HADRONIC JOURNAL 5, 1277–1366 (1982)

FOUNDATIONS OF THE HADRONIC GENERALIZATION OF THE ATOMIC MECHANICS, II: MODULAR–ISOTOPIC HILBERT SPACE FORMULATION OF THE EXTERIOR STRONG PROBLEM

Hyo Chul Myung* and **Ruggero Maria Santilli****
The Institute for Basic Research
Harvard Grounds, 96 Prescott Street
Cambridge, Massachusetts 02138, USA

Received December 3, 1981

Abstract

This paper is devoted to a first formulation of the axiomatic Hilbert space foundations of the branch of the Hadronic Mechanics dealing with the exterior treatment of strong non–Hamiltonian systems, and which admits a Lie–isotopic algebraic character. In particular, we are interested in generalizing the conventional eigenvalue equations of the Atomic Mechanics into the broadest possible equations which are permitted by an associative algebra of operators on a one–sided, modular, Hilbert space. The objective is made possible by the isotopic generalization of the associative algebra $\mathcal{A}$ of operators A, B, ... on a Hilbert space, such as the isotope $\mathcal{A}^{(T)}$ characterized by the product $A*B = ATB$, where T is a fixed, bounded, and nonsingular operator of $\mathcal{A}$. By using the T–isotopic product, we first introduce a generalization of the Hermitean conjugate, transpose, Hermitean, skew–Hermitean, unitary, skew–unitary, projection, and exponential operators. We then pass to the isotopic generalization of the notion of determinant, trace, and eigenvalue of a linear operator. Some essential properties of this modular–isotopic formulation of the Hilbert space theory are identified. The results are then applied to Hadronic Mechanics. In particular, we present the hadronic generalization of a number of postulates of Atomic Mechanics, ranging from observables, states, and their time evolution, to total probability, its conservation in time, and the expectation values. We also prove that the isotopic generalization of Schrödinger's equations presented in Paper I are equivalent to the isotopic generalization of Heisenberg's equations. We then pass to a central aspect of the exterior hadronic problem, the achievement of total conservation laws under non–Hamiltonian internal forces. For this purpose, we review the classical Birkhoffian solution of the problem, and point out its possible hadronic counterpart. As an application, the structure model of the π^0 particle proposed by one of us in 1978 is re–examined and shown to possess a modular–isotopic structure. The existence of rather intriguing and virtually endless possibilities of further developments of mathematical, theoretical, and experimental character, are self–evident.

* *Permanent address: Department of Mathematics, University of Northern Iowa, Cedar Falls, Iowa 50613*

** *Supported by the U. S. Department of Energy under contract number DE–AC02–80ER10651. A001*

II. ISOTOPES OF LINEAR OPERATIONS ON A HILBERT SPACE

2.1. Conventional Definitions. (See, for instance, Ref.s 7). Throughout this section, by a linear space V we mean a vector space over the complex number field C with addition $x + y$ and scalar multiplication λx, $x, y \in V$ and $\lambda \in C$. For a complex number λ, $\bar{\lambda}$ denotes the conjugate of λ . In the essence, we shall consider only finite-dimensional spaces.

Important linear spaces are the inner product spaces. Recall that if V is a linear space then a mapping $(,) : V \times V \to C$ is called an *inner (scalar) product* on V if it satisfies the axioms

$$(x,x) > 0 \quad \text{for all} \quad x \neq 0 \text{ in } V , \tag{2.1}$$

$$(x,y) = \overline{(y,x)} , \tag{2.2}$$

$$(x,y+z) = (x,y) + (x,z) , \tag{2.3}$$

$$(x,\lambda y) = \lambda(x,y), \ \lambda \in C \tag{2.4}$$

for all $x, y, z \in V$. Note that the inner product (x,y) is also denoted by the Dirac's notation $\langle x|y\rangle$ in other literature. A linear space V equipped with inner product is called an *inner product (Euclidean) space*.

In an inner product space V , we define the *norm* $||x||$ of each vector $x \in V$ by

$$||x|| = \sqrt{(x,x)} . \tag{2.5}$$

Thus every inner product space V turns out to be a metric space under the metric $d(x,y) = ||x - y||$. A complete inner product space in this metric is called a (complex) *Hilbert space*. An inner product space and so Hilbert spaces can be defined for the real number field. Since complex Hilbert spaces are more exclusively used in quantum physics, in this section we only treat complex Hilbert spaces.

Due to the technical problem of developing the concept of isotope, we will essentially restrict ourself to finite-dimensional Hilbert spaces. If H is a finite-dimensional Hilbert space, there exists a special type of basis $e_1, \cdots, e_n$ of H such that

$$(e_i,e_j) = \delta_{ij}, \ i, j = 1, 2, \cdots, n . \tag{2.6}$$

Such basis $e_1, \cdots, e_n$ is called an *orthonormal basis* of H . Henceforth, we fix an orthonormal basis $e_1, \cdots, e_n$ of H . Thus every vector $x \in H$ is uniquely expressed as

$$x = \alpha_1 e_1 + \cdots + \alpha_n e_n, \ \alpha_i \in C , \tag{2.7}$$

where $\alpha_i = (e_i, x)$. Given $y = \beta_1 e_1 + \cdots + \beta_n e_n$, we have

$$(x,y) = \sum_{i=1}^{n} \bar{\alpha}_i \beta_i . \tag{2.8}$$

Definition 2.1. Let x be given by (2.7). Then the *conjugate vector* $\bar{x}$ of x is defined by $\bar{x} = \bar{\alpha}_1 e_1 + \cdots + \bar{\alpha}_n e_n$. #

Definition 2.2. Let V be a linear space. A *linear operator* A on V is a mapping $A : x \to Ax$ of V into itself satisfying:

$$A(\lambda x + \mu y) = \lambda(Ax) + \mu(Ay)$$

for all $x, y \in V$ and $\lambda, \mu \in C$. If $A : V \to V$ satisfies the relation

$$A(\lambda x + \mu y) = \bar{\lambda}(Ax) + \bar{\mu}(Ay)$$

for all $x, y \in V$ and $\lambda, \mu \in C$ then A is called an *antilinear operator* on V. #

In view of Definition 2.2., *all linear operators on a Hilbert space H are assumed in this section to be defined on the entire set H*, whereas a more general definition requires only that a linear operator in H is defined on a subspace of H. Antilinear operators in a Hilbert space play a very minor role in quantum mechanics by comparison with linear operators. Thus, for most of the cases, we focus on linear operators. For the sake of generality, we introduce the following.

Definition 2.3. A *complex (associative) algebra* $\mathcal{A}$ is a linear space with addition $A+B$ and scalar multiplication λA such that there is defined an operation : $(A,B) \to AB$ of $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$ satisfying

(i) $(AB)C = A(BC)$, associativity,

(ii) $(A+B)C = AC + BC$, $A(B+C) = AB + AC$, distributivity,

(iii) $\lambda(AB) = (\lambda A)B = A(\lambda B)$, assoc. and commut. of scalar multiplic.

for all $A, B, C \in \mathcal{A}$ and complex numbers λ. The algebra $\mathcal{A}$ is called *commutative* if $AB = BA$ for all $A, B \in \mathcal{A}$. An element $1 \in \mathcal{A}$ is called an *identity element* of $\mathcal{A}$ if $1A = A1 = A$ for all $A \in \mathcal{A}$. #

An algebra $\mathcal{A}$ is in general neither commutative nor has an identity element. We are particularly interested in algebras of linear operators in a Hilbert space H. Denote by $L(H)$ the set of all linear operators on H. For $A, B \in L(H)$, as usual we define

$$(A + B)x = Ax + Bx, \tag{2.9}$$

$$(AB)x = A(Bx), \tag{2.10}$$

$$(\lambda A)x = \lambda(Ax), \; x \in H, \; \lambda \in C. \tag{2.11}$$

It is readily seen that these operations convert $L(H)$ into a complex associative algebra in the sense of Definition 2.3. Note that $L(H)$ is not commutative in general but has an identity element I which is the identity operator on H; i.e., $Ix = x$ for all $x \in H$.

Definition 2.4. Let H be a Hilbert space. By a (linear) *operator algebra* $\mathcal{A}$ of H, we mean a subalgebra $\mathcal{A}$ of $L(H)$ which contains the identity element I. Namely, $\mathcal{A}$ is a subspace of $L(H)$ such that $I \in \mathcal{A}$ and $AB \in \mathcal{A}$ for all $A, B \in \mathcal{A}$. #

Definition 2.5. Let $\mathcal{A}$ be an (associative) algebra with multiplication AB and with identity element I. An element A of is said to be *invertible* in $\mathcal{A}$ if there exists an element $A^{-1} \in \mathcal{A}$ such that $AA^{-1} = A^{-1}A = I$. In this case, call A^{-1} the *inverse* of A. An invertible linear operator A in a Hilbert space is customarily called nonsingular with inverse operator A^{-1}. #

A linear operator A on a Hilbert space H is called *bounded* if

$$||Ax|| \leq c\,||x||, \quad x \in H$$

for some fixed positive real number c. Note that if A is a bounded nonsingular linear operator then the inverse operator A^{-1} is bounded also. We assume the reader is familiar with the boundedness of operators on finite-dimensional Hilbert spaces.

2.2. Isotope of an operator algebra. The concept of isotope of an operator algebra $\mathcal{A}$ is to convert $\mathcal{A}$ into another algebra defined on the same underlying linear space as $\mathcal{A}$ where the algebra $\mathcal{A}$ can be recovered as a special case of the isotope. Therefore, it is very important in our investigation to distinguish the notions of product, inverses and identity element in $\mathcal{A}$ from those in the isotope of $\mathcal{A}$. This distinction will be the central scheme of our attempt to introduce a more generalized notion of unitary, symmetric, hermitian and exponential operators in a Hilbert space. For a general approach, we define an isotope in an arbitrary complex associative algebra.

Definition 2.6. Let $\mathcal{A}$ be a complex associative algebra with product AB and with identity element I. Let T be a fixed element of $\mathcal{A}$. The algebra with multiplication

$$A * B = ATB, \quad A, B \in \mathcal{A} \tag{2.12}$$

defined on the same underlying linear space as $\mathcal{A}$ is called the *T-homotope* of $\mathcal{A}$ and is denoted by $\mathcal{A}^{(T)}$. If T is invertible in $\mathcal{A}$ then the T-homotope $\mathcal{A}^{(T)}$ is called the *T-isotope* of $\mathcal{A}$. #

Notice that $\mathcal{A}^{(T)}$ is an associative algebra; $(A * B) * C = A * (B * C)$ for all $A, B, C \in \mathcal{A}$. The notion of homotope and isotope has been a useful tool for the structure of associative and Jordan algebras. The interested reader may be referred to Jacobson[1], Myung[2], Osborn[3], and Myung and Santilli[4]. An isotopic generalization of Lie's theorems can be found in Santilli[14].

Remark 2.1. Homotopes and isotopes are closely related to some classes of Lie-admissible algebras. Let $\mathcal{A}$ be an associative algebra with product AB. Let R, S be fixed elements of $\mathcal{A}$. Denote by $\mathcal{A}(R,S)$ the algebra with multiplication

$$A \circ B = ARB - BSA, \quad A, B \in \mathcal{A} \tag{2.13}$$

defined on the same linear space as $\mathcal{A}$. The algebra $\mathcal{A}(R,S)$ has been called the (R,S)-*mutation* of $\mathcal{A}$ and arises from a generalization of the Heisenberg equation by Santilli. Denote by $\mathcal{A}(R,S)^-$ the algebra with the commutator product

$$[A,B]^\circ = A \circ B - B \circ A, \quad A, B \in \mathcal{A}$$

defined on the linear space $\mathcal{A}(R,S)$. Then $\mathcal{A}(R,S)$ is a Lie-admissible algebra in the sense that $\mathcal{A}(R,S)^-$ is a Lie algebra; that is, $[\ ,\ ]^\circ$ satisfies the anticommutative law, $[A,B]^\circ = -[B,A]^\circ$ and the Jacobi identity, $[[A,B]^\circ, C]^\circ + [[B,C]^\circ, A]^\circ + [[C,A]^\circ, B]^\circ = 0$. However, $\mathcal{A}(R,S)$ is not in general associative; i.e., $(A \circ B) \circ C \neq A \circ (B \circ C)$ for some $A, B, C \in \mathcal{A}(R,S)$. By direct computation, one sees

$$[A, B]^\circ = A(R + S)B - B(R + S)A,$$
$$= ATB - BTA = [A, B]^*$$

which implies that the commutator $[A, B]^\circ$ in $\mathcal{A}(R,S)$ equals the commutator $[A, B]^* = A * B - B * A$ in the $T = (R + S)$-homotope $\mathcal{A}^{(T)}$, being associative. The structure of $\mathcal{A}(R,S)$ has been studied in considerable detail in relation with $\mathcal{A}^{(T)}$ by a number of authors (see Ref. 2, 3, 4 and Oehmke[5]). #

Given a homotope $\mathcal{A}^{(T)}$ of $\mathcal{A}$, define a mapping g of $\mathcal{A}^{(T)}$ into $\mathcal{A}$ by

$$g(A) = AT, \quad A \in \mathcal{A} . \tag{2.14}$$

Then g is clearly linear and satisfies

$$g(A * B) = g(A)g(B), \quad A, B \in \mathcal{A} . \tag{2.15}$$

Thus g is an algebra homomorphism of $\mathcal{A}^{(T)}$ into $\mathcal{A}$. Assume that T is invertible in $\mathcal{A}$ with inverse T^{-1}, so that $\mathcal{A}^{(T)}$ is an isotope of $\mathcal{A}$. Then g is an isomorphism of $\mathcal{A}^{(T)}$ onto $\mathcal{A}$ with inverse $g^{-1} = f : \mathcal{A} \to \mathcal{A}^{(T)}$ given by

$$f(A) = AT^{-1}, \quad A \in \mathcal{A} . \tag{2.16}$$

Though $\mathcal{A}$ has an identity element, in a homotope $\mathcal{A}^{(T)}$ there is no guarantee for the existence of identity element, unless T is invertible in $\mathcal{A}$. This leads to

Definition 2.7. Let $\mathcal{A}$ be an associative algebra with product AB and with identity element 1. Let $\mathcal{A}^{(T)}$ be the T-homotope of $\mathcal{A}$.

If $\mathcal{A}^{(T)}$ has an identity element, it is denoted by I^*; i.e., $I^* * A = A * I^* = A$ for all $A \in \mathcal{A}$. Let $\mathcal{A}^{(T)}$ have an identity element I^*. If an element A in $\mathcal{A}^{(T)}$ is invertible in $\mathcal{A}^{(T)}$, the inverse of A in $\mathcal{A}^{(T)}$ is denoted by A^{-1^*}; i.e., $A * A^{-1^*} = A^{-1^*} * A = I^*$. #

The following result is very important for our discussion and may be found in Myung[2] (or easily verified).

Theorem 2.1. Let $\mathcal{A}$ be an associative algebra with product AB and with identity element I. Let $\mathcal{A}^{(T)}$ be the T-homotope of $\mathcal{A}$ for a fixed element $T \in \mathcal{A}$. Then we have

(1) $\mathcal{A}^{(T)}$ has an identity element I^* if and only if T is invertible in $\mathcal{A}$. In this case, $I^* = T^{-1}$.

(2) If T is invertible in $\mathcal{A}$ then an element $A \in \mathcal{A}$ is invertible in $\mathcal{A}$ if and only if A is invertible in $\mathcal{A}^{(T)}$. In this case, $A^{-1^*} = T^{-1} A^{-1} T^{-1}$.

(3) If T is invertible in $\mathcal{A}$ then the set of invertible elements in $\mathcal{A}$ coincides with the set of invertible elements in $\mathcal{A}^{(T)}$. #

Another important concept for our investigation is the notion of one-sided modules for an associative algebra and its isotope.

Definition 2.8. Let $\mathcal{A}$ be an associative algebra with product AB and with identity element I. A linear space V is called a *left unital $\mathcal{A}$-module* if there is defined a mapping : $(A,x) \to Ax$ of $\mathcal{A} \times V$ into V satisfying :

(i) $A(\lambda x + \mu y) = \lambda(Ax) + \mu(Ay)$,

(ii) $(\lambda A + \mu B)x = \lambda(Ax) + \mu(Bx)$,

(iii) $(AB)x = A(Bx)$,

(iv) $Ix = x$

for all $x, y \in V$, $A, B \in \mathcal{A}$, and $\lambda, \mu \in C$. #

Let $\mathcal{A}$ be the same as in Definition 2.8 and V be a left $\mathcal{A}$ -module. Let $\mathcal{A}^{(T)}$ be the T-isotope of $\mathcal{A}$. Define a mapping $(A,x) \to A * x$ of $\mathcal{A}^{(T)} \times V$ into V by

$$A * x = (AT)x = A(Tx), \quad x \in V, \ A \in \mathcal{A} , \tag{2.17}$$

where Ax is the module action in the $\mathcal{A}$ -module V . It is easily checked that V becomes a unital left $\mathcal{A}^{(T)}$-module under the composition $A * x$, since $I^* * x = T^{-1}(Tx) = (T^{-1}T)x = Ix = x$ for all $x \in V$. Thus we can put

Definition 2.9. Let $\mathcal{A}$ and V be the same as in Definition 2.8. Then the unital left $\mathcal{A}^{(T)}$-module defined on V under the composition $A * x$ given by (2.17) is called the T-*isotope* of the $\mathcal{A}$-module V and is denoted by $V^{(T)}$. #

The definitions and results above are in particular applied to any operator algebra $\mathcal{A}$ in a Hilbert space H where Ax is simply the linear operator action of A on H . Note that if $T = I$ is the identity element, then $\mathcal{A}^{(I)} = \mathcal{A}$ and $V^{(I)} = V$. #

Our aim is to introduce a *generalized notion of unitary, hermitian,*

symmetric and exponential operators, and of eigenvalues by means of isotope. It is important to note that the hermitian conjugate and transpose of linear operators in a Hilbert space is an involutional mapping in the sense of

Definition 2.10. Let $\mathcal{A}$ be an algebra. A linear (or antilinear) mapping $i : \mathcal{A} \to \mathcal{A}$ is called an *involution* (or *anti-involution*) of $\mathcal{A}$ if $i(i(A)) = A$ and $i(AB) = i(B)\, i(A)$ for all $A, B \in \mathcal{A}$. #

The following result is useful for later discussion.

Theorem 2.2. Let $\mathcal{A}$ be an associative algebra with product AB and with identity element I . Let T be an invertible element in $\mathcal{A}$. Let i be an involution (or anti-involution) of $\mathcal{A}$. Then the mapping $i_T : \mathcal{A}^{(T)} \to \mathcal{A}^{(T)}$ defined by

$$i_T(A) = i(T)\, i(A)\, T^{-1}, \quad A \in \mathcal{A} ,$$

is an involution (or anti-involution) of the T-isotope $\mathcal{A}^{(T)}$ of $\mathcal{A}$.

Proof. The mapping i_T is the composition $f \circ i \circ g$ of the mappings

$$\mathcal{A}^{(T)} \xrightarrow{\ g\ } \mathcal{A} \xrightarrow{\ i\ } \mathcal{A} \xrightarrow{\ f\ } \mathcal{A}^{(T)}$$

where f and g are algebra isomorphisms defined by (2.16) and (2.14), respectively. Hence i_T is an involution (or anti-involution) of $\mathcal{A}^{(T)}$. #

2.3. Isotope of unitary and hermitian operators. Let H be a Hilbert space with inner product $(,)$ and let $\mathcal{A} = L(H)$ be the operator algebra of all linear operators on H, with addition $A+B$, multiplication AB and scalar multiplication λA defined by (2.9) - (2.11). We also fix T as a (bounded) nonsingular linear operator in H (so T^{-1} is bounded also). Thus the T-isotope $\mathcal{A}^{(T)}$ and $H^{(T)}$ of $\mathcal{A}$ and H are defined by Definitions 2.6 and 2.9, respectively. In other words, H is regarded as a unital left $\mathcal{A}^{(T)}$-module by the operation (2.17). The following definition is of extreme importance for our programs.

Definition 2.11. Given a linear operator A, define the linear operator A^{hT} by the relation

$$(A * x, y) = (x, A^{hT} * y), \quad x, y \in H . \tag{2.18}$$

Call A^{hT} the T-*hermitian conjugate* of A. Similarly, we define the T-*transpose* A^{tT} of A as a linear operator satisfying

$$(x, A * y) = (\bar{y}, A^{tT} * \bar{x}), \quad x, y \in H . \quad \# \tag{2.19}$$

When $T = I$ is the identity operator, the I-hermitian conjugate and I-transpose of A are simply the usual hermitian conjugate A^h and transpose A^t. In this case, (2.18) and (2.19) are given by

$$(Ax, y) = (x, A^h y), \tag{2.20}$$

$$(x, Ay) = (\bar{y}, A^t \bar{x}) . \tag{2.21}$$

Theorem 2.3. Let H be a Hilbert space, and let $\mathcal{A} = L(H)$. Then, for each $A \in \mathcal{A}$, the T-hermitian conjugate A^{hT} and the T-transpose A^{tT} exist and satisfy

$$A^{hT} = T^h A^h T^{-1} , \tag{2.22}$$

$$A^{tT} = T^t A^t T^{-1} \tag{2.23}$$

where A^h amd A^t are the usual hermitian conjugate and transpose. Furthermore,

$$(A^{hT})^{hT} = (A^{tT})^{tT} = A , \tag{2.24}$$

$$(\lambda A + \mu B)^{hT} = \bar{\lambda} A^{hT} + \bar{\mu} B^{hT} , \tag{2.25}$$

$$(A * B)^{hT} = B^{hT} * A^{hT} , \tag{2.26}$$

$$(A * B)^{tT} = B^{tT} * A^{tT} \tag{2.27}$$

for all $A, B \in \mathcal{A}$ and $\lambda, \mu \in C$.

Proof. First, note that (2.24) - (2.27) hold for A^h and A^t. In view of (2.18) and (2.20), we have

$$\begin{aligned}(A * x, y) &= (x, A^{hT} * y) = (x, A^{hT} T y) \\ &= (ATx, y) = (x, (AT)^h y) = (x, T^h A^h y)\end{aligned}$$

and this implies $A^{hT} T = T^h A^h$ and so (2.22). Similarly, $(x, A * y) = (x, AT y) = (\bar{y}, (AT)^t \bar{x}) = (\bar{y}, T^t A^t \bar{x}) = (\bar{y}, A^{tT} * \bar{x})$ and this gives (2.23). Since the mapping $i : \mathcal{A} \to \mathcal{A}$ defined by $i(A) = A^h$ or A^t is an anti-involution of $\mathcal{A}$, it follows from Theorem 2.2. that

the mapping $i_T : \mathcal{A}^{(T)} \to \mathcal{A}^{(T)}$ defined by $i_T(A) = A^{hT}$ or A^{tT} is an anti-involution of the T-isotope $\mathcal{A}^{(T)}$. This implies the relations (2.24) - (2.27). #

Definition 2.12. A linear operator S is said to be T-*symmetric* if

$$S^{tT} = S \tag{2.28}$$

and S is called T-*skew-symmetric* if

$$S^{tT} = -S \tag{2.29}$$

Similarly, a linear operator B is called T-*hermitian* (or T-*self-adjoint*) or T-*skew-hermitian* if

$$B^{hT} = B \quad \text{or} \quad B^{hT} = -B . \quad \#$$

The following result is immediate from Theorem 2.3.

Theorem 2.4. A linear operator A is T-symmetric or T-skew-symmetric if and only if

$$A = T^t A^t T^{-1} \quad \text{or} \quad A = -T^t A^t T^{-1} . \tag{2.30}$$

Similarly, B is T-hermitian or T-skew-hermitian if and only if

$$B = T^h B^h T^{-1} \quad \text{or} \quad B = -T^h B^h T^{-1} . \quad \# \tag{2.31}$$

Note that (2.31) is equivalent to:

$$(x, B * y) = (B * x, y) \quad \text{or} \quad (x, B * y) = -(B * x, y) \tag{2.32}$$

for all $x, y \in H$. The usual definition of symmetry, hermitian, etc. is recovered when $T = I$ is the identity operator.

Definition 2.13. A linear operator A is called T-*isometric* if

$$||A * x|| = ||x||, \; x \in H . \quad \# \tag{2.33}$$

If $T = I$, Definition 2.13 leads to the usual definition of isometry. By the standard linearization, one sees that (2.33) is equivalent to the relation

$$(A * x, A * y) = (x, y), \; x, y \in H . \tag{2.34}$$

Theorem 2.5. A linear operator A is T-isometric if and only if

$$A^{hT} * A = I^* \tag{2.35}$$

where $I^* = T^{-1}$ is the identity element of $\mathcal{A}^{(T)}$. Any T-isometric linear operator is bounded.

Proof. By (2.18) and (2.34), we have $(x,y) = (A * x, A * y) = (x, A^{hT} * (A * y)) = (x, (A^{hT} * A) * y) = (x, (A^{hT} * A)Ty)$ for all $x, y \in H$, since $H^{(T)}$ is a unital left $\mathcal{A}^{(T)}$-module. This gives $(A^{hT} * A)T = I$ and $A^{hT} * A = T^{-1} = I^*$. Clearly, AT is bounded and since T is bounded by the assumption, so is A. #

Definition 2.14. A linear operator U is called T-*unitary* if

$$(U * x, U * y) = (x, y) \tag{2.36}$$

for all $x, y \in H$ and U is surjective; i.e., $UH = H$. # Clearly, T-unitarity and T-isometry are equivelent for finite-dimensional Hilbert spaces.

Note that a T-unitary linear operator is T-isometric.

Theorem 2.6. A linear operator U is T-unitary if and only if

$$U^{hT} * U = U * U^{hT} = I^* \text{ or } U^{hT} = U^{-1^*} . \tag{2.37}$$

The set of all T-unitary linear operators of H forms a group with identity element I^* under the T-isotopic product " $*$ " defined by (2.12).

Proof. Assume U is T-unitary. Since U is T-isometric, by (2.35) $U^{hT} * U = I^*$. On the other hand, $(U * x, y) = (x, U^{hT} * y) = (U * x, (U * U^{hT}) * y)$. Since UT is also surjective, we have $U * U^{hT} = I^*$. Conversely, if U satisfies (2.37), U is clearly T-unitary. It remains to show that if U_1, U_2 are T-unitary then $U_1 * U_2$ is T-unitary. From (2.26), it follows that $(U_1 * U_2)^{hT} * (U_1 * U_2) = U_2^{hT} * U_1^{hT} * U_1 * U_2 = I^*$ and likewise $(U_1 * U_2) * (U_1 * U_2)^{hT} = I^*$. #

The group of all T-unitary linear operators on H is called the T-*unitary group*.

An antilinear operator U satisfying (2.36) is called T-*antiunitary*. Again, notice that if $T = I$ then the definition of T-unitary agrees with the usual definition of unitary operators U satisfying

$$U^h U = U U^h = I . \tag{2.38}$$

In fact, the T-unitary group is isomorphic to the unitary group.

Theorem 2.8. Let $U_T(H)$ and $U(H)$ be the T-unitary and the unitary group of H, respectively. Then the mapping f given by

$$f(U) = UT^{-1}, \; U \in U(H) \tag{2.39}$$

is a group isomorphism of $U(H)$ to $U_T(H)$.

Proof. Let U be unitary. Then $(UT^{-1})^{hT} * (UT^{-1}) = T^h (UT^{-1})^h T^{-1} T U T^{-1} = T^h (T^{-1})^h U^h U T^{-1} = T^{-1} = I^*$ and likewise $(UT^{-1}) * (UT^{-1})^{hT} = I^*$. Thus UT^{-1} is T-unitary. Assume U is T-unitary. Then $(UT)^h UT = T^h U^h U T = (T^h U^h T^{-1}) * (UT) = U^{hT} * (U * I) = (U^{hT} * U) * I = I^* * I = I$ and similarly $(UT)(UT)^h = I$. Hence UT is unitary. Clearly, $f(U_1 U_2) = f(U_1) * f(U_2)$ and f is an isomorphism of $U(H)$ to $U_T(H)$. #

Theorem 2.9. Suppose that W is a linear operator on H and satisfies

$$|(x,y)| = |(W * x, W * y)| \text{ for all } x, y \in H . \tag{2.40}$$

Then we have

$$W * x = e^{i\varphi(x)} U * x, \; x \in H \tag{2.41}$$

for some T-unitary or T-antiunitary operator U where $\phi(x)$ is a real valued function of x .

Proof. The proof is immediate from Theorem 2.8. and the known result that if $|(x,y)| = |(Ax,Ay)|$ for a linear operator A and $x, y \in H$ then $Ax = e^{i\phi(x)} U_1 x$ for some unitary or antiunitary operator U_1 . Since WT is a linear operator and $U = U_1 T^{-1}$ is T-unitary or T-antiunitary by Theorem 2.8, we can express as $W * x = (WT)x = e^{i\phi(x)} U_1 x = e^{i\phi(x)} (U_1 T^{-1}) * x = e^{i\phi(x)} U * x$. #

When $T = I$, Theorem 2.9. is well known for the study of symmetry properties of physical systems. Theorem 2.9 gives the interpretation that if (2.40) or (2.41) holds then the phases of all vectors can be adjusted in such a way that the T-isotopic mapping $x \to W * x$ by W is effected by either a T-unitary or T-antiunitary operator.

Theorem 2.10. If U is T-unitary and A is T-hermitian then $U * A * U^{-1^*}$ is T-hermitian.

Proof. Note $U^{-1^*} = U^{hT}$ since U is T-unitary. Thus by Theorem 2.3. $(U * A * U^{-1^*})^{hT} = (U^{-1^*})^{hT} * A^{hT} * U^{hT} = U * A * U^{-1^*}$ and hence $U * A * U^{-1}$ is T-hermitian. #

2.4. Isotope of projection operators. We discuss the isotopic generalization of a special type of linear operators, called projection operators.

Definition 2.15. A linear operator P on H is called a T-*idempotent operator* if

$$P * P = P \quad \text{or} \quad PTP = P . \quad \# \tag{2.42}$$

Consider now a subspace H_0 of H. Letting

$$H_0^{\perp} = \{x \in H \mid (x,y) = 0,\ y \in H_0\} ,$$

we have a linear space direct sum

$$H = H_0 \oplus H_0^{\perp}. \tag{2.43}$$

and any vector $x \in H$ is uniquely expressed as $x = y + z$, $y \in H_0$, $z \in H_0^{\perp}$. [Note that if H is of infinite dimension then (2.43) is still valid for a closed subspace H_0 of H.]

Definition 2.16. Given a subspace H_0 of H. Let any vector $x \in H$ be expressed as $x = y + z$ for $y \in H_0$, $z \in H_0^{\perp}$. Then the mapping $P_{H_0} : H \to H$ defined by

$$P_{H_0} * x = (P_{H_0} T)x = y \tag{2.44}$$

is linear and is called the T-*projection onto* H_0. A mapping $P : H \to H$ is called a T-*projection* if P is the T-projection onto some subspace H_0 of H; i.e., $P = P_{H_0}$. #

Theorem 2.11. A linear operator P on H is a T-projection if and only if P is T-hermitian and T-idempotent, that is, $P^{hT} = P$ and $P * P = P$.

Proof. Assume P is the T-projection onto a subspace H_0 of H. Let $x_i = y_i + z_i$, $y_i \in H_0$, $z_i \in H_0^{\perp}$, $i = 1,2$. Then $(x_1, P * x_2) = (y_1 + z_1, y_2)$ $= (y_1, y_2) = (y_1, y_2 + z_2) = (P * x_1, x_2)$ and by (2.32) $P^{hT} = P$. Clearly, $P * P = P$. Suppose that a linear operator P is T-hermitian and T-idempotent. Let $H_0 = \{y \in H \mid P * y = y\}$. Then H_0 is a subspace of H. For each vector $x \in H$ and $y \in H_0$, $x - P * x \in H_0^{\perp}$ since $(x - P * x, y)$ $= (x - P * x, P * y) = (P * (x - P * x), y) = (P * x - P * P * x, y) = 0$. But since $x = P * x + (x - P * x)$ and $P * x \in H_0$ for $x \in H$, $P = P_{H_0}$, the T-projection onto H_0. #

Note that if $T = I$, a T-projection is the usual projection. In fact, we have

Corollary 2.12. A linear operator U is a projection if and only if UT^{-1} is a T-projection.

Proof. If P is a projection then $(PT^{-1}) * (PT^{-1}) = PPT^{-1} = PT^{-1}$ and by (2.22) $(PT^{-1})^{hT} = T^h(PT^{-1})^h T^{-1} = T^h(T^{-1})^h P T^{-1} = PT^{-1}$, so PT^{-1} is T-hermitian. Conversely, assume PT^{-1} is a T-projection. Let $P_1 = PT^{-1}$, so $P = P_1 T$. Then $PP = P_1 T P_1 T = (P_1 * P_1)T = P_1 T = P$ and $P^h = (P_1 T)^h = T^h P_1^h = (T^h P_1^h T^{-1})T = P_1^{hT} T = P_1 T = P$, since P_1 is T-idempotent and T-hermitian. By Theorem 2.11, this completes the proof. #

Definition 2.16. T-projections $P_1, \cdots, P_n$ are said to be *mutually T-orthogonal* if

$$P_i * P_j = 0 \quad \text{for } i \neq j = 1,2, \cdots, n . \quad \# \tag{2.45}$$

Since the mapping $f : A \to AT^{-1}$ is an algebra isomorphism of the operator algebra $\mathcal{A} = L(H)$ onto $\mathcal{A}^{(T)}$, by Corollary 2.12 and the known result for projections, the following is immediate.

Theorem 2.13. T-projections $P_1, \cdots, P_n$ are mutually T-orthogonal if and only if the sum $P = \oplus_{i=1}^{n} P_i$ is a T-projection. In this case, we have $P = P_{H_0}$ and $P_i = P_{H_{0i}} (i = 1, \cdots, n)$ for some subspaces H_0, H_{0i} of H, and $H_0 = H_{01} \oplus \cdots \oplus H_{0n}$ as a linear space direct sum where $H_{01}, \cdots, H_{0n}$ are mutually orthogonal, i.e., $(H_{0i}, H_{0j}) = 0$ for $i \neq j$. In particular, $H_0 = H$ if and only if

$$\sum_{i=1}^{n} P_i = I^* = T^{-1} . \quad \#$$

2.5. Isotope of exponential operators. If $\mathcal{A}$ is a finite-dimensional real or complex algebra with product AB and with identity element 1 then it is a well known fact that, for each $A \in \mathcal{A}$, the exponential $e^A = \exp A$ defined by

$$e^A = \exp A = 1 + A + \frac{1}{2!} A^2 + \cdots + \frac{1}{n!} A^n + \cdots \tag{2.46}$$

exists and is a uniformly continuous function defined on $\mathcal{A}$. Besides the central role of the exponential function in the analytic and continuous group theory, it also plays an important role for the structure of some classes of nonassociative algebras (see Remark 2.3 below). If $\mathcal{A}$ is infinite dimensional, $\exp A$ does not in general exist. But the convergence of $\exp A$ can be guaranteed for some element $A \in \mathcal{A}$. For example, if A is a bounded operator in a Hilbert space then $\exp A$ is defined in $\mathcal{A} = L(H)$. For the sake of simplicity, we assume that H is a finite-dimensional Hilbert space. Let $\mathcal{A} = L(H)$.

Our aim is to develop the isotopic generalization of the following two important problems.

(1) If a linear operator $A \in \mathcal{A}$ is also a function of a real parameter t, then, for a fixed $H \in \mathcal{A}$, the solution of the equation

$$\frac{dA}{dt} = i\,[H,A] = i\,(HA - AH) \tag{2.47}$$

is expressed by

$$A(t) = A = e^{itH} A(0) e^{-itH} . \tag{2.48}$$

The equation (2.47) is recognized as the Heisenberg equation.

(2) If H is a hermitian operator then

$$U(t) = e^{itH} \tag{2.49}$$

is a unitary operator for all real numbers t.

We consider these problems in the T-isotope $\mathcal{A}^{(T)}$. We first generalize Heisenberg's equations (2.47) in the T-isotopic form according to Santilli (see Ref. 6):

$$\frac{dA}{dt} = \frac{1}{i}[A,B]^* = \frac{1}{i}(A * B - B * A) = \frac{1}{i}(ATB - BTA) \tag{2.50}$$

Let us begin by noticing $[A,B]^* = 0$ is not equivalent to the equation $[A,B] = 0$, unless T commutes in $\mathcal{A}$ with A and B. Denote by A^{*n} the n th power of A in $\mathcal{A}^{(T)}$. It is readily seen that

$$A^{*n} = (AT)^{n-1}A = A(TA)^{n-1}, \quad n = 1, 2, \cdots \tag{2.51}$$

where A^n indicates the n th power of A in $\mathcal{A}$.

Definition 2.17. We define the T-*exponential* $e_T^A = \exp_T A$ of A in the T-isotope $\mathcal{A}^{(T)}$ by

$$e_T^A = \exp_T A = I^* + A + \frac{1}{2!}A^{*2} + \cdots + \frac{1}{n!}A^{*n} + \cdots. \quad \# \tag{2.52}$$

Since the mappings $f, g \in \mathcal{A} \to \mathcal{A}^{(T)}$ defined by

$$f(A) = AT^{-1}, \quad g(A) = T^{-1}A, \quad A \in \mathcal{A} \tag{2.53}$$

are algebra isomorphisms, they map the exponential e^A in $\mathcal{A}$ to the exponential $e_T^{f(A)}$ and $e_T^{g(A)}$ of $f(A)$ and $g(A)$ in $\mathcal{A}^{(T)}$. Thus

$$f(e^{AT}) = e_T^{f(AT)} = e_T^A = e^{AT_T-1},$$

$$g(e^{TA}) = T^{-1}e^{TA} = e_T^{g(TA)} = e_T^A.$$

Hence we have the following result (also see Myung[2]).

Theorem 2.14. For every linear operator $A \in \mathcal{A}$, we have

$$e_T^A = T^{-1}e^{TA} = e^{AT}T^{-1}. \tag{2.54}$$

If $[A,B]^* = A * B - B * A = 0$ then

$$e_T^A * e_T^B = e_T^{A+B}, \tag{2.55}$$

$$e_T^{-A} = (e_T^A)^{-1^*}. \quad \# \tag{2.56}$$

By direct computation, one sees that the solution of the generalized equation (2.50) is given by

$$A = e^{itHT}A(0)e^{-itTH} \tag{2.57}$$

which can also be written as

$$A = e_T^{itH} * A(0) * e_T^{-itH}. \tag{2.58}$$

It is interesting to compare (2.57) and (2.58) with the conventional form (2.48). The solution (2.58) has an infinite series expansion by the formula

$$e_T^A * B * e_T^{-A} = B + [A,B]^* + \frac{1}{2!}[A,[A,B]^*]^* + \frac{1}{3!}[A,[A,[A,B]^*]^*]^* + \cdots. \tag{2.59}$$

Noting the fact that the T-isotopic product $A * B$ is associative, (2.59) can be proved by the same argument as in Ref. 8 (see p. 159).

For the isotopic generalization of Problem (2) above, we need

Theorem 2.15. Let H be a Hilbert space (of arbitrary dimension). The mappings f, g defined by (2.53) are bijections from the set of hermitian or skew-hermitian (symmetric or skew-symmetric) linear operators in $\mathcal{A}$ to the set of T-hermitian or T-skew-hermitian (T-symmetric or T-skew-symmetric) linear operators.

Proof. Let A be a hermitian linear operator in $\mathcal{A}$. Then $f(A)^{hT} = (AT^{-1})^{hT} = T^h(AT^{-1})^h T^{-1} = AT^{-1}$, so $f(A)$ is T-hermitian. If B is T-hermitian then, letting $A = BT$, we have $A^h = (BT)^h = (T^h B^h T^{-1})T = B^{hT}T = BT = A$ and hence A is hermitian. The same argument applies to g and to the symmetric or skew-hermitian operators. #

Since it is known that every hermitian linear operator in any Hilbert space is bounded and T is bounded by the assumption, in view of Theorem 2.15, we have

Corollary 2.16. Any T-hermitian linear operator is bounded. #

Remark 2.2. The hermitian and T-hermitian operators are not closed under the product AB and the T-isotope product $A * B$. Define the anticommutator $\{A,B\}$ and $\{A,B\}^*$ of A,B in $\mathcal{A}$ and in $\mathcal{A}^{(T)}$ by

$$\{A,B\} = AB + BA, \quad \{A,B\}^* = A * B + B * A .$$

If A and B are hermitian or T-hermitian then so is $\{A,B\}$ or $\{A,B\}^*$. This follows from Theorem 2.3. By a similar reason, if A and B are skew-hermitian or T-skew-hermitian then so is $[A,B]$ or $[A,B]^*$. #

In terms of T-isotope, we can state Problem (2) above as follows.

Theorem 2.17. Let H be an arbitrary Hilbert space. If B is a T-hermitian linear operator on H then

$$U_T(w) = e_T^{iwB} \tag{2.60}$$

are defined for all real numbers w and are T-unitary linear operators satisfying

$$U_T(0) = T^{-1} , \tag{2.61}$$

$$U_T(w_1 + w_2) = U_T(w_1) * U_T(w_2), \tag{2.62}$$

$$\text{The infinitesimal} = \frac{d}{dw}U_T(w)\Big|_{w=0} = iB . \tag{2.63}$$

Proof. Note that if A is hermitian, e^{iwA} is unitary. We use this and Theorems 2.8 and 2.15. Let B be T-hermitian. By Theorem 2.15, BT is hermitian and hence e^{iwBT} is unitary. Thus, by Theorem 2.8,

$$f(e^{iwBT}) = e_T^{iwf(BT)} = e_T^{iwB}$$

is T-unitary. The remaining relations follow from Theorem 2.14. #

Remark 2.3. The exponential e_T^A has a close relation with the exponential defined on the class of Lie-admissible algebras of the (R,S)-mutation type mentioned in Remark 2.1. Let $\mathcal{A}$ be an associative algebra with product AB and with identity element I. Let $\mathcal{A}(R,S)$ be the

(R,S)-mutation of $\mathcal{A}$ with product $A \circ B = ARB - BSA$ as in Remark 2.1. In general, the n th power of each element A in $\mathcal{A}^{(T)}$ is not definable. Given $A \in \mathcal{A}$, define $A^{\circ n}$ inductively as $A^{\circ 1} = A$, $A^{\circ n} = A^{\circ(n-1)} \circ A$. Then $\mathcal{A}(R,S)$ is called *power-associative* if $A^{\circ n} \circ A^{\circ m} = A^{\circ(m+n)}$ for all positive integers m,n and all $A \in \mathcal{A}$. If there exists an element 1° in $\mathcal{A}(R,S)$ such that $1^\circ \circ A = A \circ 1^\circ = A$ for all $A \in \mathcal{A}$ then 1° is called an *identity element* of $\mathcal{A}(R,S)$. In this case, we can define the exponential $e_\circ^A$ of A in $\mathcal{A}(R,S)$ by

$$e_\circ^A = 1^\circ + A + \frac{1}{2!} A^{\circ 2} + \cdots + \frac{1}{n!} A^{\circ n} + \cdots .$$

The important fact is that if $\mathcal{A}(R,S)$ is power-associative with identity element 1° then $1^\circ = (R - S)^{-1}$ and

$$e_\circ^A = e_{R-S}^A$$

for all $A \in \mathcal{A}$ where e_{R-S}^A is the exponential of A in the $(R - S)$-isotope $\mathcal{A}^{(R-S)}$ defined by (2.52). In other words, the exponential in some (nonassociative) Lie-admissible algebra can be realized as that in an isotope (so associative) of an associative algebra. This problem has been discussed in detail by Myung[2] and Myung and Santilli[4]. #

An important special case of isometric operators is the *Cayley transform* $c(A)$ of a linear operator A:

$$c(A) = (A - iI)(A + iI)^{-1}, \tag{2.64}$$

which plays a key role in the hermitian operator theory. The T-isotope generalization of (2.64) can be stated as

Definition 2.18. Given a linear operator $A \in \mathcal{A}$. If the linear operator $c_T(A)$ given by

$$c_T(A) = (A - iI^*) * (A + iI^*)^{-1^*} \tag{2.65}$$

is definable then $c_T(A)$ is called the T-*Cayley transform* of A. #

Since any unitary operator U on H is expressed by (2.64) for some hermitian operator A on H, in view of Theorems 2.8 and 2.15, the following is immediate.

Theorem 2.18. Any T-unitary operator on H is given by the T-Cayley transform $c_T(B)$ for some T-hermitian operator B. #

2.6. Isotope of eigenvalues, determinants, and traces.

Until now we have essentially considered the isotopic generalization of some elementary definitions and results in the linear operator theory of Hilbert spaces. On the other hand, some notion may not be preserved under the isotope in its original form. Consider, again, a finite-dimensional Hilbert space. Then, for a linear operator A on H, we define $\mathrm{Tr}\, A$ as the usual trace of A. As is well known, $\mathrm{Tr}[A,B] = 0$ in $\mathcal{A}$. But $\mathrm{Tr}[A,B]^* = \mathrm{Tr}(ATB - BTA) \neq 0$ in $\mathcal{A}^{(T)}$, in general. However, if T commutes in $\mathcal{A}$ with A and B, then $\mathrm{Tr}[A,B]^* = 0$.

For another example, given a linear operator A on H, consider the equation

$$A * x = \lambda * x = (\lambda T)x \tag{2.66}$$

for some scalar (complex number) λ and a nonzero vector $x \in H$. If one attempts to make the isotopic generalization of eigenvalues and

eigenvectors by means of (2.66), one encounters an inconsistency, namely, H *does not become a linear space over* C *under the scalar multiplication* $\lambda * x$ *given by* (2.66), since $(\lambda\mu) * x = (\lambda\mu T)\, x \neq \lambda * (\mu * x) = (\lambda\mu T^2)x$, in general.

This situation suggests to introduce *an isotopic generalization of the notion of scalars*. For this, one can proceed as follows.

Definition 2.19. We define $C^{(T)}$ as the set of all linear operators in $\mathcal{A}$ given by

$$C^{(T)} = \{\lambda I^* = \lambda T^{-1} | \lambda \in C\} . \tag{2.67}$$

We denote $\lambda_T = \lambda T^{-1}$ and call each element in $C^{(T)}$ a *T-scalar*. $C^{(T)}$ may be called the *T-isotope* of C . We also call λ_T *T-complex* or *T-real* if λ is complex or real. #

Define the addition $\lambda_T + \mu_T$ in $C^{(T)}$ as the same as in $\mathcal{A}$ or in $\mathcal{A}^{(T)}$ but the multiplication $\lambda_T * \mu_T$ in $C^{(T)}$ by

$$\lambda_T * \mu_T = \lambda_T T \mu_T = (\lambda\mu)T^{-1} . \tag{2.68}$$

Then it can be easily seen that $C^{(T)}$ is a field isomorphic to C . In fact, the mapping $\lambda \to \lambda_T = \lambda T^{-1}$ is an isomorphism of C onto $C^{(T)}$ and the inverse $\lambda_T^{-1^*}$ of λ_T and the identity element 1_T in $C^{(T)}$ is given by

$$\lambda_T^{-1^*} = \lambda_T^{-1} = \lambda^{-1}T^{-1}, \quad 1_T = I^* = T^{-1} . \tag{2.69}$$

We can now convert H into a linear space over the field $C^{(T)}$ of T-scalars by defining the T-scalar multiplication $\lambda_T * x$:

$$\lambda_T * x = (\lambda_T T)x = \lambda x, \quad x \in H , \tag{2.70}$$

namely, by defining $\lambda_T * x$ as the same as the scalar multiplication λx in the linear space H over C . It can be directly checked that this definition makes H a linear space over $C^{(T)}$, for example, $(\lambda_T * \mu_T) * x = (\lambda\mu)x = \lambda_T * (\mu_T * x)$. It is important to note that each element λ_T of $C^{(T)}$ is not in the center of $\mathcal{A} = L(H)$ but is in the center of the T-isotope $\mathcal{A}^{(T)}$; for, if $A \in \mathcal{A}$ then $A * \lambda_T = AT(\lambda T^{-1}) = \lambda A = \lambda_T * A$. Thus a T-scalar is not a scalar linear operator on H but is on the T-isotope $H^{(T)}$ (see Definition 2.9). This leads to

Definition 2.20. Given a linear operator A , a T-scalar λ_T is called a T-*eigenvalue* of A if there is a nonzero vector $x \in H$ such that

$$A * x = \lambda_T * x . \tag{2.71}$$

Then x is called a T-*eigenvector* of A corresponding to λ_T . #

From the definition, it is clear that a T-scalar λ_T is a T-eigenvalue of A with T-eigenvector x if and only if λ is an eigenvalue of AT with eigenvector x . Thus the T-eigenvalues λ_T of A are obtained by the eigenvalues λ of AT , namely, by the solutions of the equation

$$\det (AT - \lambda I) = 0 . \tag{2.72}$$

We rewrite (2.72) as $0 = \det (AT - \lambda I) = \det [(A - \lambda T^{-1})T]$ $= [\det (A - \lambda T^{-1})] \det T$. Since $\det T \neq 0$ (T nonsingular), we have $\det (A - \lambda T^{-1}) = 0$. Therefore, we have

Theorem 2.19. Let A be a linear operator on H . Then a T-scalar λ_T is a T-eigenvalue of A if and only if λ is an eigenvalue of AT . The T-eigenvalues of A are the solutions of the equation, called the T-*characteristic polynomial* of A ,

$$\det (A - \lambda_T * I^*) = 0 . \quad \# \tag{2.73}$$

The following is the T-isotope generalization of the known result for hermitian operators.

Corollary 2.20. Let B be a T-hermitian operator on H . Then the T-eigenvalues of B are all T-real.

Proof. If B is T-hermitian then, by Theorem 2.15, BT is hermitian. Hence, by the known result, the eigenvalues of BT are all real and by Theorem 2.19 the T-eigenvalues of B are all T-real. #

The discussion above and Theorem 2.19 also suggest the following T-isotope version of trace and determinant.

Definition 2.21. Let $\dim H = n$. Given a linear operator A on H , let $\lambda_T^1, \cdots, \lambda_T^n$ be the T-eigenvalues of A . The T-*trace*, $\mathrm{Tr}_T A$, of A is defined to be the T-scalar:

$$\mathrm{Tr}_T A = \lambda_T^1 + \cdots + \lambda_T^n = (\lambda^1 + \cdots + \lambda^n)T^{-1} .$$

Define the T-*determinant*, $\det_T A$, of A as the T-scalar given by

$$\det_T A = (\det AT)T^{-1} . \quad \#$$

The proof of the following result is straightforward.

Theorem 2.20. Let H be a finite-dimensional Hilbert space. For linear operators A, B, P, we have

(1) $\mathrm{Tr}_T A = (\mathrm{Tr}\ A)T^{-1}$,

(2) $\mathrm{Tr}_T (P * A * P^{-1^*}) = \mathrm{Tr}_T A$,

(3) $\det_T (A * B) = (\det_T A) * (\det_T B)$,

(4) $\det_T A^{-1^*} = (\det_T A)^{-1^*}$,

(5) $\mathrm{Tr}_T (A * B) = \mathrm{Tr}_T (B * A)$,

(6) $\det_T (\exp_T A) = \exp_T (\mathrm{Tr}_T A) . \quad \#$

Note that if $T = I$ then the T-trace and T-determinant coincide with the usual ones (multiplied by 1).

2.7. Isotopes of Hilbert spaces and C*-algebras

Our next step is to generalize the inner product $(\ ,\ )$ of H in such a way as to permit the study of the compatibility with the generalized envelope $\mathcal{A}^{(T)}$ and field $C^{(T)}$.

Definition 2.22. Let (x,y) be the inner product of a Hilbert space H and let T be an operator on H which is nonsingular and satisfies all needed topological conditions (regarding domain, range, etc. Then the T-isotopic extension of (x,y) is given by

$$(x,y)^* = (xT^h,y)I^* = I^*(x,Ty) \tag{2.74}$$

where T^h is the usual Hermitean conjugate of T.

In particular, if T, besides being invertible, is positive on H (that is, $(x,Tx) > 0$ for all $x \neq 0$ in H), then the following result can be easily proved.

Theorem 2.21. Let T be an invetible and positive operator as per Definition 2.22 under which the product $(x,y)^*$ is an inner product mapping $H \times H \to C^{(T)}$, given by (2.74). Then it satifies

$$\begin{aligned} &(x,x)^* > 0 \quad \text{for all } x \neq 0 \text{ on } H \\ &(x,y)^* = \overline{(y,x)^*} \\ &(x,y+x)^* = (x,y)^* + (x,z)^* \\ &(x,\lambda y)^* = \lambda (x,y)^* = \lambda_T (x,y)^*, \quad \lambda \in C \end{aligned} \tag{2.75}$$

where we have used the notion of T-scalar in the last expression.

The realization of the product $(x,y)^*$ we shall use in the next section can be expressed in terms of L^2-functions x and y and a (suitably selected) measure dm (we ignore the multiplication by I* here)

$$(x,Ty) = \langle x|*|y\rangle = \int \bar{x} * y \; dm = \int \bar{x} \, T \, y \, dm \tag{2.76}$$

and can be interpreted as the "expectation value" of the isotopy operator T.

We should insist, however, on the fact that, since the ordinary product has no algebraic meaning in $\mathcal{A}^{(T)}$, the inner product is now given by (2.74) and not by (x,y). This implies a number of predictable restrictions on the admissible operators T which we hope to study in a separate paper.

We are finally equipped to identify the generalized form of Hilbert spaces that will be used in the physical applications.

Definition 2.23. Let H be a Hilbert space over the field C of complex numbers, with enveloping associative algebra of operators $\mathcal{A}$ and inner product (x,y). We shall denote with $H^{(T)}$ the linear vector space over the

field $C^{(T)}$ of T-scalars with isotopic enveloping associative algebra of operators $\mathcal{A}^{(T)}$ and isotopic inner product $(x,y)^*$.

Remark 2.4. Since all the mappings

$$\begin{aligned} C &\longrightarrow C^{(T)} \\ \mathcal{A} &\longrightarrow \mathcal{A}^{(T)} \\ (x,y) &\longrightarrow (x,y)^* \end{aligned} \tag{2.77}$$

are isomorphisms we shall continue to use the name "Hilbert space" for the space $H^{(T)}$. However, we would like to stress that a considerable amount of additional studies (not conducted in this paper) are needed for the full, technical qualification of the space $H^{(T)}$ as a Hilbert space.

Note the appearance of the generalized unit I^* in the definition of inner product (2.74). It is essential to ensure that the values $(x,y)^*$ belong to $C^{(T)}$ rather than C. In turn, this is essential for $H^{(T)}$ to have the pre-requisites of a Hilbert space. Finally, note that $(x,y)^* = (x,y)$ whenever $T = I$.

Thus, $H^{(T)}$ is a meaningful generalization of H only for non-trivial (i.e., operator) isotopies.

We pass now to a few considerations for the extension of the T-trace to arbitrary dimensions. Let λ_T be a T-eigenvalue of A defined by (2.71). Thus, $(A - \lambda_T)*x = 0$ for some x in $H^{(T)}$. But $(A - \lambda_T)*x = (AT - \lambda)x = 0$. Thus, $(A - \lambda_T)$ is not invertible in $\mathcal{A}^{(T)}$ if and only if $(A - \lambda)$ is not invertible in $\mathcal{A}$. In light of theorem 2.19, this leads to the following definition of spectrum in $H^{(T)}$.

Definition 2.24. Let A be a linear operator on $H^{(T)}$. The T-spectrum, $Sp_T A$, is defined as the set of T-scalars λ_T such that $A - \lambda_T$ is not invertible in $\mathcal{A}^{(T)}$.

The following result is immediate from this and Theorem 2.19.

Theorem 2.22. For any linear operator in $H^{(T)}$,

$$Sp_T A = (SpAT)I^* \tag{2.78}$$

where Sp is the conventional spectrum of operators on H.

The following additional results can also be readily verified.

Theorem 2.23. Let A be a linear operator on $H^{(T)}$.

(1) If A is T-Hermitean, then each element of $Sp_T A$ is T-real; and

(2) If A is T-unitary, then, for each $\lambda_T \in Sp_T A$,

$$\| \lambda_T \|_T = \| \lambda \| I^* = I^*$$

Theorem 2.24. Let A be T-Hermitean in $\mathcal{OL}^{(T)}$. Then, the followings are equivalent.

(a) $Sp_T A > 0$ (i.e., $\lambda_T > 0$ for all $\lambda_T \in Sp_T A$)

(b) A is of the form $B*B^{hT}$ for some $B \in \mathcal{OL}^{(T)}$.

(c) A is of the form $B^{hT}*B^{hT}$ for some T-Hermitean operator B.

It can be seen that each of the conditions in Theorem 2.24 is equivalent to the condition

$$(x, A*x) > 0 \quad \text{for all} \quad x \in H^{(T)} \tag{2.79}$$

when, when T is positive, can be equivalently written

$$(x, A*y)^* = (x, ATy)I^* > 0 \tag{2.80}$$

This permits to call a T-hermitean operator A on $H^{(T)}$ T-positive when (2.79) is verified. In particular, a T-hermitean operator A is T-positive if and only if AT is positive in the conventional sense.

A linear operator A on $H^{(T)}$ is said to be of the T-trace class if, for any orthonormal basis $\{e_\alpha\}$ of $H^{(T)}$ the series $S = \sum_\alpha (e_\alpha, A*e_\alpha) I^*$ converges to a unique T-scalar (that is, in a way independent of the choice of the basis). In this case, the value of the series S is defined as the T-trace of A in $H^{(T)}$,

$$Tr_T A = \sum_\alpha (e_\alpha, A*e_\alpha) I^* \tag{2.81}$$

If $H^{(T)}$ is finite-dimensional, the definition above coincides with that given before in Definition 2.21. Note that not every linear operator on $H^{(T)}$ possesses a T-trace. However, for a T-positive operator (essentially those needed for Equations (2.75)), the existence of a T-trace can be proved under certain conditions that will be presented elsewere.

Let H be a (conventional) Hilbert space of arbitrary dimensions, and $\mathcal{OL}$ the (conventional) associative algebra of linear operators on H. We regard $\mathcal{OL}$ as a complex C*-algebra under the usual norm $\|A\|$, $A \in \mathcal{OL}$. Consider now the following generalization of the norm

$$\|A\|_T = \|AT\| I^* \tag{2.82}$$

Assume T to be a (conventionally) positive operator for simplicity. Then the ordering of T-scalars follows as for the ordinary case.

To show that norm (2.82) also defined a form of C*-algebras, it is sufficient to verify the following properties.

$$\|A*B\|_T \leq \|A\|_T * \|B\|_T \tag{2.83}$$

$$\|A^{hT}\|_T = \|A\|_T \tag{2.84}$$

$$\|A\|_T^{2*} \; (= \|A\|_T * \|A\|_T) = \|A^{hT}*A\|_T \tag{2.85}$$

This is left here as an (instructive) exercise to the interested reader.

We then call (2.82) the T-norm of A. The resulting algebra is here called $C^*_{(T)}$- algebra. The conventional C*-algebra is then admitted as the particular case when $I^* = I$. Additional studies are in progress and will be reported elsewhere.

III. APPLICATIONS TO THE LIE-ISOTOPIC BRANCH OF HADRONIC MECHANICS

The applications of the theory of the preceding section are virtually endless. In fact, as we shall indicate in this section, the theory permits the isotopic generalization of each and every aspect of Atomic Mechanics (the ordinary quantum mechanics) into a non-Hamiltonian form which appears to be valuable for strong non-Hamiltonian interactions as well as other physical phenomena where contact/nonpotential interactions are possible (e.g., statistical ensembles of extended molecules with inelastic collisions; electronic systems; neural systems; etc.).

In Paper I we have attempted the presentation of the Lie-isotopic generalization of Heisenberg's mechanics via only one postulate. The theory of the preceding section permits a more refined study of the results, as well as, and perhaps more importantly, the formulation of postulates encompassing other representations, such as those of Schrödinger-type.

In this section we shall therefore attempt the identification of the physical foundations of the Lie-isotopic branch of the Hadronic Mechanics, in the hope that our work results to be valuable for applications to specific systems (e.g., the deuteron, the neutron, the π^o, the nonpotential scattering theory, etc.).

A first understanding is that this section is, by far, incomplete and a considerable amount of additional work is needed.

A second understanding is that the contents of this section should be considered as of tentative nature until the theory is completed, tested in a variety of cases, and confirmed by experiments.

A third understanding is related to the primary objective of the Lie-isotopic branch of the Hadronic Mechanics, the *exterior treatment* of composite systems with non-Hamiltonian internal forces. Classically, the compatibility of total conservation laws with non-Hamiltonian forces is achieved via subsidiary constraints (Paper I). At the hadronic level, the difficulties are then two-fold. We must first reach a non-Hamiltonian generalization of Atomic Mechanics, and then we must implement it in such a way to permit total conservation laws. This section is primarily devoted to the first task, and only incidental comments will be presented for the second objective. Thus, the reader should keep in mind that the systems treated in this section are, in general, of *open non-Hamiltonian* character.

A sound knowledge of related theoretical studies is recommendable, with particular references to the research by Eder[10], Mignani[11], Okubo[12], Kapuscik[13] and others.

A sound knowledge of the Birkhoffian generalization of the classical Hamiltonian mechanics[14] appears to be also recommendable for an in-depth understanding of this section as well as for possible further developments.

In regard to numerical applications of the theory of this section, we refer specifically to the studies by Eder[10] (e.g., to represent conventional values of the spin while the magnetic moment is anomalous).In regard to the problem of structure, we refer specifically to the studies by Santilli[15] for which the Hadronic Mechanics was suggested in the first place. The reader should however be aware that the model of structure of hadrons of ref.[15] (as closed variationally nonselfadjoint systems) is considerably different than those of current trends. For instance, by recalling that the number of constituents increases in the transition from the hydrogen atom to the helium, and from the deuteron to the tritium, a fundamental condition of the model of ref.[15] is that the number of constituents increases from the π^o to the $\pi^{\pm}$ and to heavier hadrons [for a theoretical description of what is meant by "constituent" see Paper III].

3.1. Existing applications. A number of applications of the theory of the preceding section have already been identified, as indicated in Section III of Paper I. We are referring here to the isotopic generalization of

(1) Enveloping associative algebra of Heisenberg's mechanics;

(2) Lie algebras;

(3) Heisenberg's equations;

(4) Canonical commutation rules;

(5) Cononical quantization;

(6) Heisenberg's uncertainty principle;

(7) Planck's constant;

(8) Galilei's relativity;

(9) Representation theory.

It is an instructive exercise for the interested reader to reformulate each and every topic (1) through (9) within the context of the theory of the preceding section.

3.2. Hadronic-isotopic observables. We begin our study with the identification of the mathematical characteristics of observable quantities under strong non-Hamiltonian interactions. The conventional Atomic Mechanics, since it is strictly Hamiltonian in character, is inapplicable. As a result, the old characterization of observability under electromagnetic interactions via Hermitian operators is inapplicable. Once the non-Hamiltonian character of the theory of the preceding section is seen, the following generalized postulate appears plausible.

POSTULATE I. *Measurable hadronic quantities of the exterior strong problems (such as the total energy H, the total linear momentum $\vec{P}$, the total angular mementum $\vec{J}$, etc.) are characterized by T-Hermitian operators (in the sense of Definition 2.12), i.e.,*

$$H^{hT} = T^{h}H^{h}T^{-1} = H, \quad \text{etc.} \tag{3.1}$$

The generalized observability submitted here is clearly a covering of the conventional atomic one, in the sense that the latter is admitted as a particular case of a broader mathematical and physical context.

Note that the conventional observability is recovered not only when the hadronic unit $I^* = T^{-1}$ is the conventional one I, but also when I^* is a real scalar ($\neq I$). In fact, in this latter case we have

$$H^{hT} = H^{h}T^{h}T^{-1} = H^{h} \tag{3.2}$$

under which T-Hermiticity and conventional Hermiticity are equivalent.

As it will be evident later on, the transition from conventional Hermiticity to T-Hermiticity implies that physical quantities are generally *nonconserved*. The "germs" of the new mechanics are therefore already contained in Postulate I. In fact, the Atomic Mechanics was conceived to achieve a representation of the *stability of the orbit* of the electrons in the atomic structure. An objective of the Hadronic Mechanics is instead that of achieving a representation of *orbits as unstable as dynamically possible*, because we have in mind the treatment of extended hadrons moving within a medium of other hadrons. When total conservation laws are regained, the internal dynamics is left as unrestricted as possible, in the hope of achieving "global stability" via irreversible, nonconservative, and non-Hamiltonian internal dynamics, along the idea of "closed variationally nonselfadjoint systems" (Paper I).

The transition from atomic to hadronic observability therefore implies the *loss of quantum levels*, that is, the loss of Planck's constant $\hbar$, and its replacement with the operator T^{-1}. As recalled in Paper I, this generalization is submitted in the hope of achieving a more adequate representation of the processes of emission and absorption of energy for particles under conditions of mutual penetration within other particles.

But there are other reasons to suggest a departure from the original idea of Atomic Mechanics. They are given by the *clear experimental evidence according to which, in the transition from the two-body problem under electromagnetic interactions to that under strong interactions, there is the disappearance of exited states*. In fact, while the hydrogen atom and the positronium admit an infinite variety of exited states, no exited state has been experimentally established until now for the deuteron.# The same situation may occur also for other composite particles supposed to be of two-body nature, such as the π^o[15].

This drastic change in physical behaviour is, perhaps, the most forceful experimental evidence suggesting a revision of the Atomic Mechanics into a form specifically conceived for the strong interactions (an application of Hadronic Mechanics for the suppression of the atomic spectrum will be reviewed later on).

3.3. Hadronic-isotopic states. The state of an atomic system is characterized by a vector $|>$ on a Hilbert space on which Hermitian observables A act according to the familiar eigenvalue equation

$$A|> = a|> \quad . \tag{3.3}$$

It takes a minimum of five nucleons to reach a nucleus with exited states vaguely reminiscent of the quantum levels of the Atomic Mechanics, and this is a reason why the efforts here are to abandon the notion of quantum in favor of more general notions (Paper I).

This equation is intrinsically Hamiltonian in character, and, thus, inapplicable to the system considered. To lift the restriction, we introduce the following second postulate where hereon "Hilbert spaces" are referred to Def. 2.23.

POSTULATE II. *A hadronic state of the exterior strong problem is characterized by a vector $|>$ of a Hilbert space on which the hadronic observables A act according to the T-isotopic eigenvalue equations (in the sense of Definition 2.20).*

$$A * |> = a_T * |> \quad , \tag{3.4a}$$

$$A * |> = AT|> \quad , \quad a_T * |> = a_T T|> = a|> \quad . \tag{3.4b}$$

As it is the case in the Atomic Mechanics, not all observables can be measured simultaneously. Consider a second observable B with eigenvalue equation

$$B * |> = b_T * |> \quad . \tag{3.5}$$

Then we can write

$$B * A * |> = B * a_T * |> = a_T * B * |> = a_T * b_T * |> \quad , \tag{3.6}$$

$$A * B * |> = A * b_T * |> = b_T * A * |> = b_T * a_T * |> \quad ,$$

i.e.

$$A * B * |> - B * A * |> \equiv 0 \quad , \tag{3.7}$$

which can hold if and only if

$$[A,B]^* = A * B - B * A \equiv 0 \quad . \tag{3.8}$$

Lemma 3.1. A necessary and sufficient condition for two or more hadronic observables to be measurable simultaneously is that they T-commute in the sense of Equation (3.8). #

The reader should keep in mind the remark after Equation (2.50) in regard to the lack of equivalence of the commutators $[A,B]^*$ and $[A,B]$. In fact, ordinary commutativity *is not* in general sufficient to permit the simultaneous hadronic observability. It then follows that quantities which are measurable simultaneously in Atomic Mechanics are not necessarily measurable simultaneously in the Hadronic Mechanics.

Needless to say, the ordinary commutativity is a particular case of the T-commutativity. In particular, if the isotopy operator T is a scalar, ordinary commutativity and T-commutativity are equivalent.

The study of the "maximal and complete set of T-commuting observables" of a hadronic system will not be conducted here. We merely assume that this set exists, and write

$$\begin{aligned} A_1 * |> &= a_1 * |> \ , \\ A_2 * |> &= a_2 * |> \ , \\ &\vdots \\ A_n * |> &= a_n * |> \ . \end{aligned} \tag{3.9}$$

The state vector can then be labeled via the collection, say, α, of all T-eigenvalues

$$|> = \sum_{\alpha} \Psi(\alpha) * |\partial\rangle \qquad \alpha = (a_1, a_2, \cdots, a_n) \tag{3.10}$$

where

$$\Psi(\alpha) = <\alpha| * |> \tag{3.11}$$

The functions $\Psi(\alpha)$ will be called *hadronic state functions*. Note that, if one preserves the operator in the transition from the atomic to the hadronic setting, the state functions are generally different because of the isotopy in the eigenvalue equations.

Equation (3.10) identifies the existence of *hadronic representations* in much of the way as occurring in Atomic Mechanics. Our immediate objective is that of identifying at least some of these representations; identify the methods for passing from one to the other; and study some of their properties (e.g., normalization).

It is recommendable to identify the alteration (called *mutation* in Ref.[15], p. 690) of eigenvalues under isotopy. This mutation is at the basis of the physical applications of the Hadronic Mechanics, such as the anomalous magnetic moments in nuclear physics[10], the theoretical representation of nuclear irreversibility[11], the models of the structure of the pions and of the neutron[15], etc.

Suppose that the atomic state function is given by

$$\Psi = Ne^{-irs} \tag{3.12}$$

where r and s represent given physical quantities. Suppose that an operator A verifies the atomic eigenvalue law

$$A\Psi = i\frac{\partial}{\partial s}\Psi = r\Psi \ . \tag{3.13}$$

Suppose now that this law is implemented into the isotopic form with a scalar T quantity. Then we have

$$A * \Psi = i\frac{\partial}{\partial s} T\Psi = r'\Psi \quad . \tag{3.14}$$

Thus, the *mutated eigenvalue* r' is given by

$$r' = Tr \quad . \tag{3.15}$$

If $T = T(r,s)$, the mutated eigenvalue is given by the more general expression

$$r' = i\frac{\partial T}{\partial s} + Tr \tag{3.16}$$

where we have assumed that atomic and hadronic Hermiticity coincide. If this is not the case, a further, more general mutation occurs which can be computed via the rules of Section 2.6 and the explicit form of the T-quantity .

Note that, if the T-eigenvalues are used, then Eg. (3.14) can be written

$$A * \Psi = r_T' * \Psi \tag{3.17}$$

where

$$r_T' = r \ , \ r_T' = T^{-1}(i\frac{\partial T}{\partial s} + Tr) \ , \text{ etc.} \tag{3.18}$$

3.4. Hadronic-isotopic generalization of Schrödinger's representation. This representation was achieved in Paper I via the techniques of the Birkhoffian Mechanics[14]. Our objective here is that of identifying the underlying Hilbert space structure, that is, to achieve a formulation of the representation within the context of the Hilbert-isotopic theory of the preceding section.

It may be recommendable to review the most important steps. Consider an open nonselfadjoint system in first order form

$$(\dot{a}^\mu) - (\hat{\Xi}^\mu_{(t,a)}) = (\dot{a}^\mu) - (\Xi^\mu_{SA}) - (r^\mu_{NSA})$$

$$= \begin{pmatrix} \dot{\vec{r}}_k \\ \dot{\vec{p}}_k \end{pmatrix} - \begin{pmatrix} \vec{p}_k/m_k \\ \vec{F}^{SA}_k(\vec{r}) - \vec{F}^{NSA}_k(t,\vec{r},\vec{p}) \end{pmatrix}, \ a = (\vec{r},\vec{p}) \tag{3.19}$$

$$\mu = 1, 2, \cdots, 6N, \ k = 1, 2, \cdots, N$$

where the symbol "p" represents the physical linear momentum $(m\dot{\vec{r}})$. The physical angular momentum is then given by $\vec{J} = \vec{r} \times \vec{p}$. The energy H is that of the maximal selfadjoint subsystem (recall that the notion of potential energy has no physical basis for contact nonselfadjoint interactions).

By using the techniques of Ref.[14], the next step is that of constructing a representation of system (3.19) in terms of the semiautonomous Birkhoff's equations

$$\left\{ \left[\frac{\partial R_\nu(a)}{\partial a^\mu} - \frac{\partial R_\mu(a)}{\partial a^\nu} \right] \dot{a}^\nu - \frac{\partial B(t,a)}{\partial a^\mu} \right\}^{SA} = 0 \ , \tag{3.20}$$

$$\det \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \right) \neq 0$$

with underlying generalized variational principle of Pfaff's type

$$\delta A^g = \delta \int_{t_1}^{t_2} [R_\mu(a) da^\mu - B(t,a)dt] \tag{3.21}$$

$$\overset{\text{def.}}{=} \delta \int_{t_1}^{t_2} [P_k(\vec{r},\vec{p})dr^k + Q^k(\vec{r},\vec{p})dp_k - B(t,\vec{r},\vec{p})^{dt}] = 0 \ .$$

The representation is called of "semiautonomous" type to indicate that the geometric (symplectic) tensor of the theory

$$\Omega_{\mu\nu}(a) = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \tag{3.22}$$

does not depend explicitly on time in order to be consistent with the conventional geometric realizations in local coordinates of the most general possible (exact) symplectic two-forms

$$\Omega_2 = \frac{1}{2}\,\Omega_{\mu\nu}(a)da^\mu \wedge da^\nu \ . \tag{3.23}$$

The dependence of the vector field on time is then represented via the explicit time dependence of the Birkhoffian B .

Under topological restrictions inessential here, all systems (3.19) admit a representation in terms of Equations (3.20) in the local variables of the observer. This is the so-called *direct universality of Birkhoff's equations*.

Note that the Hamiltonian representations occur as a particular case of the Birkhoffians when

$$(R_\mu) \overset{\text{def.}}{=} (R^0_\mu) = (\vec{p},\vec{0}) \tag{3.24}$$

in which case the Birkhoffian and the Hamiltonian coincide.

By using the degrees of freedom characterized by the Birkhoffian guages

$$R_\mu \to R^+_\mu = R_\mu + \frac{\partial G}{\partial a^\mu} \ ; \ B \to B^+ = B - \frac{\partial G}{\partial t} \ . \tag{3.25}$$

Equations (3.20) can be turned into an equivalent form verifying the additional condition

$$\det \left(\frac{\partial R_\mu}{\partial a^\nu}\right) \neq 0 \tag{3.26}$$

under which the following Birkhoffian generalization of the Hamilton-Jacobi theory exists (Paper I).

$$\frac{\partial A^g}{\partial t} + \mathcal{B}(t,R(a)) = 0 \ , \tag{3.27a}$$

$$R_\mu(a) = \frac{\partial A^g}{\partial a^\mu} \ , \quad \mathcal{B}(t,R) = B(t,a(R)) \ . \tag{3.27b}$$

The Hamiltonian particularization is now given by the guaged case

$$R^{0+}_\mu = -\frac{1}{2}\omega_{\mu\nu}a^\nu \ , \quad (\omega_{\mu\nu}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ . \tag{3.28}$$

The use of the naive quantization rules

$$-\mathcal{B} = \frac{\partial A^g}{\partial t} \to i\frac{\partial}{\partial t} \ , \tag{3.29a}$$

$$R_\mu = \frac{\partial A^g}{\partial a^\mu} \to \frac{1}{i}\frac{\partial}{\partial a^\mu} \overset{\text{def.}}{=} \mathcal{R}_\mu \tag{3.29b}$$

finally permitted in Paper I the identification of the following generalization of Schrödinger's equation

$$i\frac{\partial}{\partial t}\Psi(t,a) = \mathcal{B}(t,R)\,\Psi(t,a)\ , \tag{3.30a}$$

$$\mathcal{R}_\mu\Psi(t,a) = \frac{1}{i}\frac{\partial}{\partial a^\mu}\Psi(t,a) = R_\mu\Psi(t,a)\ . \tag{3.30b}$$

The reader should keep in mind the necessity that the hadronic wave function Ψ depends on both, the space coordinates $\vec{r}^k$ *and* linear momentum $\vec{p}_k$ (as well as on time), as pointed out in Paper I. The reformulation of the conventional Schrödinger's equations via Birkhoff's guages (3.28) to achieve an equivalent theory in terms of the atomic wave functions $\Psi(t,\vec{r},\vec{p})$ should be kept also in mind.

Our task is now that of interpreting Equations (3.30) in terms of the isotopic Hilbert space theory of the preceding section. For this purpose we note that the classical equations (3.20) are characterized by

(1) The 2n functions $R_\mu(a)$ which essentially characterize the underlying symplectic isotopy or Lie-isotopy, that is, the replacement of the exact canonical, symplectic structure with the most general possible exact symplectic structure (*symplectic isotopy*[14])

$$\omega_2 = \frac{1}{2}\omega_{\mu\nu}da^\mu\wedge da^\nu = dp_k\wedge dr^k \to \Omega_2 = \frac{1}{2}\Omega_{\mu\nu}(a)da^\mu\wedge da^\nu \tag{3.31}$$

or, equivalently, the replacement of the fundamental Poisson brackets with the most general possible Lie realization of fundamental brackets in mechanics (*Lie isotopy*)[14]

$$\left([a^\mu,a^\nu]\right) = \begin{pmatrix} [r^i,r^j] & [r^i,p_j] \\ [p_i,r^j] & [p_i,p_j] \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = (\omega^{\mu\nu})$$

$$= \left(\left|\left|\frac{\partial R^0_\alpha}{\partial a^\beta} - \frac{\partial R^0_\beta}{\partial a^\alpha}\right|\right|^{-1}\right)^{\mu\nu} \to \quad ([a^\mu,a^\nu]^*) = (\Omega^{\mu\nu}(a)) \tag{3.32}$$

$$= \left(\left|\left|\frac{\partial R_\alpha}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\alpha}\right|\right|^{-1}\right)^{\mu\nu}\ .$$

(2) The Birkhoffian function $B(t,a)$ which is the generator of the noncanonical but generalized canonical time evolution[14]

$$a(t) = e^{t\Omega^{\mu\nu}(a)\frac{\partial B(t,a)}{\partial a^\nu}\frac{\partial}{\partial a^\mu}\Big|_{a(t_0)}}_{\hat{\mathcal{R}}} \tag{3.33}$$

(where $\hat{\mathcal{R}}$ is the conventional associative algebra);

(3) The preservation of the underlying carrier space of the Hamiltonian Mechanics (the cotangent bundle T^*M although without the conventional meaning of phase space ; it is called instead the *dynamical space* [14]).

In the transition to the Hadronic Mechnaics, the following possibilities to characterize the generalized Schrödinger's representation appear plausible.

(1*) Identification of the generalized unit operator $I^* = T^{-1}$ needed replace the conventional Heisenberg's commutation rules

$$([a^\mu,a^\nu]) = (a^\mu Ta^\nu - a^\nu Ta^\mu) = i(\omega^{\mu\nu})\,I$$

$$= i\left(\left|\left|\frac{\partial R^0_\alpha}{\partial a^\beta} - \frac{\partial R^0_\beta}{\partial a^\alpha}\right|\right|^{-1}\right)^{\mu\nu} = i\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\ ,\ T = \hbar^{-1}I \tag{3.34}$$

$$a = (r,p)$$

(where now a, R^0 , etc. are operators) with their isotopic form

$$([a^\mu,a^\nu]^*) = (a^\mu T a^\nu - a^\nu T a^\mu) = i\Omega^{\mu\nu}(a)$$

$$= i\left(\left|\left|\frac{\partial R_\alpha}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\alpha}\right|\right|^{-1}\right)^{\mu\nu}, \quad T = T(a) \quad . \tag{3.35}$$

(2*) Identification of the T-Hermitian Birkhoffian operator B which characterizes a nonunitary, T-unitary time evolution according to structure (2.60), i.e.,

$$U_T(t) = e_T^{itB} = I^* e^{iBTt} = I^* e^{iB*t} \quad . \tag{3.36}$$

(3*) Preservation of the Hilbert space of the Atomic Mechanics, although interpreted in the generalized form of Definition 2.23. It should be stressed that remarks (1*)-(3*) are introduced here on pure grounds of *similarities* with the classical cases (1)-(3). We assume the reader is familiar with the noncanonical character of the Birkhoffian time evolution (3.33), and the consequently necessary nonunitary character of the hadronic time evolution.

These occurrences render plausible the following third postulate.

POSTULATE III. *The time evolution of a hadronic state in the isotopic generalization of Schrödinger's representation is characterized by the generalized unit operator* $I^* = T^{-1}$ *and by the T-unitary time evolution induced by a T-Hermitian Birkhoffian operator* B *within the context of the isotopic formulation of the Hilbert space.*

Let $|t_0>$ be the hadronic state vector at time t_0. According to Postulate III, the state vector at a later time t is given by

$$|t> = U(t,t_0) * |t_0> = U(t,t_0)T|t_0> \tag{3.37}$$

where $U(t,t_0)$ admits representation (3.36) and, in particular, it is such that

$$U(t,t-\delta t) \cong I^* + \frac{i\delta t B}{1!} \quad . \tag{3.38}$$

Since T-unitary transformations form a Lie-isotopic group, they verify property (2.62), i.e.,

$$\begin{aligned} &U(t,t-\delta t) * U(t-\delta t,t_0) = U(t,t_0) \\ &= (I^* - i\delta t B) * U(t-\delta t,t_0) \quad . \end{aligned} \tag{3.39}$$

We can then write

$$\begin{aligned} &\lim_{\delta t\to 0} \frac{U(t,t_0) - U(t-\delta t,\, t_0)}{\delta t} = \frac{\partial}{\partial t} U(t,t_0) \\ &= -iB * U(t,t_0) \end{aligned} \tag{3.40}$$

and, by recalling (3.37) we finally have the desired isotopic interpretation of Equation (3.30a), i.e.,

$$i\frac{\partial}{\partial t}|t> = B * |t> = BT|t> \quad . \tag{3.41}$$

This essentially implies the following interpretation of the Birkhoffian operator $\mathfrak{B}$ as emerging from the naive quantization rules (3.29a)

$$\mathfrak{B} = BT \quad . \tag{3.42}$$

The Hamiltonian particularization is intriguing, inasmuch it can be written

$$i\frac{\partial}{\partial t}|t> = H * |t> = HT|t> \quad , \quad T = \hbar^{-1} \tag{3.43}$$

by therefore confirming that the conventional Schrödinger's representation has indeed an isotopic structure, although of trivial character, as it was the case for Equations (3.34).

This latter point can be made more precise by passing to the isotopic interpretation of the remaining equations (3.30b). Consider the atomic wave structure

$$\Psi(r) = Ne^{\frac{i}{\hbar}pr} = Ne^{iKr}, \quad K = \frac{p}{\hbar} \quad . \tag{3.44}$$

It can be written in the trivial isotopic form

$$\Psi(r) = Ne^{ip*r} = Ne^{ipTr} = Ne^{ik_T*r}, \quad T = \hbar^{-1} \quad . \tag{3.45}$$

Similarly, the atomic eigenvalue equation for the linear momentum

$$p\Psi(r) = -i\hbar\frac{\partial}{\partial r}\Psi(r) = K\Psi(r) \tag{3.46}$$

can be turned into the trivial isotopic form

$$p * \Psi(r) = -i\frac{\partial}{\partial r}\Psi(r) = K_T * \Psi(r) \quad . \tag{3.47}$$

By using these guidelines, we therefore postulate that the hadronic generalization of structure (3.45) is given by

$$\Psi(a) = Ne^{iR_\mu * a^\mu} = Ne^{i\alpha_{T\mu} * a^\mu} \quad . \tag{3.48}$$

The naive quantization rules (3.29b) can then be reformulated in the isotopic form

$$R_\mu * \Psi(a) = -i\frac{\partial}{\partial a^\mu}\Psi(a) = \alpha_{T\mu} * \Psi(a) \quad . \tag{3.49}$$

Under the assumption that the new unit $I^* = T^{-1}$ is known, the operator $\mathcal{R}_\mu$ originating from the naive quantization (3.29b) can be decomposed according to the rule

$$\mathcal{R}_\mu = R_\mu T \quad . \tag{3.50}$$

To see the joint equations (3.30), recall that the complete atomic wave packet for a free particle is given by

$$\Psi(t,a) = Ne^{i(p*r - H*t)} = Ne^{i(K_T * r - E_T * t)} \tag{3.51}$$

The conventional Schrödinger's equation is then justified on grounds of the identities

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi &= E_T * \Psi \quad , \\ p * \Psi &= K_T * \Psi \quad , \\ H = \frac{p^2}{2m} \rightarrow i\frac{\partial}{\partial t}\Psi &= H * \Psi \quad . \end{aligned} \tag{3.52}$$

The Hadronic generalization is straightforward. First, we shall write the *hadronic wave packet* in the form

$$\Psi(t,\vec{r},\vec{p}) = \Psi(t,a) = Ne^{i(R * a - B* t)}$$

$$= Ne^{i(\vec{P} * \vec{r} + \vec{Q} * \vec{p} - B * t)} \qquad (3.53)$$

$$= Ne^{i(\vec{\alpha}_T * \vec{r} + \vec{\beta}_T * \vec{p} - \gamma_T * t)} = Ne^{i(\rho_T * a - \gamma_T * t)}$$

by therefore recovering the form achieved in Paper I. Equations (3.30a) or (3.41) are then readily obtained via a straightforward generalization of (3.52), i.e.,

$$i\frac{\partial}{\partial t}\Psi = \gamma_T * \Psi \ ,$$

$$-i\frac{\partial}{\partial a^\mu}\Psi = \rho_T^\mu * \Psi \ , \qquad (3.54)$$

$$B = B(t,R) \to i\frac{\partial}{\partial t}\Psi = B * \Psi \ .$$

The advancement in the transition from the atomic to the hadronic wave packet is selfevident. In fact, *the atomic wave packet (3.51) represents a free particle, while the hadronic wave packet (3.53) represents a particle under the most general possible collection of local, potential, and nonpotential interactions.*

Recall that wave packet (3.51) is, in actuality, an approximation of the form

$$\Psi(t,r) = Ne^{iT\int_{t_o}^{t}(pdr - Hdt)} \ , \ T = \hbar^{-1} \ . \qquad (3.55)$$

Equivalently, we can say that the phase of the atomic wave packet is characterized by the canonical action functional.

On similar grounds, wave packet (3.53) should be considered an approximation of the more adequate form

$$\Psi(t,a) = Ne^{iT\int_{t_o}^{t}(R_\mu da^\mu - Bdt)} \qquad (3.56)$$

in which the phase is characterized by the action of Birkhoffian Mechanics, (with the understanding that the nature of T needs specific study).

To avoid possible misrepresentations, we should stress that the quantities "r" and "p" are no longer canonically conjugate in the Hadronic Mechanics. In fact, the fundamental brackets (3.34) are mutated into the isotopic form (3.35).

Intriguingly, there are new quantities which may be formally interpreted as "canonically conjugated". In fact, a comparison of the Hamiltonian and Birkhoffian actions

$$A_{Ham.} = \int_{t_o}^{t}(pdr - Hdt) \ , \ A_{Birk.} = \int_{t_o}^{t}(Rda - Bdt) \qquad (3.57)$$

reveals the possibility that the quantities "a" and "R" may be interpreted as "canonically conjugated", again, under regularity condition (3.26) to ensure their independence. In fact, we may formally introduce the "conventional" commutation rules.

$$[a^\mu, R^\nu] = a^\mu R^\nu - R^\nu a^\mu = i\delta^{\mu\nu}, \quad \mu,\nu = 1,2, \ldots, 2n \qquad (3.58)$$

with the understanding that equations (3.58) now imply

$$[r^i, p_j] = r^i p_j - p_j r^i \cong 0 \tag{3.59}$$

Note that, if the atomic operators r and p are 2n , the corresponding hadronic operators a and R are 4n.

These remarks may assist the interested researcher for further studies in the construction of the Hadronic Mechanics, as well as for specific applications.

Until now we have been interested in studying the formal structure of the isotopic generalization of Schrödinger's representation. Its explicit construction demands the selection of the identity and the Birkhoffian operators. A formal solution was given in Section IV of Paper I, and it can be reviewed according to the steps summarized below with their isotopic formulation.

STEP 1. *Identify the desired Newtonian system (3.19) and compute its Birkhoffian representation in the variables* $a = (\vec{r}, \vec{p})$ *with* $\vec{p}_k = m\dot{\vec{r}}_k$
This implies, in particular, the identification of the (2n + 1) functions

$$(R_\mu) = R_\mu(a)) = (\vec{P}_k(\vec{r},\vec{p}), \vec{Q}_k(\vec{r},\vec{p})) \; ; \; B = B(t,a) = B(t,\vec{r},\vec{p}) \; ,$$

$$\mu = 1,2, \cdot\cdot\cdot, 2n = 6N, \; k = 1,2, \cdot\cdot\cdot, N \; , \tag{3.60}$$

as well as of the energy E , angular momentum J , and all needed physical quantities

$$E = \sum_k \frac{\vec{p}_k^2}{2m_k} + V(\vec{r}) \; , \; \vec{J} = \sum_k \vec{r}_k \times \vec{p}_k \; , \text{ etc.} \tag{3.61}$$

STEP 2. *Upon verification of regularity condition (3.26) via guage (3.25), compute the dependence of the a-variables in the R-ones, i.e., identify* $a = a(R)$, *and reduce all functional dependences to the R-variables,*

$$(a^\mu) = (a^\mu(R)) = (\vec{r}^k_k(\vec{P},\vec{Q}) \; , \; \vec{p}_k(\vec{P},\vec{Q})) \; ,$$

$$B(t,a(R)) = \mathcal{B}(t,R) = \mathcal{B}(t,\vec{P},\vec{Q}) \; , \tag{3.62}$$

$$E = \sum_k \frac{\vec{p}_k^2}{2m_k} + V(\vec{r}) = \sum_k \frac{\vec{p}_k(\vec{P},\vec{Q}) \cdot \vec{p}_k(\vec{P},\vec{Q})}{2m_k} + V'(\vec{P},\vec{Q}) \; , \text{ etc.}$$

STEP 3. *Perform the naive (nonisotopic) quantization (3.29), and identify the operational form of* $a(R)$, *i.e.,*

$$\mathcal{R}_\mu = \frac{1}{i}\frac{\partial}{\partial a^\mu} \; , \; \mathcal{B} = i\frac{\partial}{\partial t} \; . \tag{3.63}$$

STEP 4. *Compute the operator image of the tensor* $\Omega^{\mu\nu}$ *and search for the operator T such that*

$$[a^\mu, a^\nu]^* = a^\mu T a^\nu - a^\nu T a^\mu = i\Omega^{\mu\nu} \tag{3.64}$$

where

$$a^\mu = (a^\mu)^{hT} \; . \tag{3.65}$$

STEP 5. *Perform the isotopic reformulation of the eigenvalue equations obtained via the naive quantization*

$$i\frac{\partial}{\partial t}\Psi(t,a) = B * \Psi(t,a) \tag{3.66}$$

where

$$B = \mathcal{B}\, T^{-1} = B^{hT} \tag{3.67}$$