}{\partial q^1} \\ \dfrac{d}{dt}\dfrac{\partial L^*}{\partial \dot{q}^2} - \dfrac{\partial L^*}{\partial q^2}\end{matrix}\right)_{SA} = \left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\ddot{q}_1 + b\dot{q}_1 + \omega^2 q_1 \\ \ddot{q}_2 - b\dot{q}_2 + \omega^2 q_2\end{pmatrix}_{NSA}\right]_{SA} = 0, \quad (3.4.20)$$

where NSA stands for non-self-adjointness.

If the ordering condition is removed, then simple calculations yield, instead of Equations (3.4.12), the equations

$$\{L_k^*(q)\} = \begin{Bmatrix}\ddot{q}_1 + b\dot{q}_1 + \omega^2 q_1 \\ \ddot{q}_2 - b\dot{q}_2 + \omega^2 q_2\end{Bmatrix} = \{F_k(q)\}, \quad (3.4.21a)$$

$$\{J_k^*(\eta)\} = \{M_k(\eta)\}, \quad (3.4.21b)$$

$$\{\tilde{J}_k^*(\tilde{\eta})\} \neq \{\tilde{M}_k(\tilde{\eta})\}, \quad (3.4.21c)$$

where the curly brackets indicate, in line with notation (3.4.10), that the enclosed quantities are considered as systems rather than term by term. We can, therefore, conclude that the elimination of ordering in this case does allow the

[21] A digression to the corresponding field theoretical case is significant here. Under the transitions

$$q_1(t) \to \varphi(x), \qquad q_2(t) \to \bar{\varphi}(x), \qquad \frac{b}{2} \to -ieA^\mu \qquad \omega^2 \to m^2 - e^2 A_\mu A^\mu, \qquad \frac{d}{dt} \to \frac{d}{dx^\mu},$$

Morse–Feshbach Lagrangian (3.4.15) becomes

$$\begin{aligned}\mathcal{L}^* &= \varphi_{\mu};\bar{\varphi}^{\mu};\ - ieA^\mu(\varphi_\mu;\bar{\varphi} - \varphi\bar{\varphi}_\mu;) - (m^2 - e^2 A_\mu A^\mu)\varphi\bar{\varphi} \\ &= (\varphi_\mu;\ + ieA_\mu\varphi)(\bar{\varphi}^\mu;\ - ieA^\mu\bar{\varphi}) - m^2\varphi\bar{\varphi},\end{aligned}$$

and represents one of the central models of contemporary gauge-invariant field theories, namely, that of the interaction of a complex scalar field with an external electromagnetic field. As a matter of fact, Lagrangian (3.4.15) was originally derived precisely through a Newtonian limit of the above field theoretical Lagrangian (Feshbach, private communication 1976). The variational behavior of the field theoretical case closely follows that of the corresponding Newtonian case [see Santilli (1977a), particularly Appendix C]. Besides its variational significance, this example has rather deep physical implications characterized by the fact that the Newtonian limit of the couplings appearing in the gauge-invariant Lagrangian of the complex scalar field in interaction with an electromagnetic field is of *nonconservative* type. The significance of the nonservative forces for our description of Newtonian systems as they actually exist in our environment is stressed in the Appendix. The above example indicates that such nonconservative forces also have a physical role within the context of elementary particle interactions. Indeed, they occur already at the level of the electromagnetic interactions of charged particles. The current unified gauge theory of weak and electromagnetic interactions preserves such a "nonconservative" character of the couplings. Therefore, "nonconservative couplings" also occur within the context of weak interactions. It is then conceivable to suppose that the same type of "nonconservative couplings" also occur for strong interactions. In this latter case, however, it is equally conceivable to suppose that the ultimate characterization of the strong interactions will demand a further generalization of the nonconservative character of the couplings. The above remarks put the analysis in this monograph in a different perspective. Indeed, the Newtonian analysis of the nonconservative forces presented in this monograph appears to be an advisable step prior to any attempt at a generalization of the couplings of the electromagnetic and weak interactions along these "nonconservative" lines. The hope to reach some insight along these lines for the problem of the strong interactions was one of my primary motivations for undertaking this task. For studies along these lines, see Santilli (1978, Vols. I, II, and III).

existence of an analytic representation even though the system is non-self-adjoint. This clearly affects the *necessity* of the conditions of self-adjointness for the existence of a Lagrangian.

The above remark illustrates the significance of the ordering in the concept of analytic representations. From now on, unless explicitly stated, all our direct or indirect analytic representations will be assumed to satisfy the ordering condition.

The subsequent Sections 3.5, 3.6, and 3.7 will be devoted to the study of the fundamental type of analytic representations, namely, the ordered *direct* analytic representations. [The indirect representations are studied in Santilli (1979)].

As indicated earlier, system (3.4.13) is a simple nonconservative generalization of system (3.4.3). The Morse–Feshbach Lagrangian (3.4.15) appears to be a generalization of Lagrangian (3.4.6). A significant question is whether a generalization of Lagrangian (3.4.4) for the representation of system (3.4.13) also exists.

An inspection of the problem[22] indicates that such a Lagrangian does indeed exist and is given by[23]

$$L = e^{bt}\tfrac{1}{2}(\dot{q}_1^2 - \omega^2 q_1^2) + e^{-bt}\tfrac{1}{2}(\dot{q}_2^2 - \omega^2 q_2^2) \tag{3.4.22}$$

[22] For a derivation of this Lagrangian, see Santilli (1979).

[23] A digression to field theory is also significant here. By performing the same transitions as in the previous cases, the Lagrangian density which corresponds to Equation (3.4.22) is given by

$$\begin{aligned}\mathscr{L} = {} & e^{-2ie\times^{\alpha}A_{\alpha}}\tfrac{1}{2}[\varphi_{,\mu};\varphi^{,\mu};-(m^2 - e^2 A_\mu A^\mu)\varphi^2] \\ & + e^{+2ie\times^{\alpha}A_{\alpha}}\tfrac{1}{2}[\bar{\varphi}_{,\mu};\bar{\varphi}^{,\mu};-(m^2 - e^2 A_\mu A^\mu)\bar{\varphi}^2].\end{aligned}$$

It also represents the complex scalar field in interaction with an external electromagnetic field, with the only difference that now the factor functions, rather than characterizing a permutation, are nontrivial. By comparing the above Lagrangian density with the conventional form of gauge theories, the reader can identify the following rather puzzling breakings.

(1) The so-called minimal coupling rule of the electromagnetic interactions, i.e., the substitutions

$$\varphi^i_{,\mu} \to \varphi^i_{,\mu};\ -ieA_\mu \quad \text{and} \quad \bar{\varphi}^i_{,\mu};\to \bar{\varphi}^i_{,\mu};\ +ieA_\mu$$

no longer hold for the Lagrangian $\mathscr{L}$.

(2) The invariance of $\mathscr{L}$ under the gauge transformations

$$\varphi \to \varphi' = e^{iew}\varphi, \qquad \bar{\varphi} \to \bar{\varphi}' \to e^{iew}\bar{\varphi}, \qquad w \in O_c,$$

which is at the basis of the customary derivation of the charge conservation law, no longer holds for $\mathscr{L}$.

(3) The invariance of $\mathscr{L}$ under translations in space–time

$$x^\mu \to x'^\mu = x^\mu + a^\mu,$$

which is at the basis of the customary derivation of the energy–momentum conservation law, no longer holds for $\mathscr{L}$.

The existence of this new Lagrangian density with the above-indicated underlying breakings is, in the final analysis, a consequence of our definition of ordered indirect analytic representations because of the freedom in the explicit form of the factor functions. The existence of two different Lagrangian densities for the representation of the same system indicates the existence of degrees of freedom, which become intimately linked to the methodology that underlies the problem of the existence of a Lagrangian. For an analysis of these and other aspects see Santilli (1978, Vols. I, II, and III).

with the underlying ordered, indirect analytic representation

$$\begin{pmatrix} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1^2} - \frac{\partial L}{\partial q_1^2} \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^2} - \frac{\partial L}{\partial q^2} \end{pmatrix}_{SA} = \left[ \begin{pmatrix} e^{bt} & 0 \\ 0 & e^{-bt} \end{pmatrix} \begin{pmatrix} \ddot{q}_1 + b\dot{q}_1 + \omega^2 q_1 \\ \ddot{q}_2 - b\dot{q}_2 + \omega^2 q_2 \end{pmatrix}_{NSA} \right]_{SA} = 0 \quad (3.4.23)$$

and factor functions

$$(h_k^i) = \begin{pmatrix} e^{bt} & 0 \\ 0 & e^{-bt} \end{pmatrix}, \qquad |h_k^i| = 1. \quad (3.4.24)$$

A few comments are now in order. Within a Newtonian context, Lagrangian (3.4.22) could be interpreted as more "natural" than Lagrangian (3.4.15) for the representation of the system considered,[24] because its explicit dependence on time directly indicates the underlying nonconservative nature of the system.[25] This aspect is somewhat hidden in Lagrangian (3.4.15) because of its lack of explicit time dependence. Indeed, the attentive reader, after an initial inspection of Lagrangian (3.4.15) *only*, might arrive at the erroneous conclusion that, in view of its invariance under time translations, the total mechanical energy is conserved.[26] An inspection of the equations of motion, however, indicates that this is not the case, owing to the presence of the nonconservative force $F = (-b\dot{q}_1, +b\dot{q}_2)$.

On practical grounds, the selection of Lagrangian (3.4.15) or (3.4.22) is a question of personal preference, because both Lagrangians lead to fully admissible analytic representations of the *same* system, with the only difference given by the explicit form and functional dependence of the factor functions.

On methodological grounds, what is significant is the existence of the *different* Lagrangians (3.4.15) and (3.4.22) for the representation of the *same* system. Indeed, this indicates a possible significance of the concept of analytic representations introduced in this section within the context of transformation theory. This aspect is investigated in Santilli (1979).

[24] This is not necessarily the case for the corresponding situation in field theory.

[25] For an elementary approach to the problem of symmetries and conservation laws, the interested reader can consult Chart A.2. For details, see Santilli (1979).

[26] It should be indicated here that the quantity related to Lagrangian (3.4.15),

$$I = \frac{\partial L^*}{\partial \dot{q}^k}\dot{q}^k - L^*,$$

is indeed conserved owing to the invariance of the Lagrangian under time translations. However, the above quantity *does not* represent the total mechanical energy [see Santilli (1979) for more details].

Another aspect of the concept of analytic representations is that it indicates the existence of a generalized form of Lagrange's equations, as is exhibited by the left-hand side of Equations (3.4.2). [This aspect is also investigated in Santilli (1979)].

Finally, the reader should be aware that the definition of analytic representation (3.4.1) implies that the actual path, the implicit functions, and the conserved quantities of the equations of motion coincide with those of their Lagrangian representation.

## 3.5 The Fundamental Analytic Theorem for Configuration Space Formulations

The fact that Lagrange's equations in class $\mathscr{C}^4$ and regular Lagrangians are self-adjoint (Theorem 3.3.1) constitutes a property of central methodological significance with implications at several levels of analytic mechanics. In this section, we shall study the implications of this property for the existence of an ordered direct analytic representation of Newtonian systems. In order to formulate and prove our main theorem, we remain with the problem of identifying a suitable region of definition of the analytic representation under consideration.

Let us recall in this respect that both Newton's equations of motion in their fundamental form and Lagrange's equations can be defined in a region $R^{2n+1}$ of the variables $t, q^k$, and $\dot{q}^k$ only, where the dependence of these equations on the accelerations $\ddot{q}^k$ can be ignored owing to their linearity. Thus, the condition that a fundamental form be of (at least) class $\mathscr{C}^2$ can be reduced to the condition that the $A_{ki}(t, q, \dot{q})$ and $B_k(t, q, \dot{q})$ functions are of class $\mathscr{C}^2$ in $R^{2n+1}$ or, equivalently, that the Lagrangian $L(t, q, \dot{q})$ is of class $\mathscr{C}^4$ in $R^{2n+1}$.

As a consequence, the ordered direct analytic representations of class $\mathscr{C}^2$ and regular Newtonian systems can be defined, at least in principle, in an arbitrarily selected region $R^{2n+1}$. However, this position is insufficient for our intent of formulating and proving our main theorem within the context of the calculus of differential forms.

Let us recall from Section 1.2 that the Converse of the Poincaré Lemma (one of the most effective tools for studying the integrability conditions in general) demands the use of a star-shaped region $R^*$ rather than an ordinary region $R$, according to the formulation of Lemma 1.2.2. Therefore, we shall restrict the analytic representations to be defined in a *star-shaped region* $R^{*2n+1}$ of the variables $(t, q, \dot{q})$, namely, an open and connected set of points $(t, q, \dot{q})$ where all points $(t, \tau q, \tau\dot{q})$, $0 \leq \tau \leq 1$, are interior points. Notice that there is no restriction on the values of the time variable, and that such a star-shaped region contains the (local) origin $q^k = 0, \dot{q}^k = 0, k = 1, 2, \ldots, n$.

We are finally equipped to formulate and prove the following important theorem.

**Theorem 3.5.1**[27] (Fundamental Analytic Theorem for Configuration Space Formulations). *A necessary and sufficient condition for a local, holonomic, generally nonconservative Newtonian system in the fundamental form*

$$A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0, \qquad k = 1, 2, \ldots, n, \tag{3.5.1}$$

*which is well defined, of (at least) class $\mathscr{C}^2$, and regular in a star-shaped region $R^{*2n+1}$ of the variables $(t, q, \dot{q})$, to admit an ordered direct analytic representation in terms of the conventional Lagrange's equations in $R^{*2n+1}$,*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \equiv A_{ki}\ddot{q}^i + B_k, \tag{3.5.2}$$

*is that the system of equations of motion is self-adjoint in $R^{*2n+1}$.*

PROOF. Since the equations of motion are of (at least) class $\mathscr{C}^2$ and regular in $R^{*2n+1}$, the Lagrangian $L$ must be (at least) of class $\mathscr{C}^4$ and regular in $R^{*2n+1}$ (Theorem 3.2.1). Then Theorem 3.3.1 applies and Lagrange's equations are self-adjoint in $R^{*2n+1}$. This proves the *necessity* of the condition of self-adjointness of the equations of motion for the existence of the ordered identifications (3.5.2) in view of the self-adjointness of their left-hand sides.

To prove *sufficiency*, we shall show that, under the conditions of self-adjointness (2.2.9) of the equations of motion in $R^{*2n+1}$, i.e.,[28]

$$A_{ij} = A_{ji}, \tag{3.5.3a}$$

$$\frac{\partial A_{ik}}{\partial \dot{q}^j} = \frac{\partial A_{jk}}{\partial \dot{q}^i}, \tag{3.5.3b}$$

$$\frac{\partial B_i}{\partial \dot{q}^j} + \frac{\partial B_j}{\partial \dot{q}^i} = 2\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\} A_{ij}, \tag{3.5.3c}$$

$$\frac{\partial B_i}{\partial q^j} - \frac{\partial B_j}{\partial q^i} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\}\left(\frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i}\right), \tag{3.5.3d}$$

$$i, j, k = 1, 2, \ldots, n,$$

[27] A list of all the most relevant references on this theorem known to me has been given in the Introduction. The formulation and proof presented here follow the field theoretical version given by Theorem 2.1 of Santilli (1977b). The most significant difference between this formulation and those of the quoted references lies in the use of the Converse of the Poincaré Lemma with consequent restriction of the region of definition to a star-shaped region. For comments on this point, see Section 3.6 and Santilli (1979).

[28] This is the form of the conditions of self-adjointness which is customarily used for the proof of the theorem. See, for instance, Davis (1929).

their reformulation (2.3.25) within the context of the calculus of differential forms apply, i.e.,[29]

$$\delta^{i_1 i_2}_{k_1 k_2} A_{i_1 i_2} = 0, \tag{3.5.4a}$$

$$\delta^{i_1 i_2 i_3}_{k_1 k_2 k_3} \frac{\partial A_{i_1 i_2}}{\partial \dot{q}^{i_3}} = 0, \tag{3.5.4b}$$

$$\delta^{i_1 i_2 i_3}_{k_1 k_2 k_3} \frac{\partial B_{i_1}}{\partial q^{i_2} \partial \dot{q}^{i_3}} = 0, \tag{3.5.4c}$$

$$\frac{1}{2} \frac{\partial}{\partial \dot{q}^{k_3}} \left( \frac{\partial B_{k_1}}{\partial \dot{q}^{k_2}} - \frac{\partial B_{k_2}}{\partial \dot{q}^{k_1}} \right) - \left( \frac{\partial A_{k_2 k_3}}{\partial q^{k_1}} - \frac{\partial A_{k_1 k_3}}{\partial q^{k_2}} \right) = 0, \tag{3.5.4d}$$

$$k_1, k_2, k_3 = 1, 2, \ldots, n,$$

and a Lagrangian $L(t, q, \dot{q})$ for ordered identifications (3.5.2) *always* exists.

Let us recall that the most general structure of the Lagrangian for the representation of regular systems is given by Equation (2.3.3), i.e.,

$$L(t, q, \dot{q}) = K(t, q, \dot{q}) + D_k(t, q)\dot{q}^k + C(t, q), \tag{3.5.5}$$

where the "kinetic term" $K$ is nonlinear in $\dot{q}^k$ and all terms $K$, $D_k$ and $C$ are of (at least) class $\mathscr{C}^4$ in a (star-shaped) region of their variables.

By substituting structure (3.5.5) in identifications (3.5.2) the problem of the existence of a Lagrangian can be reduced to the study of the set of the generally overdetermined system of partial differential equations (2.3.4a), (2.3.5), and (2.3.6) which, under the conditions of self-adjointness (3.5.3), reduce to system (2.3.7), i.e.,

$$\frac{\partial^2 K}{\partial \dot{q}^i \partial \dot{q}^j} = A_{ij}, \tag{3.5.6a}$$

$$\frac{\partial D_i}{\partial q^j} - \frac{\partial D_j}{\partial q^i} = \frac{1}{2} \left( \frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i} \right) + \left( \frac{\partial^2 K}{\partial q^i \partial \dot{q}^j} - \frac{\partial^2 K}{\partial \dot{q}^i \partial q^j} \right) \equiv Z_{ij}, \tag{3.5.6b}$$

$$\frac{\partial C}{\partial q^k} = \frac{\partial D_k}{\partial t} - B_k - \frac{\partial K}{\partial q^k} + \frac{\partial^2 K}{\partial \dot{q}^k \partial t} + \left[ \frac{\partial^2 K}{\partial q^k \partial \dot{q}^i} + \frac{1}{2} \left( \frac{\partial B_k}{\partial \dot{q}^i} - \frac{\partial B_i}{\partial \dot{q}^k} \right) \right] \dot{q}^i \tag{3.5.6c}$$

$$\equiv W_k.$$

Our proof of sufficiency consists of showing that *conditions of self-adjointness* (3.5.3) *are the integrability conditions of system* (3.5.6).

1. *Integrability conditions of Equations* (3.5.6a)

Introduce the quantities

$$T_{k_1} = \frac{\partial K}{\partial \dot{q}^{k_1}}, \tag{3.5.7}$$

[29] This is the alternative form of the conditions of self-adjointness which I have used in the proof of the theorem within a corresponding field theoretical context [Santilli (1977,b)]. It should be recalled here that Equations (3.5.3) imply Equations (3.5.4) but the inverse property does not necessarily hold. This has no bearing on our proof because the conditions of self-adjointness, in their implied form (3.5.4), will be used for the proof of sufficiency *only*.

and consider the system of first-order partial differential equations

$$\frac{\partial T_{k_1}}{\partial \dot{q}^{k_2}} - A_{k_1 k_2} = 0, \qquad k_1, k_2 = 1, 2, \ldots, n \tag{3.5.8}$$

with underlying 1-form

$$A_{k_1}^{(1)} = A_{k_1 k_2}\, d\dot{q}^{k_2}. \tag{3.5.9}$$

From the Converse of the Poincaré Lemma, reformulation (3.5.4b) of conditions of self-adjointness (3.5.3b) are the integrability conditions for Equations (3.5.8). Thus, under the assumptions of the theorem, a solution of Equations (3.5.8) always exists and is given, from Equations (1.2.30) and (1.2.25), by

$$T_{k_1} = \left[\int_0^1 d\tau\, A_{k_1 k_2}(t, q, \tau\dot{q})\right]\dot{q}^{k_2}. \tag{3.5.10}$$

The additional condition of self-adjointness (3.5.3a) or (3.5.4a) then ensures the proper symmetrization of this solution, i.e., the joint validity of Equations (3.5.8) with the additional system

$$\frac{\partial T_{k_2}}{\partial \dot{q}^{k_1}} - A_{k_1 k_2} = 0. \tag{3.5.11}$$

The next step is to consider Equations (3.5.7), i.e.,

$$\frac{\partial K}{\partial \dot{q}^{k_1}} - T_{k_1} = \frac{\partial K}{\partial \dot{q}^{k_1}} - \left[\int_0^1 d\tau\, A_{k_1 k_2}(t, q, \tau\dot{q})\right]\dot{q}^{k_2}, \tag{3.5.12}$$

with underlying 1-form

$$T^{(1)} = T_{k_1}\, dq^{k_1}. \tag{3.5.13}$$

The integrability conditions in this case are[30]

$$\begin{aligned}\delta^{i_1 i_2}_{k_1 k_2}\frac{\partial T_{i_1}}{\partial \dot{q}^{i_2}} &= \int_0^1 d\tau\, \delta^{i_1 i_2}_{k_1 k_2} A_{i_1 i_2}(t, q, \tau\dot{q}) + \left[\int_0^1 d\tau\, \tau\delta^{i_1 i_2 i_3}_{k_1 k_2 k_3}\frac{\partial A_{i_1 i_2}}{\partial \dot{q}^{i_3}}(t, q, \tau\dot{q})\right]\dot{q}^{k_3} \\ &= 0\end{aligned} \tag{3.5.14}$$

and they identically hold in view, again, of conditions (3.5.4a) and (3.5.4b). Therefore, under the assumptions of the theorem, a solution of Equations (3.5.6a) always exists and, again from Equations (1.2.25), is given by

$$K(t, q, \dot{q}) = \dot{q}^{k_1}\int_0^1 d\tau'\left\{\left[\int_0^1 d\tau\, A_{k_1 k_2}(t, q, \tau\dot{q})\right]\dot{q}^{k_2}\right\}(t, q, \tau'\dot{q}), \tag{3.5.15}$$

where the curly brackets indicate that the function of $\dot{q}^k$ resulting after integration with respect to $\tau$ must be computed along $\tau'\dot{q}^k$ prior to the integration with respect to $\tau'$.

This completes the first part of our proof of sufficiency.

[30] For the last terms of Equations (3.5.14), see Problem 1.8.

2. *Integrability conditions of Equations* (3.5.6b) *and* (3.5.6c)

We consider now, independently from each other, Equations (3.5.6b) and (3.5.6c), i.e.,

$$\frac{\partial D_{k_1}}{\partial q^{k_2}} - \frac{\partial D_{k_2}}{\partial q^{k_1}} = Z_{k_1 k_2}, \tag{3.5.16a}$$

$$\frac{\partial C}{\partial q^k} = W_k. \tag{3.5.16b}$$

The underlying differential forms are, respectively,[31]

$$Z^{(2)} = Z_{k_1 k_2}\, dq^{k_1} \wedge dq^{k_2}, \tag{3.5.17a}$$

$$W^{(1)} = W_k dq^k, \tag{3.5.17b}$$

with related integrability conditions

$$\delta^{i_1 i_2 i_3}_{k_1 k_2 k_3} \frac{\partial Z_{i_1 i_2}}{\partial q^{i_3}} = 0, \tag{3.5.18a}$$

$$\delta^{i_1 i_2}_{k_1 k_2} \frac{\partial W_{i_1}}{\partial q^{i_2}} = 0. \tag{3.5.18b}$$

As indicated in Section 2.3, upon substituting the explicit values of the $Z$ and $W$ terms, the above conditions reduce to

$$\delta^{i_1 i_2 i_3}_{k_1 k_2 k_3} \frac{\partial B_{i_1}}{\partial q^{i_2}\, \partial \dot{q}^{i_3}} = 0, \tag{3.5.19a}$$

$$\frac{1}{2}\left(\delta^{i_1 i_2 i_3}_{k_1 k_2 k_3} \frac{\partial B_{i_1}}{\partial q^{i_2}\, \partial \dot{q}^{i_3}}\right) \dot{q}^{k_2} = 0, \tag{3.5.19b}$$

and they identically hold under conditions of self-adjointness (3.5.3) in view of their reformulation (3.4.4c).

Therefore, under the assumption of the theorem, independent and sequential[32] solutions of Equations (3.5.6b) and (3.5.6c) exist and, also from Equations (1.2.25), are given by

$$D_{k_1}(t, q) = \left[\int_0^1 d\tau\, \tau Z_{k_1 k_2}(t, \tau q)\right] q^{k_2}, \tag{3.5.20a}$$

$$C(t, q) = \left[\int_0^1 d\tau\, W_k(t, \tau q)\right] q^k. \tag{3.5.20b}$$

This completes the second part of our proof of sufficiency.

3. *Compatibility of Equations* (3.5.6)

To complete our proof, we must first show that, for consistency, the right-hand sides of Equations (3.5.6b) and (3.5.6c) are independent of derivative terms. By differentiating

[31] In line with Example 1.5, the form which underlies Equations (3.5.16a) is now a 2-form due to the antisymmetric nature of the indices.

[32] During this second stage of our proof, Equations (3.5.6b) and (3.5.6c) are considered independently from each other. Therefore, Equations (3.5.6b) must be solved first because the $D$ functions appear on the right-hand sides of Equations (3.5.6c) and, as such, are assumed to be known.

these equations with respect to $\dot{q}^{k_3}$, we obtain the respective conditions

$$\frac{1}{2}\frac{\partial}{\partial \dot{q}^{k_3}}\left(\frac{\partial B_{k_1}}{\partial \dot{q}^{k_2}} - \frac{\partial B_{k_2}}{\partial \dot{q}^{k_1}}\right) - \left(\frac{\partial A_{k_1 k_3}}{\partial q^{k_2}} - \frac{\partial A_{k_2 k_3}}{\partial q^{k_1}}\right) = 0, \tag{3.5.21a}$$

$$\left[\frac{1}{2}\frac{\partial}{\partial \dot{q}^{k_3}}\left(\frac{\partial B_{k_1}}{\partial \dot{q}^{k_2}} - \frac{\partial B_{k_2}}{\partial \dot{q}^{k_1}}\right) - \left(\frac{\partial A_{k_1 k_3}}{\partial q^{k_2}} - \frac{\partial A_{k_2 k_3}}{\partial q^{k_1}}\right)\right]\dot{q}^{k_2} = 0, \tag{3.5.21b}$$

which identically hold under conditions of self-adjointness (3.5.3) in view of their redefinition (3.5.4d).

This completes the first part of our proof of compatibility and shows that Equations (3.5.4d) guarantee that the right-hand sides of Equations (3.5.6b) and (3.5.6c) are independent of derivative terms. Our proof of the theorem will be completed by showing that Equations (3.5.6) are compatible among themselves.

Since Equations (3.5.6a) must be solved first, the proof of compatibility can be reduced to the proof that Equations (3.5.6b) and (3.5.6c), under identifications (3.5.6a), are compatible among themselves. Let us rewrite these equations in the form

$$\frac{\partial D_{k_1}}{\partial q^{k_2}} - \frac{\partial D_{k_2}}{\partial q^{k_1}} = Z_{k_1 k_2}, \tag{3.5.22a}$$

$$\frac{\partial D_{k_1}}{\partial t} - \frac{\partial C}{\partial q^{k_1}} = W'_{k_1}, \tag{3.5.22b}$$

where

$$W'_{k_1} = \frac{\partial D_{k_1}}{\partial t} - W_{k_1}. \tag{3.5.23}$$

After partial differentiation with respect to $t$ and $q^{k_2}$, we can write

$$\frac{\partial^2 D_{k_1}}{\partial q^{k_2}\,\partial t} = \frac{\partial Z_{k_1 k_2}}{\partial t} + \frac{\partial^2 D_{k_2}}{\partial q^{k_1}\,\partial t}, \tag{3.5.24a}$$

$$\frac{\partial^2 D_{k_1}}{\partial q^{k_2}\,\partial t} = \frac{\partial W'_{k_1}}{\partial q^{k_2}} + \frac{\partial C_{k_1}}{\partial q^{k_2}}. \tag{3.5.24b}$$

Therefore, the *necessary* conditions for the compatibility of Equations (3.5.22) are

$$\frac{\partial Z_{k_1 k_2}}{\partial t} = \frac{\partial W'_{k_1}}{\partial q^{k_2}} - \frac{\partial W'_{k_2}}{\partial q^{k_1}}, \tag{3.5.25}$$

where we have used Equations (3.5.22b). To prove that Equations (3.5.25) are also *sufficient*, consider Equations (3.5.22) for fixed values of the indices $k_1 = k_1^\circ$ and $k_2 = k_2^\circ (\neq k_1^\circ)$. Then, in view of the continuity properties of the $Z$ and $W'$ functions, the existence theorem for linear partial differential equations[33] applies and a solution $D_{k_1^\circ}$, $D_{k_2^\circ}$, and $C$ exists. We now substitute such a solution into Equations (3.5.22) according to

$$\frac{\partial D_{k_2}}{\partial q^{k_1^\circ}} = \frac{\partial D_{k_1^\circ}}{\partial q^{k_2}} - Z_{k_1^\circ k_2}, \tag{3.5.26a}$$

$$\frac{\partial D_{k_2}}{\partial t} = W'_{k_2} + \frac{\partial C}{\partial q^{k_2}}. \tag{3.5.26b}$$

[33] See, for instance, Forsyth (1906, Vol. V, Articles 160–161).

These equations are compatible provided that

$$\frac{\partial^2 D_{k_1^\circ}}{\partial q^{k_2} \partial t} - \frac{\partial Z_{k_1^\circ k_2}}{\partial t} = \frac{\partial W'_{k_2}}{\partial q^{k_1^\circ}} + \frac{\partial^2 C}{\partial q^{k_2} \partial q^{k_1^\circ}}. \tag{3.5.27}$$

But the above conditions reduce to Equations (3.5.25) after use of Equations (3.5.22b). Thus, Equations (3.5.25) are the necessary and sufficient conditions for the compatibility of Equations (3.5.22).

We must now inspect Equations (3.5.25). By substituting the explicit values of the $Z$ and $W'$ functions from Equations (3.5.6) and (3.5.23), Equations (3.5.25) can be written

$$\left(\frac{\partial B_{k_1}}{\partial q^{k_2}} - \frac{\partial B_{k_2}}{\partial q^{k_1}}\right) - \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial B_{k_1}}{\partial \dot{q}^{k_2}} - \frac{\partial B_{k_2}}{\partial \dot{q}^{k_1}}\right) + \left[\frac{\partial}{\partial q^{k_1}}\left(\frac{\partial B_{k_2}}{\partial \dot{q}^{k_3}} - \frac{\partial B_{k_3}}{\partial \dot{q}^{k_2}}\right) + \frac{\partial}{\partial q^{k_2}}\left(\frac{\partial B_{k_3}}{\partial \dot{q}^{k_1}} - \frac{\partial B_{k_1}}{\partial \dot{q}^{k_3}}\right)\right]\dot{q}^{k_3} = 0. \tag{3.5.28}$$

In view of condition of self-adjointness (3.5.3d), they become

$$\left(\delta^{i_1 i_2 i_3}_{k_1 k_2 k_3} \frac{\partial^2 B_{i_1}}{\partial q^{i_2} \partial \dot{q}^{i_3}}\right)\dot{q}^{k_3} = 0, \tag{3.5.29}$$

and are identically verified in view of Equations (3.5.4c).

This completes the third part of our proof and shows that the Equations (3.5.4c) are not only the integrability conditions for Equations (3.5.6b) and (3.5.6c), but are also the necessary and sufficient conditions for their compatibility.[34]

[34] Our proof of compatibility of Equations (3.5.22) closely follows that of Davis (1929). It might be of some relevance to point out that Davis' crucial Theorem 2, p. 377, could be controversial in its formulation and proof. This theorem states the following.

*A necessary and sufficient condition that there exists a solution of a system of differential equations of the form* (3.5.22) *where* $C$, $Z_{k_1 k_2}$, $D_{k_1}$, $W'_{k_1}(k_1, k_2 = 1, 2, \ldots, n)$ *are functions of* $t, q^1, \ldots, q^n$, *and* $Z_{k_1 k_2} = -Z_{k_2 k_1}$, *is that Equations* (3.5.25) *hold identically in* $t, q^1, \ldots, q^{k_n}$ *for every pair of values of* $k_1$ *and* $k_2$.

The proof of the theorem then proceeds along the lines of part 3, from Equations (3.5.22) to (3.5.28), namely, of what we have called compatibility of Equations (3.5.22). The point is that Equation (3.5.25) is indeed the necessary and sufficient condition for the compatibility of Equations (3.5.22), but this aspect alone is insufficient to guarantee the existence of a solution of the same equations owing to the need for the additional closure conditions of the underlying differential forms, i.e., Equations (3.5.18). In other words, the proof of the consistency of Equations (3.5.22) under the conditions of self-adjointness demands the verification of

(a) the integrability conditions of Equation (3.5.22a), i.e., (3.5.18a);
(b) the integrability conditions of Equation (3.5.22b), i.e., (3.5.18b); and
(c) the compatibility conditions of Equations (3.5.22a) and (3.5.22b), i.e., Equation (3.5.25).

This more accurate proof was formulated, in Santilli (1977b) and indicates the effectiveness of the calculus of differential forms for the study of the integrability conditions (Davis' approach in his paper of 1929 was based on the conventional—for that time—methods of the theory of partial differential equations). Rather unpredictably, and this indicates the peculiarity of the problem of the existence of a Lagrangian, a more detailed analysis indicates that the integrability and compatibility conditions of Equations (3.5.22) are *all* ultimately equivalent to Equation (3.5.25), as clearly exhibited by their explicit forms (3.5.19) and (3.5.29). In turn, this indicates that the proof of compatibility of Equations (3.5.22) could also be considered as redundant (e.g., from the need of computing the solutions of Equations (3.5.22a) and (3.5.22b) in a sequential way).

Thus, when all the conditions of variational self-adjointness (3.5.3) are identically verified for the equations of motion, their reformulation (3.5.4) holds, Equations (3.5.6) always admit a solution, and a Lagrangian according to structure (3.5.5) always exists. *Q.E.D.*

Notice that *all* the conditions of self-adjointness (3.5.3) enter, without redundancy, into the proof of the theorem. Therefore, when the equations of motion in configuration space are non-self-adjoint, they cannot be directly represented by the conventional Lagrange's equations. It should be recalled, in this respect, that a system is non-self-adjoint when at least *one* of the conditions of self-adjointness is violated.

The reader should be aware of the fact that when the conditions of Theorem 3.5.1 are not met this does not necessarily imply that an analytic representation of the equations of motion does not exist, because this theorem deals specifically with *direct* analytic representations. Indeed, when the equations of motion are non-self-adjoint, one can seek an *indirect* analytic representation. [This aspect is considered in Santilli (1979)].

The significance of the ordering condition in Theorem 3.5.1 is now self-evident. If the ordering condition for identifications (3.5.2) is relaxed, the conditions of self-adjointness are only sufficient for the existence of a Lagrangian.[35] This point is clearly exhibited by example (3.4.20), where a permutation of the order of the equations of motion render the system non-self-adjoint. Nevertheless, a Lagrangian for its (nonordered) representation still exists.

Theorem 3.5.1 can be easily generalized to the second-order differential equations

$$F_k(t, q, \dot{q}, \ddot{q}) = 0, \qquad k = 1, 2, \ldots, n, \tag{3.5.30a}$$

$$F_k \in \mathscr{C}^2(R^{3n+1}), \qquad \left|\frac{\partial F_i}{\partial \ddot{q}^j}\right|(R^{3n+1}) \neq 0, \tag{3.5.30b}$$

which are not necessarily linear in the accelerations. This was, ultimately, the original problem considered by Helmholtz (1887). Indeed, conditions of self-adjointness (2.1.18) demand the linearity of Equations (3.5.30) in the accelerations, as shown in Section 2.2. Therefore, Theorem 3.5.1 can be equivalently formulated and proved either in terms of the Newtonian form (3.5.1) with underlying conditions of self-adjointness (3.5.3) or in terms of the more general form (3.5.30) with underlying conditions of self-adjointness (2.1.18).

Theorem 3.5.1 can also be applied to the particular case of the kinematical form of the equations of motion, i.e., Equations (2.2.2), for which

$$A_{ki} = \delta_{ki}, \qquad B_k = -f_k \tag{3.5.31}$$

This is left as an exercise for the interested reader (see Problem 3.2).

[35] The ordering condition within the context of the Inverse Problem was introduced in Santilli (1977a and b).

Another significant application of Theorem 3.5.1 is for the analytic representation of *first-order* equations, in which case the Lagrangian, according to our terminology, is *totally degenerate*, i.e.,

$$\frac{\partial^2 L}{\partial \dot{q}^i \, \partial \dot{q}^j} \equiv 0, \qquad i, j = 1, 2, \ldots, n. \tag{3.5.32}$$

This case is worked out in Chart 3.9.

The case of a bona fide *degenerate* Lagrangian, i.e.,

$$\left| \frac{\partial^2 L}{\partial \dot{q}^i \, \partial \dot{q}^j} \right| (R^{2n+1}) = 0, \qquad \frac{\partial^2 L}{\partial \dot{q}^i \, \partial \dot{q}^j} \neq 0, \tag{3.5.33}$$

will not, however, be considered in this volume. This is because these Lagrangians imply the presence of subsidiary *constraints* which, in turn, demand a careful reinspection of the variational approach to self-adjointness.[36]

A somewhat special subcase of Theorem 3.5.1 is the one-dimensional case for which identifications (3.5.2) become[37]

$$\frac{\partial^2 L}{\partial \dot{q}^2} \ddot{q} + \frac{\partial^2 L}{\partial \dot{q} \, \partial q} \dot{q} + \frac{\partial^2 L}{\partial \dot{q} \, \partial t} - \frac{\partial L}{\partial q} \equiv A(t, q, \dot{q}) \ddot{q} + B(t, q, \dot{q}). \tag{3.5.34}$$

This was, ultimately, the problem considered and solved by Darboux (1891). There is, however, a potentially misleading aspect of problem (3.5.34) which

[36] The condition of regularity is introduced in the Inverse Problem in Santilli (1977a and b). In the early references on the Inverse Problem listed in the Foreword, the only condition which is (sometimes) considered is that of continuity. My attitude is mainly precautionary in nature, and it is due to the need to specifically inspect the problem. A few remarks are in order. First of all, when the Lagrangian is degenerate, one of the central parts of the Inverse Problem, the extremal part, becomes vacuous. This is because one of the necessary conditions for an extremum, i.e., the *Legendre condition*, is violated by degenerate Lagrangians [cf., Bliss (1946, p. 23) or Rund (1966, p. 358)].

The second point is that, when the Lagrangian is degenerate, Lagrange's equations can be inconsistent (i.e., a solution does not necessarily exist) despite the required continuity condition [cf. Dirac (1964)].

The third point is that, while for regular Lagrangians the consistency of Lagrange's equations implies the consistency of their equations of variations, the corresponding case for degenerate Lagrangians is quite delicate to handle. Therefore, in line with Section 1.1, a first step which seems to be advisable prior to the extension of Theorem 3.5.1 to degenerate Lagrangians is the study of the conditions under which degenerate Lagrange's equations are consistent, to avoid the handling of systems which are ultimately vacuous on physical grounds. Oddly, to the best of my knowledge, this aspect is virtually ignored in the recent literature on degenerate systems (both physical and mathematical), with the only exception known to me being the paper by Shanmugadhasan (1973). A second advisable step is the study of the equations of variations and adjoint systems of *consistent* degenerate systems (and related aspects, e.g., the uniqueness of the adjoint). A third advisable step is the reinspection of the calculus of differential forms and the Converse of the Poincaré Lemma in particular, for the case of degenerate systems. My preliminary unpublished investigations indicate that, with a careful handling of these and other problems, the extension of Theorem 3.5.1 to class $\mathscr{C}^2$ degenerate, consistent Lagrange's equations does indeed hold.

[37] Notice that the concept of ordering is inessential for one-dimensional systems.

deserves a comment. Since identity (3.5.34) constitutes *one* linear partial differential equation in *one* unknown, the Lagrangian $L$, one might be tempted to conclude that, in view of the assumed continuity and regularity conditions, a solution always exists. This is so if and only if the right-hand side of identity (3.5.34) is self-adjoint. Indeed, two equations can be identical if and only if they both are either self-adjoint or non-self-adjoint.[38] This is the property which is ultimately at the basis of the *necessity* of the conditions of self-adjointness in Theorem 3.5.1.

If the right-hand side of identity (3.5.34) is non-self-adjoint, then such a *direct* representation is inconsistent, but one can search for an *indirect* representation of the type [see Santilli (1979)]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \equiv [h(t, q, \dot{q})(A\ddot{q} + B)_{\text{NSA}}]_{\text{SA}} \tag{3.5.35}$$

with a factor function $h$ as an *integrating factor*. It is within this broader context that Darboux's result on the "universality" of the existence of a Lagrangian for one-dimensional systems (under the assumed continuity and regularity conditions) can be formulated in the context of our approach.[39]

Another peculiarity of the one-dimensional case is due to the fact that, since system (3.5.34) is determined, it can be treated with standard existence theorems of the theory of partial differential equations without recourse to the conditions of self-adjointness. As indicated in the Introduction, this was precisely Darboux's approach to the problem.[40] Indeed, the conditions of self-adjointness become crucial whenever $n > 1$ because, in this case, the underlying system of partial differential equations for the existence of a Lagrangian is overdetermined. The point we would like to bring to the reader's attention is that, even though problem (3.5.34) can be solved with conventional techniques, the use of the conditions of self-adjointness is advisable to formulate it properly.

To close this section, we would like to indicate that the methodology which underlies the formulation and proof of Theorem 3.5.1 is ultimately *variational* in nature. There are several reasons for this. First of all, Lagrange's equations, as a variational algorithm, are the Euler equations of the *action functional* (Section 1.3). This is, ultimately, *Hamilton's Principle* (1.3.23). Thus, Lagrange's equations originate within the context of first-order variations of the action functional. This framework, per se, is insufficient for the proof of the theorem. Indeed, the conditions of self-adjointness can be derived within a variational context only by using *second-order* variations. This is implicit in the use of the

[38] Rather unpredictably, we can see in Santilli (1979) that this crucial property fails to hold if the two equations are considered in different coordinate systems.

[39] For more details, see Santilli (1979).

[40] It might be of some significance here to indicate that Darboux (1891) does not mention the prior publication by Helmholtz (1887).

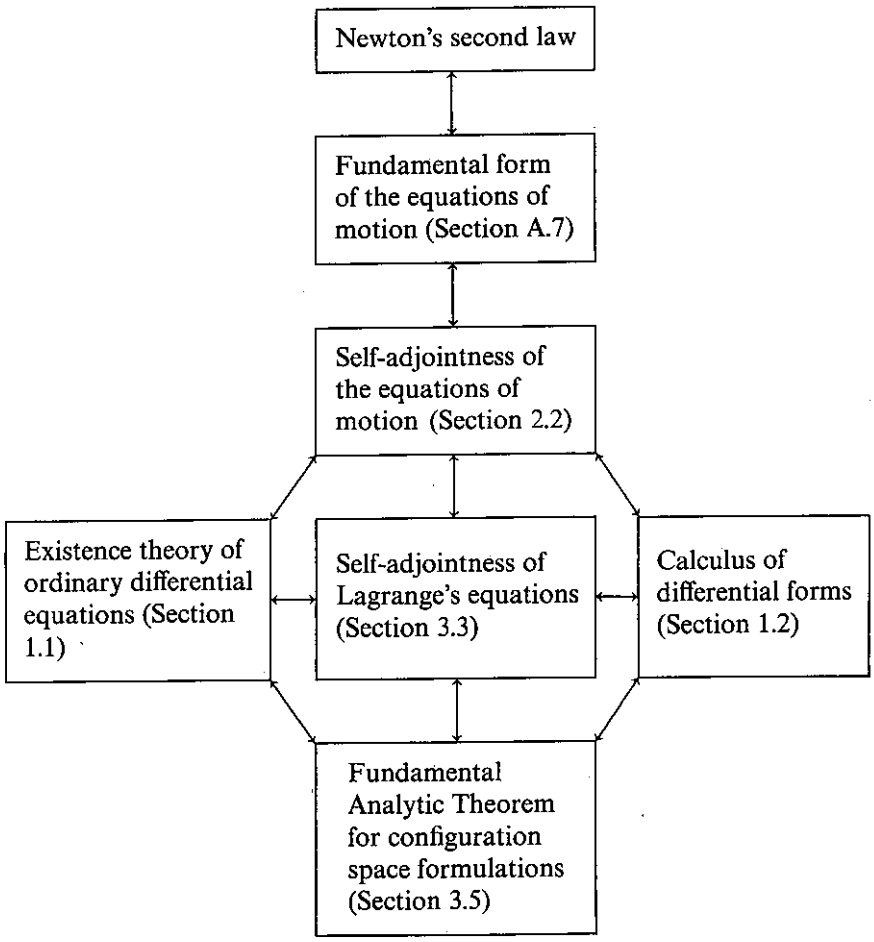



**Figure 3.1** A schematic view of the *Fundamental Analytic Theorem for configuration space formulations*. The equations of motion in their fundamental form are constructed from Newton's second law. Whenever they satisfy each of the conditions of self-adjointness, they are self-adjoint. Lagrange's equations in class $\mathscr{C}^4$ and regular Lagrangians, on the contrary, are always self-adjoint. This establishes both the necessity and the sufficiency of the conditions of self-adjointness for the existence of a Lagrangian according to the formulation and proof of Theorem 3.5.1. As a result, the conditions of self-adjointness emerge as a central mathematical tool of the Inverse Problem. The existence theory for ordinary differential equations is a prerequisite to the approach because of the need to ensure the consistency of the considered system. The calculus of differential forms is a complementary aspect to the approach in view of the fact that it constitutes one of the best arenas for the study of the integrability conditions in general, and of the existence of a Lagrangian in particular. The net result is a methodological perspective which, in view of the elemental nature of the problem of the existence of a Lagrangian, has implications at several levels of the theory considerably beyond the original objective of identifying a Lagrangian, as indicated in Santilli (1979).

equations of variations of Lagrange's equations, i.e., Jacobi's equations (3.3.7), which are of second-order variational nature, as recalled in Section 1.3.

Alternatively, the variational nature of the formulation and proof of Theorem 3.5.1 is expressed by our concept of "ordered direct analytic representation" as introduced in Section 3.4. Indeed, this concept implies the identification not only of Lagrange's equations with the equations of motion (first-order variational techniques) but also of Jacobi's equations and their adjoint system, with the equations of variations of the equations of motion and their adjoint system, respectively (second-order variational techniques), according to Equations (3.4.11).

Another point we would like to make is that the formulation and proof of Theorem 3.5.1, even though ultimately variational in nature, is insensitive to whether the action functional affords a (relative or absolute) extremum or not. This point has been crucial for the organization of this monograph in which, as indicated in the Introduction, the extremal aspect of the Inverse Problem is ignored.

Nevertheless, the Fundamental Analytic Theorem of this section indicates that, despite a rather general belief to the contrary, the *methodology* of the calculus of variations has a rather profound impact in Newtonian Mechanics which goes beyond the framework of Hamilton's Prineiple and its applications. For a schematic view, see Figure 3.1.

## 3.6 A Method for the Construction of a Lagrangian from the Equations of Motion

Our proof of Fundamental Analytic Theorem 3.5.1 provides not only the the system of partial differential equations for the construction of a Lagrangian but also one of its solutions. This result is a direct consequence of the use of the calculus of differential forms in general and the Converse of the Poincaré Lemma in particular. Indeed, it is a simple restatement of the proof of Theorem 3.5.1 to obtain the following theorem.

**Theorem 3.6.1** (A Method for the Construction of a Lagrangian).[41] *A Lagrangian for the ordered direct analytic representation of local, holonomic, generally nonconservative Newtonian systems that are well defined, of (at least) class $\mathscr{C}^2$, regular and self-adjoint in a star-shaped region $R^{*2n+1}$ of points $(t, q, \dot{q})$,*

$$A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0, \qquad k = 1, 2, \ldots, n, \tag{3.6.1}$$

*is given by*

$$L = K(t, q, \dot{q}) + D_k(t, q)\dot{q}^k + C(t, q), \tag{3.6.2}$$

[41] Santilli (1977b).

*where the $n + 2$ functions $K, D_k$, and $C$ are a solution of the linear, generally overdetermined system of partial differential equations*

$$\frac{\partial^2 K}{\partial \dot{q}^{k_1}\,\partial \dot{q}^{k_2}} = A_{k_1k_2}(t, q, \dot{q}), \tag{3.6.3a}$$

$$\frac{\partial D_{k_1}}{\partial q^{k_2}} - \frac{\partial D_{k_2}}{\partial q^{k_1}} = \frac{1}{2}\left(\frac{\partial B_{k_1}}{\partial \dot{q}^{k_2}} - \frac{\partial B_{k_2}}{\partial \dot{q}^{k_1}}\right) + \left(\frac{\partial^2 K}{\partial q^{k_1}\,\partial \dot{q}^{k_2}} - \frac{\partial^2 K}{\partial \dot{q}^{k_1}\,\partial q^{k_2}}\right)$$
$$\equiv Z_{k_1k_2}(t, q), \tag{3.6.3b}$$

$$\frac{\partial C}{\partial q^{k_1}} = \frac{\partial D_{k_1}}{\partial t} - B_{k_1} - \frac{\partial K}{\partial q^{k_1}} + \frac{\partial^2 K}{\partial \dot{q}^{k_1}\,\partial t} + \left[\frac{\partial^2 K}{\partial q^{k_1}\,\partial \dot{q}^{k_2}} + \frac{1}{2}\left(\frac{\partial B_{k_1}}{\partial \dot{q}^{k_2}} - \frac{\partial B_{k_2}}{\partial \dot{q}^{k_1}}\right)\right]\dot{q}^{k_2}$$
$$\equiv W_k(t, q), \tag{3.6.3c}$$

*given by*[42]

$$K(t, q, \dot{q}) = \dot{q}^{k_1}\int_0^1 d\tau' \left\{\left[\int_0^1 d\tau\, A_{k_1k_2}(t, q, \tau\dot{q})\right]\dot{q}^{k_2}\right\}(t, q, \tau'\dot{q}), \tag{3.6.4a}$$

$$D_{k_1} = \left[\int_0^1 d\tau\, \tau Z_{k_1k_2}(t, \tau q)\right] q^{k_2}, \tag{3.6.4b}$$

$$C = \left[\int_0^1 d\tau\, W_k(t, \tau q)\right] q^k. \tag{3.6.4c}$$

[42] It is significant here to elaborate on the fact that Equation (3.6.4b) possesses a factor $\tau$ in the integrand, while Equations (3.6.4a) and (3.6.4c) do not. Basically, this situation originates from the *order p* of the underlying differential form because the $\tau$ factor in the integrand of the solution, from Equations (1.2.25), is equal to $\tau^{p-1}$. The form that underlies Equation (3.6.3c) is clearly a 1-form. As a result, $\tau^{p-1} = 1$. This case is, therefore, straightforward. The situation for the other two sets of Equations (3.6.3) is not, however, equally transparent. First of all, since Equation (3.6.3a) is of second order, one might be tempted to assume a 2-form as the underlying form. This would yield the solution

$$K = 2\dot{q}^{k_1}\int_0^1 d\tau' \left\{\left[\int_0^1 d\tau\, \tau A_{k_1k_2}(t, q, \tau\dot{q})\right]\dot{q}^{k_2}\right\}(t, q, \tau'\dot{q})$$

which, strangely enough, produces the desired result for terms $A_{k_1k_2}$, which are independent of the velocities. In this case, we can write

$$K = A_{k_1k_2}(t, q)2\dot{q}^{k_1}\int_0^1 d\tau' \left\{\left[\int_0^1 d\tau\, \tau\right]\dot{q}^{k_2}\right\}(\tau'\dot{q}) = \tfrac{1}{2}\dot{q}^{k_1}A_{k_1k_2}(t, q)\dot{q}^{k_2}$$

by producing a "kinetic" term $K$ which is quadratic in the velocities and with the correct coefficient $\frac{1}{2}$. However, the above alternative solution is *erroneous* in the general case. Indeed, when the $A_{k_1k_2}$ terms depend on the velocities, e.g., $n = 1$, $A = \dot{q}^m$, we have

$$K = 2\dot{q}\int_0^1 d\tau' \left\{\left[\int_0^1 d\tau\, \tau^{m+1}\right]\dot{q}^{m+1}\right\}(\tau'\dot{q}) = \frac{2}{(m+2)(m+3)}\dot{q}^{m+2},$$

For illustrations of this theorem, see the example at the end of this chapter and, in particular, Example 3.6.

A few comments are in order. The first point which must be stressed is that Theorem 3.6.1 *does not* demand the knowledge of a solution of the underlying equations of motion for the computation of a Lagrangian. This point is significant for practical applications in view of the generally nonlinear nature of the considered class of equations of motion.

Secondly, under the assumptions of the theorem, there is no need to verify the consistency of system (3.6.3). Indeed, the proof of sufficiency of Theorem 3.5.1 is precisely centered on the fact that the conditions of self-adjointness *are* the integrability conditions for system (3.6.3). Therefore, for practical applications, one must verify that the given system of equations of motion is well-defined, of (at least) class $\mathscr{C}^2$, regular, and self-adjoint in a star-shaped region $R^{*2n+1}$. When such conditions are met, a solution of Equations (3.6.3) exists and is given by Equations (3.6.4).

Notice that the solutions of Equations (3.6.3) must be computed in the given order, namely, one must first solve Equation (3.6.3a) from the knowledge of the $A_{k_1k_2}$ functions of the equations of motion according to Equation (3.6.4a). The knowledge of a solution $K$ of such equations jointly with the $B_k$ terms of the equations of motion then allows the computation of the $D_k$ functions through Equation (3.6.4b), and, finally, the knowledge of the $K$, $D_k$, and $B_k$ functions allows the computation of the $C$ function through Equation (3.6.4c).

Almost needless to say, solutions (3.6.4) are *local* in nature, as is the case for all applications of the calculus of differential forms.

The reader should also recall that the velocity independence of the $Z$ and $W$ functions is guaranteed by the conditions of self-adjointness and should be aware that the integrals of Equations (3.6.4) are insensitive to the variables of the integrands other than those multiplied by the $\tau$ variables. Specifically, in the integrals of Equation (3.6.4a), the $t$ and $q^k$ variables of the $A$ functions are assumed fixed and the integration is performed only on the double $(\tau\dot{q}^k)$ dependence. Similarly, in the integrals of Equations (3.6.4b) and (3.6.4c), the $t$ variable of the integrand is assumed to be fixed and the integration is performed only in their $(\tau q^k)$ dependence.

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which is *not* a solution of Equation (3.6.3a) in view of the incorrect numerical coefficient. The use of Equation (3.6.4a), on the contrary, yields

$$K = \dot{q}\int_0^1 d\tau' \left\{\left[\int_0^1 d\tau\, \tau^m\right]\dot{q}^{m+1}\right\}(\tau'\dot{q}) = \frac{1}{(m+1)(m+2)}\,\dot{q}^{m+2},$$

which is the correct solution. This illustrates the reason that, in line with our proof of Theorem 3.5.1, Equation (3.6.3a) must be solved twice through the use of the Converse of the Poincaré Lemma. This yields the use of 1-forms twice, which, in this way, do not produce the $\tau$ factor in the integrand. The situation of Equations (3.6.3b) is somewhat the opposite. This equation is of first order, and thus one might be tempted to assume a 1-form as the underlying differential form with consequent lack of factor $\tau$ in the integrand. However, such a position would be erroneous in view of the antisymmetric nature of the equations themselves (Example 1.5).

It should be stressed here that solutions (3.6.4) *are not necessarily unique.* This is a primary reason why we have and shall always refer to the problem of the existence of "a" Lagrangian rather than "the" Lagrangian. Indeed, while the solutions of systems of *ordinary* differential equations generally depend on a number of free parameters, the solutions of systems of *partial* differential equations often depend on arbitrary *functions*, rather than parameters. As a result, system (3.6.3) can ultimately characterize a *family of Lagrangians* rather than one Lagrangian. When this is the case, each element of this family is an acceptable candidate for the analytic representation under consideration. The problem of the "degrees of freedom" of the analytic representations for a given system of equations of motion are considered in Santilli (1979).

Notice that the method for the computation of a Lagrangian according to Theorem 3.6.1 is computerizable, as intended.

We now come to a crucial as well as delicate point of our formulation and proof of Theorems 3.5.1 and 3.6.1. This is the assumed restriction that the equations of motion should be well behaved in a *star-shaped,* rather than an ordinary, region.

Before commenting on this point, let us note that on practical (although nonrigorous) grounds one can ignore any distinction between star-shaped and ordinary regions and simply verify that the system is well behaved and of (at least) class $\mathscr{C}^2$ for all the values of $q^k$ and $\dot{q}^k$ in the interval with fixed $q_0^k$ and $\dot{q}_0^k$

$$q'^k = \tau q_0^k, \qquad \dot{q}'^k = \tau \dot{q}_0^k, \qquad 0 \leq \tau \leq 1. \tag{3.6.5}$$

Notice that this interval is *closed.* This implies that the equations of motion must also be well behaved at the local origin $q^k = \dot{q}^k = 0$ as well as at the values $q^k$, $\dot{q}^k \neq 0$, $k = 1, 2, \ldots, n$. When the above conditions are met, one can compute integrals (3.6.4). On rigorous grounds, however, the notion of the star-shaped region must be used to ensure the existence of these integrals. As a matter of fact, the primary reason for restricting the formulation and proof of Theorems 3.5.1 and 3.6.1 to a star-shaped rather than an ordinary region is precisely that of ensuring the existence of the integrals of Equations (3.6.4).

The attentive reader is by now aware of the possibility that the conditions for the equations of motion to be well behaved for all values (3.6.5) can be violated in practical cases, e.g., when terms such as $\log q$, etc. appear. However this *does not* necessarily imply that in such instances a Lagrangian does not exist. Indeed, as indicated earlier, the Converse of the Poincaré Lemma has a *local* character only. This allows for redefinition of the variables, e.g., the translations

$$q^k \to q'^k = q^k + c^k, \; c^k = \text{const.} \tag{3.6.6}$$

or the more general point transformations [see Santilli (1979)]

$$q^k \to q'^k = q'^k(q), \tag{3.6.7}$$

aiming at a removal of possible divergences.

To put this situation in a different perspective, it is significant to point out that Theorem 3.6.1 ultimately provides only *one* method for the computation of a Lagrangian and that different methods for solving Equations (3.6.3) are conceivable. Indeed, the conditions of self-adjointness *do not* necessarily need a star-shaped region to be well defined, as the reader can verify with a simple inspection. Other alternatives for solving Equations (3.6.3) are, therefore, conceivable whenever the equations of motion are self-adjoint in an *ordinary* region.[43] Among these alternative approaches, the most notable is that offered by the use of the *Clauchy integral*, as outlined in Chart 3.11, which does not require the use of a star-shaped region. For further comments, see the alternative methods for the computation of a Hamiltonian in Section 3.12. This aspect is reconsidered in Santilli (1979) within the context of the transformation theory.

In conclusion, it appears that under the conditions of self-adjointness, a Lagrangian could exist in an arbitrary region, in which case the Fundamental Analytic Theorems could be formulated without restriction to a star-shaped region. The rigorous proof of this expected property would, however, bring us outside the objectives of this book owing to the need of additional mathematical tools (e.g., algebraic topology or global differential geometry). Therefore, we content ourselves with the formulation and proof of the Fundamental Analytic Theorems as given and the presentation of the methods for the computation of a Lagrangian with and without the restriction to a star-shaped region (i.e., that of Theorem 3.6.1 and of Chart 3.11).[44]

We now consider the case of the ordered direct analytic representations of Newtonian systems in their kinematical form (Problem 3.2). Under the conditions of self-adjointness, such a form reduces to the form (2.2.23). Therefore, the problem reduces to the identifications

$$\frac{\partial^2 L}{\partial \dot{q}^{k_1} \partial \dot{q}^{k_2}} \ddot{q}^{k_2} + \frac{\partial^2 L}{\partial \dot{q}^{k_1} \partial q^{k_2}} \dot{q}^{k_2} + \frac{\partial^2 L}{\partial \dot{q}^{k_1} \partial t} - \frac{\partial L}{\partial q^{k_1}} \equiv \ddot{q}_{k_1} - \rho_{k_1 k_2}(t, q)\dot{q}^{k_2} - \sigma_{k_1}(t, q), \quad (3.6.8)$$

with underlying conditions of self-adjointness (2.2.24), i.e.,

$$\rho_{k_1 k_2} + \rho_{k_2 k_1} = 0, \quad (3.6.9a)$$

$$\frac{\partial \rho_{k_1 k_2}}{\partial q^{k_3}} + \frac{\partial \rho_{k_2 k_3}}{\partial q^{k_1}} + \frac{\partial \rho_{k_3 k_1}}{\partial q^{k_2}} = 0, \quad (3.6.9b)$$

$$\frac{\partial \rho_{k_1 k_2}}{\partial t} = \frac{\partial \sigma_{k_1}}{\partial q^{k_2}} - \frac{\partial \sigma_{k_2}}{\partial q^{k_1}}. \quad (3.6.9c)$$

This is a particular case of Theorem 3.6.1, with the identifications

$$A_{k_1 k_2} = \delta_{k_1 k_2}, \qquad B_{k_1} = -\rho_{k_1 k_2}\dot{q}^{k_2} - \sigma_{k_1}. \quad (3.6.10)$$

[43] See Example 3.1 for an illustration of this point.

[44] For a recent study of star-shaped regions the reader may consult Staneck (1977) and quoted references.

From Equation (3.6.4a), we have

$$\begin{aligned} K &= \dot{q}^{k_1}\int_0^1 d\tau'\left\{\left[\int_0^1 d\tau\ \delta_{k_1k_2}\right]\dot{q}^{k_2}\right\}(\tau'\dot{q}) \\ &= \dot{q}^{k_1}\left[\int_0^1 d\tau'\ \tau'\right]\dot{q}^{k_2}\delta_{k_1k_2} \\ &= \tfrac{1}{2}\dot{q}^{k_1}\dot{q}_{k_1} \end{aligned} \tag{3.6.11}$$

by recovering in this way the familiar structure of the kinetic term. The following corollary then trivially follows.

**Corollary 3.6.1A.**[45] *A Lagrangian density for the ordered direct analytic representations of local, holonomic, generally nonconservative Newtonian systems in their kinematical form*

$$\ddot{q}_{k_1} - \rho_{k_1k_2}(t, q)\dot{q}^{k_2} - \sigma_{k_1}(t, q) = 0, \qquad k = 1, 2, \ldots, n \tag{3.6.12}$$

*which are well defined, of (at least) class $\mathscr{C}^1$, and self-adjoint in a star-shaped region $R^{*2n+1}$ of points $(t, q)$ is given by*

$$L = \tfrac{1}{2}\dot{q}^{k_1}\dot{q}_{k_1} + D_{k_1}(t, q)\dot{q}^{k_1} + C(t, q) \tag{3.6.13}$$

*where the $n + 1$ functions $D_k$ and $C$ are solutions of the linear, generally overdetermined system of partial differential equations*

$$\frac{\partial D_{k_1}}{\partial q^{k_2}} - \frac{\partial D_{k_2}}{\partial q^{k_1}} = -\rho_{k_1k_2}, \tag{3.6.14a}$$

$$\frac{\partial C}{\partial q^{k_1}} = \frac{\partial D_{k_1}}{\partial t} - \sigma_{k_1}, \tag{3.6.14b}$$

*given by*

$$D_{k_1} = -\left[\int_0^1 d\tau\ \tau\rho_{k_1k_2}(t, \tau q)\right]q^{k_2}, \tag{3.6.15a}$$

$$C = \left[\int_0^1 d\tau\left(\sigma_{k_1} + \frac{\partial D_{k_1}}{\partial t}\right)(t, \tau q)\right]q^{k_1}. \tag{3.6.15b}$$

The above corollary can be simplified further.[46] Conditions of self-adjointness (3.6.9a) and (3.6.9b) imply that the $\rho_{k_1k_2}$ functions must have the structure of a curl (Example 1.5). Therefore, under the assumed conditions, a set of functions, say $\Gamma_k(t, q)$ such that

$$\rho_{k_1k_2} = \frac{\partial \Gamma_{k_2}}{\partial q^{k_1}} - \frac{\partial \Gamma_{k_1}}{\partial q^{k_2}}, \tag{3.6.16}$$

[45] Santilli (1977b).
[46] Santilli (1977b).

exists. The conditions of self-adjointness then reduce to Equation (3.6.9c) only, the $D_k$ functions are trivially given by

$$D_{k_1} = \Gamma_{k_1}, \tag{3.6.17}$$

and Lagrangian (3.6.13) takes the form

$$L = \tfrac{1}{2}\ddot{q}^{k_1}\dot{q}_{k_1} + \Gamma_{k_1}(t, q)\dot{q}^{k_1} + C(t, q), \tag{3.6.18}$$

where the only unknown function $C$ is a solution of corresponding Equation (3.6.14c), i.e.,

$$C = q^k \int_0^1 d\tau \left(\sigma_k + \frac{\partial \Gamma_k}{\partial t}\right)(t, \tau a). \tag{3.6.19}$$

One of the best illustrations of Corollary 3.6.1A is that given by the equations of motion of a charged particle under the Lorentz force, which, as proved in Example 2.7, is variational self-adjoint. In this case Equations (3.6.12) become

$$[m\ddot{\mathbf{r}} - e\dot{\mathbf{r}} \times \mathbf{B} - e\mathbf{E}]_{i\text{th component}} = m\ddot{r}_i - e\delta_{ij}^{mn}\frac{\partial A_m}{\partial r^n}\dot{r}^j + e\left(\frac{\partial \varphi}{\partial r^i} - \frac{\partial A_i}{\partial t}\right) = 0, \tag{3.6.20}$$

$$i = x, y, z,$$

and, after computing integrals (3.6.15), one re-obtains the familiar Lagrangian

$$L = \tfrac{1}{2}m\dot{\mathbf{r}}^2 + e\mathbf{A} \cdot \dot{\mathbf{r}} - e\varphi, \tag{3.6.21}$$

which is precisely of type (3.6.18).

This concludes our analysis of the problem of constructing a Lagrangian once its existence is assured by the validity of the underlying integrability conditions. Alternative methods are presented in Chart 3.11.

## 3.7 The Implications of Nonconservative Forces for the Structure of a Lagrangian

Our analysis of the Inverse Problem for ordered direct analytic representations in configuration space would not be complete without attempting a physical interpretation of the results from the viewpoint of the nature of the acting forces (i.e., along the lines of the Appendix), and then a reinterpretation of the same results from the viewpoint of the concept of Newtonian interactions (i.e., along the lines of the Introduction).

Our first objective is the study of the following problem.

*What is the role of conservative Newtonian forces within the context of the problem of the existence of a Lagrangian?*

This problem can be more technically formulated by asking what the role of conservative forces is within the context of the integrability conditions for the existence of a Lagrangian, namely, the conditions of self-adjointness for the considered form of the equations of motion.

It has been rather surprising for me to see that, within the context of the Inverse Problem, the problem under consideration is marginal because no condition of self-adjointness is capable of restricting the acting forces to be conservative.

The possible forms of the equations of motion in configuration space are either the fundamental form or the kinematical form. The nonconservative nature of the fundamental form is clearly exhibited by the generally nonlinear dependence on the velocities which is allowed by the conditions of self-adjointness (Section 2.2). Therefore, the problem under consideration reduces to the study of the analytic representations of the kinematical forms.

But under the conditions of self-adjointness the kinematical form reduces to the form of Equation (2.2.23) (Theorem 2.2.2). Therefore, the problem under consideration further reduces to the study of the role of conservative forces within the context of the integrability conditions (3.6.9) for identifications (3.6.8). A simple inspection then indicates that *the conditions of self-adjointness* (3.6.9) *are unable to restrict the acting forces to be conservative.*

Indeed, the generally *nonconservative* nature of the represented system is clearly expressed by the admissible velocity dependence of the equations of motion. As a result, the ordered direct analytic representations of Newton's equations of motion in their kinematical form generally characterize *nonservative* systems.

This fact is remarkable, particularly in view of the simplicity of identifications (3.6.8). The functions $f_k$ of the kinematical form (i.e., the implicit functions of the system) are often proportional to the acting forces through the multiplication of the inverse of the masses.[47] One would, therefore, expect that the conditions of self-adjointness (which in this case are the conditions of the functions $f_k$ and, thus, of the acting forces) restrict these forces to be conservative. The fact that, contrary to any different belief, this is not the case, is exhibited by the nonconservative nature of identity (3.6.8).

By including in the analysis the more general fundamental form, we can conclude by saying that *the integrability conditions for the existence of a Lagrangian are insensitive to whether the acting forces are conservative or not.* As a matter of fact, this is one of the most intriguing aspects of the Inverse Problem, which ultimately dispels the rather general belief that the Lagrangian for the representation of Newtonian systems exists only when the acting forces are derivable from a potential function (Chart 3.1).

Obviously, conservative forces *are* admissible by the Inverse Problem. To see them, consider a system of particles of unit masses which is unconstrained and represents the assumed type of coordinates (e.g., Cartesian) with the $q^k$

[47] From Newton's equations of motion $m\ddot{\mathbf{r}} - \mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}}) = 0$, the implicit functions are simply given by $\mathbf{f} = \mathbf{F}/m$.

variables in a given order. Suppose that in identifications (3.6.8) the $\rho_{k_1k_2}$ functions are identically null. Then the conditions of self-adjointness (3.6.9) reduce to

$$\frac{\partial \sigma_{k_1}}{\partial q^{k_2}} - \frac{\partial \sigma_{k_2}}{\partial q^{k_1}} = 0, \tag{3.7.1}$$

and they coincide with the integrability conditions for the existence of a (conservative) potential function, i.e., Equations (A.4.7). Indeed, in view of the assumption of unit masses, the implicit functions (represented in this case by the $\sigma_k$ functions) coincide with the acting forces.

The point is that there is no need to impose the requirement that the $\rho_{k_1k_2}$ functions are identically null for identifications (3.6.8) to exist and to be consistent. As a result, conditions of self-adjointness (3.6.9), even though they do include conservative forces as a particular case, are unable to restrict all of the acting forces to be of this type.

In this way, we arrive at a most crucial point of the analysis of this monograph, which can be stated simply by saying that *Lagrange's equations can represent Newtonian systems as they actually are in physical reality, namely, generally nonconservative.*

Our second problem of this section, which is an immediate consequence of the above result, can be formulated as follows.

*What is the most general admissible form of the acting forces within the context of the analytic representations of Newtonian systems?*

For simplicity, but without loss of generality, we shall consider the case of an *unconstrained* system of $N$ particles of unit masses represented in a reference frame characterized by the $q^k$ variables, $k = 1, 2, \ldots, n = 3N$. Suppose, as a first step, that these particles are free. The equations of motion are trivially given by

$$\ddot{q}_k = 0, \qquad k = 1, 2, \ldots, n. \tag{3.7.2}$$

A Lagrangian for their analytic representation (see Equation (3.6.11)) can be written as[48]

$$L = \sum_{k=1}^{n} \tfrac{1}{2}(\dot{q}^k)^2 = L_{\text{free}}. \tag{3.7.3}$$

Our problem is to study *the most general form of coupling these free particles that is admissible by a Lagrangian representation, i.e., by Theorem* 3.6.1.

[48] By no means should this Lagrangian be considered as unique. For instance, an equally acceptable *family* of Lagrangians for the indirect representations of Equations (3.7.2) is given by

$$L = \sum_{k=1}^{n} (\dot{q}^k)^{n+2}, \qquad n = 0, 1, 2, \ldots.$$

First, let us review the most general form of Newtonian couplings,[49] i.e., the most general form of couplings that preserves the linearity in the accelerations. Such a form is given by the superposition of each of the following three classes of couplings.

I. *Time-dependent generally nonlinear couplings in the coordinates*

The equations of motion (3.7.2) are modified in this case to the form

$$\ddot{q}_k - f_k(t, q) = 0. \tag{3.7.4}$$

This class of couplings contains as a subclass [when the functions $f_k$ do not depend explicitly on time and satisfy integrability conditions (3.7.1)] the class of conservative couplings.

II. *Time-dependent generally nonlinear couplings in the velocities*

Equations of motion (3.7.2) are modified in this case to

$$\ddot{q}_k - f'_k(t, \dot{q}) = 0. \tag{3.7.5}$$

This class does not contain conservative couplings as a subclass. If couplings of classes I and II are combined, then equations of motion (3.7.2) are modified to what we have called the kinematical form, i.e.,

$$\ddot{q}_k - f_k(t, q, \dot{q}) = 0. \tag{3.7.6}$$

III. *Time, coordinates, and velocity-dependent couplings that are linear in the accelerations*

In this case the equations of motion (3.7.2) are modified to

$$A_{ki}(t, q, \dot{q})\ddot{q}^i = 0. \tag{3.7.7}$$

The collection of all couplings of classes I, II, and III then leads in a natural way to what we have called the fundamental form of the equations of motion, i.e.,

$$A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0. \tag{3.7.8}$$

It is relevant to stress that the acceleration couplings, in their most general form, demand that both the diagonal *and* off-diagonal terms of the matrix $(A)$ are non-null.

An example is useful to indicate the need, in general, for *all* of the above three classes of couplings as well as of the *nonlinearity* of couplings I and II.

Consider the case of a system of oscillators. Linear, time-independent couplings of type I produce the familiar form

$$\ddot{q}_k + c_{ki}q^i = 0. \tag{3.7.9}$$

This *conservative* formulation, however, is insufficient to represent the system as it actually occurs in our environment owing to the inevitable presence of dissipative forces. A natural generalization of Equation (3.7.9) is then given by the inclusion of time-independent linear couplings of type II by obtaining in

[49] Here the term "coupling" is used as a way of referring to the acting forces.

this way the familiar form of the coupled and damped oscillators.

$$\ddot{q}_k + b_{ki}\dot{q}^i + c_{ki}q^i = 0. \tag{3.7.10}$$

This *dissipative* formulation, however, is insufficient because, as is well known in the theory of coupled oscillators, couplings of type III also occur. The next simple generalization then yields the familiar form of the linear system with constant coefficients,[50]

$$a_{ki}\ddot{q}^i + b_{ki}\dot{q}^i + c_{ki}q^i = 0. \tag{3.7.11}$$

Notice that the acceleration couplings occur precisely because the off-diagonal as well as the diagonal elements of the matrix $(a_{ki})$ are non-null.

This indicates the need to consider *all* three classes of couplings I, II, and III for the representation of a system of oscillators. Equations of motion (3.7.11), however, still constitute an approximation of physical reality. Indeed, as is well known, they are valid only for small oscillations precisely in view of the linear nature of couplings I and II. The removal of this restriction inevitably brings the equations of motion from the linear form (3.7.11) to the fundamental form (3.7.8), where the possible presence of applied forces (which are, in practice, essential to preserve the motion for a sufficiently long period of time) can be incorporated in the $B$ terms.

This indicates the need to consider *nonlinear* couplings of types I and II. Their time dependence can then be inferred on other grounds (e.g., the variations of the parameters in time due to temperature, etc.).

Thus, the above example of a system of coupled oscillators indicates that, whenever a more accurate description of physical reality is needed, equations of motion of type (3.7.11) must be abandoned and the fundamental form of the equations of motion must be adopted. This conclusion holds irrespective of our interpretation and classification of the Newtonian couplings.

The fundamental form of the equations of motion, therefore, represents an arbitrary collection of the most general Newtonian couplings I, II, and III, although in a somewhat hidden form. Specifically, the couplings of types I (II) are represented by the time and generally nonlinear coordinate (velocity) dependence of the $A$ and $B$ terms, and the couplings of type III are represented by the non-null values of the off-diagonal, as well diagonal, terms $A_{ki}$, jointly with the indicated functional dependence of these terms.

At this point one can argue that the acceleration couplings are inessential for the equations of motion because, under the assumption of regularity, the fundamental form (3.7.8) can always be reduced to the kinematical form (3.7.6) through the identifications

$$f_k = -A_k^{-1\,i}B_i, \qquad (A_k^{-1\,i}) = (A_{ki})^{-1}. \tag{3.7.12}$$

Indeed, the reduction from Equations (3.7.8) to (3.7.6) is true from the Theorem of Implicit Functions (Section 1.1). As a consequence, the statement that Equation (3.7.6), *without* acceleration couplings can equivalently

[50] See, for instance, Symon (1960).

represent the system is also true. Thus, within the context of the theory of ordinary second-order differential equations, the acceleration couplings are indeed inessential for representing the motion.

However, within the context of the problem of the Lagrangian representation of the equations of motion, the situation is substantially different. It is precisely at this point that the integrability conditions for the existence of a Lagrangian, i.e., the conditions of self-adjointness, play a vital role.

Indeed, *Newton's equations of motion in the kinematical form* (3.7.6) *are non-self-adjoint (unless trivial forms of couplings are assumed) and, therefore, a Lagrangian for their ordered direct analytic representation, from Theorem* 3.5.1, *does not exist.* This is because the conditions of self-adjointness restrict the kinematical form to the simpler form (3.6.12), which can represent the most general form of couplings of type I, but the couplings of type II are restricted to be *linear* in the velocities, and the couplings of type III are absent.

In order to represent such equations of motion, one is forced, as one possibility, to study the indirect representations, i.e., the representations of equivalent systems of the type

$$\{h_k^i(t, q, \dot{q})[\ddot{q}_i - f_i(t, q, \dot{q})]_{\text{NSA}}\}_{\text{SA}} = 0, \tag{3.7.13}$$

and use the freedom of the factor functions $h_k^i$ to induce a self-adjoint structure.[51] If one of these equivalent self-adjoint forms of the equations of motion exists, a Lagrangian for their analytic representation exists from Theorem 3.5.1.

The point which must be stressed is that the net effect of the "integrating factors" $h_k^i$ is precisely that of retransforming the equations of motion from the kinematical form (3.7.6) to the fundamental form (3.7.8), with the consequent restoration of the acceleration couplings.

At this point, the reader is urged to inspect the conditions of self-adjointness for the fundamental form, i.e., Equations (3.5.3). It is then easy to conclude that, unlike the case of the kinematical form, *the conditions of self-adjointness for the fundamental form do allow a collection of general couplings of types I, II, and III.* This is due to the fact that such conditions allow an explicit time dependence of the equations of motion, a generally nonlinear dependence in both the coordinates and the velocities, and non-null as well as nontrivial values of the off-diagonal and diagonal elements of the matrix $(A)$.

Our findings, therefore, can be stated simply by saying that *Lagrange's equations can represent equations of motion with (local) arbitrary Newtonian forces, i.e., with an arbitrary collection of conservative, dissipative, and applied forces of types I, II, and III.*

To summarize, the acceleration couplings are not essential for representing the motion under arbitrary Newtonian forces within the context of the theory of ordinary second-order differential equations, but they are necessary within the context of their Lagrangian representations. If such acceleration couplings

[51] This problem is investigated in Santilli (1979).

are ignored within this latter context, the net effect is a considerable restriction of the type of admissible Newtonian forces.

Our third objective of this section is the study of the following problem.

*What is a general form of modification of Lagrangian* (3.7.3) *for free motion capable of representing the same system when subject to an arbitrary collection of couplings I, II, and III or, equivalently, to arbitrary Newtonian forces?*

First, it is advisable to reproduce the conventional structure of the Lagrangian for interacting Newtonian systems within the context of Theorem 3.6.1.

**Corollary 3.6.1B.**[52] *A total Lagrangian for the ordered direct analytic representation of local, holonomic, generally nonconservative, interacting Newtonian systems*

$$\ddot{q}_k + \left[\frac{\partial \Gamma_{k_1}(t, q)}{\partial q^{k_2}} - \frac{\partial \Gamma_{k_2}(t, q)}{\partial q^{k_1}}\right]\dot{q}^{k_2} - \sigma_{k_1}(t, q) = 0, \qquad (5.7.14)$$

$$k = 1, 2, \ldots, n,$$

*which are well defined, of (at least) class* $\mathscr{C}^1$, *and self-adjoint in a star-shaped region* $R^{*2n+1}$ *of points* $(t, q)$, *is given by*

$$L_{\text{tot}} = L_{\text{free}}(\dot{q}) + L_{\text{int}}(t, q, \dot{q}) = \sum_{k=1}^{n} L_{\text{free}}^{(k)} + L_{\text{int}}, \qquad (5.7.15)$$

*where*

$$L_{\text{free}}^{(k)} = \tfrac{1}{2}(\dot{q}^k)^2, \qquad (5.7.16a)$$

$$L_{\text{int}} = \Gamma_k(t, q)\dot{q}^k + C(t, q), \qquad (5.7.16b)$$

*and the function C is given by*

$$C = q^k \int_0^1 d\tau \left(\sigma_k + \frac{\partial \Gamma_k}{\partial t}\right)(t, \tau q). \qquad (5.7.17)$$

It should be stressed that within the context of the above corollary, the term $L_{\text{int}}$ can have only a *linear* dependence on the velocities.

The general case can also be derived as a reinterpretation of Theorem 3.6.1 according to the following corollary.

**Corollary 3.6.1C.**[53] *A general structure of the total Lagrangian for the ordered direct analytic representation of local, holonomic, generally nonconservative, interacting Newtonian systems in the fundamental form*

$$A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0, \qquad k = 1, 2, \ldots, n, \qquad (3.7.18)$$

*which is well defined, of (at least) class* $\mathscr{C}^2$, *regular, and self-adjoint in a star-shaped region* $R^{*2n+1}$ *of points* $(t, q, \dot{q})$, *is characterized by* $n + 1$

[52] Santilli (1977b).
[53] R. M. Santilli (1977b).

*interaction terms, n multiplicative terms, and one additive term to the Lagrangian for the free motion of each particle, according to the generalized structure*

$$L_{\text{tot}}^{\text{gen}}(t, q, \dot{q}) = \sum_{k=1}^{n} L_{\text{int, I}}^{(k)}(t, q, \dot{q}) L_{\text{free}}^{(k)} + L_{\text{int, II}}(t, q, \dot{q}), \tag{3.7.19}$$

*where the terms* $K_{\text{int, I}}^{(k)}$, $L_{\text{free}}^{(k)}$, *and* $L_{\text{int, II}}$ *admit the decompositions*

$$L_{\text{int, I}}^{(k)} = K_{\text{I}}^{(k)}(t, q, \dot{q}) + D_{\text{II}}^{(k)}(t, q)\dot{q}^i + C_{\text{I}}^{(k)}(t, q), \tag{3.7.20a}$$

$$L_{\text{free}}^{(k)} = \tfrac{1}{2}(\dot{q}^k)^2, \tag{3.7.20b}$$

$$L_{\text{int, II}} = K_{\text{II}}(t, q, \dot{q}) + D_{i\text{II}}(t, q)\dot{q}^i + C_{\text{II}}(t, q), \tag{3.7.20c}$$

*and can be expressed in terms of solutions* (3.6.4) *of Equations* (3.6.3) *by means of the identifications*

$$K(t, q, \dot{q}) = \sum_{k=1}^{\mu} L_{\text{int, I}}^{(k)} L_{\text{free}}^{(k)} + K_{\text{II}}, \tag{3.7.21a}$$

$$D_k = D_{k\text{II}}, \tag{3.7.21b}$$

$$C = C_{\text{II}}. \tag{3.7.21c}$$

Here the multiplicative interaction terms are *necessary* for the representation of the motion under forces not derivable from a potential. Indeed, when such terms are reduced to unity (jointly with the restriction that $L_{\text{int, II}}$ be linear in the velocites), the net effect is that of eliminating the acceleration couplings with consequent restriction of the types of admissible couplings, as indicated earlier. Notice that all interaction terms can now have a generally nonlinear dependence on the velocities (as well as the coordinates).

It should be stressed that structure (3.7.19) is by no means unique and other generalized forms are equally admissible. Along these lines, another significant generalized structure of the total Lagrangian is given by the following simple reinterpretation of structure (3.6.2), which emerges directly from the Fundamental Analytic Theorem[54]

$$\begin{aligned} L_{\text{tot}}^{\text{gen}}(t, q, \dot{q}) &= K(t, q, \dot{q}) + D_k(t, q)\dot{q}^k + C(t, q) \\ &= \tfrac{1}{2}[\dot{q}^i G_{ij}(t, q, \dot{q})\dot{q}^j + 2\dot{q}^i F_{ij}(t, q)q^j + q^i E_{ij}(t, q)q^j]. \end{aligned} \tag{3.7.22}$$

By inspecting the transition from the Lagrangian for free motion to the above generalized structure, we clearly see that the crucial role of representing the acceleration couplings is here played by the $G_{ij}$ tensor. Indeed, whenever the matrix $(G_{ij})$ reduces to the unit matrix $(\delta_{ij})$, Lagrangian (3.7.22) reduces to Equation (3.7.15).[55]

[54] This structure of the Lagrangian appears to be more promising than structure (3.7.19) from the viewpoint of possible quantization due to the underlying need for symmetrization. It should be recalled that, to the best of my knowledge, the problem of quantization of arbitrary Newtonian couplings is far from being solved at this time [Santilli (1978)].

[55] The extension of structure (3.7.22) to field theories is closely related to the so-called *chiral Lagrangians*. For more details, see Santilli (1977b and c).

We can, therefore, conclude that *any generalized structure of the Lagrangian is capable of representing systems with arbitrary Newtonian couplings provided that such a structure contains the representatives of the acceleration couplings.*

It might be of some significance here to point out that the Morse–Feshbach Lagrangian (3.4.15) is precisely of type (3.7.22), and *not* of type (3.7.15), despite the linearity of the velocity couplings, in view of the identifications

$$\begin{aligned}(G_{ij}) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ (F_{ij}) &= \begin{pmatrix} 0 & -b \\ +b & 0 \end{pmatrix}, \\ (E_{ij}) &= \begin{pmatrix} 0 & -\omega^2 \\ -\omega^2 & 0 \end{pmatrix}.\end{aligned} \tag{3.7.23}$$

Indeed, the represented form of the equations of motion, i.e., the right-hand side of Equation (3.4.16) is precisely a simple version of the fundamental form according to the identifications[56]

$$\begin{aligned}(A_{ij}) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ (B_k) &= \begin{pmatrix} -b\dot{q}_2 & +\omega^2 q_2 \\ +b\dot{q}_1 & +\omega^2 q_1 \end{pmatrix}.\end{aligned} \tag{3.7.24}$$

However, the $A_{ki}$ terms in this case *do not* represent genuine acceleration coupling, because their diagonal values $A_{kk}$ are null. This is reflected by the simplicity of the couplings of the equations of motion. It is, then, conceivable that any further generalization of the couplings of this system will inevitably lead to bona fide acceleration couplings, i.e., to less trivial values of the $G_{ij}$ tensor of structure (3.7.22).[57] This aspect will be illustrated in the examples at the end of this chapter, as well as in those of Santilli (1979).

The extension of the above analysis to the case of regular Newtonian systems with holonomic constraints is straightforward. Consider, first, the case of a systems of $N$ *free* particles subject to $3N - n$ holonomic constraints. Their equations of motion in configuration space can be represented by Lagrange's equations in the kinetic energy (A.4.10), i.e.,

$$\begin{aligned}L_{\text{free}} &= T(t, q, \dot{q}) \\ &= \tfrac{1}{2}\dot{q}^i Z_{ij}(t, q)\dot{q}^j + Z_k(t, q)\dot{q}^k + Z_0(t, q).\end{aligned} \tag{3.7.25}$$

[56] Let us recall from footnote 21 of p. 127 that this system is the Newtonian limit of the interactions of a complex scalar field with an external electromagnetic field, i.e., a central model of *gauge* theories. Therefore, the field equations which are represented in these theories are a generalization of structure (3.4.16). For more details, see Santilli (1977a, b, and c).

[57] Within a field theoretical context, this problem ultimately constitutes one of the central problems of contemporary theoretical high-energy physics, the study of the generalizations of the unified gauge theories of the electromagnetic and weak interactions aiming at an effective inclusion of the strong interactions. For studies along these lines, see Santilli (1978, Vols. I, II and III).

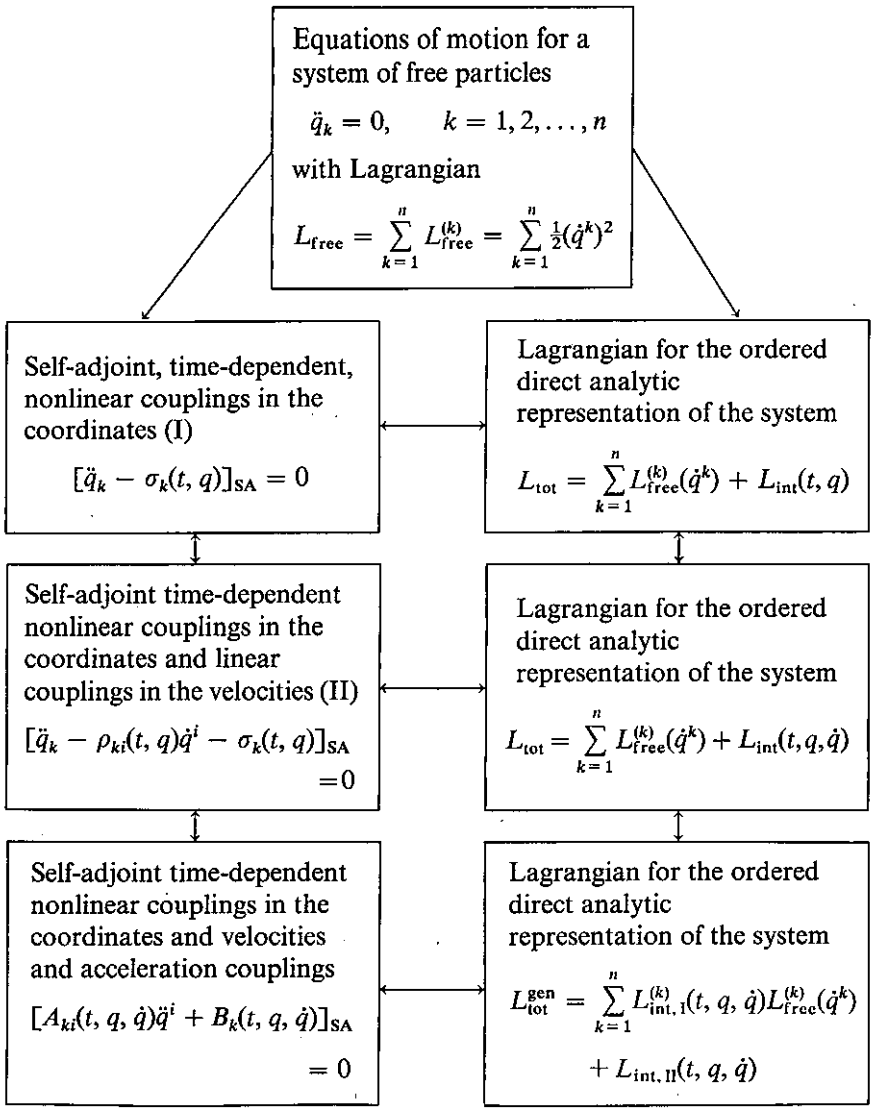


**Figure 3.2** A schematic view of *the classes of Newtonian couplings that are admissible by a Lagrangian representation.* In view of Fundamental Analytic Theorem 3.5.1, all admissible couplings must induce self-adjoint forms of the equations of motion. The simplest class of couplings, i.e., those of class I, are not necessarily conservative because the conditions of self-adjointness are unable to restrict the acting forces to be conservative, although the case is obviously admissible. The second class of couplings, i.e., those in both coordinates and velocities, is considerably restricted by the conditions of self-adjointness, because only a linear dependence on the velocities is admitted, whenever the equations of motion are written in their kinematical form. The third class of couplings is the largest admissible by a Lagrangian representation and it does allow nonlinear couplings in both the coordinates and the velocities, provided the equations of motion are written in their fundamental form. The net effect is the necessary presence of the acceleration couplings, which results in the presence of interaction terms in the total Lagrangian of both multiplicative and additive type to the free Lagrangian. Owing to the limited nature of the couplings of class II, generalized structures of the total Lagrangian emerge as necessary for a closer representation of the Newtonian physical reality.

Notice that, despite the lack of acting forces,[58] the structure of this Lagrangian closely follows structure (3.6.2) or (3.7.22), with the only functional difference characterized by the velocity independence of the $Z_{ij}$ tensor.

When all the acting forces are derivable from a potential, structure (3.7.25) is modified into the familiar form[59]

$$L_{\text{tot}} = T(t, q, \dot{q}) - U(t, q, \dot{q}). \tag{3.7.26}$$

If some of the acting forces are not derivable from a potential, then the above structure is insufficient[60] to represent the motion. The extension of the analysis conducted previously in this section then leads to the conclusion that generalized structures, e.g., of the type (3.7.22), can indeed represent regular holonomic systems with arbitrary Newtonian forces. Notice that this is equivalent to the only additional velocity dependence, in structure (3.7.25) for free motion, of the $Z_{ij}$ tensor. Other equivalent forms of representing the motion are

$$L_{\text{tot}}^{\text{gen}} = R(t, q, \dot{q})T(t, q, \dot{q}) + S(t, q, \dot{q}), \tag{3.7.27a}$$

$$L_{\text{tot}}^{\text{gen}} = R(t, q, \dot{q})[T(t, q, \dot{q}) + S'(t, q, \dot{q})], \tag{3.7.27b}$$

and they again indicate the presence of both multiplicative and additive interaction terms in the kinetic energy.

For a schematic view of the content of this section, see Figure 3.2.

## 3.8 Direct and Inverse Legendre Transforms for Conventional Analytical Representations

We consider now the transition to phase space formulations for the (ordered) direct identification

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \equiv A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}), \tag{3.8.1a}$$

$$A_{ki}, B_k \in \mathscr{C}^2(R^{2n+1}), \qquad L \in \mathscr{C}^4(R^{2n+1}), \tag{3.8.1b}$$

$$|A_{ij}|(R^{2n+1}) = \left|\frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}\right|(R^{2n+1}) \neq 0. \tag{3.8.1c}$$

The conventional *canonical prescriptions* for the characterization of the *generalized momenta* $p_k$ are

$$G_k(t, q, \dot{q}, p) = p_k - \frac{\partial L}{\partial \dot{q}^k} = 0. \tag{3.8.2}$$

[58] At this stage, even the frictional forces of the constraints are ignored.

[59] See Problems 3.3 and 3.4.

[60] Under the assumption that $T$ is the kinetic energy and $U$ the potential. When these physical restrictions are removed, then each function $T$ or $U$ can have an arbitrary functional dependence.

The transition from the Lagrangian $L$ to a new function, the *Hamiltonian* $H$, is done through the *Legendre transform*,

$$L \to H = p_k \dot{q}^k - L(t, q, \dot{q}) = H'(t, q, \dot{q}, p), \tag{3.8.3}$$

which possesses the following important properties

1. Expression (3.8.3), as written, is a function of the $3n + 1$ variables $(t, q, \dot{q}, p)$. Its reduction to a form depending only on the $2n + 1$ variables $(t, q, p)$ demands the computation of *all* the implicit functions of systems (3.8.2) with respect to $\dot{q}_k$, i.e.,

$$\dot{q}^k = \tilde{g}^k(t, q, p), \qquad k = 1, 2, \ldots, n. \tag{3.8.4}$$

When the set of all such functions exists, their substitution into Equation (3.8.3) produces the desired reduction, i.e.,

$$H(t, q, p) = p_k \tilde{g}^k - L(t, q, \tilde{g}) = H'(t, q, \tilde{g}, p). \tag{3.8.5}$$

A point of central methodological significance is that the set of *all* functions (3.8.4) exists and is unique if and only if Implicit Function Theorem 1.1.2 can be applied to prescription (3.8.2) with respect to the $\dot{q}_k$ variables everywhere in the region of interest. But from assumption (3.8.1b), the functions $G_k$ of Equation (3.8.2) are (at least) of class $\mathscr{C}^3$, and condition (1′) of Theorem 1.1.2 holds. Condition (3.8.1c) implies that the functional determinant of Equation (3.8.2) with respect to $\dot{q}_k$ is regular in $R^{2n+1}$, i.e.,[61]

$$\left|\frac{\partial G_i}{\partial \dot{q}^j}\right| = -\left|\frac{\partial^2 L}{\partial \dot{q}^i \, \partial \dot{q}^j}\right|, \tag{3.8.6}$$

and condition (2′) of Theorem 1.1.2 holds. Then the functions $\tilde{g}_k$ of Equations (3.8.4) must be such that

$$G_k(t, q, \tilde{g}, p) \equiv 0, \tag{3.8.7}$$

and this ensures the fulfillment of condition (3′) of Theorem 1.1.2. Thus, under assumptions (3.8.1b) and (3.8.1c), Implicit Function Theorem 1.1.2 holds and *all* functions $\tilde{g}^k$, $k = 1, 2, \ldots, n$ exist and are unique. Then, Hamiltonian (3.8.5) is unique.[62]

2. Assumptions (3.8.1b) and (3.8.1c) imply, from Theorem 1.1.2, that all implicit functions $\tilde{g}^k$, besides existing and being unique, are single-valued. In turn, such properties imply that Legendre transform (3.8.3) induces a one-to-one mapping

$$R^{2n+1} \to \tilde{R}^{2n+1} \tag{3.8.8a}$$

[61] In Equation (3.8.6), $p_k$ and $\dot{q}_k$ are independent variables.

[62] The implication of degeneracy should be indicated at this point. Basically, when the Lagrangian is degenerate in $R^{2n+1}$, besides the breakdown of Hilbert Differentiability Theorem 3.2.2, there is the lack of applicability of Implicit Function Theorem 1.1.2 to system (3.8.2) with respect to the $\dot{q}$'s. This implies that the set of all implicit functions (3.8.4) cannot be computed. A different methodology must then be used for the mapping to phase space. See, in this respect, Dirac (1964).

from the original region $R^{2n+1}$ of points $(t, q, \dot{q})$ to an "image" region $\tilde{R}^{2n+1}$ of points $(t, q, p)$. Furthermore, the (open) region $R^{2n+1}$ is unrestricted for transform (3.8.3). Therefore, when $R^{2n+1}$ is the space $M_t \otimes M_{(q,\dot{q})}$ of all possible values of $t$, $q_k$, and $\dot{q}_k$, its "image" region is the Kronecker product

$$\tilde{M}^{2n+1} = M_t \otimes M_{(q,p)} \tag{3.8.8b}$$

of the space $M_t$ spanned by the time variable and the *phase space* $M_{(q,p)}$ of the $2n$ *canonical conjugate variables* $q_k$ and $p_k$. Furthermore, under the above assumptions, the variables $q_k$ and $p_k$ are all independent[63] and the "image" path of $(E, \dot{E})$ is the *path* (or *path segment*) *in phase space* $\tilde{E} = \{q_k = q_k(t), p_k = p_k(t)\}$.

3. Since, under the above assumptions, $H$ can be regarded either as a function of $(t, q, \dot{q}, p)$, Equation (3.8.3), or as a function of $(t, q, p)$, Equation (3.8.5), the total differential of those two functions must coincide, i.e.,

$$\begin{aligned} dH'(t, q, \dot{q}, p) &= \dot{q}^k\, dp_k - \frac{\partial L}{\partial q^k}\, dq^k - \frac{\partial L}{\partial t}\, dt \\ &\equiv dH(t, q, p) \\ &= \frac{\partial H}{\partial q^k}\, dq^k + \frac{\partial H}{\partial p_k}\, dp_k + \frac{\partial H}{\partial t}\, dt \end{aligned} \tag{3.8.9}$$

Therefore, the following identities, involving the old and new functions[64]

$$\frac{\partial L}{\partial q^k} = -\frac{\partial H}{\partial q^k}, \tag{3.8.10a}$$

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}, \tag{3.8.10b}$$

$$\dot{q}^k = \tilde{g}^k(t, q, p) = \frac{\partial H}{\partial p_k} \tag{3.8.10c}$$

hold everywhere in their respective regions of definition.

4. Under assumptions (3.8.1b) and (3.8.1c), the new function $H(t, q, p)$ induced by the Legendre transform has the same continuity properties of $L$, but in the "image" region $R^{2n+1}$, i.e.,

$$H \in \mathscr{C}^4(\tilde{R}^{2n+1}) \tag{3.8.11}$$

[63] Notice, in this respect, that prescriptions (3.8.2) are of *nonintegrable* type, i.e., functions $G'_k(t, q, p)$ such that $G_k = \dot{G}'_k$ do not exist. Therefore, they cannot be used to decrease the number of independent phase space coordinates. This case is somewhat similar to that of nonholonomic velocity constraints (Section A.3), which, since they are nonintegrable by assumption, do not affect the independence of the $q$ coordinates. For degenerate systems, the above independence of $q_k$ and $p_k$ is lost due to the appearance of constraints. See Dirac (1964).

[64] In view of the structure of Equations (3.8.10), the variables $t$ and $q_k$ are sometimes called *passive variables*, and the variables $\dot{q}_k$ and $p_k$ are called *active variables*.

Indeed, all left-hand sides of Equations (3.8.10) are of (at least) class $\mathscr{C}^3$. Then $\partial H/\partial t$, $\partial H/\partial q^k$, and $\partial H/\partial p_k$ are of class $\mathscr{C}^3$ and, thus, $H$ is of class $\mathscr{C}^4(\tilde{R}^{2n+1})$. In particular, when $L$ does not depend explicitly on time, neither does $H$.

5. Condition (3.8.1c) implies that

$$\left|\frac{\partial^2 H}{\partial p_i\,\partial p_j}\right|(\tilde{R}^{2n+1}) \neq 0. \tag{3.8.12}$$

Indeed, by differentiating equation (5.8.10c) with respect to $\dot{q}^j$ we obtain[65]

$$\delta^i_j = \frac{\partial^2 H}{\partial p_i\,\partial p_k}\frac{\partial^2 L}{\partial \dot{q}^k\,\partial \dot{q}^j}, \tag{3.8.13}$$

where we have used Equation (3.8.2). Therefore,

$$\left|\frac{\partial^2 H}{\partial p_i\,\partial p_j}\right|(\tilde{R}^{2n+1}) = \left\{\left|\frac{\partial^2 L}{\partial \dot{q}^i\,\partial \dot{q}^j}\right|(R^{2n+1})\right\}^{-1} \tag{3.8.14}$$

and property (3.8.12) follows from assumptions (3.8.1c).[66] Thus, we can say that property (3.8.12) is the phase space "image" of condition (3.8.1c) under a Legendre transform and, as such, it can be used equivalently to define the regularity of the represented system[67]

6. Condition (3.8.1c) implies that the Hamiltonian as defined by Equation (3.8.3) cannot identically vanish. Indeed, suppose that

$$\frac{\partial L}{\partial \dot{q}^k}\dot{q}^k - L \equiv 0. \tag{3.8.15}$$

Then, differentiating with respect to $\dot{q}^j$, we obtain

$$\frac{\partial^2 L}{\partial \dot{q}^k\,\partial \dot{q}^j}\dot{q}^k = 0, \tag{3.8.16}$$

which can hold for $\dot{q}^k \neq 0$ if and only if

$$\left|\frac{\partial^2 L}{\partial \dot{q}^k\,\partial \dot{q}^j}\right| = 0. \tag{3.8.17}$$

[65] See, for instance, Rund (1966, p. 18).

[66] For degenerate Lagrangians, this procedure does not apply because all Equations (3.8.10c) cannot be defined with a conventional Legendre transform.

[67] In the following, we shall say that a Hamiltonian is *regular* or *degenerate* in a region $\tilde{R}^{2n+1}$ depending on whether condition (3.8.12) holds or not in this region. The problem of the behavior of the regularity condition of a Lagrangian or a Hamiltonian under transformations of the coordinates is studied in Santilli (1979). It is significant to recall here that the conventional canonical transformations do not necessarily preserve regularity conditions (3.8.12). As a result, the concept of regularity is customarily restricted to that of a Lagrangian only in the current literature on Analytic Mechanics. As we can see in Santilli (1979), the configuration space image of these regularity-violating transformations exists. As a result the concepts of regularity of a Lagrangian and a Hamiltonian, besides their equivalence within a fixed system of variables as indicated by Equation (3.8.14), will emerge equivalent even within the context of the transformation theory.

7. The Legendre transform (3.8.3) applies for an *arbitrary* functional dependence of the Lagrangian and, as such, is insensitive to whether the represented system is conservative or not.

Properties (1)–(7) above do not exhaust the methodological profile of the Legendre transform. A further property, which is particularly significant for our analysis, is that the Legendre transform is an *involution*, namely, a transform which, when applied twice, reproduces the original function. This implies that instead of first assigning a Lagrangian and then computing the Hamiltonian, one can inversely first assign a Hamiltonian and then compute the Lagrangian.

To avoid possible confusion, we shall call the transition from $L$ to $H$ the *direct Legendre transform*, and the transition from $H$ to $L$ the *inverse Legendre transform*.

Suppose that a Hamiltonian $H(t, q, p)$ satisfying the continuity and regularity properties

$$H \in \mathscr{C}^4(\tilde{R}^{2n+1}), \tag{3.8.18a}$$

$$\left|\frac{\partial^2 H}{\partial p_i \, \partial p_j}\right|(\tilde{R}^{2n+1}) \neq 0 \tag{3.8.18b}$$

is assigned. The inverse Legendre transform is characterized by

$$\tilde{G}^k(t, q, \dot{q}, p) = \dot{q}^k - \frac{\partial H}{\partial p_k} = 0, \tag{3.8.19a}$$

$$H \to L = \dot{q}^k p_k - H(t, q, p) = L'(t, q, \dot{q}, p), \tag{3.8.19b}$$

where Equations (3.8.19a) and (3.8.19b) are in lieu of Equations (3.8.2) and (3.8.1c), respectively.

It is easy to see that the "inverse" of properties (1)–(7) holds.

1′. The reduction of function (3.8.19b) to a form depending only on the variables $(t, q, \dot{q})$ demands the computation of *all* the implicit functions of Equation (3.8.19a), i.e.,

$$p_k = g_k(t, q, \dot{q}), \qquad k = 1, 2, \ldots, n. \tag{3.8.20}$$

Assumptions (3.8.18) then ensure the applicability of Implicit Function Theorem 1.1.2 to Equation (3.8.19a) with respect to $p_k$ and, thus, the existence, uniqueness, and single-valuedness of all implicit functions (3.8.20). In this case, instead of Equation (3.8.6), we have the functional determinant

$$\left|\frac{\partial \tilde{G}^i}{\partial p_j}\right| = \left|\frac{\partial^2 H}{\partial p_i \, \partial p_j}\right|, \tag{3.8.21}$$

and its regularity is ensured by assumptions (3.8.18b). Once all implicit functions (3.8.20) have been computed, their substitution in function (3.8.19b) produces the desired reduction, i.e.,

$$L = \dot{q}^k g_k - H(t, q, g) = L(t, q, \dot{q}). \tag{3.8.22}$$

In essence, for the direct transform one must turn prescriptions (3.8.2) "inside out" to compute functions (3.8.4). For the inverse transform an equivalent procedure applies. A point which must be stressed is that prescriptions (3.8.2) and (3.8.19a) are equivalent. More specifically, Equations (3.8.2) and (3.8.19a) are two different ways of writing the same equations. The former provide the implicit functions of the $p$ variables, while the latter express the same prescriptions but in terms of the implicit functions in the $\dot{q}$ variables. Then, Lagrangian (3.8.19b) is unique and the Legendre transform is involutive. Indeed, by applying first a direct and then an inverse Legendre transform (under the above continuity and regularity assumptions), one recovers the original function identically.[68] Notice the key role played again by regularity condition (3.8.18b) for the existence of all implicit functions (3.8.20).

2′. In view of the existence, uniqueness, and single-valuedness of all implicit functions (3.8.20), inverse transform (3.8.19) induces the one-to-one mapping

$$\tilde{R}^{2n+1} \to R^{2n+1} \tag{3.8.23}$$

of the region $\tilde{R}^{2n+1}$ of points $(t, q, p)$ to an "image" region $R^{2n+1}$ of points $(t, q, \dot{q})$. Again, the region $\tilde{R}^{2n+1}$ is unrestricted, and when $\tilde{R}^{2n+1}$ is the entire space (3.8.8b), the "image" region is the Kronecker product

$$M^{2n+1} = M_t \otimes M_q \otimes M_{\dot{q}}. \tag{3.8.24}$$

3′. The differentials of functions (3.8.19b) and (3.8.22) are again identical, i.e.,

$$\begin{aligned} dL'(t, q, \dot{q}, p) &= p_k \, d\dot{q}^k - \frac{\partial H}{\partial q^k} dq^k - \frac{\partial H}{\partial t} dt \\ &= dL(t, q, \dot{q}) \\ &= \frac{\partial L}{\partial q^k} dq^k + \frac{\partial L}{\partial \dot{q}^k} d\dot{q}^k + \frac{\partial L}{\partial t} dt. \end{aligned} \tag{3.8.25}$$

The properties

$$\frac{\partial H}{\partial q^k} = -\frac{\partial L}{\partial q^k}, \tag{3.8.26a}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \tag{3.8.26b}$$

$$p_k = g_k(t, q, \dot{q}) = \frac{\partial L}{\partial \dot{q}^k}, \tag{3.8.26c}$$

[68] We here ignore the degrees of freedom of the functions $L$ and $H$ represented by multiplicative and/or additive numerical constants.

then hold identically. In particular, Equation (3.8.26c) guarantees that original prescriptions (3.8.2) are indeed recovered.

4′. The function $L$, from Equations (3.8.26) and assumption (3.8.18a), is of class $\mathscr{C}^4$ in $R^{2n+1}$. When $H$ is independent of time, so is $L$.

5′. Condition (3.8.18b) and prescription (3.8.19a) ensure that the Lagrangian $L$ is regular in $R^{2n+1}$. Indeed, by differentiating Equation (3.8.26c) with respect to $p_j$, we obtain

$$\delta_i^j = \frac{\partial^2 L}{\partial \dot{q}^i \, \partial \dot{q}^k} \frac{\partial^2 H}{\partial p_k \, \partial p_j}. \tag{3.8.27}$$

Then Equation (3.8.14) follows and property (3.8.1c) holds from assumption (3.8.18b).

6′. Under assumptions (3.8.18), the Lagrangian cannot be identically null, for suppose that

$$\frac{\partial H}{\partial p_k} p_k - H = 0. \tag{3.8.28}$$

Then, by differentiating with respect to $p_j$, we obtain

$$\frac{\partial^2 H}{\partial p_k \, \partial p_j} p_k = 0, \tag{3.8.29}$$

which cannot hold for $p_k \neq 0$ unless assumption (3.8.18b) is violated. Notice, in this respect, that equivalent regularity condition (3.8.1c) or (3.8.18b) implies that the Lagrangian cannot be linear in the velocities and the Hamiltonian cannot be linear in the momenta.[69]

7′. The inverse transform (3.8.19) is also insensitive to the nature of the acting forces and, thus, it applies irrespective of whether the considered Hamiltonian represents a conservative, dissipative, or dynamical system.

On practical grounds, when a class $\mathscr{C}^4$ regular Lagrangian $L$ has been assigned and a direct Legendre transform is requested, the following sequential steps can be implemented.

1. Identify a region $R^{2n+1}$ which contains no zero of the functional determinant.
2. Introduce prescriptions (3.8.2), compute all the implicit functions (3.8.4), and identify the image region $\tilde{R}^{2n+1}$ of $R^{2n+1}$.
3. Introduce the direct transform (3.8.3) and compute the Hamiltonian through the use of implicit functions (3.8.4).

There is no need to inspect all other aspects [e.g., the one-to-one nature of mapping (3.8.8a), the involutive character of the transform, etc.] because they are guaranteed by the same continuity and regularity assumptions.

[69] In essence, this reflects the structure of the kinetic energy for regular systems.

If the Lagrangian is of class $\mathscr{C}^m$, $m = 2, 3$, and regular, all the properties indicated in this section apply in full without any modification. Indeed, in this case from Hilbert Differentiability Theorem 3.2.2, the accelerations $\ddot{q}_k$ are continuous for all possible paths of at least class $\mathscr{C}^1$; Lagrange's equations are continuous; the generalized momenta $p_k$, from Equation (3.8.2), are of class $\mathscr{C}^{m-1}$; and the Implicit Function Theorem 1.1.2 can be applied to prescription (3.8.2). Thus, the direct Legendre transform holds.

The case when $L$ is only of class $\mathscr{C}^1$ will not be considered. It generally implies the breakdown of the above indicated properties of the Legendre transform due to the fact that the functions $G_k$ of Equation (3.8.2) fail to possess the continuity properties needed for the applicability of Theorem 1.1.2. Then the implicit functions (3.8.4), when they exist, are not necessarily unique.

Therefore, the minimal continuity property of the Lagrangian, which we shall assume for the validity of the direct Legendre transform, is $L \in \mathscr{C}^2(R^{2n+1})$.

If a Hamiltonian $H(t, q, p)$, which is of at least class $\mathscr{C}^2$ and regular[70], is assigned and an inverse Legendre transform is requested, then steps (a), (b), and (c) can be equivalently implemented. Again, if $H$ is of class $\mathscr{C}^m$, $m < 2$ and/or degenerate, then the inverse Legendre transform in its conventional formulation does not apply.

For illustrations of the above properties, see the examples at the end of this chapter [as well as those of Santilli (1979)].

The most significant aspects studied in this section can be summarized with the following theorem.

**Theorem 3.8.1** (Direct and Inverse Legendre Transforms). *Given a Lagrangian $L(t, q, \dot{q})$ satisfying the continuity and regularity properties*

$$L \in \mathscr{C}^m(R^{2n+1}), \qquad m \geq 2, \tag{3.8.30a}$$

$$\left| \frac{\partial^2 L}{\partial \dot{q}^i \, \partial \dot{q}^j} \right| (R^{2n+1}) \neq 0, \tag{3.8.30b}$$

*in a region $R^{2n+1}$ of points $(t, q, \dot{q})$, the direct Legendre transform*

$$G_k(t, q, \dot{q}, p) = p_k - \frac{\partial L}{\partial \dot{q}^k} = 0, \tag{3.8.31a}$$

$$H = p_k \dot{q}^k - L = H(t, q, p), \tag{3.8.31b}$$

*induces a one-to-one mapping from $R^{2n+1}$ to an "image" region $\tilde{R}^{2n+1}$ of points $(t, q, p)$ and defines a unique non-null new function, the Hamiltonian $H(t, q, p)$, which satisfies the continuity and regularity properties*

$$H \in \mathscr{C}^m(\tilde{R}^{2n+1}), \qquad m \geq 2, \tag{3.8.32a}$$

$$\left| \frac{\partial^2 H}{\partial p_i \, \partial p_j} \right| (\tilde{R}^{2n+1}) \neq 0. \tag{3.8.32b}$$

[70] See footnote 67 of p. 160.

*Conversely, given a Hamiltonian $H(t, q, p)$ which satisfies properties* (3.8.32) *in a region $\tilde{R}^{2n+1}$ of points $(t, q, p)$, the inverse Legendre transform*

$$\tilde{G}^k(t, q, \dot{q}, p) = \dot{q}^k - \frac{\partial H}{\partial p_k} = 0, \tag{3.8.33a}$$

$$L = \dot{q}^k p_k - H = L(t, q, \dot{q}) \tag{3.8.34b}$$

*induces a one-to-one mapping from $\tilde{R}^{2n+1}$ to an "image" region $R^{2n+1}$ of points $(t, q, \dot{q})$ and defines a unique non-null new function, the Lagrangian $L(t, q, \dot{q})$, which satisfies continuity and regularity properties* (3.8.30).

**Corollary 3.8.1A.** *For functions that are of at least class $\mathscr{C}^2$ and regular, the Legendre transform is involutive.*

Theorem 3.8.1 provides the foundations for the equivalence of configuration and phase space formulations of class $\mathscr{C}^2$, regular, and holonomic systems which will be studied in the next section after introducing Hamilton's equations.

Corollary 3.8.1A emphasizes an aspect of central methodological significance according to which the Legendre transform characterizes the configuration and phase space formulations on equivalent footing, in the sense that each of those formulations can be assumed as "primary" and the others as "derived." This implies the existence of the following alternatives for the representation of regular holonomic Newtonian systems in terms of analytic equations.

*Alternative I*, which consists of first identifying a Lagrangian and then computing the Hamiltonian through the direct Legendre transform.

*Alternative II*, which consists of first identifying a Hamiltonian and then computing the Lagrangian through the inverse Legendre transform.[71]

In relation to alternative I, the identification of a Lagrangian for self-adjoint systems has been studied in Section 3.5 through 3.7. [The case of non-self-adjoint systems is studied in Santilli (1979)]. Once a Lagrangian has been identified with those techniques, then the methodology of this section can be used to compute the Hamiltonian.

In relation to alternative II, we shall study in the rest of this chapter [also see Santilli (1979)] the prior identification of a Hamiltonian. The methodology of this section can then be used to compute a Lagrangian.

[71] There exists another alternative, which consists of the independent identification of both the Lagrangian and the Hamiltonian. Notice that the functions $L$ and $H$ so obtained are not necessarily related by a Legendre transform (even though they represent the same system by construction) in view of the freedom of the prescriptions for the construction of the normal forms. Therefore, this third alternative is significant for the study of equivalent analytic representations [Santilli (1979)].

Predictably, the two alternatives above will turn out to be equivalent. Nevertheless, their independent study is significant (e.g., for transformation theory) because their equivalence indicates the existence of certain "degrees of freedom" of analytic formulations, which are somewhat "hidden" in the conventional approach for conservative systems.

## 3.9 The Conventional Hamilton's Equations

We consider now the *conventional analytic equations in phase space*, i.e., *Hamilton's equations*

$$\begin{cases} \dot{q}^k - \dfrac{\partial H}{\partial p_k} = 0, & \text{(3.9.1a)} \\ \dot{p}_k + \dfrac{\partial H}{\partial q^k} = 0, & \text{(3.9.1b)} \end{cases}$$

$$H \in \mathscr{C}^m(\tilde{R}^{2n+1}), \qquad m \geq 2, \tag{3.9.1c}$$

$$\left| \frac{\partial^2 H}{\partial p_i \, \partial p_j} \right| (\tilde{R}^{2n+1}) \neq 0, \tag{3.9.1d}$$

which can be derived, for instance, from prescriptions (3.8.2), properties (3.8.10c), and Lagrange's equations.

Let us recall from the Hilbert Differentiability Theorem 3.2.2 that when a path $E$ is of (at least) class $\mathscr{C}^1$ in $(t_1, t_2)$ and the Lagrangian is regular and of class $\mathscr{C}^m$, $m \geq 2$ in $R^{2n+1}$, then $E$ is also a class $\mathscr{C}^m$ in $(t_1, t_2)$.

**Theorem 3.9.1** (Relationship between Paths in Configuration and Phase Space).[72] *If the path*

$$E = \{q, p \mid q^k = q^k(t), \quad p_k = p_k(t), t \in (t_1, t_2), \quad k = 1, 2, \ldots, n\} \tag{3.9.2}$$

*in configuration space is a class* $\mathscr{C}^m(t_1, t_2)$, $m \geq 2$, *solution of Lagrange's equations*

$$\left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \right) (E) = 0, \tag{3.9.3a}$$

$$L \in \mathscr{C}^m(R^{2n+1}), \qquad m \geq 2 \tag{3.9.3b}$$

$$\left| \frac{\partial^2 L}{\partial \dot{q}^i \, \partial \dot{q}^j} \right| (R^{2n+1}), \tag{3.9.3c}$$

[72] See, for instance, Gelfand and Fomin (1963, Chapter 4).

*then the image path*

$$\tilde{E} = \left\{ q, p \,|\, q^k = q^k(t),\ p_k = p_k = p_k(t),\ t \in (t_2, t_2),\ p_k = \frac{\partial L}{\partial \dot{q}^k},\ k = 1, 2, \ldots, n \right\} \tag{3.9.4}$$

*induced by a direct Legendre transform is a class* $\mathscr{C}^m(t_1, t_2)$ *solution of Hamilton's equations* (3.9.1) *and vice versa.*

PROOF. Every class $\mathscr{C}^m (m \geq 2)$ path $E$ that satisfies Equations (3.9.3) also satisfies equations of the normal forms (3.8.4), i.e.,

$$\dot{q}^k = \tilde{g}^k(t, q, p) = \frac{\partial H}{\partial p_k}. \tag{3.9.5}$$

But Equations (3.9.3) under a direct Legendre transform become the second set of Hamilton's equations (3.9.1b). Thus, the image path $\tilde{E}$ of $E$ satisfies all Hamilton's equations. The functions $\partial H/\partial q^k$ and $\partial H/\partial p_k$ are of class $\mathscr{C}^{m-1}$ and, therefore, the path $\tilde{E}$ is of class $\mathscr{C}^m$. This proves the first part of the theorem. Conversely, if $\tilde{E}$ satisfies Equation (3.9.1b) and is of class $\mathscr{C}^m$, $m \geq 2$, then an inverse Legendre transform applied to Equations (3.9.1) generates Lagrange's equations (3.9.3). But the functions $\partial H/\partial q^k$ are of class $\mathscr{C}^{m-1}$. Thus the image path $E$ of $\tilde{E}$ satisfies Lagrange's equations and is of class $\mathscr{C}^m(t_1, t_2)$. *Q.E.D.*

The involutive nature of the Legendre transform then implies the following corollary.

**Corollary 3.9.1A.** *Lagrange's equations* (3.8.3) *and Hamilton's equations* (3.9.1) *are equivalent when their respective Lagrangian and Hamiltonian functions are connected by a Legendre transform.*

The equivalence property of the above corollary can also be proved in several other ways, but we shall not indulge in their analysis at this time.

In essence, Theorems 3.8.1 and 3.9.1 and Corollaries 3.8.1A and 3.9.1A prove that, for the study of regular Newtonian systems, one can equivalently use the analytic equations in either configuration space or phase space.

The problem of the existence of a Lagrangian has been investigated in Section 3.5. When a Lagrangian exists, the consequent existence of a Hamiltonian has been stressed in Section 3.8. However, one of our objectives is the study of the problem of the existence of a Hamiltonian per se, i.e., *independently* from the existence of a Lagrangian. This latter problem demands a more effective formulation of Hamilton's equations.

Introduce the $2n$-component contravariant vector

$$a^\mu = \begin{cases} q^\mu, \mu = 1, 2, \ldots, n, \\ p_{\mu-n}, \mu = n+1, n+2, \ldots, 2n, \end{cases} \tag{3.9.6}$$

which now spans a phase space by assumption. Then Hamiltonian $H$ can be written $H(t, q, p) = H(t, a) = H(t, a^\mu)$, and through the use of the contravariant form[73]

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & +1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{3.9.7}$$

we can write

$$\begin{pmatrix} +\dfrac{\partial H}{\partial p_k} \\ -\dfrac{\partial H}{\partial q^k} \end{pmatrix} = \left(\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}\right). \tag{3.9.8}$$

Therefore, Hamilton's equations (3.9.1) can be written in the form

$$\dot{a}^\mu - \omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} = 0, \qquad \mu = 1, 2, \ldots, 2n, \tag{3.9.9}$$

which we shall call the *contravariant normal form of the analytic equations in phase space.*

By introducing the inverse matrix

$$(\omega_{\mu\nu}) = (\omega^{\mu\nu})^{-1} = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ +1_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{3.9.10}$$

and the covariant $2n$-vector

$$a_\mu \equiv \omega_{\mu\nu}a^\nu = \begin{cases} -p_\mu, \mu = 1, 2, \ldots, n, \\ q^{\mu-n}, \mu = n+1, n+2, \ldots, 2n, \end{cases} \tag{3.9.11}$$

we obtain, from Equations (3.9.9), the system

$$\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu} = 0, \tag{3.9.12}$$

which we shall call the *covariant normal form of the analytic equations in phase space.*

Trivially, Equation (3.9.12) in conventional notation reads

$$\begin{aligned} -\dot{p}_k - \frac{\partial H}{\partial q_k} &= 0, \\ +\dot{q}^k - \frac{\partial H}{\partial p_k} &= 0, \end{aligned} \tag{3.9.13}$$

[73] This introduces a symplectic structure in the space $\tilde{M}_{(q,p)} = \tilde{M}_{(a)}$. See the Charts of Chapter 2.

and, as such, they are the conventional Hamilton's equations, only written in a unified notation[74]

By introducing the notation

$$H'(t, a_\mu) = H'(t, \omega_{\mu\nu}a^\nu) = H(t, a^\mu),$$

$$\left(\frac{\partial H'}{\partial a_\mu}\right) = \begin{pmatrix} -\dfrac{\partial H}{\partial p_k} \\ \dfrac{\partial H}{\partial q^k} \end{pmatrix} = -\left(\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}\right), \tag{3.9.14}$$

Equations (3.9.9) and (3.9.12) can also be written in the factorized form

$$\omega^{\mu\nu}\left(\dot{a}_\nu - \frac{\partial H}{\partial a^\nu}\right) = 0, \tag{3.9.15a}$$

$$\omega_{\mu\nu}\left(\dot{a}^\nu + \frac{\partial H'}{\partial a_\nu}\right) = 0, \tag{3.9.15b}$$

which more transparently exhibits the "cross-wise" correspondence of the contravariant (covariant) tensor $\omega^{\mu\nu}(\omega_{\mu\nu})$ with the covariant (contravariant) normal[75] form of the canonical equations.

## 3.10 Self-Adjointness of the Conventional Hamilton's Equations

We are now equipped to study the self-adjointness of the conventional Hamilton's equations.

Consider the one-parameter family $\tilde{\tau}^1_{(t,a)}$ of *contravariant paths*

$$\tilde{E}(w) = \{a^\mu | a^\mu = a^\mu(t; w); t \in (t_1, t_2), w \in 0_\varepsilon\} \tag{3.10.1}$$

which are solutions of the covariant canonical equations

$$F_\mu(\tilde{E}) = \left(\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right)(\tilde{E}(w)) = 0, \qquad \mu = 1, 2, \ldots, 2n. \tag{3.10.2}$$

Suppose that the family $\tilde{\tau}^1_{(t,a)}$ satisfies all necessary theorems of the existence theory in such a way that the *contravariant variations*

$$\eta^\mu = \left.\frac{\partial a^\mu}{\partial w}\right|_{w=0}, \qquad \dot{\eta}^\mu = \left.\frac{\partial \dot{a}^\mu}{\partial w}\right|_{w=0}, \tag{3.10.3}$$

exist and are continuous in $(t_1, t_2)$.

[74] Notice that Equation (3.9.12) directly exhibits the fundamental symplectic structure $\omega$ (Chart 2.3), while Equations (3.9.13) do not, even though the two systems coincide. This difference in the way of writing Hamilton's equations, besides its geometrical significance, will be crucial for the study of the self-adjointness properties.

[75] Here, we stress the point that Hamilton's equations, from the viewpoint of the theory of ordinary differential equations, constitute a system of $2n$ first-order differential equations in the *normal form* (in the sense of footnote 6, Chapter 1).

By computing the total derivative of Equations (3.10.2) with respect to $w$ and by setting $w = 0$, we obtain the equations

$$M_\mu(\eta) = \frac{dF_\mu}{dw}\bigg|_{w=0} = \omega_{\mu\nu}\dot{\eta}^\nu - \sigma_{\mu\nu}\eta^\nu = 0, \tag{3.10.4}$$

where

$$\sigma_{\mu\nu} = \frac{\partial^2 H}{\partial a^\mu\, \partial a^\nu}\bigg|_{w=0}, \tag{3.10.5}$$

which are the *equations of variations of Hamilton's equations.* If the forms $M_\mu(\eta)$ are computed along an (admissible) variation $\eta^\mu$ which is not necessarily a solution of Equation (3.10.4), we shall call them the *variational forms of Hamilton's equations.*

**Theorem 3.10.1** (Self-Adjointness of Hamilton's Equations).[76] *Under the assumption that the Hamiltonian $H(t, a^\mu)$ is of (at least) class $\mathscr{C}^2$ and regular in a region $\tilde{R}^{2n+1}$ of points $(t, a^\mu)$, the covariant normal form of Hamilton's equations is always self-adjoint in $R^{2n+1}$.*

PROOF. The equations of variations (3.10.4), under the assumed continuity conditions, always coincide with the adjoint system (Section 2.1)

$$\tilde{M}_\mu(\eta) = -\dot{\eta}^\nu\omega_{\nu\mu} - \eta^\nu\sigma_{\nu\mu} = 0 \tag{3.10.6}$$

everywhere in $\tilde{R}^{2n+1}$. *Q.E.D.*

In essence, the identity

$$M_\mu(\eta) \equiv \tilde{M}_\mu(\eta), \qquad \eta \in \mathscr{C}^2(t_1, t_2), \qquad \mu = 1, 2, \ldots, 2n \tag{3.10.7}$$

is based on the following assumptions.

1. The form $(\omega_{\mu\nu})$ is independent of the path $a^\alpha$ and of the time $t$. This guarantees that in the construction of the adjoint, the identities

$$\frac{d}{dt}(\eta^\nu\omega_{\nu\mu}) \equiv \dot{\eta}^\nu\omega_{\nu\mu} \tag{3.10.8}$$

   hold.
2. The form $(\omega_{\mu\nu})$ is antisymmetric. This guarantees the identities of the following terms of Equations (3.10.4) and (3.10.6).

$$\omega_{\mu\nu}\dot{\eta}^\nu = -\dot{\eta}^\nu\omega_{\nu\mu}. \tag{3.10.9}$$

[76] This theorem, to the best of my knowledge, is not treated in the available literature.

3. The Hamiltonian is of (at least) class $\mathscr{C}^2$ $(\tilde{R}^{2n+1})$. This guarantees the last identities needed for Equations (3.10.7), i.e.,

$$\begin{aligned} \sigma_{\mu\nu} &= \frac{\partial^2 H}{\partial a^\mu \, \partial a^\nu} \\ \equiv \sigma_{\nu\mu} &= \frac{\partial^2 H}{\partial a^\nu \, \partial a^\mu} \end{aligned} \tag{3.10.10}$$

Alternatively, Theorem 3.10.1 can be proved by verifying that Equations (3.10.2) satisfy the conditions of self-adjointness (2.7.20), i.e., for a covariant normal form

$$\omega_{\mu\nu}\dot{a}^\nu - \Xi_\mu(t, a^\alpha) = 0, \tag{3.10.11}$$

that is,

$$\frac{\partial \Xi_\mu}{\partial a^\nu} = \frac{\partial \Xi_\nu}{\partial a^\mu} \tag{3.10.12}$$

(see Theorem 2.7.3). But the functions $\Xi_\mu$ for Equations (3.10.2) are given by

$$\Xi_\mu = \frac{\partial H}{\partial a^\mu}, \tag{3.10.13}$$

thus condition (3.10.12) trivially reduce to the "commutativity" property (3.10.10).

The study of the self-adjointness of the contravariant normal form (3.9.15a) with respect to the covariant vector $a_\alpha$ can be done with simple modifications of the above procedure. It is left as an exercise for the interested reader.

Notice that the self-adjointness of Hamilton's equations can be proved with the minimal continuity conditions that $H \in \mathscr{C}^2(\tilde{R}^{2n+1})$. This should be compared with the corresponding aspect in configuration space for which the minimal continuity property is $L \in \mathscr{C}^4$ $(R^{2n+1})$.

By introducing the quantities

$$\eta^\mu = \begin{cases} \left.\dfrac{\partial q^\mu}{\partial w}\right|_{w=0} = \alpha^\mu, \ \mu = 1, 2, \ldots, n, \\ \left.\dfrac{\partial p_{\mu-n}}{\partial w}\right|_{w=0} = \beta_\mu, \ \mu = n+1, n+2, \ldots, 2n, \end{cases} \tag{3.10.14}$$

Equation (3.10.4) can be written in conventional notation:

$$\dot{\alpha}^k = \frac{\partial^2 H}{\partial q^i \, \partial p_k}\alpha^i + \frac{\partial^2 H}{\partial p_i \, \partial p_k}\beta_i, \tag{3.10.15a}$$

$$\dot{\beta}_k = -\frac{\partial^2 H}{\partial q^i \, \partial q^k}\alpha^i - \frac{\partial^2 H}{\partial p_i \, \partial q^k}\beta_i. \tag{3.10.15b}$$

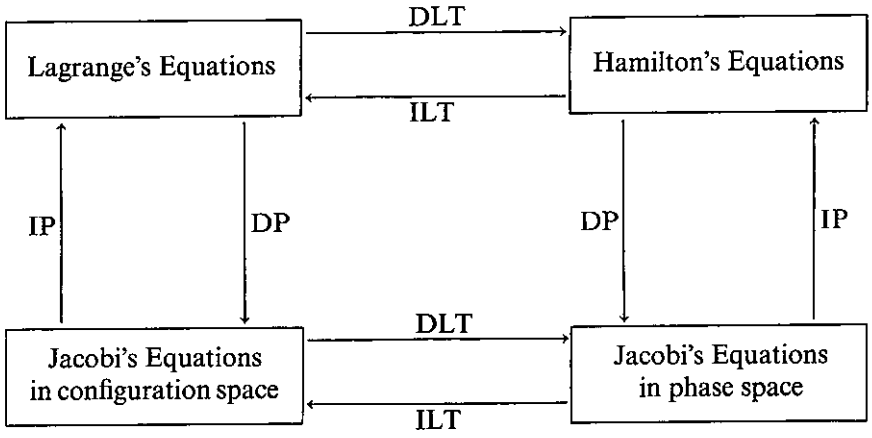


**Figure 3.3** A schematic view of the *fundamental analytic equations of Newtonian Mechanics, their equations of variations, and their relationships*, where DLT = direct Legendre transform, ILT = inverse Legendre transform, DP = derivation with respect to the parameters, and IP = integration with respect to the parameters. Notice that the above diagram is closed and invertible. The analysis of this book is centered on the fact that the *joint use* of the analytic equations and their equations of variations, with the underlying self-adjointness properties (Theorem 3.3.1 and 3.10.1), allows the study of the necessary and sufficient conditions for the existence of a Lagrangian or Hamiltonian (Theorems 3.5.1 and 3.12.1).

Furthermore, by introducing the function

$$\Omega(t, \alpha, \beta) = \frac{1}{2}\left(\frac{\partial^2 H}{\partial q^i\, \partial q^j}\alpha^i\alpha^j + 2\frac{\partial^2 H}{\partial q^i\, \partial p_j}\alpha^i\beta_j + \frac{\partial^2 H}{\partial p_i\, \partial p_j}\beta_i\beta_j\right)$$
$$= \Omega(t, \eta), \tag{3.10.16}$$

Equation (3.10.4) can be written in the form

$$\dot{\eta}^\mu - \omega^{\mu\nu}\frac{\partial \Omega}{\partial \eta^\nu} = 0, \qquad \mu = 1, 2, \ldots, 2n, \tag{3.10.17}$$

which exhibits a striking resemblance to the conventional Hamilton's equations.

Equations (4.10.15) or (3.10.17) (sometimes referred to as *Jacobi's equations in phase space*) emerge within the framework of the calculus of variations for the so-called *accessory problem in canonical formulations.*[77]

It should be noted that Jacobi's equations in configuration and phase space can also be related through a direct or inverse Legendre transform. The verification of this important property is left to the interested reader (Problem 3.7).

[77] See, for instance, Bliss (1946).

Finally, the reader should be aware that the direct proof of the self-adjointness of Hamilton's equations demands the use of the tensor formulation, e.g., the formulation of Hamilton's equations in the form (3.10.2). It is an instructive exercise for the interested reader to see that the proof of this central property of Hamilton's equations for the conventional form (3.9.1) is rather laborious. This fact has a number of implications for variational principles (see Charts 3.6 and 3.7), as well as for the transformation theory [see Santilli (1978)].

For a schematic view of the fundamental analytic equations, see Figure 3.3.

## 3.11 The Concept of Analytic Representation in Phase Space

The fundamental and kinematical forms of Newton's equations of motion in configuration space, even though they are particularly significant for the problem of their Lagrangian representation, do not exhaust all possible forms of writing the equations of motion. Clearly, within the context of the problem of the Hamiltonian representation, the forms of the equations of motions that are particularly significant are the first-order forms.

Let us recall from the analysis of Sections 2.4 and 2.5 that, starting from the equations of motion in configuration space as they naturally arise from Newton's second law, i.e., the equations of the form

$$A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0, \tag{3.11.1a}$$

$$A_{ki}, B_k \in \mathscr{C}^1(R^{2n+1}), \tag{3.11.1b}$$

$$|A_{ij}|(R^{2n+1}) \neq 0, \tag{3.11.1c}$$

one can reduce these equations to first-order forms, without using Lagrange's equations, by introducing the generally *noncanonical prescriptions* (2.4.2) in $n$ new variables $y_k$, i.e.,

$$G_k(t, q, \dot{q}, y) = \alpha_{ki}(t, q, y)\dot{q}^i + \beta_k(t, q, y) = 0, \tag{3.11.2a}$$

$$\alpha_{ki}, \beta_k \in \mathscr{C}^1(\tilde{R}^{2n+1}), \quad |\alpha_{ij}|(\tilde{R}^{2n+1}) \neq 0. \tag{3.11.2b}$$

The reduction of Equations (3.11.1) in $\ddot{q}^k$ to equations in $\dot{y}_k$ then renders the system of $n$ second-order equations equivalent to a system of $2n$ first-order equations of the type

$$\begin{aligned} &\alpha_{ki}(t, q, y)\dot{q}^i + \beta_k(t, q, y) = 0, \\ &\alpha'^i_k(t, q, y)\dot{y}_i + \beta'_k(t, q, y) = 0, \end{aligned} \tag{3.11.3}$$

which can be written in the general *covariant form* (2.5.3)

$$C_{\mu\nu}(t, a)\dot{a}^\nu + D_\mu(t, a) = 0, \tag{3.11.4a}$$

$$C_{\mu\nu}, D_\mu \in \mathscr{C}^1(\tilde{R}^{2n+1}), \tag{3.11.4b}$$

$$|C_{\mu\nu}|(\tilde{R}^{2n+1}) \neq 0, \tag{3.11.4c}$$

$$(a^\mu) = (q^k, y_k), \tag{3.11.4d}$$

or in the equivalent *contravariant normal form* (2.5.6)

$$\dot{a}^\mu - \Xi^\mu(t, a) = 0, \qquad \Xi^\mu \in \mathscr{C}^1(\tilde{R}^{2n+1}), \tag{3.11.5a}$$

$$\Xi^\mu = -C^{\mu\nu}D_\nu, \qquad (C^{\mu\nu}) = (C_{\mu\nu})^{-1}, \tag{3.11.5b}$$

or in the equivalent *general contravariant* form (2.6.25) or *covariant normal form* (2.5.14), i.e.,

$$C'^{\mu\nu}(t, a_\alpha)\dot{a}_\nu + D'^{\mu}(t, a_\alpha) = 0, \tag{3.11.6a}$$

$$\dot{a}_\mu - \Xi'_\mu(t, a_\alpha) = 0, \tag{3.11.6b}$$

$$\Xi'_\mu, C'^{\mu\nu}, D'^{\mu} \in \mathscr{C}^1(\tilde{R}^{2n+1}), \qquad |C'^{\mu\nu}|(\tilde{R}^{2n+1}) \neq 0, \tag{3.11.6c}$$

$$C'^{\mu\nu}(t, a_\alpha) = C^{\mu\nu}(t, a^\beta(a_\alpha)) = C'^{\mu\nu}(t, \omega_{\alpha\beta}a^\beta), \text{ etc.}, \tag{3.11.6d}$$

$$(a_\alpha) = (\omega_{\alpha\beta}a^\beta) = (-y_k, q^k). \tag{3.11.6e}$$

For the reader's convenience, let us also recall that: (1) at this stage the variables $y_k$ *do not* necessarily coincide with the generalized momenta $p_k$; (2) the form which directly arises from the reduction of system (3.11.1) with identifications (3.11.4d) is the *covariant* form (3.11.4a) in the *contravariant* variables $(a^\mu)$; (3) the application of the Theorem on the Implicit Functions to form (3.11.4) then naturally produces the *contravariant* normal form (3.11.5); (4) the remaining two forms (3.11.6a) and (3.11.6b) can be constructed through the use of identifications as in Equation (3.11.6e); and (5) the prime in the above expressions indicates a transition in the functional dependence of the considered functions from the contravariant to the covariant variables, e.g., Equation (3.11.6d).

Our problem is that of extending the concept of the analytic representation of Section 3.4 to the representation of the above first-order forms in terms of Hamilton's equation. This can be done according to the following definition.

**Definition 3.11.1.** A class $\mathscr{C}^1$, regular, holonomic system of Newton's equations of motion in a first-order contravariant (covariant) form admits an *analytic representation in terms of the contravariant (covariant) Hamilton's equation* in a region $\tilde{R}$ of the variables $(t, a)$ when there exist $4n^2$ functions $h'^\mu_\nu$ $(h^\nu_\mu)$ which are of (at least) class $\mathscr{C}^1$ in $\tilde{R}$ and whose matrix $(h)$ is regular in $\tilde{R}$, such that the conventional contravariant (covariant) Hamilton's equations coincide with the equations of motion up to the equivalence transform induced by such a matrix $(h)$, i.e.,

$$\omega^{\mu\nu}\dot{a}_\nu - \frac{\partial H'}{\partial a_\mu} \equiv h^\mu_\nu(C'^{\nu\rho}\dot{a}_\rho + D'^{\nu}), \tag{3.11.7a}$$

$$\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu} \equiv h^\nu_\mu(C_{\nu\rho}\dot{a}^\rho + D_\nu), \tag{3.11.7b}$$

$$\mu = 1, 2, \ldots, 2n,$$

or, equivalently, when the equations of motion coincide with the contravariant (covariant) Hamilton's equations up to an equivalence transform induced by the inverse matrix $(h^{-1}) = (h)^{-1}$, i.e.,

$$h'^{-1\mu}_{\nu}\left(\omega^{\nu\rho}\dot{a}_{\rho} - \frac{\partial H'}{\partial a_{\nu}}\right) \equiv C'^{\mu\rho}\dot{a}_{\rho} + D'^{\mu}, \tag{3.11.8a}$$

$$h^{-1\nu}_{\mu}\left(\omega_{\nu\rho}\dot{a}^{\rho} - \frac{\partial H}{\partial a^{\nu}}\right) \equiv C_{\mu\rho}\dot{a}^{\rho} + D_{\mu}. \tag{3.11.8b}$$

The representation is called *direct* (*indirect*) when the matrix $(h)$ is (is not) the unit matrix, and it is called *ordered* when identification (3.11.7) or (3.11.8) holds not only for the left-hand and right-hand sides considered as systems, but also member by member for all the values of the index $\mu = 1, 2, \ldots, 2n$ in a given ordering. When the latter ordering requirement is not met, the representation is termed *nonordered*.

The most significant case is again that of an "ordered direct analytic representation" which, in this case, consists of the identifications of the *normal* forms

$$\omega^{\mu\nu}\dot{a}_{\nu} - \frac{\partial H'}{\partial a_{\mu}} \equiv \omega^{\mu\nu}\dot{a}_{\nu} - \Xi'^{\mu}(t, a_{\alpha}), \tag{3.11.9a}$$

$$\omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial H}{\partial a^{\mu}} \equiv \omega_{\mu\nu}\dot{a}^{\nu} - \Xi_{\mu}(t, a^{\alpha}). \tag{3.11.9b}$$

A variational structure equivalent to that of Equations (3.4.11) then holds. For illustrations, see the examples at the end of this chapter [as well as those of Santilli (1979)].

Notice that: (a) the *factor functions* of Equations (3.11.7) or (3.11.8) have been selected in such a way as to preserve the contravariant or covariant nature of the system; (b) the maximal functional dependence of these functions is $h = h(t, a)$ because any dependence on the derivatives $\dot{a}$ would alter the first-order nature of the systems; and (c) in identities (3.11.7) and (3.11.8) there is not only the preservation of the overall tensor nature of the systems (e.g., contravariant systems are identified with contravariant analytic equations), but also the preservation of their functional dependence (e.g., the contravariant Hamilton's equations in the *covariant* variables $a_{\alpha}$ are identified with the contravariant equations of motion in the same covariant variables).

A comparative analysis with the framework of Section 3.4 reveals not only predictable similarities between the concepts of analytic representations in configuration and phase space, but also significant differences. In the former case, the implicit functions of the system are the functions $f_k(t, q, \dot{q})$, the underlying form is the kinematical form, and the ordered *direct* analytic representation of this form is a *particular case* of the more general representation of the fundamental form. In the latter case, the implicit functions of the system are the functions $\Xi_{\mu}(t, a)$, the underlying forms are normal forms

(3.11.5) and (3.11.6b), and their ordered *direct* representations in terms of Hamilton's equations *are not* a particular case of a more general representation. This difference is ultimately due to a difference in the structure of Lagrange's and Hamilton's equations. Indeed, unlike the case of Lagrange's equations, Hamilton's equations *are* equations in their normal form. As such, they can only directly represent systems in such a form. This difference between Lagrange's and Hamilton's equations will imply, as we shall see in the next section, a considerable simplification of the Inverse Problem in the transition from configuration to phase space formulations.

Another considerable difference between the analytic representations in configuration and phase space is constituted by their impact on the underlying representation spaces. In the former case, the underlying space (at the level of the equations of motion in their second-order form) is the configuration space of the generalized coordinates $q^k$. The existence of an analytic representation of such equations of motion in terms of Lagrange's equations *does not* affect the nature of such a space. In the latter case, the original underlying space of the equations of motion in their first-order form is the $2n$-dimensional space of the variables $(a^\mu) = (q^k, y_k)$, which *is not necessarily* a phase space. However, if an analytic representation of such equations exists, such a space is a phase space with $y_k \equiv p_k$. Therefore, the existence of an analytic representation in the $a$-variables implies the presence of a *canonical structure*. This occurrence has several geometrical implications.

As a final remark, we would like to point out that the concept of analytic representation according to Definition 3.11.1 indicates the existence of generalized forms of the conventional Hamilton's equations, i.e., the left-hand sides of identifications (3.11.8). [This aspect is studied in Santilli (1979)].

## 3.12 The Fundamental Analytic Theorem for Phase Space Formulations and a Method for the Independent Construction of a Hamiltonian

We are now equipped to formulate and prove the following important theorem

**Theorem 3.12.1** (Fundamental Analytic Theorem for Phase Space Formulations).[78] *A necessary and sufficient condition for a local, holonomic, generally nonconservative Newtonian system in the covariant normal form*

$$\begin{aligned} &\omega_{\mu\nu}\dot{a}^\nu - \Xi_\mu(t, a^\alpha) = 0, \\ &\mu = 1, 2, \ldots, 2n, \end{aligned} \tag{3.12.1}$$

[78] Despite a rather laborious search, I have been unable to identify any reference to this theorem in the mathematical or physical literature within the context of the variational approach to self-adjointness.

*which is well defined and of (at least) class $\mathscr{C}^1$ in a star-shaped region $\tilde{R}^{2n+1}$ of the variables $(t, a^\alpha)$, to admit an ordered direct analytic representation in terms of the covariant Hamilton's equations in $\tilde{R}^{2n+1}$*

$$\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu} \equiv \omega_{\mu\nu}\dot{a}^\nu - \Xi_\mu \tag{3.12.2}$$

*is that the covariant normal form is self-adjoint in $\tilde{R}^{2n+1}$.*

PROOF. Conditions of self-adjointness (2.7.20) are the integrability conditions of the differential form that underlies identifications (3.12.2). *Q.E.D.*

Explicitly, identifications (3.12.2) hold if and only if

$$\frac{\partial H}{\partial a^\mu} - \Xi_\mu(t, a^\alpha) = 0. \tag{3.12.3}$$

This is an overdetermined system of first-order partial differential equations in the unknown $H$. The underlying differential form is

$$\Xi^{(1)} = \Xi_\mu \, da^\mu. \tag{3.12.4}$$

In view of the Converse of the Poincaré Lemma 1.2.2, the integrability conditions for the existence of a primitive form are given by

$$\delta^{\nu_1\nu_2}_{\mu_1\mu_2} \frac{\partial \Xi_{\nu_1}}{\partial a^{\nu_2}} = 0, \qquad \mu_1, \mu_2 = 1, 2, \ldots, 2n, \tag{3.12.5}$$

and they coincide with conditions of self-adjointness of form (3.12.1) according to Theorem 2.7.3.

It is significant to compare the simplicity of the proof of Theorem 3.12.1 with the rather involved proof of Theorem 3.5.1. The problem of constructing the Hamiltonian, once its existence is guaranteed by Theorem 3.12.1, is also considerably simpler than the corresponding problem of the computation of a Lagrangian. Indeed, a simple reformulation of the proof of Theorem 3.12.1, as well as use of Equation (1.2.25), leads to the following theorem.

**Theorem 3.12.2** (A Method for the Construction of a Hamiltonian). *A Hamiltonian for the ordered direct analytic representation of holonomic, generally nonconservative Newtonian systems in the covariant normal form* (3.12.1), *which is well defined, of (at least) class $\mathscr{C}^1$, and self-adjoint in a star-shaped region $\tilde{R}^{2n+1}$ of points $(t, a^\alpha)$, is a solution of the overdetermined system of first-order partial differential equations* (3.12.3) *given by*

$$H = a^\mu \int_0^1 d\tau \, \Xi_\mu(t, \tau a^\alpha). \tag{3.12.6}$$

For illustrations of the above theorem, see the examples at the end of this chapter [as well as those of Santilli (1979)].

A few comments are in order. The first point which must be stressed is that Theorems 3.12.1 and 3.12.2 establish a methodology for the existence of analytic representations of Newtonian systems which, according to our derivation, is *independent* of any prior knowledge of a Lagrangian. Basically, according to the remarks at the end of Section 3.8, Theorem 3.6.1 is in line with the approach of first identifying a Lagrangian and then computing a Hamiltonian through a direct Legendre transform. Theorem 3.12.2 is along the lines of the opposite approach, namely, that of first identifying a Hamiltonian and then computing a Lagrangian by means of an inverse Legendre transform.

A rather intriguing implication of the Inverse Problem is that *the above two methods do not necessarily lead to the same Lagrangians and Hamiltonians*, in the sense that, given a Newtonian system in the self-adjoint fundamental and normal forms with corresponding Lagrangian $L$ and Hamiltonian $H$, the function $H(L)$ is not necessarily the Legendre transform of $L(H)$. This property is due to the fact that prescriptions (3.11.2) for the construction of a normal form *do not* necessarily coincide with canonical prescriptions (3.8.2) related to a given $L$. As a result, the emerging Hamiltonian is not necessarily the Legendre transform of $L$. For illustrations, see the examples at the end of this chapter.

This situation clearly offers a first hint of the significance of the Inverse Problem for transformation theory, which is investigated in more detail in Santilli (1979).

Given a system of Newton's equations of motion in their "natural" form, i.e., the fundamental form (3.11.1), the computation of a Hamiltonian according to the above independent method can be carried out according to the following steps.

1. *Construct a covariant normal form which incorporates arbitrary functions.* This can be achieved by using prescriptions (3.11.2), where the $n^2 + 1$ functions $\alpha_{ki}$ and $\beta_k$ are, at this stage, arbitrary. As a result, the implicit functions of the system will depend on these functions, i.e., $\Xi_\mu = \Xi_\mu(t, a^\alpha, \alpha_{ki}, \beta_k)$.
2. *Impose conditions of self-adjointness* (3.12.5) *to remove the functional arbitrariness of step* 1. This can be done by solving Equations (3.12.5) in the unknown functions $\alpha_{ki}$ and $\beta_k$ and, therefore, by reducing the $\Xi_\mu$ functions to a known functional dependence. Verify that the normal form so obtained is well defined and of at least class $\mathscr{C}^1$ in a star-shaped region of the variable.
3. *Compute the Hamiltonian by using integral* (3.12.6). There is no need to inspect the consistency of Equation (3.12.3) because the proof of Theorem 3.12.2 is centered precisely on the point that Equations (3.12.5) *are* the integrability conditions of Equation (3.12.3).

The possible non-uniqueness of the Hamiltonian then emerges in a self-evident way. Notice that a Hamiltonian does not necessarily exist within the

context of the above method. The problem of whether the method can be implemented to "always" yield a Hamiltonian is studied in Santilli (1979).

The remarks of Section 3.6 regarding the nature and use of a star-shaped rather than an ordinary region extend trivially to Theorem 3.12.2. In this respect, another method for the computation of $H$ (when its existence is ensured by the conditions of self-adjointness), which does not require the use of a star-shaped region, is given by the *Cauchy integral*[79]

$$H(t, a^\alpha) = \int_{a_0}^{a^\mu} da^\mu \, \Xi_\mu(t, a^\alpha = a_0^1, \ldots, a^{\mu-1} = a_0^{\mu-1}, a^\mu, a^{\mu+1}, \ldots, a^{2n}) \tag{3.12.7}$$

Clearly, if covariant normal form (3.12.1) violates only *one* of conditions of self-adjointness (3.12.5), then it is non-self-adjoint and a Hamiltonian for its ordered direct representation (3.12.2) *does not* exist. Again, if the ordering condition of Theorem 3.12.1 or 3.12.2 is removed, the condition of self-adjointness is only *sufficient* for the existence of a Hamiltonian.

The extension of the theorems of this section to contravariant normal forms is left as an exercise for the interested reader.

The crucial role of the $(\omega_{\mu\nu})$ matrix for the construction of the covariant normal form should be emphasized here. Indeed, if one uses, say, the tensor $\delta_{\mu\nu}$ as the lowering tensor of the $a^\mu$ variables, then the emerging normal form takes the structure

$$\delta_{\mu\nu}\dot{a}^\nu - \Xi_\mu(t, a^\alpha) = 0 \tag{3.12.8}$$

and, as such, it cannot be directly represented with Hamilton's equations. This indicates the need for using a symplectic structure in the space of the $a^\mu$ variables which, in turn, is deeply linked to the self-adjointness of Hamilton's equations and their canonical structure (Chart 3.18). In the final analysis,

*Theorem 3.12.1 provides necessary and sufficient conditions in local coordinates for a vector field to be globally Hamiltonian*, in the sense of Chart 2.3.

[79] By comparing the use of the Poincaré approach for the computation of a Hamiltonian, Equation (3.12.6), and the use of the Cauchy approach, Equation (3.12.7), one might have the impression that the freedom in the presence of the initial point $a_0^\mu$ exists only for the latter approach. This is not the case, because Equation (3.12.6) can be modified into the form

$$H = (a^\mu - a_0^\mu) \int_0^1 d\tau \, \Xi_\mu(t, \tau a^\alpha + (1 - \tau) a_0^\alpha).$$

I would like to express may appreciation to E. Engels for bringing this point to my attention. A corresponding method for the construction of a Lagrangian with the use of the Cauchy approach (and, thus, without the use of a star-shaped region) is given in Chart 3.11. In conclusion, the computation of a Lagrangian or a Hamiltonian with the use of the Cauchy approach constitutes a valuable alternative to that offered by the Converse of the Poincaré Lemma, particularly when the system considered is self-adjoint but violates the condition of being well behaved in a star-shaped region.

The extension of the analysis to the case of locally Hamiltonian vector fields is done in Santilli (1979) via the use of *Birkhoff's equations.*

The analysis of Section 3.7 can also be easily extended to the framework of this section with the equivalent result that *the conventional Hamilton's equations can represent Newton's equations of motion with arbitrary Newtonian forces, i.e., couplings of types I, II, and III.* The insensitivity of the conditions of self-adjointness to the nature of the acting forces is, within this context, even more evident owing to the considerable "degrees of freedom" that often occur in prescriptions (3.11.2). In this respect, it should be noted that, unlike the case of Equation (3.7.1), conditions of self-adjointness (3.12.5) *cannot* be interpreted as integrability conditions for the existence of a potential function, i.e., for the acting forces to be conservative. Indeed, such conditions are explicitly given by the $2n \times 2n$ expressions

$$\begin{pmatrix} \left(\frac{\partial \Xi_i}{\partial q^j}\right) & \left(\frac{\partial \Xi_i}{\partial p_j}\right) \\ \left(\frac{\partial \Xi_{i+n}}{\partial q^j}\right) & \left(\frac{\partial \Xi_{i+n}}{\partial p_j}\right) \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial \Xi_j}{\partial q^i}\right) & \left(\frac{\partial \Xi_{j+n}}{\partial q^i}\right) \\ \left(\frac{\partial \Xi_j}{\partial p_i}\right) & \left(\frac{\partial \Xi_{j+n}}{\partial p_i}\right) \end{pmatrix} \tag{3.12.9}$$

The statement then becomes self-evident, for instance, when prescriptions (3.11.2) reduce to the simple identities $\dot{q}_k = y_k$, by indicating in this way a possible explicit dependence of the $\Xi_\mu$ functions on the velocities. Obviously, conservative forces also occur as a subcase of conditions (3.12.9), e.g., when the off-diagonal elements are all identically null.

The analysis of the structure of the Hamiltonian capable of representing nonconservative systems can be conducted along lines parallel to those of Section 3.7 and leads to equivalent results. This analysis can also be carried out by studying the Legendre transform of the generalized structures of the Lagrangian introduced in Section 3.7. Consider, for instance, the case of a system of free particles subject to holonomic constraints and represented by the kinetic energy (3.7.25). Let $\tilde{T}(t, q, p)$ be the "image" of $T(t, q, \dot{q})$ under a Legendre transform. If all acting forces are derivable from an additive interaction term to the kinetic energy according to structure (3.7.26), the Hamiltonian will have the conventional structure

$$H_{\text{tot}} = \tilde{T} + \tilde{U}. \tag{3.12.10}$$

However, if some of the acting forces are not derivable from a potential, then the phase-space image of structure (3.7.27a) can be written

$$H_{\text{tot}}^{\text{gen}} = \tilde{R}\tilde{T} + \tilde{S}. \tag{3.12.11}$$

The nonconservative nature of the system is thus again reflected by the presence of terms which multiply the Hamiltonian for the free motion.

For a schematic view of the content of this section, see Figure 3.4. For a schematic view of the Inverse Problem, see Figure 3.5.

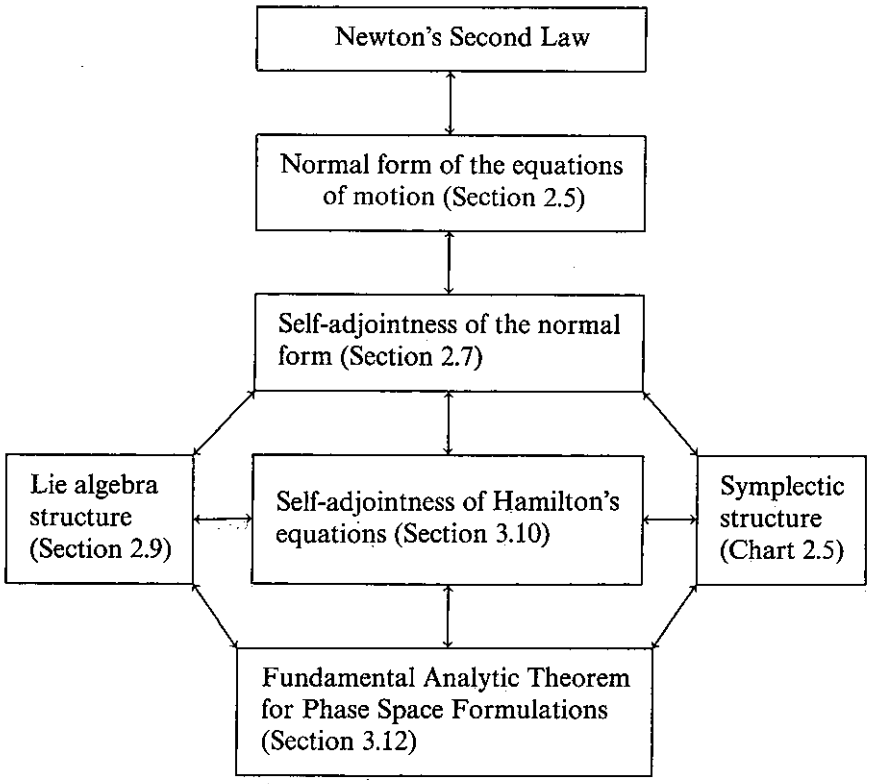


**Figure 3.4** A schematic view of the *Fundamental Analytic Theorem for Phase Space Formulations* treated independently from the corresponding theorem in configuration space. The equations of motion in their (first-order) normal form are constructed from Newton's second law through a set of prescriptions for the characterization of new variables. Whenever, for a suitable selection of such prescriptions, they satisfy each of the conditions of self-adjointness, they are self-adjoint. Hamilton's equations with class $\mathscr{C}^2$ Hamiltonians, on the contrary, are always self-adjoint. Similar to the corresponding situation in configuration space (Figure 3.1), the conditions of self-adjointness for first-order forms emerge again as the central mathematical tool for the independent treatment of the Inverse Problem in phase space. However, the underlying conditions are different from the corresponding ones in configuration space. Furthermore, such conditions now acquire a direct algebraic and geometric meaning because, from an algebraic viewpoint, they guarantee that the analytic brackets of the theory satisfy the Lie algebra axioms and, from a geometric viewpoint, they guarantee the presence of a symplectic structure in the space of the $(q, p)$ variables. The emerging methodological context is, therefore, centered on a synthesis of primitive concepts of three interrelated disciplines, i.e., analytic mechanics, abstract algebras, and differential geometry (Chart 3.18), with implications particularly for transformation theory, symmetries, and conserved quantities of systems with arbitrary couplings [as can be seen in Santilli (1979)].

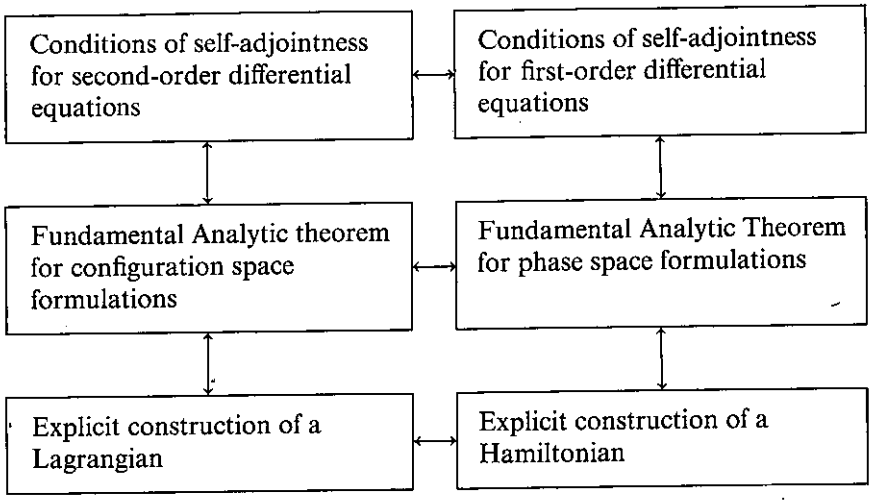


**Figure 3.5** A schematic view of the *Inverse Problem in Newtonian Mechanics.* The problem consists of the independent methods for the construction of a Lagrangian or Hamiltonian from given equations of motion. The methods are centered on the concept of variational self-adjointness which establishes the corresponding Fundamental Analytic Theorems (Figures 3.1 and 3.4) and then allows the construction of the desired functions. The net result is a considerable broadening of the representational capabilities of the analytic equations, or of Hamilton's principle, which now emerge as being able to represent all Newtonian systems admitting a self-adjoint form of the equations of motion (Chart 3.2). The independent treatment of the problem in configuration and phase space reveals deep conceptual and technical differences, despite the ultimate equivalence of the two approaches [which is investigated in more detail in Santilli (1979)]. In particular, the independent computation of a Lagrangian and a Hamiltonian for the same system produces functions that turn out to be not necessarily related by a Legendre transform. This is an indication that the integrability conditions for the existence of a Lagrangian or Hamiltonian have a direct impact for transformation theory resulting in degrees of freedom of these functions which are not expected to be of conventional type. Despite their structural differences, Lagrange's and Hamilton's equations can be written in a formally unified way (Chart 3.7). This implies a formal unification of the two aspects of the Inverse Problem. Variational principles can then be generalized not only to induce the conventional analytic equations, but also the necessary and sufficient conditions for their existence (Charts 3.3 through 3.7).

**Chart 3.1** The Controversy on the Representation of Nonconservative Newtonian Systems with the Conventional Hamilton's Principle

Nonconservative Newtonian systems are known to be representable by the following *modified* form of Hamilton's principle with external terms

$$\delta \int_{t_1}^{t_2} dt(T + W) = 0, \qquad W = \mathscr{F}_k \cdot \mathbf{r}^k, \qquad \delta W = \mathscr{F}_k \cdot \delta \mathbf{r}^k \qquad (1)$$

[see, for instance, Goldstein (1950, pp. 38–40)].

A problem which has been controversial for over a century in the physical literature is whether nonconservative Newtonian systems can be represented by the *conventional* Hamilton's principle without external terms (1.3.23), i.e.,

$$\delta \int_{t_1}^{t_2} dt\, L(t, q, \dot{q}) = 0. \tag{2}$$

To the best of my knowledge, this controversy, somewhat inherited from contrasting statements dating back to the nineteenth century, reached a climactic stage in the early 1930s as a result of the following corollary of a theorem by Bauer (1931):

*The equations of motion of a dissipative linear dynamical system with constant coefficients are not given by a variational principle.*

The above corollary prompted the publication of Bateman's disproof of 1931 according to a prolongation method we shall outline later in Chart 3.13. (Bauer's paper was submitted as a Harvard note in March 21, 1931 and Bateman's rebuff was submitted as a Caltec note in June 17 of the same year.)

Despite the contribution by Bateman (which he properly published in *The Physical Review*), the controversy did not end, but was taken up again by J. L. Synge (1932) and other authors, and subsequently resulted n negative positions in more recent textbooks on the subject.

For instance, C. Lanczos, in his textbook on variational principles (1949), states on page xxi:

*Forces of a frictional nature, which have no work function, are outside the realm of variational principles.*

Similarly, on p. 19-7 of Vol. II of the Feynman Lectures [Feynman et al., (1966)], we read

*The principle of least action only works for conservative systems—where all forces can be gotten from a potential function.*

It was unfortunate that the Inverse Problem of the calculus of variations, which was already well established in mathematical circles in the late 1920s, as indicated in the Introduction, had not propagated into the physical literature.

As we shall see in this book and in Santilli (1979), the Inverse Problem allows the resolution of this controversy. In particular, the arena of representational capabilities of the conventional Hamilton's principle without external terms for nonconservative Newtonian systems will emerge as being of considerable magnitude.

**Chart 3.2** The Arena of Applicability of Hamilton's Principle

Consider a class $\mathscr{C}^2$, regular, unconstrained Newtonian system of $N$ particles in the Euclidean space of its experimental detection with Cartesian coordinates $r^{ka}$, $k = 1, 2, \ldots, N$, $a = x, y, z$:

$$A_{kajb}(t, \mathbf{r}, \dot{\mathbf{r}})\ddot{r}^{jb} + B_{ka}(t, \mathbf{r}, \dot{\mathbf{r}}) = 0,$$
$$A_{kajb}, B_{ka} \in \mathscr{C}^2(R), \quad |A|(R) \neq 0, \qquad k = 1, 2, \ldots, n, a = x, y, z. \tag{1}$$

The assumed continuity and regularity conditions ensure the existence and uniqueness of the implicit functions (Chart 1.1)

$$\ddot{r}^{ka} = f^{ka}(t, \mathbf{r}, \dot{\mathbf{r}}), \qquad f^{ka} = -A^{kajb}B_{jb} \in \mathscr{C}^2(R), \qquad (A^{kajb}) = (A_{kajb})^{-1}. \tag{2}$$

We can say that a nonconservative Newtonian system [Equation (1)] admits a representation within the considered variables $\{t, \mathbf{r}\}$ in terms of Hamilton's principle

$$\delta \int_{t1}^{t2} dt\, L(t, \mathbf{r}, \dot{\mathbf{r}}) = -\int_{t_1}^{t2} dt \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{r}^{ka}} - \frac{\partial L}{\partial r^{ka}} \right) \delta r^{ka} = 0 \tag{3}$$

when the totality of solutions of Lagrange's equations

$$L_{ka}(\mathbf{r}) = \frac{d}{dt} \frac{\partial L}{\partial \dot{r}^{ka}} - \frac{\partial L}{\partial r^{ka}} = 0 \tag{4}$$

and of equations of motion (1) coincide.

From the uniqueness of the implicit functions and the existence theory of Section 1.1, we can equivalently say that equations of motion (1) admit a representation within the considered variables $\{t, \mathbf{r}\}$ in terms of principle (3) when the system of implicit functions of Lagrange's equations (4) coincides with that of the equations of motion.

By using our concept of analytic representation, Definition 3.4.1, we can equivalently say that equations of motion (1) admit a representation within the considered variables $\{t, \mathbf{r}\}$ in terms of principle (3) when there exists a class $\mathscr{C}^2$ regular matrix of elements $h_{ka}^{jb}(t, \mathbf{r}, \dot{\mathbf{r}})$ such that the following ordered identifications hold

$$L_{ka}(r) \equiv h_{ka}^{jb}(A_{jbic}\ddot{r}^{ic} + B_{jb}), \qquad k = 1, 2, \ldots, n, \qquad a = x, y, z. \tag{5}$$

The Fundamental Analytic Theorem for configuration space representations, Theorem 3.5.1, implies that, whenever a class $\mathscr{C}^2$ and regular matrix $(h)$ exists such that the right-hand side of Equations (5) is well behaved and self-adjoint in a star-shaped region of the variables, a representation of system (1) in terms of principle (3) holds. The Fundamental Analytic Theorem for phase space formulations, Theorem 3.12.1, yields the same result (via the use of the inverse Legendre transform) with the additional possibility of reducing the minimal continuity condition to that of class $\mathscr{C}^1$ only.

The above remarks are already sufficient to indicate that the arena of representational capabilities of Hamilton's principle in Newtonian mechanics is rather broad indeed. When the matrix $(h)$ of representation (5) is the unit matrix and the Lagrangian has the conventional structure $L = T(\dot{\mathbf{r}}) - U(t, \mathbf{r}, \dot{\mathbf{r}})$, one recovers the representation in terms of Hamilton's principle of the Newtonian systems with (local) forces derivable from a potential including, of course, the case of the Lorentz force. However, when the matrix $(h)$ of representation (5) is not the unit matrix, the representation in terms of Hamilton's principle of a considerably broad class of nonconservative systems is allowed. This is essentially due to the fact that Hamilton's principle, as a mathematical algorithm, holds for Lagrangians of *arbitrary* functional dependence in their variables [a property which is crucial for the consistency of the calculus of variations (Section 1.3)], while the integrability conditions for the existence of a Lagrangian, the conditions of variational self-adjointness, are insensitive as far the physical nature of the acting forces is concerned (Section 3.7).

Representations (5) with nontrivial factor matrices ($h$) are indirect according to our terminology (Section 3.4). As such, they are studied in Santilli (1979). It is significant here to indicate that, as formulated within a fixed system of the variables $\{t, \mathbf{r}\}$, they are not "universal", namely, a factor matrix ($h$) allowing representation (5) does not necessarily exist for an arbitrarily given system (1). As seen in Santilli (1979), the conditions of variational self-adjointness essentially characterize a class of non-conservative Newtonian systems (1) which admit representation (5).

When analytic representation (5) does not exist within the fixed system of variables $\{t, \mathbf{r}\}$, this does not mean that Hamilton's principle is necessarily unable to represent the system at hand. The reason is that one can still seek a representation of an equivalent system in a new set of variables $\{t', \mathbf{r}'\}$ induced by the class $\mathscr{C}^1$ and invertible transformations

$$\mathbf{r}^k \to \mathbf{r}'^k = \mathbf{r}'^k(t, \mathbf{r}, \dot{\mathbf{r}}), \qquad t \to t' = t'(t, \mathbf{r}, \dot{\mathbf{r}}), \tag{6}$$

with underlying indirect analytic representations

$$L'_{ka}(\mathbf{r}') = h'^{jb}_{ka}(t', \mathbf{r}', \dot{\mathbf{r}}')\,[A'_{jbic}(t', \mathbf{r}', \dot{\mathbf{r}}')\ddot{r}'^{ic} + B'_{jc}(t', \mathbf{r}', \dot{\mathbf{r}}')]. \tag{7}$$

In this case, by construction, the totality of solutions of Lagrange's equations in the new variables is equivalent to (rather than coincidental with) the totality of solutions of system (1).

In conclusion, the analysis of this first volume allows the formulation and proof of the following lemma

**Lemma.** *Hamilton's principle is capable of representing all class $\mathscr{C}^2$, regular, unconstrained Newtonian systems, admitting an equivalent form which is well behaved and self-adjoint in a star-shaped region of the local variables.*

A primary objective of Santilli (1979) is the study of this lemma within the context of the equivalence transformations induced by factor matrices and transformations of the local variables.

**Chart 3.3** Generalization of Hamilton's Principle to Include the Integrability Conditions for the Existence of a Lagrangian

Let $E_0$ be an *admissible possible path* of the action functional $A(E) = \int_{t1}^{t2} dt\, L(t, E, E')$, i.e., an element of the $\infty^{2n}$ family of solutions $q^k(t; u, v)$ of Lagrange's equations in $L(t, q, \dot{q})$ which is of at least class $\mathscr{C}^2$ (Section 1.3) (we here tacitly implement our continuity and regularity restrictions on the Lagrangian). Let $E' = E_0 + \delta^1 E_0$ be an *admissible varied path with fixed end points,* i.e., a varied path which is also of class $\mathscr{C}^2$ and which satisfies end-point conditions (1.3.15). By integrating by parts and using these conditions, the contemporaneous second-order variation of the action, Equation (1.3.30), can be written

$$\delta^2 A|_{E_0} = \tfrac{1}{2}\delta^1(\delta^1 A)|_{E_0} = -\int_{t1}^{t2} dt\, L_k(E_0)\delta^2 q_0^k - \int_{t1}^{t2} dt\, J_k(\delta^1 E_0)\delta^1 q_0^k. \tag{1}$$

$L_k(E_0) = 0$ stands for Lagrange's equations (1.3.21) and $J_k(\delta^1 E_0) = 0$ represents the associated Jacobi's equations (1.3.25). Since $E_0$ is a possible path, Lagrange's equations are verified and the first term of

Equation (1) is identically null. However, since the varied path $E'$ is not necessarily a possible path in the sense of Equations (1.3.26), the variations $\delta^1 E_0$ are not necessarily a solution of Jacobi's equations and variation (1) is not necessarily null.

Let $\tilde{E}' = E_0 + \tilde{\delta}^1 E_0$ be another admissible varied path with fixed end points different from $E'$. The two first-order variations of Equation (1) can be differentiated between themselves, and we can write

$$\begin{aligned} \tfrac{1}{2}\tilde{\delta}^1(\delta^1 A)\,|_{E_0} &= -\int_{t_1}^{t_2} dt\, J_k(\delta^1 E_0)\tilde{\delta}^1 q_0^k, \\ \tfrac{1}{2}\delta^1(\tilde{\delta}^1 A)\,|_{E_0} &= -\int_{t_1}^{t_2} dt\, \tilde{J}_k(\tilde{\delta}^1 E_0)\delta^1 q_0^k. \end{aligned} \tag{2}$$

But the variations $\delta^1$ and $\tilde{\delta}^1$ are trivially commutative. By interpreting the system $\tilde{J}_k(\tilde{\delta}^1 E_0)$ as the adjoint system of Jacobi's system $J_k(\delta^1 E_0)$ (Section 3.3) and by recalling the fundamental condition of self-adjointness, Equation (2.1.15), we obtain th following *generalization of Hamilton's principle to include the conditions of variational self-adjointness.*

$$\delta^1 A(E)\,|_{E_0} = -\int_{t_1}^{t_2} dt\, L_k(E_0)\delta^1 q_0^k = 0, \tag{3a}$$

$$\lim_{\tilde{\delta}^1 = \delta^1} \tfrac{1}{2}[\delta^1(\tilde{\delta}^1 A) - \tilde{\delta}^1(\delta^1 A)]_{E_0}$$

$$= \lim_{\tilde{\delta}^1 = \delta^1} \int_{t_1}^{t_2} dt[\tilde{\delta}^1 q_0^k J_k(\delta^1 E_0) - \delta^1 q_0^k \tilde{J}_k(\tilde{\delta}^1 E_0)]_{E_0} = 0, \tag{3b}$$

which is equivalent to the following lemma.

**Lemma.** *Newtonian systems representable by the action functional $A(E)$ evolve according to a possible path $E_0$ along which the first-order variation with fixed end points of the action is null and self-adjoint.*

This generalization of Hamilton's principle clearly incorporates in the variational algorithm not only Lagrange's equations, but also the necessary and sufficient conditions for their existence. In turn, this indicates the need of the inclusion in the variational principles of the second-order variation of the action.

In view of the generalizations of Hamilton's principle of the subsequent charts, it should be stressed that for principle (3) the variations $\delta^1 E_0$ and $\tilde{\delta}^1 E_0$ need not necessarily be the solutions of Jacobi's equations.

*Nota Bene.* The fact that the limit $\tilde{\delta}^1 = \delta^1$ of the left-hand side of Equation (3b) is null, is trivial. This is not the case for the right-hand side of the same expression. Indeed, by using Equation (2.1.9), this limit implies that

$$\eta^i M_i(\eta) - \eta^i \tilde{M}_i(\eta) = \frac{d}{dt} Q(\eta, \eta) \equiv 0, \tag{4}$$

which is not the case for an arbitrary system. However, property (4) holds if and only if the system is self-adjoint, i.e., when all conditions of self-adjointness are varified (Problem 3.5). It is in this sense that the generalization of Hamilton's principle presented in this chart contains all the integrability conditions for the existence of a Lagrangian or, equivalently, of an action functional.

**Chart 3.4** Generalization of Hamilton's Principle to Include Lagrange's Equations and Their Equations of Variation

Let $E_0$ be an admissible possible path and $E' = E_0 + \delta E_0$ be an admissible varied path in the sense recalled in Chart 3.3. The computation of Lagrange's equations along $E'$ and the use of the Taylor expansion leads to the important formula

$$L_k(E') = L_k(E_0) + \sum_{s=1}^{m} J_k^s(\delta^s E) + 0_L^{m+1},\ J_k^s(\delta^s E) = \frac{1}{s}\,\delta^1(J_k^{s-1}(\delta^{s-1}E)) \quad (1)$$

This is a generalization to an arbitrary order $s = 1, 2, 3, \ldots$ of the property of Section 3.3 according to which Jacobi's forms are the variational forms of Lagrange's equations, i.e., $J_k^1(\delta^1 E) = \delta^1 L_k(E)$. The equations $J_k^s(\delta^s E) = 0$ will be called the *equations of variation of order s of Lagrange's equations.*

Suppose that the varied path $E'$ satisfies fixed end-point condition (1.3.15) to all orders in the variations. Then, by using these conditions and by extending the integration by parts of Equation (1) of Chart 3.3 to higher order, we can write

$$\begin{aligned}
\delta^1 A|_{E_0} &= -\int_{t_1}^{t_2} dt\, L_k(E_0)\delta^1 q_0^k = 0,\\
\delta^2 A|_{E_0} &= -\int_{t_1}^{t_2} dt\, L_k(E_0)\delta^2 q_0^k - \int_{t_1}^{t_2} dt\, J_k^1(\delta^1 E_0)\delta^1 q_0^k \neq 0, \qquad (2)\\
&\vdots\\
\delta^s A|_{E_0} &= -\int_{t_1}^{t_2} dt\, L_k(E_0)\delta^s q_0^k - \sum_1^s{}_i \int_{t_1}^{t_2} dt\, J_k^i(\delta^i E_0)\delta^{s-i} q_0^k \neq 0.
\end{aligned}$$

Suppose, finally, that $E'$ is also a possible path ($\neq E_0$), i.e., $E' = E'_0$ is another element of the $\infty^{2n}$ family of solutions of Lagrange's equations. We then reach the following *generalization of Hamilton's principle to a variation of the action of arbitrary order* $s = 1, 2, 3, \ldots$.

$$\delta^s A|_{E_0} = -\sum_1^s{}_i \int_{t_1}^{t_2} dt\, J_k^i(\delta^i E_0)\delta^{s-i} q_0^k = 0, \qquad J_k^0 = L_k, \qquad \delta^0 E_0 = E_0, \qquad (3)$$

which is equivalent to the following lemma.

**Lemma.** *Newtonian systems representable by the action functional* $A(E)$ *evolve according to a possible path* $E_0$ *and a possible varied path* $E'_0$ *along which the contemporaneous variation of order* $s = 1, 2, 3, \ldots$ *of* $A(E)$ *is null.*

The case $s = 1$ trivially recovers the conventional Hamilton's principle with underlying Lagrange's equations. The cases $s > 1$ represent generalizations, where the underlying equations are the equations of variation of order $s$ of Lagrange's equations. Thus, our generalizations, to be applicable,

demand the validity of Lagrange's equations as well as their equations of variation of arbitrary order. Besides being regular, the Lagrangian $L$ is here tacitly assumed to be of class $\mathscr{C}^\infty$.

**Chart 3.5** Generalization of Hamilton's Principle to Include Lagrange's Equations, Their Equations of Variation, and the End-Point Contributions

The generalizations of Hamilton's principle of Charts 3.3 and 3.4 are restrictive because they deal with *contemporaneous* variations, i.e., those occurring at a fixed value of time (Section 1.3). In this chart, we introduce an extension of the generalization of Chart 3.4 to *noncontemporaneous* variations, i.e., those which also imply a variation of time (Section 1.3). The corresponding extension of the generalization of Chart 3.3 is left as an exercise for the interested reader. Other types of generalizations of Hamilton's principle suggested by the methodology of the Inverse Problem will be introduced in subsequent charts, as well as in the charts of Santilli (1979), depending on the aspect considered.

By using an iterative procedure, the noncontemporaneous variations of the action functional can be written (Section 1.3)

$$\begin{aligned}
\hat{\delta}^1 A|_E &= -\int_{t_1}^{t_2} dt\, L_k(E)\delta^1 q^k + (\mathrm{EPC})^1, \\
\hat{\delta}^2 A|_E &= -\int_{t_1}^{t_2} dt\, L_k(E)\delta^2 q^k - \int_{t_1}^{t_2} dt\, J_k^1(\delta^1 E)\delta^1 q^k + (\mathrm{EPC})^2, \qquad (1) \\
&\vdots \\
\hat{\delta}^s A|_E &= -\int_{t_1}^{t_2} dt\, L_k(E)\delta^s q^k - \sum_{i=1}^{s} \int_{t_1}^{t_2} dt\, J_k^i(\delta^i E)\delta^{s-i} q^k + (\mathrm{EPC})^s,
\end{aligned}$$

where $(\mathrm{EPC})^s$ is the *end-point contribution of order s* which is given, for the cases $s = 1$ and $s = 2$, by Equations (1.3.40) and (1.3.43), respectively.

By using the lemma of Chart 3.4, we then reach the *generalization of Hamilton's principle to include Lagrange's equations, their equations of variation, and the end-point contributions*

$$\hat{\delta}^s A|_{E_0\, \varepsilon_0} = (\mathrm{EPC})^s, \qquad s = 1, 2, \ldots, \qquad (2)$$

which is equivalent to the following lemma.

**Lemma.** *Newtonian systems representable by the action functional $A(E)$ evolve according to a path $E_0$ along which the noncontemporaneous variations of order $s = 1, 2, 3, \ldots$ with variable end points of the action, when computed along a possible varied path, are equal to the end-point contributions of the same order.*

The case $s = 1$ recovers the known Weiss' (Holder's) principle, as recalled in Section 1.3. The cases $s > 1$ constitute generalizations to higher orders. Their significance is indicated in Santilli (1979) in relation to the problem of symmetries and conserved quantities of Newtonian systems with arbitrary couplings.

**Chart 3.6** Generalization of Hamilton's Principle to Include a Symplectic Structure

By performing a direct Legendre transformation of the Lagrangian (Section 3.8), the *action functional in phase space* can be written from Equation (2) of Chart 3.1:

$$\tilde{A}(\tilde{E}) = \int_{t1}^{t2} dt[\dot{q}^k p_k - H(t, q, p)]_{\tilde{E}}, \qquad \tilde{E} = \{q, p\}. \tag{1}$$

It constitutes the basic functional of the canonical formulation of the calculus of variations [see, for instance, Bliss (1946, Chap. III)]. Since the generalized momenta $p_k$ are independent of the coordinates $q^k$, they must be independently varied. This implies the doubling of the number of variations of configuration space formulations. Elementary calculations then yield *Hamilton's principle for phase space formulations*

$$\delta^1 \tilde{A}|_{\tilde{E}_0} = \int_{t1}^{t2} dt \left[ \left( \dot{q}^k - \frac{\partial H}{\partial p_k} \right) \delta^1 p_k - \left( \dot{p}_k + \frac{\partial H}{\partial q^k} \right) \delta q^k \right]_{\tilde{E}_0} = 0, \tag{2}$$

which now leads to Hamilton's, rather than to Lagrange's equations.

Principle (2) has been subjected to a number of critical examinations. As we see in Santilli (1979), a first problematic aspect is due to the inability of principle (2) of being form-invariant under phase space transformations more general than the canonical transformations. A second problematic aspect is the lack of "symmetry" of action (1) in the derivatives. This implies the rather peculiar situation whereby the action cannot be written in terms of the (covariant or contravariant) tensor notation $a = (q, p)$, while the end result of the principle, Hamilton's equations, can. A third problematic aspect is the inability of introducing a symplectic structure directly in the integrand of action (1), contrary to the established fact that the geometry that underlies canonical formulations is the symplectic geometry. In turn, this has led to the general lack of use of variational principles in the study of Newtonian systems from a geometrical profile [as Abraham and Marsden (1967) put it, variational principles, after their function of inducing the canonical equations, "do not have a crucial role in the theory" (p. 129), contrary to their role for other aspects (e.g., derivation of the Hamilton–Jacobi equations and quantization].

This situation demands the study of whether the conventional Hamilton's principle in phase space can be implemented into a form allowing a direct geometrical treatment. This objective can be achieved by introducing the *extended action functional in phase space*

$$\begin{aligned} \tilde{A}^e(\tilde{E}) &= \int_{t1}^{t2} dt[\dot{q}^k p_k - H]_{\tilde{E}} - \tfrac{1}{2} | q^k p_k |_{t1}^{t2} (\tilde{E}) \\ &= \int_{t1}^{t2} dt[\tfrac{1}{2}(\dot{q}^k p_k - q^k \dot{p}_k) - H]_{\tilde{E}} \\ &= \int_{t1}^{t2} dt[\tfrac{1}{2} \dot{a}^\mu \omega_{\mu\nu} a^\nu - H(t, a)]_{\tilde{E}}, \\ a^\mu &= \{q^k, p_k\}, \qquad \dot{a}^\mu \omega_{\mu\nu} a^\nu = \dot{a}_\mu \omega^{\mu\nu} a_\nu, \end{aligned} \tag{3}$$

which now directly exhibits the presence of the fundamental symplectic form $\omega$. The variations of the $q^k$ and $p_k$ variables can now be unified into the variations of the $a^\mu$ variables. We reach, in this way, the following *generalization of Hamilton's principle to include a symplectic structure*

$$\begin{aligned}\delta^1 A^e|_{\tilde{E}_0} &= \delta^1 \int_{t_1}^{t_2} dt[\tfrac{1}{2}\dot{a}^\mu \omega_{\mu\nu} a^\nu - H]_{\tilde{E}_0} \\ &= -\int_{t_1}^{t_2} dt\left[\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right]_{\tilde{E}_0} \delta a^\mu \\ &= 0,\end{aligned} \tag{4}$$

which now leads directly to Hamilton's equations in their (covariant or contravariant) normal form (Section 3.9). Since these equations, when explicitly written, coincide with the conventional nontensorial form of Hamilton's equations, principles (2) and (4) characterize the same Newtonian system and, thus, the same path. In this sense, they are equivalent. However, our generalization (4) is clearly preferable over the conventional form (2) from the viewpoint of the transformation theory, the differential geometry and, inevitably, the Lie algebra structure [which is elaborated upon in Santilli (1979)].

The configuration space image of principle (4) yields equivalent results. By using an inverse Legendre transform (Section 3.8), we reach the *extended action in configuration space*

$$\begin{aligned}A^e(E) &= \int_{t_1}^{t_2} dt\, L(t, q, \dot{q})_E - \frac{1}{2}\left|\frac{\partial L}{\partial \dot{q}^k} q^k\right|_{t_1}^{t_2}(E) \\ &= A(E) + g(E)\end{aligned} \tag{5}$$

with corresponding generalized principle

$$\delta^1 A^e|_{\tilde{E}_0} = -\int_{t_1}^{t_2} dt\, L_k(E_0)\delta q^k = 0, \qquad \delta^1 g = 0. \tag{6}$$

The underlying equations are, therefore, the conventional Lagrange's equations, as expected. Thus, principle (6) is equivalent to Hamilton's principle in configuration space. Notice that in principle (6) the additive term produces no contribution under contemporaneous variations with fixed end points because, trivially, it is (constant and) computed at end points.

In conclusion the variational principles with fixed end points of the extended action in configuration and phase space are equivalent to the conventional corresponding forms of Hamilton's principle.

To avoid possible misrepresentations, it should be noted that the above equivalence of the actions $A(E)$ and $A^e(E)$ holds only at the level of variational *principles*, but not at the level of variational *problems* (Section 1.3). Indeed, the transition from $A(E)$ to $A^e(E)$ characterizes the transition from an ordinary extremal problem of single integral path functionals to a subcase of the extension known as the *Problem of Bolza* [see, for instance, Bliss (1946, part II)].

It should also be indicated that the above equivalence of the variational principles for $A(E)$ and $A^e(E)$, strictly speaking, holds only at the level of contemporaneous variations with fixed end points. The corresponding principles for the case of noncontemporaneous variations with variable end

points are still equivalent in the sense that they still characterize the same system, but the additive term of our extended action in configuration space now implies a modification to the end-point contribution. This is left to the interested reader. Notice that the generalizations of Hamilton's principles introduced in Charts 3.3, 3.4 and 3.5 can be reformulated in terms of our extended action. These further generalization appear to be significant for the study of the transformation theory of Newtonian systems with arbitrary (local) forces, as can be seen in Santilli (1979).

**Chart 3.7** Generalization of Hamilton's Principle for the Unified Treatment of the Inverse Problem in Configuration and Phase Space

As is familiar by now, the independent formulations of the Inverse Problem in configuration and phase space exhibit rather significant differences due to the differences in the underlying analytic equations (Lagrange's and Hamilton's equations) and their integrability conditions (i.e., the conditions of self-adjointness for systems of second-order and first-order equations).

Nevertheless, the analytic formulations in configuration and phase space are known to be equivalent. The corresponding Inverse Problems are, therefore, expected to be equivalent, too. This situation creates the problem whether there exists a variational algorithm which exhibits such equivalence. A formal answer to this problem is given by the following lemma.

**Lemma.** *Let* $R^{2m+1}$ *be a* $(2m + 1)$ *space spanned by time* $t$ *and* $m$ *independent variables* $y^k$, *and let*

$$A(E) = \int_{t_1}^{t_2} dt\, F(t, y, \dot{y})_E \tag{1}$$

*be the action along* $E = \{y^k\}$. *Then the self-adjoint variational principle with fixed end points* (*Chart* 3.3)

$$\delta^1 A|_{E_0} = -\int_{t_1}^{t_2} dt\, F_k(E_0)\delta y_0^k = 0, \tag{2a}$$

$$\lim_{\tilde{\delta}1 = \delta 1} \tfrac{1}{2}[\tilde{\delta}^1(\delta^1 A) - \delta^1(\tilde{\delta}^1 A)]_{E_0}$$

$$= \lim_{\tilde{\delta}1 = \delta 1} \int_{t_1}^{t_2} dt[\tilde{\delta}^1 y_0^k J_k(\delta^1 E_0) - \delta^1 y_0^k \tilde{J}_k(\tilde{\delta}^1 E_0)]_{E_0} = 0 \tag{2b}$$

*can characterize the Inverse Problem in either configuration or phase space, depending on the assumed realization of the* $y^k$ *and* $F$ *functions.*

Assume $m = n$, $y^k = q^k$, and $F = L(t, q, \dot{q})$. Then Equation (2a) yields Lagrange's equations

$$F_k(E_0) = \left(\frac{d}{dt}\frac{\partial F}{\partial \dot{y}^k} - \frac{\partial F}{\partial y^k}\right)_{E_0} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k}\right)_{E_0} = 0, \tag{3}$$

while Equation (2b) characterizes their integrability conditions.

Assume now $m = 2n$, $\{y^k\} = \{a^\mu\}$, and $F = \frac{1}{2}\dot{a}^\mu \omega_{\mu\nu} a^\nu - H(t, a)$. Then Equation (2a) now yields Hamilton's equations

$$F_k(\tilde{E}_0) = \left(\frac{d}{dt}\frac{\partial F}{\partial \dot{y}^\mu} - \frac{\partial F}{\partial y^\mu}\right)_{\tilde{E}_0} = \left(\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right)_{\tilde{E}_0} = 0, \tag{4}$$

while Equation (2b) characterizes their integrability conditions.

Notice that, while for realization (3) the function $F$ is (generally) regular, for realization (4) the function $F$ is totally degenerate because it is linear in the first-order derivatives.

A central difference between realizations (3) and (4) is that, while for the former case the action is the conventional form, for the latter case the action is the extended form introduced in Chart 3.6.

We are now in a position to better elaborate the insufficiency of the conventional Hamilton's principle within the context of the Inverse Problem. Let us recall that the identification of the self-adjointness of Hamilton's equations demands, for its proper treatment, the use of tensor formulations. To state it explicitly, when Hamilton's equations are written in the tensor form of Equation (4), their self-adjointness can be established easily (Section 3.10), though this is not the case when the same equations are written in the conventional notation, i.e., Equations (3.9.1). At the level of variational principles, this situation implies the inability to reach a second-order generalization inclusive of the integrability conditions for the existence of a Hamiltonian, i.e., the canonical formulation of our generalization of Hamilton's principle of Chart 3.3 cannot be formulated directly. In turn, this situation can be reduced to the insufficiency of the conventional action in phase space, Equation (1) of Chart 3.6. On the contrary, if the extended action in phase space, Equation (3) of Chart 3.6, is assumed, then the implementation of the first-order principle with a second-order algorithm directly expressing the integrability conditions for the existence of a Hamiltonian is possible, as indicated by the lemma of this chart.

In conclusion, the methodology of the Inverse Problem provides additional indications of insufficiency of the conventional action in phase space, besides those of algebraic and geometrical nature recalled in Chart 3.6. In turn, this provides additional confirmation of the known deep relationship between the analytic, algebraic, and geometric profiles in the sense that the insufficiency of the conventional Hamilton's principle in phase space from, say, a geometric profile (lack of a symplectic structure) has a precise image within an analytic setting (lack of a variational algorithm expressing the integrability conditions for the existence of a Hamiltonian).

The interrelations of the analytic, algebraic, and geometric approaches to Newtonian Mechanics are studied in more detail in Santilli (1979). For a preview, see Chart 3.18.

**Chart 3.8** Self-Adjointness of First-Order Lagrange's Equations

**Theorem.** *Under the assumptions that the Lagrangian $L(t, q, \dot{q})$ is of (at least) class $\mathscr{C}^3$ and is* totally degenerate, *i.e.,* $\partial^2 L/\partial \dot{q}^i\, \partial \dot{q}^j = 0$, $i, j = 1,$

2, . . . , $n$, *in a region* $R^{2n+1}$ *of points* $(t, q, \dot{q})$, *the* first-order Lagrange's is equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \equiv \frac{\partial^1 L}{\partial \dot{q}^k\, \partial q^i}\dot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k\, \partial t} - \frac{\partial L}{\partial q^k} = 0, \qquad (1)$$

$$k = 1, 2, \ldots, n$$

*are always self-adjoint in* $R^{2n+1}$.

PROOF From Equations (3.3.8) and (3.3.9), the equation of variations of Equation (1) is

$$J_k(\eta) = \left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{q}^k\, \partial q^i}\right) - \frac{\partial^2 L}{\partial q^k\, \partial q^i}\right]\eta^i + \left(\frac{\partial^2 L}{\partial \dot{q}^k\, \partial q^i} - \frac{\partial^2 L}{\partial q^k\, \partial \dot{q}^i}\right)\dot{\eta}^i = 0, \quad (2)$$

and, under the assumed continuity conditions, always coincides with the adjoint system

$$\tilde{J}_k(\eta) = \eta^i\left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{q}^i\, \partial q^k}\right) - \frac{\partial^2 L}{\partial q^i\, \partial q^k}\right] - \frac{d}{dt}\left[\eta^i\left(\frac{\partial^2 L}{\partial \dot{q}^i\, \partial q^k} - \frac{\partial^2 L}{\partial q^i\, \partial \dot{q}^k}\right)\right] = 0 \qquad (3)$$

everywhere in $R^{2n+1}$. *Q.E.D.*

The above theorem can also be proved by showing that Equations (1) satisfy *all* conditions of self-adjointness (2.2.32) for first-order systems. This analysis is simplified if one notes that, from the condition of total degeneracy, the Lagrangian must be *linear* in the velocities, i.e., of the type

$$L = \gamma_k(t, q)\dot{q}^k + \delta(t, q). \qquad (4)$$

Equations (1) then become

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \equiv \left(\frac{\partial \gamma_k}{\partial q^i} - \frac{\partial \gamma_i}{\partial q^k}\right)\dot{q}^i + \left(\frac{\partial \gamma_k}{\partial t} - \frac{\partial \delta}{\partial q^k}\right) = 0, \qquad (5)$$

and Equations (2.2.32) are readily verified under the identifications

$$X_{ij} = \frac{\partial \gamma_i}{\partial q^j} - \frac{\partial \gamma_j}{\partial q^i}, \qquad (6a)$$

$$Y_i = \frac{\partial \gamma_i}{\partial t} - \frac{\partial \delta}{\partial q^i}. \qquad (6b)$$

*Nota Bene*: The above theorem indicates that the variational approach to the self-adjointness of the *regular* Lagrange equations as given in Theorem 3.3.1 extends to the case of *totally degenerate* Lagrange equations without any additional technical difficulty. The reader should again be warned that this is not the case for the transition to "bona fide" *degenerate* Lagrangians (see footnote 36 of page 138 for more details).

**Chart 3.9** The Fundamental Analytic Theorem for First-Order Equations of Motion in Configuration Space

**Theorem.** *A necessary and sufficient condition for a system of first-order equations of motion in configuration space*

$$F_k(t, q, \dot{q}) = 0, \qquad k = 1, 2, \ldots, n \tag{1}$$

*which is well defined and of (at least) class $\mathscr{C}^2$ in a star-shaped region $R^{*2n+1}$ of the variables $(t, q, \dot{q})$ to admit an ordered direct analytic representation in terms of the conventional Lagrange's equations in a first-order Lagrangian $L(t, q, q)$ in $R^{*2n+1}$*

$$L_k(q) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \equiv F_k(t, q, \dot{q}), \tag{2}$$

*is that the system is everywhere self-adjoint in $R^{*2n+1}$, i.e., that each and all the conditions*

$$\begin{gathered} \frac{\partial F_k}{\partial \dot{q}^i} = -\frac{\partial F_i}{\partial \dot{q}^k}; \\ \frac{\partial F_i}{\partial q^k} = \frac{\partial F_k}{\partial q^i} - \frac{d}{dt}\frac{\partial F_k}{\partial q^i}; \\ i, k = 1, 2, \ldots, n \end{gathered} \tag{3}$$

*is everywhere satisfied in $R^{*2n+1}$.*

PROOF[80] The necessity of the condition of self-adjointness follows from the theorem of Chart 3.8. For the sufficiency of this condition, assume

$$L(\tau) = -q^i F_i(t, \tau q, \tau \dot{q}). \tag{4}$$

Then, by using conditions (3) and after some simple algebra, we can write

$$\begin{aligned} L_k(\tau q) &= F_k + q^i \frac{\partial F_i}{\partial q^k} - \frac{d}{dt}\, q^i \frac{\partial F_i}{\partial \dot{q}^k} \\ &= F_k + \gamma^i \frac{\partial F_k}{\partial \gamma^i} + \dot{\gamma}^i \frac{\partial F_k}{\partial \dot{\gamma}^i} \\ &= \frac{d}{d\tau}\left[\tau F_k(t, \tau q, \tau \dot{q})\right]; \\ \gamma^i &= \tau q^i \end{aligned} \tag{5}$$

By integrating Equation (4) with respect to $\tau \in (0, 1)$, i.e.,

$$L(t, q, \dot{q}) = -q^k \int_0^1 d\tau\, F_k(t, \tau q, \tau \dot{q}), \tag{6}$$

identifications (2) then follow and, under conditions (3), a Lagrangian always exists. *Q.E.D.*

[80] The above given proof had been formulated following a private communication by Lovelock and Anderson (1976) concerning the studies by these authors within the context of the cohomological approach by Horndeski (1974).

Notice that Equation (6) provides a method for the computation of a Lagrangian (once its existence is assured by the verification of the condition of the theorem). In line with the remarks following Theorem 3.6.1, if the integral of Equation (6) is not well defined, then the above method for the proof of the theorem does not hold.

Conditions (3) directly follow from conditions (2.1.18) under the absence of dependence on the accelerations. Notice that this "direct form" of the conditions of self-adjointness, rather than the "reduced form", (2.2.32), enters into the proof of the theorem. The proof under these latter conditions is left to the interested reader.

The Lagrangian $L(t, q, \dot{q})$ is called *first order* in the theorem to indicate the order of the maximal derivative of its functional dependence. Therefore, the extension of the theorem of this chart to second-order equations of motion *is not* equivalent to the Fundamental Analytic Theorem 3.5.1, because it deals with analytic representations of such systems with *second-order Lagrangians*, i.e., $L = L(t, q, \dot{q}, \ddot{q})$, which are now, again, totally degenerate (i.e., linear, in this case, in the accelerations). Within such a context, the extension does indeed hold (see Problem 3.6). As a matter of fact, the reader can verify that the extension of the theorem of this chart holds for a totally degenerate Lagrangian of arbitrary order.

It is of some relevance to indicate that the transition from the above Newtonian framework to that of relativistic field theories leads to the corresponding theorem for the analytic representation of *spinorial field equations.*[81]

**Chart 3.10** A Unified Treatment of the Conditions of Self-Adjointness for First-, Second-, and Higher-Order Ordinary Differential Equations

**Theorem 1.** *A necessary and sufficient condition for a system of n ordinary differential equations of order m,*

$$F_k(t, q^{(0)}, q^{(1)}, \ldots, q^{(m)}) = 0,$$

$$q^{(\alpha)} = \frac{d^\alpha q}{dt^\alpha}, \begin{cases} k = 1, 2, \ldots, n, \\ m = 1, 2, 3, \ldots, \\ \alpha = 0, 1, 2, \ldots, m, \end{cases} \tag{1}$$

*which is of class $\mathscr{C}^{2m}$ in a region R of the variables $(t, q^{(0)}, q^{(1)}, \ldots, q^{(m)})$ to be self-adjoint in R is that all the conditions*

$$\sum_{\alpha=\beta}^{2m} (-1)^\alpha \binom{\alpha}{\beta} \frac{d^\alpha}{dt^\alpha} \left( \frac{\partial F_i}{\partial q^{j(\alpha+\beta)}} \right) - \frac{\partial F_j}{\partial q^{i(\beta)}} = 0, \begin{cases} \beta = 0, 1, 2, \ldots, 2m, \\ i, j = 1, 2, \ldots, n, \end{cases} \tag{2}$$

*are everywhere satisfied in R.*

The proof of the theorem[82] can be conducted by using a generalization of the methods of Section 2.1, and is left to the interested reader.

[81] For the (spinorial) field theoretical case, see Santilli (1978, Vol. I).

[82] Lovelock and Anderson (1976) and Horndeski (1974).

**Theorem 2.** *Under the assumption that the m-order Lagrangian* $L(t, q^{(0)}, q^{(1)}, \ldots, q^{(m)})$ *is of at least class* $\mathscr{C}^{2m+2}$ *and regular in a region of its variables, the m-order Lagrange's equations along class* $\mathscr{C}^{2m+3}$ *paths*

$$L_k^{(m)}(q) = \sum_{\alpha=0}^{m} (-1)^\alpha \frac{d^\alpha}{dt^\alpha} \frac{\partial L}{\partial q^{k(\alpha)}} = 0 \begin{cases} k = 1, 2, \ldots, n, \\ m = 1, 2, 3, \ldots \end{cases} \tag{3}$$

*are always self-adjoint in R.*

This second theorem can be proved by showing that Equations (3), under the assumed conditions, satisfy each of the conditions of self-adjointness (2), and it also is left to the interested reader.

For $m = 1$, Equations (2) reproduce conditions of self-adjointness (2.1.18), and Theorem 2 of this chart coincides with Theorem 3.3.1 according to the first proof of Problem 3.1.

Theorem 2 extends trivially to the case of *totally degenerate m-order Lagrangians,* i.e., Lagrangians that are linear in $q^{(m)}$. The case of degenerate Lagrangians is excluded on precautionary grounds along the remarks of footnote 36 of page 138. Theorem 2, for the case of totally degenerate first-order Lagrangians, coincides with the theorem of Chart 3.9.

Notice the unified treatment offered by Theorems 1 and 2 of this chart.

### Chart 3.11 Engels' Methods for the Construction of a Lagrangian

A central aspect of the Inverse Problem is the construction of a Lagrangian once the integrability conditions for its existence are verified. Therefore, is is important to review the available methods for this construction, other than that of Section 3.6. In this chart, we review two methods introduced by Engels (1975 and 1978).

Consider a second-order system of ordinary differential equations in the fundamental form,

$$F_k = A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0, \tag{1}$$

which satisfies all the conditions of Theorem 3.5.1 and thus, in particular, all the conditions of self-adjointness (3.5.3). The problem under consideration is that of constructing a regular first-order (Chart 3.9) Lagrangian $L(t, q, \dot{q})$ for the ordered direct analytic representations of system (1) in terms of the conventional (second-order) Lagrange's equations.

The method of Chart 3.9 for totally degenerate first-order Lagrangians can be extended to higher order. For system (1), this method yields a totally degenerate second-order Lagrangian given by

$$L_0(t, q, \dot{q}, \ddot{q}) = -q^k \int_0^1 d\tau\, F_k(t, \tau q, \tau\dot{q}, \tau\ddot{q}) \tag{2}$$

This Lagrangian is not suitable for our objective because it demands the use of third-order Lagrange's equations. The problem is now turned into that of identifying an equivalence transformation of $L_0$ which removes its dependence on the accelerations:

$$L_0 \to L = L_0 + \frac{d}{dt} G(t, q, \dot{q}) \exists \frac{\partial L}{\partial \ddot{q}^k} = 0, \qquad k = 1, 2, \ldots, n. \tag{3}$$

By using Equations (1) and (2), this condition can be written

$$\int_0^1 d\tau(\tau q^k)A_{k1}(t, \tau q, \tau\dot{q}) - \frac{\partial G}{\partial \dot{q}^i} = 0. \tag{4}$$

The solution

$$G(t, q, \dot{q}) = \int_0^1 \int_0^1 d\tau\, d\tau'(\tau q^k)A_{ki}(t, \tau q, \tau\tau'\dot{q})\dot{q}^i \tag{5}$$

is then ensured by conditions (3.5.3b).

Therefore, a regular first-order Lagrangian for the representation of system (1) is given (up to gauge transformations) by

$$L(t, q, \dot{q}) = -q^k \int_0^1 d\tau\, F_k(t, \tau q, \tau\dot{q}, \tau\ddot{q}) + \frac{d}{dt}\int_0^1 \int_0^1 d\tau\, d\tau'(\tau q^k)A_{ki}(t, \tau q, \tau\tau'\dot{q})\dot{q}^i \tag{6}$$

see Engels (1978).

Another method can be introduced as follows. Consider first a solution $\Phi_j = T_j + \chi_j(t, q)$ of Equations (3.5.11), where $T_j$ is any particular solution of Equations (3.5.11), e.g.,

$$T_j = \int_{c_j}^{\dot{q}_j} d\dot{q}^j A_{ji}(\dot{q}_1 = c_1, \ldots, \dot{q}_{j-1} = c_{j-1}, \dot{q}_j, \dot{q}_{j+1}, \ldots, \dot{q}_n), \tag{7}$$

and the $\chi$'s are, at this point, arbitrary (but of class $\mathscr{C}^3$) functions. Introduce, then, the functions

$$\Gamma_i = B_i - \frac{\partial T_i}{\partial q^j}\dot{q}^j - \frac{\partial T_i}{\partial t}. \tag{8}$$

By using the conditions of self-adjointness, it is then possible to prove that the functions $\chi_i$ can always be selected in such a way that

$$\begin{aligned} \frac{\partial \Phi_i}{\partial q^j} + \frac{\partial \Gamma_j}{\partial \dot{q}^i} &= 0, \\ \frac{\partial \Phi_i}{\partial \dot{q}^j} - \frac{\partial \Phi_j}{\partial \dot{q}^i} &= 0, \\ \frac{\partial \Gamma_i}{\partial q^j} - \frac{\partial \Gamma_j}{\partial q^i} &= 0. \end{aligned} \tag{9}$$

It then follows that

$$\Phi_i = \frac{\partial L}{\partial \dot{q}^i}, \qquad \Gamma_i = -\frac{\partial L}{\partial q^i}, \qquad F_i = \frac{d}{dt}\Phi_i + \Gamma_i. \tag{10}$$

Therefore, a regular first-order Lagrangian for the representation of system (1) is given (again, up to gauge transformations) by

$$L(t, q, \dot{q}) = \int_{cj}^{qj} d\dot{q}^j \, \Phi_j(t, q, \dot{q}_1 = c_1, \ldots, \dot{q}_{j-1} = c_{j-1}, \dot{q}_j, \dot{q}_{j+1}, \ldots, \dot{q}_n)$$
$$- \int_{dj}^{qj} dq \; \Gamma_j(t, q, = d_1, \ldots, q_{j-1}$$
$$= d_{j-1}, q_j, q_{j+1}, \ldots, q_n, \dot{q} = c). \tag{11}$$

see Engels (1975).

The reformulation of Equations (7) and (11), in terms of the Converse of the Poincaré Lemma, is left as an exercise for the interested reader.

In conclusion, there exist at least three methods for the construction of a Lagrangian (i.e., that of Section 3.6 and the two additional methods of this chart). For cases of simple equations of low dimensionality, they are equivalent from a computational viewpoint. However, for more complex functional structures and/or higher dimensionality, one method may result to be more manageable than the other owing to the differences in the integrals to be computed. In general, method (6) appears to be the most straightforward. Notice that all three methods considered appear to be computerizable.

## Chart 3.12 Mertens' Approach to Complex Lagrangian Structures

The Lagrangians that originate within the context of the Inverse Problem for the representation of nonconservative Newtonian systems generally exhibit a rather complex structure, as indicated by Equations (3.7.19) or (3.7.22). It then follows that the explicit computation of Lagrange's equations is, in these cases, a rather laborious task. This aspect is compounded by the fact that the difference in the two main terms of Lagrange's equations,

$$\frac{d}{dt}(\text{grad}_{\dot{q}} L) - \text{grad}_q L = 0, \tag{1}$$

generally implies the cancellation of complex terms which are in this way redundantly computed twice.

To simplify the computational process, therefore, it is significant to ask whether Equation (1) can be written in a form which avoids the indicated cancellations. This problem was solved by R. Mertens in a note of 1976.[83] Mertens' result is that the known form of Lagrange's equations in terms of Christoffel symbols (see Chart A.14), besides its geometrical significance, has precisely the practical advantage of avoiding the indicated cancellations of form (1).

Let us illustrate Mertens' findings with the simple Lagrangian

$$L(q, \dot{q}) = \tfrac{1}{2}\dot{q}^i M_{ij}(q)\dot{q}^j, \qquad M_{ij} = M_{ji}. \tag{2}$$

[83] Mertens (1976).

Equation (1) then becomes

$$M \cdot \ddot{q} + \dot{M} \cdot \dot{q} - \text{grad}_q L = 0. \tag{3}$$

The cancellations occur here between the terms $\dot{M} \cdot \dot{q}$ and $\text{grad}_q L$. Write the $i$th component of the difference $\dot{M} \cdot \dot{q} - \text{grad}_q L$ in the form

$$\left[\left(\frac{\partial M_{ij}}{\partial q^k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial q^i}\right)\dot{q}^k\right]\dot{q}^j = A_{ij}\dot{q}^j. \tag{4}$$

It is then straightforward to prove that

$$\begin{aligned} A_{ij} &= \left(\frac{\partial M_{ij}}{\partial q^k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial q^i}\right)\dot{q}^k \\ &= \frac{1}{2}\left(\frac{\partial M_{ij}}{\partial q^k} + \frac{\partial M_{ik}}{\partial q^j} - \frac{\partial M_{jk}}{\partial q^i}\right)\dot{q}^k \\ &= [jk; i]\dot{q}^k, \end{aligned} \tag{5}$$

where $[jk; i]$ is the Christoffel symbol of the first kind (Chart A.14). Equation (1) can then be written

$$M_{ij}\,\ddot{q}^j + [jk, i]\dot{q}^k\dot{q}^j = 0 \tag{6}$$

by recovering, in this way, a subcase of Equation (4) of Chart A.14 for Lagrangian (2).

The point is that the computation of the term $[jk; i]\dot{q}^k\dot{q}^j$ in lieu of the term $\dot{M} \cdot \dot{q} - \text{grad}_q L$ has precisely the practical advantage of avoiding the indicated cancellations. As a result, form (6) of Lagrange's equations is preferable over the conventional form (1) for explicit computations in several cases of complex Lagrangian structures.

The extension of the analysis to Lagrangian structures more general than form (2) and to the inclusion of external terms is left as an exercise for the interested reader.

### Chart 3.13 Bateman's Approach to the Inverse Problem[84]

**Theorem.** *Given a linear, second-order, homogeneous system of n ordinary second-order differential equations,*

$$a_{ki}(t)\ddot{q}^i + b_{ki}(t)\dot{q}^i + c_{ki}(t)q^i = 0, \tag{1}$$

*which does not admit an ordered direct analytic representation in terms of the conventional Lagrange equations, there always exist a prolongation of Equation* (1) *into a system of* $2n$ *second-order ordinary equations in the variables* $q^k$ *and a new set* $y^k$ *for which such an analytic representation exists.*

Bateman's proof of the theorem was based on conventional techniques of the theory of differential equations. The following proof is a simple elaboration of Dedecker's analysis with the use of Section 2.1. Since Equation (1) does not admit an ordered, direct analytic representation by assumption, from Theorem 3.5.1 they must be non-self-adjoint. But such

[84] Bateman (1931). A treatment of *Bateman's prolongation method* is given by Dedecker (1949), who also treated the nonlinear case.

equations are linear and homogeneous. Therefore, they are equivalent to their equations of variations, say $M_k(\eta) = 0$. The adjoint system $\tilde{M}_k(\tilde{\eta}) = 0$ now does not coincide with $M_k(\eta) = 0$. However, the system of $2n$ equations of variations $\{M_k(\eta), \tilde{M}_k(\tilde{\eta}^k)\} = 0$ is self-adjoint and, as a consequence, a Lagrangian for its representation exists. The theorem then follows by performing the prolongation of system (1) into a $2n$-dimensional system, where the second set of $n$-equations are equivalent to the adjoint $\tilde{M}_k(\tilde{\eta}) = 0$.

The case discussed by Bateman is the damped oscillator,

$$\ddot{q} + 2k\dot{q} + n^2 q = 0, \tag{2}$$

which does satisfy the conditions of the theorem. The extension found by Bateman is then given by

$$\begin{aligned} \ddot{q} + 2k\dot{q} + n^2 q &= 0, \\ \ddot{y} - 2k\dot{y} + n^2 y &= 0. \end{aligned} \tag{3}$$

The reader can now verify our proof because the second equation of the above prolongation is precisely equivalent to the adjoint equation of variation of the first equation.

*Nota Bene*: System (3) is equivalent to system (3.4.13), which can be represented by Morse–Feshbach Lagrangian (3.4.15). This point is significant because, as indicated in the footnotes of Section 3.4, the Morse–Feshbach Lagrangian constitutes the Newtonian limit of the Lagrangian densities of the gauge theories. Therefore, the structure of the Lagrangian representations of the recent unified gauge theories of weak and electromagnetic interactions can ultimately be seen, to a considerable extent, from the viewpoint of Bateman's prolongation approach.

**Chart 3.14** Douglas' Approach to the Inverse Problem

Consider a system of second-order ordinary differential equations $F_k(t, q, \dot{q}, \ddot{q}) = 0, k = 1, 2, \ldots, n$, which satisfies all the conditions of the Theorem on Implicit Function 1.1.2. Then such system can be equivalently written in what we have called the kinematic form:

$$\ddot{q}^i = f^i(t, q, \dot{q}), f^i \in \mathscr{C}^1(R_{2n+1}) \tag{1}$$

The same Theorem 1.1.2 then allows one to replace the accelerations $\ddot{q}^i$ with the implicit functions $f^i$ everywhere in the region of definition of the system.

By using this latter property, Douglas (1941) studied the generally overdetermined system of partial differential equations,

$$\frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^i} f^i + \frac{\partial^2 L}{\partial \dot{q}^k \partial q^i} \dot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k \partial t} - \frac{\partial L}{\partial q^k} = 0, \tag{2}$$

in the unknown function $L$, where the implicit function $f^i$ is given. The analysis was conducted within the context of the conventional theory of partial differential equations, with particular reference to the so-called *Riqier theory*, without any use of the conditions of self-adjointness, as can be inferred from the same starting point, i.e., Equation (2).

The analysis was centered on the reduction of second-order system (2) to an equivalent first-order system derivable as follows.

The *Lagrange derivative* $L_k = (d/dt)(\partial L/\partial \dot{q}^k) - \partial L/\partial q^k$ satisfies the identities

$$\begin{cases} \dfrac{\partial L_i}{\partial \dot{q}^j} + \dfrac{\partial L_j}{\partial \dot{q}^i} + 2\dfrac{dL_{ij}}{dt} + \dfrac{\partial f^k}{\partial \dot{q}^j} L_{ik} + \dfrac{\partial f^k}{\partial \dot{q}^i} L_{jk} = 0, & \text{(3a)} \\ \dfrac{d}{dt}\left(\dfrac{\partial L_i}{\partial \dot{q}^j} - \dfrac{\partial L_j}{\partial \dot{q}^i}\right) - 2\left(\dfrac{\partial L_i}{\partial q^j} - \dfrac{\partial L_j}{\partial q^i}\right) + \dfrac{1}{2}\dfrac{\partial f^k}{\partial \dot{q}^i}\left(\dfrac{\partial L_j}{\partial \dot{q}^k} + \dfrac{\partial L_k}{\partial q^j}\right) \\ \qquad - \dfrac{1}{2}\dfrac{\partial f^k}{\partial \dot{q}^j}\left(\dfrac{\partial L_i}{\partial \dot{q}^k} + \dfrac{\partial L_k}{\partial \dot{q}^i}\right) + L_{ik} G_j^k - L_{jk} G_i^k = 0, & \text{(3b)} \end{cases}$$

$$G_j^k \equiv \frac{d}{dt}\frac{\partial f^k}{\partial \dot{q}^j} - 2\frac{\partial f^k}{\partial q^j} - \frac{1}{2}\frac{\partial f^m}{\partial \dot{q}^j}\frac{\partial f^k}{\partial \dot{q}^m},$$

$$L_{ij} \equiv \frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}, \qquad \text{(3c)}$$

Under the assumption that the Lagrange derivatives are null, i.e., that they reduce to Lagrange's *equations* $L_k = 0$, the above identities reduce to the following linear system in the unknown $L_{ij}$.

$$\begin{cases} \dfrac{\partial L_{ij}}{\partial \dot{q}^k} - \dfrac{\partial L_{ik}}{\partial \dot{q}^j} = 0, \\ L_{ik} G_j^k - L_{jk} G_i^k = 0, \\ \dfrac{d}{dt} L_{ij} + \dfrac{1}{2}\dfrac{\partial f^k}{\partial \dot{q}^j} L_{ik} + \dfrac{1}{2}\dfrac{\partial f^k}{\partial \dot{q}^i} L_{jk}, \\ L_{ij} - L_{ji} = 0, \end{cases} \qquad \text{(4)}$$

which Douglas, proved to be equivalent to the original system (2). Douglas then entered into a detailed study of the problem for the case $n = 2$.

Despite the minimal dimensionality assumption (the case $n = 1$ being trivial) Douglas' analysis turned out to be considerably involved to the point of apparently discouraging subsequent investigations by other researchers.

In essence, Douglas' approach is an extension of Darboux's approach (1891) to the case of two-dimensional systems.

It should be indicated here that the concept of "analytic representation" used in this chart and that of Section 3.4 do not coincide. For an elaboration of this point as well as additional remarks, see Santilli (1979).

**Chart 3.15** Rapoport's Approach to the Inverse Problem

Consider the *initial value Newtonian problem*

$$F_k \equiv A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0, \qquad \text{(1a)}$$

$$q^i(t_1) = b_1^i, \qquad q^i(t_2) = b_2^i, \qquad \text{(1b)}$$

and suppose that it satisfies the conditions of Theorem 3.5.1. Then it can be turned into the *initial value variational problem*

$$A(q) = \int_{t1}^{t2} dt\, L(t, q, \dot{q}), \tag{2a}$$

$$q^i(t_1) = b_1^i, \quad i \quad q^i(t_2) = b_2^i, \tag{2b}$$

where $L$ provides a direct analytic representation of system (1a). Let $q'$ be an admissible (in the sense of Section 3.3) varied path. Then we can write

$$A = A(q') - A(q) \qquad \delta^1 A = \int_{t1}^{t2} dt\, \delta^1 L = \int_{t1}^{t2} dt\, F_k \delta^1 q^k = 0. \tag{3}$$

The problem of the construction of a function $L$ capable of satisfying the above relation under the presence of initial value conditions was studied by Rapoport (1938). He basically transformed the single integral of Equation (3) into a double integral of the type

$$S(q, q') = \int_{t1}^{t2} dt \int_0^1 d\tau\, F_k[t_1 q' + \tau(q - q'), \dot{q}' + \tau(\dot{q} - \dot{q}'), \ddot{q}' + \tau(\ddot{q} - q')]\,(q^k - \dot{q}'^k), \tag{4}$$

which he proved to be equal to the difference between the desired functional $A$ computed along the paths $q$ and $q'$, i.e., $S(q, q') = A(q) - A(q')$. We shall content ourselves with an illustration of Equation (4). For more details, the reader is referred to the article by Rapoport.

Consider the Newtonian initial value problem

$$\ddot{q} + kq = 0, \qquad q(t_1) = b_1, \qquad q(t_2) = b_2. \tag{5}$$

Then, in view of the relations

$$q'(t_s) = q(t_s), \qquad s = 1, 2, \tag{6}$$

Equation (4) yields

$$\begin{aligned} S(q, q') &= \int_{t1}^{t2} dt \int_0^1 d\tau\{\ddot{q}' + \tau(\ddot{q} - \ddot{q}') + k^2[q' + \tau(q - q')]\}(q - q') \\ &= \frac{1}{2}\int_{t1}^{t2} dt[\ddot{q}' + \ddot{q} + k(q' + q)](q - q') \\ &= \frac{1}{2}\int_{t1}^{t2} dt[\dot{q}'^2 - \dot{q}^2 + k^2(q' - q)] + \left|\frac{1}{2}(\dot{q}' + \dot{q})(q - q')\right|_{t2}^{t1} \\ &= -\int_{t1}^{t2} dt \left|\frac{1}{2}(\dot{q}^2 - k^2 q^2)\right|_{q=q(t)}^{q=q'(t)} \end{aligned} \tag{7}$$

by producing in this way the correct expression of the Lagrangian for problem (5).

Rapoport's approach is significant because it ultimately provides an alternative method for the construction of a Lagrangian (once its existence is ensured by the conditions of self-adjointness). Notice, however, that again the system must be well defined in a *star-shaped* region for the integral with respect to $\tau$ of Equation (4) to exist.

**Chart 3.16** Vainberg's Approach to the Inverse Problem[85]

The branch of functional analysis which is particularly significant for the Inverse Problem is that dealing with nonlinear operators. A few basic definitions, given below, are useful to outline this profile. The reader is, however, urged to consult the quoted references for all technical details.

A generally nonlinear system of $n$ second-order ordinary differential equations in the $n$ generalized coordinates $q$, i.e., $F(q) = 0$, is characterized by a generally nonlinear *operator* $F$ acting on the function space $Q(q)$ of the $q$'s. In such (linear) space $Q(q)$, we can define the *scalar product* $\{q_1, q_2\} \equiv \int_{\bar{R}} q_1(t) q_2(t) dt$, $\bar{R} \in (t_1, t_2)$; the *norm* $\|q\| = \{q, q\}^{1/2}$, and the *distance* $D = \|q_1 - q_2\|$; $q, q_1, q_2 \in Q(q)$. This characterizes what is (often) called a *pre-Hilbert space.* Such a space is turned into a *Hilbert space* by adding to it all the missing limit elements. A *line* joining two paths $q_1$ and $q_2$ is a one-parameter family of paths $q(t; \tau)$, $\tau \in (0, 1)$ such that $q(t; 0) = q_1$ and $q(t; 1) = q_2$. The *circulation* of $F(q)$ along a line is defined by

$$A = \int_{\bar{R}} \int_{q_1}^{q_2} F(q) \delta q \, dt = \int_{\bar{R}} \int_0^1 F(q(\tau)) \frac{\partial q}{\partial \tau} d\tau \, dt. \tag{1}$$

If the circulation is independent of the path, we can compute it along $q_1 = 0$ and $q_2 = \tau q$ and write

$$A(q) = \int_{\bar{R}} \int_0^1 F(\tau q) q \, d\tau \, dt = \int_{\bar{R}} L(q) dt,$$

$$L(q) = q \int_0^1 F(\tau q) d\tau. \tag{2}$$

The functional $A(q)$ is called the *potential* of the operator $F(q)$ which, in turn, is called the *gradient* of the functional. When a functional $A(q)$, such that its gradient generates the operator $F(q)$ exists, then $F(q)$ is called a *potential operator.* Thus, within this context, the problem of the integrability conditions for the existence of a Lagrangian is turned into the necessary and sufficient conditions for a generally nonlinear operator to be a potential operator. The derivation of these latter conditions demands the introduction of the concept of a differential of a nonlinear operator, which can be achieved through the so-called *Frechét differential*

$$\lim_{\varepsilon \to 0} \frac{F(q + \varepsilon\eta) - F(q)}{\varepsilon} = F'_q \eta = \left. \frac{\partial F(q + \varepsilon\eta)}{\partial \varepsilon} \right|_{\varepsilon = 0} \tag{3}$$

where the term $F'_q$ is called the *Frechét derivative* of $F$. The Frechét differential of a functional is then given by

$$\lim_{\varepsilon \to 0} \frac{A(q + \varepsilon\eta) - A(q)}{\varepsilon} = \int_{\bar{R}} L'_q \eta \, dt \tag{4}$$

The *Gateau differential* of $F(q)$ is given by

$$\lim_{\varepsilon \to 0} \frac{F(q + \varepsilon\eta) - F(q)}{\varepsilon} = VF(q, \eta), \tag{5}$$

[85] Vainberg (1964). See also Atherton and Homsey (1975), and quoted references.

and is generally nonlinear in $\eta$, while the Frechét differential is always linear in $\eta$. Thus, a linear (uniformly continuous) Gateau differential is a Frechét differential.

**Theorem** [Vainberg (1964)]. *Suppose the following conditions are fulfilled:*

1. *$F$ is an operator from $Q(q)$ into the conjugate space $Q^*(q)$.*
2. *$F$ has a linear Gateau differential $VF(q, \eta)$ at every point of the ball $B$: $\|q - q_0\| < r$.*
3. *The functional $\{VF(q, \eta), \tilde{\eta}\}$ is continuous at every point of $B$.*

*Then, in order that the operator $F$ be potential in $B$, it is necessary and sufficient that the bilinear functional $\{VF(q, \eta), \tilde{\eta}\}$ be symmetrical for every $q$ in $B$, i.e.,*

$$\{VF(q, \eta), \tilde{\eta}\} = \{VF(q, \tilde{\eta}), \eta\}, \qquad \eta, \tilde{\eta} \in Q. \tag{6}$$

**Chart 3.17** Tonti's Approach to the Inverse Problem[86]

As recalled in the foreword, Vainberg's approach to the Inverse Problem within the context of functional analysis (Chart 3.16) was so abstract that it remained either inaccessible or unknown to researchers in applied mathematics for a considerable time.

E. Tonti recognized the significance of Vainberg's approach to the Inverse Problem and reformulated the approach in a form directly applicable to practical problems for single-integral as well as multiple-integral path functionals.

We consider here, for simplicity, the case of single-integral path functionals. The Frechét differential can be explicitly written

$$\lim_{\varepsilon \to 0} \frac{F(q + \varepsilon\eta) - F(q)}{\varepsilon} = F'_q\eta = \frac{\partial F}{\partial q}\eta + \frac{\partial F}{\partial \dot{q}}\dot{\eta} + \frac{\partial F}{\partial \ddot{q}}\ddot{\eta}. \tag{1}$$

According to Vainberg's theorem, a necessary and sufficient condition for an operator $F(q)$ to be potential, i.e., the gradient of a functional, can then be explicitly written

$$\begin{aligned} \int_{\bar{R}} dt[\tilde{\eta}(F'_q\eta) - \eta(\widetilde{F'_q\eta})] = \int_{\bar{R}} dt \Bigg\{ & \tilde{\eta}\left[\frac{\partial F}{\partial q} + \frac{\partial F}{\partial \dot{q}}\frac{d}{dt} + \frac{\partial F}{\partial \ddot{q}}\frac{d^2}{dt^2}\right]\eta \\ & - \eta\left[\left(\frac{\partial F}{\partial q} - \frac{d}{dt}\frac{\partial F}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial F}{\partial \ddot{q}}\right)\right. \\ & \left. + \left(-\frac{\partial F}{\partial \dot{q}} + 2\frac{d}{dt}\frac{\partial F}{\partial \ddot{q}}\right)\frac{d}{dt} + \frac{\partial F}{\partial \ddot{q}}\frac{d^2}{dt^2}\right]\tilde{\eta}\Bigg\} = 0 \end{aligned} \tag{2}$$

[86] Tonti (1968).

This condition, in turn, holds if and only if all the relations

$$\frac{\partial F_i}{\partial \ddot{q}^j} = \frac{\partial F_j}{\partial \ddot{q}^i},$$

$$\frac{\partial F_i}{\partial \dot{q}^j} + \frac{\partial F_j}{\partial \dot{q}^i} = 2\frac{d}{dt}\frac{\partial F_j}{\partial \ddot{q}^i}, \tag{3}$$

$$\frac{\partial F_i}{\partial q^j} - \frac{\partial F_j}{\partial q^i} = \frac{d}{dt}\left(\frac{d}{dt}\frac{\partial F_j}{\partial \ddot{q}^j} - \frac{\partial F_j}{\partial \dot{q}^i}\right)$$

are verified identically in the domain of definition of $F$. Equations (3) can then be used as the explicit form of the integrability conditions for the existence of a path functional.

*Nota Bene*: Tonti's analysis is particularly significant for the study of this monograph. It essentially indicates that the formulation of the Inverse Problem within the context of the functional analysis is equivalent to our variational approach to self-adjointness. Indeed, when explicitly computed, Frechét differential (1) *coincides* with variational forms (2.1.4), Vainberg's concept of potential operator, Equation (2); *coincides* with the concept of self-adjoint variational forms, Definition 2.1.2, and the explicit forms of the integrability conditions identified by Tonti, Equations (3); and *coincides* with Helmholtz's conditions of self-adjointness, Equations (2.1.18).

For brevity, we leave to the interested reader the proof that the geometrical approach by Klein (1962) or the cohomology approach by Hordneski (1974) are also equivalent to the variational approach to self-adjointness used in this monograph, in the sense that the integrability conditions for the existence of an action functional or a Lagrangian, when explicitly computed, either coincide with those obtainable with the variational approach or are equivalent to them.

In conclusion, it appears that the methodology of the Inverse Problem can be formulated in a variety of different mathematical methods which, however, result to be equivalent to the variational approach to self-adjointness.

**Chart 3.18** Analytic, Algebraic, and Geometrical Significance of the Conditions of Variational Self-Adjointness

The analytic, algebraic, and geometric approaches to Newtonian systems are known to be deeply interrelated. Therefore, it is significant to reinspect these interrelationships within the context of the Inverse Problem for phase space formulations.

Consider a class $\mathscr{C}^\infty$, regular Newtonian system in the general first-order form (Section 2.5)

$$C_{\mu\nu}(t, a)\dot{a}^\nu + D_\mu(t, a) = 0, \qquad \mu = 1, 2, \ldots, 2n. \tag{1}$$

The conditions of variational self-adjointness for this system are (Section 2.7)

$$C_{\mu\nu} + C_{\nu\mu} = 0, \tag{2a}$$

$$\frac{\partial C_{\mu\nu}}{\partial a^{\sigma}} + \frac{\partial C_{\nu\sigma}}{\partial a^{\mu}} + \frac{\partial C_{\sigma\mu}}{\partial a^{\nu}} = 0, \tag{2b}$$

$$\frac{\partial C_{\mu\nu}}{\partial t} = \frac{\partial D_{\mu}}{\partial a^{\nu}} - \frac{\partial D^{\nu}}{\partial a^{\mu}}. \tag{2c}$$

The methodological significance of the above conditions, depending on the profile considered, can be formulated as follows.

Analytic Significance. Conditions (2), when applied to a normal form, guarantee the existence of a Hamiltonian. *Indeed, for a normal form we have* $C_{\mu\nu} = \omega_{\mu\nu}$ *and* $D_{\mu} = -\Xi_{\mu}$. *The conditions for self-adjointness in this case are the integrability conditions for the existence of a Hamiltonian* (*Theorem* 2.12.1).

Algebraic Significance. Conditions (2) guarantee the existence of a Lie algebra structure. *In particular, conditions* (2a) *and* (2b) *are equivalent to the Lie algebra laws, as indicated in Section* 2.9.

Geometrical Significance. Conditions (2) guarantee the existence of symplectic or contact structure. *By keeping into account the Charts of Chapter* 2, *conditions* (2a) *and* (2b) *guarantee a symplectic structure for the autonomous case while the full set of conditions* (2) *guarantees a contact structure for the nonautonomous case.*

We can, therefore, reach the conclusion that the conditions of variational self-adjointness for (class $\mathscr{C}^{\infty}$, regular, local) Newtonian systems in their first-order general form constitute a symbiotic characterization of certain fundamental aspects of analytic mechanics, abstract algebras, and differential geometry.

## EXAMPLES

*Example 3.1*

We shall illustrate the *independent* application of the Inverse Problem for configuration and phase space formulations with a simple example.

Consider the case of a particle under a drag force represented by the equation of motion in the *kinematical* form

$$\ddot{q} - f(t, q, \dot{q}) \equiv (\ddot{q} + \gamma\dot{q})_{\mathrm{NSA}} = 0, \qquad \dot{q} > 0. \tag{1}$$

This equation is, trivially, of class $\mathscr{C}^{\infty}$ and regular, but *non-self-adjoint*. A Lagrangian for its direct analytic representation in terms of the conventional Lagrange's equation

does not exist (see the remarks after Theorem 3.5.1). However, the same equation of motion can be written in the equivalent *self-adjoint fundamental* form

$$A\ddot{q} + B \equiv e^{\gamma t}\ddot{q} + e^{\gamma t}\gamma\dot{q} = 0, \tag{2}$$

which satisfies all the conditions of Theorems 3.5.1 and 3.6.1. A Lagrangian for the direct representation

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right)_{\text{SA}} = [e^{\gamma t}(\ddot{q} + \gamma q)_{\text{NSA}}]_{\text{SA}} \tag{3}$$

then exists. Equations (3.6.3) in this case are

$$\frac{\partial^2 K}{\partial \dot{q}^2} = e^{\gamma t},$$

$$\frac{\partial C}{\partial q} - \frac{\partial D}{\partial t} = -e^{\gamma t}\gamma\dot{q} - \frac{\partial^2 K}{\partial \dot{q}\,\partial t}, \tag{4}$$

and solutions (3.6.4) become

$$K = \dot{q}\int_0^1 d\tau' \left[\dot{q}\int_0^1 d\tau\, e^{\gamma t}\right](\tau' q) = e^{\gamma t}\tfrac{1}{2}\dot{q}^2,$$

$$D = C = 0, \tag{5}$$

yielding the Lagrangian[87]

$$L = e^{\gamma t}\tfrac{1}{2}\dot{q}^2. \tag{6}$$

A simple inspection then proves the validity of the above computation. Indeed, from structure (3.1.3) of Lagrange's equations we have

$$\frac{\partial L}{\partial \dot{q}} = e^{\gamma t}\dot{q}, \qquad \frac{\partial^2 L}{\partial \dot{q}^2} = e^{\gamma t}, \qquad \frac{\partial^2 L}{\partial \dot{q}\,\partial t} = e^{\gamma t}\gamma\dot{q},$$
$$\frac{\partial^2 L}{\partial \dot{q}\,\partial q} = \frac{\partial L}{\partial q} = 0 \tag{7}$$

and this verifies representation (3).

Notice that Lagrangian (6) is of the generalized type (3.7.19) or (3.7.22) or (3.7.27), where the presence of a term which *multiplies* the Lagrangian for the free motion, i.e., $\frac{1}{2}\dot{q}^2$, is *necessary* because the acting force is not derivable from a potential.

This concludes a first use of the Inverse Problem for configuration space formulations.

We now turn to the *independent* application of the inverse problem for phase space formulations. By following the guidelines given after Theorem 3.12.2, we introduce the following prescription for the reduction of Equation (1) to an equivalent system of two first-order differential equations,

$$G = \alpha(t, q, y)\dot{q} + \beta(t, q, y) = 0,$$
$$\alpha, \beta \in \mathscr{C}^1(R^3), \qquad \alpha(R) \neq 0, \tag{8}$$

[87] This and several other Lagrangian have been identified without the use of the Inverse Problem. See, for instance, Denman and Buch (1972) and quoted papers.

where the functions $\alpha$ and $\beta$ are at this moment undetermined. After substituting in Equation (1), we obtain the system

$$\begin{aligned} &\dot{q} - g_1(t, q, y) = 0, \qquad g_1 = -\beta/\alpha, \\ &-\dot{y} - g_2(t, q, y) = 0, \qquad g_2 = \frac{(\partial g_1/\partial q)g_1 + (\partial g_1/\partial t) + \gamma g_1}{\partial g_1/\partial y}, \end{aligned} \tag{9}$$

which can be represented with the notation

$$\begin{aligned} &\omega_{\mu\nu}\dot{a}^\nu - \Xi_\mu(t, a) = 0, \\ &(\omega_{\mu\nu}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad (a^\mu) = \begin{pmatrix} q \\ y \end{pmatrix}, \qquad (\Xi_\mu) = \begin{pmatrix} g_2 \\ g_1 \end{pmatrix}. \end{aligned} \tag{10}$$

The conditions of self-adjointness for this form, Equations (3.12.5) or, explicitly, (3.12.9), now give rise only to the condition

$$\frac{\partial g_2}{\partial y} = \frac{\partial g_1}{\partial q}. \tag{11}$$

All the functions $\alpha$ and $\beta$ that, through Equations (9), satisfy the above condition, imply the self-adjointness of system (10) and, from Theorem 3.12.2, the existence of corresponding Hamiltonians. This gives an indication of the existence of a *family of different Hamiltonians* that are all capable of representing the considered system. However, we are interested, at this stage, in identifying *one* Hamiltonian. To simplify the computation we assume the reduced form of prescription (8),

$$G = \dot{q} + \beta(y) = 0, \tag{12}$$

where $\beta$ is again unknown but it is independent of $t$ and $q$. Equation (11) then becomes

$$\frac{\partial g_2}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\gamma\beta}{\partial\beta/\partial y}\right) = \frac{\partial g_1}{\partial q} = -\frac{\partial\beta}{\partial q}, \tag{13}$$

admitting as a solution

$$\beta = ce^{y+d}, \qquad c, d = \text{const.} \tag{14}$$

By assuming the values $c = -1$ and $d = 0$, for convenience, system (10) becomes

$$(\omega_{\mu\nu}\dot{a}^\nu - \Xi_\mu)_{\text{SA}} = \begin{pmatrix} -\dot{y} - \gamma \\ +\dot{q} - e^y \end{pmatrix}_{\text{SA}} = 0. \tag{15}$$

Such a system satisfies all the conditions of Theorem 3.12.2. A Hamiltonian then exists and, from Equation (3.12.6), is given by

$$\begin{aligned} H^* &= a^\mu \int_0^1 d\tau\, \Xi_\mu(t, \tau a) = a^1 \int_0^1 d\tau\, \Xi_1(\tau a) + a^2 \int_0^1 d\tau\, \Xi_2(\tau a) \\ &= q\int_0^1 d\tau\, \gamma + p\int_0^1 d\tau\, e^{\tau p} = \gamma q + e^p - 1, \end{aligned} \tag{16}$$

where we have used the identification $y = p$ because we now have a canonical structure. It is significant here to indicate that alternative method (3.12.7) can also be applied, yielding

$$\begin{aligned} H^* &= \int_0^{a^\mu} da^\mu \, \Xi_\mu = \int_0^{a^1} da^1 \, \Xi_1 + \int_0^{a_2} da^2 \, \Xi_2 \\ &= \int_0^q dq \, \gamma + \int_0^p dp \, e^p = \gamma q + e^p - 1. \end{aligned} \tag{17}$$

A simple inspection then verifies that the above Hamiltonian does indeed provide a representation of system (1) in phase space, because

$$\left(\omega_{\mu\nu} a^\nu - \frac{\partial H^*}{\partial a^\mu}\right)_{\text{SA}} = \begin{pmatrix} -\dot{p} - \gamma \\ +\dot{q} - e^p \end{pmatrix}_{\text{SA}} = 0. \tag{18}$$

The third part of this example deals with the necessary comparison of the above independent applications of the Inverse Problem.

By using the methodology of Section 3.8 and, as expected, from the analysis of Section 3.12, we immediately see that *Lagrangian* (6) *and Hamiltonian* (16) *are not related by a Legendre transform.* Rather than being a drawback of the Inverse Problem, this indicates its richness, particularly from the viewpoint of transformation theory, because the resultant Lagrangian and Hamiltonian represent, by construction, the *same* system.

First, let us see whether the framework of Lagrangian (6) has a phase space image which is consistent with the Inverse Problem. By using a direct Legendre transform, from Lagrangian (6) we can write the Hamiltonian

$$\begin{aligned} p - \frac{\partial L}{\partial \dot{q}} &= p - e^{\gamma t}\dot{q} = 0, \\ H &= e^{-\gamma t}\tfrac{1}{2}p^2. \end{aligned} \tag{19}$$

From the form of canonical prescription (19a), we can then identify the functions $\alpha$ and $\beta$ of Equation (8) as

$$\alpha = -e^{\gamma t}, \qquad \beta = p. \tag{20}$$

A simple inspection shows that the above values do satisfy condition (11), yielding the self-adjoint covariant normal form

$$(\omega_{\mu\nu}\dot{a}^\nu - \Xi_\mu)_{\text{SA}} = \begin{pmatrix} -\dot{p} - 0 \\ +\dot{q} - e^{-\gamma t}p \end{pmatrix}_{\text{SA}} = 0. \tag{21}$$

The use of Equation (3.12.6) then yields the Hamiltonian

$$H = p\int_0^1 d\tau \, e^{-\gamma t\tau} p = e^{-\gamma t}\tfrac{1}{2}p^2, \tag{22}$$

which coincides with that obtained by using a Legendre transform of Lagrangian (6). This indicates that the phase space image of the analytic framework of representation (3) is indeed consistent with the Inverse Problem.

Finally, we inspect the configuration space image of analytic representation (15) with Hamiltonian (16). First, an inverse Legendre transform yields

$$\dot{q} - \frac{\partial H^*}{\partial p} = \dot{q} - e^p,$$

$$L^* = \dot{q}p - H^* = \dot{q} \ln \dot{q} - \gamma q - (\dot{q} - 1). \tag{23}$$

The *new* Lagrangian representation is now given by

$$\left(\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right)_{SA} = \left[\frac{1}{\dot{q}}(\ddot{q} + \gamma\dot{q})_{NSA}\right]_{SA}. \tag{24}$$

A simple inspection shows that, despite the presence of a new factor term as compared with Equation (3), the right-hand side of the above identification is indeed self-adjoint.

A point of considerable methodological significance is that *representation* (24) *is not well defined in a star-shaped region* because of its divergent character at $\dot{q} = 0$.

The reader is urged, at this point, to verify that *solutions* (3.6.4) *do not hold for representation* (24). However, the underlying system of equations for the construction of a Lagrangian, Equations (3.6.3), is given by

$$\frac{\partial^2 K}{\partial \dot{q}^2} = \frac{1}{\dot{q}},$$

$$\frac{\partial C}{\partial q} - \frac{\partial D}{\partial t} = \gamma, \tag{25}$$

and it is consistent from the self-adjointness of the right-hand side of Equation (24). Therefore, it can be integrated with methods *other than* that of Equations (3.6.4). Indeed, the use of the second method of Chart 3.11 yields, as a solution of Equations (25),

$$K = \dot{q} \ln \dot{q}, \qquad D = -1, \qquad C = -\gamma q + 1, \tag{26}$$

with underlying Lagrangian

$$L^* = K + D\dot{q} + C = \dot{q} \ln \dot{q} - \gamma q - (\dot{q} - 1), \tag{27}$$

which coincides with that obtained through an inverse Legendre transform of Hamiltonian (16). This illustrates the statement of Section 3.6 to the effect that, under the conditions of self-adjointness, a Lagrangian or Hamiltonian is expected to exist without any restriction to a star-shaped region.

*Example 3.2*

The analysis of the previous example can be extended to the more general *non-self-adjoint* equation of motion

$$(\ddot{q} + \gamma\dot{q} + \omega^2 q)_{NSA} = 0, \qquad \omega^2 - \frac{\gamma^2}{4} > 0, \tag{1}$$

which represents the one-dimensional, linear, damped oscillator.

The method of reducing this equation to an equivalent self-adjoint form is studied in Santilli (1979). Therefore, we ignore, at this stage, the problem of constructing a

Lagrangian and search for a Hamiltonian representation. Explicit calculations indicate that prescription (8), with values (20) of the previous example, also leads to a self-adjoint covariant normal form of the equation of motion that satisfies all the conditions of Theorem 3.12.2. The Hamiltonian, then, exists and is given by

$$H = e^{-\gamma t}\tfrac{1}{2}p^2 + e^{+\gamma t}\tfrac{1}{2}\omega^2 q^2. \tag{2}$$

This yields, by using an inverse Legendre transform, the Lagrangian

$$L = e^{\gamma t}\tfrac{1}{2}(\dot{q}^2 - \omega^2 q^2), \tag{3}$$

which produces the representation

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right)_{\mathrm{SA}} = [e^{\gamma t}(\ddot{q} + \gamma\dot{q} + \omega^2 q)_{\mathrm{NSA}}]_{\mathrm{SA}}. \tag{4}$$

The reader can now verify that, *because* of the factor $e^{\gamma t}$, the above equivalent equation of motion is self-adjoint. If such a self-adjoint form were known, one could have first used Theorem 3.6.1 for the computation of Lagrangian (3) and then computed Hamiltonian (2) through a direct Legendre transform with equivalent results. But the above value of the prescription, which induces a self-adjoint normal form, *is not* unique. A more elaborate solution then leads to the Hamiltonian[88]

$$H = \ln q - \ln[\cos(\omega pq)] - \tfrac{1}{2}\gamma pq, \tag{5}$$

with corresponding Lagrangian

$$L = \frac{2\dot{q} + \gamma q}{2q\sqrt{\omega^2 - \gamma^2/4}}\, tg^{-1}\left(\frac{2\dot{q} + \gamma q}{2q\sqrt{\omega^2 - \gamma^2/4}}\right) - \tfrac{1}{2}\ln(\dot{q}^2 + \gamma q\dot{q} + \omega^2 q^2), \tag{6}$$

which can be considered as the generalization of corresponding functions (16) and (23) of the previous example. Notice that for both Examples 3.1 and 3.2, the knowledge of the Lagrangian *alone* is insufficient for establishing an analytic representation in line with Definition 3.4.1.

*Example 3.3*

Consider the class $\mathscr{C}^2$ regular system of linear, coupled, and damped oscillators

$$\begin{aligned} &a_{ki}(t)q^i + b_{ki}(t)\dot{q} + c_{ki}(t)\ddot{q}^i = 0, \\ &a_{ki}, b_{ki}, c_{ki} \in \mathscr{C}^2(R_t), \qquad |C_{ki}(R_t)| \neq 0. \end{aligned} \tag{1}$$

The system is s[illegible] adjoint if and only if conditions (2.1.17) hold, i.e.,

$$\begin{aligned} c_{ij} - c_{ji} &= 0, \\ b_{ij} + b_{ji} &= 2\dot{c}_{ij}, \\ a_{ij} - a_{ji} &= -\dot{b}_{ij} + \ddot{c}_{ij}. \end{aligned} \tag{2}$$

Therefore, the particular system

$$\begin{aligned} &a_{ki}(t)q^i + \dot{c}_{ki}(t)\dot{q}^i + c_{ki}(t)\ddot{q} = 0, \\ &a_{ki} = a_{ik}, \qquad c_{ki} = c_{ik} \end{aligned} \tag{3}$$

[88] Havas (1957).

is self-adjoint. To determine the Lagrangian we use Theorem 3.6.1 which, in this case, yields the equations

$$\begin{aligned} \frac{\partial^2 K}{\partial \dot{q}^i\, \partial \dot{q}^j} &= c_{ij}(t), \\ \frac{\partial D_i}{\partial q^j} - \frac{\partial D_j}{\partial q^i} &= 0, \\ \frac{\partial D_i}{\partial t} - \frac{\partial C}{\partial q^i} &= a_{ik}(t)q^k + \frac{\partial K}{\partial \dot{q}^i} - \frac{\partial^2 K}{\partial q^i\, \partial t}, \end{aligned} \tag{4}$$

with a solution

$$\begin{aligned} K &= \tfrac{1}{2}c_{ij}(t)\dot{q}^i q^j, \\ D_k &= 0, \qquad k = 1, 2, \ldots, n, \\ C &= -\tfrac{1}{2}a_{ij}(t)q^i q^j. \end{aligned} \tag{5}$$

Therefore, one admissible Lagrangian is given by

$$L(t, q, \dot{q}) = \tfrac{1}{2}[\dot{q}^i c_{ij}(t)\dot{q}^j - q^i a_{ij}(t)q^j]. \tag{6}$$

*Example 3.4*

A Lagrangian for the self-adjoint system

$$\left\{c_{ki}(t, q)\ddot{q}^i + \left[\frac{\partial c_{ki}(t, q)}{\partial q^j} - \frac{\partial c_{ij}(t, q)}{\partial q^k}\right]\dot{q}^i\dot{q}^j + \frac{\partial c_{ki}(t, q)}{\partial t}\dot{q}^i + a_{ki}(t)q^i\right\}_{\text{SA}} = 0 \tag{1}$$

is

$$L = \tfrac{1}{2}[\dot{q}^i c_{ij}(t, q)\dot{q}^j - q^i a_{ij}(t)q^j]. \tag{2}$$

*Example 3.5*

Consider the radial equation for a particle subject to a central force field,

$$\left[m\ddot{r} - \frac{M^2}{mr^3} + \frac{\partial V(r)}{\partial r}\right]_{\text{SA}} = 0. \tag{1}$$

This equation is self-adjoint and can be directly represented with the well-known Lagrangian

$$L = \tfrac{1}{2}m\left(\dot{r}^2 + \frac{M^2}{m^2r^2}\right) - v(r). \tag{2}$$

However, a Lagrangian capable of satisfying the *direct* representation of the equivalent form

$$\left[m^2r^3\ddot{r} - M^2 + mr^3\frac{\partial V(r)}{\partial r}\right]_{\text{NSA}} = 0 \tag{3}$$

*does not* exist because the equivalent system is non-self-adjoint. In particular, Equations (3.6.3) become inconsistent. This can be seen by first computing the solution of Equation (3.6.4a), i.e.,

$$K = \tfrac{1}{2}m^2r^3\dot{r}^2. \tag{4}$$

Then, Equation (3.6.3c), i.e.,

$$\frac{\partial D}{\partial t} - \frac{\partial C}{\partial r} = mr^3 \frac{\partial V}{\partial r} - M^2 - \tfrac{3}{2}m^2 r^2 \dot{r} \tag{5}$$

is now *inconsistent* due to the appearance of the velocity $\dot{r}$ on the right-hand side. Similarly, one can see that a Lagrangian capable of satisfying the direct representation of the equivalent non-self-adjoint form of the harmonic oscillator (for $q \neq 0$), given by

$$\left(\frac{\ddot{q}}{q} + \omega^2\right)_{\text{NSA}} = 0, \tag{6}$$

*does not* exist.

*Example 3.6*

In this example, we illustrate in more detail the "mechanics" of the construction of a Lagrangian from the equations of motion according to the method of Section 3.6 for the case of more than one dimension. The two-dimensional, nonlinear, nonconservative system

$$\begin{aligned} &(\dot{q}_1 + 2\dot{q}_2)\ddot{q}_2 + 2(\dot{q}_1 + \dot{q}_2)\ddot{q}_2 + q_2\dot{q}_2 - q_1\dot{q}_2 + q_1 q_2 + \tfrac{1}{2}q_2^2 = 0, \\ &2(\dot{q}_1 + \dot{q}_2)\ddot{q}_1 + (2q_1 + \dot{q}_2)\ddot{q}_2 + q_1\dot{q}_1 - q_2\dot{q}_1 + q_1 q_2 + \tfrac{1}{2}q_1^2 = 0 \end{aligned} \tag{1}$$

satisfies all the conditions of Theorem 3.6.1, namely, the system is (a) of class $C^\infty$, (b) regular, and (c) well behaved and self-adjoint in a star-shaped region. Thus, a Lagrangian for its ordered direct analytic representation in terms of the conventional Lagrange's equations exists in terms of solutions (3.6.4). System (1) is the fundamental form

$$A_{ki}(\dot{q})\ddot{q}^i + B_k(q, \dot{q}) = 0, \qquad k = 1, 2. \tag{2a}$$

$$(A_{ij}) = \begin{pmatrix} (\dot{q}_1 + 2\dot{q}_2) & 2(\dot{q}_1 + \dot{q}_2) \\ 2(\dot{q}_1 + \dot{q}_2) & (2\dot{q}_1 + \dot{q}_2) \end{pmatrix} \tag{2b}$$

$$(B_k) = \begin{pmatrix} (q_2 - q_1)\dot{q}_2 + q_1 q_2 + \tfrac{1}{2}q_2^2 \\ (q_1 - q_2)\dot{q}_1 + q_1 q_2 + \tfrac{1}{2}q_1^2 \end{pmatrix}. \tag{2c}$$

The use of Equation (3.6.4a) then yields

$$\begin{aligned} K &= (\dot{q}_1\dot{q}_2)\int_0^1 d\tau' \left\{ \int_0^1 d\tau \begin{pmatrix} \tau(\dot{q}_1 + 2\dot{q}_2) & 2\tau(\dot{q}_1 + \dot{q}_2) \\ 2\tau(\dot{q}_1 + \dot{q}_2) & \tau(2\dot{q}_1 + \dot{q}_2) \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \right\} (\tau'\dot{q}) \\ &= (\dot{q}_1\dot{q}_2)\int_0^1 d\tau' \left[ \begin{pmatrix} \tfrac{1}{2}(\dot{q}_1 + 2\dot{q}_2) & (\dot{q}_1 + \dot{q}_2) \\ (\dot{q}_1 + \dot{q}_2) & \tfrac{1}{2}(2\dot{q}_1 + \dot{q}_2) \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \right] (\tau'\dot{q}) \\ &= (\dot{q}_1\dot{q}_2)\int_0^1 d\tau' \begin{pmatrix} \tfrac{1}{2}\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \dot{q}_2^2 \\ \dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \tfrac{1}{2}\dot{q}_2^2 \end{pmatrix} (\tau'\dot{q}) \\ &= (\dot{q}_1\dot{q}_2)\int_0^1 d\tau' \begin{pmatrix} \tau'^2(\tfrac{1}{2}\dot{q}_1 + 2\dot{q}_1\dot{q}_2 + \dot{q}_2^2) \\ \tau'^2(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \tfrac{1}{2}\dot{q}_2^2) \end{pmatrix} \\ &= (\dot{q}_1\dot{q}_2) \begin{pmatrix} \tfrac{1}{6}\dot{q}_1^2 + \tfrac{2}{3}\dot{q}_1\dot{q}_2 + \tfrac{1}{3}\dot{q}_2^2 \\ \tfrac{1}{3}\dot{q}_1^2 + \tfrac{2}{3}\dot{q}_1\dot{q}_2 + \tfrac{1}{6}\dot{q}_2^2 \end{pmatrix} \\ &= \tfrac{1}{6}(\dot{q}_1^3 + \dot{q}_2^3) + \dot{q}_1^2\dot{q}_2 + \dot{q}_1\dot{q}_2^2, \end{aligned} \tag{3}$$

which is the "kinetic term" of Lagrangian (3.6.2). To compute the $D_k$ terms of the same structure, we must first compute the $Z_{ki}$ terms of Equation (3.6.3b). Since function (3) does not depend on the $q$'s, we have

$$(Z_{ij}) = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial B_1}{\partial \dot{q}^2} - \frac{\partial B_2}{\partial \dot{q}^1}\right) \\ \frac{1}{2}\left(\frac{\partial B_2}{\partial \dot{q}^1} - \frac{\partial B_1}{\partial \dot{q}^2}\right) & 0 \end{pmatrix} = \begin{pmatrix} 0 & (q_2 - q_1) \\ (q_1 - q_2) & 0 \end{pmatrix}. \tag{4}$$

Equation (3.6.4b) then becomes

$$\begin{aligned} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} &= \left[\int_0^1 d\tau\, \tau \begin{pmatrix} 0 & Z_{12}(\tau q) \\ Z_{21}(\tau q) & 0 \end{pmatrix}\right]\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ &= \left[\int_0^1 d\tau \begin{pmatrix} 0 & \tau^2(q_2 - q_1) \\ \tau^2(q_1 + q_2) & 0 \end{pmatrix}\right]\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{3}(q_2 - q_1) \\ \frac{1}{3}(q_1 - q_2) & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3}(q_2^2 - q_1 q_2) \\ \frac{1}{3}(q_1^2 - q_1 q_2) \end{pmatrix}. \end{aligned} \tag{5}$$

We can thus construct the second term of Lagrangian (3.6.2) according to

$$\begin{aligned} D_k(q)\dot{q}^k &= (\dot{q}_1 \dot{q}_2)\begin{pmatrix} \frac{1}{3}(q_2^2 - q_1 q_2) \\ \frac{1}{3}(q_1^2 - q_1 q_2) \end{pmatrix} \\ &= \tfrac{1}{3}(q_2^2 \dot{q}_1 + q_1^2 \dot{q}_2) - \tfrac{1}{3} q_1 q_2 (\dot{q}_1 + \dot{q}_2). \end{aligned} \tag{6}$$

To compute the last term $C$ of the same structure we must first compute the $W_k$ terms of Equation (3.6.3c). Since $K$ is independent of the $q$'s and the $D_k$'s are independent of time, we have

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} -B_1 + \frac{1}{2}\left(\frac{\partial B_1}{\partial \dot{q}^k} - \frac{\partial B_k}{\partial \dot{q}^1}\right)\dot{q}^k \\ -B_2 + \frac{1}{2}\left(\frac{\partial B_2}{\partial \dot{q}^k} - \frac{\partial B_k}{\partial \dot{q}^2}\right)\dot{q}^k \end{pmatrix} = \begin{pmatrix} -q_1 q_2 - \frac{1}{2}q_2^2 \\ -q_1 q_2 - \frac{1}{2}q_1^2 \end{pmatrix}. \tag{7}$$

Notice that the above terms are also independent of the velocities as guaranteed by the self-adjointness of the equations of motion. The use of Equation (3.6.4c) then yields the desired result

$$\begin{aligned} C &= (q_1 q_2)\int_0^1 d\tau \begin{pmatrix} W_1(\tau q) \\ W_2(\tau q) \end{pmatrix} \\ &= (q_1 q_2)\int_0^1 d\tau \begin{pmatrix} -\tau^2(q_1 q_2 + \frac{1}{2}q_2^2) \\ -\tau^2(q_1 q_2 + \frac{1}{2}q_1^2) \end{pmatrix} \\ &= -(q_1 q_2)\begin{pmatrix} \frac{1}{3}(q_1 q_2 + \frac{1}{2}q_2^2) \\ \frac{1}{3}(q_1 q_2 + \frac{1}{2}q_1^2) \end{pmatrix} \\ &= -\tfrac{1}{2}(q_1^2 q_2 + q_1 q_2^2). \end{aligned} \tag{8}$$

By summing up the various terms, the Lagrangian is given by

$$\begin{aligned} L &= K(\dot{q}) + D_k(q)\dot{q}^k + C(q) \\ &= \tfrac{1}{6}(\dot{q}_1^3 + \dot{q}_2^3) + \dot{q}_1^2\dot{q}_2 + \dot{q}_1\dot{q}_2^2 \\ &\quad + \tfrac{1}{3}(q_2^2\dot{q}_1 + q_1^2\dot{q}_2) - \tfrac{1}{3}q_1q_2(\dot{q}_1 + \dot{q}_2) - \tfrac{1}{2}(q_1^2q_2 + q_1q_2^2). \end{aligned} \tag{9}$$

The interested reader is urged, at this point, to verify, through the proper use of Lagrange's equations in their full form (3.1.3), that Lagrangian (9) does indeed provide an ordered direct analytic representation of system (1).

## Problems

3.1 Establish alternative proofs for the self-adjointness of Lagrange's equations by writing them in the form

$$F_k(q) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} = 0,$$

$$A_{ki}\ddot{q}^i + B_k = \left(\frac{\partial^2 L}{\partial \dot{q}^k\,\partial \dot{q}^i}\right)\ddot{q}^i + \left(\frac{\partial^2 L}{\partial \dot{q}^k\,\partial q^i}\dot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k\,\partial t} - \frac{\partial L}{\partial q^k}\right) = 0,$$

and by proving that they satisfy *all* conditions of self-adjointness (2.1.18) and (2.2.9), respectively.

3.2. By using Theorems 2.2.2 and 3.5.1, prove the fundamental analytic theorem for kinematical forms, i.e.,

**Theorem.** *A necessary and sufficient condition for a local, holonomic, generally non-conservative Newtonian system in the kinematical form*

$$\ddot{q}_k - f_k(t, q, \dot{q}) = 0, \qquad k = 1, 2, \ldots, n,$$

*which is well defined and of (at least) class $\mathscr{C}^2$ in a star-shaped region $R^{*2n+1}$ of points $(t, q, \dot{q})$, to admit an ordered direct analytic representation in terms of the conventional Lagrange's equations in $R^{*2n+1}$,*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \equiv \ddot{q}_k - f_k(t, q, \dot{q}), \qquad k = 1, 2, \ldots, n,$$

*is that the system of equations of motion is self-adjoint in $R^{*2n+1}$.*

3.3 Prove the *Inverse of the D'Alembert Principle* (A.4.9) in configuration space, namely, that the equations of motion

$$\begin{aligned} A_{ki}(t, q)\ddot{q}^i + B_k(t, q, \dot{q}) &= Z_{ki}(t, q)\ddot{q}^i + \left[\frac{\partial Z_{kj}(t, q)}{\partial q^i} - \frac{1}{2}\frac{\partial Z_{ij}(t, q)}{\partial q^k}\right]\dot{q}^i\dot{q}^j \\ &\quad + \left[\frac{\partial Z_k(t, q)}{\partial q^i} - \frac{\partial Z_i(t, q)}{\partial q^k} + \frac{\partial Z_{ki}(t, q)}{\partial t}\right]\dot{q}^i \\ &\quad \times \frac{\partial Z_k(t, q)}{\partial t} - \frac{\partial Z_0(t, q)}{\partial q^k} - \frac{\partial V(q)}{\partial q^k} \\ &= 0 \end{aligned}$$

admit an ordered, direct analytic representation in Lagrange's equations with Lagrangian

$$\begin{aligned} L &= T(t, q, \dot{q}) - V(q) \\ &= \tfrac{1}{2}\dot{q}^i Z_{ij}(t, q)\dot{q}^j + Z_k(t, q)\dot{q}^k + Z_0(t, q) - V(q). \end{aligned}$$

3.4 Study the generalization of Problem 3.3 to the case of forces derivable from a potential $U(t, q, \dot{q})$.

3.5 Prove the crucial relation (3b) of Chart 3.3 for a generalization of Hamilton's principle to include the integrability conditions for the existence of a Lagrangian, by proving the following theorem.

**Theorem.** *A necessary and sufficient condition for the self-adjointness of a system of differential equations is that the quantity $Q(\eta, \tilde{\eta})$ of Definition* 2.3.1 *is conserved, i.e.,*

$$\lim_{\tilde{\eta} \to \eta} \frac{d}{dt} Q(\eta, \tilde{\eta}) = 0.$$

3.6 Newton's equations of motion in their fundamental (second-order) form can also be represented with *second-order Lagrangians* $L(t, q, \dot{q}, \ddot{q})$, which are totally degenerate (i.e., linear) in the accelerations and corresponding second-order Lagrange's equations. By using the methodology of Chart 3.10, prove the following theorem.

**Theorem.** *A necessary and sufficient condition for a Newtonian system of second-order equations*

$$F_k = A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k = 0, \qquad k = 1, 2, \ldots, n,$$

*which is well defined, regular and of at least class $\mathscr{C}^3$ in a star-shaped region $R^{*3n+1}$ of the variables $(t, q, \dot{q}, \ddot{q})$ to admit the ordered direct analytic representation in terms of totally degenerate second-order Lagrange's equations in $R^{*3n+1}$,*

$$-\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}^k} + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \equiv A_{ki}\ddot{q}^i + B_k,$$

*is that the system is self-adjoint in $R^{*2n+1}$, i.e., that each of the conditions* (2.1.18) *are satisfied everywhere in $R^{*3n+1}$. In particular, establish a method for the computation of this higher-order Lagrangian.*

3.7 Prove that the equations of variations of Hamilton's equations can be derived through a direct Legendre transform of the equations of variations of Lagrange's equations and vice versa.

3.8 By following the procedure in Example 3.1, construct an analytic representation of the non-self-adjoint system with an "inverted" drag force,

$$\ddot{q} + \gamma\dot{q}^{-1} = 0, \qquad \dot{q} \neq 0,$$

by first identifying a self-adjoint covariant normal form which satisfies all the conditions of Theorem 3.12.2, then by computing a Hamiltonian by means of Equations (3.12.6), and then by computing a Lagrangian through an inverse Legendre transform. Ascertain whether such a Lagrangian characterizes an analytic representation which is or is not well defined in a star-shaped region of its variables.

3.9 The Newtonian limit of the field theoretical *sine-Gordon equation* can be written

$$\ddot{q} + \sin q = 0.$$

Prove that this equation satisfies all the conditions of Theorem 3.6.1 and compute a Lagrangian. The given equation of motion with related Lagrange's equation is highly nonlinear. Show that the associated Jacobi's equations are linear and compute their solution.

3.10 Compute the factor term $h$ of analytic representation (3.4.1a) induced by Lagrangian (5) of Example 3.2.

3.11 Compute a Lagrangian for the self-adjoint system

$$\sin(\dot{q}_1\dot{q}_2)(\dot{q}_2^2\ddot{q}_1 + \dot{q}_1\dot{q}_2\ddot{q}_2) + q_2 = 0,$$
$$\sin(\dot{q}_1\dot{q}_2)(\dot{q}_1^2\ddot{q}_2 + \dot{q}_1\dot{q}_2\ddot{q}_1) + q_1 = 0.$$

APPENDIX

# Newtonian Systems

## A.1 Newton's Equations of Motion

As indicated in the Introduction, present-day theoretical physics provides a variety of methodological formulations to represent the physical world. Each formulation can be identified from the salient characteristics of the considered system such as, for instance, whether the system is

1. discrete or continuous,
2. nonrelativistic or relativistic,[1]
3. classical or quantum mechanical.

In this book, we study some methodological foundations of only one class of physical systems: the discrete, nonrelativistic, classical systems also known as *Newtonian systems*. Therefore, we shall restrict our attention only to those systems for which the continuous, relativistic, and quantum mechanical aspects can be ignored.

Within such a framework, the equations describing the motion of $N$ particles in a Cartesian reference frame are the celebrated *Newton's equations*[2]

$$\dot{\mathbf{p}}_k = \mathscr{F}_k(t, \mathbf{r}, \dot{\mathbf{r}}), \qquad k = 1, 2, \ldots, N, \tag{A.1.1}$$

[1] This terminology, even though almost universally accepted, is not immune from criticism, due to the fact that what is commonly referred to as a "nonrelativistic" ("relativistic") system is in actuality a *Galilei (Einstein) relativistic system*. See, in this respect, Chart A.1.

[2] Newton (1687). For historical notes see, for instance, Dugas (1950).

where $\mathscr{F}_k$ is the total force acting on the particle $k$, $\mathbf{p}_k$ is the linear momentum, and $\dot{\mathbf{p}}_k$ is its total time derivative.[3]

In this Appendix, we classify systems (A.1.1) from a physical profile and point out a general form of the equations for the (constrained) motion.

## A.2 Constraints

The study of Newtonian systems demands the characterization not only of all the acting forces but also of all the possible *constraints*, namely, restrictions on the dynamical evolution. Several types of constraints exist, e.g., those restricting the degrees of freedom of coordinates, velocities, and accelerations. To avoid possible confusion, it is important to classify such constraints and properly identify the related terminology.

It should be mentioned in this respect that, regrettably, there does not seem to be a universally accepted terminology in the existing literature and sometimes identical terms are used with different meanings by different authors. This has led to a voluminous and often contrasting literature on the problem of the identification of the class of Newtonian systems that are derivable from a variational principle.[4]

Consider a system of $N$ particles in a three-dimensional Cartesian reference frame. Suppose that certain means (hinges, strings, etc.) restrict the possible values of the coordinates $\mathbf{r}^i$, velocities $\dot{\mathbf{r}}^i$, and accelerations $\ddot{\mathbf{r}}^i$.

The first classification of such constraints which is significant for our analysis can be formulated in terms of whether:

I.a. all constraints can be expressed in terms of equalities, e.g.,

$$\phi_s(t, \mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}) = 0, \qquad s = 1, 2, \ldots, r; \tag{A.2.1}$$

I.b. all constraints can be expressed in terms of inequalities, e.g.,

$$\phi_s(t, \mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}) \geq 0, \qquad s = 1, 2, \ldots, r; \tag{A.2.2}$$

I.c. some of the constraints are of type I.a and others are of type I.b.

A particle[5] that can only move along a sphere of radius $R$ is subject to a constraint expressible in terms of an equality, i.e.,

$$|\mathbf{r}| - R = 0, \tag{A.2.3}$$

and thus this constraint is of type I.a.

A particle placed on the surface of a sphere of radius $R$ is subject to a constraint which is only expressible in terms of an inequality, i.e.,

$$|\mathbf{r}| - R \geq 0. \tag{A.2.4}$$

Thus the constraint in this case is of type I.b.

[3] We shall denote with script letters, e.g., $\mathscr{F}_k$, $\mathscr{V}$, $\mathscr{T}$, etc. quantities in a Cartesian reference frame.

[4] See, for instance, Rund (1966, Section 5.5).

[5] By *particle* we mean a massive point.

A second significant classification is given in terms of whether:

II.a. all constraints do not depend explicitly on time;
II.b. all constraints do depend explicitly on time;
II.c. some constraints are of type II.a and others are of type II.b.

For instance, constraints (A.2.3) and (A.2.4) are of type II.a. A typical example of a constraint of type II.b occurs in the case of a bead sliding on a moving wire.

The third classification is given in terms of whether:

III.a. all constraints either are expressed by, or can be reduced to, equalities involving the positions and, possibly, time, i.e.,

$$\phi_s(t, \mathbf{r}) = 0, \qquad s = 1, 2, \ldots, r; \tag{A.2.5}$$

III.b. no constraint can be reduced to the form (2.2.5);
III.c. some constraints are of type III.a and others are of type III.b.

In accordance with the most widely accepted (although not universally accepted) terminology, constraints III.a (III.b) will be called *holonomic (nonholonomic)*[6] and those of type II.a (II.b) will be termed *scleronomous (rheonomous)*[7] *constraints.*

Thus, Equation (A.2.3) represents a scleronomous holonomic constraint. Expression (A.2.4), if modified for the case of the moving sphere, would represent a rheonomous nonholonomic constraint. Those are simple examples for which, apparently, there is no major differentiation in the existing literature as far as the terminology is concerned.

However, according to our terminology, velocity-dependent contraints of the type

$$\phi_s(t, \mathbf{r}, \dot{\mathbf{r}}) = 0, \qquad s = 1, 2, \ldots, r \tag{A.2.6}$$

can be either holonomic or nonholonomic depending on whether they are integrable or not, that is, whether or not there exist functions $\phi'_s(t, \mathbf{r})$ such that

$$\phi_s(t, \mathbf{r}, \dot{\mathbf{r}}) = \frac{d}{dt}\phi'_s(t, \mathbf{r}) = 0. \tag{A.2.7}$$

When this is the case, constraints (A.2.6) are equivalent to the conventional holonomic coordinate constraints

$$\phi'_s(t, \mathbf{r}) = 0, \qquad s = 1, 2, \ldots, r. \tag{A.2.8}$$

This is so because of the property that, for consistency, a constraint must be obeyed at all times. Thus, when Equation (A.2.8) holds, Equation (A.2.7) also holds, and vice versa.

For instance, if a system of two particles is subject to the velocity constraint

$$\dot{\mathbf{r}}^{(1)} - \dot{\mathbf{r}}^{(2)} = 0, \tag{A.2.9}$$

[6] Von Hertz (1894).
[7] Boltzmann (1922).

then, for consistency, such a constraint is equivalent to the coordinate constraint

$$\mathbf{r}^{(1)} - \mathbf{r}^{(2)} - \mathbf{c} = 0, \tag{A.2.10}$$

which is holonomic.

More generally, according to case III.a, all constraints which depend on time $t$, positions $\mathbf{r}$ and their time derivatives, and which are integrable (nonintegrable) to coordinate constraints of type (A.2.5) are *holonomic* (*nonholonomic*).[8]

One of the most common types of velocity constraints is characterized by a linearity in the velocity, i.e., it can be expressed in the form

$$\phi_s(t, \mathbf{r}, \dot{\mathbf{r}}) = \mathbf{\Gamma}_{si}(t, \mathbf{r}) \cdot \dot{\mathbf{r}}^i + \Lambda_s(t, \mathbf{r}) = 0, \tag{A.2.11}$$
$$s = 1, 2, \ldots, r.$$

Such constraints admit the equivalent differential form

$$\mathbf{\Gamma}_{si}(t, \mathbf{r}) \cdot d\mathbf{r}^i + \Lambda_s(t, \mathbf{r})dt = 0. \tag{A.2.12}$$

Suppose, now, that Equation (A.2.11) can be written

$$\phi_s(t, \mathbf{r}, \dot{\mathbf{r}}) = \frac{d}{dt}\phi'_s(t, \mathbf{r}) = \frac{\partial \phi'_s}{\partial \mathbf{r}^i} \cdot \dot{\mathbf{r}}^i + \frac{\partial \phi'_s}{\partial t}. \tag{A.2.13}$$

In this case, they are integrable because they are reducible through integration to the coordinate constraint

$$\phi'_s(t, \mathbf{r}) = 0. \tag{A.2.14}$$

The necessary and sufficient conditions for the integrability of Equation (A.2.13) are

$$\begin{aligned} \frac{\partial^2 \phi_s}{\partial \mathbf{r}^i \, \partial \mathbf{r}^j} - \frac{\partial^2 \phi_s}{\partial \mathbf{r}^j \, \partial \mathbf{r}^i} &= 0, \\ \frac{\partial^2 \phi_s}{\partial \mathbf{r}^i \, \partial t} - \frac{\partial^2 \phi_s}{\partial t \, \partial \mathbf{r}^i} &= 0, \end{aligned} \tag{A.2.15}$$

and by using the right-hand side of Equation (A.2.11) they can be written[9]

$$\frac{\partial \Gamma_{sia}}{\partial r^{jb}} - \frac{\partial \Gamma_{sjb}}{\partial r^{ia}} = 0, \tag{A.2.16a}$$

$$\frac{\partial \Gamma_{sia}}{\partial t} - \frac{\partial \Lambda_s}{\partial r^{ia}} = 0. \tag{A.2.16b}$$

[8] For a different meaning of the term "nonholonomic" see, for instance, Gelfand and Fomin (1963), page 48. According to the terminology of these authors, Equation (A.2.6) are nonholonomic, irrespective of whether they are integrable or nonintegrable. In principle, this is also an acceptable terminology. The point we want to stress here is that the reader should be fully aware of the significance of the terms "holonomic" or "nonholonomic" assumed by each author in order to avoid some considerable confusion when studying the methodology for Newtonian systems, particularly within the framework of the variational approach.

[9] See Rund (1966, Section 5.5). See also Section 1.2.

Thus, according to our terminology, velocity constraints of type (A.2.11) are holonomic (nonholonomic) depending on whether all Equations (A.2.16) hold (do not hold). If Equations (A.2.16) hold only for a subset of constraints (A.2.11), then we have a case of constraints of type III.c.

A typical example of a nonintegrable (and, thus, nonholonomic) constraint of type (A.2.11) is a vertical disk of radius $R$ rolling without skidding on a horizontal plane $(x, y)$. The speed $|\dot{\mathbf{r}}|$ at the center of the disk is restricted by the equality[10]

$$|\dot{\mathbf{r}}| - R\dot{\alpha} = 0, \tag{A.2.17}$$

where $\dot{\alpha}$ is the angular velocity of the disk. The projection of Equation (A.2.17) in the $(x, y)$ plane can be written

$$\begin{aligned} dx - (R \sin\theta)d\alpha = 0, \\ dy + (R \cos\theta)d\alpha = 0, \end{aligned} \tag{A.2.18}$$

where $\theta$ is the angle between the $x$ axis and the projection of the symmetry axis of the disk in the $(x, y)$ plane. The above equations do not satisfy integrability conditions (A.2.16). Thus, the constraint is nonholonomic.

## A.3 Generalized Coordinates

We now restrict our attention to *holonomic systems* of $N$ particles, namely, to Newtonian systems whose constrains are *all* holonomic. Suppose that the number of such independent[11] constraints is $3N - n$. When all such constraints are either assigned or reduced to the coordinate form

$$\phi_s(t, \mathbf{r}) = 0, \qquad s = 1, 2, \ldots, 3N - n \leq 3N, \tag{A.3.1}$$

they can be used to express the $n$ actual degrees of freedom in terms of a set of $n$ new independent variables, termed *generalized coordinates*,[12] and customarily denoted with the symbols $q^1, q^2, \ldots, q^n$.[13]

Such new independent variables can, in general, be assumed to be $n$ independent functions of the positions $\mathbf{r}^i$ and time $t$, i.e.,

$$q^k = q^k(t, \mathbf{r}). \tag{A.3.2}$$

Such functions determine $n$ Cartesian coordinates as functions of the $q$'s, the remaining $3N - n$ Cartesian coordinates and time. Constraint Equation (A.3.1) then allow the determination of the residual $3N - n$ Cartesian coordinates. Therefore, from the combined use of Equations (A.3.1) and

[10] See, for instance, Goldstein (1950, page 13).

[11] The independence of the constraints (A.3.1) can be expressed, for instance, by the conditions that the $(3N - n)x(3N)$ matrix $(\partial\phi_s/\partial r^{ia})$, $s = 1, 2, \ldots, 3N - n, i = 1, 2, \ldots, N, a = 1, 2, 3$, has rank $3N - n$.

[12] Thomson and Tait (1879).

[13] For a first geometrical significance of the upper or lower indices, see Charts A.11 through A.14. For a more advanced treatment, see Charts 2.1 through 2.5.

(A.3.2), we can express all Cartesian coordinates as functions of the $q$'s and of time, i.e.,

$$\mathbf{r}^i = \mathbf{r}^i(t, q) = \mathbf{r}^i(t, q^r, \ldots, q^n). \tag{A.3.3}$$

In this way, the constraint equations (A.3.1) are obeyed identically by Equation (A.3.3). As a result, in the representation of holonomic systems in terms of generalized coordinates, as we shall see more clearly in the next section, we expect no representative of the forces of constraints. This fact can be seen at this point from the property that the generalized coordinates can be varied arbitrarily without conflicting with Equation (A.3.1), while this is not the case for the Cartesian coordinates. Explicitly, jointly with the $q$'s, all the differentials

$$dq^k = \frac{\partial q^k}{\partial \mathbf{r}^i} \cdot d\mathbf{r}^i + \frac{\partial q^k}{\partial t}\, dt \tag{A.3.4}$$

are independent and, once interpreted as virtual displacements, are consistent with Equation (A.3.1). On the contrary, the $3N$ differentials $dr^{ia}$, to be consistent with the constraint equations, cannot all be independent. This can be seen from the time derivative of Equation (A.3.3),

$$\dot{\mathbf{r}}^i = \frac{\partial \mathbf{r}^i}{\partial q^k}\, \dot{q}^k + \frac{\partial \mathbf{r}^i}{\partial t}, \tag{A.3.5}$$

whose differential version,

$$d\mathbf{r}^i = \frac{\partial \mathbf{r}^i}{\partial q^k}\, dq^k + \frac{\partial \mathbf{r}^i}{\partial t}\, dt, \tag{A.3.6}$$

allows only a subset of the differentials $d\mathbf{r}^i$ to be independent. As a result, all trajectories in the space of generalized coordinates are admissible by the constraint equations, and, as such they do not activate the forces of constraints.

The $n$-dimensional space $M_{(q)}$ characterized by the generalized coordinates will be termed *configuration space*. The space $M_{(q)}$ will be the representation space of our analysis. It will be used with the understanding that, when the systems are unconstrained, the set of $n = 3N$ variables $q^1, q^2, \ldots, q^{3N}$ represents any set of conventional coordinates (e.g. Cartesian, cylindrical, spherical, etc.) in a given ordering.

## A.4 Conservative Systems

A class of Newtonian systems of central significance from a methodological viewpoint is that for which all the acting forces $\mathscr{F}_1, \ldots, \mathscr{F}_N$ are *conservative*, that is, they depend only on the positions of the particles, and their virtual work is the total differential of a (single-valued) function $-\mathscr{V}(\mathbf{r})$, i.e.,

$$\mathscr{F}_i(\mathbf{r}) \cdot d\mathbf{r}^i = -d\mathscr{V}. \tag{A.4.1}$$

We shall use the term *conservative systems* to describe all Newtonian systems with acting forces satisfying condition (A.4.1). Then for such systems

$$\mathscr{F}_i = -\frac{\partial \mathscr{V}}{\partial \mathbf{r}^i}. \tag{A.4.2}$$

The function $\mathscr{V}(\mathbf{r})$ of equations (A.4.1) will be called the *potential energy function* (or *potential* for short) of the acting forces (or of the system). When Equation (A.4.1) holds, then along any closed curve in the space of the $r$'s,[14]

$$\oint \mathbf{F}_i \cdot d\mathbf{r}^i = -\oint d\mathscr{V} \equiv 0. \tag{A.4.3}$$

Suppose, for simplicity, that the masses $m_i$ are constant. Let the *kinetic energy* and its total time derivative be represented by the familiar forms

$$\mathscr{T} = \tfrac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}^i, \tag{A.4.4a}$$

$$\frac{d\mathscr{T}}{dt} = m_i \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}^i = \mathscr{F}_i \cdot \dot{\mathbf{r}}^i. \tag{A.4.4b}$$

Then, by using Equation (A.4.2), we can write[15]

$$\frac{d}{dt}(\mathscr{T} + \mathscr{V}) = \mathbf{F}_i \cdot \dot{\mathbf{r}}^i + \frac{\partial \mathscr{V}}{\partial \mathbf{r}^i} \cdot \dot{\mathbf{r}}^i \equiv 0. \tag{A.4.5}$$

Thus, for a conservative system, the sum of the kinetic and potential energies is a constant, i.e.,

$$\mathscr{E}_T = \mathscr{T} + \mathscr{V} = \text{const.} \tag{A.4.6}$$

The value $\mathscr{E}_T$ is then determined by the initial conditions.

The necessary and sufficient conditions[16] for the left-hand side of Equation (A.4.1) to be an exact differential are (in Cartesian coordinates)

$$\frac{\partial \mathscr{F}_{ia}}{\partial r^{jb}} - \frac{\partial \mathscr{F}_{jb}}{\partial r^{ia}} = 0, \tag{A.4.7}$$

$$i, j = 1, 2, \ldots, N, \qquad a, b = 1, 2, 3.$$

Notice that Equation (A.4.7) can be used as a criterion to ascertain whether a given set of Newtonian forces $\mathscr{F}_1(\mathbf{r}), \mathscr{F}_2(\mathbf{r}), \ldots, \mathscr{F}_N(\mathbf{r})$ is conservative or not.

[14] For a more rigorous treatment, see Example 1.4.

[15] For a first elementary approach to conservation laws, see Chart A.2. For a more detailed treatment, see Santilli (1979).

[16] For a more technical formulation, see Example 1.4, Theorem 3.5.1, and Section 3.7.

Consider now a conservative $N$-dimensional system in a Cartesian reference frame subject to $3N - n$ independent holonomic coordinate constraints. The equations of motion are

$$\begin{cases} \dot{\mathbf{p}}_i = \mathscr{F}_i(\mathbf{r}) = -\dfrac{\partial \mathscr{V}}{\partial \mathbf{r}^i}, & \text{(A.4.8a)} \\ \phi_s(t, \mathbf{r}) = 0, & \text{(A.4.8b)} \end{cases}$$

$$i = 1, 2, \ldots, N, \qquad s = 1, 2, \ldots, 3N - n.$$

If $d\mathbf{r}^i$ represents a *virtual displacement* of the particle $i$, that is, an infinitesimal displacement compatible with the *acting forces*[17] and constraints, then the virtual work $\hat{\mathscr{F}}_i \cdot d\mathbf{r}^i$ of the forces of constraints $\hat{\mathscr{F}}_i$ is null, and we have the so-called *D'Alembert's principle*:

$$\begin{cases} \left(\dot{\mathbf{p}}_i + \dfrac{\partial \mathscr{V}}{\partial \mathbf{r}^i}\right) \cdot d\mathbf{r}^i = 0, & \text{(A.4.9a)} \\ \phi_s(t, \mathbf{r}) = 0. & \text{(A.4.9b)} \end{cases}$$

In view of constraint (A.4.8b), the system has $n$ actual degrees of freedom. Suppose they are represented by the generalized coordinates $q^1, q^2, \ldots, q^n$. Then properties (A.3.2) and (A.3.3) allow the reformulation of the kinetic energy in configuration space according to

$$\mathscr{T}(\dot{\mathbf{r}}) = \tfrac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}^i = \tfrac{1}{2} m_i \left(\frac{\partial \mathbf{r}^i}{\partial q^j} \dot{q}^j + \frac{\partial \mathbf{r}^i}{\partial t}\right)^2$$

$$\equiv \tfrac{1}{2} Z_{ij}(t, q) \dot{q}^i \dot{q}^j + Z_k(t, q) \dot{q}^k + Z_0(t, q) = T(t, q, \dot{q}), \quad \text{(A.4.10)}$$

where

$$Z_{ij} = m_k \frac{\partial \mathbf{r}_k}{\partial q^i} \cdot \frac{\partial \mathbf{r}^k}{\partial q^j}, \quad \text{(A.4.11a)}$$

$$Z_k = m_i \frac{\partial \mathbf{r}_i}{\partial q^k} \cdot \frac{\partial \mathbf{r}^i}{\partial t}, \quad \text{(A.4.11b)}$$

$$Z_0 = \tfrac{1}{2} m_i \frac{\partial \mathbf{r}^i}{\partial t} \cdot \frac{\partial \mathbf{r}^i}{\partial t}. \quad \text{(A.4.11c)}$$

For the case of the potential energy, we have simply

$$\mathscr{V}(\mathbf{r}) = \mathscr{V}[\mathbf{r}(t, q)] = V(t, q). \quad \text{(A.4.12)}$$

[17] We shall term "acting" force any force $\mathscr{F}_i (\neq 0)$ whose work $\mathscr{F}_i \cdot d\mathbf{r}^i$ for either virtual or actual displacement is non-null.

D'Alembert's principle (A.4.9) can then be written

$$\left(\dot{\mathbf{p}}_i + \frac{\partial \mathscr{V}}{\partial \mathbf{r}^i}\right) \cdot d\mathbf{r}^i = \left[\frac{d}{dt}\left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}^i}{\partial q^j}\right) - m_i \mathbf{r}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}^i}{\partial q^j}\right) + \frac{\partial V}{\partial q^j}\right] \delta q^j$$

$$= \left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}^j}\left(\tfrac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}^i\right) - \frac{\partial}{\partial q^j}\left(\tfrac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}^i\right) + \frac{\partial V}{\partial q^j}\right] \delta q^j$$

$$= \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^j} - \frac{\partial (T - V)}{\partial q^j}\right] \delta q^j$$

$$= \left[\frac{d}{dt}\frac{\partial (T - V)}{\partial \dot{q}^j} - \frac{\partial (T - V)}{\partial q^j}\right] \delta q^j$$

$$= 0, \tag{A.4.13}$$

where we have used Equations (A.4.10) and (A.4.12). The variation in the $q$'s is denoted by the symbol $\delta q_j$.

By introducing the *Lagrangian*

$$L = L(t, q, \dot{q}) = T(t, q, \dot{q}) - V(t, q), \tag{A.4.14}$$

the necessary and sufficient conditions for the validity of principle (A.4.13) (in view of the independence and arbitrariness of variations $\delta q^i$, can be expressed in terms of the (Euler–) *Lagrange's equations in configuration space*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^j} - \frac{\partial L}{\partial q^j} = 0, \qquad j = 1, 2, \ldots, n. \tag{A.4.15}$$

Under the assumption[18]

$$\left| \frac{\partial^2 L}{\partial \dot{q}^i \, \partial \dot{q}^j} \right| \neq 0, \tag{A.4.16}$$

and the prescriptions for the characterization of $n$ new variables termed *generalized momenta*

$$p_k = \frac{\partial L}{\partial \dot{q}^k}, \tag{A.4.17}$$

one can introduce the *Hamiltonian* through the *Legendre transform*

$$H(t, q, p) := \dot{q}^k p_k - L. \tag{A.4.18}$$

Equation (A.4.15) can then be mapped into *Hamilton's equations*

$$\begin{cases} \dot{q}^k - \dfrac{\partial H}{\partial p_k} = 0, & \text{(A.4.19a)} \\[2ex] \dot{p}_k + \dfrac{\partial H}{\partial q^k} = 0, & \text{(A.4.19b)} \end{cases}$$

[18] According to the notation of Equation (1.1.6) and (1.1.8), matrices and determinants are denoted with the symbols ( ) and | |, respectively.

which act on the space $\tilde{M}_{(q,p)}$ of the canonically conjugate variables $q^k$ and $p_k$, termed *phase space*.

For scleronomous holonomic conservative systems the Hamiltonian does not depend explicitly on time; it represents the total energy of the system, and thus it is a constant of the motion, i.e.,

$$\frac{dH(q, p)}{dt} = 0. \tag{A.4.20}$$

Let us stress the fact that when the Hamiltonian $H$ depends only on the elements $(q, p)$ of the phase space, this by no means guarantees that the system is conservative or that $H$ represents the total energy, even though property (A.4.20) still holds. Indeed, in this volume we shall encounter several cases in which functions $H(q, p)$ represent truly nonconservative systems.

Another aspect we would like to stress is the fact that the equations of motion of conservative systems, irrespective of whether or not they contain holonomic constraints, are not necessarily linear in the coordinates. For instance, a Lagrangian (A.4.14) with a potential of the type

$$V(q) = c_k q^k + c_{ij} q^i q^j + c_{ijk} q^i q^j q^k + \cdots \tag{A.4.21}$$

represents a truly conservative system, but Lagrange's equations are nonlinear in $q^k$.

Equations (A.4.15) and (A.4.19) will often be referred to as the *analytic equations in configuration* and *phase space*, respectively. When Newtonian systems (A.1.1) can be expressed in terms of such equations, we shall say that they admit an *analytic representation.*

Our derivation of Lagrange's equations through the use of D'Alembert's principle is the one most widely used in the current literature. However, such a derivation is not immune to criticisms,[19] owing to the "static" nature of D'Alembert's principle.

Whatever approach is used for deriving Lagrange's equations, it must be compatible with one of their central features, namely, the lack of any representative of the forces of constraints. For principle (A.4.13), this is a consequence of the assumption that the displacements $d\mathbf{r}^i$ are compatible with the constraints, and thus the work of the forces of constraints is null.

## A.5 Dissipative Systems

Conservative systems often constitute a simplification of physical reality, since they imply, for instance, that motion is frictionless even in the presence of constraints. As is well known, constraints inevitably imply the presence of frictional forces and the motion in an actual environment necessarily implies a resistance due to the medium.

[19] See, for instance, Kilmister (1967, page 34).

A second significant class of Newtonian systems is represented by *dissipative systems*, namely, systems that are subject to an arbitrary collection of conservative and dissipative forces and whose total energy is monotonically[20] nonincreasing in time, i.e.,

$$\mathscr{E}_T|_{t_1} \geq \mathscr{E}_T|_{t_2}, \qquad t_1 < t_2. \tag{A.5.1}$$

Energy can be dissipated in an endless variety of ways, and dissipative forces can be characterized in an equally endless variety of ways. For instance, when a particle moves in a medium, a dissipative force is any drag or resisting force which opposes the motion. In this case, if the total energy is equal to the kinetic energy, all resisting forces due to the medium for which the kinetic energy monotonically tends to zero as $t \to \infty$, i.e.,

$$\lim_{t\to\infty} \mathscr{T} = 0, \tag{A.5.2}$$

can be considered to be dissipative forces.[21]

A large class of dissipative forces is expressible as a series in the velocity, say

$$\mathscr{F}^d = |\mathscr{F}^d| = b_0 + b_1 v + b_2 v^2 + \cdots, \tag{A.5.3}$$

where the coefficients depend, in general, on positions and on time. Thus the functional dependent of $\mathscr{F}^d$ is

$$\mathscr{F}^d = \mathscr{F}^d(t, \mathbf{r}, \dot{\mathbf{r}}). \tag{A.5.4}$$

Dissipative forces are usually opposite in direction to the velocity, in which case their Cartesian components can be written as

$$\mathscr{F}^d_a = -\mathscr{F}^d \frac{\dot{r}_a}{|\dot{\mathbf{r}}|}, \qquad a = 1, 2, 3. \tag{A.5.5}$$

Several types of dissipative forces can be approximated with expressions simpler than the form (A.5.3), such as the following.

1. *Dissipative forces independent of the velocity.*
   They are proportional to the normal force $\mathscr{F}_\perp$ between the surfaces in contact and are independent of contact area and velocity

   $$\mathscr{F}^d = b_0 = b_0(\mathbf{r}, \mathscr{F}_\perp), \tag{A.5.6}$$

   where the space dependence arises from possible variations in the coefficient of friction and the time dependence arises from possible variations in the normal force as well as the positions.

[20] A function $f(x)$ is said to be *monotonic* (*strictly monotonic*) in an interval $(x_1^0, x_2^0)$ if for every pair of points $x_1, x_2 \in (x_1^0, x_2^0)$ such that $x_1 < x_2$, the relation $f(x_1) \leq f(x_2)$ (or $f(x_1) < f(x_2)$) holds.

[21] According to Birkhoff (1927, page 31), a force $f_k^d$ is dissipative when $f_k^d \dot{q}^k \geq 0$. Then the energy integral is monotonically nonincreasing. When $f_k^d \dot{q}^k > 0$, the energy integral decreases toward some limiting value in a strictly monotonic sense.

2. *Dissipative forces proportional to the velocity.*
   They are usually opposite in direction to $\dot{\mathbf{r}}$, i.e.,

$$\mathscr{F}^d = -b_r \dot{\mathbf{r}}. \tag{A.5.7}$$

   Resistive forces of this type occur when a system is slowly moving in a fluid, or a magnet is slowly moving near a conducting sheet.
3. *Dissipative forces proportional to higher powers of the velocity.*
   When the motion in a given medium is not slow, the resistive force can depend on higher powers of $\dot{\mathbf{r}}$, e.g.,

$$\mathscr{F}^d = b_m v^m, \qquad m = \text{fixed}, \tag{A.5.8}$$

   where for a given range of speed, $m = 2$, for a higher speed $m = 3$, etc.

Usually, dissipative forces do not affect the degrees of freedom of a system. For instance, if dissipative forces are considered for a system of $N$ particles with $3N - n$ independent holonomic constraints, and thus $n$ degrees of freedom, the number of generalized coordinates remains unchanged. It is then possible to represent dissipative forces in terms of generalized coordinates. This is achieved by considering the virtual work done by the dissipative forces, which must be the same in Cartesian and in generalized coordinates, i.e.,

$$\begin{aligned} dW^d &= \mathscr{F}_i^d \cdot d\mathbf{r}^i \\ &= b_{im}(V_i^i)^{m-r} V_{ia}\, dr^{ia} \\ &= f_k^d(t, q, \dot{q})\delta q^k, \end{aligned} \tag{A.5.9}$$

where we have assumed for simplicity that the series (A.5.3) can be truncated at the power $M$ of the speed, and where $f_k^d$ represents the components of the so-called *generalized force.*

For a significant class of dissipative systems, it is possible to introduce the so-called *power function* $P^d(t, q, \dot{q})$, which represents the dissipative force in configuration space through the relations[22]

$$f_k^d = \frac{\partial P^d}{\partial \dot{q}^k}. \tag{A.5.10}$$

Suppose the dissipative forces satisfy the relations[23]

$$\frac{\partial \mathscr{F}_{ia}^d}{\partial v^{jb}} = \frac{\partial \mathscr{F}_{jb}^d}{\partial v^{ia}}. \tag{A.5.11}$$

Then the following exact differential occurs.

$$\mathscr{F}_{ia}^d\, dv^{ia} = dp(v), \tag{A.5.12}$$

[22] See, for instance, Wells (1967, Chapter 6).

[23] To avoid possible confusion, we shall sometimes add a superscript $c$, $d$, $e$, or $a$ to the symbol of a force to denote, specifically, conservative, dissipative, applied (or external), and acting (or total) force, respectively. Notice the mathematical similarity between Equation (A.5.11) and (A.4.7). For a more rigorous treatment, see Example 1.4.

which can also be written

$$f_k^d \delta \dot{q}^k = \delta P, \qquad P(t, q, \dot{q}) = p(v), \tag{A.5.13}$$

yielding relation (A.2.1).

A typical example of a power function in configuration space is

$$P^d = \frac{1}{m+1} b_k (\dot{q}^k)^{m+1}, \qquad m = 1, 2, 3, \ldots, \tag{A.5.14}$$

with corresponding dissipative forces

$$f_k^d = b_k (\dot{q}^k)^m \qquad \text{(no summation)}. \tag{A.5.15}$$

Suppose that the $f_k^d$ force is a function of position and time only:

$$f_k^d = f_k^d(t, q). \tag{A.5.16}$$

Then, as in case (A.2.5), we have

$$P^d = f_k^d \dot{q}^k, \tag{A.5.17a}$$

$$f_k^d = \frac{\partial P^d}{\partial \dot{q}^k}. \tag{A.5.17b}$$

Notice that all conservative forces can be expressed by means of power function (A.5.17a). Clearly, $P^d$ has the dimension of power.

Expression (A.5.14) with $m = 2$ is the celebrated *Rayleigh's dissipation function.*[24] More generally, this function can be defined as a homogeneous quadratic form in $\dot{q}^k$, i.e.,

$$P^d = \tfrac{1}{2} \dot{q}^i b_{ij} \dot{q}^j, \tag{A.5.18}$$

where the coefficients $b_{ij}$ are negative definite functions of the $q$'s and time. Then the components of the dissipative forces are

$$f_k^d = b_{ki}(t, q) \dot{q}^i. \tag{A.5.19}$$

In all the cases considered above, the dissipative force is opposite to the direction of motion. But dissipative forces can also have an arbitrary orientation with respect to the velocity of the particle. To illustrate this point, consider a magnetic pole[25] which moves with velocity $\dot{\mathbf{r}}$ in the vicinity of a conducting grill. As a result of the variation of magnetic flux, the grill acquires local currents. The induced magnetic field will oppose the motion of the magnetic pole through a force which is not opposite to $\dot{\mathbf{r}}$, but opposite to the component of $\dot{\mathbf{r}}$ along the direction perpendicular to the grill.

[24] Notice that the analytic representations of dissipative systems studied in this book are *without* the Rayleigh function.

[25] For example, one pole of a sufficiently long permanent magnet, so that it can be considered as isolated.

Consider, now, a holonomic conservative system with forces represented in configuration space by

$$f_k^c = -\frac{\partial V}{\partial q^k}, \tag{A.5.20}$$

and with total energy $\mathscr{E}_T = T + V$. If an arbitrary dissipative force $f_k^d(t, q, \dot{q})$ is applied to it, the rate of change of $\mathscr{E}_T$ is

$$\frac{d\mathscr{E}_T}{dt} = f_k^d \dot{q}^k. \tag{A.5.21}$$

If $f_k^d$ can be expressed in terms of a power function $P^d$, Equation (A.5.21) becomes

$$\frac{d\mathscr{E}_T}{dt} = \frac{\partial P^d}{\partial \dot{q}^k} \dot{q}^k. \tag{A.5.22}$$

Finally, if $P^d$ is homogeneous of degree two in the $q$'s, then we simply have

$$\frac{d\mathscr{E}_T}{dt} = 2P^d, \tag{A.5.23}$$

that is, the rate of change of the energy is twice the power function. Since $P^d$ is definitely negative (or null) by assumption, condition (A.5.1) of the rate of variation of the energy with time is verified.

Clearly, as for conservative systems, dissipative systems can also be represented by either linear or nonlinear differential equations.

From a methodological viewpoint, dissipative Newtonian systems constitute a significant complement to conservative systems, since they generally imply "nonconservation" not only of energy but also of linear momentum, angular momentum, or any other physical quantity. Ultimately, the class of dissipative systems is broader than that of conservative systems, i.e.,

$$\{\text{conservative systems}\} \subset \{\text{dissipative systems}\}, \tag{A.5.24}$$

because the former class occurs as a particular subclass of the latter at the limit when all dissipative forces are null. Thus, it is expected that the methodology for dissipative systems implies a suitable generalization of that for conservative systems.

In this book, we study the problem of the representation of holonomic Newtonian systems in general (and, thus, holonomic dissipative systems in particular) in terms of the (conventional) Lagrange's equations without external terms. In this way, we remove the major simplification made in the transition from Newton's equations of motion (A.4.8) to Lagrange's equations (A.4.15), namely, that the motion is frictionless despite the presence of constraints. The reader should be aware that, despite this broadening of the considered physical context, our treatment is still restrictive because we exclude *nonlocal forces*. Nevertheless, local forces not derivable from a potential are known to constitute a good approximation of nonlocal dissipative forces and, as such, are sufficient for our objectives.

## A.6 Dynamical Systems

*Dynamics* is the branch of classical mechanics that studies the motion of a system of particles subject to "arbitrary" Newtonian forces and constraints.

Clearly, the conservative and dissipative forces considered so far do not exhaust all possible Newtonian forces. Without any claim to completeness, it is sufficient for our needs to introduce a third class of forces, the *applied forces.*

We shall call an applied force any external force of a nonimpulsive nature[26] with an arbitrary functional dependence on positions, velocities, and time. The above definition includes a rather large variety of forces not necessarily derivable from a potential $V(q)$ or a generalized potential $U(t, q, \dot{q})$, such as logarithmic forces, periodical forces, etc. However, applied forces derivable from a potential are not excluded.

We shall define a *dynamical system* as a system of particles subject to an arbitrary collection of conservative, dissipative, and applied forces. Clearly, in a dynamic system the energy can vary arbitrarily in time, i.e.,

$$\mathscr{E}_T|_{t_1} \gtreqless \mathscr{E}_T|_{t_2}, \qquad t_1 < t_2. \tag{A.6.1}$$

A typical example of a truly dynamical[27] system is the damped and forced oscillator. In practical applications within a Newtonian framework,[28] the conservative approximation of the oscillatory motion generated by an elastic force is usually valid for periods of time in which there is no appreciable variation (in relation to the desired approximation) of the amplitude and frequency. For comparably longer periods of time (or higher approximation), the inevitable presence of dissipative forces will result in a progressive damping of the motion up to the configuration of null amplitude (or energy). Thus, the preservation of the motion for a long period of time demands the use of applied forces. The rate of variation of the energy then depends on the relationship of the rate of supplied energy with the rate of dissipated energy, thus it can vary arbitrarily in time. The dependence of the energy on time, however, can only be continuous if possible discontinuities in the velocities are excluded.[29]

From now on we shall assume that the acting forces in Cartesian coordinates have an arbitrary dependence[30] on time, coordinates, and velocities,[31] i.e., $\mathscr{F}_i = \mathscr{F}_i(t, \mathbf{r}, \dot{\mathbf{r}})$, and they represent an arbitrary collection of conservative, dissipative, and applied forces. We shall also assume that all

[26] This restriction is introduced to avoid discontinuities in the velocities. See Chart A.3.

[27] The term "truly" dynamical is used here in the sense that the total acting force includes conservative, dissipative, and applied forces.

[28] We stress here the exclusion of non-Newtonian (e.g., quantum mechanical) frameworks.

[29] Clearly, discontinuities in the coordinates $\mathbf{r}^i(t)$ or $q^k(t)$ are not admissible in any Newtonian system, irrespective of whether or not it contains impulsive forces.

[30] Certain continuity restrictions are introduced in Chapters 1, 2, and 3.

[31] We shall tacitly exclude a possible dependence on the accelerations or higher-order derivatives of the coordinates.

constraints are holonomic. As a more general formulation of the virtual work (A.5.9), we have

$$dW = \mathscr{F}_i \cdot d\mathbf{r}^i$$
$$= \mathscr{F}_i \cdot \frac{\partial \mathbf{r}^i}{\partial q^j} \delta q^j = f_j \delta q^j. \tag{A.6.2}$$

The *generalized forces*

$$f_j = \mathscr{F}_i \cdot \frac{\partial \mathbf{r}^i}{\partial q^j} \tag{A.6.3}$$

then represent the configuration space "image" of the forces $\mathscr{F}_i$.

The definition of *total physical energy* for a dynamical system with dissipative and applied forces should be recalled. It is, in essence, given by the sum of the kinetic energy and the potential energy of all forces derivable from a potential.[32] A central aspect of the study of dynamical systems is then given by identification of the *variation* of this energy in time.

Notice the generalization of the concept of rate of variation of the energy in the transition from dissipative to dynamical systems. Indeed, from expressions (A.5.1) and (A.6.1), we see that while for the former the energy is nonincreasing in time (*dissipation*), for the latter the energy can vary arbitrarily in time (*nonconservation*). This is a consequence of the fact that dissipative systems are a subclass of dynamical systems. By recalling Equation (A.5.24), we shall then write the inclusion properties

$$\{\text{conservative systems}\} \subset \{\text{dissipative systems}\} \subset \{\text{dynamical systems}\} \tag{A.6.4}$$

with an example provided by the above-indicated case of damped and forced oscillators:

$$\{\text{harmonic oscillators}\} \subset \{\text{damped oscillators}\} \subset \{\text{damped and forced oscillators}\}. \tag{A.6.5}$$

It should be indicated here that the broadening of our framework to include arbitrary Newtonian forces implies a conceptual as well as a technical modification of the theory. In particular, while the conventional methodology of Newtonian systems is basically a theory of "conservative systems," the methodology of dynamical systems is eminently a theory of "nonconservative systems." This fact, however, does not exclude the study of symmetries and conserved quantities. Only their emphasis is changed, as seen in Santilli (1979).

Our definition of "dynamical" systems is in line with the conventional definition of "dynamics" as recalled at the beginning of this section. The reader should be aware, in this respect, that the term "dynamical" systems

[32] See, for instance, Symon (1960).

is often restricted in currently available literature to that of "conservative" systems. This is certainly a proper terminology, because conservative systems are also dynamical systems. However, they constitute, strictly speaking, only a sublcass of the class of dynamical systems and often a crude approximation of physical reality. Clearly, on methodological grounds, a deeper insight can be gained by including all (local) Newtonian forces, namely, conservative, dissipative, and applied.

## A.7 The Fundamental Form of the Equations of Motion in Configuration Space

We shall now identify a general form of the equations of motion in configuration space for holonomic systems with arbitrary Newtonian forces. Such a form is used in the text for the study of the problem of the existence of a representation of a given dynamical system in terms of conventional analytic equations.

Consider a holonomic dynamical system in Cartesian coordinates represented in terms of the D'Alembert principle

$$\begin{cases} [\dot{\mathbf{p}}_i - \mathscr{F}_i(t, \mathbf{r}, \dot{\mathbf{r}})] \cdot d\mathbf{r}^i = 0, & \text{(A.7.1a)} \\ \phi_s(t, \mathbf{r}) = 0. & \text{(A.7.1b)} \end{cases}$$

Under the assumption that the masses are constants, the above system can be written in configuration space as

$$[A_{ij}(t, q)\ddot{q}^j + B_i(t, q, \dot{q})]\delta q^i = 0, \tag{A.7.2}$$

with

$$A_{ij} = Z_{ij}, \tag{A.7.3a}$$

$$B_i = Z_{ij}\dot{q}^j + Z_i - \frac{1}{2}\frac{\partial Z_{jk}}{\partial q^i}\dot{q}^k\dot{q}^j - \frac{\partial Z_k}{\partial q^i}\dot{q}^k - \frac{\partial Z_0}{\partial q^i} - f_i, \tag{A.7.3b}$$

where we have used Equations (A.4.11) and (A.6.3).

In view of the arbitrariness and independence of the $\delta q$'s, a necessary and sufficient condition for the validity of principle (A.7.2) is that each of the equations

$$A_{ij}(t, q)\ddot{q}^j + B_i(t, q, \dot{q}) = 0, \qquad i = 1, 2, \ldots, n \tag{A.7.4}$$

holds. The above equations, however, do not represent the most general form of Newton's equations of motion in configuration space because they do not necessarily account for the so-called "*acceleration couplings*"[33] or for possible equivalence transformations.

[33] These couplings occur, for instance, within the systems of coupled, forced and damped oscillators (see Section 3.7).

By taking into consideration these latter aspects, we obtain the equations

$$A_{ij}(t, q, \dot{q})\ddot{q}^j + B_i(t, q, \dot{q}) = 0, \tag{A.7.5}$$

in which there is an additional dependence of the $A_{ij}$ terms on the generalized velocities $\dot{q}^i$. We assume (A.7.5) to be our *fundamental form of the equations of motion in configuration space.*

A central aspect of Equation (A.7.5) is their linearity in the accelerations. This is ultimately a consequence of the same structure of Newton's equations (A.1.1).

Independent of that, equations of the type (A.7.5) constitute the most general form of ordinary differential equations that are representable by Lagrange's equations. Indeed, the explicit form of these equations is

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} &= \frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}\ddot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i\, \partial q^j}\dot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i\, \partial t} - \frac{\partial L}{\partial q^i} \\ &= A'_{ij}(t, q, \dot{q})\ddot{q}^j + B'_i(t, q, \dot{q}) \\ &= 0,\end{aligned} \tag{A.7.6}$$

where

$$A'_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}, \tag{A.7.7a}$$

$$B'_i = \frac{\partial^2 L}{\partial \dot{q}^i\, \partial q^j}\dot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i\, \partial t} - \frac{\partial L}{\partial q^i}, \tag{A.7.7b}$$

that is, Lagrange's equations are also always linear in the accelerations. By inspecting Equation (A.7.6), we then see that the most general form of differential equations they are capable of representing is precisely the fundamental form (A.7.5).

In this Appendix we have introduced the classification of Newtonian systems into (1) holonomic or nonholonomic, from the nature of the constraints, and into (2) conservative, dissipative, or dynamical, from the type of acting force. We have then restricted our analysis to holonomic constraints only and included all (local nonimpulsive) Newtonian forces. Finally, we have obtained a general form of the equations of motion in configuration space for such a class of systems, i.e., Equation (A.7.5).

A central objective of this monograph is the identification of the necessary and sufficient conditions for Equation (A.7.5) to admit an analytic representation in terms of Lagrange equation (A.7.6). For this purpose, the above classifications of Newtonian systems are insufficient and other methodological aspects must be taken into consideration. This task is initiated in Chapter 1 and completed in the Chapters 2 and 3.

**Chart A.1** Galilean Relativity

The *representation* (or *carrier*) *space* for unconstrained Newtonian systems is the Kronecker product $E_{3,1}(\mathbf{r}, t) = E_3(\mathbf{r}) \times E_1(t)$ of the three-dimensional *Euclidean space* $E_3(\mathbf{r})$ representing the coordinate of the particles and the one-dimensional space $E_1(t)$ representing time. Space $E_3(\mathbf{r})$ is homogeneous and isotropic, while space $E_1(t)$, being one-dimensional, is only homogeneous. These characteristics of the representation space express certain basic assumptions of Newtonian Mechanics. First of all, the homogeneous and isotropic character of $E_3(\mathbf{r})$ guarantees the existence of *inertial frames*. Indeed, under such assumptions, a free particle which is at rest at a given time with respect to a given frame remains at rest at all subsequent times and the frame is inertial. All reference frames considered in this volume are inertial. The independence of $E_3(\mathbf{r})$ and $E_1(t)$ expressed by the Kronecker product $E_3(\mathbf{r}) \times E_1(t)$ and the homogeneous character of $E_1(t)$ allow compliance with another basic postulate of Newtonian Mechanics, namely, the assumption that *time is absolute*, or, alternatively, that the same event, if measured by two inertial frames $A$ and $A'$, occurs at the same time, i.e., $t \equiv t'$. This implies the acceptance of infinite velocities for signals and/or particles. Throughout our analysis, the absolute nature of time is tacitly assumed.

The *Galilean Relativity Principle* states that for all inertial frames, the laws of Newtonian Mechanics are the same. Thus, starting with one inertial frame, one can equivalently study the same motion from a second inertial frame. The procedure can be iterated an infinite number of times by indicating in this way that there is no inertial frame of reference in Newtonian Mechanics which is "absolute," that is, preferable over all other inertial frames, because all inertial frames are equivalent.

This demands the study of the transformations connecting two inertial frames which do not alter the laws of Newtonian Mechanics. Consider Newton's second law in the (unprimed) inertial frame $A$ for a particle with constant (inertial) mass $m$, i.e., $m\ddot{\mathbf{r}} = \mathbf{F}$. Let $m'\ddot{\mathbf{r}}' = \mathbf{F}'$ be the law for the same motion as seen from a second inertial frame, $A'$. The assumption that the mass must be the same for both $A$ and $A'$ implies $m = m'$. The assumption that the work done by the acting force must be the same for both $A$ and $A'$ then restricts all possible *linear* coordinate transformations to the so-called *orthogonal transformations* (or *rotations*)

$$r'^a = R^a_b r^b, \qquad RR^T = R^T R = 1, \qquad R = (R^a_b). \tag{1}$$
$$a = 1, 2, 3.$$

The admissible nonlinear transformations are then given by the solutions of the equations

$$\frac{d^2}{dt^2} r'^a = R^a_b \frac{d^2}{dt^2} r^b, \tag{2}$$

which are

$$\dot{r}'^a = R^a_b r^b + v^a, \qquad v^a = \text{const.}, \tag{3a}$$

$$r'^a = R^a_b r^b + v^a t + r^a_0, \qquad r^a_0 = \text{const.} \tag{3b}$$

If $m$ is at the origin of $A$, then $r^a_0(v^a)$ represents the coordinates (velocities) of such an origin within frame $A'$. Notice that the admissible relative motion between $A$ and $A'$ is that with constant velocity $v$.

The assumption that time is absolute implies that the duration of an event in $A$ is the same as that in $A'$, i.e., $\Delta t = \Delta t'$, thus $t' = t + t_0$, with $t_0 = \text{const.}$

The set of all transformations so obtained,

$$\begin{cases} \mathbf{r}' = R\mathbf{r} + \mathbf{v}t + \mathbf{r}_0, \\ t' = t + t_0, \end{cases} \tag{4}$$

are called *Galilei transformations*[33a] and represent the largest set of linear transformations in $E_{3.1}(\mathbf{r}, t)$ that preserve the laws of Newtonian Mechanics. Such transformations form a 10-parameter group called the *Galilei group* $G_{3.1}$, where

1. 3 parameters (angles) characterize an arbitrary rotation $\mathbf{r}' = R\mathbf{r}$;
2. 3 parameters (the velocity $\mathbf{v}$) characterize the transformation $\mathbf{r}' = \mathbf{r} + \mathbf{v}t$ called *Galilei boost*;
3. 4 parameters (the positions $\mathbf{r}_0$ and the time $t_0$) characterize the translations in space $\mathbf{r}' = \mathbf{r} + \mathbf{r}_0$ and in time $t' = t + t_0$.

As a final comment of speculative nature, the reader should be aware that the systems studied in this monograph *are not* form-invariant under the Galilei transformations because their forces, besides being not derivable from a potential and explicitly dependent on time, are nonlinearly dependent on coordinates and velocities. This creates the problem whether the Galilei relativity as currently known (and outlined in this chart) needs a suitable generalization to become applicable to nonconservative systems in general (rather than to conservative and Galilei form-invariant systems in particular). This problem is identified and treated in details by Santilli (1978). The same references present the conjecture of a generalization of the Galilei relativity for nonconservative systems based on the so-called Lie-admissible algebras (which are algebraic coverings of the Lie algebras directly applicable in Newtonian Mechanics for forces not derivable from a potential via the brackets of the time evolution law of a suitable generalization of Hamilton's equation). Possible relativistic extensions are considered too.

*Nota Bene*: According to the assumptions of Section A.1, our analysis is restricted to those systems for which all relativistic effects can be ignored. Therefore, all velocities considered in this volume are tacitly assumed to be much smaller than the velocity of light. Within such a framework, Newtonian Mechanics and the underlying Galilean Relativity Principle are in agreement with physical reality. In this respect, one aspect of our analysis calls for clarification. The Lagrangians $L(t, q, \dot{q})$ we deal with are functions generally defined for all values $-\infty \leq t, q^k, \dot{q}^k \leq +\infty$. This implies possible values of the velocities $\dot{q}^k$ for which relativistic effects should be taken into consideration. We eliminate such possibilities by considering the Lagrangians and related analytic representations over a region $R^{2n+1}$ of values $(t, q, \dot{q})$ selected in such a way as to avoid relativistic effects. Therefore, throughout our analysis we tacitly assume that all regions of definition of the Lagrangian are consistent with the assumptions of Section A.1. This is compatible with the methodology used for the integrability conditions for the existence of a Lagrangian, i.e., the calculus of differential forms and the Converse of the Poincaré Lemma (Section 1.2). Indeed, this methodology can be formulated in the neighborhood of a point of the variables

[33a] Galilei (1638). For contemporary accounts, see, for instance, Landau and Lifshitz (1960, Section I.3) or Mann (1974, Section I.1.C). For the Galilean group see, for instance, Levy-Leblond (1971).

$(t, q, \dot{q})$ satisfying certain restrictions (of being star-shaped). In particular, the value of $\dot{q}^k$ can be arbitrarily selected. In conclusion, there is no need for including divergent values of the velocities for the rigorous formulation and treatment of the problem of the existence of a Lagrangian.

**Chart A.2** Ignorable Coordinates and Conservation Laws

Let $L(t, q, \dot{q}) = T(t, q, \dot{q}) - V(t, q)$ be the Lagrangian of a *conservative* system. Each of the $2n + 1$ quantities $t$, $q$, and $\dot{q}^k$ is said to be *ignorable* when the Lagrangian does not depend explicitly on it.

Conservation law of the total energy. *When the time is ignorable, the total energy* $\mathscr{E}_T = \dot{q}^k\, \partial L/\partial \dot{q}^k - L = T + V$ *is constant.*

PROOF By using Lagrange's equations, the total time derivative of the Lagrangian can be written

$$\frac{dL}{dt} = \frac{d}{dt}\left(\dot{q}^k \frac{\partial L}{\partial \dot{q}^k}\right). \tag{1}$$

Thus

$$\frac{d}{dt}\left(\dot{q}^k \frac{\partial L}{\partial \dot{q}^k} - L\right) = \frac{d}{dt}\mathscr{E}_T \equiv 0. \tag{2}$$

*Nota Bene*: The quantity $\dot{q}^k\, \partial L/\partial \dot{q}^k - L$ *does not,* in general, represent the physical total energy unless the system is scleronomous, holonomic and conservative, in which case $\dot{q}^k\, \partial L/\partial \dot{q}^k = 2T$. Notice that when $L$ does not depend explicitly on time, it is invariant under time translations (i.e., $t \to t' = t + t_0$, $t_0 = \text{const.}$)

Conservation law of the generalized momentum. *If the generalized coordinate* $q^k$ *is ignorable then the generalized momentum* $p_k$ *is constant.*

PROOF From the definition $p_k = \partial L/\partial \dot{q}^k$, the Lagrange equations, and the assumption that $q^k$ is ignorable, it follows that

$$\dot{p}_k = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} = \frac{\partial L}{\partial q^k} \equiv 0. \tag{3}$$

*Nota Bene*: The generalized momentum $p_k$ *does not,* in general, coincide with the physical linear momentum unless the system is conservative and without constraints. If this is the case, the above conservation law does not imply that the "total" momentum is constant unless all coordinates are ignorable (i.e., when, trivially, all particles are free). For conservative systems (only) the condition $\dot{p}_k = 0$ implies that the $k$ component of the acting force is null. Notice also that when $L$ does not depend on $q^k (k = \text{fixed})$ it is invariant under translations in the $k$ coordinate (i.e., $q^k \to q'^k = q^k + q_0^k$, $q_0^k = \text{const.}$).

Conservation law of the angular momentum.[34] *Consider a conservative system of N particles with total angular momentum* $\mathbf{m}^T = \sum_i \mathbf{m}_i$, *and*

[34] See, for instance, Whittaker (1904), Goldstein (1950), and Mann (1974). A study of the problem of symmetries and conserved quantities for Newtonian systems with arbitrary forces is provided in Santilli (1979).

*without constraints. Let* $\mathbf{m}_i \cdot \mathbf{n}$ *be the component of the angular momentum* $\mathbf{m}_i$ *along an axis in space with unit vector* $\mathbf{n}$. *Represent the system in cylindrical coordinates* $q^i = r^i$, $q^{i+1} = \theta^i$, $q^{i+2} = n^i$, $i = 1, 2, \ldots, N$, *where* $n^i$ *is the component of* $\mathbf{r}^i$ *along the* $\mathbf{n}$ *axis. Then, if the angle* $q^{i+1} = \theta^i$ *is ignorable, the component* $\mathbf{m}_i \cdot \mathbf{n}$ *of the angular momentum* $\mathbf{m}_i$ *is constant.*

PROOF For $q^{i+1} = \theta^i$, $p_{i+1} = \partial L/\partial \dot{q}^{i+1} = m_i r^2 \dot{\theta}^i = \mathbf{m}_i \cdot \mathbf{n}$. The proof is then similar to that for the conservation of the generalized momentum $p_{i+1}$.

*Nota Bene*: Again, the above conservation law does not imply that the "total" angular momentum is constant unless "all" angles in all three possible representations of the systems in cylindrical coordinates along three orthogonal axes are ignorable (e.g., when $\mathscr{V}(\mathbf{r}^1, \ldots, \mathbf{r}^N) = \mathscr{V}(r^1, \ldots, r^N)$. For the considered system, the condition $\partial L/\partial \theta^i = 0$ implies that the $\theta^i$ component of the acting torque is null. When the system is not conservative, the generalized momentum does not coincide with the linear momentum, and $\partial L/\partial \dot{\theta}^i$ does not represent the component $\mathbf{m}_i \cdot \mathbf{n}$ of the physical angular momentum $\mathbf{m}_i$. Notice that when the Lagrangian (of either a conservative or nonconservative system) does not depend on an angle, then it is invariant under rotations about that angle.

*Caution*: The conservation laws given above do not allow the identification of all possible constants of the motions. For instance, for a free particle with $\mathbf{p} = m\dot{\mathbf{r}}$, the quantity $m\mathbf{r} - \mathbf{p}t$ is also a constant of the motion resulting from the invariance of the system under translations and the Galilean transformations $\mathbf{r} \to \mathbf{r}' = \mathbf{r} - \mathbf{v}t$.

### Chart A.3 Impulsive Motion[35]

Consider a Newtonian system of particles with constant masses without constraints. The motion is said to be *impulsive* when sudden changes of dynamical quantities occur because of forces $\mathscr{F}$ of large intensity acting in a small period of time (called *impulsive forces*). For instance, we have an impulsive motion when there is no appreciable change of the positions $\mathbf{r}(t)$ in the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ but a large variation of the velocities $\dot{\mathbf{r}}(t)$ in the same interval of time occurs. This impulsive motion is customarily interpreted in mathematical formulations by saying that the coordinates $\mathbf{r}(t)$ are continuous, while the velocities $\dot{\mathbf{r}}(t)$ are discontinuous at $t_0$.

By introducing the *impulse*,

$$\mathscr{P} = \int_{t_0-\varepsilon}^{t_0+\varepsilon} dt\, \mathscr{F}, \tag{1}$$

the equations of motion can be written

$$m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_i^0) = \mathscr{P}_i, \tag{2}$$

where $\dot{\mathbf{r}}(\dot{\mathbf{r}}^0)$ represents the velocities prior to (after) the application of the impulse.

*Nota Bene*: Impulsive motions present several problematic aspects if one attempts to represent them with conventional analytic formulations. This is due, for instance, to the fact that the Lagrangian $L$ customarily depends on

[35] See, for instance, Pars (1965, Chapter XIV).

the velocities, $L = L(\mathbf{r}, \dot{\mathbf{r}})$, and, thus, the inclusion of impulsive forces implies the appearance of discontinuities in the functional dependence of the Lagrangian. In turn, such discontinuities have implications at all levels of the methodology, e.g., the existence theory of solutions, variational principles, etc. Therefore, we shall exclude, from here on, the representation of impulsive motion whenever considering analytic formulations. More specifically, we shall restrict our study of analytic formulations to systems whose coordinates, velocities, and accelerations are all continuous functions of time in the interval considered.

### Chart A.4 Arrow of Time and Entropy

From the viewpoint of *time inversion*,

$$t \to -t, \tag{1}$$

Newton's equations of motion can be classified into "symmetry-preserving" and "symmetry-violating" equations. Conservative systems, such as the case of a particle under an elastic force $-kx$, i.e.,

$$m\ddot{x} + kx = 0, \tag{2}$$

constitute a class of systems which is *time-inversion symmetry-preserving.* This implies that for each process there exists a corresponding equivalent time-reversed process. Dissipative systems, on the contrary, such as the case of a particle moving under the action of a drag force $-b\dot{x}$, i.e.,

$$m\ddot{x} + b\dot{x} = 0, \tag{3}$$

are, generally, *time-inversion symmetry-violating.* In this case, a time inversion leads to a nonequivalent motion, such as

$$m\ddot{x} - b\dot{x} = 0. \tag{4}$$

As a result of this property, dissipative systems and, more generally, all systems with velocity-dependent forces (e.g., dynamical systems) that are not invariant under time inversion demand, for their consistent description, a specified direction of time or "time's arrow."

*Nota Bene*: The time-inversion symmetry violation is *not* in contradiction with the *Principle of Causality* according to which a cause must precede the effect, because such a principle holds in Newtonian Mechanics for each specified direction of time.

The above behavior of Newtonian systems under time inversion has profound implications from a thermodynamic profile. One of the most significant aspects of the thermodynamical description of systems of particles is the existence of *irreversible processes.* According to Planck's definition, they are processes which, once performed, leave the world in an altered state with no experimental process capable of restoring the initial state. Such irreversibility has been related to a priviledged direction of time, namely, that for which Clausius's *entropy* is nondecreasing. More specifically, the *second law of thermodynamics* states that for any isolated system, the variation $dS$ of the entropy $S$ in the time $dt(\geq 0)$ must be non-increasing, i.e., $dt > 0 \leftrightarrow dS \geq 0$. Eddington vividly expressed such connection by saying that *entropy is time's arrow.*

Thermodynamics is today a sophisticated and fascinating discipline with a direct or indirect impact on virtually all other branches of Physics. A variety of excellent easily identifyable treatises are available to the interested student. Among the contributions by the "founding fathers" of thermodynamics, Boltzmann's contribution were and still are crucial. In essence, they identify and correlate the following layers of descriptions of the physical reality.

Newtonian description (Newton's equations of motion)
Probabilistic description (kinetic equations)
Thermodynamical description (entropy)

Boltzmann's approach is useful here to emphasize that the content of this book deals only with the first layer of description, namely, the Newtonian description.

It is also significant to indicate that Newtonian systems with time-inversion symmetry-violating forces, i.e., dissipative or nonconservative systems, have a rather natural place within the context of Boltzmann's analysis. This is not the case for conservative systems precisely in view of their time-inversion symmetry-preserving character. This problem can be expressed in terms of the so-called *Loschmidt's paradox*: when a system is invariant under time inversion, the time-reversed process is physically admissible and the original state can be recovered contrary to the second law of thermodynamics.

For a treatment of Loschmidt's paradox see, for instance, Prigogine (1973).

**Chart A.5** Gauss' Principle of Least Constraint[36]

Consider a system of $N$ particles with (constant) masses $m_i$ without constraints in a Cartesian reference frame. Introduce the quantity

$$Q(\ddot{\mathbf{r}}) = \tfrac{1}{2}m_i\left(\ddot{r}_i - \frac{\mathscr{F}_i}{m_i}\right)^2. \tag{1}$$

Gauss' Principle of Least Constraint *states that the system evolves in such a way that the quantity $Q$ affords a minimum for the actual accelerations.*

PROOF Let $\mathbf{r}$ be the actual accelerations and $\ddot{\mathbf{r}}^i + \Delta\ddot{\mathbf{r}}^i$ represent possible accelerations. Then

$$Q(\ddot{\mathbf{r}} + \Delta\ddot{\mathbf{r}}) - Q(\ddot{\mathbf{r}}) = \tfrac{1}{2}m_i(\Delta\ddot{\mathbf{r}}^i)^2 + (m_i\ddot{\mathbf{r}}_i - \mathscr{F}_i), \qquad \Delta\ddot{\mathbf{r}}^i > 0, \tag{2}$$

unless $\Delta\ddot{\mathbf{r}}^i = 0$, $i = 1, 2, \ldots, N$.

*Significance*: The condition that the (first-order) variation $\delta Q$ of $Q$ be null, i.e.,

$$\delta Q(\ddot{\mathbf{r}}) = 0 \tag{3}$$

is sufficient to derive Newton's equations of motion.

[36] See, for instance, Pars (1965, page 42).

**Chart A.6** The Gibbs–Appel Equations[37]

Consider a Newtonian system of $N$ particles (with constant masses) in a Cartesian reference frame. The quantity

$$G = \tfrac{1}{2} m_i \ddot{\mathbf{r}}^i \tag{1}$$

is called the *Gibbs function*. When expressed in terms of generalized coordinates, it can be written

$$G = G(t, q, \dot{q}, \ddot{q}) = \tfrac{1}{2} \ddot{q}^i Z_{ij}(t, q) \ddot{q}^j + Z'_k(t, q, \dot{q}) \ddot{q}^k + Z'_0(t, q, \dot{q}), \tag{2}$$

where $Z_{ij}$ is the same quantity as that of the kinetic energy and $Z'_k$ and $Z'_0$ are functions which depend on the forms of the constraints. The Gibbs function is instrumental in formulating the following theorem.

**Theorem.** *The accelerations of the system are such that the quantity*

$$G' = G - f_i \ddot{q}^i \tag{3}$$

*always assumes a minimum value when considered as a function of* $\ddot{q}^i$.

*Nota Bene*: In the above theorem, coordinates $q^i$ and velocities $\dot{q}^i$ are assumed to be constants. The proof then can be given by noting that the total variation $\Delta G(\ddot{q})$ is always positive unless $\Delta \ddot{q}^i = 0$.

Necessary conditions for $G(\ddot{q})$ to afford a minimum at $\ddot{q}$ are the equations

$$\frac{\partial G'}{\partial \ddot{q}^k} = f_k, \qquad k = 1, 2, \ldots, n, \tag{4}$$

called *Gibbs–Appel equations.* They are equivalent to Newton's equations of motion, they hold for both holonomic and nonholonomic constraints, and they are closely related to Gauss's principle of least constraint (Chart A.5).

**Chart A.7** Virial Theorem[38]

The so-called *Clausius's virial* for a Newtonian system of $N$ particles in Cartesian space subject to (nonimpulsive) forces $\mathscr{F}_i$ is the time average

$$\mathscr{V} = -\langle \tfrac{1}{2} \mathbf{r}^i \cdot \mathscr{F}_i \rangle = -\frac{1}{t_0} \int_0^{t_0} dt \, \tfrac{1}{2} \mathbf{r}^i \cdot \mathscr{F}_i. \tag{1}$$

The *virial theorem* states that under the assumption that the particles move in a closed region and the velocities are bounded, the time average of the kinetic energy is equal to the virial, i.e.,

$$\langle \mathscr{T} \rangle = \langle \tfrac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}^i \rangle = -\langle \tfrac{1}{2} \mathbf{r}^i \cdot \mathscr{F}_i \rangle. \tag{2}$$

For *central force fields*, the term $\mathbf{r}^i \cdot \mathscr{F}_i$ becomes equal to the potential energy. Thus, the virial theorem for central force fields states that the time average of the kinetic energy is equal to, but opposite, half the time average of the potential energy.

[37] See, for instance, Pars (1965, Chapters XII and XIII).
[38] See, for instance, Lindsay (1941).

*Significance*: The virial theorem is particularly useful from a statistical profile, e.g., in the kinetic theory of gases where it leads in a natural way to Boyle's law for perfect gases.

### Chart A.8 Liouville's Theorem for Conservative Systems[39]

Consider a conservative holonomic Newtonian system of $N$ particles, represented by the Hamiltonian $H(t, p, q) = T(t, q, p) + V(t, q)$ and Hamilton's equations

$$\dot{q}^k = \frac{\partial H}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H}{\partial q^k}, \qquad k = 1, 2, \ldots, n. \tag{1}$$

Suppose that the Hamiltonian $H$ satisfies all needed continuity conditions. Each point of the phase space $M_{(q,p)}$ represents a possible state of the system and one and only one possible path crosses through each of those points.

**Liouville's Theorem for Conservative Systems** *states that the phase space volume occupied by a conservative holonomic Newtonian system of particles is constant in time (i.e., particles in phase space move as an "incompressible fluid").*

Let $V$ be the volume in phase space. By using Gauss's divergence theorem we have

$$\frac{dV}{dt} = \int_V dq^1 \cdots dq^N \, dp_1 \cdots dp_N \left( \frac{\partial \dot{q}^k}{\partial q^k} + \frac{\partial \dot{p}_k}{\partial p_k} \right), \tag{2}$$

and in view of Equations (1) and the continuity of $H$, we can write

$$\frac{dV}{dt} = \int_V dq^1 \cdots dq^N \, dp_1 \cdots dp_N \left( \frac{\partial^2 H}{\partial q^k \, \partial p_k} - \frac{\partial^2 H}{\partial p_k \, \partial q^k} \right) \equiv 0. \tag{3}$$

*Nota Bene*: Liouville's Theorem holds also for time-dependent Hamiltonians.

*Significance*: Liouville's Theorem is particularly important for statistical mechanics and, thus, for any large collection of either microscopic or macroscopic particles (such as a plasma or a galaxy, respectively). The statistical properties of such systems may be specified at any time $t$ through the density $\rho(t, q, p)$ of points per unit volume of phase space. Liouville's theorem then implies that the density $\rho$ remains constant in the neighborhood of possible trajectories in phase space. *Statistical equilibrium* can then be defined as the distribution for which the density $\rho$ is uniform along possible phase space trajectories.

### Chart A.9 Generalizations of Liouville's Theorem to Dynamical Systems

Two generalizations of Liouville's theorem to dynamical systems can be formulated depending on the assumed form of the canonical equations.

[39] See, for instance, Tolman (1938).

*Global Approach*: Suppose that a holonomic Newtonian system admits a representation in terms of the Hamiltonian $H(t, q, p)$ and the conventional Hamilton's equations

$$\dot{q}^k = \frac{\partial H}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H}{\partial q^k}, \qquad k = 1, 2, \ldots, n, \tag{1}$$

Gauss's divergence theorem (Chart A.14) applies and we can state the following theorem.

**Liouville's Theorem for the Global Approach to Holonomic Newtonian Systems.** *The volume $V$ in the phase space $M(q, p)$ occupied by a holonomic Newtonian system of particles represented in terms of the conventional Hamilton's equations is* constant *in time, i.e.,*

$$\frac{dV}{dt} = \int_V dq^1 \cdots dq^N \, dp_1 \cdots dp_N \left( \frac{\partial^2 H}{\partial q^k \, \partial p_k} - \frac{\partial^2 H}{\partial p_k \, \partial q^k} \right) \equiv 0. \tag{2}$$

*Nota Bene*: The Hamiltonian $H(t, q, p)$ can have here an arbitrary functional dependence on $(t, q, p)$. The dependence $H = T + U$ may occur as a particular case when all the acting forces are derivable from a potential. Theorem (2) is clearly insensitive to the functional dependence of $H$ and it centrally depends on Equations (1) and the tacit requirement that $H$ possess continuous partial derivatives of at least second order. For the construction of a Hamiltonian capable of representing a (self-adjoint) holonomic Newtonian system through Equations (1), see Chapter 3.

*External Approach*: Suppose that a holomic Newtonian system is represented in terms of the Hamiltonian $H^+(t, q, p^+)$ and the Hamilton equations with external forces

$$\dot{q}^k = \frac{\partial H^+}{\partial p_k}, \qquad \dot{p}_k^+ = -\frac{\partial H^+}{\partial q^k} + f_k, \tag{3}$$

where the $q$'s are the same as those of Equations (1) and the $p^+$'s are new variables.

Let $M^+$ denote the space spanned by the $2n$ variables $q$ and $p^+$. Gauss's divergence theorem still applies for the space $M^+$, but we now have the following theorem.

**Liouville's Theorem for the External Approach to Holonomic Newtonian Systems.** *The volume $V^+$ in the space $M^+$ of variables $(q, p^+)$ occupied by a holonomic Newtonian system of particles represented in terms of the Hamilton's equations with external forces* is not constant *in time unless $\partial f_k / \partial p_k^+ \equiv 0$, i.e.,*[40]

$$\begin{aligned} \frac{dV^+}{dt} &= \int_{V^+} dq^1 \cdots dq^N \, dq_1^+ \cdots dp_N^+ \left( \frac{\partial \dot{q}^k}{\partial q^k} + \frac{\partial \dot{p}_k^+}{\partial p_k^+} \right) \\ &= \int_{V^+} dq^1 \cdots dq^N \, dp_1^+ \cdots dp_N^+ \left( \frac{\partial^2 H^+}{\partial q^k \, \partial p_k^+} - \frac{\partial^2 H^+}{\partial p_k^+ \, \partial q^k} + \frac{\partial f_k}{\partial p_k^+} \right) \\ &= \int_{V^+} dq^1 \cdots dq^N \, dp_1^+ \cdots dp_N^+ \frac{\partial f_k}{\partial p_k^+} \neq 0. \end{aligned} \tag{4}$$

[40] Lichtenberg, Stehle, and Symon (1956).

*Nota Bene*: If $f_k = f_k(t, q)$ in Equation (4) and the system is conservative, then the above two generalizations of Liouville's Theorem are equivalent. Thus, these two theorems actually differentiate when the acting forces (either dissipative or applied or both) are velocity-dependent. It must be stressed that these two theorems are not in contradition, because the spaces $M$ and $M^+$ *do not* coincide. In actuality, the above two theorems are complementary to each other in the sense that they express complementary views: if the system is represented as a whole with one single function $H$ (global approach), the volume in the space $M(q, p)$ is constant; but if the system is only *partially* represented with a function $H^+(q, p^+)$ and external $p^+$-dependent forces $f_k$ occur, the volume in the related $M^+(q, p^+)$, space is no longer constant due to the presence of external terms. The reformulation of the two theorems in terms of the densities $\rho$ and $\rho^+$ is here left to the interested reader.

The above properties give a clear indication of the methodological differences which result from the use of Equations (1) or (3) for the representation of Newtonian systems, as indicated in the Introduction.

## Chart A.10 The Method of Lagrange Undetermined Multipliers[41]

Consider a holonomic conservative system represented by the Lagrangian $L(\mathbf{r}, \dot{\mathbf{r}}) = T(\dot{\mathbf{r}}) - \mathscr{V}(\mathbf{r})$. Suppose that the system is subject to $k$ nonholonomic constraints of the (nonintegrable) velocity type,

$$\phi_s(t, \mathbf{r}, \dot{\mathbf{r}}) = \Gamma_{si}(t, \mathbf{r}) \cdot \dot{\mathbf{r}}^i + \Lambda_s(t, \mathbf{r}), \qquad s = 1, 2, \ldots, k. \tag{1}$$

Suppose also that $\mathscr{F}_i$ are the forces necessary for the system to satisfy such constraints. Then the equations of motion can be written

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{r}}^i} - \frac{\partial L}{\partial \mathbf{r}^i} = \mathscr{F}_i, \qquad i = 1, 2, \ldots, N. \tag{2}$$

But the work generated by $\mathscr{F}_i$ for all displacements $d\mathbf{r}^i$ which satisfy constraints (1) at a given (fixed) time $t$ must be null by assumption, i.e., $\mathscr{F}_i \cdot d\mathbf{r}^i = 0$ for all $d\mathbf{r}^i$ such that $\mathscr{F}_{si} \cdot d\mathbf{r}^i = 0$ at fixed $t(dt = 0)$. Then a necessary condition for such work to be null is that $\mathscr{F}_i = {}^s\Gamma_{si}$, where the $^s$'s are termed *Lagrange undetermined multipliers*. Therefore, the equations of motion of the system are given by the set of $N + k$ equations

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\mathbf{r}}^i} - \dfrac{\partial L}{\partial \mathbf{r}^i} = {}^s\Gamma_{si}, \\ \phi_s(t, \mathbf{r}, \dot{\mathbf{r}}) = 0 \end{cases} \tag{3}$$

in the $3N + k$ unknowns $\dot{\mathbf{r}}^i$ and $^s$.

Suppose, now, that constraints (1) are integrable, i.e., they satisfy conditions (A.2.16). Then Equations (3) are equivalent to the equations

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L^{(\alpha)}}{\partial \dot{\mathbf{r}}^i} - \dfrac{\partial L^{(\alpha)}}{\partial \mathbf{r}^i} = 0, \\ \dfrac{d}{dt}\dfrac{\partial L^{(\alpha)}}{\partial \dot{\alpha}^s} - \dfrac{\partial L^{(\alpha)}}{\partial \alpha^s} = 0 \end{cases} \tag{4}$$

[41] See, for instance, Rund (1966, Section 5.5).

in the Lagrangian

$$L^{(\alpha)} = L + \alpha^s \phi_s, \tag{5}$$

with $\dot{\alpha}^s = -I^s$. Indeed, Equations (4) can be written

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L^{(\alpha)}}{\partial \dot{\mathbf{r}}^i} - \dfrac{\partial L^{(\alpha)}}{\partial \mathbf{r}^i} = -\dot{\alpha}^s \Gamma_s + \alpha^s \left[ \left( \dfrac{\partial \Gamma_{si}}{\partial \mathbf{r}^k} - \dfrac{\partial \Gamma_{sk}}{\partial \mathbf{r}^i} \right) \cdot \dot{\mathbf{r}}^k - \left( \dfrac{\partial \Gamma_{si}}{\partial t} - \dfrac{\partial \Lambda_s}{\partial \mathbf{r}^i} \right) \right], \\ \dfrac{d}{dt}\dfrac{\partial L^{(\alpha)}}{\partial \dot{\alpha}^s} - \dfrac{\partial L^{(\alpha)}}{\partial \alpha^s} = -\dfrac{\partial L^{(\alpha)}}{\partial \alpha^s} = -\phi_s = 0, \end{cases} \tag{6}$$

and, in view of Equations (A.2.16), they coincide with Equations (3) for $\dot{\alpha}^s = -I^s$.

*Nota Bene*: Equations (3) are customarily derived on an empirical basis without methodological backing (e.g., variational). On somewhat pragmatic grounds they are used because they are known to be correct. In the transition to the configuration space of generalized coordinates, the forces of constraints are lost (Section A.4). By comparing Equations (A.4.15) with Equations (3) or (4), we see that when only the actual path is needed, both types of equations can be used. However, when the computation of the forces of constraints is required, then only the latter equations should be used. If a holonomic constraint is assigned in the coordinate form, it can be equivalently written in velocity form (1) through a time derivative (see Section A.2). Notice that the transition from Equations (3) to the equivalent form (4) is not trivial on methodological grounds, because it represents the transition from analytic equations *with* external terms in the Euclidean space $E_3(\mathbf{r})$ to analytic equations *without* external terms in the space $E_3(\mathbf{r})^3 \times E_k(\alpha)$ of the Cartesian coordinates and the multipliers $\alpha^s$. It should finally be indicated that Equations (3) constitute, strictly speaking, a system of second-order ordinary differential equations subject to a system of first-order subsidiary constraints. This class of systems will not be studied in this book.

### Chart A.11 Geometric Approach to Newtonian Systems

In the main text of this book, we study the analytic approach to Newtonian systems. Such an approach is undoubtedly essential to analyze several basic aspects and advisable as a first step for the uninitiated reader, but it does not exhaust all possible methodological alternatives. In this series of charts at the end of each chapter, we often touch upon a second significant approach to Newtonian Mechanics, the geometric approach, with the intention of providing the interested reader with the elements for a broader methodological horizon, but without any claim of completeness or mathematical rigor. Therefore, the uninitiated reader is urged to study the quoted references or some equivalent sources.

There exist several motivations for the analysis of Newtonian systems from a geometrical profile, ranging from a purely aesthetic need to specific methodological tools not provided by the conventional analytic approach. It is sufficient, in this respect, to mention that the geometric approach provides the means for characterizing the "essential features" of a dynamical system in a way independent from the selected type of coordinates. There are, however, deeper motivations for recommending the study of the geometric aspects. They are ultimately connected to the fact that within each selected coordinate system there exists an infinite variety

of equivalent equations of motion, all able to characterize the same *actual path* (i.e., the solution of a Newtonian initial value problem). This fact will be analyzed in more detail later. Thus, the actual path can be considered as an "essential feature" of a dynamical system. However, when such a system is represented in terms of Newton's (or analytic) equations, then the study of the class of equivalent Newtonian initial value problems becomes essential for any in-depth study of the framework. In turn, this aspect is intimately linked to the so-called transformation theory.

Again, the geometric approach provides means for characterizing the class of equivalent differential equations without any specific reference to an individual element of the class. There exist several aspects of the geometric approach which are relevant for our analysis. In these charts, we touch only on those of major significance. In Charts A.12 through A.16, we consider only the tensor calculus for linear and nonlinear coordinate transformations and the representation of the equations of motion in curvilinear coordinates. The concepts of manifold and symplectic geometry are introduced in the charts of Chapter 2. We hope that the presentation of the geometric approach jointly with the conventional analytic approach, rather than as a disjoint discipline per sé, will render it more accessible to the uninitiated reader, and that it will be more effective.

**Chart A.12** Tensor Calculus for Linear Coordinate Transformations[42]

*Tensor calculus*, one of the first steps toward a geometric analysis of physical systems, is concerned with the behavior of physical quantities under coordinate transformations. In this chart, we restrict our attention to linear transformations. The nonlinear case is considered in Chart A.13.

Consider an $n$-dimensional Euclidean space $E_n$ with coordinates $q_i$, $i = 1, 2, \ldots, n$, for a point $P \in E_n$. Let the linear transformation

$$\bar{q}_k = a^i_k q_i, \tag{1}$$

be an orthogonal transformation, i.e., a transformation satisfying the conditions $aa^T = a^T a = 1$, where $a = (a^i_j)$ and $a^T$ is the transpose of $a$.

A set of $n^r$ quantities $T_{i_1 i_2 \ldots i_r}$ is termed an *affine tensor of rank r* if these quantities transform, under transformations (1), according to the law

$$\bar{T}_{j_1 j_2 \ldots j_r} = \frac{\partial \bar{q}_{j_1}}{\partial q_{i_1}} \frac{\partial \bar{q}_{j_2}}{\partial q_{i_2}} \cdots \frac{\partial \bar{q}_{j_r}}{\partial q_{i_r}} T_{i_1 i_2 \ldots i_r}, \tag{2}$$

with inverse

$$T_{i_1 i_2 \ldots i_r} = \frac{\partial q_{i_1}}{\partial \bar{q}_{j_2}} \frac{\partial q_{i_2}}{\partial \bar{q}_{j_2}} \cdots \frac{\partial q_{i_2}}{\partial \bar{q}_{j_2}} \bar{T}_{j_1 j_2 \ldots j_r}. \tag{3}$$

An *affine vector* $A_i$ in $E_n$ is an affine tensor of rank $r = 1$ with transformation laws (and inverse)

$$\bar{A}_j = \frac{\partial \bar{q}_j}{\partial q_i} A_i, \qquad A_i = \frac{\partial q_i}{\partial \bar{q}_j} \bar{A}_j. \tag{4}$$

[42] See, for instance, Lovelock and Rund (1975).

An *affine scalar* $A(q)$ (or *affine invariant*) is an affine tensor of rank $r = 0$ with transformation law

$$\bar{A}(\bar{q}) = A(a). \tag{5}$$

If an affine tensor vanishes in one coordinate system it vanishes in any other reference frame obtained through orthogonal transformations. The set of all affine tensors of rank $r$ constitutes a (linear) vector space over the field of real numbers.

*Nota Bene*: For the affine (or *orthogonal* or *Cartesian*) framework there is no distinction between coordinates with upper or lower indices, i.e., $q_i = q^i, i = 1, 2, \ldots$. The configuration space $M_{(q)}$ with generalized coordinates $q^i$ used in the text refers primarily to this context. The set of all orthogonal transformations in $E_n$ forms a group called the orthogonal group $O(n)$ (which includes inversions, i.e., the transformation $\bar{q}_i = -q_i$).

**Chart A.13** Tensor Calculus for Nonlinear Coordinate Transformations[43]

The linear transformations of Chart A.12 are often inadequate (e.g., for the case of curvilinear coordinate systems or of tensor calculus on manifolds). As a generalization of the affine framework, consider the transformations in a Euclidean space $E_n$ with points $P = (q^1, \ldots, q^n)$

$$\bar{q}^i = \bar{q}^i(q), \tag{1}$$

which are *not* assumed to be necessarily linear in $q^i$. Suppose that the functions $\bar{q}^i(q)$ possess continuous partial derivatives of the second order in a region $R_n \in E_n$ and that everywhere in such a region transformations (1) are invertible. A *scalar* (or *invariant*) in $R_n$ is any quantity $A(q)$ such that $\bar{A}(\bar{q}) = A(q)$. The gradient vector $A_i = \partial A/\partial q^i$ transforms under (1) according to

$$\bar{A}_i = \frac{\partial q^j}{\partial \bar{q}^i} A_j, \tag{2}$$

and is termed a *covariant vector.* A set of quantities $A^i(q)$ is termed a *contravariant vector* when it transforms under Equation (1) according to the law

$$\bar{A}^i = \frac{\partial \bar{q}^i}{\partial q^j} A^j. \tag{3}$$

The coefficients of transformation laws (2) and (3) must be evaluated at the point $P$. Three types of tensors of rank 2 can now be distinguished. They

[43] See, for instance, Misner, Thorne, and Wheeler (1973) or Lovelock and Rund (1975).

are the *contravariant tensor*, $T^{ij}$, the *covariant tensor* $T_{ij}$ and the *tensor of mixed type* $T^i_j = T^i_j$, defined by the corresponding transformation laws

$$\bar{T}^{ij} = \frac{\partial \bar{q}^i}{\partial q^k}\frac{\partial \bar{q}^j}{\partial q^l} T^{kl},$$

$$\bar{T}^i_j = \frac{\partial \bar{q}^i}{\partial q^k}\frac{\partial q^l}{\partial \bar{q}^j} T^k_l,$$

$$\bar{T}_{ij} = \frac{\partial q^i}{\partial \bar{q}^k}\frac{\partial q^j}{\partial \bar{q}^l} T_{kl}. \tag{4}$$

In addition, a quantity $\Gamma^i_{jk}$, termed *connection*, can be introduced for which

$$\bar{\Gamma}^{i'}_{j'k'} = \frac{\partial \bar{q}^{i'}}{\partial q^i}\frac{\partial q^j}{\partial \bar{q}^{j'}}\frac{\partial q^k}{\partial \bar{q}^{k'}}\Gamma^i_{jk} + \frac{\partial q^k}{\partial \bar{q}^{j'}}\frac{\partial^2 \bar{q}^{i'}}{\partial q^k\,\partial q^l}\frac{\partial q^l}{\partial \bar{q}^{k'}}. \tag{5}$$

Such a connection allows the quantities $dA^i + \Gamma^i_{jk}A^j\,dq^k$ to transform as contravariant vectors. This permits the introduction of the *covariant derivatives*

$$\begin{cases} A^i_{;j} = \dfrac{\partial A^i}{\partial q^j} + \Gamma^i_k\,A^k_j, \\ A_{i;j} = \dfrac{\partial A_i}{\partial q^j} + \Gamma^k_{ij}A_k, \end{cases} \tag{6}$$

for which

$$A^i_{\ ;j;k} - A^i_{\ ;k;j} = A^l R^i_{lkj} + (\Gamma^l_{ji} - \Gamma^l_{ki})A^i_{\ ;l},$$

the quantity

$$R^j_{ihk} = \frac{\partial \Gamma^j_{ih}}{\partial q^k} - \frac{\partial \Gamma^j_{ik}}{\partial q^h} + \Gamma^j_{mk}\Gamma^m_{ih} - \Gamma^j_{mh}\Gamma^m_{ik}, \tag{7}$$

is termed the *curvature tensor* of the connection.

*Nota Bene*: Unlike the case of linear transformations (Chart A.12), the covariant and contravariant quantities do not coincide within the context of nonlinear transformations. This is ultimately due to the fact that the terms $\partial q^i/\partial \bar{q}^j$ and $\partial \bar{q}^i/\partial q^j$ are not, in general, equal. If, however, transformations (1) are linear, then: (a) such identification is possible; (b) the distinction between covariant and contravariant quantities is lost; and (c) the connection becomes identically null (together with its curvature tensor). We must stress that the connection $\Gamma^i_{hk}$ is, at this point, arbitrary in the sense that it can be characterized by any set of $n^3$ numbers and law (5). It should be mentioned that such a connection is often called an *affine connection*, although the context is not the conventional affine framework of Chart A.12.

### Chart A.14 Dynamical Systems in Curvilinear Coordinates[44]

Consider a Newtonian system of $N$ particles with constant masses subject to scleronomous holonomic constraints and generalized forces $f_k(t, q, \dot{q})$. Its kinetic energy can be written

$$T(q, \dot{q}) = \tfrac{1}{2}\dot{q}^i Z_{ij}(q)\dot{q}^j, \tag{1}$$

[44] See, for instance, Rund (1966, Appendix 2).

where the quantities $Z_{ij}$ are given by Equation (A.4.11). Then we can write

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^i} - \frac{\partial T}{\partial q^i} = Z_{ik}\ddot{q}^k + \frac{1}{2}\left(\frac{\partial Z_{ik}}{\partial q^j} + \frac{\partial Z_{ji}}{\partial q^k} - \frac{\partial Z_{jj}}{\partial q^k}\right)\dot{q}^j\dot{q}^k. \tag{2}$$

By introducing the so-called *Christoffel symbols of the first kind,*

$$[ji, k] = \frac{1}{2}\left(\frac{\partial Z_{ik}}{\partial q^j} + \frac{\partial Z_{ji}}{\partial q^k} - \frac{\partial Z_{ji}}{\partial q^{k'}}\right) \tag{3}$$

the equation of motion can be written

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^i} - \frac{\partial T}{\partial q^i} - f_i = Z_{ik}\ddot{q}^k + [ji, k]\dot{q}^j\dot{q}^k - f_i = 0. \tag{4}$$

By introducing the *Christoffel symbols of the second kind,*

$$\{{}^{i}_{jk}\} = Z^{il}[jk; l] \tag{5}$$

with the "inverse"

$$[jk; l] = Z_{il}\{{}^{i}_{jk}\}, \tag{6}$$

where the matrix $(Z^{ij})$ is the inverse of $(Z_{ij})$, the equation of motion can be written

$$Z_{ik}[\ddot{q}^k + \{{}^{k}_{jl}\}\dot{q}^j\dot{q}^l] - f_i = 0 \tag{7}$$

In *Riemannian geometry,* the quantity within the square brackets is termed the *covariant derivative* of $\dot{q}^k$ with respect to $t$, and denoted by

$$\frac{D\dot{q}^k}{Dt} = \ddot{q}^k + \{{}^{k}_{jl}\}\dot{q}^j\dot{q}^l. \tag{8}$$

Then the equations of motion can be written in the concise form

$$Z_{ik}(q)\frac{D\dot{q}^k}{Dt} - f_i = 0. \tag{9}$$

*Significance*: Equations (4) or (7) or (9) are particularly useful from a geometric (i.e., Riemannian) profile both per sé and as an intermediate step prior to the general theory of gravitation.

*Nota Bene*: Equations (4), (7), or (9) can be interpreted as the equations of motion of scleronomous holonomic systems in a curvilinear coordinate system with affine connection $\Gamma^i_{jl} = \{{}^{i}_{jl}\}$ or, equivalently, in a curved space with metric tensor $q_{ij} = \frac{1}{2}Z_{ij}$. When the elements $g_{ij}(q)$ characterize a nonsingular, symmetrjc, covariant tensor field over a differentiable manifold (see Chart 2.1), the representation space can be interpreted as an $n$-dimensional Riemannian space.

At the limit when $g_{ij} = \frac{1}{2}m_i\delta_{ij}$ (i.e., when the system is unconstrained), the Christoffel symbols vanish identically and Equation (4), (7), or (9) coincides with Newton's equations in Cartesian coordinates.

## EXAMPLES

*Example A.1*

A significant example of a Newtonian conservative system without constraints is provided by an isolated system of $N$ particles moving in a vacuum under their mutual gravitational forces. This system can be represented by the Lagrangian

$$L(\dot{\mathbf{r}}, \mathbf{r}) = \tfrac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}^i - G \sum_{\substack{i,j=1 \\ i<j}}^{n} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|},$$

where $G$ is the universal constant of gravitation. The study of this system, particularly for the case $N > 2$, falls within the framework of the many-body formulations indicated in the Introduction.

*Example A.2*

One of the simplest examples of systems with conservative forces and scleronomous holonomic coordinate constraints is given by the plane pendulum. In this case, the coordinate constraint arises from mechanical means (inextensible rod of length $l$ and hinge), which force the particle to move in a vertical circle of radius $l$. If one takes the generalized coordinate $q$ to be the angle between the rod and the vertical, the system is described by the Lagrangian

$$L(q, \dot{q}) = \tfrac{1}{2} m l^2 \dot{q}^2 - mgl(1 - \cos q)$$

when frictional forces are ignored.

*Example A.3*

An example of a system with conservative forces and rheonomous holonomic coordinate constraints is given by a particle which moves under the force of gravity along a straight line, rotating in a vertical plane with a constant angular velocity $\dot{\varphi}$. The Lagrangian (for $q = r =$ coordinate along the straight line) is

$$L(t, q, \dot{q}) = \tfrac{1}{2} m(q^2 \dot{\varphi}^2 + \dot{q}^2) + mgq \cos \dot{\varphi} t.$$

*Example A.4*

Another simple example of a conservative system is the one-dimensional harmonic oscillator with the Lagrangian

$$L(x, \dot{x}) = \tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} k x^2$$

and the (linear) equation of motion

$$\ddot{x} + \omega_0^2 x = 0, \qquad \omega_0^2 = \frac{k}{m}$$

which holds for small oscillations when damping forces are ignored.

*Example A.5*

A typical example of a nonlinear conservative system is the anharmonic oscillator with the Lagrangian

$$L(x, \dot{x}) = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}k_1x^2 - \tfrac{1}{3}k_2x^3.$$

The equation of motion

$$\ddot{x} + \omega_0^2 x + \omega_1^2 x^2 = 0,$$

$$\omega_0^2 = \frac{k_1}{m}, \qquad \omega_1^2 = \frac{k_2}{m}$$

is nonlinear (in the coordinate) and provides a higher approximation than that of the linear case, when damping forces are again ignored.

*Example A.6*

A typical example of one-dimensional linear dissipative systems is the oscillator with conservative (elastic) force $\mathscr{F}^c = -m\omega_0^2 x$ and damping force $\mathscr{F}^d = -2m\beta_0\dot{x}$. The equation of motion is

$$\ddot{x} + 2\beta_0\dot{x} + \omega_0^2 x = 0.$$

If one assumes $\mathscr{P}^d = -m\beta_0\dot{x}^2$ as the power function, then the rate of change of energy is

$$\frac{d\mathscr{E}_T}{dt} = 2\mathscr{P}^d = \mathscr{F}^d\dot{x} = -2m\beta_0\dot{x}^2.$$

*Example A.7*

An example of a nonlinear dissipative system occurs when the conservative and dissipative forces are

$$\mathscr{F}^c = -m\omega_0^2 x - m\omega_1^2 x^2 \quad \text{and} \quad \mathscr{F}^d = -2m\beta_0\dot{x} - 2m\beta_1\dot{x}^2,$$

respectively. The equation of motion is

$$\ddot{x} + 2\beta_0\dot{x} + 2\beta_0\dot{x}^2 + \omega_0 x + \omega_1 x^2 = 0$$

and the rate of change of the energy is

$$\frac{d\mathscr{E}_T}{dt} = -2m\beta_0\dot{x}^2 - 2m\beta_1\dot{x}^3.$$

Notice that this system is nonlinear in both the position and the velocity.

*Example A.8*

An example of a linear dissipative system in more than one dimension is given by the equations

$$a_k(t)q^k + b_k(t)\dot{q}^k + c_k(t)\ddot{q}^k = 0 \quad \text{(no summation)}$$
$$k = 1, 2, \ldots, n,$$

which represent a system of damped oscillators. The oscillators, however, are decoupled.

*Example A.9*

A more significant example of a (linear) dissipative system of arbitrary dimension is given by a system of coupled and damped oscillators with the equations

$$a_{ki}(t)q^i + b_{ki}(t)\dot{q}^i + c_{ki}(t)\ddot{q}^i = 0,$$

which play a central role in the theory of small oscillations with damping.

*Example A.10*

A significant example of a truly dynamic system in one dimension is the damped and forced oscillator with conservative force $-m\omega_0^2 x$, dissipative force $-2m\beta_0 x$, and applied force $f^e(t, x, \dot{x})/m$. Its equation of motion is

$$\ddot{x} + 2\beta_0\dot{x} + \omega_0^2 x = f^e(t, x, \dot{x}).$$

*Example A.11*

The system of coupled, damped, and forced oscillators with the equations of motion

$$a_{ki}(t)q^i + b_{ki}(t)\dot{q}^i + c_{ki}(t)\ddot{q}^i = f_k^e(t, q, \dot{q})$$

is a significant example of a dynamic system in more than one dimension. Notice that this system can be written in the form of fundamental equations of motion (A.7.5) by putting

$$A_{ki} = c_{ki},$$

$$B_k = a_{ki}q^i + b_{ki}\dot{q}^i - f_k^e.$$

Notice the appearance of the so-called *acceleration couplings* due to generally non-null values of the off-diagonal terms of the matrix $(c_{ij})$.

## Problems

A.1 Prove the following relations in the transition from the Cartesian to the generalized space of coordinates.

$$\frac{\partial \dot{\mathbf{r}}^i}{\partial \dot{q}^j} = \frac{\partial \mathbf{r}^i}{\partial q^j},$$

$$\frac{d}{dt}\frac{\partial \dot{\mathbf{r}}^i}{\partial \dot{q}^j} = \frac{\partial \dot{\mathbf{r}}^j}{\partial q^j},$$

$$\ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}^i}{\partial q^j} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}^j}\left(\tfrac{1}{2}\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}^i\right) - \frac{\partial}{\partial q^i}\left(\tfrac{1}{2}\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}^i\right),$$

which are used for the Lagrangian formulation of D'Alembert's principle (A.4.9).

A.2 Prove that acceleration-dependent forces generally violate the postulate of Newtonian Mechanics according to which the total acceleration of a particle is the (vector) sum of the accelerations produced by each individual acting force.

A.3 Perform the transition to configuration space for D'Alembert's principle (A.7.1) in the case of masses variable in time.

A.4 Compute the equations of motion of Examples A.2 and A.3.

A.5 Compute the energy rate of variation for Examples A.9, A.10, and A.11.

A.6 Examples A.2 and A.3 constitute only an approximation of the corresponding physical systems. Construct more realistic models with the inclusion of dissipative forces and identify their equations of motion.

A.7 The case of a massive charged particle moving (nonrelativistically) in an electromagnetic field constitutes a somewhat hybrid mixture of a discrete system (the particle) and a continuous system (the electromagnetic field). Assume the total energy of this system to be that of its discrete part, namely, the kinetic energy of the particle. Compute the rate of variation of this energy and identify the system as conservative, dissipative, or dynamic.

A.8 Prove the Gibbs–Appel equations of Chart A.6. (Hint: Assume coordinates and velocities to be constant.)

A.9 Prove the virial theorem of Chart A.7.

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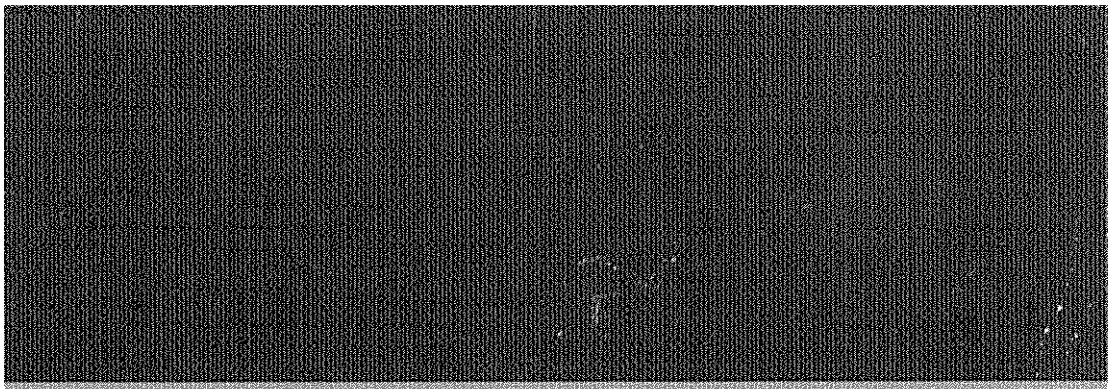

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