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ANTIGRAVITY

Ruggero Maria Santilli
The Institute for Basic Research
P. O. Box 1577, Palm Harbor FL 34682, USA
Fax 1-813-934 9275, E-mail ibrrms@pinet.aip.org

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Abstract

Recent advances in the integral, axiom-preserving isotopies and isodualities of the Minkowskian and Riemannian geometries permit a quantitative formulation of antigravity conceived as the reversal of the gravitational attraction, which is here submitted as a scientific curiosity for young minds. According to these advances, the *interior* gravitational problem of matter is characterized by the isoriemannian geometry and that of antimatter by a novel antiautomorphic map known as isoduality. The corresponding *exterior* gravitational problems of matter and antimatter in vacuum are characterized by the conventional Riemannian geometry and its isodual. The covering isogeometries permit quantitative studies on the *origin* (rather than the description) of the gravitation itself, which yield the identification of the gravitational and electromagnetic interactions in the structure of matter (i.e., the electromagnetic origin of mass plus short range contributions). The understanding of the origin of gravitation then yields the capability to reverse the gravitational force exactly as it occurs for the Coulomb force. In particular, antigravity emerges as the projection of the isodual gravitational field of antimatter in the gravitational field of matter (or viceversa). While antigravity is prohibited by conventional geometries, the covering isogeometries and related isoduals predict that antiparticles experience a repulsive force in the gravitational field of matter. The proposed antigravity is fully testable with current technology, e.g., via the comparison of suitable interferometric measures on thermal beams of neutrons and antineutron in the gravitational field of Earth.

1. INTRODUCTION

The search for antigravity dates back to the beginning of physics (see, e.g., the comprehensive review by Nieto and Goldman [1a]) and includes attempts which have been even patented (see [1b,c] and quoted literature). All existing studies deal either with a *reduction* of the gravitational attraction or with *mechanical* means to bypass gravity. None of them deals with *the conception of antigravity as the reversal of the attractive character of the gravitational field.* This is due to the fact that the Riemannian geometry (see, e.g., [2]) permits no known possibility of reversing the attractive character of gravitation.

To put it clearly, by its very nature antigravity is "beyond Einsteinian theories", that is, it requires theories of gravitation structurally more general than Einstein's gravitation. The understanding is that if antigravity is experimentally established via the tests proposed in this note or other approaches, Einstein's gravitation is not "violated", but "inapplicable", simply because Einstein did not formulate his theory to study antigravity.

Antigravity therefore requires the identification of the arena of conception and applicability of Einstein gravitation, with the understanding that its assumption as being universally valid under whatever conditions exist in the universe has no scientific value.

As clearly identified in his limpid writings, Einstein conceived his gravitation for the *exterior* gravitational problem of *matter* in vacuum. The first arena of *inapplicability* (and not "violation") of Einstein's gravitation is therefore the exterior problem of *antimatter* in vacuum. After all, antimatter did not exist at the time of Einstein's conception of his gravitation.

Our first condition for the study of antigravity is therefore the assumption of the Riemannian geometry and Einstein's gravitation as being valid for the *exterior problem of matter in vacuum* and the search for a different geometry and gravitational theory for the *exterior problem of antimatter in vacuum.*

The known unresolved problematic aspects afflicting Einstein's gravitation have no bearing for this note owing to the dominance of the Riemannian geometric profile over the explicit form of the field equations which can be defined in it. As such, these problematic aspects will be ignored hereon.

The second condition is that the reversal of the sign of gravity is evidently linked to the problem of the *origin* (rather than "description") of the gravitational field. As such, it requires geometries which are first suitable to represent actual interior conditions, and then capable of reducing the gravitational field to primitive interactions originating mass itself.

Now, interior dynamical problems, such as missiles in atmosphere, are arbitrarily nonlinear *in the velocities*, nonlocal–integral in various variables (e.g., dependent on the shape of the body) and non-(first-order)-Lagrangians (variationally nonselfadjoint systems [3,4]). The assumption of the local-differential-Lagrangian Riemannian geometry as being *exactly* valid for interior problems also has no scientific value.

As an example, interior problems such as gravitational collapse are not given by a large yet finite set of isolated points, but in the physical reality they are composed of extended charge-distributions/wavepackets/wavelengths of hadrons in conditions of total mutual penetration and compression in large number into small regions of space. The consequential emergence of the indicated nonlinear-nonlocal-nonlagrangian conditions is then beyond credible doubts.

Our second condition for the study of antigravity is therefore the search for suitable *covering geometries* for the quantitative representation of nonlinear-nonlocal-nonlagrangian interior gravitational problems of matter and, separately, of antimatter, as well as for the complete reduction of the gravitational field to the primitive fields originating matter itself, the electromagnetic interactions plus short range contributions. It is evident that the latter *interior* generalizations must be able to reproduce the preceding representations of the *exterior* gravitational problems of matter and antimatter in vacuum.

Our third condition for the study of antigravity is that, whatever geometric description of the gravitational field of antimatter is assumed, that description must recover all other experimentally established behaviour of antimatter, e.g., under electromagnetic interactions.

In different terms, the *gravitational* characteristics of antiparticles are theoretically and experimentally unsettled at this writing. However, their behaviour under *electromagnetic* interactions is fully established. Any theory of antigravity based on antimatter must therefore recover the conventional electromagnetic behaviour of antiparticles in its entirety.

The only known mathematical formulations which meet all the above requirements are given by the so-called *isotopies and isodualities of the Riemannian geometry*. The general lines of isotopies were first introduced by this author in 1978 [3] and then studies in its various aspects in [4-15]. The first specific application to gravitation was done in 1988 [7] and then studied in detail in [8,10,11]. The most recent isotopies and isodualities of gravitation have been studied in [14] and those of the tangent Minkowskian geometry in [15]. Independent reviews can be found in [16-20]. Topological aspects of isomanifolds and isotensors defined on them are studied in [21].

The above studies have identified the following four primary geometries at the foundations of this note [10,11]:

Riemannian geometry, for the exterior gravitational problem of matter

in vacuum;

Isodual Riemannian geometry, for the exterior gravitational problem of antimatter in vacuum,

Isoriemannian geometry, for the interior gravitational problem of matter; and

Isodual isoriemannian geometry, for the interior gravitational problem of antimatter.

In this note we shall study only the **exterior problem of antigravity,** that in vacuum, and therefore limit our analysis to the Riemannian geometry and its isodual (a detailed nonlinear-nonlocal-nonlagrangian treatment of the interior problem is available in Vol. II of ref.s [4]). The local tangent geometries of our study are given by the conventional **Minkowskian geometry** for the case of matter and the **Isodual Minkowskian geometry** for the case of antimatter [6,15].

Intriguingly, the theory of antigravity submitted in this paper is reducible to the interplay between two primitive symmetries, the conventional Poincaré symmetry P(3.1) for matter in the tangent plane and its isodual image $P^d(3.1)$ for antimatter also in the tangent plane, with isotopic generalizations $\hat{P}(3.1)$ and $\hat{P}(3.1)$ which have resulted to be directly universal for all possible exterior and interior gravitational problems of matter and antimatter, respectively (see [5] for the original proposal, [6,10,11] for detailed studied, [13] for recent advances and [20] for an independent review).

More particularly, the lack of antigravity in contemporary gravitation appears to be due to its lack of a universal symmetry, such as the Poincaré symmetry of the special relativity. The identification of the universal isopoincaré symmetry for the conventional gravitational field of matter then leads naturally and uniquely to the isodual isopoincaré symmetry for antimatter. Antigravity then follows uniquely and unambiguously.

2. BASIC NOTIONS ON ISOTOPIES AND ISODUALITIES

2.1. Isotopies and isodualities of the unit. The isotopies [3] are maps of any given linear-local-potential theory into nonlinear-nonlocal-nonpotential forms which are such to preserve the original axioms.

The fundamental isotopies are the liftings of the basic unit, in our case the four dimensional unit of the Riemannian geometry $I = \text{diag.}(1, 1, 1, 1)$, into a positive-definite four dimensional matrix $\hat{1}$ whose elements have the most general possible dependence on local coordinates x, their derivatives of arbitrary order $\dot{x}$, $\ddot{x}$, ..., the local density μ, temperature τ, frequency ω, index of refraction n, and any other needed quantity [3,4,10,11]

$$I = \mathrm{diag.}\,(1, 1, 1, 1) > 0 \quad \rightarrow \quad \hat{1}(t, x, \dot{x}, \ddot{x}, \mu, \tau, n, \omega, \ldots) > 0\,, \tag{2.1a}$$

$$\hat{1} = \mathrm{diag.}\,(n_1^2, n_2^2, n_3^2, n_4^2) > 0, \qquad n_\mu > 0, \quad \mu = 1, 2, 3, 4, \tag{2.1b}$$

The lifting is called an "isotopy" because $\hat{1}$ preserves the original axioms of I. Diagonal form (2.1b() is then always possible from the positive–definiteness of $\hat{1}$ (although not necessary). Lifting (2.1) is a short–hand notation of the lifting of the unit in *each* of the space–time direction $I_\mu \rightarrow n_\mu^2$.

The lifting of the unit demands, for consistency, a corresponding, compatible lifting of all associative products A×B among generic quantities A, B, into the *isoproduct* [loc. cit.]

$$A \times B \quad \rightarrow \quad A \hat{\times} B = A\,T\,B, \qquad T = \text{fixed}, \tag{2.2}$$

where T is called the *isotopic element*, whose isotopic character is ensured by the preservation of the original associativity of A×B, A*(B*C) ≡ (A*B)*C, under which $\hat{1} = T^{-1}$ is the correct left and right unit of the theory called the *isounit*, $\hat{1}\hat{\times}A \equiv A\hat{\times}\hat{1} \equiv A$.

The *isodualities*, first introduced in [6] and then studied in [8–11], are antiautomorphic maps of any given conventional or isotopic structure based on the conjugation of the isounit

$$\hat{1} \rightarrow \hat{1}^d = (T^d)^{-1} = -\hat{1} < 0, \tag{2.3a}$$

$$A \hat{\times} B = A\,T\,B \ \rightarrow \ A \hat{\times}^d B := A\,T^d B = -A\,T\,B\,. \tag{2.3b}$$

The above liftings are subdivided into various classes (called *Kadeisvili's classes* [18]) depending on the topological characteristics of the isounit $\hat{1}$ or isotopic element T. In this paper we shall assume that all isounits are sufficiently smooth, bounded, nowhere singular, Hermitean and positive–definite (Class I). The isodual isounit $\hat{1}^d$ will then have the same characteristics except being negative–definite (Class II).

More general methods are called *genotopies* and *genodualities* [10,11] which are used for particles and antiparticles, respectively, in open nonconservative conditions and are characterized by isounits no longer Hermitean. These latter methods will not be considered in this note for brevity.

2.2. Isotopies and isodualities of fields. The isotopies of the unit $I \rightarrow \hat{1}$ and of the product $A\times B \rightarrow A\hat{\times}B$ demand the lifting of conventional fields F(a,+,×) of real numbers R, complex number C and quaternions Q with generic elements a, conventional sum + and product a×b := ab, into the *isofields* studied in detail in

[9]

$$F(a,+,*) \ \rightarrow \ \hat{F}(\hat{a},+,\hat{\times}),\ \hat{a} = a\hat{1}\,,\ \hat{a}\hat{\times}\hat{b} = \hat{a}\,T\,\hat{b} = (a\,b)\hat{1},\ \ \hat{1} = T^{-1}\,, \tag{2.4}$$

with elements $\hat{a}$ called *isonumbers* and *isonorm* $\hat{|}\hat{a}\hat{|} = |aTa|^{\frac{1}{2}}\hat{1} > 0$.

The above isotopies implies the generalization of all conventional operations for numbers, such as $\hat{a}\hat{/}\hat{b} = (a/b)\hat{1}$, $\hat{a}^{\hat{n}} = \hat{a}\hat{\times}\hat{a}\hat{\times}\ldots\hat{\times}\hat{a}$ (n–times), etc. In particular, the isounit preserves all original axioms, such as $\hat{1}^{\hat{n}} \equiv \hat{1}, \hat{1}^{\hat{\frac{1}{2}}} \equiv \hat{1}, \hat{1}\hat{/}\hat{1} \equiv \hat{1}$, etc. (see [9] for brevity). One should recall that the isoproduct of an isonumber by any quantity coincides with the conventional product, $\hat{n}\hat{\times}A \equiv n\times A = nA$. This implies that the actual numbers emerging for experiments are the conventional ones.

The *isodual isofields* are given by the conjugation [loc. cit.]

$$\hat{F}^d(\hat{a}^d,+,\hat{\times}^d),\ \hat{a}^d = a\hat{1}^d\,,\quad \hat{a}^d\,\hat{\times}^d\,\hat{b}^d := \hat{a}^d\,T^d\,\hat{b}^d\,,\quad \hat{1}^d = -\hat{1}, T^d = -T, \tag{2.5}$$

where the elements $\hat{a}^d$ are called *isodual isonumbers*. In this case the *isodual isonorm* is given by

$$\hat{|}\hat{a}^d\hat{|}^d = |aTa|\hat{1}^d < 0\,, \tag{2.6}$$

and it is *negative-definite*. This implies that all numbers which are conventionally positive, become negative under isoduality. One should however remember that the underlying unit is negative definite. This yields an equivalence between positive numbers +m when referred to the conventional unit +I with their negative image −m when referred to the negative unit −I.

Note that isodualities imply different generalizations of all operations, such as $\hat{a}^d\hat{/}^d\hat{a}^d = (a/b)\hat{1}^d = -\hat{a}\hat{/}\hat{b}$, $a^{d\ \hat{n}d} = \hat{a}^d\hat{\times}^d\hat{a}^d\hat{\times}^d\hat{a}^d$ (n–times) etc.. The isodual isounit also preserves all original axioms, such as $\hat{1}^{d\ \hat{n}d} \equiv \hat{1}^d, \hat{1}^{d\ \hat{\frac{1}{2}}d} \equiv \hat{1}^d, \hat{1}^d\hat{/}^d\hat{1}^d \equiv \hat{1}^d$, etc. [9] and, therefore, it is a fully acceptable unit. One should also note that the isodual isoproduct of an isodual number by any quantity coincides with the *conventional* product, $\hat{n}^d\hat{\times}^dA \equiv n\times A = nA$. This may be a reason why isodual numbers were discovered only recently.

This note is restricted to the exterior geometric studies in vacuum. As such as shell study conventional geometries defined over conventional real numbers R(n,+,×) and isodual geometries defined over isodual numbers $R^d(n^d,+,\times^d)$. The more general isonumbers and their isodual are used for the problem of the origin of gravitation and will not be considered here.

Operator formulations will be based on the conventional field C(c,+,×) of complex numbers c with the usual operations + and × and related additive unit 0 and multiplicative unit +I. For isodual aspects we shall use the isodual field

$C^d(c^d,+,\times^d$ of *isodual complex numbers*

$$c^d = \bar{c}\, 1^d = -(n_1 - i \times n_2)\, 1^d = -\bar{c}, \tag{2.7}$$

where $\bar{c}$ is the conventional complex conjugation. The *isodual norm* in this case is given by

$$|c^d|^d = |c \times \bar{c}|\, 1^d = -n_1^2 - n_2^2 < 0, \tag{2.8}$$

where $|c|$ is the conventional norm. For other properties we refer the reader for brevity to [9], Sect. 9B.

The axiom-preserving character of the lifting $F(a,+,\times) \rightarrow \hat{F}(\hat{a},+,\hat{\times})$ is easily seen from the local isomorphism $F(a,+,\times) \approx \hat{F}(\hat{a},+,*)$ resulting from the positive-definiteness of $\hat{1}$. Yet the liftings of conventional numbers are not trivial. As an illustration, traditional statements such as "two multiplied by two equals four" have no mathematical meaning under isotopies, evidently because of the need to identify the underlying unit and multiplication. In fact, for $\hat{1} = 3$, "two multiplied by two equals twelve". Also, numbers conventionally considered as being prime are no longer necessarily prime under change of the unit and viceversa. As an example, for the preceding case of isounit $\hat{1} = 3$, the number 6 is prime. It is hoped that these occurrences illustrate the nontriviality of the isotopies of numbers irrespective of any physical applications.

2.3. Isotopy and isodualities of Minkowskian and Riemannian spaces. The liftings $I \rightarrow \hat{1}$, $A \times B \rightarrow A \hat{\times} B$ and $F \rightarrow \hat{F}$ require the isotopies of metric and pseudo-metric spaces $S(x,g,R)$ with Hermitean metric g over R, into the *isospaces* first introduced in [5] and then studies in details in [8,10,11]

$$\hat{S}(x,\hat{g},\hat{R}), \quad \hat{g} = Tg, \quad \hat{1} = T^{-1}, \qquad x^2 = (x^t \hat{g}\, x)\, \hat{1} \in \hat{R}. \tag{2.9}$$

where $\hat{g} = Tg$ is called the *isometric*. The deformation of the metric $g \rightarrow \hat{g} = Tg$ while the basic unit is deformed of the *inverse* amount $\hat{1} = T^{-1}$ then ensures the preservation of the original axioms of $S(x,g,R)$. Thus, *isospaces are locally isomorphic to the original spaces by construction,* $\hat{S}(x,\hat{g},\hat{R}) \approx S(x,g,R)$ [loc. cit.].

The *isodual isospaces*, first introduced in [6] and then studied in [8,10,11], are given by the structures

$$\hat{S}^d(x,\hat{g}^d,\hat{R}^d), \quad \hat{g}^d = T^d g, \quad \hat{1}^d = (T^d)^{-1} = -\hat{1}, \quad x^{\hat{2}d} = (x^t \hat{g}^d\, x)\, \hat{1}^d \in \hat{R}, \tag{2.10}$$

where $\hat{g}^d = -Tg$ is called the *isodual isometric*. Then, *isodual isospaces are anti-isomorphic to the original space.*

A property which is fundamental for this note is that $x^{2d} \equiv x^2$, i.e., *all line elements, whether conventional or isotopic, are invariant under isoduality.* This property implies the new *universal invariance of physical laws under isoduality* studied and illustrated in detail in [10,11].

The isotopies and isodualities most important for this paper are those of the Minkowskian and Riemannian spaces and of the related geometries, studies in details in the recent papers [14,15] respectively. We here limit ourselves to recall that, given a conventional Minkowski space $M(x,\eta,R)$ with local coordinates $x = (x^\mu) = (r, x^4)$, $x^4 = c_o t$, $\mu = 1, 2, 3, 4$, where c_o is the speed of light in vacuum, and metric $\eta = \mathrm{diag.}\,(1, 1, 1, -1)$ over the reals $R(n,+,\times)$, the isominkowski spaces are given by

$$\hat{M}(x,\hat{\eta},\hat{R}): \quad \hat{\eta} = T\,\eta, \; T = \mathrm{diag.}\,(n_1^{-2}, n_2^{-2}, n_3^{-2}, n_4^{-2}) > 0, \quad \hat{1} = T^{-1}, \tag{2.11a}$$

$$x^{\hat{2}} = (x\, n_1^{-2} x + y\, n_2^{-2} y + z\, n_3^{-2} z - t\, c_o^2\, n_4^{-2} t)\, \hat{1} \in \hat{R}(n,+,*). \tag{2.11b}$$

The effectiveness of the isotopies for interior problems is then transparent. For instance, one has a *direct representation of the locally varying speed of light* $c = c_o/n_4 = c(x, \dot{x}, \ddot{x}, \mu, \tau, n, \ldots)$, *where* n_4 *is the local index of refraction, without altering the original axioms of the geometry for the constant speed* c_o *in vacuum.*

Similarly, for locally varying speeds of light (e.g., in our atmosphere) one evidently loses the fundamental light cone. The isominkowskian geometry permits the reconstruction of such a cone in isospace, called *isolight cone* (see, e.g., [15]) which results to *coincide* with the conventional one at the abstract level. In fact, in correspondence with the deformation of each component of the light cone $l_\mu \rightarrow n_\mu^{-2}$, the corresponding unit is deformed of the *inverse* amount $l_\mu \rightarrow n_\mu^{-2}$, thus preserving the axioms of the original cone in isospace.

Step-by-step, axiom-preserving isotopies of the special relativity into the so-called *isospecial relativity* then follow (see monographs [10] for the classical profile and monographs [11] for the operator counterpart).

In this note we shall use the *isodual Minkowskian space* given by

$$M^d(x,\eta^d,R^d), \quad \eta^d = -\eta, \quad x^{2d} = (x^\mu\, \eta^d_{\mu\nu}\, x^\nu)\, I^d \in R^d(n^d,+,\times^d). \tag{2.12}$$

The difference with the conventional alternatives of the Minkowski metric $\eta_1 = \mathrm{diag.}\,(1, 1, 1, -1)$ or $\eta_2 = \mathrm{diag.}\,(-1, -1, -1, 1)$ should be pointed out. Conventionally, one assumed either η_1 or η_2 *but always defined over the conventional field of real number* $R(n,+,\times)$. On the contrary, isodualities imply the transition from either one of these two metrics η_k to the other $-\eta_k$ *under the condition of its definition in the isodual field* $R^d(n^d,+,\times^d)$. The important consequence is that *the separation changes sign in the former case,* $x^2(\eta_L) = -x^2(\eta_2)$, while *the same separation does not change sign in the latter case,* $x^2(\eta_k) \equiv x^{2d}(\eta^d{}_k)$ because all separation must be defined on the base field.

As we shall see, the above occurrences are the geometric foundations for our isodual representation of antiparticles and its compliance with available experimental evidence on electromagnetic interactions.

Consider now conventional Riemannian spaces $\Re(x,g,R)$ here assumed in (3+1)–dimension with local coordinates x with nondegenerate, Hermitean metric g(x) and conventional separation $x^2 = x^\mu g_{\mu\nu} x^\nu$ over the reals R(n,+,×). The *isoriemannian spaces* [7] are given by the isostructures $\hat{\Re}(x,\hat{g},\hat{R})$ with the same local coordinates x , in which the original metric g is lifted into an arbitrarily generalized metric $\hat{g} = Tg$, while jointly lifting the unit into the inverse of the deformation,

$$\hat{\Re}(x,\hat{g},\hat{R}): \quad \hat{g} = T(x, \dot{x}, \ddot{x}, \mu, \tau, \omega, n, ...) g(x), \quad T = T^\dagger > 0, \; \hat{1} = T^{-1}, \tag{2.13a}$$

$$x^{\hat{2}} = (x^t g T x) T^{-1} \in \hat{R}. \tag{2.13b}$$

The *Riemannian–isotopic geometry,* or *isoriemannian geometry* for short, was suggested by this author [7] as the geometry of isospaces $\hat{\Re}(x,\hat{g},\hat{R})$. Its primary function is to provide nonlocal–integral generalizations of Riemannian geometry for interior problems while preserving the abstract axioms of the conventional geometry in vacuum. thus permitting a geometric unification of exterior and interior problems.

The effectiveness of the isotopies for interior gravitational problems is then also transparent. In fact, without altering the original Riemannian axioms at the abstract level, one has a representation *directly in the isometric* of: the experimentally established inhomogeneity and anisotropy of interior physical media; the locally varying speed of light $c = c_0/n_4$; the internal effects which are arbitrarily nonlinear in the velocities, nonlocal–integral in various quantities and non–first–order–Lagrangians and other interior features simply beyond any descriptive capability of the conventional Riemannian geometry.

Note that for T = constant, we have the identity $x^{\hat{2}} \equiv x^2$. Thus, to have a nontrivial isoriemannian space one must select a nontrivial 4×4–dimensional isotopic element $T \neq I$.

The *isodual isoriemannian geometry* [10,11,14] is the image of the isoriemannian geometry under isoduality $\hat{1} \to \hat{1}^d = -\hat{1}$. It is a geometry on isodual isospaces $\hat{\Re}^d(x,\hat{g}^d,\hat{R}^d)$ over isodual isoreals and separation $x^{2d} \equiv x^2$ which, as such, possesses a *negative–definite norm* and therefore possesses negative–definite physical quantities.

In this note we shall study antigravity via the *conventional Riemannian geometry* for the exterior problem of matter in vacuum, and the *isodual Riemannian geometry* for the exterior problem, of antimatter. The covering isoriemannian geometry and its isodual will be used in a subsequent paper to study antigravity from the viewpoint of the *origin* of the gravitational itself in matter and antimatter.

2.4. Isotopies and isodualities of Lie's theory. The preceding liftings demand a corresponding compatible lifting of all branches of Lie's theory, including the lifting of enveloping associative algebras, Lie algebras, Lie groups, transformation and representation theories, symmetries, etc., which was first submitted by this author in [3], expanded and applied in monographs [4,10,11] and then studied by a number of authors (see the mathematical studies by Sourlas and Tsagas [16], Lômus, Paal and Sorgsepp [17] and Kadeisvili [18,20]).

The emerging generalized Lie theory is subdivided into the following branches:

> **Lie's theory** with trivial unit I > 0 for point particles in vacuum;

> **isodual Lie's theory** with isodual unit $I^d = -I < 0$ for point–antiparticles in vacuum;

> **Lie-isotopic theory** with isounit $\hat{1} > 0$ for extended particles within physical media; and

> **isodual Lie-isotopic theory** with isodual isounits $\hat{1}^d = -\hat{1} < 0$ for extended antiparticles within physical media.

The above generalizations are centrally dependent on the *Lie–isotopic second theorem*

$$[X_i \hat{,} X_j] = X_j T X_j - X_i T X_i = \hat{C}_{ij}{}^k T X_k \tag{2.14}$$

where the $\hat{C}$'s are the so–called *structure isofunctions*, whose verification of the abstract Lie algebra axioms is instructive. The related *Lie–isotopic transformation groups* on $\hat{S}(x,\hat{g},\hat{R})$ are given by

$$x' = \hat{\Lambda}(\hat{w}) * x = \hat{\Lambda}(\hat{w}) T x, \tag{2.15a}$$

$$\hat{\Lambda}(\hat{w}) = \prod^*{}_k \hat{e}^{\,i \hat{w}_k * X_k} = \left(\prod_k e^{\,i X_k T w_k} \right) \hat{1}. \tag{2.15b}$$

where $\hat{e}^A$ is the isoexponentiation in the isoenvelope, $\hat{e}^A = \hat{1} + A/1! + A*A/2! + ...$ The verification of the abstract Lie group axioms by structure (2.15) is also instructive.

The *isodual Lie–isotopic second theory* [10,11] is then given by the conjugation of the basis $X^d{}_k = -X^d{}_k$ and the form

$$[X_i \hat{,} X_j]^d = X^d_j T^d X^d_j - X^d_i T^d X^d_i = -(X_j T X_j - X_i T X_i) =$$
$$= \hat{C}^d{}_{ij}{}^k T^d X^d_k = -\hat{C}_{ij}{}^k T X_k \qquad (2.16)$$

The related *isodual Lie-isotopic transformation groups* on $\hat{S}^d(x,\hat{g}^d,\hat{R}^d)$ are given by

$$x' = \hat{A}^d(\hat{w}^d)\, T^d\, x, \qquad (2.17a)$$

$$\hat{A}^d(\hat{w}^d) = \prod{}^*_k \hat{e}^{\,i^d \hat{w}^d_k T^d X^d_k} = \{ \prod_k e^{\,i X_k T w_k} \} 1^d. \qquad (2.17b)$$

Note the necessity of generalizing the unit for the very definition of the isodual Lie theory. In fact, even the *isodual second theory* on $R^d(x,x^d)$ requires a nontrivial generalization of the unit and multiplication

$$[X_i \hat{,} X_j]^d = X^d_j (-1) X^d_j - X^d_i (-1) X^d_i = -(X_j T X_j - X_i T X_i) =$$
$$= C^d{}_{ij}{}^k (-1) X^d_k = -C_{ij}{}^k T X_k \qquad (2.18)$$

The nontriviality of the Lie-isotopic theory is clearly illustrated by *the appearance of an arbitrary, nonlinear, integro-differential element* T *in the exponent of the group structure,* Eq.s (2.15b). An an illustration, *the conventional, linear-local-canonical realizations of the space-time symmetries* SO(3) *and* SO(3.1) *are turned into nonlinear-nonlocal-noncanonical coverings* $S\hat{O}(3)$ *and* $S\hat{O}(3.1)$, *but always in such a way to verify the local isomorphisms* $S\hat{O}(3) \approx SO(3)$ *and* $S\hat{O}(3.1) \approx SO(3.1)$.

For instance, consider the infinite family of Class I ($\hat{1} > 0$) but otherwise *arbitrary nonlinear-nonlocal-noncanonical deformations* of the Minkowski line element

$$x^2 = x^\mu \eta_{\mu\nu} x^\nu \rightarrow x^{\hat{2}} = x^\mu \hat{\eta}_{\mu\nu} x^\nu, \qquad (2.19a)$$

$$\hat{\eta} = T\eta = \text{diag.}(T_{11}, T_{22}, T_{33}, -T_{44}), \quad T_{\mu\mu}(x, \dot{x}, \ddot{x}, \mu, \tau, n, \ldots) > 0. \qquad (2.19b)$$

Then, the Lie-isotopic theory permits the construction of their *universal symmetry*, that is, the symmetry valid for all infinitely possible deformed line elements $x^{\hat{2}}$ as the isotopes $S\hat{O}(3.1)$ of the Lorentz symmetry SO(3.1) of the original Minkowskian line element x^2.

The *isorotations* $S\hat{O}(3)$ in the (1, 2)-plane were first computed in [6] and are given by

$$x' = x \cos(T_{11}^{\frac{1}{2}} T_{22}^{\frac{1}{2}} \theta_1) - y\, T_{11}^{-\frac{1}{2}} T_{22}^{\frac{1}{2}} \sin(T_{11}^{\frac{1}{2}} T_{22}^{\frac{1}{2}} \theta_1)$$

$$y' = x\, T_{11}^{\frac{1}{2}} T_{22}^{-\frac{1}{2}} \sin(T_{11}^{\frac{1}{2}} T_{22}^{\frac{1}{2}} \theta_1) + y \cos(T_{11}^{\frac{1}{2}} T_{22}^{\frac{1}{2}} \theta_1), \qquad (2.20)$$
$$z' = z, \quad x'^4 = x^4$$

where θ_1 is a conventional Euler angle. The most general possible isorotation in three-dimension is then obtained via the isotopies of the conventional combination of rotations on all three Euler angles $\{\theta_1, \theta_2, \theta_3\}$ (see [11], Ch. 6 for details). Note the nonlinear-nonlocal character of isotransforms (2.20) originating from the unrestricted functional dependence of the T_{kk} elements.

The *Lorentz-isotopic transforms* $S\hat{O}(3.1)$ were first computed in [5] and are given by the isorotations plus the *isoboosts* in the (3, 4)-plane

$$x' = x^1, \quad y' = y,$$
$$z' = z \cosh(T_{33}^{\frac{1}{2}} T_{44}^{\frac{1}{2}} v) - x^4 T_{33}^{\frac{1}{2}} T_{44}^{\frac{1}{2}} \sinh(T_{33}^{\frac{1}{2}} T_{44}^{\frac{1}{2}} v)$$
$$= \hat{\gamma}(x^3 - \beta x^4), \qquad (2.21)$$
$$x^4 = z\, T_{33}^{\frac{1}{2}} T_{44}^{-\frac{1}{2}} \sinh(T_{33}^{\frac{1}{2}} T_{44}^{\frac{1}{2}} v) + x^4 \cosh(T_{33}^{\frac{1}{2}} T_{44}^{\frac{1}{2}} v)$$
$$= \hat{\gamma}(x^4 - \hat{\beta} x^3)$$

where v is the conventional Lorentz parameter (speed) and

$$\beta = v / c_0, \quad \hat{\beta} = v_k T_{kk}^{\frac{1}{2}} / c_0 T_{44}^{\frac{1}{2}}, \qquad (2.22a)$$

$$\sinh(T_{33}^{-\frac{1}{2}} T_{44}^{-\frac{1}{2}} v) = \hat{\beta}\hat{\gamma}, \qquad (2.22b)$$

$$\cosh(T_{33}^{\frac{1}{2}} T_{44}^{\frac{1}{2}} v) = \hat{\gamma} = | 1 - \hat{\beta}^2 |^{-\frac{1}{2}}, \qquad (2.22c)$$

The verification that isotransforms (2.20) and (2.21) do indeed leave invariant isoseparation (2.19a) is instructive. The physical relevance is equally transparent. For instance, the assumption $T_{44} = 1/n_4^2$ where n_4 is the ordinary local index of refraction implies the replacement of the speed in vacuum c_0 of the conventional Lorentz transforms with the physical speed $c = c_0/n_4$ of the covering isolorentz transforms, plus corresponding deformations of the space components required by both Lorentz and isolorentz invariance. The understanding is that the isotopic element T admits other physical applications, some of which are studied below.

Note that the expectation that isotopic theories should predict one value of the isotopic element is exactly the same as the expectation that Einstein's gravitation should predict only one mass. On the contrary, one of the reasons for the physical interest of Einstein's gravitation is that it permits *all infinitely possible masses.* Along exactly the same line, one of the reasons for the physical interest of isotopic theories is that they admit *infinitely different isotopic elements for each given mass.* In fact, each given mass can be realized in an infinite number of different ways, with different size, density, temperature, chemical composition, etc. The power of the isotopies is to reduce all these infinitely different interior conditions to one single geometric entity: the isotopic element T.

At any rate, applications show that the knowledge of the interior conditions completely identify the isotopic element T [11].

The extension to the *isopoincaré symmetry* $\hat{P}(3.1) = SO(3.1)\times\hat{T}(3.1)$ is given by adding the *isotranslations*

$$x' = x + x^{\circ} B^{2}(x, \dot{x}, \ddot{x}, \mu, \tau, n, ...), \qquad (2.23)$$

where x° represents the conventional translation parameters and the B's are certain nonlinear–nonlocal functions derived from each given isotopic element T via simple rules (see ref.s [10,11] for brevity).

An important application of the above results is the following

Theorem 2.1 (Universality of the isopoincare´ symmetry in gravitation [10,11]): *Consider all infinitely possible (3+1)–dimensional Riemannian spaces $\Re(x,g,R)$ with local coordinates x and Hermitean nonsingular metrics g(x) over the field of real numbers $R(n,+,\times)$ under the isotopic decomposition of the metric $g(x) = T(x)\eta$, where η is the local Minkowskian metric. Consider the reinterpretation of the spaces $\Re(x,g,R)$ as isominkowskian spaces $\hat{M}(x,\hat{\eta},\hat{R})$, $\hat{\eta} = T(x)\eta \equiv g(x)$, over the isofield of real isonumbers $\hat{R}(\hat{n},+,*)$ with gravitational isounit $\hat{1}(x) = [T(x)]^{-1}$. Then the universal symmetry of all infinitely possible Riemannian line elements in $\Re(x,g,R)$ is given by the isopoincaré symmetry $\hat{P}(3.1)$ of Class I constructed as the isosymmetry of the line elements in the equivalent isominkowskian space $\hat{M}(x,\hat{\eta},\hat{R})$ with respect to the gravitational isounit $\hat{1}(x) = [T(x)]^{-1} > 0$.*

In fact, the decomposition $g(x) \doteq T(x)\eta$ is always possible for all infinite (3+1)–dimensional Riemannian metrics from their local Minkowskian character. The theorem then follows from the Lie–isotopic theory [3,4,10,11,21].

Corollary 2.1.A: *Under the conditions of the theorem the isopoincaré symmetry $\hat{P}(3.1)$ of all Riemannian line elements is locally isomorphic to the conventional Poincaré symmetry P(3.1) of the Minkowskian line element, $\hat{P}(3.1) \approx P(3.1)$.*

In fact, the conditions of nondegeneracy plus that of locally Minkowskian character imply $T(x) > 0$. The local isomorphism $\hat{P}(3.1) \approx P(3.1)$ then follows (see, again, ref.s [loc. cit.] for details).

The interested reader can verify that the isosymmetries (2.20), (2.21), (2.23) leave invariant all possible gravitational line elements, such as those of Schwartzschild, Krasner, Schucking–Heckmann, Growdy, Lifshitz–Khalatnikov and any other types [2]. In particular there is nothing to compute but just plug the $T_{\mu\mu}$ elements of the factorization $g(x) = T(x)\eta$ in the isotransforms. The invariance of the metric is then guaranteed by the Lie–isotopic theory.

Note the *loss* of the curvature/Riemannian space in the process and the return to a flat space $\hat{M}(x,\hat{\eta},\hat{R})$ locally isomorphic to the Minkowski space $M(x,\eta,R)$, only realized in its most general possible axiom–preserving form.

The above universal invariance is not a mere mathematical curiosity, because it permits the reinspection of gravitation from its foundations and its rigorous construction from a universal symmetry, much along the way in which the special relativity was constructed from the Poincaré symmetry. Also, the methods permit *the unification of the special and general relativity at the abstract isotopic level and their realization depending on the selected value of the isounit* [10,11].

For completeness we mention that the extension of Theorem 2.1 to Kadeisvili Class IV (admitting singular isotopic elements) is under study to permit the incorporation of the gravitational horizon, occurring for the zeros of the space–component of the isotopic elements $T(r) = 0$, and of gravitational collapse all the way to a geometric singularity, occurring for the zeros of the space–component of isounit $\hat{1}(r) = 0$.

In its simplest possible form, the antigravity submitted in this paper can be studied in a purely flat exterior space without gravitation via the use of the conventional Poincaré symmetry P(3.1) with unit I = diag. (1, 1, 1, 1) and the *isodual Poincaré symmetry* $P^{d}(3.1)$ with isodual unit $I^{d} = -I$, the latter symmetry being given by the conventional Poincaré transforms with isodual parameters $w_k{}^{d} = -w_k$ defined on the isodual isofield $R^{d}(n^{d},+,\times^{d})$ and , thus with negative–definite norm. This implies that physical quantities in isodual spaces are *negative–definite* (see Sect. 3).

The second level of study of antigravity can be done at the exterior gravitational level in vacuum via the use of the isopoincaré symmetry $\hat{P}(3.1)$ with gravitational isounit $\hat{1}(x) = [T(x)]^{-1}$ for matter, and the *isodual isopoincaré symmetry* $\hat{P}^{d}(3.1)$ with isodual gravitational isounit $\hat{1}^{d}(x) = -[T(x)]^{-1}$ for antimatter, the latter symmetry being given by rules (2.16)–(2.18), or merely by using the isodual parameters $w^{d}{}_{k} = -w_k$ in isotransforms (2.21)–(2.23).

But the restriction of the isotopic element T in deformations (2.19) to a sole dependence on the local coordinates x is *grossly unwarranted* for the isotopic theory again, because such a functional dependence is unrestricted. To put it clearly: why work on a Riemannian space with metric g(x) when one can work in the covering isoriemannian space with metric $\hat{g}(x, \dot{x}, \ddot{x}, \mu, \tau, ...)$? This brings in a natural way the third level of study of antigravity in the *interior gravitational problem of matter and antimatter* in which the isopoincaré symmetry $\hat{P}(3.1)$ and its isodual $\hat{P}^d(3.1)$ remains formally unchanged, but now defined with respect to isounits with arbitrary functional; dependence $\hat{1}(x, \dot{x}, \ddot{x}, \mu, \tau, n, \dots) > 0$ and their isoduals $\hat{1}^d = -\hat{1}$, respectively.

We finally mention that the above three levels of investigations are unified in the isopoincaré symmetry of Class III (Union of I and II), although this profile will not be presented to avoid excessively abstract geometric lines.

The minimal knowledge of the Lie-isotopic theory for the understanding of this note is, e.g., that of review [20] whose content is hereon assumed.

2.5. Isotopies and isodualities of functional analysis. It is easy to see that all conventional functions (trigonometric functions, etc.), special functions (Legendre polynomials, etc.), distributions (Dirac's δ, etc.), integral transforms (Fourier or Laplace transforms, etc.) are *inapplicable* under isotopies and isodualities. However, they have shown to admit simple yet unique and intriguing generalizations which we cannot possibly review here for brevity [10,11].

Since this paper is restricted to the exterior problem, we only need the *isodual functional analysis*, that is, the analysis reconstructed with respect to the isodual unit $-I$ which is evidently trivial and will be assumed as known.

Lesser trivial is the isoduality of Hilbert spaces. Let $\mathcal{H}$ be a conventional Hilbert space with states $|\psi>, |\phi>, \dots$ and inner product $<\psi|\phi>$ over the field $C(c,+,\times)$ of complex numbers c.

The *isodual Hilbert space* is the space $\mathcal{H}^d$ of states $<\psi|$, $<\psi|$, etc. equipped with the isodual inner product over the isodual field $C^d(c^d,+,\times^d)$ of isodual complex numbers $c^d = -\bar{c}$

$$\mathcal{H}^d: \quad <\psi|^d\phi> = <\psi|(-I)|\phi> I^d \in C^d(c^d,+,\times^d). \tag{2.24}$$

It is evident that *the inner products of $\mathcal{H}$ and $\mathcal{H}^d$ coincide by construction*, as it is the case for the separation in isospace and their isoduals. Nevertheless, $\mathcal{H}$ and $\mathcal{H}^d$ do not coincide. Note also the different between the conventional *dual* Hilbert space and its *isodual*. Under the conventional duality we have the maps $|\psi> \to <\psi| = (|\psi>)^\dagger$. Under isodualities we do preserve the conventional duality, but add the definition of the space in the isodual field $C^d(c^d,+,\times^d)$.

Isodual Hilbert spaces imply that the conventional eigenvalues equations for Hermitean operators $H|\psi> = E|\psi>$, $H = H^\dagger$, are mapped in the antiautomorphic form

$$<\psi|T^d H = -<\psi|H = <\psi|T^d E^d = -<\psi|(-E). \tag{2.25}$$

Similarly, we have the *isodual unitary operators*

$$U^d T^d U^{\dagger d} = U^{d\dagger} T^d U^d = -U^\dagger U = -U U^\dagger = I^d = -I, \tag{2.26}$$

The *isodual expectation values* are then *negative-definite*

$$<H>^d = \frac{<\psi|T^d H T^d|\psi> I^d}{<\psi|T^d|\psi> I^d} = -<H> \in R^d(n^d,+,\times^d) \tag{2.27}$$

as necessary because of their definition over the isodual field.

The reader can then easily construct the remaining parts of *isodual quantum mechanics* (see [11] for details).

3. ISODUAL REPRESENTATION OF ANTIPARTICLES UNDER ELECTROMAGNETIC INTERACTIONS

In the recent paper [12], we have shown the viability of representing antiparticles under electromagnetic interactions by essentially showing that *charge conjugation and isoduality are equivalent.* A knowledge of these results is essential for the understanding of this note.

Recall that antiparticles were first identified in the negative-energy solutions of conventional relativistic equations over ordinary fields and were originally conjectured by Stückelberg, Feynman and others to evolve backward in time (see, e.g., [23]). This representation had to be abandoned because antiparticles under such characterization do not behave in a way compatible with physical evidence *when realized in our space-time with the familiar unit* +1. This occurrence forced the study of alternative approaches, such as Dirac's celebrated "hole theory" and other models in second quantization.

Studies [12] have shown that antiparticles with negative-definite energy and evolution backward in time are indeed fully physical *when expressed in isodual spaces with negative-definite unit* -1, thus eliminating the need to conjecture an infinite sea of hypothetical undetectable particles or the use of second quantization for theories requiring semiclassical treatments (recall that antiparticle tracks are macroscopic, thus detectable classically and without any quantization, let alone second quantization).

The first step of studies [12] is that *our isodual representation of antiparticles is embedded in the very structure of conventional relativistic equations without any modification, and only with the appropriate re-interpretation of the negative-energy solutions.*

The best case is that of the conventional Dirac equation (see, e.g., [24]

$$\{ \gamma^{\mu} [p_{\mu} + e A_{\mu}(x) / c_o] + i m \} \psi(x) = 0 , \tag{3.1a}$$

$$\{ \gamma^{\mu} , \gamma^{\nu} \} = \gamma^{\mu} \gamma^{\nu} + \gamma^{\nu} \gamma^{\mu} = 2 \eta^{\mu\nu} , \tag{3.1b}$$

where the gamma matrices have the form ([24], p. 135)

$$\gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} , \quad \gamma^4 = i \begin{pmatrix} I_s & 0 \\ 0 & -I_s \end{pmatrix} , \tag{3.2}$$

I_s = diag. (1, 1) and σ_k are Pauli's matrices.

A mere inspection of the above familiar forms clearly indicates the intrinsic existence in Dirac equation of an isodual structure. In fact, the gamma matrices can be *identically* rewritten

$$\gamma^k = \begin{pmatrix} 0 & \sigma_k \\ \sigma^d_k & 0 \end{pmatrix} , \quad \gamma^4 = i \begin{pmatrix} I_s & 0 \\ 0 & I^d_s \end{pmatrix} , \tag{3.3a}$$

$$\sigma^d_k = - \sigma_k , \qquad I^d_s = - I_s = - \mathrm{Diag.}\ (1, 1). \tag{3.3b}$$

namely, they are characterized by the tensorial product of the regular irreducible isoreps of SU(2) with conventional unit I = diag. (1, 1) multiplied by the *isodual regular irreducible isorep of* $SU^d(2)$ *with isodual unit* $I^d_s = - I_s$. We thus have the property:

Lemma 3.1 [12]: *Dirac's gamma matrices are <isoselfdual>, that is, invariant under isoduality.*

The implications of this property are far reaching because it requires a revision of the treatment of the *conventional* Dirac equation into a form that must also be isoselfdual at each and every step, evidently to avoid internal inconsistencies.

To begin, recall that the Dirac equation is usually realized in the Kronecker product of the conventional (3+1)-dimensional Minkowski space $M(x,\eta,R)$ with unit I = diag. (1, 1, 1, 1) for the orbital part, and of the internal two-dimensional SU(2) space $S(s,\delta,C)$, $\delta = I_s^{-1}$ = diag. (1, 1), for the spin part, resulting in the total space $S_{Tot}{}^{Dirac} = M(x,\eta,R) \odot S(s,\delta,C)$, where $\odot$ represents the Kronecker product. But this space is not isoselfdual. As a result, the carrier space of the conventional Dirac equation, to be fully compatible with the algebraic properties of Dirac's gamma matrices, must have the isoselfdual structure

$$S_{Tot}{}^{Dirac} = \{ M(x,\eta,R) \odot S(s,\delta,C) \} \odot \{ M^d(x,\eta^d,R^d) \odot S^d(s,\delta^d,C^d) \} . \tag{3.4}$$

Still in turn, the statement that the Dirac equation is invariant under the conventional spinorial covering of the Poincaré symmetry $\mathcal{P}(3.1) = SL(2.C) \times T(3.1)$ is also not compatible with the algebraic structure of Dirac's gamma matrices. The correct invariance is given by the isoselfdual structure

$$\mathcal{P}(3.1) \odot \mathcal{P}^d(3.1) = \{ SL(2.C) \times T(3.1) \} \odot \{ SL^d(2.C^d) \times T^d(3.1) \} . \tag{3.5}$$

The above reinterpretation implies that the *total unit* of the conventional Dirac equation has the structure

$$I^{Tot}{}_{Dirac} = \{ I_{orb.} \odot I_{spin} \} \odot \{ I^d{}_{orb} \odot I^d{}_{spin} \} , \tag{3.6a}$$

$$I_{orb} = \mathrm{diag.}\ (1, 1, 1, 1), \quad I_{spin} = \mathrm{diag.}\ (1, 1). \tag{3.6b}$$

which is also isoselfdual, as it must be, thus being the correct unit of the symmetry characterized by Dirac's gamma matrices.

This leads in a unique and unambiguous way to the *isotopic interpretation of antiparticles.* In fact, the positive-energy solutions must evidently be interpreted with respect to the conventional unit $I_{orb} \times I_{spin}$. However, the negative-energy solutions must be necessarily interpreted with respect to the different unit $I^d{}_{orb} \times I^d{}_s = (-I_{orb}) \times (-I_s)$.

A fundamentally novel interpretation of antiparticles then follows. In fact, the orbital quantities emerge as being characterized over the isodual field $R^d(n^d,+,\times^d)$ with isodual unit $I^d{}_{orb}$ while their intrinsic characteristics are defined over an isodual complex field $C^d(c^d,+,\times^d)$ with isounit $I^d{}_{spin}$, both fields having a *negative-definite* norm.

The above re-interpretation implies the complete elimination of the conjecture of an infinite sea of hypothetical undetectable antiparticle with the "hole" being the physically observed particle, as well as other conjectures of infinite states in second quantization.

Under the above *semiclassical* analysis, if a particle has energy $E = |E| > 0$, time $t = |t| > 0$, linear momentum p, etc., its antiparticles emerge as an individual entity with *negative-definite energy* $E^d = |E|^d = - E < 0$, *moving backward in time* $t^d = |t|^d < 0$, and possessing similar isodual conjugates of the remaining quantities.

It is an instructive exercise to verify that all historical objections which lead Dirac to conjecture the "hole theory" are resolved by their isodual interpretation.

In inspecting this occurrence one should keep in mind that a negative-definite energy referred to a negative-definite unit is fully equivalent to a positive-definite energy referred to a positive-definite unit, and the same occurrence holds for time and all other physical quantities.

The study of the *operator formulation* of the isodual interpretation of antiparticles confirms the above results [11,12]. To have a technical understanding of this occurrence, one should be aware that isotopic methods have identified a novel antiautomorphic image of quantum mechanics submitted by this author under the name of *isodual quantum mechanics* [11]. The latter is essentially the image of the conventional mechanics under isoduality $I \rightarrow I^d = -I$. As such, it is defined in terms of isodual Hilbert spaces $\mathcal{H}^d$ over isodual complex fields $C^d(c^d,+,\times^d)$, possessing an isodual enveloping operator algebra, isodual eigenvalue equations, etc.

Then the *total Hilbert space* of the *conventional* Dirac equation can be written

$$\mathcal{H}^{Tot}{}_{Dirac} = \{\mathcal{H}_{orb} \odot \mathcal{H}_{spin}\} \odot \{\mathcal{H}^d{}_{orb} \odot \mathcal{H}^d{}_{spin}\}. \qquad (3.7)$$

and results to be isoselfdual, as it must be for compatibility with Dirac's gamma matrices. We then have the following important

Theorem 3.1 (Equivalence of charge conjugation and isoduality [11,12]): *The conventional Dirac equation for a particle of charge* e *under external electromagnetic interactions with potential* $A_\mu(x)$ *on the Hilbert space* $\mathcal{H}_{orb}\times\mathcal{H}_{spin}$ *is mapped under isoduality into its antiparticle, that is, the particle with charge* $e^d = -e$ *under the external potentials* $A^d{}_\mu(x) = -A_\mu(x)$ *characterized by the adjoint Dirac equation on the isodual Hilbert space* $\mathcal{H}^d{}_{orb}\times\mathcal{H}^d{}_{spin}$

$$\{[\gamma_\mu \eta^{\mu\nu}(p_\nu - e\times A_\nu) + i\times m]|\psi>\}^d =$$

$$= [\gamma^d{}_\mu \eta^{d\,\mu\nu}(p^d{}_\nu - e^d\times^d A^d{}_\nu) + i^d\times^d m^d]|\psi>^d =$$

$$= <\psi|[\gamma_\mu \eta^{\mu\nu}(p^d{}_\nu - e^d A_\nu) + i\,m^d] = 0. \qquad (3.7)$$

The proof is elementary and merely requires Lemma 3.1 plus the properties of the isodual quantum mechanics (this implies the understanding that *all* conventional multiplication $A\times B$ have no meaning under isoduality and must be replaced by their isodual image $A^d\times^d B^d$, with similar occurrences holding for *all* other operations).

Note that the equivalence of charge conjugation and isoduality with consequential elimination of infinite seas of undetectable antiparticles can be directly derived from the properties of the spinorial covering of the isodual Poincaré symmetry $\mathcal{P}^d(3.1) = SL^d(2.C^d\times T^d(3.1)$ on isodual Hilbert spaces (see [11] for details).

As an example, the isodual $T^d(3.1)$ of the translational symmetry $T(3.1)$ is characterized by the antiautomorphic map of the exponentiation

$$\psi(x) = e^{\,i\,p_\mu \eta^{\mu\nu} x_\nu} = e^{\,i\times(\mathbf{k}\times\mathbf{r} - E\times t)} \Rightarrow$$

$$\Rightarrow \psi^d(x) = \{e^{\,i\,p_\mu \eta^{\mu\nu} x_\nu}\}^d = e^{\,i^d\times^d(\mathbf{k}^d\times^d \mathbf{r}^d - E^d\times^d t^d)}$$

$$= e^{\,i\times[-(-\mathbf{k})\times(-\mathbf{r}) + (-E)\times(-t)]}. \qquad (3.7)$$

thus yielding negative-energy solutions including *negative rest mass*, evolving backward in time although defined with respect to a negative-definite unit. A similar result holds for the spin.

A further notion important for these studies is that of *isoselfdual particle* which, physically, is a bound state of a particle and its antiparticle such as the π° and, mathematically, is represented via isoselfdual structures of type (3.7). The peculiarity of these bound states is that they admit *positive energies and times* when studied in our space-time, and *negative energies and times* when studied in the isodual space-time. For details underlying these results, we refer the interested reader to Sect. 7.8, Vol. II, ref. [11].

Additional properties of the isodual representation of antiparticles directly relevance for antigravity are presented in Sect. 5.

We indicated earlier that the hypothesis of antiparticles moving backward in time is by no means new, because it was studied by Stückelberg, Feynman and others since the very discovery of antiparticles [23].

Similarly, by no means the assumption of physical particles as having negative rest mass is new. For instance, comprehensive studies along these lines were conducted by Recami and Ziino (see [25] and papers quoted therein) who showed via conventional relativistic arguments that <relativity requires antiparticles to be formally associated with *negative* rest masses>. These independent results are directly relevant for the antigravity of the Sect. 5.

The sole novelty of our treatment is the *re-interpretation* of time evolving backward and negative rest masses via *isodualities.*

4. ISODUAL CHARACTERIZATION OF ANTIPARTICLES UNDER GRAVITATIONAL INTERACTIONS.

The interior problem of gravitation has been studied in the recent paper [14] via the isotopies and isodualities of the Riemannian geometry. In conventional gravitational studies matter and antimatter are assumed to belong to the same geometry, that is, to the same universe. According to studies [14] antimatter belongs instead to a new geometry and therefore characterizes a hitherto unknown universe called *isodual universe.*

These studies also permit a novel cosmological conception of the universe which, in its limit conditions, is composed of equal amounts of matter and antimatter, thus resulting in null total physical characteristics, including null total energy, time, linear momentum, angular momentum, etc.

A main result of these studies is that *an antiparticle in the gravitational field of antimatter experiences an attractive interaction identical to the corresponding case for particle-matter.*

Stated in different terms, the isodual conjugation per se does not produce antigravity, because it preserves the attractive character of ordinary gravitation for matter.

The treatment of the above result via the isoriemannian geometry and its isodual is much beyond the limited scope of this note. However, the property can also be seen via the conventional Riemannian geometry in spaces $\Re(x,g,R)$ and the *dual Riemannian geometry* in isodual spaces $\Re^d(x,g^d,R^d)$. The basic results can be expressed via the following exterior limit of the interior Theorem 3.2 of ref. [14]:

Theorem 4.1 (Exterior gravitational treatment of antiparticles in vacuum [14]). *The exterior problem of antimatter in vacuum is characterized by the following properties of the isodual Riemannian geometry*

Basic unit	$\hat{1} \to I^d = -I,$
Isotopic element	$T \to T^d = -T = -I,$
Metric	$g \to g^d = -g,$
Connection coefficients	$\Gamma_{klh} \to \Gamma^{d1}{}_{klh} = -\Gamma^1{}_{klh},$

Curvature tensor	$R_{lijk} \to R^d{}_{lijk} = -R_{lijk},$	
Ricci tensor	$R_{\mu\nu} \to R^d{}_{\mu\nu} = -R_{\mu\nu},$	
Ricci scalar	$R \to R^d = R,$	(4.1)
Einstein tensor	$G_{\mu\nu} \to G^d{}_{\mu\nu} = -G_{\mu\nu},$	
Electromagnetic potentials	$A_\mu \to A^d{}_\mu = -\hat{A}_\mu,$	
Electromagnetic field	$F_{\mu\nu} \to F^d{}_{\mu\nu} = -F_{\mu\nu},$	
Elm energy-mom. tensor	$T_{\mu\nu} \to T^d{}_{\mu\nu} = -T_{\mu\nu},$	
Stress-energy tensor	$\hat{t}_{\mu\nu} \to t^d{}_{\mu\nu} = -t_{\mu\nu},$	

The proof require the knowledge of the *isodual calculus,* e.g., $(d/dx)^d = I^d d/dx = -d/dx$, etc. Note the achievement of gravitational *attraction* for *antimatter-antimatter* systems via a *negative* curvature, although referred to a *negative-definite unit.*

Ant instructive aspect is the *necessity* of using isodual spaces to reach negative-energy solutions of the gravitational field equations. In fact, in the conventional formulation of gravitation, antiparticles do admit the conjugation of their charge, but their energy-momentum tensor can only be positive-definite. In turn, this permits, apparently for the first time, a full geometric equivalence between relativistic field equations and gravitation, that is, it permits the existence and quantitative treatment of negative-energy solutions at the full gravitational level. As is well known, such solutions are notoriously absent in conventional gravitational theories [2].

The importance of the latter result is evident. In fact, if antiparticles originated in the negative-energy solutions of the relativistic field equations, no compatible study of their gravitational field can possibly be done if the gravitational theory is restricted to admit only positive-energy solutions.

The study also permits the novel notion of *negative curvature* (see, e.g., [26]) although defined with respect to a negative-definite unit. This is the mechanism which yields attraction for the gravitational field of an antiparticle in the gravitational field of antimatter.

The most important result of isotopic methods for the study of antigravity is the elimination of the now vexing problem of "unification" of the gravitational and electromagnetic fields, and its replacement with the "identification" of the two fields in the ultimate origin of matter, plus second-order corrections due to weak and strong interactions when applicable [22].

Stated in different terms, isogeometries and their isoduals permit quantitative studies of the "origin" of the gravitational field of matter and antimatter, respectively. These studies result in the complete elimination of the matter tensor $M^{\mu\nu}$ in the field equations and their replacement with the fields originating matter itself.

By recalling the experimentally established fact that the mass of elementary particles has a primary electromagnetic origin with weak and strong corrections when applicable. Recall that a macroscopic mass is nothinbg but an aggregate of elementary particles plus internal interactions of the same nature. One can then see that the *exterior* gravitational field (i.e., the field at large distances) of a given mass is fully identified with the electromagnetic field originating that mass, while the *interior* gravitational field (i.,e., the field at short distances) is identified with the electromagnetic and short range fields originating the same mass. Regrettably, we are forced to refer the interested reader to studies [10,11,22] to avoid an excessive length.

As shown in the next section, antigravity is a mere consequence of the identification of gravitational and electromagnetic interactions.

5. ANTIGRAVITY AND ITS POSSIBLE EXPERIMENTAL VERIFICATION.

To understand the proposed antigravity, it is necessary to understand deeper the comparative behaviour of particles and antiparticles under isodualities.

Consider two electrons of negative charges $-e$ in our space–time, that is, with respect to the trivial unit $+1$. Their interaction is characterized by the familiar *repulsive* Coulomb law

$$F = K \times (-e) \times (-e) / |r|^2 . \tag{5.1}$$

Under isoduality the the electrons are mapped into positron and we have

$$F^d = K^d \times^d (-e)^d \times^d (-e)^d /^d |r|^{2d} = -F. \tag{5.2}$$

where one should note the isoduality of *all* operations. But the latter law is referred to the isodual unit $1^d = -1$. As such, it still represents the Coulomb *repulsion* among equal charges.

In order to have *attraction*, we have to project the antiparticle in the field of the particle, i.e.,

$$F' = K \times (-e) \times (-e)^d / |r|^2 = -F \tag{5.3}$$

and this does now yield an *attraction* because it is referred to our space–time, that is, with respect to the unit $+1$. Similarly, the projection of the particle in the field of the antiparticle yields

$$F^d = K^d \times^d (-e)^d \times^d (-e) /^d |r|^{2d} = F \tag{5.4}$$

But the basic unit is now -1, thus yielding again attraction.

DECAY OF MUON μ^- IN MINKOWSKI SPACE | **DECAY OF ANTIMUON μ^+ IN ISODUAL MINKOWSKI SPACE**

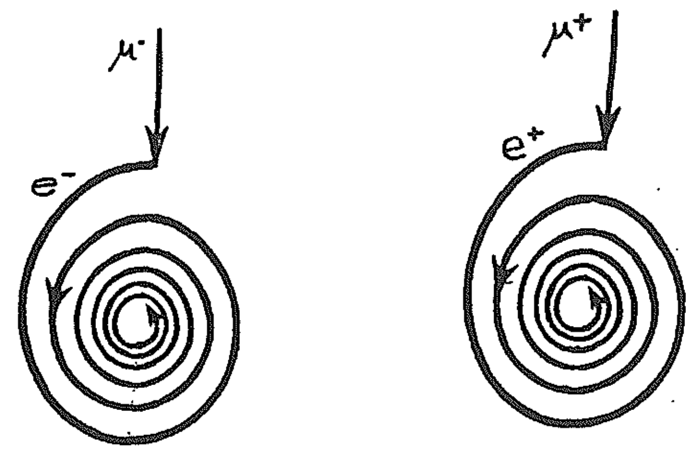

FIGURE 1: An understanding of the antigravity proposed in this paper requires the knowledge that *the trajectory of a particle with charge* q *under a magnetic field* **B** *in Minkowski space* M(x,η,R) *coincides with the trajectory of the antiparticle with charge* $q^d = -q$ *under the magnetic field* $\mathbf{B}^d = -\mathbf{B}$ *in isodual Minkowski space* $M^d(x,\eta^d,R^d)$. The same property evidently holds under arbitrary electromagnetic fields (Theorem 5.1 below). At first, this may be considered contrary to experimental evidence because visual observation indicate that, e.g., the trajectory of the positron e^+ originating from the decay of a muon μ^+ under an external magnetic field spirals in the direction opposite to that of the electron e^- from the decay of μ^-. The key point is that the trajectory of the latter is *the projection of the trajectory of the antiparticle in our space–time and not the actual trajectory in the appropriate isodual space–time* (Corollary 5.1.A below). The mechanism of antigravity is essentially the same, only referred to masses and gravitational fields.

A deeper understanding of the representation of antiparticles with isoduality can be reached via the following property illustrated in Fig. 1.

Theorem 5.1 (Identity of the trajectories of particles and antiparticles [11]): *The trajectory of a particle under an electromagnetic field with potentials $A_\mu(x)$ in Minkowski space $M(x,\eta,R)$ and that of the corresponding antiparticle under the isodual potential $A_\mu{}^d(x)$ in isodual Minkowski space $M^d(x,\eta^d,R^d)$ coincide.*

A deeper understanding of the behaviour of antiparticles in the field of particles and viceversa can be understood via the following

Corollary 5.1.A: *An antiparticle reverses its trajectory when projected from its isodual space to the the space of the particle, and viceversa.*

In the transition to gravitation, the predictions of the isotopic theory are fully equivalent to the above electromagnetic behavior. In fact, for the case of a particle of mass m_1 (>0) in the field of a second mass m_2 (>0) we have Newton's gravitational law

$$F = -G \times m_1 \times m_2 / |r|^2 < 0, \tag{5.5}$$

where we have introduced the minus sign for compatibility with the Coulomb law (5.1).

In the transition to antimatter represented via isodualities the Newtonian gravitational force remains attractive because the positive masses are now mapped into negative values [12,25] $m_1{}^d = -m_1$ and $m_2{}^d = -m_2$ yielding the isodual law

$$F^d = -G^d \times^d m^d{}_1 \times^d m^d{}_2 /^d |r|^{2d} = F > 0, \tag{5.6}$$

which represents *attraction* because referred to the unit −1.

However, if we project the antiparticle with mass $m_2{}^d = -m_2$ (< 0) in the field of the particle with mass m_1 (> 0), we do have *repulsion* because Newton's gravitational law now reads

$$F' = -G \times m_1 \times m_2{}^d / |r|^2 = F > 0, \tag{5.7}$$

and is referred to the same unit +1 of law (5.5).

Similarly, if we project the particle of mass m_1 (>0) in the field of the antiparticle $m_2{}^d = -m_2$ (<0) we also have the law

$$F^d = -G^d \times^d m_1 \times m_2{}^d /^d |r|^{2d} = -F < 0, \tag{5.7}$$

which again represents *attraction* because referred to the unit − 1.

The above results imply the following

Proposition 5.1 [11]: *The isodual representation of antiparticles identifies gravitation and electromagnetism in the exterior problem.*

In fact, the admission of negative masses for antiparticles renders the Coulomb and Newton laws fully equivalent, and the same holds for more general formulations. Note that such a full equivalence is absent in current theories of gravitation precisely because of the lack of antigravity via isoduality.

This occurrence confirms the expectation indicated in the preceding section that antigravity is a consequence of the resolution of the vexing problem of "unification" of the gravitational and electromagnetic field via their "identification" in the "origin" of matter and antimatter and, thus, of their gravitational fields.

We can therefore conclude this paper with the following:

BASIC HYPOTHESIS: Antigravity is given by the projection of the gravitational field of antimatter in the gravitational field of matter, or viceversa.

It is an instructive exercise for the interested reader to prove that none of the known "arguments against antigravity", such as those by Morrison and others of ref. [1a], apply to the above antigravity because of the role of isoduality.

The above hypothesis can be experimentally verified with current technology at very low energies (the experimental problems are multiplied at the very high energies, say, of contemporary accelerators). In fact, neutron interferometric techniques have now reached a precision sufficient to measure gravitational effects. The above basic hypothesis is therefore experimentally testable, e.g., via the comparison of interferometric measures of thermal beams of neutrons and antineutrons in the gravitational field of Earth following a sufficiently long flight, such as that in the one km long netron tunnel at the JINR in Dubna. Under the assumption of energies as low as possible, the experiment therefore consists in three interferometric measures after said long flight: first, the calibration of the no-deviation point, e.g., via optical means; second, the measure of the downward deviation of neutrons caused by gravity which is measurable after a sufficiently long flight; and, third, the establishing whether the displacement of antineutrons due to gravity is downward or upward (see Vol.s II and III of ref.s [11] for detailed theoretical and experimental studies).

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REFERENCES

1. M. M. Nieto and T. Goldman, Physics Reports **205**, 221 (1991) and **216**, 343 (1992) [1a]; D. H. Childress, *The Antigravity Handbook*, Network Unlimited Press, Stelle, Illinois (1993) [1b]; B. C. Ebershaw, *Antigravity Propulsion Devices*, D. & B. Associates (1980) [1c].
2. P. G. Bergmann, *Introduction to the Theory of Relativity* (with *Preface* by Albert Einstein), Dover Publ., New York, 1942; W. Misner, K. S. Thorne and A. Wheeler, *Gravitation*, Freeman, San Francisco (197).
3. R. M. Santilli, *Hadronic J.* **1** (1978), 223; Addendum, *ibidem* **1** (1978), 1279; and *Phys. Rev. D* **20** (1979), 555.
4. R. M. Santilli, *Foundations of Theoretical Mechanics*, Vol. I: *The Inverse Problem in Newtonian Mechanics* (1978), Vol. II: *Birkhoffian Generalization of Hamiltonian Mechanics* (1982), Springer-Verlag, Heidelberg .
5. R. M. Santilli, *Lett. Nuovo Cimento* **37** (1983), 545.
6. R. M. Santilli, *Hadronic J.* **8** (1985), 25 and 36.
7. R. M. Santilli, *Hadronic J. Suppl.* **4A** (1988), 407.
8. R. M. Santilli, *Algebras, Groups and Geometries* **8** (1991), 266 and 275.
9. R. M. Santilli, *Algebras, Groups and Geometries* **10** (1993), 273.
10. R. M. Santilli, *Isotopic Generalization of Galilei's and Einstein's Relativities*, Vol. I: *Mathematical Foundations*, Vol. II: *Classical Isotopies*, First Edition (1991); Second edition, Academy of Sciences of Ukraine, Kiev, (1994).
11. R. M. Santilli, *Elements of Hadronic Mechanics*, Vol. I: *Mathematical Foundation* (1993), Vol. II: *Theoretical Foundations* (1994), Vol. III: *Experimental Verifications* (to appear), Academy of Sciences of the Ukraine, Kiev.

12. R. M. Santilli, Isodual representation of antiparticles, *Comm. Theor. Phys.* (1994), in press.
13. R. M. Santilli, Nonlinear, nonlocal, noncanonical, axiom-preserving isotopies and isodualities of the Poincaré symmetry, *J. Moscow Phys. Soc.* (1994), in press.
14. R. M. Santilli, in *Analysis, Geometry and Groups: A Riemann Legacy Volume* (H. M. Srivastava and Th. M. Rassias, Editors), Hadronic Press, Palm Harbor, Florida, 1993.
15. R. M. Santilli, in *The Mathematical Legacy of Hanno Rund* (J. V. Kadeisvili, Editor), Hadronic Press, Palm Harbor, Florida, 1993.
16. D. S. Sourlas and G. T. Tsagas, *Mathematical Foundations of the Lie-Santilli Theory*, Academy of Sciences of Ukraine, Kiev (1993).
17. J. Lôhmus, E. Paal and L. Sorgsepp, *Nonassociative Algebras in Physics*, Hadronic Press, Palm Harbor, Florida (1994).
18. J. V. Kadeisvili, *Santilli's Lie-isotopic Generalization of Contemporary Algebras, Geometries and Relativities*, second edition, Academy of Sciences of Ukraine, Kiev (1994).
19. A. K. Aringazin, A. Jannussis, D. F. Lopez, M. Nishioka and B. Veljanoski, *Santilli's Isotopies of Galilei's and Einstein's Relativities*, Kostarakis Publisher, Athens, Greece (1991).
20. J. V. Kadeisvili, in *Analysis, Geometry and Groups: A Riemann Legacy Volume* (H. M. Srivastava and Th. M. Rassias, Editors), Hadronic Press, Palm Harbor, Florida (1993).
21. D. S. Sourlas and G. T. Tsagas, "Isomanifolds" and "Isotensors on isomanifolds", *Algebras Groups and Geometries* **11** (1994, to appear).
22. R. M. Santilli, *Ann. Phys.* **83** (1974), 108.
23. P. A. M. Dirac, *The Principles of Quantum Mechanics*, Fourth edition, Oxford University Press, Oxford, 1958; R. P. Feynman, B. Leighton and M. Sands, *The Feynman Lectures in Physics*, Addison-Wesley, Reading MA, 1964
24. E. Corinaldesi and F. Strocchi, *Relativistic Wave Mechanics*, North Holland, Amsterdam (1963).
25. E. Recami and G. Ziino, Nuovo Cimento **33A**, 205 (1976)
26. K. A. Loubani, *Fundamental Groups of Manifolds of Negative Curvature*, Ph. .D. Thesis, King Saud University, Riyadh, Saudi Arabia.