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GALILEI-ISOTOPIC SYMMETRIES

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ABSTRACT

We construct the most general known classical, nonlinear and nonlocal generalization of the conventional Galilei transformations, as well as the corresponding classical realization of the infinite family of Galilei–isotopic symmetries $\hat{G}$ (3.1) proposed in preceding works, under the condition that they result to be all locally isomorphic to the conventional Galilei symmetry. The symmetries $\hat{G}$ (3.1) are then used to characterize the largest possible class of nonlinear, nonlocal and nonhamiltonian Newtonian systems which still verify the conservation laws of all ten, total, physical quantities, as preparatory grounds for subsequent operator studies for the hadronic structure. The method for the explicit construction of the space–time isosymmetries $\hat{G}$ (3.1) from given Galilei–noninvariant equations of motion is outlined.

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As well known, the conventional *Galilei symmetry* G(3.1) (see, e.g., ref.s [1,2]) can be defined as the largest Lie group of *linear* transformations leaving invariant the separations

$$t_a - t_b = \text{inv.},$$
$$(r_{ia} - r_{ib})\,\delta_{ij}\,(r_{ja} - r_{jb}) = \text{inv. at } t_a = t_b, \qquad (1)$$
$$i,j = 1,2,3 \ (= x,y,z), \qquad a = 1,2,\dots,N$$

in $\Re_t \times T^*E(r,\delta,\Re)$, where $\Re_t$ represents time, $E(r,\delta,\Re)$ is the conventional Euclidean space, and T^*E its cotangent bundle (phase spacer), with metric $\delta = \text{diag.}(1,1,1)$ over the reals $\Re$. The explicit form of the *Galilei* transformations is given by the familiar expressions

$$t' = t + t^\circ, \quad \text{translations in time} \qquad (2a)$$
$$r'_{ia} = r_{ia} + r^\circ_i, \quad \text{translations in space} \qquad (2b)$$
$$r'_{ia} = r_{ia} + t^\circ v^\circ_i, \quad \text{Galilei bosts} \qquad (2c)$$
$$r'_a = R(\theta)\, r_a, \quad \text{rotations.} \qquad (2d)$$

A classical realization of G(3.1) (for the case of non-null masses $m_a \neq 0$, $a = 1,2,\dots,N$ herein assumed) is characterized by the (ordered sets of) parameters

$$w = (w_k) = (\theta_i, v^\circ_i, r^\circ_i, t^\circ), \quad k = 1,2,\dots 10, \quad i = 1,2,3 \quad (3)$$

and generators

$$X = (X_k) = (J_i, G_i, P_i, H), \qquad (4a)$$
$$J_i = \textstyle\sum_a \epsilon_{ilm}\, r_{ia}\, p_{ia}, \quad P_i = \sum_a p_{ia}, \qquad (4b)$$
$$G_i = \textstyle\sum_a (m_a r_{ia} - t p_{ia}), \quad H = p_{ia}\, p_{ia}/2m_a + V(r_{ab}), \qquad (4c)$$
$$r_{ab} = r_a - r_b, \quad i = 1,2,3, \quad k = 1,2,\dots,10, \quad a,b = 1,2,\dots,N$$

with canonical realization of the Lie algebra **G(3.1)** via the conventional Poisson brackets

$$[J_i, J_j] = \epsilon_{ijk} J_k, \quad [J_i, P_j] = \epsilon_{ijk} P_k, \tag{5a}$$

$$[J_i, G_j] = \epsilon_{ijk} G_k, \quad [J_i, H] = 0, \tag{5b}$$

$$[G_i, P_j] = \delta_{ij} M, \qquad [G_i, H] = P_i, \tag{5c}$$

$$[P_i, P_j] = [G_i, G_j] = [P_i, H] = 0, \tag{5d}$$

$$M = \sum_a m_a, \tag{5e}$$

Casimir invariants

$$C_0 = I, \qquad C_1 = P^2 - 2MH, \quad C_2 = (MJ - G \wedge P)^2, \tag{6}$$

and canonical realization of the group structure G(3.1)

$$G(3.1): \quad a' = g(w)\, a = \{ \exp [w_k\, \omega^{\mu\nu} (\partial_\nu X_k) (\partial_\mu)] \}\, a \tag{7}$$

$$\partial_\mu = \partial / \partial a^\mu, \quad a = (a^\mu) = (r_i{}^a, p_i{}^a), \ \mu = 1,2,...,6N,$$

where $\omega^{\mu\nu}$ is the canonical Lie tensor [1,2].

In our original submission of the *Lie-isotopic theory* [3], we proposed the construction of a step-by-step generalization of classical Hamiltonian mechanics , under the name of Birkhoffian *mechanics*, which possesses a Lie-isotopic structure. In the same memoir, we recommended the construction of the consequential, expected existence of the *Galilei-isotopic symmetries* $\hat{G}(3.1)$ and related *Galilei-isotopic relativities*. [The proposal was made as a particular case of the yet more general Lie-admissible formulations not considered here].

After much additional research, the proposal was worked out in detail in monographs [4,5] which presented the most general possible formulation of the Galilei-isotopic symmetries $\hat{G}(3.1)$ on the conventional space $\Re_t \times T^*E(r,\delta,\Re)$ and related covering relativities, which are *nonlinear and nonhamiltonian*, yet *local-differential*. This was due to the fact that the underlying geometry, the conventional *symplectic geometry*, is strictly local, thus preventing any topologically consistent treatment of nonlocal-integral interactions.

Despite the appearance of monograph [5] (submitting the generalized relativities beginnign with the title of Chap. 6), the Galilei-isotopic symmetries remained ignored in the subsequent years. The studies were therefore continued by this author alone.

A first subsequent advancement was made in ref. [6] with the introduction of the notion of *isospace*, such as the *Euclidean-isotopic space* $\hat{E}(r,g,\hat{\Re})$ (where g is a nonsingular and Hermitean generalization of δ, called *isometric*, and $\hat{\Re} = \Re\,\hat{1}$, $\hat{1} = g^{-1}$, is the *isofield* of real *isonumbers* $\hat{N}$ with ordinary sum and isomultiplication $\hat{N}_1 * \hat{N}_2 = (N_1 N_2)\,\hat{1}$).

Ref. [6] also presented the *isolinear transformation theory on isospaces*. We are here referring to transformations $r' = \hat{T}(w)*r = \hat{T}(w)\, g(r,\dot{r},\ddot{r},..)\, r$, where g is fixed, $r,r' \in \hat{E}(r,g,\hat{R})$, which evidently coincide at the abstract level with the conventional transformations, $r' = T(w)\, r$, by construction.

The above notions permitted the presentation in the subsequent paper [7] of the most general known *isolinear formulation of the isorotational subgroup* $\hat{O}(3)$ of G(3.1).

The main advance of Refs.[6,7] over the original proposals [3,5] was essentially that of constructing the isotopes $\hat{O}(3)$ by preserving the formal linearity of O(3), while having in reality an intrinsically nonlinear theory.

In fact, *a theory can be said to be isolinear when it is formally linear in an isotopic space, say $\hat{E}(r,g,\hat{R})$, but intrinsically nonlinear when referred to the conventional space, say $E(r,\delta,\Re)$*. The reader should keep in mind that, under sufficient topological conditions, *all nonlinear theories in an ordinary metric space can be cast into an identical isolinear form on an isospace* [6].

The arbitrariness of the isotopic element g in the isomodular action $\hat{T}(w)*r$ then implied the *direct universality of the nonlinearity of isorotations* $\hat{O}(3)$ of ref. [7], in the sense of containing as particular case, all conceivable nonlinear generalizations of O(3) (universality), directly in the frame of the observer (direct universality).

The classical Birkhoffian realization of $\hat{O}(3)$ for nonhamiltonian systems was touched in ref. [7], but only in *local approximation*, because the underlying classical geometry was still the conventional symplectic geometry.

A second advancement over the original proposal [3,5] has been made in note [8] of this series, via the identification of the true geometry underlying the Lie-isotopic algebras, under the proposed name of *symplectic-isotopic geometry*.

In essence, the dichotomy existing prior to ref. [8] was that, on the one hand, the abstract formulation of the Lie-isotopic algebras readily permitted the treatment of nonlocal-integral terms in a topologically consistent way, via their incorporation in the isounit $\hat{1}$ of the theory. On the other hand, the symplectic geometry prevented the treatment of nonlocal-integral term owing to its local-differential structure recalled earlier.

As a result, while operator realizations of Lie-isotopic algebras could incorporate (nonlinear as well as) nonlocal interactions, their classical Birkhoffian counterpart could only allow (nonlinear, yet) local-differential settings.

In fact, the construction of the isotopes $\hat{O}(3)$ of ref. [7] was *nonlinear and nonlocal* in its abstract [e.g. matrix) presentations, but then restricted to *local* forms when passing to the classical Birkhoffian realization.

In turn, the lack of a topologically consistent, nonlocal theory at the primitive Newtonian setting created predictable restrictions in the operator formulation of the theory.

Note [8] resolved this problem via: 1) the realization of the symplectic two-forms Ω_2 in *isocotangent bundle* $T^*\hat{E}_2(r,G,\mathfrak{R})$; 2) their restriction to the two-forms $\hat{\Omega}_2$ that are exact and factorizable as in the coordinate-free expression

$$\hat{\Omega}_2 = \omega_2 \times T_2 = d(\theta_1 \times T_1), \quad T_1, T_2 > 0, \tag{8}$$

or in the corresponding realization in the local chart a

$$\begin{aligned}\hat{\Omega}_2 &= [\tfrac{1}{2}\omega_{\mu\sigma} \times T_{2\sigma\nu}(a)]\, da^\mu \wedge da^\nu \\ &= d[R^\circ \times T_1(a)]_\mu\, da^\mu, \quad R^\circ = (p,0)\end{aligned} \tag{9}$$

where T_1 = diag. $(g, g) > 0$ (T_2= diag. $(G, G) > 0$) is the isotopic element of the underlying isospace $T^*E_1(r,g,\mathfrak{R})$ ($T^*\hat{E}_2(RG,\mathfrak{R})$) with isounit $\hat{1}_1 = T_1^{-1}$ ($\hat{1} = T_2^{-1}$) and ω_2 (θ_1) is the canonical two-form (one-form); and 3) by incorporating all the nonlocal-integral terms in the isotopic elements T_1 and T_2.

Topological consistency is achieved because the symplectic-isotopic geometry is structurally characterized by the

conventional canonical forms and related conventional local-differential topology. The emerging geometry is then insensitive to the topology of its isounit, provided that it is positive-definite.

In fact, the Lie algebra brackets characterized by two-forms (9) are given by the following expression among two functions A, B $\in T^*\hat{E}$ [1]

$$[A\hat{,} B] = (\partial_\mu A)\, \omega^{\mu\sigma} \times \hat{1}_2{}^{\sigma\nu} (\partial_\nu B), \tag{10a}$$

$$\omega^{\mu\sigma} \times \hat{1}_2{}^{\sigma\nu} = (|\omega_{\alpha\beta}|^{-1})^{\mu\sigma} (|T_{2\rho\tau}|^{-1})^{\sigma\nu}, \quad \hat{1}_2 > 0 \tag{10b}$$

In this way, note [8] identified a classical realization of the Lie-isotopic product which exhibits the explicit presence of the isounit $\hat{1}_2$. This illustrates the reasons why the symplectic-isotopic geometry is the *bona-fide* geometry underlying the Lie-isotopic algebras.

The above results permitted the achievement of the first *nonlinear and nonlocal classical realization of the isorotation, symmetries* $\hat{O}(3)$ [8], which were then extended to the *Euclidean-isotopic symmetries* $\hat{E}(3)$ in the subsequent note [9].

In this note we shall introduce the most general known nonlinear and nonlocal, classical realization of the Galilei-isotopic symmetries $\hat{G}(3.1)$ which can be introduced as follows.

DEFINITION: The general nonlinear and nonlocal, classical realization of the Galilei-isotopic symmetries $\hat{G}(3.1)$ is given by the Lie-isotopic groups of the most general possible transformations on $\hat{\mathfrak{R}}_t \times T^*\hat{E}_2(r,g,\mathfrak{R})$

$$t_a - t_b = \text{inv.}, \tag{11a}$$

$$(r_{ka} - r_{kb})\, B_k{}^2(r,p,\ldots)\, (r_{ka} - r_{kb}) = \text{inv. at } t_a = t_b, \tag{11b}$$

$$t_a, t_b \in \hat{\mathfrak{R}}_t, \qquad r_a, r_b \in T^*\hat{E}_2(r,G\mathfrak{R}) \tag{11c}$$

where $\hat{\mathfrak{R}}_t$ is an isotopic lifting of the conventional field $\mathfrak{R}_t$, here called isotime field, with explicit structure

$$\hat{\Re}_t = \Re\,\hat{1}_t, \quad \hat{1}_t = B_4^{-2}(r,p,...), \quad B_4 > 0, \qquad (12)$$

$T^*\hat{E}_2(r,G,\hat{\Re})$ is the isocotangent bundle for symplectic-isotopic two-forms with isometrics G diagonabilzable to the form

$$G = \text{diag.}\,(B_1^2, B_2^2, B_3^2), \qquad B_k = B_k(r,p,...) > 0, \qquad (13)$$

and the four functions B_1, B_2 B_3 and B_4 besides being independent and positive-definite, are arbitrary nonlinear and nonlocal (e.g., integral) functions on all possible, or otherwise needed local variables and quantities.

The reason for the additional lifting $\Re_t \Rightarrow \hat{\Re}_t$ in the trasition from the Euclidean-isotopic symmetries $\hat{E}(3)$ of note [9] to the Galilei-isotopiç symmetries $\hat{G}(3.1)$ of this note, will be evident shortly, although their ultimate meaning will appear in the study of the nonrelativistic limit of the Poincaré-isotopic symmetries to be presented in future notes of this series.

At this point we merely recall from the results of ref. [6] that the use of the isotime field does not affect the physical time. In fact, isofields possess the conventional sum, but have the isotopic multiplication

$$\hat{t}_1 * \hat{t}_2 = (t_1 t_2)\,\hat{1}_t. \qquad (14)$$

As a result, we have identities of the type $\hat{t} * A \equiv tA$ which justify the setting of the measurement theory with respect to the ordinary time.

LEMMA 1: The general nonlinar and nonlocal, classical realization of the Galilei-isotopic symmetries $G(3.1)$ on $\hat{\Re}_t \times T^*\hat{E}_2(r,G,\hat{\Re})$ as per the above Definition, can be written

$$t' = t + t^\circ \hat{B}_4^{-2}(t^\circ), \qquad \text{iso-time translations} \quad (15a)$$

$$r_i' = r_i + r^\circ_i \hat{B}_i^{-2}(r^\circ), \qquad \text{iso-space translations} \quad (15b)$$

$$r_i' = r_i + t^\circ \hat{B}_i^{-2}(v^\circ), \quad \text{iso-Galilei boosts} \qquad (15c)$$

$$r' = \hat{R}(\theta) * r, \qquad \text{isorotations,} \qquad (15d)$$

where the $\hat{B}$-functions are generally nonlinear and nonlocal in all possible local variables and quantities to be identified shortly. Moreover, the Galilei-isotopic symmetries $\hat{G}(3.1)$ are characterized by the Lie-isotopic brackets (10) underlying the exact symplectic-isotopic two-forms, with explicit expression

$$[A\hat{,} B] = \frac{\partial A}{\partial r_k} B_k^{-2} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} B_k^{-2} \frac{\partial B}{\partial r_k} \qquad (16)$$

and possess the following structure:
1) the conventional parameters (3), i.e.

$$w = (w_k) = (\theta_i, r^\circ_i, v^\circ_i, t^\circ), \quad k = 1,2,...10, \qquad (17)$$

2) the conventional generators (4), but now defined on isospace $\hat{\Re}_t \times T^*\hat{E}_2(r,G,\hat{\Re})$, i.e.

$$J_i = \sum_a \epsilon_{ijk}\, r_{ia} p_{ia}, \qquad P_i = \sum_a p_{ia}, \qquad (18a)$$

$$G_i = \sum_a (m_a\, r_{ia} - t\, p_{ia}), \qquad (18b)$$

$$H = p_{ka} B_k^2 p_{ka} / 2m_a + V(r_{ab}), \qquad (18c)$$

$$r_{ab} = |r_a - r_b|^{\hat{}} = \{(r_{ka} - r_{kb}) B_k^2 (r_{ka} - r_{kb})\}^{\frac{1}{2}}, \qquad (18d)$$

3) the Lie-isotopic algebra $\hat{G}(3.1)$

$$[J_i \hat{,} J_j] = \epsilon_{ijk} B_k^{-2} J_k, \quad (J_i \hat{,} P_j] = \epsilon_{ijk} B_j^{-2} P_k, \qquad (19a)$$

$$[J_i \hat{,} G_j] = \epsilon_{ijk} B_j^{-2} G_k, \quad [J_i \hat{,} B] = 0, \qquad (19b)$$

$$[G_i \hat{,} P_j] = \delta_{ij} M B_j^{-2}, \quad [G_i \hat{,} B] = 0, \qquad (19c)$$

$$[P_i \hat{,} P_j] = [G_i \hat{,} G_j] = [P_i \hat{,} B] = 0, \qquad (19d)$$

4) the Lie-isotopic group $G(3.1)$

$$r' = \{\prod_k \exp [\, w_k\, \omega^{\mu\sigma} \times 1_2^{\sigma\nu} (\partial_\nu X_k) (\partial_\mu) \hat{1}_2\} * r, \qquad (20)$$

and

5) the explicit expressions of the $\hat{B}_i$ functions

$$\hat{B}_i^{-2}(r^\circ) = B_i^{-2} + r^\circ_j [B_i^{-2} \hat{,} P_j] / 2! + r^\circ_m r^\circ_n [[B_i^{-2} \hat{,} P_m \hat{,} P_n / 3! + \ldots \qquad (21a)$$

$$\hat{B}_i^{-2}(v^\circ) = B_i^{-2} + v^\circ_j (B_i^{-2} \hat{,} G_j] / 2! + v^\circ_m v^\circ_n [[B_i^{-2} \hat{,} G_m] \hat{,} G_n] + \ldots \qquad (21b)$$

while $\hat{B}_4^{-2}(t^\circ)$ is the solution of the algebraic equation

$$r(t + t^\circ \hat{B}_4^{-2}) = \{\exp [\, t^\circ \omega^{\mu\sigma} \times \hat{1}_2^{\sigma\nu} (\partial_\mu H) (\partial_\nu)\} \, r. \qquad (22)$$

The infinite family of Galilei-isotopic symmetries so constructed result to be all locally isomorphic to the conventional Galilei symmetry under the conditions of sufficient smoothness, nonsingularity and positive-definitness of the isounits. Finally, all isosymmetries $\hat{G}(3.1)$ can approximate the conventional symmetry $G(3.1)$ as close as desired whenever the isounits approach the conventional unit, and they all admit the conventional symmetry as a particular case by construction.

• The proof of the above Lemma is straightforward. As now familiar, the Lie-isotopic theory preserves, by central condition, the parameters and generators of the conventional symmetries, and this illustrates points 1) and 2). The Lie-isotopic algebra $\hat{G}(3.1)$ can then be readily computed via the use of brackets (16), and this proves point 3). The exponentiation to the Lie-isotopic group $\hat{G}(3.1)$ then follows uniquely via the use of the general theory. The application of such exponentiations to the local coordinates then yields the explicit forms (15) of the nonlinear and nonlocal Galilei-isotopic transformations with explicit form (21) of the $\hat{B}$-functions. Finally, the isounit of the time isofield, $\hat{B}_4^{-2}(t^\circ)$ is provided by the solution of e=Eq. (23).

Note in this latter respect that, for $\hat{1}_2 = I$ one recovers the conventional Galilei symmetry $G(3.1)$, in which case $\hat{B}_4 = 1$ in Eq. (23), which provides the canonical representation of linear translations in time. However, for $\hat{1}_2 \neq I$, Eq.s (22) evidently can hold no longer for $t' = t + t^\circ$. The lifting to form (15a) then follows.

Explicit examples will be worked out in the subsequent notes. The preceding results evidently include those for the Euclidean-isotopic symmetries $\hat{E}(3)$ [9], as well as of the isorotational symmetries [8].

The proof of the local isomorphisms $\hat{G}(3.1) \approx G(3.1)$ for the case of constant $\hat{B}$-quantities can be done via the use of redefinitions (3) of ref. [8]. For the general case, one can note that, locally (that is, at a given point of space-time) the $\hat{B}$-functions can also be approximated with constants, and the preceding proof locally applies. The global case is not considered because it is beyond the scope of this note.

It is an instructive exercise for the interested reader to prove that the infinite family of isosymmetries $\hat{G}(3.1)$ so constructed do indeed verify the conditions of the Definition above and, in particular, do constitute isosymmetries of invariants (11).

COROLLARY 1A: In the particular case of constant isometrics, we have

$$\hat{\mathfrak{R}}_{t\times T^*\hat{E}_1}(r,g,\hat{\mathfrak{R}}) \equiv \hat{R}_{t\times T^*\hat{R}_2}(r,G,\hat{\mathfrak{R}}) \equiv \mathfrak{R}_{t\times T^*\hat{E}}(r,g,\hat{\mathfrak{R}}), \qquad (23a)$$

$$g = G = \mathrm{diag.}\,(b_1^2, b_2^2, b_3^2), \quad b_k = \mathrm{cost.} > 0; \qquad (23b)$$

the $\hat{B}$-quantities coincide with the diagonal elements of the isounits,

$$\hat{B}_i^{-2}(r^\circ) \equiv \hat{B}_i^{-2}(v^\circ) \equiv B_i^{-2} \equiv b_i^{-2}, \qquad \hat{B}_4^{-2}(t^\circ) \equiv 1. \tag{24}$$

and the general Galilei-isotopic transformations (15) become linear, i.e., they assume the simplified form of the linear Galilei-isotopic transformations

$$t' = t + t^\circ, \tag{25a}$$

$$r'_i = r_i + r^\circ_i b_i^{-2}, \tag{25b}$$

$$r'_i = r_i + t^\circ v^\circ_i b_i^{-2}, \tag{25c}$$

$$r' = \hat{R}(\theta) * r. \tag{25d}$$

The above properties, whose proof is trivial, have important implications from a relativity viewpoint. In fact, they imply that the *Galilei-isotopic symmetries can indeed preserve inertial frames,* but, of course, in their linear particularization.

The isocasimirs of $\hat{G}(3.1)$ for the case of constant isometrics are given by

$$\hat{C}_o = \hat{1}_2, \quad \hat{C}_1 = P^{\hat{2}} - MH\hat{1}_2, \qquad \hat{C}_2 = (MJ - G \wedge P)^{\hat{2}}, \tag{26}$$

where the squares are evidently of isotopic type (as a necessary condition to be in the center of the underlying isoassociative algebra [3,5]).

Isocasimirs (26) also hold locally for the general case of arbitrary dependence of the isometrics. The problem of the isocasimirs for the global case requires a study of the isoscalar extensions of the Galilei-isotopic symmetries, isoassociative envelopes in classical realization and their neutral elements. As such, this study will be conducted at some later time.

The application of the Galilei-isotopic symmetries $\hat{G}(3.1)$ to the characterization of closed nonselfadjoint systems [5,10] can now be formulated. In fact, the use of Theorems 1 and 2 of Ref. [8] readily yields the following

LEMMA 2: The subclass of closed nonselfadjoint systems, Eq. (3) of Ref. [10], is invariant under the Galilei-isotopic symmetries $\hat{G}(3.1)$ when they can be consistently written in isospace $\hat{R}_t \times T^*\hat{E}_2(r,G,\hat{\mathfrak{R}})$ and admit the representation in terms of the symplectic-isotopic or, equivalently, Lie-isotopic Birkhoffian representation [8]

$$\omega_{\mu\sigma}\, T_{2\sigma\nu}(a)\, \dot{a}^\nu = \partial_\mu H, \qquad \dot{a}^\mu = \omega^{\mu\sigma}\, \hat{1}_2{}^{\sigma\mu}(a)\, \partial_\nu H, \tag{27a}$$

$$H = p_{ia}\, G_{ij}(r,p,..)\, p_{ja}/2m_a + V(r_{ab}), \tag{27b}$$

$$r_{ab} = \{(r_{ia} - r_{ib})\, G_{ij}(r,p,..)\, (r_{ja} - r_{jb})\} \tag{27c}$$

in which case all total quantities (18) are not subsidiary constraints, but first integrals of the equations of motion.

It should be understood that the imposition of the Galilei-isotopic invariance *restricts* the closed nonselfadjoint systems, Eqs. (3) of Ref. [10], from the general class with subsidiary constraints (3b), to that particular subclass in which the total quantities are automatically conserved in virtue of the equations of motion.

We close this note with a few remarks on the problem of the *explicit construction of the Galilei-isotopic symmetries.*

As well known, in the conventional canonical treatment of mechanics, the Galilei symmetry G(3.1) is preassigned. Physical systems are then restricted to those which are G(3.1)-invariant. This evidently results in severe limitations in the class of systems admitted, which are essentially given by the linear, local and Hamiltonian systems of the perpetual-motion type.

In the covering Birkhoffian mechanics, the situation is reversed [3,4]. In fact, one considers, first, the equations of motion as provided by the experimental evidence, and then searches for their space-time symmetries.

The Birkhoffian realization of the Lie-isotopic theory [3,4,8] has been conceived also for the explicit construction of the isotopic covering $\hat{G}(3.1)$ of G(3.1) from given, G(3.1)-noninvariant equations of motion, with the consequentia result of substantially broadening the class of admitted systems, while preserving the conventional class as a particular case.

The rules for the explicit construction of the covering $\hat{G}(3.1)$ symmetries from given equations of motion are so simple, to appear trivial. In fact, one merely has to write the system considered in the symplectic-isotopic form (27) on $\hat{\mathfrak{R}}_t \times T^*\hat{E}_2 r,G,\hat{\mathfrak{R}})$. This provides the fundamental isounit $\hat{1}_2$ which characterizes the Lie-isotopic structure (20) of $\hat{G}(3.1)$. The rest of the isosymmetry (20) is characterized by the *conventional parameters* w_k and by the *conventional generators* X_k (only properly written in $\hat{\mathfrak{R}}_t \times T^*\hat{E}_2(r,G,\hat{\mathfrak{R}})$).

Note finally that, under a sufficient smoothness of the isounit (generally assured by its positive-definiteness), the existence and convergence of the infinite expansions (20) is guaranteed by those of the conventional structure (7). The reader should, however, not easily expect summable series, as illustrated by the sum into a *trascendental function* of the original proposal [3].

The first (and perhaps most important) examples of $\hat{G}(3.1)$-invariant systems are the two-body and three-body, closed nonselfadjoint systems of Ref.. [10]. Additional explicit examples will be given later on in this series, following the identification of the relativities underlying the isosymmetries $\hat{G}(3.1)$.

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