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Relativistic Hadronic Mechanics: Nonunitary, Axiom-Preserving Completion of Relativistic Quantum Mechanics

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The most majestic scientific achievement of this century in mathematical beauty, axiomatic consistency, and experimental verifications has been special relativity with its unitary structure at the operator level and canonical structure at the classical level, which has turned out to be exactly valid for point particles moving in the homogeneous and isotropic vacuum (exterior dynamical problems). In recent decades a number of authors have studied nonunitary and noncanonical theories, here generally called deformations, *for the representation of broader conditions, such as extended and deformable particles moving within inhomogeneous and anisotropic physical media (interior dynamical problems). In this paper we show that nonunitary deformations, including q-, k-, quantum-, Lie-isotopic, Lie-admissible, and other deformations, even though* mathematically *correct, have a number of problematic aspects of* physical *character when formulated on conventional spaces over conventional fields, such as lack of invariance of the basic space-time units, ambiguous applicability to measurements, loss of Hermiticity-observability in time, lack of invariant numerical predictions, loss of the axioms of special relativity, and others. We then show that the classical noncanonical counterparts of the above nonunitary deformations are equally afflicted by corresponding problems of physical consistency. We also show that the contemporary formulation of gravity is afflicted by similar problematic aspects because Riemannian spaces are noncanonical deformations of Minkowskian spaces, thus having noninvariant space-time units. We then point out that new mathematical methods, called* isotopies, genotopies, hyperstructures and their isoduals, *offer the possibilities of constructing a nonunitary theory, known as* relativistic hadronic mechanics *which: (1) is as axiomatically consistent as relativistic quantum mechanics, (2) preserves the abstract axioms of special relativity, and (3) results in a completion of the conventional mechanics much along the celebrated Einstein–Podolski–Rosen argument. A number of novel applications are indicated, such as a geometric unification of the special and general relativity via the isominkowskian geometry in which the two relativities are differentiated via the invariant basic*

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unit, while preserving conventional Riemannian metrics, Einstein's field equations, and related experimental verifications; a novel operator form of gravity verifying the axioms of relativistic quantum mechanics under the universal isopoincaré symmetry; a new structure model of hadrons with conventional massive particles as physical constituents which is compatible with composite quarks and with established unitary classifications; and other novel applications in nuclear physics, astrophysics, theoretical biology, and other fields. The paper ends with the proposal of a number of new experiments, some of which may imply new practical applications, such as conceivable new forms of recycling nuclear waste. The achievement of axiomatic consistency in the study of the above physical problems has been possible for the first time in this paper thanks to mathematical advances that recently appeared in a special issue of the Rendiconti Circolo Matematico Palermo, *and in other journals, identified in the Acknowledgments.*

1. OUTLINE OF DEFORMATIONS

In 1948, the American mathematician A. A. Albert[1] introduced the notion of *Jordan admissible and Lie-admissible algebras* as generally nonassociative algebras U with elements a, b, c, and abstract product ab which are such that the attached algebras U^+ and U^-, which are the same vector spaces as U equipped with the products $\{a, b\}_U = ab + ba$ and $[a, b]_U = ab - ba$, are Jordan and Lie algebras, respectively. Albert then studied the algebra with product

$$(A, B) = p \times A \times B + (1-p) \times B \times A \tag{1.1}$$

where p is a parameter, A, B are matrices or operators hereon assumed to be Hermitian, and $A \times B$ is the associative product. It is easy to see that the above product is indeed jointly Jordan- and Lie-admissible because $\{A, B\}_U = A \times B + B \times A$ and $[A, B]_U = (1-2p) \times (A \times B - B \times A)$.

As part of his Ph.D. studies in theoretical physics, Santilli[2] introduced in 1967 a stronger notion of Lie admissibility which is Albert's definition,[1] plus the condition that the algebras U admit Lie algebras in their classification. This refinement is recommendable for physical application because Albert was primarily interested in the *Jordan content* of a given algebra (for $p = 0$ product (1.1) becomes that of a *commutative Jordan algebra*), while possible physical applications are evidently enhanced by a well-defined *Lie content*. In fact, product (1.1) does not admit a (finite) value of p under which it recovers the Lie product and, therefore, it cannot be used for possible generalizations of current physical theories.

Santilli[2a] therefore introduced the realization

$$(A, B) = p \times A \times B - q \times B \times A \tag{1.2}$$

with related time evolution in the following infinitesimal and finite forms ($\hbar = 1$):

$$i\, dA/dt = p \times A \times H - q \times H \times A \tag{1.3a}$$

$$A(t) = e^{iq \times t \times H} \times A(0) \times e^{-ip \times t \times H} \tag{1.3b}$$

where p and q are finite parameters with non-null values $p \pm q$, A, B are Hermitian matrices or operators; and $A \times B$ is also the associative product. It is easy to see that product (1.2) is Jordan-admissible, Lie-admissible, and admits Lie algebras as particular (nondegenerate) cases for $p = q$ ($\neq 0$).

Refinement[2] turns out to be insufficient in physical applications because, as we shall see shortly, the parameters p and q become *operators* under the time evolution of the theory. Santilli[3a, b] therefore introduced in 1978 the broader condition of *general Lie-admissibility* which is the notion of Ref. 1 plus the condition that the algebra U admits *Lie-isotopic* (rather then Lie) algebras in its classification.

The latter notion was realized via the *general Lie-admissible* product (first introduced in Ref. 3b, p. 719)

$$(A, B) = A \times P \times B - B \times Q \times A \tag{1.4}$$

with time evolution in infinitesimal and finite forms (Ref. 3b, pp. 741, 742)

$$i\, dA/dt = A \times P \times H - H \times Q \times A \tag{1.5a}$$

$$A(t) = e^{iH \times Q \times t} \times A(0) \times e^{-it \times P \times H} \tag{1.5b}$$

where P and Q are generally nonhermitian matrices or operators with nonsingular and Hermitian sum $P + Q$ admitting of parametric values p and q as particular cases. The conventional Heisenberg's equations are evidently recovered for $P = Q = 1$.

Note that the P and Q operators must be sandwiched in between the elements A and B to characterize an algebra as commonly understood in mathematics. In fact, the script $P \times A \times B - Q \times B \times A$ would be acceptable for P and Q parameters, but it would violate the right distributive and scalar laws for P and Q operators (see Refs. 3a, 3b for details).

In the latter case the algebras U admit Lie algebras for $P = Q = 1$, and the attached antisymmetric algebra U^- is not characterized by the traditional product $[A, B] = A \times B - B \times A$, but rather by the product (first introduced in Ref. 3b, p. 725)

$$[A, \hat{}B]_U = A \hat{\times} B - B \hat{\times} A = A \times T \times B - B \times T \times A, \qquad T = P - Q = T^\dagger \tag{1.6}$$

called *Lie-isotropic*, because verifying the Lie axioms, although in a more general way, with the product $A \hat{\times} B = A \times T \times B$ called *isoassociative* because it is more general than the conventional associative product $A \times B$, yet preserves associativity, $A \hat{\times} (B \hat{\times} C) \equiv (A \hat{\times} B) \hat{\times} C$.

According to the above results, the *nonassociative* algebra U with product (A, B), Eq. (1.4), can be replaced by an algebra $\hat{\xi}$ with *isoassociative* product $A \hat{\times} B = A \times T \times B$, in the characterization of the attached antisymmetric algebra[3a, 3b, 3d]

$$[A, \hat{,} B]_U = (A, B) - (B, A) \equiv [A, \hat{,} B]_{\hat{\xi}} = A \hat{\times} B - B \hat{\times} A \tag{1.7}$$

The latter property permitted a step-by-step lifting of the conventional formulation of Lie theory in terms of the isoassociative product $A \hat{\times} B$, including enveloping algebras, Lie algebras, Lie groups, Lie symmetries, transformation and representation theory, etc.,[4] called today *Lie–Santilli isotheory* (see Ref. 5 and papers quoted therein).

As a particular case of the broader Lie-admissible formulations, Santilli[3] therefore studied the Lie-isotopic time evolution in infinitesimal and finite forms for $T = T^{\dagger}$ (first introduced in Ref. 3b, p. 752)

$$i\, dA/dt = [A, \hat{,} H]_{\hat{\xi}} = A \hat{\times} H - H \hat{\times} A = A \times T \times H - H \times T \times A \tag{1.8a}$$

$$A(t) = e^{iH \times T \times t} \times A(0) \times e^{-iH \times T \times H} \tag{1.8b}$$

which admit conventional quantum equations for $\hat{T} = 1$.

The latter theory was called *isotopic*[3] in the Greek sense of being *axiom-preserving*, because the deformation is still Lie, yet of a more general nature, while the preceding theory (1.5) was called *genotopic*, in the Greek sense of being *axiom-inducing*, because the Lie axioms are replaced by the covering Lie-admissible axioms.

No operator theory has sufficient depth without well-defined classical foundations. For this reasons, Santilli conducted extensive studies on the classical counterparts of the preceding theories reported in Refs. 3d, 3e. In essence, the classical action underlying the Lie-isotopic theories resulted in the most general possible, first-order Pfaffian action in phase space

$$A = \int_{t_1}^{t_2} dt[R_\mu(b)\, db^\mu/dt + H(t, b)] \tag{1.9}$$

$$b = \{b^\mu\} = \{r^k, p_k\}$$

$$R = \{R_\mu\} = \{A_k(r, p), B^k(r, p)\}, \qquad \mu = 1, 2, \ldots, 6, \quad k = 1, 2, 3$$

whose variations yield the well-known *Birkhoff's* equations in the following covariant and contravariant forms (see Ref. 3d for historical notes and references)

$$\Omega_{\mu\nu}(b) \frac{db^\nu}{dt} = \frac{\partial H(t, b)}{\partial b^\mu} \tag{1.10a}$$

$$\frac{db^\mu}{dt} = \Omega^{\mu\nu}(b) \frac{\partial H(t, b)}{\partial b^\nu} \tag{1.10b}$$

with (nowhere degenerate) covariant and contravariant tensors

$$\Omega_{\mu\nu} = \partial R_\nu / \partial b^\mu - \partial R_\mu / \partial b^\nu \tag{1.11a}$$

$$\Omega^{\mu\nu}(b) = (|\Omega_{\alpha\beta}|^{-1})^{\mu\nu} \tag{1.11b}$$

The ensuing mechanics, called *Birkhoffian mechanics* in Ref. 3d, and *Birkhoff–Santilli isomechanics* in various references (see, e. g., Ref. 5 and papers quoted therein), was said to be isotopic because it preserves the main axioms of conventional Hamiltonian mechanics although realized in their most general possible form, i.e.: (1) derivability from the most general possible first-order action (analytic isotopy); (2) characterization by the most general possible, regular symplectic structure in local coordinates (analytic isotopy),

$$\Omega = \Omega_{\mu\nu}(b)\, d\hat{b}^\mu \wedge db^\nu \tag{1.12}$$

and (3) characterization by the most general possible regular (unconstrained) brackets verifying the Lie axioms (algebraic isotopy)

$$[A, B]^* = \frac{\partial A}{\partial b^\mu} \Omega^{\mu\nu}(b) \frac{\partial B}{\partial b^\nu} \tag{1.13}$$

Conventional classical Hamiltonian mechanics is admitted as a particular case at all levels for $R = R^\circ = (p, 0)$, as one can easily verify.

One may consult Ref. 3d for additional aspects, including: the unified treatment via the *conditions of variational self-adjointness*; the use of the *isotopies of Lie's theory*; the proof of the "direct universality" of the mechanics, i.e., its capability to represent all well-behaved local-differential nonconservative Newtonian systems (universality) in the given b-coordinates of the experimenter (direct universality); and other aspects.

Since Eqs. (1.8) and (1.10) have the most general possible (unconstrained and regular) Lie structures, the former were introduced in Ref. 3d as the operator image of the latter.

References 3a, 3e were devoted to the study of the classical counterpart of Lie-admissible equations (1.5). Conventional Newtonian forces are divided into variationally *self-adjoint* (SA) and *non-self-adjoint forces* (NSA), $F_k(t, b) = F_k^{SA} + F_k^{NSA}$. The SA forces are represented in terms of a conventional potential $U(t, b)$ via the techniques of the inverse problem of Ref. 3d. The NSA forces are represented via the *algebraic tensor* of the theory, according to the equations of Refs. 3a, 3e

$$\frac{db^\nu}{dt} - S^{\mu\nu}(t, b)\frac{\partial H(t, b)}{\partial b^\mu} \equiv m\, dv_k/dt - F_k^{SA}(t, b) - F_k^{NSA}(t, b) \qquad (1.14a)$$

$$(S^{\mu\nu}) = (\omega^{\mu\nu}) + (s^{\mu\nu}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (F^{NSA}/(\partial H/\partial p)) \end{pmatrix} \qquad (1.14b)$$

where $\omega^{\mu\nu}$ is the familiar canonical Lie tensor and $S_{\mu\nu}$ is a Lie-admissible tensor because

$$S^{\mu\nu}(t, b) - S^{\nu\mu}(t, b) = 2\omega^{\mu\nu} \qquad (1.15)$$

Consequently, the brackets of the time evolution

$$dA/dt = (A, H) = \frac{\partial A}{\partial b^\mu} S^{\mu\nu}(t, b)\frac{\partial H}{\partial b^\nu} \qquad (1.16)$$

are Lie-admissible,

$$(A, B) - (B, A) = 2[A, B] \qquad (1.17)$$

with a compatible lifting of the symplectic two-form (1.12) called *symplectic-admissible.*

The emerging mechanics was called *Birkhoff-admissible mechanics* in Ref. 3e and it is called *Birkhoff–Santilli genomechanics* in the literature.[5] Note its very simple direct universality for all possible Newtonian systems, owing to general solution (1.14b), which should be compared with the rather complex direct universality of Birkhoff's equations (1.10).

We should also recall that the Lie-admissible equations (1.14) were constructed[3a, 3e] along the original Hamilton's equations, those with *external terms* here denoted F_k^{NSA}. The important point is that the numbers of independent functions in the external terms F_k^{NSA} and in the Lie-admissible tensor $S^{\mu\nu}$ coincide.

Reformulation (1.14) is required by the fact that the brackets of Hamilton's equations with external terms violate the condition to form any algebra, let alone the Lie algebras, thus preventing the construction of a covering mechanics. On the contrary, brackets (1.16), first of all, verify all conditions to characterize an algebra, and, second, that algebra becomes Lie-admissible, i.e., a covering of the algebraic structure of conventional Hamiltonian mechanics.

Note also that *Lie-isotopic equations* (1.10) *are structurally reversible,* that is, they are reversible for reversible Hamiltonians. On the contrary, *Lie-admissible equations* (1.14) *are structurally irreversible,* that is, they are irreversible even for reversible Hamiltonians. These main characteristics will persist throughout the analysis of this paper.

As such, the Lie-admissible equations are particularly suited for an *axiomatization of irreversibility,* that is, its representation via the structure of the theory, rather than the addition of symmetry breaking terms in a time-symmetric Lagrangian or Hamiltonian.

Since Eqs. (1.5) and (1.14) have the maximal possible (unconstrained and regular) Lie-admissible structures, the former were assumed in Refs. 3e to be the operator image of the latter. For additional aspects, the reader may inspect Ref. 3e.

Classical and operator Lie-admissible structures and their Lie-isotopic particularizations were then studied in a variety of mathematical and physical papers; see, e.g., Ref. 5 and additional papers.[6] A comprehensive bibliography up to 1984 can be found in Ref. 7, and that on subsequent works in Ref. 5d.

In 1985 Biedenharn[8a] and Macfarlane[8b] introduced the so-called q-deformations which were followed by a large number of papers in the field (see, e.g., Ref. 9). Still more recently, other types of deformations of relativistic quantum formulations appeared in the literature under the name of k-deformations (see, e.g., Ref. 10). Comprehensive studies were also conducted in the field known under the name (somewhat misleading) of *quantum groups* (see, e.g. Ref. 11).

The latter deformations are essentially reducible to the following types:

(1) Deformations of enveloping associative algebras

$$A \times B \to A \hat{\times} B = q \times A \times B \qquad (1.18)$$

(2) Deformations of the Lie product

$$A \times B - B \times A \to A \times B - q \times A \times B \qquad (1.19)$$

(3) Deformations of the structure constants

$$X_i \times X_j - X_j \times X_i = C_{ij}^k X_k \rightarrow X_i \times X_j - X_j \times X_i = C_{ij}^k X_k \qquad (1.20)$$

and numerous others studied in Sect. 2 and 3.

One can easily see that deformations (1.18) and (1.19) are particular cases of the Lie-admissible deformations (1.5), while alteration of the *structure constants*, Eq. (1.20), are true deformations as intended in mathematics. Because of this disparity, Ref. 2 suggested the name of *mutations* for *alterations of the structure of the Lie product*, with the intent of preserving the name of *deformations* for structure such as (1.20).

Nevertheless, the term "deformations" is now widely used and will be kept in this paper to avoid misinterpretations. We shall therefore call "deformation" any alteration of the structure of classical or quantum mechanics, thus including *q*-, *k*-, quantum-deformations, the deformations of Lie-isotopic and Lie-admissible type, as well as any deviation from the conventional linear, local, canonical, or unitary structure.

Ironically, by the time Biedenharn's and Macfarlane's papers[8] appeared, Santilli had already abandoned this line of inquiry because of insurmountable problematic aspects of *physical* character preliminarily reported by Lopez in Ref. 12.

Despite the appearance of the latter papers and the passing of time, the problematic aspects of deformations of classical and quantum formulations have not yet propagated in the literature, thus rendering their additional study recommendable.

The ultimate problem addressed in this paper is the following. On the one hand, the main characteristics of conventional classical and quantum formulations are those of being *canonical* and *unitary*, respectively. On the other hand, advancements in interior problems, e.g., the classical representation of the irreversibility of the structure of Jupiter, requires a *noncanonical* theory[3d, 3e] or the operator representation of a black hole structure requires a *nonunitary* theory.[13]

But, as outlined in the next section, the above classical and quantum deformations possess a number of rather serious, problematic aspects of physical nature, even though they possess an undeniable *mathematical* beauty (which perhaps accounts for the large number of papers in the field).

Above all, noncanonical-nonunitary theories violate the axioms of the special relativity, thus creating the considerable problems of identifying new axioms, proving their axiomatic consistency and, after that, establishing them experimentally.

The main problem considered in this paper is therefore the achievement of theories which are structurally noncanonical at the classical level and nonunitary at the operator level, yet formulated in such a way to be as axiomatically consistent as conventional mechanics and, above all, capable of preserving the abstract axioms of especial relativity.

As we shall see, contemporary formulations of quantum gravity are afflicted by similar problematic aspects because *Riemannian spaces are deformations of he Minkowski space which are noncanonical at the classical level and nonunitary at the operator level.* Therefore, quantum gravity suffers from essentially the same problematic aspects of the preceding nonunitary theories.

The primary objective of this paper is of *methodological* character. As such, applications and verifications will only be indicated for further studies elsewhere with the understanding that it would be unreasonable to expect their joint detailed treatment here.

2. PROBLEMATIC ASPECTS OF QUANTUM DEFORMATIONS, CLASSICAL DEFORMATIONS AND GRAVITY

As is well known, a necessary condition to exit the class of equivalence of quantum mechanics is that the map from quantum to deformed formulations must be *nonunitary*

$$U \times U^{\dagger} \neq I \qquad (2.1)$$

when referred to conventional Hilbert spaces $\mathscr{H}$ with inner product and normalization

$$\langle \psi | \phi \rangle \in C(c, +, \times), \qquad \langle \psi | \psi \rangle = 1 \qquad (2.2)$$

where $C(c, +, \times)$ represents the conventional field of complex numbers c with familiar sum $+$, multiplication $\times$, and related additive unit 0 and multiplicative unit 1.

It is evident that, to be nontrivial, quantum deformations must be a nonunitary images of conventional quantum setting, otherwise they are mere equivalent quantum mechanical forms. Note that this includes not only *q*-, *k*-, and quantum-deformations, but also all Lie-admissible and Lie-isotopic formulations (1.2)–(1.8). As an example, we have the following

Lemma 1. The general Lie-admissible time evolution (1.5) and its Lie-isotopic particularization (1.8) are nonunitary on $\mathscr{H}$ over C.

$t = 0$ are no longer generally Hermitian at subsequent times, and the considered quantum deformations do not possess unambiguous observables.

As is well known, the numerical predictions of quantum mechanics are the result of data elaboration via special functions (and transforms). The predictions of quantum deformations are also the result of special functions although of new type specifically built per each case considered, the so-called q-, k-, quantum-special functions of Ref. 8–11. But nonunitary deformations are not form invariant under their time evolution and so are the related special functions. Problematic aspect (3) then follows because the lack of invariance of deformed special functions evidently implies the lack of invariant numerical predictions.

For instance, q-special functions at the initial time $t = 0$ no longer generally apply at a later time t because the q parameter becomes a Q operator under nonunitary transforms, according to the rule

$$q \times A \times B \to q \times U \times A \times A \times B \times U^{\dagger} = \hat{A} \times \hat{Q} \times \hat{B} \tag{2.6a}$$

$$U \times U^{\dagger} \neq I, \qquad Q = q \times (U \times U^{\dagger})^{-1}$$

$$\hat{A} = U \times A \times U^{\dagger}, \qquad \hat{B} = U \times B \times U^{-1} \tag{2.6b}$$

and a similar situation holds in the other cases. ∎

It should be stressed that Theorem 1 applies, specifically, to *nonunitary* deformations computed on a *conventional* Hilbert space over *conventional* fields. If the deformations are *unitary*, no problematic aspect evidently arises when computed over a conventional Hilbert space over C.

Similarly, if the deformations are *nonunitary* and computed over *suitably generalized* Hilbert space and fields, then consistency can be regained under certain conditions studied in Sec. 3.

The problematic aspects of the above "No-Go Theorem" are serious per se. Yet, additional problematic aspects are implied by consequences (1), (2), and (3). For instance, it is known that the probabilities of quantum mechanics are deeply linked to the invariance of the unit and its decomposition. The lack of invariance of the unit under nonunitary transforms then implies the following property (where the computation on conventional Hilbert spaces over conventional fields is assumed hereon):

Corollary 1.A. Nonunitary quantum deformations do not possess invariant probabilities.

Recall that the physical laws of quantum mechanics are unique in their definition and invariant under the time evolution of the theory. By recalling the several alternative possibilities of defining q-, k-, quantum-, and other

special functions (e.g., the numerous q-exponentiations existing in the literature[8–10] and their lack of invariance in time, we have the following:

Corollary 1.B. Nonunitary quantum deformations do not possess unique and invariant physical laws.

Recall that the causality of quantum mechanics follows from the unitarity of its time evolution. We therefore have the additional:

Corollary 1.C. Nonunitary quantum deformations violate causality.

But the problematic aspect considered particularly serious by this author is the following one of evident derivation from Theorem 1:

Corollary 1.D. Nonunitary quantum deformations violate the axioms of the special relativity.

The above occurrence can be easily illustrated by noting that, e.g., the deformed Minkowski spaces of Ref. 10 are not compatible with the Lorentz transforms, or that the corresponding deformed Poincaré symmetry is not isomorphic to the conventional symmetry. These occurrences create the sizable problems identified in Sec. 1 (which are inherent in relativistic deformations[10] of: (a) identifying new relativistic axioms which replace the Einsteinian ones; (b) proving their axiomatic consistency; and, after that, (c) establishing them experimentally.

By no means does the above analysis exhausts all physical problematic aspects of the deformations of quantum mechanics currently under study. For completeness, we mention that the rather old addition of an "imaginary potential" $iV(r)$ to a (Hermitian) Hamiltonian H_0, $H = H_0 + iV(r)$, which is frequently use in nuclear physics to represent dissipation, implies the deformation of the basic brackets from a bilinear to a triple form,

$$[A, H_0] = A \times H_0 - H_0 \times A \to [A, H, H^{\dagger}] = A \times H^{\dagger} - H \times A \tag{2.7}$$

By recalling that the brackets of the time evolution must be, for consistency, the brackets of the underlying algebras and symmetries, generalizations (2.7) imply the loss of *all* algebras as commonly understood, let alone the loss of all Lie algebras (e.g., the $SU(2)$-spin symmetry cannot be even defined, let alone treated, with triple systems). Under these conditions, familiar physical terms such as "protons and neutrons with spin 1/2" have no mathematical or physical meaning of any known nature (for more details on the problematic aspects of generalization (2.7), see Ref. 3b.

Proof. Heisenberg's time evolution in finite form has a *bimodular Lie structure*, in the sense of being characterized by an action to the right, here denoted $U^> = \exp\{iH \times t\}$, and an action to the left, here denoted ${}^<U = \exp\{-it \times H\}$,

$$A(t) = U^> \times A(0) \times {}^<U = e^{iH\times t} \times A(0) \times e^{-it\times H} \tag{2.3}$$

The unitarity of the evolution follows from the familiar conjugation

$${}^<U = (U^>)^\dagger \tag{2.4}$$

under which we have the law

$$U \times U^\dagger = U^\dagger \times U = U^> \times {}^<U = {}^<U \times U^> = U^> \times (U^>)^\dagger = (U^>)^\dagger \times U^>$$
$$= {}^<U \times ({}^<U)^\dagger = ({}^<U)^\dagger \times {}^<U = I \tag{2.5}$$

The general Lie-admissible law (1.5) violates, first, condition (2.4) and then each condition (2.5) because of the lack of commutativity of P and Q with H. The Lie-isotopic time evolution (1.8) verifies condition (2.4) but violates conditions (2.5), again because of the lack of general commutativity of T and H. Therefore, time evolutions (1.5) and (1.8) are nonunitarity. The same occurs for particular cases such as q-deformations (1.19). ∎

Needless to say, the corresponding transformation theory of the classical Lie-admissible (1.14) and Lie-isotopic equations (1.10) are *noncanonical*, as studied in detail in Refs. 3d, 3e.

Even though Lie's theory is preserved, we essentially have a similar situation for deformations (1.20), in fact, to be nontrivial, the deformation of the structure constants $C^k_{ij} \to D^k_{ij}$ must be produced at the classical level by *noncanonical* transforms with *nonunitary* image at the operator level (the reader may inspect the noncanonical deformation of the Minkowski space and of the Casimir invariants of the Poincaré symmetry of Ref. 10).

The general loss of unitarity then has the following serious problematic aspects of physical character:

Theorem 1. All possible nonunitary deformations of quantum mechanics computed on conventional Hilbert spaces over conventional fields, including q-, k-, quantum-, Lie-isotopie, and Lie-admissible and other deformations, have the following physical problematic aspects: (1) they lack the invariance of the unit, thus lacking unambiguous applications to measurements; (2) they lack the preservation of Hermiticity in time, thus lacking unambiguous observables; and (3) they lack invariant special functions and transforms, thus lacking invariant numerical predictions.

Proof. The unit of a quantum theory is the unit I of the enveloping associative operator algebra ξ with generic elements $A, B,\ldots$ and conventional associative product $A \times B$,

$$I \times A = A \times I \equiv A, \qquad \forall A \in \xi \tag{2.1}$$

It is well known that, by definition, the above unit is not invariant under nonunitary transforms

$$I \to I' = U \times I \times U^\dagger \neq I \tag{2.2}$$

and it is not generally preserved under the time evolution, e.g.,

$$i\, dI/dt = (I, H) = I \times P \times H - H \times Q \times I \neq 0 \tag{2.3}$$

Problematic aspect (1) then follows because the considered quantum deformations cannot be unambiguously applied to measurements, e.g., it is not possible to measure distances with a (stationary) meter of varying length.

Under a nonunitary transform, the familiar associative modular action of the Schrödinger's representation $H \times |\psi\rangle$, where H is an operator Hermitian at the initial time $t = 0$, becomes

$$U \times H \times |\psi\rangle = U \times H \times U^\dagger \times (U \times U^\dagger)^{-1} \times U \times |\psi\rangle$$
$$= \hat{H} \times \hat{T} \times |\hat{\psi}\rangle \tag{2.4a}$$

$$U \times U^\dagger \neq I, \qquad \hat{T} = (U \times U^\dagger)^{-1}, \qquad |\hat{\psi}\rangle = U \times |\psi\rangle, \qquad \hat{H} = U \times H \times U^\dagger \tag{2.4b}$$

By noting that $\hat{T}$ is Hermitian, $\hat{T} = (U \times U^\dagger)^{-1} = \hat{T}^\dagger$, the initial condition of Hermiticity of H on $\mathscr{H}$, $\langle\psi| \times \{H \times |\psi\rangle\} \equiv \{\langle\psi| \times H^\dagger\} \times |\psi\rangle$, when applied to the Hilbert space $\mathscr{H}$ with states $|\hat{\psi}\rangle$, $|\hat{\phi}\rangle$, etc. requires the action of the transformed operator (2.4) on a *conventional* inner product, resulting in the expressions

$$\langle\hat{\psi}| \times \{\hat{H} \times \hat{T} \times |\hat{\psi}\rangle\} \equiv \{\langle\hat{\psi}| \times \hat{T} \times \hat{H}^\dagger\} \times |\hat{\psi}\rangle,$$
$$\text{i.e., } \hat{H}^\dagger = \hat{T}^{-1} \times \hat{H} \times \hat{T} \neq \hat{H} \tag{2.5}$$

As such, Hermiticity is not preserved under nonunitary transforms formulated on conventional spaces $\mathscr{H}$ over conventional fields C, because of the lack of general commutativity of $\hat{T}$ and $\hat{H}$. By recalling that the time evolution of the considered class of deformations is nonunitary, problematic aspect (2) follows because operators which are Hermitian at the initial time

The same situation occurs in statistical mechanics when collisions are represented via the deformation of the Liouville equation with an external term

$$i\,d\rho/dt = [\rho, H] \to i\,d\rho/dt = [\rho, H, C] = \rho \times H - H \times \rho + C \qquad (2.8)$$

In addition to the loss of all algebras, and, therefore, of all possible symmetries as currently understood, theories (2.7) and (2.8) do not have an invariant unit, thus suffering from most of the problematic aspects of Theorem 1 (see Ref. 14 for additional studies on the problematic aspects of statistical equations with external terms).

Similarly, the deformation of the *linearity* of quantum mechanics into *nonlinear theories* (hereon referred to *nonlinearity in the wavefunctions*), e.g., of the type[15]

$$H(x, p, \psi, \ldots) \times \psi = E \times \psi \qquad (2.9)$$

even though *mathematically* impeccable, has serious problematic aspects of *physical* nature, such as the violation of the superposition principle. As such, nonlinear theories cannot be used for consistent studies of composite systems, besides having the problematic aspects of Theorem 1 whenever the time evolution is nonunitary (see Ref. 16 for detailed studies on the problematic aspects caused by nonlinearity).

Additional deformations of quantum mechanics are based on *nonassociative envelopes*, e.g., *Weinberg's theory*[17a] which can be reformulated in the methods of this paper via the general Lie-admissible structure,

$$[A, \hat{\ } B] = (A, B) - (B, A) = \text{Lie} \qquad (2.10a)$$

$$(A, B) = \frac{\partial A}{\partial \psi_k} \frac{\partial B}{\partial \psi_k^\dagger} = \text{nonassociative Lie-admissible} \qquad (2.10b)$$

Even though of impeccable mathematical beauty, the latter theory violates Okubo's[6b] "*No-Go Theorem*" on deformations with *nonassociative* envelopes, under which there is the loss of the equivalence of the Heisenberg-type and Schrödinger-type representations and other problematic aspects. In addition, *Weinberg's theory possesses no unit at all in the envelope* (i.e., there is no nontrivial quantity E such that for product (2.10b) $(E, A) \equiv (A, E) \equiv A$ for all possible generators A), thus having physically unsettled aspects more serious than those of Theorem 1 (for detailed studies of the latter problematic aspects see Ref. 18).

Note that the attempt of Ref. 17 to reconstruct Weinberg's theory with an *associative* envelope with the brackets $awb - bwa = (a_{ir})(w_{rs})(b_{sj}) - (b_{ir})(w_{rs})(a_{sj})$ is precisely along our rule (1.7) which turns Weinberg's nonlinear theory into our Lie-isotopic theory.[3b]

Yet another group of deformations of quantum mechanics affected by Theorem 1 is that of the so-called *squeezed states*[19] which are also generally nonunitary images of conventional theories. As such, they suffer the same problematic aspects of Theorem 1.

A further important type of quantum deformations is *Prigogine's nonunitary statistics*[20] introduced to attempt a reconciliation of the irreversibility of classical and quantum worlds. Being nonunitary, this theory too is affected by Theorem 1. However, unlike other deformations, Prigogine's nonunitary statistics may only requires its isotopic formulation on appropriate spaces and fields to achieve invariance and axiomatic consistency, as shown in the next section.

Needless to say, the same problematic aspects exist for the more general Lie-admissible statistics in its first formulation submitted by Fronteau *et al.*[61] (see Sec. 3.12 for its current mathematical formulation).

There is little doubt that *problematic aspects in deformed quantum formulations must have corresponding problematic aspects in their classical counterpart.* Recall that the Birkhoff-admissible equations (1.10) are the classical counterpart of the operator Lie-admissible equations (1.5), and the Birkhoff's equations (1.10) are the classical counterpart of the operator Lie-isotopic equations (1.8).

Recall also that the fundamental unit of classical theories is the unit $\hat{I} = \text{Diag}(1, 1, 1)$ of the Euclidean space which represents the units of the three Cartesian coordinates (say, 1 cm) in dimensionless form. We then have the following:

Theorem 2. All noncanonical deformations of classical Hamiltonian mechanics formulated on conventional spaces over conventional fields, including the classical image of q-, k-, quantum-deformation, the Birkhoffian-admissible and other deformations, do not possess invariant units, with consequential problematic aspects in their application to measurements.

Proof. The admitted transformation theories are *noncanonical* by assumption, e.g., they leave invariant the Birkhoff's (1.11) or Birkhoff-admissible tensor (1.15). As such, they do not leave invariant the canonical tensor $\omega_{\mu\nu}$. In particular, a map from the Hamiltonian to the Birkhoffian mechanics is given precisely by noncanonical transforms $b = \{r, p\} \to b'(b) = \{r', p'\}$ (see Ref. 3e for details) such that

$$\omega_{\mu\nu} \to \omega'_{\mu\nu} = \frac{\partial b^\alpha}{\partial b'^\mu} \omega_{\alpha\beta} \frac{\partial b^\beta}{\partial b'^\nu} = \Omega_{\mu\nu}(a') \qquad (2.11)$$

But the canonical tensor represents the fundamental space units of the theory,

$$(\omega_{\mu\nu}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{2.12}$$

and this establishes the inability of noncanonical theories on conventional spaces over conventional fields to have invariant basic units, with ensuing problematic aspects in measurements. ∎

We then have the following evident implication,

Corollary 2.A. The relativistic versions of all noncanonical classical theories, including the classical image of q-, k-, quantum-, and other deformations, the Birkhoffian and Birkhoffian-admissible mechanics, and other theories, violate the axioms of classical relativistic mechanics.

This illustrates the reasons why, after conducting the rather laborious classical studies of Refs. 3d, 3e, Santilli had to re-start from the beginning and identify a new form of generalized classical mechanics with invariant fundamental units. Intriguingly, a necessary condition resulted in the preservation of the abstract relativistic axioms, as we shall see in the next section.

By no means should the reader dismiss Theorem 2 following a possible impression that it has marginal implications, because the lack of invariance of the unit implies rather deep axiomatic inconsistencies which generally remain undetected by a nonexpert in the field.

As an illustration, the lack of conservation of the basic unit implies a corresponding lack of conservation of the base fields. Thus, starting from a theory defined at the initial time on conventional numbers, the same theory has to be defined at a later time on new *yet unknown numbers*. The ambiguities of noncanonical theories in their application to actual measurements are then beyond scientific doubts.

Another important class of theories with serious problematic aspects of *physical* character is given by the conventional formulation of *gravity*, i.e., that on conventional curved spaces over conventional fields (see, e.g., Ref. 21 and contributions quoted therein). In fact, we have the following:

Theorem 3. The basic unit of all (nowhere degenerate, real valued, and symmetric) geometries with non-null curvature over conventional fields is not invariant under the symmetries of the line element with consequential problematic aspects in applications to measurements for both classical and quantum formulations.

Proof. Let $E(x, \delta, R)$ and $\Re(y, g, R)$ be n-dimensional Euclidean and Riemannian spaces, respectively, with the same signature $(+, +, \ldots, +)$, basic unit $I = \mathrm{diag}(1, 1, \ldots, 1)$, metrics $\delta = \delta_{ij}) = \mathrm{Diag}(1, 1, \ldots, 1)$ and $g(y) = (g_{ij}) = g^t$, and local coordinates $x = \{x^k\}$, $y = \{y^k\}$, $i, j, k = 1, 2, \ldots, n$, over the reals $R = R(n, +, \times)$.

The transformation $x \to y(x)$ for which the Euclidean metric is mapped into the Riemannian metric,

$$\delta_{ij} \to g_{ij}(y) = \frac{\partial y^r}{\partial x^i} \delta_{rs} \frac{\partial y^s}{\partial x^j} \tag{2.13}$$

is *noncanonical*. Therefore, the symmetries of the Riemannian line elements $y^2 = y^t g y$ are necessarily *noncanonical*. As such, these symmetries do not generally preserve the basic unit I at the classical level. The symmetries of the same line element in operator formulation are then necessarily *nonunitary* for consistency (see next section), and this proves the lack of invariance of the basic unit also for operator theories. The same proof evidently applies for indefinite signatures $(+, +, \ldots, -, -, \ldots)$. ∎

To understand the implications of the above theorem, recall that for the $(3+1)$-dimensional Minkowskian and Riemannian geometries the basic unit is given by $I = \mathrm{Diag}(\{1, 1, 1\}, 1)$, where the first three components represent the space units (say 1 cm) in dimensionless form, and the fourth component represents the time unit (say 1 sec), also in dimensionless form. The above theorem establishes that *curvature implies the lack of invariance of the fundamental space-time units*, thus activating the problematic aspects of Theorem 1.

In different terms, the adoption of the conventional Riemannian space $\Re(x, g, R)$ over conventional fields R implies the adoption of a *noncanonical* theory, thus suffering from the problematic aspects of all noncanonical theories (Theorem 2). In fact, a particular case of the Birkhoffian mechanics is that in which the Euclidean metric δ is replaced precisely by a Riemannian metric.[3d]

Therefore, *the problems in the quantization of gravity are not necessarily due to Einstein's (or other) field equations, but rather to their referral to a manifold in which the basic unit is not invariant.* This identifies a novel alternative in both classical and quantum gravity considered in the next section, that of preserving Einstein's (or other) field equations identically and searching instead for a formulation of the Riemannian spaces in which the unit is invariant (see the Sec. 3.11).

A detailed study of the lack of invariance of numerical predictions of nonunitary deformations can be found in Ref. 22b, App. 4.E, where it is shown that the numerical predictions of deviations from the conventional uncertainties of squeezed states and other theories are not unique in their definition at the initial time and their numerical value is not preserved by the time evolution of the theory. Besides, when formulated in an axiomatically correct way, *nonunitary theories preserve Heisenberg's uncertainty*, as shown in the next section. Note that these problematic aspects also apply to quantum gravity.

This completes our study of deformations of quantum mechanics with the understanding that, by no means, do the above lines exhaust all classical and quantum deformations available in the literature. They are merely intended to identify primary classes. The author would be grateful to colleagues who care to bring to his attention additional important deformations.

In summary, the problematic aspects studied in this section confirm the majestic mathematical beauty and physical consistency of relativistic classical and quantum mechanics, and suggest caution before exiting from their *canonical and unitary* structure with *associative* enveloping algebra and *invariant* fundamental unit.

3. A POSSIBLE RESOLUTION OF THE PROBLEMATIC ASPECTS

3.1. Foundations

According to overwhelming experimental evidence, relativistic quantum mechanics is *exactly valid* for the so-called *exterior dynamical problems*, such as the structure of atoms, the electroweak interactions at large, and others.

Despite these achievements, Prigogine *et al.*,[20] Ellis *et al.*,[13] and Santilli *et al.*[3, 22] have suggested the study of broader theories for the more general *interior dynamical problems*, such as the structure of stars, quasars, black holes, and others in which hadrons are under "contact interactions," i.e., at mutual distances equal to or smaller than their charge radius. The latter physical conditions are expected to imply novel nonlinear and nonlocal interactions which, being of contact type, are nonhamiltonian and therefore, nonunitary.

In short, the *nonunitary* character of new operator theories appears to be uncompromisable for possible advances, with the corresponding *noncanonical* classical counterpart. The problem considered in this section is therefore how to reach a theory as axiomatically consistent as conventional classical and quantum mechanics yet having a noncanonical and nonunitary structure.

Above all, the problem consists in reaching a noncanonical and nonunitary theory for *interior* dynamical conditions while preserving the abstract axioms of the relativistic quantum mechanics for *exterior* conditions. Moreover, to represent a smooth transition from interior to exterior conditions, all nonunitary formulations must admit conventional formulations as particular cases under a smooth limit, a condition which is assumed hereon.

The above problems were studied at length by this author. Their solution appears to be possible thanks to certain rather crucial *mathematical* advances which only recently appeared in the special issue[23] of *Rendiconti Circolo Matematico Palermo* entirely dedicated to the mathematical issues herein considered. This section is dedicated to the outline of the essential *physical* aspects.

We have attempted to render this section minimally self-sufficient. Nevertheless, its technical understanding requires a technical knowledge of at least the special issue.[23] The noninitiated reader should be aware that the studies herein presented are based on novel *mathematics*. Any appraisal based on conventional mathematics is therefore afflicted by a host of inconsistencies.

As in Secs. 1 and 2, we shall first identify the axiomatically correct operator nonunitary theory and then study the corresponding classical noncanonical counterpart.

3.2. Nonunitary Image of Quantum Mechanics

The main problem of the earlier operator formulations[3] of Lie-isotopic and Lie-admissible deformations is that they are computed on *conventional* Hilbert spaces over *conventional* fields, thus sufferings the problematic aspects studied in Sec. 2.

By recalling the need to preserve nonunitarity for advances, the only possible alternative for consistency is therefore their formulation on *generalized* Hilbert spaces and fields. However, in order not to exit from the axioms of special relativity, the latter generalizations have to be axiom-preserving. This is the main idea of Santilli's[3a] *isotopies*, although its realization in an axiomatically consistent form is possible only following the mathematical advances that recently appeared in Ref. 23, as illustrated below.

The best way of identifying the needed mathematical structure is by subjecting to nonunitary transforms the main aspects of conventional quantum structures (see later on for other maps). This approach yields the following nonunitary image of the algebraic structure of quantum mechanics (unit, enveloping operator algebra ξ, and attached Lie algebra L):

$$I \to \hat{I} = U \times I \times U^{\dagger} = \hat{I}^{\dagger} \neq I, \qquad \hat{T} = (U \times U^{\dagger})^{-1} = \hat{T}^{\dagger} \tag{3.1a}$$

$$\xi: A \times B \to \hat{\xi}: U \times A \times B \times U^{\dagger} = \hat{A} \times \hat{T} \times \hat{B} = \hat{A} \hat{\times} \hat{B} \tag{3.1b}$$

$$L \approx \xi^{-}: [A, B] = A \times B - B \times A \to \hat{L}: [\hat{A}, \hat{\ } \hat{B}] = \hat{A} \hat{\times} \hat{B} - \hat{B} \hat{\times} \hat{A}$$
$$= \hat{A} \times \hat{T} \times \hat{B} - \hat{B} \times \hat{T} \times \hat{A} \tag{3.1c}$$

where $\hat{A} = U \times A \times U^{\dagger}$, etc. The nonunitary image of states and inner product of the Hilbert space is then given by

$$|\psi\rangle \to |\hat{\psi}\rangle = U \times |\psi\rangle \tag{3.2a}$$

$$\mathscr{H}: \langle \phi | \psi \rangle \to \hat{\mathscr{H}}: \langle \phi | \times U^{\dagger} \times U^{\dagger -1} \times U^{-1} \times U \times |\psi\rangle$$
$$= \langle \hat{\phi} | \times \hat{T} \times |\hat{\psi}\rangle = \langle \hat{\psi} | \hat{\times} | \hat{\psi} \rangle \tag{3.2b}$$

where one should keep in mind that $\hat{I}$ and $\hat{T}$ are Hermitian. As such, they are hereon assumed to be diagonal and positive-definite (see Refs. 22 and 23 for other possibilities).

It is then easy to see that the above liftings are axiom-preserving and thus isotopic in the sense of Ref. 3a. In fact, $\hat{\xi}$ is still associative because $(\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{C} = \hat{A} \hat{\times} (\hat{B} \hat{\times} \hat{C})$, and posseses the left and right unit $\hat{I}$,

$$\hat{I} \hat{\times} \hat{A} = \hat{A} \hat{\times} \hat{I} \equiv \hat{A}, \qquad \forall \hat{A} \in \hat{\xi} \tag{3.3}$$

Thus, $\hat{\xi}$ is locally isomorphic to ξ, yet it is structurally broader, as desired.

Similarly, the generalized Lie product $[\hat{A}, \hat{\ } \hat{B}]$ (first proposed in Ref. 3b p. 725) is still Lie, as one can verify, and $\hat{L}$ can be proved to be locally isomorphic to L (for positive-definite $\hat{I} > 0$).[22] Thus the lifting $L \to \hat{L}$ is *nonunitary yet axiom-preserving*, as desired.

Finally, the deformed composition $\langle \hat{\phi} | \times \hat{T} \times | \hat{\psi} \rangle$ is still inner and, therefore, $\mathscr{H}$ is still Hilbert. The lifting $\mathscr{H} \to \hat{\mathscr{H}}$ is therefore an isotopy, with $\hat{\mathscr{H}}$ broader than $\mathscr{H}$, as desired.

Nonunitary structures (3.1) and (3.2) imply the following *Heisenberg-isotopic formulations* (first introduced by Santilli in Ref. 3b, p. 752), here considered in one dimension for simplicity:

$$i\, d\hat{A}/dt = [A, \hat{\ } \hat{H}]_{\hat{\xi}} = \hat{A} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{A} = \hat{A} \times \hat{T} \times \hat{H} - \hat{H} \times \hat{T} \times \hat{A} \tag{3.4a}$$

$$[\hat{p}, \hat{\ } \hat{r}] = \hat{p} \hat{\times} \hat{r} - \hat{r} \hat{\times} \hat{p} = \hat{p} \times \hat{T} \times \hat{r} - \hat{r} \times \hat{T} \times \hat{p} = i \times \hat{I} \tag{3.4b}$$

$$[\hat{p}, \hat{\ } \hat{p}] = [\hat{r}, \hat{\ } \hat{r}] = 0 \tag{3.4c}$$

and the following *Schrödinger-isotopic counterpart for the energy* (first identified by Myung and Santilli[24] and Mignani[25] *and for the linear momentum* (first identified by Santilli[22, 23a])

$$\hat{H} \hat{\times} |\hat{\psi}\rangle = \hat{H} \times \hat{T} \times |\hat{\psi}\rangle = E\, |\hat{\psi}\rangle \tag{3.5a}$$

$$\hat{p} \hat{\times} |\hat{\psi}\rangle = P \times \hat{T} \times |\hat{\psi}\rangle = -i \hat{T} \times \hat{\nabla}\, |\hat{\psi}\rangle \tag{3.5b}$$

where we have assumed for simplicity that $\hat{T}$ is independent of r so that $U \times \nabla\, |\psi\rangle = U \times \nabla \times U^{\dagger} \times (U \times U^{\dagger})^{-1} \times U \times |\psi\rangle = \hat{T} \times \hat{\nabla}\, |\hat{\psi}\rangle$ (see later on for an arbitrary dependence).

The above structures do permit the resolution of the problem of Hermiticity of Sec. 2, because now the condition of Hermiticity reads

$$\langle \hat{\psi} | \times \hat{T} \times \{ H \times \hat{T} \times |\hat{\psi}\rangle \} = (\langle \hat{\psi} | \times \hat{T} \times H^{\dagger} \} \times \hat{T} \times |\hat{\psi}\rangle \tag{3.6a}$$

$$H^{\hat{\dagger}} = \hat{T}^{-1} \times \hat{T} \times H^{\dagger} \times \hat{T} \times T^{-1} \equiv H^{\dagger} \tag{3.6b}$$

Thus, starting from an operator H which is Hermitian at the initial time, the nonunitarily transformed operator $\hat{H} = U \times H \times U^{\dagger}$ remain Hermitian under nonunitary transforms. However, a necessary condition is that Hermiticity *is not* computed in the conventional Hilbert space $\mathscr{H}$, but rather in the above-defined generalized Hilbert space.

Despite this encouraging result, deformations (3.1)–(3.3) and related dynamical equations (3.4) and (3.5) are still far from physical consistency, because *they are not invariant under additional nonunitary transforms*, for which we have

$$W \times W^{\dagger} \neq I, \qquad \hat{Z} = (W \times W^{\dagger})^{-1} \tag{3.7a}$$

$$\hat{I} \to \hat{I}' = W \times \hat{I} \times W^{\dagger} \neq \hat{I} \tag{3.7b}$$

$$\hat{\xi}: \hat{A} \times \hat{T} \times \hat{B} \to \hat{\xi}': W \times (\hat{A} \hat{\times} \hat{B}) \times W^{\dagger} = \hat{A} \times \hat{Z} \times \hat{T}' \times \hat{Z} \times \hat{B}' \neq \hat{A}' \times \hat{T} \times \hat{B}' \tag{3.7c}$$

$$\hat{L}: \hat{A} \times \hat{T} \times \hat{B} - \hat{B} \times \hat{T} \times \hat{A} \to \hat{L}': \hat{A}' \times \hat{Z} \times \hat{T}' \times \hat{Z} \times \hat{B}' - \hat{B}' \times \hat{Z} \times \hat{T}' \times \hat{Z} \times \hat{A}'$$
$$\neq \hat{A}' \times \hat{T} \times \hat{B}' - \hat{B}' \times \hat{T} \times \hat{A}' \tag{3.7d}$$

It then follows that a theory with dynamical equations (3.4) and (3.5) is not physically consistent when formulated on generalized Hilbert space (3.2), because the "No-Go" Theorem 1 still applies.

3.3. Isofields and Isohilbert Spaces

Extensive studies of all possible alternatives conducted since Ref. 3 have established that the above problematic aspects are due to the fact that a nonunitary transform cannot be consistently applied only to *part* of the quantum formalism, while the remaining formalism stays conventional. In fact, as shown in the recent works,[23] *the isotopic theory apparently achieves the same axiomatic consistency of conventional quantum mechanics when the entirety of the mathematical structure of quantum mechanics, without exception, is subjected to an isotopic map with the same generalized unit.*

A primary objective of this section is to indicate the problematic aspects which emerge for nonunitary theories in isotopic treatment whenever *any* aspect of quantum mechanics is not subject to isotopy.

To begin, transforms (3.1a) imply the generalization of the basic unit of the theory. The definition of *generalized* structures (3.1)–(3.5) on a *conventional field* $C(c, +, \times)$ is, therefore, bound to imply axiomatic inconsistencies. This is due to the fact that the latter is still defined with respect to the *conventional* unit 1 while the former has a *generalized* unit $\hat{I}$.

To achieve a consistent formulation of the above nonunitary theory, the conventional fields of numbers have to be generalized into a form admitting of $\hat{I}$, rather than of I, as their left and right unit. This study has been conducted in detail in Ref. 26, resulting in the *isofields* $\hat{C} = \hat{C}(\hat{c}, +, \hat{\times})$ which are rings of elements $\hat{c} = c \times \hat{I}$, $\hat{I} \notin C$, called *isocomplex numbers* or, in general, *isonumbers*, equipped with the following *isotopic sum and multiplication*:

$$\hat{c}_1 + \hat{c}_2 = (c_1 + c_2) \times \hat{I}, \qquad \hat{c}_1 \hat{\times} \hat{c}_2 = \hat{c}_1 \times \hat{T} \times \hat{c}_2 = (c_1 \times c_2) \times \hat{I} \tag{3.8}$$

under which the quantity $\hat{I} = \hat{T}^{-1}$ is the correct left and right multiplicative unit, $\hat{I} \hat{\times} \hat{c} = \hat{c} \hat{\times} \hat{I} \equiv \hat{c}$, $\forall c \in \hat{C}$, called *isounit*, while $\hat{T}$ is called the *isotopic element*. The additive unit remains the conventional quantity $\hat{0} = 0$, $\hat{c} + 0 = 0 + \hat{c} \equiv \hat{c}$, $\forall \hat{c} \in \hat{C}$. It is easy to see that, under these conditions, $\hat{C}$ satisfies all axioms of a field. The lifting $C \to \hat{C}$ is therefore an isotopy, as desired.

It is evident that all operations of fields are generalized for isofields in a simple yet unique and significant way. For instance, conventional squares $c^2 = c \times c$ have no sense for $\hat{C}$ and must be lifted into the *isosquare* $\hat{c}^{\hat{2}} = \hat{c} \hat{\times} \hat{c}$, with corresponding *isopower* $\hat{c}^{\hat{n}} = \hat{c} \hat{\times} \hat{c} \hat{\times} \cdots \hat{\times} \hat{c}$; square roots $c^{1/2}$ are lifted into the *isosquare roots* $\hat{c}^{\widehat{1/2}} = c^{1/2} \times \hat{I}^{1/2}$; quotients a/b are lifted into the *isoquotient* $a \hat{/} b = (a/b) \times \hat{I}$; the norm $|c|$ is lifted into the *isonorm* $\uparrow \hat{c} \uparrow = |c| \times \hat{I}$; etc.

The isofield $\hat{R} = \hat{R}(\hat{n}, +, \hat{\times})$ of *isoreal numbers* $\hat{n} = n \times \hat{I}$, $n \in R(n, +, \times)$, is evidently a particular case of $\hat{C}$. For subsequent needs one should note that the isoproduct of an isonumber $\hat{n}$ by a quantity Q is conventional, $\hat{n} \hat{\times} Q = n \times \hat{I} \times \hat{T} \times Q = n \times Q$. Even though the numbers are generalized, the numbers predicted by the theory are therefore conventional, as we shall see (for detailed studies of the isoreal, isocomplex, isoquaternionic, and isooctonionic numbers we refer the interested reader to Ref. 26).

As is well known, Hilbert spaces are defined over fields. Part of the problematic aspects of Theorem 1 is that nonunitary theories are defined on a conventional Hilbert space over *conventional fields*. It is easy to see that the use of *generalized* Hilbert space (3.2) over a *conventional* field $C(c, +, \times)$ is bound to be axiomatically inconsistent.

In fact, the modular action $\hat{H} \hat{\times} |\hat{\psi}\rangle = \hat{H} \times \hat{T} \times |\hat{\psi}\rangle$ and composition $\langle \hat{\psi} | \hat{\times} | \hat{\psi} \rangle = \langle \hat{\psi} | \times \hat{T} \times | \hat{\psi} \rangle$ possess the generalized unit $\hat{I} = \hat{T}^{-1}$, because that is the only quantity such that $\hat{I} \hat{\times} |\hat{\psi}\rangle \equiv |\hat{\psi}\rangle$. Their referral to a field $C(c, +, \times)$ with conventional unit I is then inconsistent.

To achieve axiomatic consistency, the new Hilbert space $\hat{\mathscr{H}}$ must be referred to the isofield $\hat{C}$ with the same basic unity $\hat{I}$. As a consequence, the deformed Hilbert space must have the structure of an isonumber $c \times \hat{I}$. This leads in a unique and unambiguous way to the *isohilbert space* characterized by the following *isoinner product and isonormalization*[22a, 23b, 24]

$$\hat{\mathscr{H}}: \langle \hat{\phi} \uparrow \hat{\psi} \rangle = \langle \hat{\phi} | \hat{\times} | \hat{\psi} \rangle \times \hat{I} = \langle \hat{\phi} | \times \hat{T} \times | \hat{\psi} \rangle \times \hat{T}^{-1} \in \hat{C} \tag{3.9a}$$

$$\langle \hat{\psi} | \hat{\times} | \hat{\psi} \rangle = \langle \hat{\psi} | \times \hat{T} \times | \hat{\psi} \rangle = I \tag{3.9b}$$

Note that *isohermiticity coincides with conventional Hermiticity* in view of property (3.6). As a result, *all conventional quantum mechanical observables are preserved for the above isohilbert spaces over isofields.*

The conventional unitary transforms on $\mathscr{H}$ over C are lifted under isotopies into the *isounitary transforms* on $\hat{\mathscr{H}}$ over $\hat{C}$

$$\hat{U} \hat{\times} \hat{U}^{\dagger} = \hat{U} \times \hat{T} \times \hat{U}^{\dagger} = \hat{U}^{\dagger} \hat{\times} \hat{U} = \hat{U}^{\dagger} \times \hat{T} \times \hat{U} = \hat{I} = \hat{T}^{-1} \tag{3.10}$$

The conventional theory of linear operators on $\mathscr{H}$ must then be subjected to a compatible lifting on $\hat{\mathscr{H}}$ over $\hat{C}$ which is studied in Ref. 22a. We here merely mention the correct form of the *isoeigenvalue equations*

$$\hat{H} \hat{\times} |\hat{\psi}\rangle = \hat{H} \times \hat{T} \times |\hat{\psi}\rangle = \hat{E} \times |\hat{\psi}\rangle = (E \times \hat{I}) \times \hat{T} \times |\hat{\psi}\rangle = E \times |\hat{\psi}\rangle$$
$$\hat{H}^{\hat{\dagger}} = \hat{H} \in \hat{\xi}, \qquad \hat{E} \in \hat{R}, \qquad E \in R \tag{3.11}$$

and of the *isoexpectation values*

$$\langle \hat{H} \rangle = \frac{\langle \hat{\psi} | \times \hat{T} \times \hat{H} \times \hat{T} \times | \hat{\psi} \rangle}{\langle \hat{\psi} | \times \hat{T} \times | \hat{\psi} \rangle} \tag{3.12}$$

The reader can easily prove that the *isoeigenvalues of isohermitian operators are isoreal*, and that *the isoeigenvalues and isoexpectation values* *of the same operator coincide.* Note that the final numbers of isoeigenvalue equations (3.11) are conventional, and so are the isoexpectation values (because the isounits cancel in the quotient). Equations (3 11) and (3.12) therefore confirm that the final numbers of the theory are conventional. Note the necessity for each and every multiplication to require the sandwiching of the isotopic operator $\hat{T}$ in order to have $\hat{I}$ as the isounit (for these and all other aspects, see Ref. 22a).

The nontriviality of the isotopies here considered is illustrated by the fact that *the isoeigenvalues of an operator are generally different from its conventional eigenvalues.* In fact, starting from the expression $H \times |\psi\rangle = E_0 \times |\psi\rangle$, we have $\hat{H} \hat{\times} |\hat{\psi}\rangle = E \times |\hat{\psi}\rangle$, where the operator $\hat{H}$ is the same, but the eigenvalues E_0 and E are different. This result should not be surprising to the attentive reader because the theory under consideration is a *nonunitary* image of the quantum theory, and such transforms are known *not* to preserve the eigenvalues.

In actuality, Eqs. (3.11) establish that *the same Hermitian operator generally possesses an infinite class of different sets of eigenvalues, one per each selected unit,* thus disproving a rather popular belief that a Hermitian operator possesses a unique set of eigenvalues.

3.4. Isolinearity, Isolocality, Isounitarity

The current definition of (operator) *isotopies*, originally submitted in Ref. 3a but completed only in the recent special issue,[23] is that of *maps of any given linear, local and unitary, mathematical or physical structure into the most general possible nonlinear, nonlocal, and nonunitary forms which are capable of restoring linearity, locality, and unitarity in isospaces over isofields.* An understanding of these basic aspects is essential for this paper.

As we shall see better later on, the quantum mechanical representation of *exterior* systems (particles at large mutual distances compared to wavelengths) requires the knowledge of *two* quantities, the Hamiltonian H and the assumption of the trivial value I for the basic unit. Similarly, the isotopic representation of *interior* systems (particles at mutual distances equal to or smaller than their wavelengths) also requires the knowledge of *two* quantities, the Hamiltonian H representing all conventional exterior interactions and, this time, a nontrivial unit $\hat{I}$ representing *interior* nonlinear, nonlocal, and nonhamiltonian effects due to overlapping of the wavepackets (which occurs also for point-like charges).

The Hamiltonian is conventional and it is only rewritten in isospace. It is therefore time to begin acquiring more knowledge on the isounit, with the understanding that, as is the case for the Hamiltonian, its explicit and unique form can be solely fixed by the physical conditions considered

regarding shape, density, and other typical interior characteristics usually ignored in the Hamiltonian.

Besides the positive-definiteness, isotopic theories leave unrestricted the functional dependence of the isounit $\hat{I}$ and isotopic element $\hat{T}$, which can therefore depend on coordinates r, wavefunctions ψ, their derivative of arbitrary order, the local density μ of the considered interior problem, its local temperature τ, and any needed additional quantity,

$$\hat{I} = \hat{I}(r, p, \dot{p}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, ...) > 0, \qquad \hat{T} = \hat{T}(r, p, \dot{p}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, ...) > 0 \tag{3.13}$$

Moreover, the latter dependence is unrestricted in topological character, that is, it can be *arbitrarily nonlinear* in the wavefunctions or in any other quantity, *nonlocal*, e.g., of integral type, or of other types as well as of any other admissible character, e.g., discrete in time and/or space. As an illustration, the isounit used in some of the applications (see later on Secs. 3.7 and 3.15 for more details) is of the type

$$\hat{I} = \mathrm{Diag}(n_1^2, n_2^2, n_3^2, n_4^2) \times \exp(tN(\psi_\uparrow/\hat{\psi}_\downarrow + \partial\psi_\downarrow/\partial\hat{\psi}_\downarrow + \cdots) \times \int dv\, \psi_\uparrow^\dagger(r) \times \psi_\downarrow(r) \tag{3.14}$$

where the quantities n_1^2, n_2^2, n_3^2 represent the *extended, nonspherical, and deformable* shapes of the hadron considered; n_4^2 represents its *density*; the terms in the exponent $\psi_\downarrow/\hat{\psi}_\downarrow$, $\partial\psi_\downarrow/\partial\hat{\psi}_\downarrow$, etc., represent a typical nonlinearity, and the integral $\int dv\, \psi_\uparrow^\dagger(r) \times \psi_\downarrow(r)$ in the exponent represents a typical nonlocality due to mutual penetration and wave-overlapping of the charge distributions of the hadrons considered. A system of particles is evidently represented by an isounit which is the tensorial product of isounits of type (3.14).

Whenever the hadrons considered are perfectly spherical and perfectly rigid, $n_1^2 = n_2^2 = n_3^2 = 1$, the representation of their density is ignored, $n_4^2 = 1$, and the mutual distances are such as to imply no appreciable overlapping of the wavepackets, $\int dv\, \psi_\uparrow^\dagger(r) \times \psi_\downarrow(r) = 0$; then $\hat{I} \equiv I$, the considered nonunitary structure collapses into a unitary form, and conventional quantum mechanics is recovered identically, in accordance with our fundamental condition of Sec. 3.1.

As a result of the above occurrences, the lifting of conventional into isotopic eigenvalue equations is highly nonlinear as well as nonlocal and nonunitary:

$$H(r,p)\times|\psi\rangle = E\times|\psi\rangle \to H\hat{\times}|\hat{\psi}\rangle$$
$$= H(r,p)\times \hat{T}(r,p,\dot{p},\psi,\partial\psi,\partial\partial\psi,\mu,\tau,\dots)\times|\hat{\psi}\rangle$$
$$= E\times|\hat{\psi}\rangle \tag{3.15}$$

As a matter of fact, the above lifting is "directly universal" for all systems with conserved Hamiltonians (see Sec. 3.12 for nonconserved Hamiltonians), namely, it can represent all possible well-behaved, nonlinear, nonlocal, and nonunitary generalizations of the conventional eigenvalue equation (universality) in the coordinates of the observer (direct universality).

The first aspect to understand here is that, despite the above generality, the mapped theory does indeed satisfy *linearity in isospace over isofields*, called *isolinearity*.(22a) In fact, the lifted theory satisfies all familiar linearity conditions,

$$\hat{A}\hat{\times}(\hat{c}\hat{\times}|\hat{\psi}\rangle + \hat{d}\hat{\times}|\hat{\psi}\rangle) = \hat{c}\hat{\times}\hat{A}\hat{\times}|\hat{\psi}\rangle + \hat{d}\hat{\times}\hat{A}\hat{\times}|\hat{\psi}\rangle \tag{3.16a}$$

$$(\hat{c}\hat{\times}\hat{A} + \hat{d}\hat{\times}\hat{B})\hat{\times}\hat{\psi}\rangle = \hat{c}\hat{\times}\hat{A}\hat{\times}|\hat{\psi}\rangle + \hat{d}\hat{\times}\hat{B}\hat{\times}|\hat{\psi}\rangle \tag{3.16b}$$

$$(\hat{A}\hat{\times}\hat{B})\hat{\times}|\hat{\psi}\rangle = \hat{A}\hat{\times}(\hat{B}\hat{\times}|\hat{\psi}\rangle) \tag{3.16c}$$

As one can see, *the recovering of linearity in isospace is ensured by the embedding of all nonlinear terms in the isounit*. One can also prove that *any nonlinear theory can always be identically rewritten in an isotopic form*. In fact, all possible nonlinear theories (2.9) always admit the factorization of the nonlinear terms, which can then be assumed as the isotopic element of the theory

$$H(r,p,\psi,\dots)\times|\hat{\psi}\rangle = H_0(r,p)\times\hat{T}(r,p,\psi,\dots)\times|\hat{\psi}\rangle$$
$$= H_0\hat{\times}|\hat{\psi}\rangle, \qquad \hat{T} = H_0^{-1}\times H \tag{3.17}$$

where H_0 is the maximal (Hermitian) operator representing the total energy. A first advantage of the above identical reformulation is *the recovering of the superposition principle in isospace, thus permitting a consistent study of composite systems under nonlinear interactions*.(22b)

A similar occurrence holds for nonlocality. In fact, under the condition that all nonlocal terms are embedded in the isotopic element, isotopic theories verify *locality in isospace*, called *isolocality*.(23b) In fact, the theory is *everywhere local except at the isounit*.

A similar occurrence also holds for nonunitarity. In fact, any possible (well-behaved) nonunitary transform with the same "magnitude" $\hat{I}$ can always be identically reformulated in an *isounitary form on isospace over isofields*, called *isounitarity*,(23b) according to the rules

$$W\times W^{\dagger} = \hat{I} \neq I, \qquad W = \hat{W}\times\hat{T}^{1/2} \tag{3.18a}$$

$$W\times W^{\dagger} = \hat{W}\times\hat{T}\times\hat{W}^{\dagger} = W^{\dagger}\times W = \hat{W}^{\dagger}\times\hat{T}\times W = \hat{I} \tag{3.18b}$$

Note that the actions $H\times|\psi\rangle$ and $H\hat{\times}|\psi\rangle$ coincide at the abstract level. We can therefore state that *nonlinearity, nonlocality, and nonunitarity are not irreducible properties because they can be made to disappear at the abstract level under isotopies*.

The recovering of a classical canonical structure in phase space, called *isocanonicity*, is studied in Sec. 3.8.

3.5. Isotopic Realization of "Hidden variables" and "Completion" of Quantum Mechanics

At this intermediate stage of our analysis we can temporarily define the *isotopies of quantum mechanics* as a theory with dynamical equations (3.4) and (3.5) defined with respect to the isoenveloping operator algebras $\hat{\xi}$ on isohilbert spaces $\hat{\mathscr{H}}$ over isofields $\hat{C}$ with common isounit $\hat{I}$.

A fundamental property is that, in view of the positive-definiteness of $\hat{I}$ and $\hat{T}$, the isotopic theory coincides with quantum mechanics at the abstract realization-free level by conception(3b) and realization,(23b) because at the abstract level I and $\hat{I}$, ξ and $\hat{\xi}$, L and $\hat{L}$, C and $\hat{C}$, $\mathscr{H}$ and $\hat{\mathscr{H}}$, etc., coincide (see Ref. 22b when the isounit is no longer positive-definite).

To avoid misrepresentations, we should therefore stress that by no means do the above isotopies constitutes a "new theory." In fact, a new theory can only be claimed under *structurally novel axioms*. On the contrary, the above isotopic theory preserves the conventional abstract axioms by conception and construction. As such, *the isotopies merely provide new realizations of the abstract axioms of quantum mechanics, with the conventional realization recovered identically as a particular case for* $\hat{I} = I$.

In different terms, conventional quantum mechanics holds under the (necessary) assumption that the basic unit has the trivial value I. The studies herein reported have established that such an assumption is unnecessary, and that the same axioms also hold for arbitrary positive-definite units $\hat{I}$. Thus, *the isotopies identify the infinite class of realizations of the same quantum axioms characterized by all infinitely possible isounits* $\hat{I}$ with the quantum unit $\hat{I} = I$ as particular case.

It is understood that *theories with different isounits are mathematically equivalent but physically different*, otherwise it would be like pretending that nonunitary theories are physically equivalent to the unitary ones. Alternatively, we can say that the unit I is fixed in quantum mechanics and the same must be for *each* isounit $\hat{I}$ of the isotopic realizations.

The abstract identity of the isotopic and conventional operator theories is illustrates by the following *new invariance law of the Hilbert space* for $\hat{T}$ independent from the integration variables, here introduced apparently for the first time,

$$\langle \phi | \psi \rangle \equiv \langle \phi | \psi \rangle \times \hat{T} \times \hat{T}^{-1} = \langle \phi | \times \hat{T} \times | \psi \rangle \times \hat{I} = \langle \phi \uparrow \psi \rangle \tag{3.19}$$

and called *iso-self-scalarity*. The above invariance confirms the preservation of the original quantum axioms, as desired, for the preservation of Einstein's axioms and as otherwise needed for axiomatic consistency under nonunitary transforms.

Note that invariance (3.19) has remained undetected in this century. This should not be surprising because its identification required the prior discovery of *new numbers*, those with arbitrary unites.[26] In fact, invariance (3.19) cannot be defined via the conventional theory of numbers, that with the sole unit $+1$.

It is intriguing to note that *the isotopic theory here considered constitutes an explicit and concrete realization of the theory of "hidden variables"* λ (see, e.g., Ref. 27). In fact, we can rewrite Eq. (3.11) in the form

$$\hat{H} \hat{\times} |\hat{\psi}\rangle = \hat{H} \times \lambda \times |\hat{\psi}\rangle = \hat{E} \hat{\times} |\hat{\psi}\rangle = (E \times \lambda^{-1}) \times \lambda \times |\hat{\psi}\rangle = E_\lambda \times |\hat{\psi}\rangle$$
$$\hat{E} \in \hat{R}, \qquad E \in R \tag{3.20}$$

which does evidently provide said concrete and explicit realization of the "hidden variables" λ actually in the more general form of "hidden operators" $\lambda(r, p, \dot{p}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, \dots) = \hat{T} > 0$.

The "hidden" character is an evident consequence of the preservation of the quantum mechanical axioms. Note the nontriviality of the realization. In fact, the eigenvalues of a Hermitian operator turn out to be different for different "hidden operators" λ.

In fact, the axiomatic structure of conventional eigenvalue expressions is given by the *modular, associative action of an operator on a state* $H \times |\psi\rangle$, for which $(A \times B \times C) \times |\psi\rangle = A \times ((B \times C) \times |\psi\rangle) = (A \times B) \times (C \times |\psi\rangle)$. These axiomatic properties are preserved for the isotopies here considered because in the latter case we have a *modular isoassociative action of an operator on an isostate* $\hat{H} \hat{\times} |\hat{\psi}\rangle$ for which the preceding properties are preserved in isospaces in view of (3.15c). The important point from which the realization of "hidden variables" follows is that the two axioms "$H \times |\psi\rangle$" and "$\hat{H} \hat{\times} |\hat{\psi}\rangle$" coincide at the abstract, realization-free level on all grounds.

We can therefore state that *concrete and explicit realizations of "hidden variables" are indeed admitted by the abstract axioms of quantum mechanics, provided that they are realized in a nonunitary axiom-preserving way.*

Moreover, *the isotopies of quantum mechanics constitute a "completion" of quantum mechanics intriguingly along the celebrated argument by Einstein, Podolsky, and Rosen.*[28] In fact, the conventional unitary realization can be "completed" into an axiom-preserving nonunitary-isounitary form with evident structural broadening.

In particular, von Neumann's theorem[29] and Bell's inequalities[30] *do not* apply, trivially, because the considered theory is nonunitary. This reinforces the connection with the EPR argument, because *the nonunitarily transformed Bell's inequalities (the only ones applying under isotopies) can indeed admit a classical image in interior problems (only) owing to the arbitrariness in their classical limit under no unitary transforms* (see Ref. 22b, App. 4.C).

In different terms, in exterior problems in vacuum von Neumann's theorem and Bell's inequalities apply as is the case for all of quantum mechanics. In interior problems the situation is different under nonunitary-isotopic transforms, because they evidently imply a *necessary alteration of the upper boundaries* of the inequalities which now can admit a classical counterpart (see Ref. 22b, Appendix 4.C, for details and proofs, including the image of Pauli's matrices under nonunitary transforms as isorepresentations of the isotopic $S\hat{U}(2)$ symmetry).

A reason for the still unresolved controversies in these issues is that, in order to be nontrivial, any realization of "hidden variables" or "completion" of quantum mechanics must be *outside the class of equivalence of quantum mechanics*, that is, they must have a nonunitary structure. The isotopies then emerge as the sole known methods capable of formulating them in an axiom–preserving way.

The reader should meditate a moment on the implications of the above results. For instance, the above realization of "hidden variables" and "completion" of quantum mechanics imply that *discrete time theories* (see, e.g., Ref. 31) *are compatible with the abstract axioms of quantum mechanics, provided that they are realized in their isotopic form* (i.e., via the embedding of all discrete terms in the isounit of the theory).

Virtually all applications and verifications of the isotopic theories outlined in Sec. 3.14 are, strictly speaking, realizations and verifications of the theory of "hidden variables" and of the EPR "completion" of quantum mechanics.

Almost needless to say, we have considered here only the "isotopic" realization of "hidden operators" and "completion" of quantum mechanics, without any claim that it is unique, while encouraging the identification of inequivalent realizations.

3.6. Isotopies of Differential Calculus, Lie's Theory and Functional Analysis

Despite all the preceding studies (conducted since the proposal[3b] of 1978 and completed by the early 1990's), the isotopic theories were still afflicted by axiomatic inconsistencies of rather subtle origin which escaped prolonged efforts at their resolutions.

We come in this way to the crucial role of the special issue of *Rendiconti*.[23] In essence, lengthy studies on all possible alternatives indicated that the inconsistencies originated where one would expect them the least, in the *ordinary differential calculus*. Even though ignored because of protracted use over centuries, the dependence of the ordinary differential calculus on the basic unit I is rather fundamental because the differential calculus acts on rings of functions *defined over conventional fields*. Such a dependence becomes nontrivial for generalized units because in this case $d\hat{I} \neq 0$. The use of the conventional differential calculus for theories with generalized units is then bound to be inconsistent.

One should note that this is not a mere mathematical curiosity, because the issue directly affects the basic dynamical equations which, when defined via the conventional differential calculus in the time and space derivatives as in Eqs. (3.4) and (3.5), escape all efforts to achieve invariance.

Santilli therefore introduced in Ref. 23b the *isotopies of the differential calculus*, or *isodifferential calculus* for short, which is essentially based on the following simple, yet unique and unambiguous, *isodifferentials* and *isoderivatives*

$$\hat{d}r^k = \hat{I}^k_i \times dr^i, \qquad \hat{d}r_k = \hat{T}^i_k \times dr_i \tag{3.21a}$$

$$\hat{\partial}/\hat{\partial}r^k = \hat{T}^i_k \times \partial/\partial r^i, \qquad \hat{\partial}/\hat{\partial}r_k = \hat{I}^k_i \times \partial/\partial r_i \tag{3.21b}$$

$$\hat{d}p_k = \hat{T}^i_k \times dp_i, \qquad \hat{d}p^k = \hat{I}^k_i \times dp^i \tag{3.21c}$$

$$\hat{\partial}/\hat{\partial}p_k = \hat{I}^k_i \times \partial/\partial p_i, \qquad \hat{\partial}/\hat{\partial}p^k = \hat{T}^i_k \times \partial/\partial p^i \tag{3.21d}$$

with basic properties

$$\hat{\partial}r^i/\hat{\partial}r^j = \delta^i_j, \qquad \hat{\partial}r_i/\hat{\partial}r^j = \hat{T}^j_i, \qquad \hat{\partial}r^i/\hat{\partial}r_j = \hat{I}^i_j, \text{ etc.} \tag{3.22}$$

and other axiom-preserving properties here omitted for brevity.[23b]

It should be noted that other definitions, such as $\hat{d}r = d(\hat{I} \times r) = (r \times \partial\hat{I}/\partial r + \hat{I}) \times dr = \hat{I}' \times dr$, $\hat{I}' = r \times \partial\hat{I}/\partial r + \hat{I}$, lead to inconsistencies because they imply *the alteration of the basic unit under the operation of differential*, $\hat{I} \to \hat{I}' \neq \hat{I}$. This would imply the loss of the systems considered because of the lack of homomorphic map under differentiation. In fact, the original ring of functions would be mapped into another ring with a different unit with evident problematic aspects of various nature which are in general detected only following in-depth inspection.

An important confirmation of the axiom-preserving character of the isodifferential calculus (as formulated above with an invariant isounit) will be indicated in the next section.

The next isotopies needed for axiomatic consistencies are those of Lie's theory. In fact, the use of *conventional* symmetries for *nonunitary* theories also leads to serious inconsistencies, because Lie's theory is notoriously constructed with respect to the conventional unit $I = \text{diag}(1, 1, \ldots, 1)$, while the theories considered have the isounit $\hat{I} \neq I$.

This occurrence required the construction of the *step-by-step isotopies of Lie's theory*, including *the isotopies of enveloping associative algebras, Lie algebras, Lie groups, Lie symmetries, transformation and representation theory, etc.*, which were first proposed by Santilli in,[3a] studied in detail in Refs. 3b–3d, 4, 22, 23b and numerous other contributions, and are nowadays called the *Lie–Santilli isotheory* (see Refs. 5, 23c and papers quoted therein). The latter theory is essentially the reconstruction of all aspects of the conventional formulation of Lie's theory with respect to the generalized unit $\hat{I}$.

Regrettably, we cannot possibly review the latter, rather vast studies and are forced to refer the interested reader to Refs. 22a, 22b, 5c, 5e, 5f, and 23c. We merely mention that by no means was the Lie–Santilli isotheory conceived to discover new Lie algebras, because all these algebras (over a field of characteristic zero) are known from Cartan's classification.

By recalling that the current formulation of Lie's theory is strictly linear, local, and unitary, the Lie–Santilli isotheory is specifically intended to provide the broadest possible *nonlinear, nonlocal, and nonunitary realizations* of known Lie algebras and groups, according to the following main lines:

(a) the *universal isoassociative enveloping algebra* $\hat{\xi}$ with isounit $\hat{I}$ and isotopic product $\hat{A} \hat{\times} \hat{B}$ as characterized by the *isotopic Poincaré–Birkhoff–Witt theorem* (first formulated in the original proposal[3a, 3d] (see also Refs. 5c, 6g) with infinite-dimensional basis

$$\hat{\xi}: \hat{I}, \quad X_k, \quad \hat{X}_i \hat{\times} \hat{X}_j, \quad i \leq j, \quad X_i \hat{\times} \hat{X}_k, i \leq j \leq k, \ldots, \quad i, j, k = 1, 2, \ldots, N \tag{3.23}$$

from which we have the unique and unabiguous *isoexponentiation*

$$e_{\hat{\xi}}^{iw \times X} = \hat{e}^{iw \times X} = \hat{I} + w \times X/1! + (w \times X) \hat{\times} (w \times X)/2! + \cdots$$
$$= (e^{iX \times T \times w}) \times \hat{I} = \hat{I} \times (e^{iw \times T \times X}) \tag{3.24}$$