

## Elaboration of the recently proposed test of Pauli's principle under strong interactions

Christos N. Ktorides\*

*Harvard University, Science Center, Cambridge, Massachusetts 02138*

Hyo Chul Myung

*University of Northern Iowa, Department of Mathematics, Cedar Falls, Iowa 50613*

Ruggero Maria Santilli

*Harvard University, Department of Mathematics, Cambridge, Massachusetts 02138*

(Received 4 January 1979; revised manuscript received 30 January 1980)

The primary objective of this paper is to stimulate the experimental verification of the validity or invalidity of Pauli's principle under strong interactions, according to a proposal which has recently appeared in the literature. For this objective, we first outline the most relevant steps in the evolution of the notion of particle, from the classical notion of massive point, to the quantum-mechanical notion of massive, spinning, and charged particle under electromagnetic interactions, as characterized by the Poincaré symmetry and as experimentally established. We then recall recent studies according to which this latter notion of particle might still need suitable implementations when referred to the additional presence of strong interactions. By recalling that no experimental evidence of direct, or final or unequivocal character is available at this moment on the value of the spin under strong interactions, the following hypothesis of these studies is recalled. It consists of the idea that the spin as well as other intrinsic characteristics of extended, massive, particles under electromagnetic interactions at large distances are subjected to a mutation under additional strong interactions at distances smaller than their charge radius. These dynamical effects can apparently be conjectured to account for the nonpointlike nature of the particles, their necessary state of penetration to activate the strong interactions, and the consequential emergence of broader forces which imply the breaking of the  $SU(2)$ -spin symmetry. Among the rather numerous technical problems which must be studied to reach a quantitative assessment of these ideas, in this paper we study a characterization of the mutated value of the spin via the transition from the associative enveloping algebra of  $SU(2)$  to a nonassociative Lie-admissible form. The departure from the original associative product then becomes directly representative of the breaking of the  $SU(2)$ -spin symmetry, the presence of forces more general than those derivable from a potential, and the mutated value of the spin. In turn, such a departure of the spin from conventional quantum-mechanical values implies the inapplicability of Pauli's exclusion principle under strong interactions, because, according to this hypothesis, particles that are fermions under long-range electromagnetic interactions are no longer fermions under these broader, short-range, forces. The case of nuclear physics is considered in detail. It is stressed that, in this case, possible deviations from Pauli's exclusion principle can at most be very small. A class of nuclei for the test considered is selected. It consists of all nuclei whose volume lies below the value predicted by the proportionality law of the nuclear volume with the total number of nucleons. These experimental data establish that, for the nuclei considered, nucleons are in a partial state of penetration of their charge volumes although of small statistical character. In turn, this state of penetration of the charge volumes activates the model of breaking of the  $SU(2)$ -spin symmetry reviewed in this paper.

The primary purpose of this paper is to stimulate the experimental resolution of the problem, recently proposed by one of the authors, of whether Pauli's principle is valid under strong interactions in the same quantitative measure as it is in the atomic structure, or deviations are experimentally detectable at the nuclear and hadronic levels.

The line of presentation which has been selected to achieve this objective is as direct as possible and consists of the study of whether a deviation from the conventional value of the spin of hadrons under strong interactions is conceivable, plausible, and quantitatively treatable. In turn, this possible deviation from the value of the spin im-

plies a corresponding deviation from Pauli's principle. The idea is that the most effective way to stimulate the experimental resolution of the problem is to study the plausibility of the inapplicability of Pauli's principle under strong interactions.

The hypothesis of a conceivable deviation from conventional values of the spin of hadrons under strong interactions is conjectural at this time, on both theoretical and experimental grounds. Again, this recent hypothesis is studied in this paper for the specific intent of stimulating the resolution of the problem considered at the experimental level as the only way for the sound conduction of physical studies.

We have attempted to write this paper in a self-

contained way, to be readable without a prior knowledge of the Birkhoffian and Lie-admissible generalization of mechanics.

### I. INTRODUCTION

The notion of "particle" has been the subject of a rather remarkable evolution since the beginning of this century. It is an easy prediction that this evolution has not reached a terminal stage with the currently accepted notion of "elementary particle," and that we are simply at one point in time of a continuing scientific process. Without any claim of completeness, we would like here briefly to recall the following three, compatible, lines of evolution.

(i) *Evolution of Galilei type.* The first nontrivial implementation occurred in the early part of this century in the transition from the classical notion of massive point by Galilei,<sup>1</sup> Newton,<sup>2</sup> and other authors to its quantum-mechanical extension, with particular reference to the de Broglie conception of wave structure.<sup>3</sup> This latter notion of particle resulted in being more adequate for the microscopic world. Nevertheless, it soon revealed considerable limitations. Indeed, clear experimental evidence of the atomic spectra subsequently forced the acceptance (not without initial skepticism) of a second nontrivial step: the addition of the intrinsic angular momentum or spin by Uhlenbeck and Goudsmit.<sup>4</sup> This second implementation allowed the achievement, via a body of contributions ranging over the first part of this century, of the nonrelativistic model of the atomic structure as it is known nowadays. More recent studies by a number of authors have identified the fundamental role of the underlying relativity, the Galilei relativity,<sup>5</sup> for the proper characterization of the notion of particle. These efforts have finally produced the notion of nonrelativistic, quantum-mechanical, massive, charged, and spinning particle of the contemporary physical literature.

(ii) *Evolution of Einstein type.* The advent of Einstein's special relativity<sup>6</sup> implied a number of additional nontrivial steps in the evolution of the notion of particle. First, there was the achievement of the notion of classical relativistic particle<sup>7</sup> and then that of quantum-mechanical relativistic type<sup>8</sup> as well as that of Dirac's field-theoretical character.<sup>9</sup> The role of the underlying relativity was in this case identified by a pioneering paper by Wigner<sup>10</sup> (which preceded the studies of the corresponding Galilean case). In this way, and thanks to a body of contributions which are still forthcoming, we have reached the notion of relativistic, quantum-mechanical, massive, spinning, and charged particle which, for the case of the

*electromagnetic interactions*, has been experimentally established on rather solid grounds. The existence of still another line of evolution of the notion of particle, this time of gravitational character, should be kept in mind.

(iii) *Evolution of technical character.* Despite the physical consistency of the notions of particles along lines 1 and 2, much remained to be done to reach a full technical characterization. A remarkable physical and mathematical maturity has been reached in this respect for both the Galilei and Einstein approaches via the use of the symplectic geometry of Souriau,<sup>11</sup> Guillemin and Sternberg,<sup>12</sup> and other mathematicians. Despite a rather considerable increase of the technical difficulties, remarkable progress has also been achieved in the quantum field-theoretical, axiomatic approach<sup>13</sup> and the constructive field theory.<sup>14</sup>

Therefore we can conclude by saying that nowadays we possess a notion of (classical and quantum mechanical, discrete and continuous, relativistic and nonrelativistic) massive, charged, and spinning particle *under electromagnetic interactions* which not only possesses unequivocal experimental verifications, but has also reached a remarkable maturity of mathematical formulation. In particular, we can nowadays claim to have achieved the solution of the problem of the atomic structure because we possess a theoretically and experimentally consistent quantitative characterization of the atomic constituents which, in this case, are precisely massive, charged, and spinning particles under (long range) electromagnetic interactions.

Despite these achievements, the currently available notion of particle is expected to constitute only a given stage in a continuing scientific process and to exhibit its own limitations when applied to yet broader physical arenas, such as the joint electromagnetic and strong interactions experienced by a particle within the core of a neutron star or, along rather similar lines, by a constituent of a hadron. In particular, our lack of achievement until now in arriving at the final solution of the problem of the hadronic structure might well be due to the fact that we simply do not possess at this time a theoretically and experimentally established notion of particle under strong interactions.

This latter aspect has been recently studied by Santilli<sup>15,16</sup> who has pointed out the possible inapplicability of the contemporary notion of particle under strong interactions, and the possible need for new physical and mathematical generalizations. Subsequently, the problem has been studied by a number of authors, in particular, see Ref. 17.

The most salient aspects of these studies are

be summarized as follows. The Galilei and Einstein notions of particle essentially characterize a particle under action-at-a-distance interactions, that is, with forces derivable from a potential (as typical of the electromagnetic interactions), via an articulated body of compatible mathematical tools of analytic, algebraic, and geometric character.

At the classical discrete level, these tools are expressible via Hamilton's equations without external terms (analytic profile), with underlying Lie algebra (algebraic profile), and symplectic structure (geometric profile), i.e.,

$$\dot{a}^\mu - \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu} = 0, \quad a = (r, p),$$

$$H = T + V, \quad \mu = 1, 2, \dots, 2n, \quad (1.1a)$$

$$[A, B]_{cl} = \frac{\partial A}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial B}{\partial a^\nu} = \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k} \frac{\partial A}{\partial p_k}, \quad (1.1b)$$

$$\omega_2 = \frac{1}{2} \omega_{\mu\nu} da^\mu \wedge da^\nu = dp_k \wedge dr^k = dR_1^0, \quad (1.1c)$$

$$\omega_{\mu\nu} = \frac{\partial R_\nu^0}{\partial a^\mu} - \frac{\partial R_\mu^0}{\partial a^\nu}, \quad R^0 = (p, 0),$$

$$\omega^{\mu\nu} = (\|\omega_{\mu\nu}\|^{-1})^{\mu\nu} \quad (1.1d)$$

The corresponding quantum-mechanical formulation in terms of operators  $\hat{a} = (\hat{r}, \hat{p})$  and  $\hat{H}$  is furnished by Heisenberg's equations

$$\dot{\hat{a}}^\mu - \frac{1}{i} [\hat{a}^\mu, \hat{H}] = 0, \quad [\hat{a}^\mu, \hat{a}^\nu] = i\omega^{\mu\nu}, \quad \hbar = 1, \quad (1.2)$$

It is known that formulations (1.1) can represent only *part* of Newtonian mechanics in a direct way, that is, without equivalence transformations. In fact, Newtonian forces are generally nonderivable from a potential. This occurrence creates the need of a generalization of the Hamiltonian mechanics, first, for *local*, generally nonpotential forces, and second, for nonlocal unrestricted forces.

A generalization of the Hamiltonian mechanics capable of achieving direct universality for local Newtonian systems is provided by the *Birkhoffian mechanics*, as treated in details in monographs<sup>18</sup> (see also the original proposal<sup>19</sup> and the more recent review<sup>20</sup>). The terms "direct universality" express the capability of the Birkhoffian mechanics of representing, under sufficient smoothness conditions, all local Newtonian systems (universality) in the coordinates of their experimental detection (direct universality).

On mathematical grounds the generalization of the Hamiltonian into the Birkhoffian mechanics is characterized by the following well-defined transition:

(1) from the conventional Hamilton's variational principle in phase space (Hamilton's equations) to

the most general possible variational principle for first-order systems (Birkhoff's equations);

(2) from the realization of the Lie-algebra product in terms of the Poisson brackets to the most general possible realization of the Lie-algebra product in Newtonian mechanics; and

(3) from the fundamental symplectic structure to the most general possible (but exact) symplectic structure (in local coordinates), that is, by the generalization of Eqs. (1.1) into the form for the autonomous case (see Refs. 18-20 for the non-autonomous case)

$$\dot{a}^\mu - \Omega^{\mu\nu}(a) \frac{\partial B(a)}{\partial a^\nu} = 0, \quad \det(\Omega^{\mu\nu}) \neq 0, \quad (1.3a)$$

$$[A, B]^* = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu} \frac{\partial B}{\partial a^\nu}, \quad (1.3b)$$

$$\Omega_2 = \frac{1}{2} \Omega_{\mu\nu} da^\mu \wedge da^\nu = dR_1, \quad (1.3c)$$

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}, \quad R_\mu \neq R_\mu^0,$$

$$\Omega^{\mu\nu} = (\|\Omega_{\mu\nu}\|^{-1})^{\mu\nu} \quad (1.3d)$$

A generalization of the Hamiltonian mechanics is predictably expected to lead to a generalization of the Heisenberg mechanics. In fact, the following quantization of Eqs. (1.3) has been proposed in Ref. 15 (see Ref. 20 for a more recent account)

$$\dot{\hat{a}}^\mu - \frac{1}{i} [\hat{a}^\mu, \hat{B}]^* = 0, \quad [\hat{a}^\mu, \hat{a}^\nu]^* = i\bar{\Omega}^{\mu\nu}(\hat{a}), \quad (1.4a)$$

$$[\hat{A}, \hat{B}]^* = \hat{A}\hat{C}\hat{B} - \hat{B}\hat{C}\hat{A}. \quad (1.4b)$$

As indicated earlier, Eqs. (1.3) have been proposed for the case of local systems.<sup>18</sup> A generalization of the Birkhoffian mechanics for nonlocal systems has been identified in Ref. 19 (see Ref. 20 for a review). It is characterized by the transitions

(a) from Hamilton's equations without external terms to the equations originally conceived by Hamilton, those with external terms;

(b) from the Lie algebras to the covering Lie-admissible algebras; and

(c) from the symplectic geometry to the covering symplectic-admissible geometry; according to the equations<sup>15,20</sup>

$$\dot{a}^\mu - \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu} - F^\mu = \dot{a}^\mu - S^{\mu\nu}(a) \frac{\partial H}{\partial a^\nu} = 0,$$

$$\det(S^{\mu\nu}) \neq 0, \quad (1.5a)$$

$$(A, B) = \frac{\partial A}{\partial a^\mu} S^{\mu\nu}(a) \frac{\partial B}{\partial a^\nu}, \quad (1.5b)$$

$$S_2 = S_{\mu\nu}(a) da^\mu \otimes da^\nu$$

$$= \frac{1}{2} (S_{\mu\nu} - S_{\nu\mu}) da^\mu \wedge da^\nu + \frac{1}{2} (S_{\mu\nu} + S_{\nu\mu}) da^\mu \times da^\nu$$

$$\equiv \Omega_2 + T_2, \quad dS_2 \neq 0, \quad d\Omega_2 = 0, \quad (1.5c)$$

$$S_{\mu\nu} - S_{\nu\mu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}, \quad \det(S_{\mu\nu} - S_{\nu\mu}) \neq 0, \quad (1.5d)$$

$$S^{\mu\nu} - S^{\nu\mu} = \left( \frac{\partial R'_\nu}{\partial a^\mu} - \frac{\partial R'_\mu}{\partial a^\nu} \right)^{-1}, \quad S^{\mu\nu} = (\|S_{\mu\nu}\|^{-1})^{\mu\nu}. \quad (1.5e)$$

Equations (1.5a), called *Hamilton-admissible equations*, have been proved to be directly universal for the most general (unconstrained) Newtonian systems known at this time, the nonlocal variationally non-self-adjoint systems, that is, the systems with a superposition of local and nonlocal forces derivable and nonderivable from a potential.<sup>20</sup> The local part of the system derivable from a potential is represented via the "Lie content" (or the symplectic content), that is, by the antisymmetric component of the product (two-form), while the departure from these familiar formulations (given by the symmetric part of the product or of the two-form) represents the nonlocal component of the system as well as the forces nonderivable from a potential.

A quantum-mechanical version of Eqs. (1.5) has also been identified, and it is given by<sup>15,20</sup>

$$\dot{\bar{a}}^\mu - \frac{1}{i}(\bar{a}^\mu, \bar{H}) = 0, \quad (\bar{a}^\mu, \bar{a}^\nu) = i\bar{S}^{\mu\nu}(\bar{a}), \quad (1.6a)$$

$$(\bar{A}, \bar{B}) = \bar{A}\bar{R}\bar{B} - \bar{B}\bar{S}\bar{A}, \quad \bar{R} \neq \pm \bar{S}, \quad \bar{R}, \bar{S} = \text{fixed}. \quad (1.6b)$$

The possible relevance of formulations (1.3)–(1.6) for the strong interactions is the following.<sup>15,20</sup> According to established experimental evidence, the size (charge radius) of all hadrons is approximately the same, and it coincides with the range of the strong interactions. A necessary condition for hadrons to activate the strong interactions is, therefore, that they enter into a state of mutual penetration (or overlapping of the wave packets). This is along the lines of the rather old expectation that the strong interactions are nonlocal, in which case they belong to the arena of applicability of the generalized Lie-admissible formulations (1.5) and (1.6). Nevertheless, nonlocal forces are known to be well approximated by local nonpotential forces (e.g., polynomial expansions in the velocities). As a result, under a local approximation, the strong interactions are expected to belong to the arena of applicability of the generalized Lie formulations (1.3) and (1.4).

The possible relevance of formulations (1.3)–(1.6) for the notion of particle can further be put into focus via the following considerations. The conventional notion of particle under action at a distance already mentioned (self-adjoint forces) can be quantitatively characterized via the applicable relativity, Galilei's or Einstein's (special). In turn, these relativities are rather crucially de-

pendent on Lie's theory in its conventional formulation compatible with Hamilton's and Heisenberg's equations. In particular, the integrated version of Hamilton's (Heisenberg's) equations is the time component of the classical (quantum-mechanical) Galilei relativity according to the familiar canonical realization for the classical case

$$G_1(t): a(t) = \exp\left(t\omega^{\mu\nu} \frac{\partial H}{\partial a^\nu} \frac{\partial}{\partial a^\mu}\right) a(0). \quad (1.7)$$

The quantum-mechanical extension is given by the unitary transformation

$$G_1(t): \bar{a}(t) = e^{i\bar{H}t} \bar{a}(0) e^{-i\bar{H}t}, \quad \bar{H} = \bar{H}^\dagger. \quad (1.8)$$

Owing to the deep link between dynamical equations, relativities, and the notion of particle, Santilli conjectured that the replacement of the dynamical equations with covering equations implies the replacement of conventional relativities with covering relativities for more general forces and dynamical conditions.<sup>19</sup> In fact, he first identified the integration of Eqs. (1.3) and (1.4) into the Birkhoffian form<sup>19,20</sup>

$$G_1^*(t): a(t) = \exp\left(t\Omega^{\mu\nu} \frac{\partial B}{\partial a^\nu} \frac{\partial}{\partial a^\mu}\right) a(0), \quad (1.9)$$

with quantum-mechanical image given by the (non-unitary) unitary-admissible transformations<sup>15,20</sup>

$$G_1^*(t): \bar{a}(t) = e^{i\bar{B}t} \bar{a}(0) e^{-i\bar{C}t},$$

$$\bar{B}^\dagger = \bar{B}, \quad [\bar{B}, \bar{C}] \neq 0$$

and then the integrations of Eqs. (1.5) and (1.6) into the Hamilton-admissible form<sup>19,20</sup>

$$\hat{G}_1(t): a(t) = \exp\left(tS^{\mu\nu} \frac{\partial H}{\partial a^\nu} \frac{\partial}{\partial a^\mu}\right) a(0), \quad (1.10)$$

with quantum-mechanical image given by the broader unitary-admissible form<sup>15,20</sup>

$$\hat{G}_1(t): \bar{a}(t) = e^{i\bar{H}t} \bar{a}(0) e^{-i\bar{R}t}, \quad \bar{H} = \bar{H}^\dagger, \quad (1.11)$$

$$[\bar{H}, \bar{R}] \neq 0, \quad [\bar{H}, \bar{S}] \neq 0$$

Santilli therefore conjectured that structures (1.8) and (1.9) are the time components of classical and quantum-mechanical coverings of Galilei's relativity, respectively, with 10-parameter extensions

$$G^*(3.1): \begin{cases} a(t) = \exp\left(\theta_k \Omega_k^{\mu\nu} \frac{\partial X_k}{\partial a^\nu} \frac{\partial}{\partial a^\mu}\right) a(0), & (1.12a) \\ \bar{a}(t) = e^{i\theta_k \bar{X}_k} \bar{a}(0) e^{-i\theta_k \bar{C}_k}, & (1.12b) \end{cases}$$

$$\bar{X}_k \in g(3.1), \quad \bar{C}_k \in A(g(3.1))$$

where  $g(3.1)$  is Galilei's algebra and  $A(g(3.1))$  its associative envelope (see below), and the  $\theta$ 's are the conventional parameters. Structures (1.12)

were called the *Lie-isotopic coverings*<sup>15,19,20</sup> of the canonical and unitary realizations of Galilei's relativity, respectively.

Along similar lines, structures (1.10) and (1.11) are interpreted as the time components of still more general, classical, and quantum-mechanical coverings of Galilei's relativity, with 10-parameter extensions

$$\hat{G}(3.1): \begin{cases} a(t) = \exp\left(\theta_k S_k^{\mu\nu} \frac{\partial X_k}{\partial a^\nu} \frac{\partial}{\partial a^\mu}\right) a(0), & (1.13a) \\ \bar{a}(t) = e^{i\theta_k \bar{X}_k \bar{S}_k} \bar{a}(0) e^{-i\theta_k \bar{R}_k \bar{X}_k}, & (1.13b) \end{cases}$$

$$\bar{X}_k \in g(3.1), \quad \bar{R}_k, \bar{S}_k \in A(g(3.1)),$$

called *Lie-admissible genotopic coverings* of the canonical and unitary realization of Galilei's relativity, or *Galilei-admissible relativity*.<sup>15,19,20</sup>

For completeness, it should be indicated here that the generalized dynamical Eqs. (1.3)–(1.6) and relativities (1.8)–(1.13) were derived by Santilli after working out the rudiments of corresponding generalizations of Lie's first, second, and third theorems, following a generalization of the Poincaré-Birkhoff-Witt theorem to nonassociative enveloping algebras studied by Ktorides.<sup>21</sup> For a review of these studies we refer the interested reader to Ref. 20, Sec. 1.2. The initiation of the representation theory of Lie-admissible algebras of operators on bimodular Hilbert spaces has been conducted in Ref. 22. The study of the integrability conditions of the generalized equations into unitary-admissible structures (as a generalization of the known Nelson's integrability conditions) has been initiated in Ref. 23.

Predictably, numerous technical problems must be resolved before reaching a quantitative assessment of the proposed generalizations of Galilei's relativity, with particular reference to their capability of characterizing a covering notion of particle. This paper is devoted to the study of the "spin" part of the Galilei-admissible relativity, i.e., the structure<sup>24</sup>

$$\begin{aligned} \text{su}(2)^\wedge: \bar{A}^1 &= e^{i\theta_k \bar{J}_k \bar{S}_k} \bar{A} e^{-i\theta_k \bar{R}_k \bar{J}_k}, \\ \bar{J}_k &\in \text{su}(2), \quad \bar{R}_k, \bar{S}_k \in A(\text{su}(2)). \end{aligned} \quad (1.14)$$

More specifically, we shall study the proposal<sup>15,20</sup> according to which, when a hadron performs the transition from

- (1) motion in vacuum under long-range electromagnetic interactions to
- (2) motion under strong interactions, that is, conditions of overlapping of its wave function with that of other hadrons (e.g., for a proton inside the core of a star),

we have the transition from

- (1') the exact SU(2)-spin-symmetry to
- (2') the broken SU(2)-spin symmetry under variationally non-self-adjoint (local or nonlocal) forces, which is quantitatively treatable via the SU(2)-admissible structure (1.14);

as well as the transition from

- (1'') the conventional magnitude of the spin computed via the associative Lie-admissible envelope  $A(\text{su}(2))$  (Ref. 25)

$$\begin{aligned} \|\bar{J}\|_{A(\text{su}(2))}^2 \psi &= (\bar{J}_1 \bar{J}_1 + \bar{J}_2 \bar{J}_2 + \bar{J}_3 \bar{J}_3) \psi \\ &= S(S+1) \psi, \quad S=0, \frac{1}{2}, 1, \dots \\ \bar{J}_i \bar{J}_j &= \text{associative} \end{aligned} \quad (1.15)$$

to

- (2'') a mutated value of the magnitude of the spin computed via the *nonassociative Lie-admissible envelope*  $U(\text{su}(2))$

$$\begin{aligned} \|\bar{J}\|_{U(\text{su}(2))}^2 \psi &= (\bar{J}_1 \circ \bar{J}_1 + \bar{J}_2 \circ \bar{J}_2 + \bar{J}_3 \circ \bar{J}_3) \psi = f(S) \psi, \\ \bar{J}_i \circ \bar{J}_j &= \bar{J}_i \bar{R}_j \bar{J}_j - \bar{J}_j \bar{S}_i \bar{J}_i = \text{nonassociative}, \end{aligned} \quad (1.16)$$

where the *departure* of the generalized product  $\bar{J}_i \circ \bar{J}_j$  from the conventional associative product  $\bar{J}_i \bar{J}_j$  is a measure of the SU(2)-spin symmetry-breaking forces.

In particular, our studies will be conducted for the case when the elements  $\bar{R}$  and  $\bar{S}$  belong to the field (i.e., are free parameters) in which case we have a *flexible Lie-admissible algebra*<sup>26-28</sup> with product

$$\bar{J}_i \circ \bar{J}_j = \lambda \bar{J}_i \bar{J}_j + \mu \bar{J}_j \bar{J}_i, \quad (1.17)$$

also known as the  $(\lambda, \mu)$  mutation of the associative algebra, according to the original proposals.<sup>29</sup>

The motivation of this article pointed out at the beginning now comes to light. In fact, features 2, 2', and 2'' imply the possibility that the spin of hadrons is mutated in the transition from the physical conditions of its current experimental detection (motion in vacuum under long-range electromagnetic interactions only) to the *different* physical conditions of motion "within hadronic matter" (e.g., under conditions of overlapping of the wave packets). In turn, a possible deviation from conventional values of the spin implies a corresponding inapplicability of Pauli's exclusion principle and other conventional laws. Still in turn, this confirms the need for the experimental verification of the validity or invalidity of Pauli's exclusion principle under strong interactions in a way independent from its known, exact validity for the electromagnetic interactions.

Our studies will be conducted as follows. In Sec. II we shall work out the necessary mathemat-

ical tools for an initial, yet quantitative treatment of the physical problem considered. Although not sufficiently emphasized in the physical literature, a rather crucial part of Lie's theory for the conventional treatment of spin (as well as other physical quantities) is the universal enveloping associative algebra  $A(\mathfrak{su}(2))$ . In fact, this algebra is essential for the definition of the magnitude  $\vec{J}^2 = J_i J_i$  of the spin. In addition,  $A(\mathfrak{su}(2))$  is essential for a number of technical problems, such as the transition from the Lie algebra  $\mathfrak{su}(2)$  to the Lie group  $SU(2)$ . As is well known, the basis of the enveloping associative algebra is characterized by the Poincaré-Birkhoff-Witt theorem.<sup>25</sup> When a Lie symmetry is broken, its enveloping associative algebra and related Poincaré-Birkhoff-Witt theorem cannot be used for physical calculations (because they would imply both an exact Lie algebra as well as an exact Lie group).<sup>19,20</sup> In this case, the use of a nonassociative envelope is possible, but then it becomes important to identify the new basis, that is, to work out the generalization of the Poincaré-Birkhoff-Witt theorem to a nonassociative envelope. Section II is devoted to the mathematical study of this problem, as a continuation of the studies initiated in Ref. 21. Section III is devoted to the application of the mathematical tools of Sec. II to the treatment of the spin of hadrons under conditions of penetration within hadronic matter and expected forces more general than the familiar potential forces. The paper concludes with remarks on the experimental resolution of the physical problem considered.

A few additional comments appear advisable. First, we would like to indicate that our studies apply to both the Birkhoffian and the Lie-admissible generalization of Heisenberg's equations. In fact, Eqs. (1.4b) admit a nonassociative envelope according to the rule of nonassociative Lie admissibility<sup>15</sup>

$$[\vec{A}, \vec{B}]^* = (\vec{A}, \vec{B}) - (\vec{B}, \vec{A}), \quad \vec{C} = \vec{R} + \vec{S}. \quad (1.18)$$

Second, we would like to indicate that the use of the theory of nonassociative algebras in general, and that of the Lie-admissible algebras in particular, is not a technical virtuosity, but it is actually needed for practical physical calculations. This is the case not only for nonpotential forces (as elaborated earlier), but also for conventional forces derivable from a potential, although of nonlinear character.

The latter comment calls for a brief elaboration. It refers to the known theorems of inconsistency of Heisenberg's quantization.<sup>20,30</sup> In essence, the envelope of Hamilton's equations is *not* an associative envelope, as in Heisenberg's equations. Instead, it is given by a nonassociative Lie-admis-

sible envelope with product

$$A \cdot B = \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k}. \quad (1.19)$$

As a result, the quantization for nonlinear systems

$$\left( \begin{array}{l} \text{Hamilton's equations} \\ \dot{A} = [A, H]_{cl} = A \cdot H - H \cdot A \\ A \cdot H = \text{nonassociative} \end{array} \right) \quad \rightarrow \quad \left( \begin{array}{l} \text{Heisenberg's equations} \\ \dot{\vec{A}} = \frac{1}{i} [\vec{A}, \vec{H}] = \frac{1}{i} (\vec{A}\vec{H} - \vec{H}\vec{A}) \\ \vec{A}\vec{H} = \text{associative} \end{array} \right),$$

despite its use for over half a century, is *inconsistent* because it violates the structure of the envelope. For brevity, we refer the interested reader to Refs. 20 and 30.

The above remark serves to illustrate the need of the theory of nonassociative algebras for physical calculations. It appears that, as a *necessary* condition to preserve the structure of Hamilton's equations, the quantum-mechanical equations should be constructed within the framework of nonassociative enveloping algebras. This is exactly the case of the proposed generalizations (1.4) and (1.6), as well as of the flexible case worked out by Okubo<sup>31</sup> for the quantum-mechanical profile, and by Ktorides<sup>32</sup> for the quantum-field-theoretical profile.

In conclusion, the theory of nonassociative algebras has lately emerged as being useful to achieve a consistent quantization of conventional (although nonlinear) systems with local forces derivable from a potential, and this situation simply persists for more general forces. In this sense, it appears that the relevance of the nonassociative Lie-admissible enveloping algebra of Sec. II goes beyond the spin aspect of an elementary particle. In the present study however, we shall focus our attention on the spin only.

## II. LIE-ADMISSIBLE UNIVERSAL ENVELOPING MUTATION ALGEBRAS

Given a Lie algebra  $L$  over an arbitrary field  $F$ . For a pair  $(\lambda, \mu)$  of fixed scalars  $\lambda, \mu \in F$ , we construct a flexible Lie-admissible algebra  $U_{\lambda, \mu} = U(L)_{\lambda, \mu}$  with unit element 1 such that  $L$  is isomorphically embedded into the attached Lie algebra  $U_{\lambda, \mu}^-$  and  $U_{\lambda, \mu}$  is generated by 1 and  $L$ . The algebra  $U_{\lambda, \mu}$  is in general nonassociative and is called a universal enveloping  $(\lambda, \mu)$ -mutation algebra of  $L$ . We also prove an analog of the well-known Poincaré-Birkhoff-Witt theorem for the al-

gebra  $U_{\lambda, \mu}$ . Though the discussion in this section applies to arbitrary Lie algebras, in the remainder of this paper any of the applications will be restricted to the Lie algebra  $su(2)$ .

#### A. Preliminaries

We recall some identities which are relevant to Lie admissibility. For a nonassociative algebra  $A$ , define  $A^-$  to be the algebra with multiplication  $[x, y] = xy - yx$  defined on the same vector space as  $A$ . Then  $A$  is said to be *Lie-admissible* if  $A^-$  is a Lie algebra, that is,  $A^-$  satisfies the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

The associative algebras are clearly Lie admissible. Various types of nonassociative Lie-admissible algebras, which arise in both algebraic and physical contexts, are discussed in Myung<sup>26, 27</sup> and Santilli.<sup>15, 19</sup>

Denote  $S(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y)$  where  $(x, y, z) = (xy)z - x(yz)$  is the associator in  $A$ . In any algebra, by a direct computation we have

$$[xy, z] + [yz, x] + [zx, y] = S(x, y, z).$$

From this we obtain

$$S(x, y, z) - S(x, z, y) = [[x, y], z] + [[y, z], x] + [[z, x], y],$$

which also holds in any algebra. Thus an algebra  $A$  is Lie admissible if and only if  $A$  satisfies the identity

$$S(x, y, z) = S(x, z, y). \quad (2.1)$$

This identity is called the *Lie-admissible law*. In particular, an algebra satisfying  $S(x, y, z) = 0$  for all  $x, y, z$  is Lie admissible.

Suppose that  $A$  is Lie admissible and satisfies third power-associativity  $x^2x = xx^2$  or  $(x, x, x) = 0$ . Albert<sup>23</sup> has shown that the linearization of  $(x, x, x) = 0$  implies

$$[xy + yx, z] + [yz + zy, x] + [zx + xz, y] = 0,$$

or equivalently

$$S(x, y, z) + S(x, z, y) = 0.$$

This with (2.1) implies  $S(x, y, z) = 0$ . Therefore, we have the following lemma.

**Lemma 2.1.** An algebra  $A$  of characteristic  $\neq 2$  satisfying  $x^2x = xx^2$  is Lie admissible if and only if  $A$  satisfies  $S(x, y, z) = 0$ .

There are some well-known algebras which satisfy third power-associativity. Recall that an algebra  $A$  is called *flexible* if the flexible law  $(xy)x$

$= x(yx)$  holds in  $A$ . The identity  $(xy)y = xy^2$  [or  $x(xy) = x^2y$ ] is called the *right* [or *left*] *alternative law*. Therefore, it is evident from Lemma 2.1 that any power-associative, flexible, right or left alternative algebra is Lie-admissible if and only if it satisfies  $S(x, y, z) = 0$ . While it is known that every right (or left) Lie-admissible algebra is power-associative,<sup>34</sup> flexible Lie-admissible algebras need not be power-associative.<sup>35</sup> Okubo<sup>36</sup> has recently constructed a simple, flexible Lie-admissible algebra which is not power-associative.

Let  $A^*$  denote the algebra with multiplication  $x \cdot y = \frac{1}{2}(xy + yx)$  defined on the same vector space as  $A$ . Let us recall that  $A$  is called *Jordan admissible* if  $A^*$  satisfies the Jordan identity

$$(x \cdot x) \cdot (y \cdot x) = (x \cdot x) \cdot y \cdot x.$$

Schafer<sup>37</sup> has shown that an algebra of characteristic  $\neq 2$  is flexible Jordan admissible if and only if  $A$  is flexible and satisfies the identity  $x^2(yx) = (x^2y)x$ . An algebra satisfying the latter identities is called a *noncommutative Jordan algebra*. An algebra can be both Lie and Jordan admissible. While these algebras of nonflexible type have been discussed in Myung,<sup>27</sup> the algebras we discuss in this paper form a class of flexible algebras which are both Lie and Jordan admissible. Note that an associative algebra is clearly flexible Lie and Jordan admissible. Let  $A$  be a flexible Lie-admissible algebra over a field of characteristic  $\neq 2$ . It was shown by Myung<sup>28</sup> that, for every  $x \in A$ , the linear mapping  $[, x]: y \rightarrow [y, x]$  is a derivation of  $A$ , that is,

$$[yz, x] = y[z, x] + [y, x]z, \quad (2.2)$$

for all  $x, y, z \in A$ . Conversely, let  $A$  be an algebra satisfying the identity (2.2). Setting  $y = x$  in (2.2), we have that  $[xz, x] = x[z, x]$  which implies the flexible law  $(xy)x = x(yx)$ . Since the mapping  $[, x]: y \rightarrow [y, x]$  is a derivation of  $A$ , it is also a derivation of the algebra  $A^-$ . Thus the Jacobi identity holds in  $A^-$  and so  $A^-$  is a Lie algebra. Therefore,  $A$  is flexible Lie admissible if and only if  $A$  satisfies (2.2).

#### B. Construction of the algebra

One of most significant flexible Lie-admissible algebras which arose in physical contexts is the  $(\lambda, \mu)$ -mutation algebra of an associative algebra. As indicated earlier, let  $B$  be an associative algebra over a field  $F$  of characteristic  $\neq 2$ . Let  $\lambda, \mu$  be two fixed independent scalars. Then the algebra  $B(\lambda, \mu)$ , called the  $(\lambda, \mu)$ -mutation of  $B$ , is defined on the same vector space as  $B$  but with multiplication given by

$$a \circ b = \lambda ab + \mu ba, \quad a, b \in B$$

where  $ab$  indicates the associative product in  $B$ . Denote  $[x, y]^0 = x \circ y - y \circ x$  and  $(x, y, z)^0 = (x \circ y) \circ z - x \circ (y \circ z)$  as the commutator and the associator in  $B(\lambda, \mu)$ . Then it can be easily computed that

$$[x, y]^0 = (\lambda - \mu)[x, y], \tag{2.3}$$

$$(a, b, c)^0 = \lambda \mu [[c, a], b] = \frac{\lambda \mu}{(\lambda - \mu)^2} [[c, a]^0, b]^0. \tag{2.4}$$

Setting  $a = c$  in (2.4) implies that  $B(\lambda, \mu)$  is flexible. If we put  $a^2 = c$ , then (2.4) implies the Jordan identity  $(a, b, a^2)^0 = 0$  in  $B(\lambda, \mu)$ . Thus  $B(\lambda, \mu)$  is flexible Jordan admissible. Also, it follows from (2.4) that  $(a, b, c)^0 + (b, c, a)^0 + (c, a, b)^0 = 0$ , since  $B$  is Lie admissible. Therefore, by lemma 2.1,  $B(\lambda, \mu)$  is also Lie admissible. It is also clear that the mapping  $[ , x]: y - [y, x]$  is a derivation both in  $B$  and  $B(\lambda, \mu)$ .

Let  $L$  be a Lie algebra over a field  $F$ . Let  $T(L)$  be the tensor algebra on  $L$ . By definition,

$$T(L) = F1 \oplus L_1 \oplus L_2 \oplus \dots \oplus L_n \oplus \dots,$$

where  $L_0 = F1$  and  $L_n = L \otimes \dots \otimes L$  ( $n$  times). Thus  $T(L)$  is a universal associative algebra generated by  $L$  and the unit element 1 in the sense that if  $f$  is a linear mapping of  $L$  into any associative algebra  $B$  with unit element 1, then  $f$  can be extended to a unique homomorphism  $f'$  of  $T(L)$  into  $B$  such that  $f'(1) = 1$ . Let  $R$  be the ideal of  $T(L)$  generated by the elements

$$[ab] - a \otimes b + b \otimes a, \quad a, b \in L$$

and let  $A = A(L) = T(L)/R$ . If  $j$  denotes the natural homomorphism of  $T(L)$  onto  $A(L)$ , then  $(A, j)$  becomes a *universal enveloping algebra* of  $L$  in the sense that if  $B$  is any associative algebra and  $f$  is a homomorphism of  $L$  into the Lie algebra  $B^-$ , then there exists a unique homomorphism  $f'$  of  $A$  into  $B$  such that  $f = f'j$ . Then the calculated Poincaré-Birkhoff-Witt theorem (PBW theorem) states that, given an ordered basis for  $L$ , the cosets of 1 and the standard monomials in this basis form a basis for  $A(L)$ . It is clear from the definition that  $j$  maps  $F$  isomorphically into  $A$ , so  $A$  contains the scalars. It is also immediate from the PBW theorem that  $j$  is injective on  $L$  and so is a faithful representation of  $L$ . Let  $\{u_i | i \in \Lambda\}$  be an ordered basis for  $L$  and identify  $u_i$  with  $j(u_i)$ . Write the product in  $A$  as  $xy$ . What the PBW theorem then amounts to is that the elements, called *standard monomials*,

$$1, u_{i_1} u_{i_2} \dots u_{i_r}, \quad i_1 \leq i_2 \leq \dots \tag{2.5}$$

form a basis for  $A$ . It can be shown that<sup>30</sup> any associative algebra with a basis consisting of all the standard monomials in (2.5) is a universal enveloping algebra of  $L$ .

Let  $T(L)(\lambda, \mu)$  be the  $(\lambda, \mu)$  mutation of the tensor algebra  $T(L)$  of  $L$ . For subspaces  $H, K$  of  $T(L)$ , as usual, denote  $H \circ K$  as the subspace spanned by the elements  $h \circ k$ ,  $h \in H, k \in K$ . Since the associative operation in  $T(L)$  is the tensor product " $\otimes$ ",

$$h \circ k = \lambda h \otimes k + \mu k \otimes h.$$

We define  $L'_n$  for  $n = 0, 1, 2, \dots$  inductively as

$$L'_0 = F, \quad L'_1 = L, \quad L'_2 = L_2, \quad L'_n = \sum_{i=1}^{n-1} L'_i \circ L'_{n-i}.$$

If we set

$$T_{\lambda, \mu} = T(L)_{\lambda, \mu} = F1 \oplus L'_1 \oplus L'_2 \oplus \dots \oplus L'_n \oplus \dots,$$

then  $T_{\lambda, \mu}$  is a subalgebra of  $T(L)(\lambda, \mu)$  since  $L'_i \circ L'_j \subseteq L'_{i+j}$ . In fact, it is readily seen that  $T_{\lambda, \mu}$  is precisely the subalgebra of  $T(L)(\lambda, \mu)$  generated by 1 and  $L$ . Thus  $T_{\lambda, \mu}$  inherits much of the algebraic properties in  $T(L)(\lambda, \mu)$ . In particular,  $T_{\lambda, \mu}$  is flexible and both Lie and Jordan admissible. Let  $I$  be the ideal of  $T_{\lambda, \mu}$  generated by the elements

$$[ab] - [a, b]^0 = [ab] - (\lambda - \mu)(a \otimes b - b \otimes a), \quad a, b \in L$$

where  $[ab]$  is the product in  $L$  and

$$[a, b]^0 = a \circ b - b \circ a = (\lambda - \mu)(a \otimes b - b \otimes a)$$

[by (2.3)] is the commutator in  $T_{\lambda, \mu} = T(L)_{\lambda, \mu}$ . We now form the quotient algebra

$$U_{\lambda, \mu} = U(L)_{\lambda, \mu} = T(L)_{\lambda, \mu} / I.$$

Let  $i$  be the natural homomorphism of  $T_{\lambda, \mu}$  onto  $U_{\lambda, \mu}$ , then clearly  $i$  maps  $F$  isomorphically into  $U_{\lambda, \mu}$  and hence  $U_{\lambda, \mu}$  contains the scalars. Note that  $U_{\lambda, \mu}$  is flexible Lie admissible and Jordan admissible, but not in general associative because of (2.4). Following Ktorides,<sup>21</sup> we call  $U_{\lambda, \mu}$  a *universal enveloping  $(\lambda, \mu)$ -mutation algebra* ( $UE(\lambda, \mu)$ -MA) of  $L$ . The present approach is different from Ktorides's original one but is in essence the same algebra as Ktorides constructed. In the remainder of this section we will discuss a generalization of the PBW theorem to the algebra  $U(L)_{\lambda, \mu}$ .

C. A generalization of the Poincaré-Birkhoff-Witt theorem

Ktorides<sup>21</sup> has obtained an analog of the PBW theorem for the algebra  $U_{\lambda, \mu}$ . A significance of this study from the mathematical point of view lies in the fact that the PBW theorem is extended for the first time to a flexible (nonassociative) Lie-admissible algebra. From the physical point of view the algebra  $U_{\lambda, \mu}$  resulted in the first construction of a possible nontrivial theory of quantum-



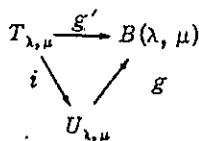
mechanical interacting fields.<sup>32</sup> When  $L$  is taken to be the Lie algebra  $\mathfrak{su}(3)$ , an extension of the Gell-Mann-Okubo mass formula to the algebra  $U_{\lambda,\mu}$  is possible.<sup>40,21</sup>

In the present paper we obtain even a closer analog of the PBW theorem for the algebra  $U_{\lambda,\mu}$  by showing that the natural homomorphism  $i: T_{\lambda,\mu} \rightarrow U_{\lambda,\mu}$  is injective on  $L$ . Thus  $L$  is faithfully represented into a nonassociative but Lie-admissible algebra. As we shall see shortly, this is an immediate consequence of the following theorem which is fundamental in our discussion.

**Theorem 2.2.** Let  $L$  be a Lie algebra over a field  $F$ , and  $B$  be any associative algebra with unit element over  $F$ . If  $f$  is a homomorphism of  $L$  into  $B$ , then there exists a unique homomorphism  $g$  of  $U_{\lambda,\mu}$  into  $B(\lambda, \mu)$  such that

$$\begin{aligned} g'([ab] - (\lambda - \mu)(a \otimes b - b \otimes a)) &= \frac{1}{\lambda - \mu} f([ab]) - (\lambda - \mu)g'(a \otimes b - b \otimes a) \\ &= \frac{1}{\lambda - \mu} [f(a), f(b)] - (\lambda - \mu)(g'(a)g'(b) - g'(b)g'(a)) \\ &= \frac{1}{\lambda - \mu} [f(a), f(b)] - \frac{1}{\lambda - \mu} [f(a), f(b)] = 0, \end{aligned}$$

since  $g$  is a homomorphism of  $T(L)$  into  $B$  and the restriction of  $g'$  to  $L$  is  $[1/(\lambda - \mu)]f$ . Thus  $g'$  vanishes on the generators of  $I$  and since  $g'$  is an algebraic homomorphism, we have  $g'(I) = 0$ . Therefore, there exists a unique homomorphism  $g$  of  $U_{\lambda,\mu} = T_{\lambda,\mu}/I$  into  $B(\lambda, \mu)$  such that the following diagram is commutative:



Thus  $gi = g'$  and the restriction of this to  $L$  gives  $gi = [1/(\lambda - \mu)]f$ . The uniqueness of  $g$  follows from the fact that  $i(L)$  generates the algebra  $U_{\lambda,\mu}$ . This completes the proof.

Theorem 2.2 generalizes theorem 2.2 of Ktorides.<sup>21</sup> In theorem 2.2, if one takes  $f$  to be a faithful representation of  $L$ , then it follows from (2.6) that  $i$  is injective on  $L$ . Note that  $L$  always has a faithful representation. Thus we have the following corollary.

**Corollary 2.3.** The natural homomorphism  $i: T_{\lambda,\mu} \rightarrow U_{\lambda,\mu}$  is injective on  $L$ .

Corollary 2.3 is a crucial step to obtain an analog of the PBW theorem for  $U_{\lambda,\mu}$ , whereas each element  $a \in L$  can be identified with the coset  $a + I$ . Also, corollary 2.3 resolves a question raised by Myung.<sup>27</sup> In view of this identification,  $U_{\lambda,\mu}$  is

$$gi = \frac{1}{\lambda - \mu} f, \tag{2.6}$$

where  $i$  is the natural homomorphism of  $T_{\lambda,\mu}$  into  $U_{\lambda,\mu}$  and  $B(\lambda, \mu)$  is the  $(\lambda, \mu)$ -mutation of  $B$  with  $\lambda, \mu$  distinct and both nonzero.

*Proof.* Since the tensor algebra  $T(L)$  is a universal associative algebra generated by 1 and  $L$ , as noted earlier the linear mapping  $[1/(\lambda - \mu)]f: L \rightarrow B$  can be extended to a unique homomorphism  $g'$  of  $T(L)$  into  $B$ , that is, the restriction of  $g'$  to  $L$  is  $[1/(\lambda - \mu)]f$ . Since  $g'$  is also a homomorphism of  $T(L)(\lambda, \mu)$  into  $B(\lambda, \mu)$ ,  $g'$  induces a homomorphism of  $T_{\lambda,\mu}$  into  $B(\lambda, \mu)$  [note that  $T_{\lambda,\mu}$  is the subalgebra of  $T(L)(\lambda, \mu)$  generated by 1 and  $L$ ].

We now examine the action of  $g'$  on the generators of the ideal  $I$ . For  $a, b \in L$ , we have

a flexible Lie-admissible algebra generated by 1 and  $L$ , and theorem 2.2 reads as follows: If  $f$  is a representation of any Lie algebra  $L$  into an associative algebra with unit element then the representation  $[1/(\lambda - \mu)]f$  of  $L$  into the flexible Lie-admissible algebra  $B(\lambda, \mu)$  can be extended to a unique homomorphism of  $U_{\lambda,\mu}$  into  $B(\lambda, \mu)$ . In view of this, calling  $U_{\lambda,\mu}$  a "universal" enveloping  $(\lambda, \mu)$ -mutation algebra of  $L$  is justified. Let  $\{u_i | i \in \Lambda\}$  be an ordered basis for a Lie algebra  $L$ . Since this basis and 1 generate  $U_{\lambda,\mu}$ , every element in  $U_{\lambda,\mu}$  is written as a linear combination of nonassociative monomials

$$1, u_{i_1} \circ u_{i_2} \circ \dots \circ u_{i_r}, \quad i_k \in \Lambda$$

in some association of the product. At this point it should be noted that the algebra  $U_{\lambda,\mu}$  also satisfies the identity (2.4). Let  $f$  be any representation of  $L$  into an associative algebra  $B$ . Then by theorem 2.2 there exists a unique homomorphism  $g$  of  $U_{\lambda,\mu}$  into  $B(\lambda, \mu)$  such that

$$g(u_{i_1} \circ u_{i_2} \circ \dots \circ u_{i_r}) = \frac{1}{(\lambda - \mu)^r} f(u_{i_1}) * f(u_{i_2}) * \dots * f(u_{i_r}),$$

where  $x * y$  indicates the product in  $B(\lambda, \mu)$ . In particular, if  $B$  is the universal enveloping algebra  $A(L)$  of  $L$  then, since  $f$  can be taken to be the identity mapping on  $L$ , there exists a unique homomorphism  $g$  of  $U_{\lambda,\mu}$  into  $A(L)(\lambda, \mu)$  such that

$$g(u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_r}) = \frac{1}{(\lambda - \mu)^r} u_{i_1} * u_{i_2} * \cdots * u_{i_r}, \quad (2.7)$$

where both sides have the same type of association.

Suppose that  $L$  is abelian; that is,  $[LL] = 0$ . Then it is readily seen that the universal enveloping algebra  $A$  of  $L$  is a polynomial algebra which is, of course, commutative and associative. Thus  $A(\lambda, \mu)$  is also commutative and associative by (2.4) and  $x * y = (\lambda + \mu)xy$ , so (2.7) becomes

$$g(u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_r}) = \left(\frac{\lambda + \mu}{\lambda - \mu}\right)^r u_{i_1} u_{i_2} \cdots u_{i_r}.$$

In particular, this implies that  $g$  is an isomorphism and hence  $U_{\lambda, \mu}$  is commutative and associative.

*Corollary 2.4.* If  $L$  is an abelian Lie algebra then the  $UE(\lambda, \mu)$ -MA  $U_{\lambda, \mu}$  of  $L$  is a commutative associative algebra.

Let  $B$  be an associative algebra with unit element 1. For elements  $b, a_1, a_2, \dots, a_k \in B$ , using the fact that the linear mapping  $[\cdot, x]: B \rightarrow B$  is a derivation of  $B$ , it can be easily seen that

$$[b, a_1 a_2 \cdots a_k] = \sum_{i=0}^{k-1} a_1 a_2 \cdots a_i [b, a_{i+1}] a_{i+2} \cdots a_k, \quad (2.8)$$

where  $a_1 a_2 \cdots a_i = 1$ , if  $i=0$  and  $a_{i+2} \cdots a_k = 1$ , if  $i=k-1$ .

*Lemma 2.5.* Let  $b, c, a_1, \dots, a_k$  be elements in an associative algebra with unit element. Then  $[[b, a_1 a_2 \cdots a_k], c]$  is a sum of products each of which contains exactly two commutators or one double commutator in  $b, c, a_1, a_2, \dots, a_k$ .

Let  $\{u_i | i \in A\}$  be an ordered basis for a Lie algebra  $L$ . A monomial  $u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_r}$  in  $T(L)_{\lambda, \mu}$  (or in  $U_{\lambda, \mu}$ ) in certain association is called a *standard monomial* in  $T(L)_{\lambda, \mu}$  (or in  $U_{\lambda, \mu}$ ) if  $i_1 \leq i_2 \leq \cdots \leq i_r$  and  $r$  is the *degree* of the monomial. Since the product is not associative, an ordered subset  $\{u_{i_1}, \dots, u_{i_r}\}$  with  $i_1 \leq \cdots \leq i_r$  gives

$$\begin{aligned} (x \circ u_i) \circ (y \circ u_j) &= ((x \circ u_i) \circ u_j) \circ y - \lambda \mu [[y, x \circ u_i], u_j] = ((x \circ u_i) \circ u_j) \circ y - \lambda \mu [x \circ [y, u_i] + [y, x] \circ u_i, u_j] \\ &= ((x \circ u_i) \circ u_j) \circ y - \lambda \mu x \circ [[y, u_i], u_j] - \lambda \mu [x, u_j] \circ [y, u_i] \\ &\quad - \lambda \mu [[y, x], u_j] \circ u_i - \lambda \mu [y, x] \circ [u_i, u_j]. \end{aligned}$$

Now,  $u_i$  and  $u_j$  in  $((x \circ u_i) \circ u_j) \circ y$  can be interchanged via the previous calculation. As before, the remaining terms are linear combinations of standard monomials modulo  $R$  of degree  $\leq m+n+1$ . Note that  $(x \circ u_i) (u_j \circ y)$  is a monomial of degree  $m+n+2$ . Therefore, we have proved the following generalization of the PBW theorem for  $U_{\lambda, \mu}$ .

*Theorem 2.6.* Every element in the universal

enveloping  $(\lambda, \mu)$ -mutation algebra  $U_{\lambda, \mu}$  of a Lie algebra  $L$  is a linear combination of  $I$  cosets of 1 and  $T_{\lambda, \mu}$  monomials, and  $R$  cosets of 1 and  $T(L)$  monomials.

rise to distinct standard monomials in  $U_{\lambda, \mu}$ . For example,

$$\begin{aligned} &((u_{i_1} \circ u_{i_2}) \circ u_{i_3}) \circ u_{i_4}, \quad (u_{i_1} \circ u_{i_2}) \circ (u_{i_3} \circ u_{i_4}), \\ &(u_{i_1} \circ (u_{i_2} \circ u_{i_3})) \circ u_{i_4}, \quad \text{etc.} \end{aligned}$$

To obtain an analog of the PBW theorem for  $U_{\lambda, \mu}$ , we first need to develop a machinery to interchange two basis elements in a monomial. Let  $x, y$  be monomials in  $T_{\lambda, \mu}$  and let  $u, v$  be basis elements for  $L$ . Then

$$\begin{aligned} (u \circ x) \circ v &= (x \circ u) \circ v + [u, x]^0 \circ v \\ &= (x \circ u) \circ v + (\lambda - \mu)[u, x] \circ v \end{aligned}$$

and by (2.8), the last term is a linear combination of monomials of degree  $\leq 1 + \text{degree } x$  modulo  $R$ . A similar observation can be made for a product of the form  $(x \circ u) \circ (y \circ v)$ . Therefore, it suffices to consider the following two types of products in  $T_{\lambda, \mu}$ :

$$(x \circ u_i) \circ u_j, \quad (x \circ u_i) \circ (u_j \circ y),$$

where  $x$  and  $y$  are monomials of degree  $m$  and  $n$ , and  $u_i, u_j$  are basis elements. For this we make repeated use of (2.4) and the fact that the mapping  $[\cdot, a]$  is a derivation both in  $T_{\lambda, \mu}$  and  $T(L)$ . First, we compute

$$\begin{aligned} (x \circ u_i) \circ u_j &= x \circ (u_i \circ u_j) + \lambda \mu [[u_j, x], u_i], \\ &= x \circ (u_j \circ u_i) + x \circ [u_i, u_j]^0 \\ &\quad + \lambda \mu [[u_j, x], u_i] \\ &= (x \circ u_j) \circ u_i + x \circ [u_i, u_j]^0 \\ &\quad - \lambda \mu [[u_i, x], u_j] + \lambda \mu [[u_j, x], u_i]. \end{aligned}$$

In the last expression, by (2.8) the second term can be written as a sum of standard monomials modulo  $R$  of degree  $\leq m+1$  and by Lemma 2.5, the last two terms are a linear combination of standard monomials modulo  $R$  of degree  $\leq m$ . Similarly, using the fact that the mapping  $[\cdot, a]: b \rightarrow [b, a]$  is a derivation of  $T(L)$  and  $T_{\lambda, \mu}$ , we compute

Let  $A_{\lambda, \mu} = A(L)_{\lambda, \mu}$  subalgebra of  $A(L)(\lambda, \mu)$  generated by 1 and  $L$ , where  $A(L)$  is the universal enveloping algebra of  $L$ . Notice that  $A_{\lambda, \mu}$  does not equal  $A(L)(\lambda, \mu)$  in general. By (2.7), we see that

enveloping  $(\lambda, \mu)$ -mutation algebra  $U_{\lambda, \mu}$  of a Lie algebra  $L$  is a linear combination of  $I$  cosets of 1 and  $T_{\lambda, \mu}$  monomials, and  $R$  cosets of 1 and  $T(L)$  monomials.

$g$  is a homomorphism of  $U_{\lambda,\mu}$  onto  $A_{\lambda,\mu}$ . Due to (2.4), the algebras  $U_{\lambda,\mu}$  and  $A_{\lambda,\mu}$  are far from associative and an explicit description of the standard monomials in  $U_{\lambda,\mu}$  or in  $A_{\lambda,\mu}$  seems quite difficult. Let  $K$  be the kernel of  $g$ . Thus the quotient algebra  $\bar{U}_{\lambda,\mu} = U_{\lambda,\mu}/K$  is isomorphic to  $A_{\lambda,\mu}$ . We have seen that if  $L$  is abelian, then  $K=0$ ; however, we have no example for  $K \neq 0$ .

Let  $L$  be a finite-dimensional semisimple Lie algebra over an algebraically closed field  $F$  of characteristic 0. Let  $Z(A) = Z(A(L)) = \{x \in A(L) \mid xy = yx \text{ for all } y \in A(L)\}$  be the center. Then  $Z(A)$  is a subalgebra of  $A(L)$  and an integral domain.<sup>25</sup> Let  $(x, y)$  be the Killing form of  $L$ . Let  $u_1, u_2, \dots, u_n$  be a basis for  $L$  and let  $(u_i, u_j) = \beta_{ij}$ ,  $i, j = 1, 2, \dots, n$ . Since the matrix  $(\beta_{ij})$  is a nonsingular symmetric matrix, it has an inverse matrix  $(\alpha_{ij})$  which is symmetric also. Then it is shown that the Casimir element

$$\Omega = \sum_{i,j=1}^n \alpha_{ij} u_i u_j$$

is in the center  $Z(A)$  of  $A(L)$ . Since  $(\alpha_{ij})$  is symmetric and  $u_i * u_j + u_j * u_i = (\lambda + \mu)(u_i u_j + u_j u_i)$ , in terms of the  $(\lambda, \mu)$ -mutation operation we have

$$\Omega' = \frac{1}{\lambda + \mu} \sum_{i,j=1}^n \alpha_{ij} u_i * u_j.$$

Therefore we have that  $\Omega \in A(L)_{\lambda,\mu}$  and by (2.4)  $\Omega$  is also contained in the center of the nonassociative algebra  $A(L)_{\lambda,\mu}$ . In view of this, we call  $\Omega' = (\lambda + \mu)\Omega$  the Casimir  $(\lambda, \mu)$ -mutation element of  $L$ .

For the purpose of our discussion, let  $\mathfrak{su}(2)$  be the Lie algebra of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

over the complex field  $C$ . Letting

$$u_1 = \frac{1}{2}i\sigma_1, \quad u_2 = -\frac{1}{2}i\sigma_2, \quad u_3 = \frac{1}{2}i\sigma_3,$$

we have the Lie multiplication

$$[u_1, u_2] = u_3, \quad [u_2, u_3] = u_1, \quad [u_3, u_1] = u_2,$$

and this implies the Killing form  $(u_i, u_j) = -2\delta_{ij}$ ,  $i, j = 1, 2, 3$ . Thus the Casimir element of  $\mathfrak{su}(2)$  is

$$\begin{aligned} \Omega' &= -\frac{1}{2}(u_1 * u_1 + u_2 * u_2 + u_3 * u_3) = (\lambda + \mu)\Omega \\ &= -\frac{1}{2}(\lambda + \mu)(u_1^2 + u_2^2 + u_3^2). \end{aligned} \quad (2.9)$$

In this case, it is shown that<sup>41</sup> the center of the algebra  $A(\mathfrak{su}(2))$  is precisely  $C[\Omega]$ , the polynomial algebra in  $\Omega$  over  $C$ , and thus the center of  $A(\mathfrak{su}(2))_{\lambda,\mu}$  is  $C[\Omega']$ . If  $f$  is an irreducible representation of  $\mathfrak{su}(2)$  acting on a vector space  $V$ , then  $f(\Omega)$  acts on  $V$  as a scalar and so does  $f(\Omega')$  on  $V$ . Then we have

$$f(\Omega') = (\lambda + \mu)f(\Omega).$$

### III. APPLICATION TO THE MUTATION OF SPIN $\frac{1}{2}$

In this paper we are primarily interested in massive, extended, strongly interacting particles which possess spin  $\frac{1}{2}$  under electromagnetic interactions, such as the nucleons. We shall therefore apply the mathematical methods of Sec. 2 to a speculative study of the possible mutation of the spin  $\frac{1}{2}$ . The hope is that such an application will result in being of assistance for the needed experimental resolution of the problem identified in Ref. 15, whether the spin of extended particles is preserved or mutated under strong interactions. For this purpose, we shall also briefly comment on the available proposals for experimental tests.

To supplement the discussion of the Introduction, it may be valuable to elaborate in more detail on some physical reasons for the expected mutation of the spin, as well as on its computation via the Lie-admissible algebras. We shall follow the traditional attitude of the exact  $SU(2)$  case, in the sense of first presenting the basic ideas for the case of the angular momentum and then extending them to the case of the spin.

The class of physical systems under consideration at the Newtonian limit is constituted by the most general, local class  $C^\infty$  and regular systems in a three-dimensional Euclidean space with coordinates  $r^{ka}$ ,  $k=1, 2, \dots, n$   $a=x, y, z$ . These systems can be written

$$\{[m_k \ddot{r}_{ka} - f_{ka}(t, \vec{r}, \dot{\vec{r}})]_{SA} - F_{ka}(t, \vec{r}, \dot{\vec{r}})\}_{NSA} = 0, \quad (3.1)$$

where SA (NSA) stands for variational self-adjointness (non-self-adjointness), that is, the property that the forces  $f_{ka}$  ( $F_{ka}$ ) verify (violate) the integrability conditions for the existence of a potential energy  $U(t, \vec{r}, \dot{\vec{r}})$ . This is, in essence, the technical language of the inverse problem of the calculus of variations<sup>18</sup> to express the fact that Newtonian systems, as they occur in our environment, exhibit the presence of long-range forces derivable from a potential (e.g., the Lorentz force), as well as contact nonconservative forces which are not derivable from a potential (e.g., drag forces quadratically dependent on the velocities). In particular, the Newtonian limit of the forces which are used for the breaking of the  $SU(2)$  angular momentum/spin symmetry is given precisely by the non-self-adjoint forces.

A subclass of systems (3.1) (called nonessentially non-self-adjoint<sup>15</sup>) verifies the integrability conditions for the existence of a Hamiltonian representation without changing the coordinates and time variables of its experimental detection. This is the first restriction on the class of non-self-adjoint forces which is needed for the type of  $SU(2)$  symmetry breaking we are referring to here. The