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SOME REALIZATIONS OF HADRONIC MECHANICS

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Abstract

In this paper, we review some realizations of hadronic mechanics recently achieved by a number of authors, and present further developments. The examples presented in this paper are as follows: (1) a generalization of M. Gasperini's work on the Lie–isotopic lifting of conventional gauge theory, (2) an introduction of gauge field by the Lie–isotopic lifting of the Hilbert space, (3) extension of the Dirac–Myung–Santilli Delta Function to field theory, (4) relations between some models of field theory and the Lie–isotopic or the Lie–admissible approach, (5) remarks on the noncanonical commutation relations of the energy–momentum operators, (6) an application of the Lie–admissible lifting to the unstable state.

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1. INTRODUCTION

According to Santilli et al. [1], hadronic mechanics can be defined as follows: *Hadronic mechanics is a generalization of quantum mechanics currently under construction for the description of hadrons as closed systems with conventional, potential interactions as well as non–local/non–Hamiltonian internal interactions due to the mutual penetration of the wave packets of the constituents. The closed–exterior branch of hadronic mechanics is constructed via an isotopic lifting of the enveloping associative operator algebras, of the fields and of the Hilbert spaces of quantum mechanics.*

Santilli [2–5] laid the foundations for the formulation of hadronic mechanics as a generalization of quantum mechanics for extended deformable particles. A main idea is that the hadrons and their components are extended particles, subject to contact–non-potential interactions for which conventional quantum mechanics is inapplicable. The conventional Lie theory should then be replaced by a suitable generalization, such as the Lie–isotopic covering of the Lie theory. For the exterior treatment of closed systems (like a hadron) with extended constituents and non-potential interactions, the Lie–isotopic theory appears to be suitable. Since gauge theory is applied to the hadronic components of matter, it is then justified to investigate the possibility of a Lie–isotopic generalization of the conventional gauge theory. Along this line of study, M. Gasperini [7] recently suggested a possible Lie–isotopic generalization of conventional gauge theory. Starting from the Lie–isotopic lifting of a continous group of transformations [5], he defined isotopic covariant derivatives, field strengths and also interpreted the generalized theory as a gauge theory with variable effective coupling constant and a gauge potential minimally coupled to the geometry of a Riemann–Cartan space with an antisymmetric connection. Using the same techniques as that of M. Gasperini, we shall show that, in the Lie–isotopic lifting of gauge theory, variations including contraction may occur spontaneously in the Lie–algebra

in which the gauge potentials have the values. We shall also construct a Lagrangian density of quarks and gauge fields to find that the generalized theory can be interpreted as a gauge theory in a curved space–time, for example, in space–time with constant curvature and modified gauge potentials. Furthermore, we shall introduce a gauge field via the isotopic lifting of the Hilbert space. The above–mentioned aspects will be treated in Chapter 2. In Chapter 3, we consider the hadronic–isotopic generalization of the Dirac delta function proposed by Myung and Santilli [8]. In Chapter 4, we shall briefly sketch relations between some models of field theory and the Lie–isotopic or the Lie–admissible approach. Santilli [10] found the Lie–admissible covering of Heisenberg's algebra and, with Myung [11], derived the corresponding Lie–admissible Schrödinger's equations. Mignani [12] derived the same equations via a different approach. Recently, A. Jannussis et al. [13] studied the time non-canonical character of hadronic mechanics by making use of the Lie–admissible formulation of the Schrödinger equation. They began their study on the problem of the equivalence of the energy operator with the Hamiltonian. They are interested in the following general question: Is it possible that the equivalence can be generalized in a functional way? . In Chapter 5, we shall generalize their ideas to include not only the time component but also the space components. We also generalize the Lie–admissible formulation of the Schrödinger equation by Santilli and Myung and by Mignani. By making use of the generalized equation, we shall obtain in the Lie–admissible commutator, an interesting form of an admissilbe element. In Chapter 6, it is shown that the Lie–admissible lifting of the Schrödinger equation is suitable for the description of the decay of the unstable state and the corresponding evolution operator becomes essentially non–unitary. These studies are mainly based on recent work by the present author, which have been recently published.

2. GAUGE THEORY

(a) A GENERALIZATION OF M. GASPERINI'S WORK ON THE LIE–ISOTOPIC LIFTING OF CONVENTIONAL GAUGE THEORY.

M. Gasperini [7] recently tried to formulate the generalized theory, following the Lie–isotopic lifting of the Lie theory investigated in great detail by Santilli [21], especially in the work of Myung and Santilli [8]. Following M. Gasperini, we shall summarize some results of the Lie–isotopic lifting of conventional gauge theory for later use. Let us give an invertible and hermitian operator T. The enveloping Lie algebra of a theory with associative product AB and unit I is generalized by introducing the isotopic (associative) product $A*B = ATB$ and a new unity $I = T^{-1}$, such that $A*I = I*A = A$. We define the isotopic generalization of hermitian conjugate, $A^{\dagger}$, and inverse, A^{-1}, of an operator A

$$\overset{\circ}{A^{\dagger}} = T^{\dagger} A^{\dagger} \overset{\circ}{I} , \tag{2.1}$$

$$\overset{\circ}{A^{-1}} = \overset{\circ}{I} A^{-1} \overset{\circ}{I} , \tag{2.2}$$

respectively.
The Lie–isotopic lifting G of the compact group G is represented by the following transformation:

$$\Psi' = \overset{\circ}{U} * \Psi , \tag{2.3}$$

where

$$\overset{\circ}{U} = \overset{\circ}{I} \exp(-i\theta^{k} * X_{k}) = \exp(-i X_{k} * \theta^{k}) \overset{\circ}{I} , \tag{2.4}$$

θ^{k} is a function of x, X_{k} is a matrix representation of the generators of group G satisfying

$$[X_{i} , X_{j}] = i c_{ij}{}^{k} X_{k} , \tag{2.5}$$

$c_{ij}{}^{k}$ are the structure constants of the Lie algebra of G and the notation $[X_{i}, X_{j}]$ shall denote hereon the **conventional** Lie product. It is shown [7] that U is a T–unitary operator, that is

$$\overset{\circ}{U}{}^{\overset{\circ}{\dagger}} * \overset{\circ}{U} = \overset{\circ}{I} . \tag{2.6}$$

Then we obtain from (2.3) an invariant relation

$$\Psi^{\dagger} * \Psi = \Psi'^{\dagger} * \Psi' . \tag{2.7}$$

In analogy with ordinary gauge theory, we introduce the isotopic covariant derivative D_{μ} by imposing the following transformation rules

$$\overset{\circ}{D}{}'_{\mu} * \overset{\circ}{U} * \Psi = \overset{\circ}{U} * \overset{\circ}{D}_{\mu} * \Psi , \tag{2.8}$$

where D_μ are given by

(2.9) $$\overset{\circ}{D}_\mu = (\partial_\mu - igA^k_\mu * X_k)\overset{\circ}{I} ,$$

A^k_μ are gauge potentials.
From (2.8) we obtain the transformation rules for A^k_μ

(2.10) $$A'^i_\mu * X_i = \overset{*}{U}A^i_\mu * X_i \overset{*}{U}^{-1} - \frac{1}{g}(\partial_\mu \overset{*}{U})\overset{*}{U}^{-1} ,$$

where $U = UI$, g is the coupling constant of the gauge field. We define the isotopic gauge field strengths $F_{\mu\nu}$ for the gauge potential in what follows:

(2.11) $$\overset{\circ}{F}_{\mu\nu} * \Psi = -\frac{1}{ig}(\overset{\circ}{D}_\mu * \overset{\circ}{D}_\nu - \overset{\circ}{D}_\nu * \overset{\circ}{D}_\mu) * \Psi .$$

From (2.8) and (2.11) we have the transformation rule for $F_{\mu\nu}$

(2.12) $$\overset{\circ}{F}'_{\mu\nu} = \overset{\circ}{U} * \overset{\circ}{F}_{\mu\nu} * \overset{\circ}{U}^{-1} .$$

Let us take notice of the minimal coupling term of (2.9)

(2.13) $$A^k_\mu * X_k = A^k_\mu T X_k .$$

We assume that T has a following form

(2.14) $$T = f(x)S ,$$

where $f(x)$ is a function of x, S is an operator independent of x.

Then

(2.15) $$A^k_\mu * X_k = f(x)A^k_\mu(SX_k) .$$

We shall study two cases for SX_k

(a) S is in the center of the algebra of the original Lie group G

(2.16) $$[X_i , S] = 0.$$

The operator S is assumed to be a transformation matrix of the vector space with a set of basis vectors (X_i), then a new set of basis Y_i is related to X_i by

(2.17) $$[Y_i , Y_j] = is_{ij}{}^k(\varepsilon)Y_k ,$$

in terms of S which is considered to have a parameter ϵ, and (X_i), (2.17) can be written

$$[S(\varepsilon)_i^r X_r \,,\; S(\varepsilon)_j^s X_s] = i s_{ij}{}^k(\varepsilon) S(\varepsilon)_k^t X_t \,, \tag{2.18}$$

$$S_i^r S_j^s c_{rs}{}^t S^{-1}{}_t^k = s_{ij}{}^k \,, \tag{2.19}$$

where

$$S(\varepsilon=1)_j^i = \delta_j^i \,, \tag{2.20}$$

and

$$\det(S(\varepsilon=0)) = 0 \,, \tag{2.21}$$

that is, S (0) is singular.

When $\epsilon \to 0$, the limit $s_{ij}{}^k$ may or may not exist. When the limit

$$\lim_{\varepsilon\to 0} s_{ij}{}^k(\varepsilon) = s_{ij}{}^k(0) \,, \tag{2.22}$$

exists and is well defined, the new structure constants $s_{ij}{}^k(0)$ characterize a Lie algebra that may or may not be isomorphic with the original algebra [22], that is Inonu–Wigner contraction. As is seen above, by the introduction of T(or S), variations including contraction may appear spontaneously in the Lie–algebra (X_i) in which the gauge potentials have the values. In

this case, we have the Lie–algebra (Y_i) which may or may not be isomorphic with the original algebra (X_i). This is a remarkable fact which should be noticed in the Lie–isotopic lifting of conventional gauge theory. The isotopic covariant derivative D_μ becomes

$$\mathring{D}_\mu = (\partial_\mu - i g f(x) A_\mu^k Y_k)\mathring{I} \,, \tag{2.23}$$

where (Y_k) may not be isomorphic with the original algebra (X_k).

(b) S is not in the center of the original Lie–algebra

$$[X_i \,,\; S] = R_i \neq 0. \tag{2.24}$$

We write down the commutator $[SX_i, SX_j]$ symbolically

$$[SX_i \,,\; SX_j] = S^2[X_i \,,\; X_j] - S(R_iX_j - R_jX_i) \,. \tag{2.25}$$

(i) $R_i = X_i$, then the commutator (2.25) becomes

$$[SX_i \,,\; SX_j] = iS(S-1)c_{ij}{}^k S^{-1}(SX_k) \,. \tag{2.26}$$

(ii) R_i = number, then

$$[SX_i \,,\; SX_j] = iS^2 c_{ij}{}^k S^{-1}(SX_k) - R_i(SX_j) + R_j(SX_i) \,, \tag{2.27}$$

in this part, (R_i) must be pure imaginary.

Judging from the commutators (2.26), (2.27), we obtain the new structure constants which seem to be different from the originals. So the new algebra (Y_i) may not be isomorphic with the original algebra.

We have shown in these cases (a), (b) that the new Lie–algebra may appear in the generalized theory introduced by the Lie–isotopic lifting of conventional gauge theory. The possibility of appearance of new algebra in the Lie–isotopic lifting of gauge theory was not noticed in the work of M. Gasperini.

By making use of (2.9) and (2.11), we obtain in the field Strengths $F_{\mu\nu}$ which are defined by

$$\mathring{F}_{\mu\nu} = \overset{*}{F}_{\mu\nu}\mathring{I} , \tag{2.28}$$

$$\overset{*}{F}_{\mu\nu} = \overset{*}{F}{}^{i}_{\mu\nu} * X_i , \tag{2.29}$$

$$\begin{aligned} \overset{*}{F}{}^{i}_{\mu\nu} * X_i = {} & (\partial_\mu A^i_\nu - \partial_\nu A^i_\mu) * X_i + A^i_\alpha(\delta^\alpha_\nu \partial_\mu T - \delta^\alpha_\mu \partial_\nu T) X_i \\ & - i g A^j_\mu A^k_\nu T (X_j T X_k - X_k T X_j) . \end{aligned} \tag{2.30}$$

This is given by M. Gasperini [7]. We shall rewrite (2.30) by making use of our formalism including (2.14), (2.17)

$$\overset{*}{F}{}^{i}_{\mu\nu} = \nabla_\mu A^i_\nu - \nabla_\nu A^i_\mu + g f(x) s_{jk}{}^{i} A^j_\mu A^k_\nu , \tag{2.31}$$

where

$$\nabla_\mu A^i_\nu = \partial_\mu A^i_\nu - A^i_\alpha \Gamma_{\mu\nu}{}^{\alpha} ,$$

$$\Gamma_{\mu\nu}{}^{\alpha} = \frac{1}{2}(\delta^\alpha_\mu \partial_\nu f - \delta^\alpha_\nu \partial_\mu f) f^{-1} .$$

The field strengths in our formalism have the characteristic that they contain the new structure constants $s_{ij}{}^{k}$ which are induced spontaneously by the introduction of the Lie–isotopic lifting and may be different from the original ones $c_{ij}{}^{k}$. The difference between Gasperini's work, other's work [23] and ours lies in this characteristic. The formula (2.31) may be interpreted as the field strengths with a variable effective coupling constant $gf(x)$ and a gauge potential minimally coupled to the geometry of a Riemann—Cartan space with an antisymmetric connection $\Gamma_{\mu\nu}{}^{\alpha}$. However, we can express the formula (2.31) in another form as follows:

$$\overset{*}{F}{}^{i}_{\mu\nu} = f(x)^{-1} \overset{*}{H}{}^{i}_{\mu\nu} , \tag{2.32}$$

where $H^i_{\mu\nu}$ are given by

$$\overset{*}{H}{}^{i}_{\mu\nu} = \partial_\mu B^i_\nu - \partial_\nu B^i_\mu + g s_{jk}{}^{i} B^j_\mu B^k_\nu , \tag{2.33}$$

and

$$B^i_\mu = f(x) A^i_\mu . \tag{2.34}$$

Hereafter it is assumed that we are in a curved space–time with the metric tensor $g_{\mu\nu}$ which is defined as

$$g_{\mu\nu} = f(x)^{-1}\eta_{\mu\nu} \quad , \tag{2.35}$$

where $\eta_{\mu\nu}$ is the Minkowski metric.

Then

$$\overset{*}{F}{}^{i\mu\nu} = g^{\mu\rho}g^{\nu\sigma}\overset{*}{F}{}^{i}_{\rho\sigma} = f\eta^{\mu\rho}\eta^{\nu\sigma}\overset{*}{H}{}^{i}_{\rho\sigma} = f\overset{*}{H}{}^{i\mu\nu} \quad . \tag{2.36}$$

The Lagrangian density for the gauge field will be given by

$$L_{GF} = -\frac{1}{4}\sum_i \overset{*}{F}{}^{i\mu\nu}\overset{*}{F}{}^{i}_{\mu\nu} = -\frac{1}{4}\sum_i \overset{*}{H}{}^{i\mu\nu}\overset{*}{H}{}^{i}_{\mu\nu} \quad . \tag{2.37}$$

The quark field carries colour and flavor indices as well as a Dirac index, and these indices are summed over in Ψ. We shall propose that the Lie–isotopic lifting of the Lagrangian density for quark field interacting with the gauge field is given by

$$L_Q = i\bar{\Psi}*\Gamma^{\mu}*\mathring{D}_{\mu}*\Psi - m_q\bar{\Psi}*\Psi \quad , \tag{2.38}$$

where m_q is a quark mass, and Γ_μ are the Lie–isotopic lifting of Dirac matrices being assumed to obey the following rules:

$$\Gamma_\mu = \gamma_\mu\mathring{I} \ , \quad \bar{\Psi} = \Psi^{\dagger}\Gamma_0 \quad , \tag{2.39}$$

$$\Gamma_\mu*\Gamma_\nu + \Gamma_\nu*\Gamma_\mu = 2g_{\mu\nu}\mathring{I} \ , \tag{2.40}$$

$$\Gamma^{\mu} = g^{\mu\nu}\Gamma_\nu \quad . \tag{2.41}$$

In terms of Dirac matrices γ_μ, L_Q can be expressed as

$$L_Q = i\bar{\bar{\Psi}}\gamma^{\mu}\overset{*}{D}_{\mu}\Psi - m_q\bar{\bar{\Psi}}\Psi \quad , \tag{2.42}$$

where

$$\bar{\bar{\Psi}} = \Psi^{\dagger}\gamma_0 \quad , \tag{2.43}$$

and D_μ is given by

$$\mathring{D}_\mu = \overset{*}{D}_\mu\mathring{I} \quad .$$

Dirac matrices γ_μ obey the following rules:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} \; . \tag{2.44}$$

Then the total Lagrangian density $L = L_{GF} + L_Q$ is

$$L = i\bar{\bar{\Psi}}\gamma^\mu(\partial_\mu - igB^i_\mu Y_i)\Psi - m_q \bar{\bar{\Psi}}\Psi - \frac{1}{4}\sum_i \overset{*}{H}{}^{i\mu\nu}\overset{*}{H}{}^i_{\mu\nu} \tag{2.45}$$

If we consider the action, we must take $\sqrt{-g}$ into consideration. An example of our space–time will be a space–time with constant curvature. In this case, $f(x)$ can be determined. Our Lagrangian density differs from the original one in the fact that we have B^i_μ and Y_i (or $s_{jk}{}^i$), respectively. These discussions are mainly based on [14.15].

(b) A FORMULATION OF GAUGE FIELDS VIA THE ISOTOPIC LIFTING OF THE HILBERT SPACE.

Let T be an operator that preserves appropriate properties and is nonsingular, Hermitian. Following [8], we shall introduce the isotopic lifting $\mathring{H}$ of the Hilbert space H of quantum mechanics. Let vectors be $\mathring{\phi}, \mathring{\psi}, \ldots$, the inner product will be defined as

$$(\mathring{\phi} \mathring{|} \mathring{\psi}) = (\mathring{\phi}|T|\mathring{\psi}) = (\mathring{\phi}|T\mathring{\psi}) = (T\mathring{\phi}|\mathring{\psi}) \in C \; , \tag{2.46}$$

and normalization

$$(\mathring{\phi} \mathring{|} \mathring{\phi}) = 1 , \tag{2.47}$$

where all symbols without upper small circle denote the corresponding quantity in H.

Following [24, 8], we define the associative–isotopic lifting of the enveloping associative algebra E of Hermitian operator A, B on C whose composition is given by the simple associative product AB. Following [24], we subject E to an isotpic lifting $\mathring{E}$ characterized by the product $A{*}B = ATB$ and the new unity $\mathring{I} = T^{-1}$, that is to say, $\mathring{I}{*}A = A{*}\mathring{I} = A$. Following [24], we shall define the action of the algebra E on the space $\mathring{H}$, which is characterized by the modular isotopic form $A{*}\psi = AT\psi$. We shall also define [24] the operation of *isotopic, linear, Hermitian, adjoint* in what follows:

$$(A{*}\mathring{\psi} \mathring{|} \mathring{\phi}) = (\mathring{\psi} \mathring{|} A^{\mathring{\dagger}}{*}\mathring{\phi}) \; . \tag{2.48}$$

From (3) we have

$$A^{\mathring{\dagger}} = A^{\dagger} \; . \tag{2.49}$$

Next, by making use of [24.8], we shall define *isotopic, linear, unitary* operator in what follows:

$$(\mathring{U}{*}\mathring{\psi} \mathring{|} \mathring{U}{*}\mathring{\phi}) = (\mathring{\psi}|\mathring{\phi}) \; . \tag{2.50}$$

From (2.48, 49, 50), we obtain

(2.51) $$\mathring{U}^{\dagger} * \mathring{U} = \mathring{U} * \mathring{U}^{\dagger} = \mathring{I} .$$

The Lie–isotopic lifting G of the compact group G is represented by the following transformation:

(2.52) $$\mathring{\psi}' = \mathring{U} * \mathring{\psi} ,$$

where U is *isotopic, linear, unitary* operator given by

(2.53) $$\mathring{U} = \mathring{I}\exp(-i\theta^{k} * X_{k}) = \exp(-i\theta^{k} * X_{k})\mathring{I} ,$$

θ^k is a function of x, X_k is a matrix representation of the generators of group G satsifying

(2.54) $$[X_{i} , X_{j}] = ic_{ij}{}^{k} X_{k} ,$$

$c_{ij}{}^{k}$ are the structure constants of the Lie algebra of G.
If we set

(2.55) $$\rho = \mathring{\psi}^{\dagger} * \mathring{\psi} ,$$

From (2.51, 52, 53), we obtain

(2.56) $$\rho' = \rho ,$$

that is to say

(2.57) $$\mathring{\psi}'^{\dagger} * \mathring{\psi}' = \mathring{\psi}^{\dagger} * \mathring{\psi} .$$

Next, we introduce a Lie–isotopic lifting of the exterior derivative in what follows:

(2.58) $$\mathring{d} = d\,\mathring{I} ,$$

where d is the ordinary exterior derivative.
The operation of $\mathring{d}$ on $\mathring{\psi}$ is assumed as follows:

(2.59) $$\mathring{d} * \mathring{\psi} = d\mathring{\psi} .$$

The operation of $\mathring{d}$ on ρ by making use of (2.59)

(2.60) $$\mathring{d} * \rho = (\mathring{d} * \mathring{\psi}^{\dagger}) * \mathring{\psi} + \mathring{\psi}^{\dagger}(\mathring{d} * T)\mathring{\psi} + \mathring{\psi} * (\mathring{d} * \mathring{\psi}) .$$

If we assume

(2.61) $$\mathring{d} * T = V^{\dagger} * T + T * V ,$$

where V is given by

(2.62) $$V = F\mathring{I} ,$$

where F is a I—form,
then we substitute (2.61) into (2.60), we obtain

(2.63) $$\mathring{d}*\rho = (\mathring{D}*\mathring{\psi})^{\dagger}*\mathring{\psi} + \mathring{\psi}^{\dagger}*(\mathring{D}*\mathring{\psi}) ,$$

where $\mathring{D}$ is given by

(2.64) $$\mathring{D}*\mathring{\psi} = \mathring{d}*\mathring{\psi} + V*\mathring{\psi} .$$

We set the postulate that under the transformation (2.52), the formula (2.63) should be invariant. Then

(2.65) $$(\mathring{d}*\rho)' = \mathring{d}*\rho .$$

From (2.65) we have the transformation law for $\mathring{D}*\mathring{\psi}$ in what follows:

(2.66) $$\mathring{D}'*\mathring{U}*\mathring{\psi} = \mathring{U}*\mathring{D}*\mathring{\psi} .$$

From (2.66) $\mathring{D}$ is given by

(2.67) $$\mathring{D} = \mathring{d} - iA^{k}*X_{k}\mathring{I} ,$$

where in connection with (2.62) F is given by

(2.68) $$F = -iA^{k}*X_{k} ,$$

and A^k is I—form.

From (2.66) we also obtain the transformation rules for A^k_μ which is defined as $A^k = A^k dx^\mu$

(2.69) $$A'^{i}_{\mu}*X_{i} = UA^{i}_{\mu}*X_{i}U^{-1} - (\partial_{\mu}U)U^{-1} ,$$

where $\mathring{U} = U\mathring{I}$.

A^k_μ can be identified as gauge potentials, we define the isotopic gauge field strengths $F_{\mu\nu}$ for the gauge potentials in what follows:

(2.70) $$\mathring{F}_{\mu\nu}*\mathring{\psi} = i(\mathring{D}_{\mu}*\mathring{D}_{\nu} - \mathring{D}_{\nu}*\mathring{D}_{\mu})*\mathring{\psi} ,$$

from (2.66) and (2.70) we have the transformation rules for $F_{\mu\nu}$

(2.71) $$\mathring{F}'_{\mu\nu} = \mathring{U}*\mathring{F}_{\mu\nu}*\mathring{U}^{-1} ,$$

where $\overset{\circ}{D}_\mu$ are defined as $\overset{\circ}{D} = \overset{\circ}{D}_\mu dx^\mu$.

We usually accept the simplifying hypothesis that T is in the center of the algebra of the original Lie group G,

$$(2.72) \qquad [T\,,\,X_i] = 0.$$

In our case the commutator (2.72) does not necessarily hold.

3. EXTENSION OF THE DIRAC–MYUNG–SANTILLI DELTA FUNCTION TO FIELD THEORY.

We consider the hadronic–isotopic generalization of the Dirac delta function proposed by Myung and Santilli, and we introduce its extension to field theory. Some of the first implications are identified, such as the capability of removing the singularities of the propagation function on the light cone. As is well known, the Dirac delta function is at the ultimate foundations of quantum mechanics, inasmuch as it is the central representative of the point–like description of particles which is inherent in the mechanics itself.

In a recent paper [8], Myung and Santilli proposed the following hadronic–isotopic generalization of the delta function

$$(3.1) \qquad \delta^*(x) = \frac{1}{2\pi}\int \exp(iz*x)\,dz = \frac{1}{2\pi}\int \exp(izTx)\,dz\ ,$$

where T is an operator generally restricted to be nonsingular and Hermitian. Generalization (3.1), hereon referred to as the

Dirac–Myung–Santilli delta function, was proposed to achieve a representation of hadrons as extended–deformable charge distributions.

We shall initiate the studies on the extension of the Dirac–Myung–Santilli delta function to field theory. For this purpose, let us recall the definition of the Dirac delta function in a Minkowski space–time with local coordinates x^μ, $\mu = 0, 1, 2, 3$

$$(3.2) \qquad \delta(x) = \frac{1}{(2\pi)^4}\int \exp(ik_\mu x^\mu)\,dk\ ,$$

where $dk = dk_0 dk_1 dk_2 dk_3$ and $k_\mu x^\mu$ is Lorentz–invariant.

Following [8], we shall therefore introduce the following hadronic–isotopic lifting of structure (3.2)

$$(3.3) \qquad \delta^*(x) = \frac{1}{(2\pi)^4}\int \exp(ik_\mu * x^\mu)\,dk = \frac{1}{(2\pi)^4}\int \exp(ik_\mu T x^\mu)\,dk\ ,$$

where T is assumed to be, for simplicity, a (scalar) function of Lorentz–invariant quantities.

We shall show that generalization (3.3) offers a number of intriguing possibilities in field theory, such as that of removing light–cone singularities of propagators, or at least rendering them more manageable.

Consider the invariant distance with the Minkowski metric

$$(3.4) \qquad \tau^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2\ .$$

T is assumed to be a function of τ, that is to say,

$$(3.5) \qquad T = f(\tau)\ .$$

If we set

$$x'^{\mu} = Tx^{\mu} = f(\tau)x^{\mu} , \tag{3.6}$$

we have for the invariant square

$$x'_{\mu}\, x'^{\mu} = f^2(\tau)\, x_{\mu}x^{\mu} = f^2(\tau)\,\tau^2 . \tag{3.7}$$

Then it follows from (3.7) that the space–time of x'^{μ} differs from the Minkowski space–time.

If we consider the D'Alembert equation of scalar fields $\psi(x'^{\mu})$ in x'^{μ} space–time

$$\Box\psi(x') = 0, \tag{3.8}$$

the plane wave solution is of course given by

$$\psi = \exp(ik_{\mu}x'^{\mu}) . \tag{3.9}$$

Eq. (3.8) written in terms of x^{μ} will be slightly complicated, but the plane wave solution will be

$$\psi = \exp(ik_{\mu}Tx^{\mu}) = \exp(ik_{\mu}fx^{\mu}) , \tag{3.10}$$

where we have used (3.6)

The Dirac delta function with argument $x_{\mu}x^{\mu}$ is often used in field theory [25], so we consider it with argument $x'_{\mu}x'^{\mu}$

$$\delta(x'_{\mu}x'^{\mu}) = 4\pi D(x') , \tag{3.11}$$

where $D(x')$ is a propagation function in field theory. From (3.7) and (3.11), $D(x')$ is expressed by

$$D(x') = \frac{1}{4\pi}\delta(f^2\tau^2) . \tag{3.12}$$

If we assume that $f(\tau)$ has the following form:

$$f^2(\tau) = \frac{h(\tau)}{\tau^2} , \tag{3.13}$$

and that $h(\tau)$ obeys the following conditions:
the equation $h(\tau) = 0$ has the roots s_i and $h'(s_i)$ don't vanish, where $h'(z) = \frac{dh}{dz}$,
then D of (3.12) is expressed as

$$D(x') = \frac{1}{4\pi}\sum_{i}\frac{1}{|h'(s_i)|}\,\delta(\tau - s_i) . \tag{3.14}$$

If s_i don't vanish, from (15) the singularities of the propagation function D don't lie on the light cone. So the field theory with (3.12) is not ordinary field theory. As seen above, $\delta^*(x)$ play an important role to modify the ordinary field theory, especially for escape out from the singularity on the light cone of the propagation function. This chapter is mainly based on [17].

4. RELATIONS BETWEEN SOME MODELS OF FIELD THEORY AND THE LIE–ISOTOPIC OR THE LIE–ADMISSIBLE APPROACH.

We shall define the Lie–isotopic lifting of the product $\phi \cdot \phi$

(4.1) $$\phi * \phi = \phi T \phi .$$

The Lie–isotopic lifting of the polynomials of ϕ will be defined as

(4.2) $$\phi * \phi * \cdots\cdots * \phi = \overset{\circ}{I} (T\phi)^n .$$

By making use of (4.2) we shall study the Lie–isotopic or Lie–admissible lifting of some simple Lagrangian densities.

We shall consider a free scalar field ϕ with mass m, of which Lagrangian density is given by

(4.3) $$L = -\frac{1}{2}(\partial^\mu \phi \partial_\mu \phi + m^2 \phi\phi) .$$

The Lie–isotopic lifting of (4.3) will be given by

(4.4) $$L = -\frac{1}{2}(\partial^\mu \phi * \partial_\mu \phi + m^2 \phi * \phi)$$

$$= -\frac{1}{2}(\partial^\mu \phi T \partial_\mu \phi + m^2 \phi T \phi) .$$

From (4.4) we obtain the equation for ϕ

(4.5) $$\partial^\mu (T \partial_\mu \phi) - m^2 T\phi = 0 .$$

If we assume the commutativity between T and ∂_μ

(4.6) $$[T , \partial_\mu] = 0 ,$$

we obtain the Klein–Gordon equation for ϕ and $T\phi$

(4.7) $$(\partial^\mu \partial_\mu - m^2)\phi = 0 ,$$

(4.8) $$(\partial^\mu \partial_\mu - m^2) T\phi = 0 .$$

The momentum operator for ϕ will be defined by (4.4) through ordinary procedure

$$\Pi = T\partial_0\phi \tag{4.9}$$

So the commutation relation for Π and ϕ at equal time is given by

$$[\phi(x) \quad , \quad \Pi(x')] = i\delta(\mathrm{x} - \mathrm{x}'), \text{ at } x_0 = x_0' , \tag{4.10}$$

where x, x′ are the space components of x, x', respectively. From (4.9) and (4.10) we have

$$[\phi(x) \quad , \quad T\pi(x')] = i\delta(\mathrm{x} - \mathrm{x}'), \text{ at } x_0 = x_0', \tag{4.11}$$

where π is the momentum operator derived from (4.3).

The relation (4.11) is the commutation relation often appeared in the Lie–isotopic lifting of quantum mechanics. From (4.11) we obtain

$$[\phi(x) , \pi(x')] = iT^{-1}\delta(\mathrm{x} - \mathrm{x}'), \text{ at } x_0 = x_0', \tag{4.12}$$

where T^{-1} operatoes on x' parts only.

Here we would like to mention that by choosing T appropriate we can have a possibility of avoiding divergences appearing in ordinary quantum field theory.

We shall consider a Lagrangian density of a scalar field with self–interaction, in ordinary case we have the Lagrangian density of the following form:

$$L = -\frac{1}{2}(\partial^\mu\phi\partial_\mu\phi \quad + \quad m^2\phi^2) \quad + \quad U(\phi), \tag{4.13}$$

or equivalently

$$L = \frac{1}{2}\phi(\partial^\mu\partial_\mu\phi \quad - \quad m^2)\phi \quad + \quad U(\phi). \tag{4.14}$$

As for $U(\phi)$, we often consider a polynomial, exponential or their combination. For example, the Lie–isotopic lifting of the exponential is given by [25]

$$\exp(\phi) \rightarrow T^{-1}\exp(T\phi) \quad . \tag{4.15}$$

Then the Lie–isotopic lifting of U will be

$$U(\phi) \rightarrow T^{-1}U(T\phi). \tag{4.16}$$

The Lie–isotopic lifting of the Lagrangian density (4.14) is given by making use of (4.16)

$$L = \frac{1}{2}\phi T(\partial^\mu\partial_\mu \quad - \quad m^2)\phi \quad + \quad T^{-1}U(T\phi). \tag{4.17}$$

Here we introduce another invertible, hermitian operator R in addition to T. Using R, we make the Lie–isotopic lifting of U

(4.18) $$U(\phi) \rightarrow R^{-1}U(R\phi).$$

We make the Lie–isotopic lifting of (4.14) by making use of T and R as follows:

(4.19) $$L = \frac{1}{2}\phi T(\partial^{\mu}\partial_{\mu} - m^{2})\phi + R^{-1}U(R\phi).$$

(4.20) $$T \text{ or } R = F(\partial^{\mu}\partial_{\mu}),$$

(4.21) $$T \text{ or } R = K(\partial^{k}\partial_{k}),$$

where indices k denote the space components.
Comparison (4.4) and (4.19) with some models in field theory will be seen in [18].

5. REMARKS ON THE NONCANONICAL COMMUTATION RELATIONS OF THE ENERGY–MOMENTUM OPERATORS.

Recently, A. Jannussis et al. [13] studied time noncanonical character of hadronic mechanics by making use of the Lie–admissible formation of the Schrödinger equation by Santilli and Myung [11] and by Mignani [12]. They began their study to adopt as an example the equivalence of the energy operator $E = i\hbar\partial_t$ with the Hamiltonian i.e.,

(5.1) $$i\hbar\partial_t \rightarrow H.$$

They are interested in the following general question: Is it possible that law (5.1) can be generalized in a functional way? e.g., of the form

(5.2) $$F(i\hbar\partial_t, t, \ldots) = f(H, \ldots.):$$

Once relation (5.2) is accepted, they are led to the simple result from the solution with respect to $i\hbar\partial_t$

(5.3) $$i\hbar\partial_t \rightarrow H'(H, \ldots.) = HT(H, t, \ldots.),$$

where the new operator H' is in general non–hermitian and non–convervative.

In the relation (5.2), they distinguish mainly two classes of problems: First, whether the commutator

(5.4) $$[F(i\hbar\partial_t, t, \ldots), t] = i\hbar \text{ or } \neq i\hbar,$$

presupposing that the right–hand side operator f in (5.2) is hermitian. In the case of the value $i\hbar$ they naturally say that the theory is canonical if between the operators q and p the canonical commutation relations also hold.

When the commutator (5.4) is a function of time, the theory by its nature will be noncanonical in time. When the operator in the right–hand side in (5.2) is non–hermitian, the theory is again noncanonical. The theory of the Lie–admissible Schrodinger representation belongs to this latter class. In this chapter, we shall treat at the same time the commutator like (5.4), not only of t but also of the space components $q_r (r = 1,2,3)$. Then we generalize the Lie–admissible formulation of the Schrödinger equation and by making use of the generalized equation, we shall obtain in the Lie–admissible commutator an interesting form of an admissible element T as appeared in (5.3).

Let's consider the four–vectors $p_\mu (\mu: 0,1,2,3)$ of the energy–momentum operators corresponding to the four–coordinates q_μ of the space–time. In the canonical formalism we define p_μ in what follows:

(5.5) $$p_\mu \rightarrow i\hbar\eta_{\mu\nu}\partial_\nu \quad ,$$

where p_0 corresponds to the Hamiltonian, p_r corresponds to the momentum operators, $\eta_{\mu\nu} = (1, -1, -1, -1)$ (the Minkowski metric). Then the canonical commutation relations are given by

(5.6) $$[p_\mu \ , \ q_\nu] = i\hbar\eta_{\mu\nu} \quad .$$

From (5.5) we obtain

(5.7) $$p_\mu\Psi = i\hbar\eta_{\mu\nu}\partial_\nu\Psi \quad ,$$

where the equation for $\mu = 0$ corresponds to the Schrödinger equation. The Lie–admissible formulation of the equations (5.7) are of the forms

(5.8) $$i\hbar\eta_{\mu\nu}\partial_\nu\Psi = p_\mu T\Psi \quad ,$$

(5.9) $$-\ i\hbar\eta_{\mu\nu}\bar{\Psi}\overleftarrow{\partial}_\nu = \bar{\Psi}R\overleftarrow{p}_\mu \quad ,$$

for forward and backward motion, respectively.
According to A. Jannussis et al., we also ask the following question: Is it possible that law (5.5) can be generalized in a functional way? e.g., of the forms

(5.10) $$F_\mu(i\hbar\eta_{\mu\nu}\partial_\nu \ , \ q_\sigma, \ldots) = f_\mu(p_\mu, \ldots) .$$

From (5.8), using the transformation

(5.11) $$T\Psi = \Phi \quad ,$$

we obtain the forms

(5.12) $$i\hbar\eta_{\mu\nu}\partial_\nu T^{-1}\Phi = p_\mu\Phi \quad ,$$

The equation (5.12), according to (5.10), corresponds to

When the commutator (5.4) is a function of time, the theory by its nature will be noncanonical in time. When the operator in the right–hand side in (5.2) is non–hermitian, the theory is again noncanonical. The theory of the Lie–admissible Schrodinger representation belongs to this latter class. In this chapter, we shall treat at the same time the commutator like (5.4), not only of t but also of the space components $q_r (r = 1,2,3)$. Then we generalize the Lie–admissible formulation of the Schrödinger equation and by making use of the generalized equation, we shall obtain in the Lie–admissible commutator an interesting form of an admissible element T as appeared in (5.3).

Let's consider the four–vectors $p_\mu (\mu: 0,1,2,3)$ of the energy–momentum operators corresponding to the four–coordinates q_μ of the space–time. In the canonical formalism we define p_μ in what follows:

$$p_\mu \rightarrow i\hbar\eta_{\mu\nu}\partial_\nu \quad , \tag{5.5}$$

where p_0 corresponds to the Hamiltonian, p_r corresponds to the momentum operators, $\eta_{\mu\nu} = (1, -1, -1, -1)$ (the Minkowski metric). Then the canonical commutation relations are given by

$$[p_\mu \quad , \quad q_\nu] = i\hbar\eta_{\mu\nu} \quad . \tag{5.6}$$

From (5.5) we obtain

$$p_\mu \Psi = i\hbar\eta_{\mu\nu}\partial_\nu \Psi \quad , \tag{5.7}$$

where the equation for $\mu = 0$ corresponds to the Schrödinger equation. The Lie–admissible formulation of the equations (5.7) are of the forms

$$i\hbar\eta_{\mu\nu}\partial_\nu \Psi = p_\mu \mathrm{T}\Psi \quad , \tag{5.8}$$

$$- i\hbar\eta_{\mu\nu}\bar{\Psi}\overleftarrow{\partial}_\nu = \bar{\Psi}\mathrm{R}\overleftarrow{p}_\mu \quad , \tag{5.9}$$

for forward and backward motion, respectively.
According to A. Jannussis et al., we also ask the following question: Is it possible that law (5.5) can be generalized in a functional way? e.g., of the forms

$$F_\mu (i\hbar\eta_{\mu\nu}\partial_\nu \; , \; q_\sigma , \ldots) = f_\mu (p_\mu , \ldots .) . \tag{5.10}$$

From (5.8), using the transformation

$$\mathrm{T}\Psi = \Phi \quad , \tag{5.11}$$

we obtain the forms

$$i\hbar\eta_{\mu\nu}\partial_\nu \mathrm{T}^{-1}\Phi = p_\mu \Phi \quad , \tag{5.12}$$

The equation (5.12), according to (5.10), corresponds to

(5.13) $$F_\mu(i\hbar\eta_{\mu\nu}\partial_\nu,\ q_\sigma,\dots) = i\hbar\eta_{\mu\nu}\partial_\nu T^{-1},$$

and the commutation relations like (5.4) take the forms

(5.14) $$[i\hbar\eta_{\mu\nu}\partial_\nu T^{-1},\ q_\rho] = i\hbar\eta_{\mu\rho}T^{-1}.$$

As the operator T in general is non—hermitian, we can say that the Lie—admissible formulation of the equations (5.8) are noncanonical theory with respect to q_μ.

We can obtain the corresponding rules for (5.9) by means of

(5.15) $$\bar{\Psi}R = \bar{\Phi}\ ,\quad \bar{\Psi} = \bar{\Phi}R^{-1},$$

then we have

(5.16) $$F_\mu(i\hbar\eta_{\mu\nu}\partial_\nu,\ q_\rho,\dots) = -i\hbar R^{-1}\eta_{\mu\nu}\partial_\nu,$$

and

(5.17) $$[F_\mu(i\hbar\eta_{\mu\nu}\partial_\nu,\ q_\rho,\dots),\ q_\rho] = -i\hbar\eta_{\mu\rho}R^{-1}.$$

Next we shall study the Lie—admissible commutators as follows:

(5.18) $$(i\hbar\eta_{\mu\nu}\partial_\nu,\ q_\rho) \equiv i\hbar\eta_{\mu\nu}(\partial_\nu T q_\rho - q_\rho R\partial_\nu)$$
$$= i\hbar\eta_{\mu\nu}[(\partial_\nu T)q_\rho + T\delta_{\nu\rho} + (Tq_\rho - q_\rho R)\partial_\nu].$$

If we assume that T and R don't contain ∂_μ (for all μ), and T=R, we have from (5.18)

(5.19) $$(i\hbar\eta_{\mu\nu}\partial_\nu,\ q_\rho) = i\hbar\eta_{\mu\nu}[(\partial_\nu T)q_\rho + T\delta_{\nu\rho}].$$

We consider the case for

(5.20) $$(\partial_\nu T)q_\rho + T\delta_{\nu\rho} = \delta_{\nu\rho},$$

this case may be interesting, because the Lie—admissible commutators (5.19) correspond exactly to the conventional commutators (5.6). The solution for (5.20) will be classified into two cases

(a) $\upsilon \neq \rho$, in this case Eqs. (5.20) become

(5.21) $$(\partial_\nu T)q_\rho = 0,$$

the solution for (5.21) is given by

(5.22) $$T = \text{const}.$$

(b) $\upsilon = \rho$, in this case Eqs. (5.20) are given by

(5.23) $$(\partial_\nu T)q_\nu + T = 1,$$

For every υ, the solutions are given by

$$T(q_\nu) = 1 + aq_\nu^{-1} , \qquad (5.24)$$

where a is constant.

For $\nu = 0$, the form of $T(q_\nu)$ was given by A. Jannussis et al., the solutions (5.24) are all asymmetrical i.e.

$$T(-q_\nu) \neq T(q_\nu) . \qquad (5.25)$$

For q_ν large enough, $T(q_\nu)$ approach to unity. Then in this case the Lie–admissible (or exactly Lie–isotopic) commutators (5.19) correspond to the ordinary (canonical) commutators. It follows from the standpoint of the space components of (5.24) that in the microscopic region like the interior or the neighborhood of hadron we may have to use the Lie–admissible formulation of the noncanonical commutation relations, though in the range of the atomic structure we can use the canonical commutation relations. This discussion is mainly based on [19].

6. A LIE–ADMISSIBLE REPRESENTATION OF UNSTABLE STATES

In this chapter, we shall show that the Lie–admissible lifting of the Schrodinger equation is suitable for the description of decay of the unstable state and the corresponding evolution operator becomes essentially non–unitary.

Santilli and Myung [11] derived the Lie–admissible Schrödinger equations. Mignani [12] independently found the same equations by studying the nonpotential scattering theory. The Lie–admissible Schrodinger equations are given by (5.8) and (5.9) with $\mu = 0$, the Hamiltonian H describes all forces derivable from a potential; the operators T and R represent the nonpotential forces, and their value $T \neq R$ represent nonconservation. We shall start by assuming that an unstable particle can be

described by vectors in a Hilbert space H_U which can be embedded in a larger Hilbert space H such that $H = H_U + H_D$ where H_D is the Hilbert space of the decay products. Let P denote the projection into the H_U and let $Q = 1 - P$ denote the projection onto H_D.

Recently, E. Recami et al. [26] studied unstable states and non–Hermitian Hamiltonian by means of potential scattering and proposed the Lie–admissible lifting of the Schrödinger equation. According to them, T and R in (5.8) and (5.9) with $\mu = 0$ are given by

$$T = 1 + iaH^{-1}P , \qquad (6.1)$$

$$R = 1 - iaPH^{-1} , \qquad (6.2)$$

respectively, where q is a quantity derived from the potential scattering [27].

In this chapter, we shall generalize the proposal of E. Recami et al., and study the following T:

$$T = 1 - i\lambda H^{-1}A , \qquad (6.3)$$

where λ is a parameter geing assumed to be positive, A represents not only P but also Q. From (5.8) and (6.3), we obtain the evolution operator U_t as follows:

$$U_t\Psi_0 = \Psi , \qquad (6.4)$$

$$U_t = \exp(-iHt - \lambda At) , \qquad (6.5)$$

where Ψ_0 is the initial state, we used the unit $\hbar = 1$.

The variation of the norm squared of Ψ with respect to time is given by

$$\frac{d}{dt}||\Psi||^2 = -2\lambda\langle\Psi|A|\Psi\rangle \quad , \tag{6.6}$$

where we refer to [28].
We can represent A in terms of unstable state or decayed state in what follows:

$$A = |\phi\rangle\langle\phi| \, , \text{ or } |\Phi\rangle\langle\Phi| \, , \tag{6.7}$$

where we used the Dirac notation, $|\phi\rangle$, $|\Phi\rangle$ denote unstable and decayed states, respectively.
Using (6.7), we have

$$\langle\Psi|A|\Psi\rangle = ||\langle\phi|\Psi\rangle||^2 \text{ or } ||\langle\Phi|\Psi\rangle||^2 \tag{6.8}$$

The r.h.s. of (6.8) represents the probability that at time t the unstable state remains undecayed or it is decayed, respectively. Then from (6.6) and (6.8), we obtain

$$\frac{d}{dt}||\Psi||^2 = -2\lambda||\langle\phi|\Psi\rangle||^2 < 0, \tag{6.9}$$

for the case where at time t, the unstable state remains undecayed.

Another formula is given by

$$\frac{d}{dt}||\Psi||^2 = -2\lambda||\langle\Phi|\Psi\rangle||^2 < 0, \tag{6.10}$$

for the case where at time t, the unstable state has already decayed. In other cases we have

$$\frac{d}{dt}||\Psi||^2 = 0. \tag{6.11}$$

For the case (6.9) and (6.10) hold, U_t is often called completely non–unitary.
Next, we shall consider the case where λ in (6.5) takes large value. From (6.4) with (6.5) we obtain

$$\Psi = \lim_{n\to\infty}\{\exp(-iHt/n)\exp(-\lambda At/n)\}^n\Psi_0 \, . \tag{6.12}$$

For (6.7), $\exp(-\lambda At/n) = 1 - \{1 - \exp(-\lambda t/n)\}|\phi\rangle\langle\phi|$,
or

$$\exp(-\lambda At/n) = 1 - \{1 - \exp(-\lambda t/n)\}|\Phi\rangle\langle\Phi| \, . \tag{6.13}$$

Then, if we set $\lambda \to \infty$,

$$\exp(-\lambda At/n) \to 1 - |\phi\rangle\langle\phi| \quad , \tag{6.14}$$

or

$$\exp(-\lambda At/n) \to 1 - |\Phi\rangle\langle\Phi| \, . \tag{6.15}$$

If we consider

$$\lim_{\lambda\to\infty}\Psi = \lim_{\lambda\to\infty}\lim_{n\to\infty}\{\exp(-iHt/n)\exp(-\lambda At/n)\}^n\Psi_0 \, , \tag{6.16}$$

under the assumption of the interchange of the order of the limits, we obtain for (6.14) or (6.15)

(6.17) $$\lim_{\lambda\to\infty} \Psi = \lim_{n\to\infty}\{\exp(-iHt/n)Q\}^n \Psi_0 \ ,$$

or

(6.18) $$\lim_{\lambda\to\infty} \Psi = \lim_{n\to\infty}\{\exp(-iHt/n)P\}^n \Psi_0 \ ,$$

respectively.
In deriving (6.17) or (6.18), we have used the following relations:

(6.19) $$P = |\phi\rangle\langle\phi| , \quad Q = |\psi\rangle\langle\psi| , \quad P + Q = 1.$$

There have been many investigations of a mathematically rigorous formulation of the problem of decay of an unstable particle in the usual quantum mechanical formalism [29, 30, 31, 32]. The investigations mentioned above start with the study of (6.17) or (6.18). So, as is seen above, the Lie–admissible lifting of the Schrödinger equation (or, of the evolution operator) seems to be useful in the study of the problem of unstable particle. This discussion is mainly based on [20].

7. YUKAWA'S BILOCAL THEORY FROM A VIEWPOINT OF THE LIE–ADMISSIBLE APPROACH.

Yukawa [33] as early as in 1950 proposed a nonlocal field theory. He began to study a simple bilocal field theory and showed the possibility of intrinsically extended elementary particle. In this section, we shall relate the Lie–admissible lifting of

the commutation relations with the equations of Yukawa's bilocal field theory. We shall define the Lie–admissible lifting of the commutation relation for A and B in what follows:

(7.1) $$[A \ , \ B]^* \equiv ATB - BRA \ ,$$

where we impose the following condition on T and R

(7.2) $$R^\dagger = -T \ , \quad T^\dagger = -R \ .$$

We shall set

(7.3) $$A = U(\text{field operator}),$$

(7.4) $$B = x^\mu(\text{the coordinates operators}),$$

and

(7.5) $$T = x^\mu - U^{-1}x^\mu U,$$

then

(7.6) $$R = -x^\mu + Ux^\mu U^{-1}.$$

From (7.1,3,4,5,6) we obtain

(7.7) $$[U \ , \ x^\mu]^* = [[U \ , \ x^\mu], \ x^\mu].$$

The Lie–admissible lifting of the commutation relation will be generally expressed as

$$[A\ ,\ B]^* = S, \tag{7.8}$$

where A, B and S are all hermitian.
Then from (7.7) and (7.8), we have

$$[U\ ,\ x^\mu]^* = S(U,x^\mu). \tag{7.9}$$

Next we consider the case for $A = U$, $B = p_\mu$ (the four—momentum operators). In this case, the coresponding Lie—admissible elements must be replaced by

$$T \to T^* = p_\mu - U^{-1}p_\mu U\ , \tag{7.10}$$
$$R \to R^* = -p_\mu + Up_\mu U^{-1}.$$

The Lie—admissible lifting of the commutation relation for U and p_μ be given by

$$[U\ ,\ p_\mu]^* \equiv UT^*p_\mu - p_\mu R^*U = S', \tag{7.11}$$

then from (7.10) and (7.11)

$$[U\ ,\ p_\mu]^* = [[U\ ,\ p_\mu],\ p_\mu] = S'(U,p_\mu), \tag{7.12}$$

where S' is hermitian.
We shall consider two possibilities of the Lie—admissible lifting of the commutation relations as follows:

$$[U\ ,\ x^\mu]^{**} \equiv UT^*x^\mu - x^\mu R^*U = [[U\ ,\ p_\mu],\ x^\mu], \tag{7.13}$$

$$[U\ ,\ p_\mu]^* \equiv UTp_\mu - p_\mu RU = [[U\ ,\ x^\mu],\ p_\mu]. \tag{7.14}$$

If we set

$$S = \lambda^2 U\ ,$$

$$S' = -\Lambda^2 U,$$

$$[U\ ,\ x^\mu]^{**} = 0,$$

we have Yukawa's bilocal theory. Details of our approach in connection with Yukawa's bilocal theory and other nonlocal theories including relativistic oscillator model will be seen in [34, 35]. This section is based on [34].

9. CONCLUDING REMARKS.

As seen above, we studied examples as realization of hadronic mechanics which were applications of the Lie—isotopic or the Lie—admissible approach. These trials are only a little part of the profound program which Santilli and his collaborators published mainly in the *Hadronic Journal*. It will be necessary and important for us to develop concrete realization of their program, so that we can build up the foundations of hadronic mechanics.

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REFERENCES

[1] R. Mignani, H. C. Myung and R. M. Santilli, Hadronic J. 6, 1873 (1983).

[2] R. M. Santilli, Hadronic J. 2, No. 6 (1979), Ibid., 3, No. 1 (1979), Ibid., 4, No. 2 (1980), Ibid., 4, No. 3 (1980).

[3] R. M. Santilli, Hadronic J. 1, 223 (1978).

[4] R. M. Santilli, *Lie–admissible approach to the hadronic structure*, Vol. 2, (Hadronic Press, Nonantum, MA., 1982).

[5] R. M. Santilli, Lett. Nuovo Cimento, 33, 145 (1982); 37, 545 (1983); 38, 509 (1983).

[6] R. M. Santilli, Ref. [4], chapter 1,3,5.

[7] M. Gasperini, Hadronic J. 6, 935 and 1462 (1983).

[8] H. C. Myung and R. M. Santilli, Hadronic J. 5, 1277 (1982).

[9] Private communication with the present author.

[10] R. M. Santilli, Hadronic J. 1, 574 (1978).

[11] H. Myung and R. M. Santilli, Hadronic J. 5, 1277 and 1367 (1982).

[12] R. Mignani, Hadronic J. 5, 1120 (1982).

[13] A. Jannussis, G. Brodimas, A. Pierris and E. Ioannidou, Hadronic J. 6, No. 6 (1983) (Proceedings of the First Workshop on Hadronic Mechanics).

[14] M. Nishioka, Nuovo Cimento A (submitted).

[15] M. Nishioka, Lett. Nuovo Cimento (submitted).

[16] M. Nishioka, Lett. Nuovo Cimento (in press) 1984.

[17] M. Nishioka, Lett. Nuovo Cimento (in press) 1984.

[18] M. Nishioka, Hadronic J. 7, No. 2 (1984).

[19] M. Nishioka, Lett. Nuovo Cimento (in press) 1984.

[20] M. Nishioka, Lett. Nuovo Cimento (in press) 1984.

[21] R. M. Santilli, *Foundations of Theoretical Mechanics II* (Springer–V., 1982).

[22] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*

[23] C. Mukku and W. A. Sayed, Phys. Lett., 82B, 382 (1979).

[24] H. C. Myung and R. M. Santilli, Hadronic J., 5, 1367 (1982).

[25] H. C. Myung: A generalization of hermitian and unitary operators in a Hilbert space, preprint. 1983.

[26] E. Recami, W. A. Rodriguez, Jr. and P. Smrz, Hadronic J., 6, 1773 (1983).

[27] A. Agodi, M. Baldo and E. Recami, Ann. Physics, 77, 157 (1973).

[28] E. B. Davies, *One–Parameter Semigroups*, (Academic Pr., London) 1976.

[29] L. P. Horwitz, J. P. Marchand, The Rockey Mountain J., 1, 225 (1971).

[30] K. Sinha, Helv. Phys. Acta, 45, 619 (9172).

[31] B. Misra, K. Sinha, ibid., 50, 99 (1977).

[32] E. B. Davies, *Quantum Theory of Open Systems*, (Academic Pr., London) 1976.

[33] H. Yukawa, Phys. Rev., 77, 219 (1950), ibid., 80, 1047 (1950).

[34] M. Nishioka, Proceedings of the Second Workshop on Hadronic Mechanics held at Villa Olmo, Como, Italy, August 1–3, 1984.

[35] M. Nishioka, Nuovo Cimento A (submitted).

[36] M. Nishioka and R. M. Santilli, IBR–P–1–84.

[37] R. M. Santilli, Lett. Nuovo Cimento, 38, 509, (1983).

[38] C. Itzykson and J–B Zuber, *Quantum Field Theory* (McGraw–Hill, N.Y.) 1980.

[39] K. Fujimura, T. Kobayashi and M. Namiki, Prog. Theor. Phys., 43, 73 (1970).

[40] R. Hofstadter, *Electron Scattering and Nuclear and Nucleon Structure*, (North–Holland Pub., N.Y.) 1970.