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ABSTRACT

As a necessary additional step in preparation of our operator studies of closed nonhamiltonian systems, in this note we consider the algebraic structure of the original equations proposed by Lagrange's and Hamilton's, those with external terms representing precisely the contact nonpotential forces of the interior dynamical problem. We show that the brackets of the theory violate the conditions to characterize any algebra. Nevertheless, when properly written, they characterize a covering of the Lie-isotopic algebras called *Lie-admissible algebras*. It is indicated that a similar occurrence exists for conventional operator treatments, e.g. for nonconservative nuclear cases characterized by nonhermitean Hamiltonians. This occurrence then prevents a rigorous treatment of basic notions, such as that of angular momentum and spin spin, which are centrally dependent on the existence of a consistent algebraic structure. The emergence of the Lie-admissible algebras is therefore expected to be unavoidable for any rigorous operator treatment of open systems with nonlinear, nonlocal and nonhamiltonian external forces.

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This series of notes cannot provide a sufficient Newtonian basis for operator formulations of the nonselfadjoint systems, without the identification of the algebraic structure of the *original Hamilton's equations* [1], those with external terms

$$\dot{r}_{ka} = \partial H / \partial p_{ka}, \quad (1a)$$

$$\dot{p}_{ka} = - \partial H / \partial r_{ka} + F_{ka}, \quad (1b)$$

$$H = p_{ka}\, p_{ka} / 2m_a + V(r), \quad F_{ka} = F_{ka}(t,r,p), \quad (1c)$$

$$k = 1, 2, 3\ (= x, y, z), \quad a = 1, 2, ..., N$$

as well as of the most general possible *nonautonomous Birkhoff's equation* [2,3] in $T^*E(r,\delta,\Re)$ with local coordinates $a = (a^\mu) = (r, p) = (r_k, p_k)$, i.e.

$$\dot{a}^\mu = \Omega^{\mu\nu}(t,a)\, [\, \partial_\nu B(t,a) + \partial_t R_\nu(t,a)\,], \quad \mu, \nu = 1, 2, ..., N, \quad (2a)$$

$$\Omega^{\mu\nu} = (\,|\,\Omega_{\alpha\beta}|^{-1})^{\mu\nu}, \quad \Omega_{\mu\nu} = \partial_\mu R_\nu(t,a) - \partial_\nu R_\mu(t,a), \quad (2b)$$

These studies were originally conducted in Refs. [4-6], and then continued in Refs. [7,8]. A comprehensive presentation can be found in ref.s [9]. It is important here to review the central ideas of these studies and identify the necessity of their complementary role with respect to the Lie-isotopic research of the preceding notes (10-13).

To begin, the conventional Poisson brackets [A,B] of Hamilton's equations without external terms are generalized for Eqs. (1) in a form, say A×B, which is explicitly given by

$$A \times B = [A,B] + (\partial A / \partial p_{ka})\, F_{ka}. \quad (3)$$

It is easy to see that the above brackets do not characterize an algebra as commonly understood (see, e.g., ref. [13] and quoted literature) because they violate the right scalar and right distributive laws, i.e.

$$\alpha \times (B \times C) = A \times (\alpha \times B) = (\alpha \times A) \times B, \quad (4a)$$

$$(A \times B) \times \alpha \neq A \times (B \times \alpha) \neq (A \times \alpha) \times B, \qquad (4b)$$

and

$$(A + B) \times C = A \times C + B \times C, \qquad (5a)$$

$$A \times (B + C) \neq A \times B + A \times C, \qquad (5b)$$

In different terms, *in the transition from the contemporary Hamilton's equations to their original form with external terms, we have the loss, not only of the Lie algebras, but more precisely of all consistent algebraic structures.*

Exactly the same situation occurs for the quantum mechanical treatment of nonconservative forces via nonhermitean Hamiltonians $H \neq H^{\dagger}$ [8]. In fact, under these conditions, the conventional Heisenberg's brackets among operators A, H, ... on a Hilbert space, $[A,H] = AH - HA$, over a complex field C are generalized into a form, say $A \tilde{\times} H$, which is evidently defined by the new equations of motion

$$i \dot{A} = A \tilde{\times} H = AH - H^{\dagger} A \quad \hbar = 1, \qquad (6)$$

Again, *the nonconservative Heisenberg's brackets $A \tilde{\times} H$, not only lose the Lie algebra character of conventional quantum mechanics, but do not characterize any consistent algebra, because they violate the right scalar and right distributive laws,* as the reader is encouraged to verify.

This is not a mere mathematical occurrence, because it carries rather deep physical implications. For instance, the notion of angular momentum can be consistently defined in conventional (classical and quantum) Hamiltonian mechanics, and treated via its underlying Lie symmetry O(3). In the transition to Hamilton's equations with external terms and their operator counterpart (6), we have lost all Lie algebras, let alone that of the rotational symmetry, with the consequential inability to provide a truly consistent, quantitative treatment of the angular momentum.

In fact, it would be fundamentally inconsistent to use one product $A \tilde{\times} H$ for the time evolution, and a *different product* [A,H] for the characterization of physical quantities such as the angular momentum. This is due to the well known ancient rule of dynamics

whereby *the product of the algebra characterizing a given theory, whether classically or operationally, must coincide with that of the time evolution law.*

One therefore has the insidious occurrence, which is rather widespread in the physical literature (particularly that of nuclear physics) whereby notions centrally dependent on the existence and consistency of underlying Lie symmetries, such as that of angular momentum and spin, continues to be used in conjunction with equations of motion of type (6), although they have lost all mathematical foundations for their existence.

To put it explicitly, a statement to the effect that, say, a particle described by Eqs. (6) has spin one, is fundamentally inconsistent, mathematically, because of the loss of any algebra, and, physically, because the spin of particles in open nonconservative conditions is ultimately unknown to this writing.

Exactly the same situation occurs for the nonautonomous Birkhoff's equations. In fact, Birkhoff's brackets $[A \hat{,} B]$ for the autonomous case [3,10],

$$[A \hat{,} B] = (\partial_{\mu} A)\, \Omega^{\mu\nu}(a)\, (\partial_{\nu} B), \qquad (7)$$

have to be generalized for Eqs. (2) in the form [6]

$$A \odot B = (A \hat{,} B) + (\partial_{\mu} A)\, \Omega^{\mu\nu}\, (\partial_t R_{\mu}) \qquad (8)$$

Equivalently, one can say that for the case of time-dependent R-functions, Birkhoff's equations can be expressed with the $(2N+1) \times (2N+1)$ contact tensor of Eqs. (2b) which, being odd-dimensional, do not admit a consistent contravariant (Lie) counterpart [3].

The reader should therefore be aware that *the Galilei-isotopic relativities studied in* Refs. *[3,7,11] are inapplicable to the nonautonomous Birkhoff's equations, because of the loss of a consistent algebraic structure, let alone the loss of their Lie-isotopic character:*

The above occurrences evidently create the problem of identifying the relativities which are directly applicable to open, nonconservative, nonautonomous, Newtonian systems, such as oscillator with a time-dependent applied force, etc.).

In turn, the above relativities cannot be identified without first reformulating Eqs. (1) and (2) in an analytically *identical* way (to avoid the alteration of the equations of motion) which is however admitting a consistent algebraic structure.

This problem signals the birth of the Lie-admissible algebras [14] in physics. In fact, on one hand, the consistent brackets, say, (A,B), cannot be antisymmetric, to permit the representation of the time-rate-of-variation of the energy

$$\dot{H} = (H,H) = (\partial H / \partial p_{ka}) F_{ka} = v_{ka} F_{ka}, \quad (9)$$

while, on the other hand, Lie algebras cannot be abandoned [4], but must be admitted as a particular case for null nonselfadjoint forces, i.e.

$$(A,B)\big|_{F_{ka}=0} \equiv [A,B]. \quad (10)$$

This problem was studied in Refs. [4-6] and reinspected in ref. [7], where it was pointed out that conditions (9,10) identify uniquely the so-called *general Lie-admissible algebras.*

According to Albert [14], an algebra **U** with (abstract) elements a,b,c,...and (generally nonassociative, abstract) product ab over a field **F** is called a *Lie-admissible algebra,* when the attached algebra $\mathbf{U}^-$, which is the same vector space as **U**, but equipped with the product

$$\mathbf{U}^- : \; [a,b]_{\mathbf{U}} = ab - ba, \quad (11)$$

is Lie.

A first classification of Lie-admissible algebras was conducted in ref. [7]. Evidently, all *associative algebras* **A** are Lie-admissible, resulting in the familiar Lie product $[a,b]_{\mathbf{A}} = ab - ba$, where now ab is associative.

All *Lie algebras* **L** with (abstract) product ab are also Lie-admissible, because $[a,b]_{\mathbf{L}} = 2[A,B]_{\mathbf{A}}$, where now ab is non-associative. Thus, Lie algebras are contained in Lie-admissible algebras in a two-fold way, first, in the classification and, second,

as the attached antisymmetric algebras.

A first *bona-fide* generalization of the Lie algebras is provided by the so-called flexible *Lie-admissible algebras*, which are Lie-admissible algebras **U** verifying the *flexibility law* $(ab)a = a(ba)$ for all elements $a \in U$.

The most general possible algebras of the type considered are called *general Lie-admissible algebras* ***U*** [7], which (besides the right and left distributive and scalar laws to qualify as an algebra) verify no condition other than the Lie-admissibility law (11).

The first classical realization of the Lie-admissible algebras in physics was introduced in ref. [6] and then worked out in more details in ref. [7]. Let A, B. ...be (nonsingular, sufficiently smooth) functions in $\Re_t \times T^*E(r,\delta,\Re)$. Then the brackets

$$(A,B) = (\partial_\mu A)\, S^{\mu\nu}(t,a)\, (\partial_\nu B) \quad (12)$$

over the reals $\Re$ characterize a Lie-admissible algebra **U** when the attached antisymmetric brackets

$$\mathbf{U}^- : \; [A,B]_{\mathbf{U}} = (A,B) - (B,A) \quad (13)$$

are Lie or, equivalently, when the attached antisymmetric tensor

$$S^{\mu\nu} - S^{\nu\mu} = \Omega^{\mu\nu} = \text{Lie}. \quad (14)$$

is Birkhoffian.

Now, the direct way of writing brackets (3) in an algebraically consistent way is by introducing the tensor in $\Re_t \times T^*E(r\delta,\Re)$

$$S^{\mu\nu}(t,a) = \omega^{\mu\nu} + s^{\mu\nu}(t,a), \quad (15)$$

where $\omega^{\mu\nu}$ is the (totally antisymmetric) canonical Lie tensor [8], and $s^{\mu\nu}$ is the totally symmetric tensor

$$s = (s^{\mu\nu}) = \text{diag.}\,(0, s), \quad s = F/(\partial H/\partial p) \quad (16)$$

The brackets (A,B), when written in form (12) with the S=tensor given by Eq.(15) first of all, verify both right and left scalar and distributive laws, and, secondly, they characterize a Lie-admissible algebra because the attached brackets are Lie

$$(A,B) - (B,A) = 2\,[A,B], \quad S^{\mu\nu} - S^{\nu\mu} = 2\omega^{\mu\nu}. \quad (17)$$

Finally, the equations of motion are not altered when rewritten in terms of tensor (15), i.e.

$$\dot{a}^{\mu} = S^{\mu\nu}\, \partial H/\partial a^{\nu} = (a^{\mu}, H), \quad (18)$$

or, more explicitly

$$\dot{r}_{ka} = \partial H/\partial p_{ka}, \quad (19a)$$

$$\dot{p}_{ka} = -\partial H/\partial r_{ka} + s_{kajb}\, \partial H/\partial p_{jb} = -\, \partial H/\partial R_{ka} + F_{ka}. \quad (19b)$$

In particular, the brackets (A,B) preserve the time-rate-of-variation of the Hamiltonian

$$\dot{H} = (H,H) = v_{ka} F_{ka}, \quad (20)$$

as desired.

The first operator realization of Lie-admissible algebras were introduced in ref. [8], and will emerge rather forcefully in the operator treatment of the theory.

The regaining of a consistent mathematical structure carries rather deep physical implications.

As an example, Eq. *(1) do not admit a consistent exponentiation into a finite form.* On the contrary, when written in their equivalent Lie-admissible form (18) they can be easily exponentiated into the form

$$a' = \exp|_{A}\, (t^{o} S^{\mu\nu} \partial_{\nu} H)\, (\partial_{\mu})\, a, \quad (21)$$

which constitutes an intriguing generalization of the notion of Lie-isotopic symmetry [9], known as a *Lie-admissible symmetry* (7,9).

In fact, structure (21) leaves invariant the equations of motion in exactly the same way as it occurs for the conventional Lie and Lie-isotopic cases , i.e., we can write the invariance law for all vector-fields $\Gamma(t,a)$ represented by Eqs.(18)

$$\Gamma'(t,a') =$$

$$\{ \exp|_{A}\, (t^{o} S^{\mu\nu} (\partial_{\nu} H)\, (\partial_{\mu}) \}\, \Gamma(t,a) = \Gamma(t,a'), \quad (22)$$

(see ref. [9] for details and examples). Structure (22) is, therefore, a *bona-fide* symmetry.

The physical differences with the conventional approach are, however, rather deep. In fact, *in the conventional Lie and Lie-isotopic symmetries characterized by the Hamiltonian as generator represent the conservation of the energy.* In the more general case under consideration here, we can say that *the broader Lie-admissible symmetry characterized by the Hamiltonian as generator represents the time-rate-of-variation of the energy*

$$\dot{H} = H(t,a) - \{\exp|_{A}\, (t^{o} S^{\mu\nu} (\partial_{\nu} H)\, (\partial_{\mu})\}\, H(t,a) = v_{ka} F_{ka}. \quad (23)$$

Moreover, exponentiation (18) admits the following explicit form

$$\{ \exp|_{A}\, (t^{o} S^{\mu\nu} (\partial_{\nu} H)\, (\partial_{\mu})\}\, A =$$

$$A + t^{o}(A,H)/1! + t^{o2}((A,H),H)/2! + \ldots\ldots \quad (24)$$

namely, *symmetries* (18) *admit non-Lie, Lie-admissible algebras in the neighborhood of the identity.* This signals the possibility of generalizing the entire Lie's and Lie-isotopic theories in a yet more general Lie-admissible theory [7,9].

The mathematical and physical covering character of the Lie-admissible formulations over the Lie-isotopic and Lie formulations is then evident.

By recalling that the symmetry characterized by the Hamiltonian as generator is the time component of the Galilei and of the Galilei-isotopic relativities, symmetry (22) can then be considered as the time component of conceivable, still more general relativities, tentatively called *Lie-admissible relativities* [7,9], for open nonconservative systems in which the form-invariance characterizes , this time, the time-rate-of-variation of the Galilean quantities. The understanding is that the studies on Lie-admissibility are considerably less advanced than the corresponding Lie-isotopic case, and so much remains to be done. For additional comments, see Figure 1.

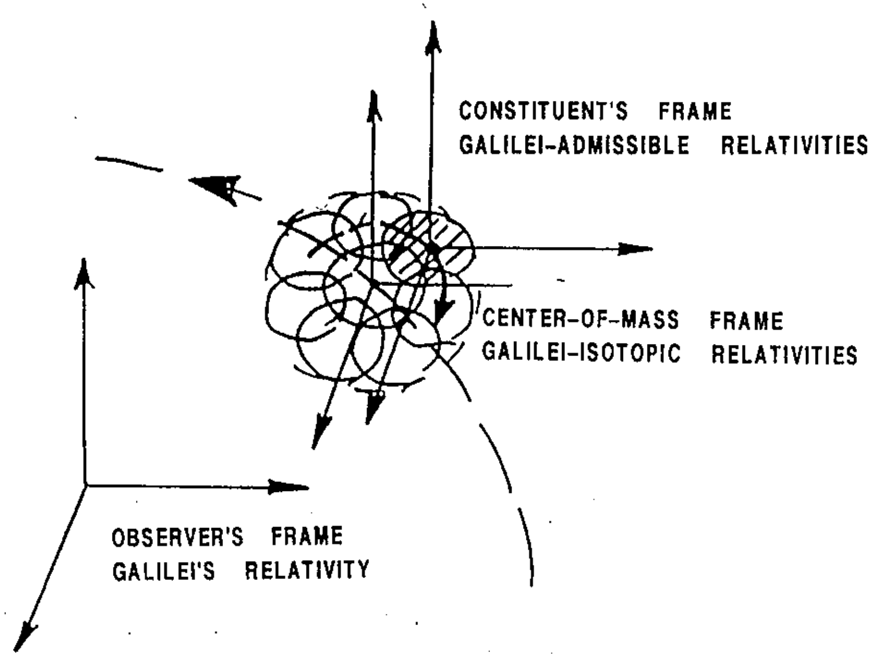

FIGURE 1: A schematic view of the three most important frames and related methodological tools that are recommendable for a comprehensive description of closed nonselfadjoint systems, such as Jupiter [9]. First, we have the external, inertial, observer's frame and related, conventional, Galilei's relativity which describes the center-of-mass trajectory. The reader should be aware that, for such a Galilean setting, Jupiter can only be a structureless massive point. Second, we have the frame at rest with the center-of-mass of Jupiter and the related Galilei-isotopic relativities. In this case, Jupiter is indeed represented as possessing an extended structure which verifies all

external, conventional, total conservation laws and symmetries; yet it admits nonlinear, nonlocal and nonhamiltonian internal forces [11]. Finally, we have the frame at rest with respect to one individual constituent, while considering the rest of the system as external. In this latter case, we have the broader, Lie-admissible methodology of this note. The reason for their emergence as a necessary complement of the Lie-isotopic treatment is the following. Lie-isotopic algebras have an antisymmetric product and, as such, are structurally set for the characterization of the *closed* character of systems and their total conservation laws [3,7]. In the preceding note [12] we have indeed presented certain applications of the Galilei-isotopic relativities for the characterization of isoparticles in open-nonconservative conditions caused by external selfadjoint and nonselfadjoint interactions. However, in this latter case, the generators such as that of the time evolution cannot possess a direct physical meaning, and generally consists of first integrals. Also, and perhaps most importantly, it was stressed in ref. [12) that the applications are specifically restricted to *semiautonomous* open systems, i.e., those with Birkhoffian representations of the type R = R(a) and B = B(t,a) [3,10]. For the case of full non-autonomous, open systems, the Galilei-isotopic techniques are inapplicable, as indicated in the text of this note. The *only* known, mathematically consistent algebras capable of characterizing the latter systems are precisely the Lie-admissible algebras. Since an individual constituent of a closed nonselfadjoint systems is generally nonautonomous, the necessary complementarity of the Lie-admissible algebras then follows. Note that Lie-admissible algebras permit the representation of systems in such a way that all generators have a direct physical meaning (see Refs. [7,9] for details and examples). Also, note that the Galilei-isotopic symmetries for open semiautonomous systems with generators lacking physical meaning [12], can be rewritten in an identical Lie-admissible form with a direct physical meaning of the generators (see, again, Refs. [7,9] for details and examples). The latter algebras can also be obtained from the viewpoint of the *classes of equivalence of the frames considered.* The external, inertial, observer's frame possesses its own class of equivalence, evidently given by all possible inertial frames, as characterized by the *linearity* of the conventional Galilei relativity. In the transition to the representation of Jupiter's structure as it appears to direct experimental evidence (with a nonlinear, nonconservative and nonhamiltonian interior dynamics), we need an intrisically *nonlinear* theory to prevent excessive approximation of the type of the perpetual motion in a physical environment. In this latter case, the center-of-mass frame of the system is evidently *noninertial*, because inertial frame does not exist in our physical reality [11]. The Galilei-isotopic relativities then characterize the class of noninertial systems that are equivalent to the center-of-mass frame. It is geometrically possible to show that such (infinite) class *does not* contain the frames of individual constituents, e.g., because they are generally in unstable orbits, while the center-of-mass frame of the system represent the globally stability of the system. In turn, the identification of the class of frames equivalent to (each) constituent's frame can be best done via methods structurally set for nonconservative conditions, the Lie-admissible methods [7,9].

The identification of the algebraic structure of the nonautonomous Birkhoff's equations (2) is now easy. It was originally identified in ref. [7] and then studied in details in ref. [9]. Introduce the generalized tensor

$$\hat{S}^{\mu\nu}(t,a) = \Omega^{\mu\nu}(a) + T^{\mu\nu}(t,a), \qquad (25)$$

where $\Omega^{\mu\nu}$ is the (totally antisymmetric) Birkhoff's tensor (2b), and $T^{\mu\nu}$ is given by the totally symmetric form

$$T = (T^{\mu\nu}) = \mathrm{diag.}(0,\sigma), \quad \sigma = (\partial_t R) / (\partial_\mu B). \qquad (26)$$

The generalized brackets

$$(A\hat{,} B) = (\partial_\mu A)\ \hat{S}^{\mu\nu}(t,a)\ (\partial_\nu B) \qquad (27)$$

are then algebraically consistent and Lie-admissible, as one can see. In particular, the transition from brackets (12) to (27)

$$(A,B) \Rightarrow (A\hat{,} B), \qquad (28)$$

characterizes a *Lie-admissible isotopy*. For further studies, we refer the interested reader to ref. [9], where one can see the elements for a further generalization of Birkhoffian mechanics into

a covering discipline, tentatively called *Birkhoffian-admissible mechanics*, including the indication of the underlying generalized geometric structure under the name of *symplectic-admissible geometry*.

We can therefore close this note with the view that, by no means, the Galilei-isotopic relativities of Refs. [7,9,11] can be considered as the final relativities, because physics is a discipline that will never admit terminal theories.

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