

**PROBLEMATIC ASPECTS OF q-DEFORMATIONS  
AND THEIR ISOTOPIC RESOLUTION**

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**Abstract**

By recalling that q-deformations have an impeccable mathematical structure, we outline their numerous problematic aspects of physical nature which essentially emerge whenever attempting dynamical applications, thus implying evolution in time. We outline Santilli's initiation of q-deformations back in 1967 via isotopies and genotopies of classical and quantum mechanics. We show how they permit an axiomatic reformulation of q-deformations which leaves the results unchanged while avoiding their problematic aspects. We finally point out applications and experimental verifications which would be generally precluded to q-deformations without their consistent axiomatic reformulation.

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**I: PROBLEMATIC ASPECTS OF q-DEFORMATIONS**

The so-called q-deformations (see, e.g., ref.s [1] and quoted literature) include a great variety of deformations of Quantum Mechanical (QM) formalisms whose *mathematical* consistency is impeccable. Nevertheless, the q-deformations are afflicted by a number of rather serious problematic aspects of *physical* character which emerge whenever dynamical applications are attempted, thus implying the evolution in time.

The latter problematic aspects and their resolution, together with a number of applications, have been studied in great details by Santilli in the three volumes [2] on Hadronic Mechanics (HM). They were also presented by this author [3] at the *International Conference on Symmetry Methods in Physics* held this past July at the J.I.N.R. in Dubna. This paper is an extended version of note [3]. By following ref.s [2], we classify q-deformations into the following primary types:

**I) Deformation of the enveloping associative algebra.** Let  $\xi(L)$  be the universal enveloping associative algebra of a Lie algebra L with elements A, B, ... and conventional associative product AB over a field  $F(\alpha, +, \times)$  with generic elements  $\alpha$ , conventional sum + and product  $\alpha\beta := \alpha\beta$ . This first type is characterized by the following generalization of the associative product AB

$$A B \Rightarrow A * B = q A B, \quad (1.1)$$

where q is an element of the base field (or a function).

**II) Deformation of the Lie product.** Let L be a Lie algebra in quantum mechanical realization on a Hilbert space  $\mathcal{H}$  over a field  $F(\alpha, +, \times)$  with fundamental commutation rules  $r p - p r = i (\hbar = 1)$ . This second type of q-deformation is based on the generalized commutators

$$r p - p r \Rightarrow r p - q p r = i f(q, \dots) \quad (1.2)$$

where  $f(q, \dots)$  is a sufficiently smooth, bounded and nonsingular function.

**III) Deformation of the structure constants.** Let L be an n-dimensional Lie algebra with ordered basis  $X_i$ , envelope  $\xi(L)$  and commutation rules  $[X_i, X_j] = C_{ij}^k X_k$  over a field  $F(\alpha, +, \times)$ . This third type of deformations, which contains the Hopf algebras and others, is based on the preservation of the original product Lie  $X_i X_j - X_j X_i$ , while deforming this time the structure constants

$$X_i X_j - X_j X_i = C = C_{ij}^k X_k \Rightarrow X_i X_j - X_j X_i = F_{ij}^k(q, \dots) X_k, \quad (1.3)$$

where the functions  $F_{ij}^k$  are also sufficiently smooth, bounded and nonsingular.

Numerous other q-deformations exist in the literature (such as the deformation of creation-annihilation operators of the above Types I, II, III) [1]

which can be derived via the techniques of the above three deformations. Others can be reduced to a combination of the above three types, or are given by to a combination of one of the above types with QM structures (such as the combination of *deformed* commutators (1.2) and *conventional* Heisenberg equations for the time evolution).

Again, the mathematical consistency of all the above q-deformations is undeniable. However, when considered for physical applications they require the necessary use of the dynamical time evolution, in which case the following problematic aspects emerge as identified in detail in Vol. I of ref.s [2] (see also [3]):

**A) General loss of the Hermiticity/observability of the Hamiltonian and of other physical quantities.** q-deformations imply a *nonunitary time evolution*, as necessary for Types I, II, III from the lack of canonicity of the commutation rules, and demonstrable, e.g., via quantization of the corresponding, classical, noncanonical theories (see below for more details). In turn, nonunitary time evolutions imply the following generalization of the structure of the enveloping associative algebra first identified in ref. [4]

$$\xi: AB \Rightarrow \xi: A^*B^* = A^*QB^*, \quad A^* = UAU^\dagger, B^* = UBU^\dagger, \quad (1.4a)$$

$$UU^\dagger \neq I, \quad Q = (UU^\dagger)^{-1}, \quad (1.4b)$$

which evidently also applies to the product qAB. Still in turn, the above structure implies the loss of the Hermiticity/observability of the Hamiltonian and of other physical quantities. This is due to the fact that q-deformations are defined on a *conventional Hilbert space*, while the preservation of Hermiticity under lifting (1.4) demands the joint deformation of the base field and of the Hilbert space (see later on Lemma 3.1).

**B) General loss of the measurement theory and consequential lack of applicability to experiments.** q-deformations are deformations of the basic associative product AB and/or of Planck's constant  $\hbar = 1$ , and/or of structure constants, without a corresponding redefinition of the unit as done in Santilli's isotopic theories [2]. Therefore, *q-deformations are theories without a left and right unit which remains invariant under the time evolution.*

This occurrence is transparent in lifting (1.1) which deforms the product  $AB \Rightarrow A^*B = qAB$  without jointly deforming the unit as done in the foundations of hadronic mechanics [2,3,4]:

$$I \Rightarrow \mathbb{1} = q^{-1}. \quad (1.5)$$

The lack of basic unit can also be established for deformations of Types II and III, e.g., under time evolution with ensuing nonunitary structure, and unification of all envelopes into isotopic form (1.4). The loss of the unit then implies the

evident loss of the measurement theory, owing to the necessary existence of a well defined, left and right unit for the very concept of measurement.

It should be indicated that the problem of the unit is much deeper of what may appear at a first inspection. QM theories have in actuality *two different units*, the unit  $\hbar = 1$  of the *field* and the unit  $I = \text{diag.} (1, 1, \dots)$  of the *envelope*. Under deformations (1.1) we evidently have the loss of the unit I of the envelope.

In regards to the unit of the field we have two alternatives. The first is to keep the theory with one single associative product  $A^*B = qAB$  which would then apply also to numbers  $\alpha^*\beta = q\alpha\beta$ . In this first case one has evidently the loss also of the unit of the field. The q-deformations are then theories on a Hilbert space defined over a *commutative ring without unit*. The lack of applicability to experiments is then transparent.

The second alternative, which is that followed by the current literature [1], is to define deformations (1.1) on a Hilbert space defined over a *conventional field* which, as such, does possess the unit. This evidently implies that *deformations (1.1) are theories with two different associative multiplications, one for the envelope and one for the field*. The problem is that the differentiation of these two multiplications leads to the lack of observability of the physical quantities because it prevents the needed lifting of the underlying Hilbert space and related field.

In summary, rather deep technical reason related to the preservation of the observability at all times demand the unification of the associative product of the envelope with that of the field, as well as the unification of their unit (see Vol. I of ref.s [2] for a detailed treatment).

**C) General lack of uniqueness of mathematical structures, such as Gaussian distributions, with consequential lack of uniqueness of physical laws.** One of the strengths of quantum mechanics is the *uniqueness* of its mathematical structure (such as the exponentiation and related Gaussian) which evidently implies the known uniqueness of its physical laws (such as the uniqueness of Heisenberg's uncertainties as derivable from the unique Gaussian distribution). This uniqueness can be mathematically traced to the uniqueness of the basic unit of the theory, Planck's constant, as well as to the existence of a right and left unit of the universal enveloping operator algebra  $\xi(L)$ .

The mathematical implications of the general lack of the basic unit implies that *q-deformations do not possess a consistent formulation of the Poincaré-Birkhoff-Witt theorem which is applicable at all times*. In fact, a necessary condition for the very formulation of the theorem is the existence and uniqueness of a left and right unit (see Jacobson [5]).

This means *the lack of existence of a unique, infinite-dimensional basis for the envelopes of q-deformations* and, therefore, *the lack of existence of a unique form of exponentiation*. In fact, q-deformations are known for the variety of their possible "exponentiations".

Even though mathematically correct (and intriguing), the above occurrences have rather severe physical consequences identified by Santilli [2], such as *the lack of uniqueness of a Gaussian distribution with consequential lack of uniqueness of the generalized uncertainties*. A similar situation occurs for other physical laws.

It should be stressed that the above occurrences *are not* referred to different physical laws for different  $q$ -deformations, which would be physically acceptable, but to different physical laws which can be introduced in *each*  $q$ -deformation.

**D) General loss of special functions under time evolution.** As well known,  $q$ -deformations are formulated at a fixed value of time, and so are the related  $q$ -special functions [1]. But under time evolution the  $q$ -number is replaced by the operator  $Q$ . The inapplicability of the  $q$ -special functions under time evolution is then consequential.

Again, this occurrence is fully acceptable on mathematical grounds. However, its physical implications are rather serious, such as the impossibility of performing a partial  $q$ -wave-analysis at all times.

**E) General loss of the fundamental axioms of Einstein's special and general relativities.** Even though not fully identified in the literature, all  $q$ -deformations imply a structural departure from *all* basic axioms of the special and general relativities, as established by the noncanonicity of the commutation rules, the nonunitary character of the time evolution, the deformation of the structure constants of the Poincaré symmetry, etc.

Again, this occurrence can be mathematically intriguing, but it carries rather serious physical problems in the compliance with physical reality which must be addressed prior to any physical application.

The reader can derive numerous additional problematic aspects as a consequence of the above primary ones.

In the following we shall review Santilli's origination of  $q$ -deformations back in 1967 because it provides significant insights in their appropriate treatment, and then his axiomatization of  $q$ -deformations which avoids all the preceding problematic aspects. After achieving a physically consistent reformulations, we shall then point out numerous physical applications of  $q$ -deformations which would be otherwise precluded.

## 2: ORIGIN OF $q$ -DEFORMATIONS

When studying the axiomatic structure of quantum mechanics, the first and most fundamental task is the identification of the algebra characterized by the commutator  $[A,B] = AB - BA$ , the Lie algebra [5]. Similarly, when studying  $q$ -deformations, the identification of the algebra characterized by the "commutator"  $[A,B]_q = AB - qBA$  is an evident pre-requisite for the achievement of a consistent

axiomatization.

The algebra characterized by the product  $[A,B]_q$  was first introduced by the American mathematician Albert [6] back in 1948, via the following notions:

**Lie-admissibility:** a (generally nonassociative) algebra  $U$  with elements  $a, b, c, \dots$  and (abstract) product  $ab$  over a field  $F$  is said to be *Lie-admissible* when the attached algebra  $U^-$ , which is the same vector space as  $U$  but equipped with the product  $[a,b]_U = ab - ba$ , is a Lie algebra. A Lie-admissible (or any other) algebra  $U$  is said to be *flexible* when it verifies the weaker form of associativity  $a(ba) = (ab)a$  for all  $a, b, c \in U$ .

**Jordan-admissibility:** the algebra  $U$  is said to be *Jordan-admissible* if the attached algebra  $U^+$ , which is the vector space  $U$  equipped with the product  $[a,b]_U = ab + ba$ , is a (commutative) Jordan algebra [7]. An algebra  $U$  is said to be *noncommutative Jordan algebra* when the product  $ab$  is noncommutative but verifies Jordan axiom  $(ab)a^2 = a(ba^2)$ .

The first introduction of  $q$ -deformations in the mathematical and physical literature was done by the physicist Santilli [8] back in 1967 as part of his Ph.D. in physics at the University of Turin, Italy. In fact, in [8], p. 573, one can see the first appearance of the product

$$(a,b) = \lambda ab - \mu ba = \rho [a,b] + \sigma (a,b), \quad \lambda = \rho + \sigma, \mu = \rho - \sigma \in F, \quad (2.1)$$

which, for  $ab$  associative, was introduced as characterizing an algebra  $U$  which is Lie-admissible, Jordan-admissible, flexible as well as noncommutative Jordan. Moreover, product (2.1) was introduced as the  $(\lambda, \mu)$ -mutation of a generic (not necessarily associative) algebra  $U$  with product  $ab$ , in order to distinguish it from *deformations* of an algebra as conventionally understood in mathematics.

In fact, formulation of Type III are true "deformations", but formulations of Types I and II are not thus justifying the term "mutations". Nevertheless, the term "deformation" is now entered in the literature and will be kept in this paper to avoid confusion.

It is evident that the  $q$ -deformation  $[A,B]_q = AB - qBA$  is a particular case of Santilli's mutation for  $\lambda = 1, \mu = q$  and  $ab$  associative.

One should note the virtually complete silence in the entire literature [1] on the above origin of  $q$ -deformations. This is rather odd because Albert's notion of Lie-admissibility, or the emergence of the still open Jordan's legacy alone, should be reason for their quotation.

To clarify the priority of product (2.1) we recall that Albert presented in [6] an abstract (and relatively short) treatment of Lie-admissibility, with more emphasis on the Jordan-admissibility because of its greater interest in the mathematics of the time. In fact, the sole explicit realization of the product in Albert's paper is given by the known realization of noncommutative Jordan algebras [6,7]

$$(a,b) = \lambda ab - (1 - \lambda) ba, \quad (2.2)$$

for ab associative. The point is that q-deformations are a particular case of Santilli's mutation (2.1) and not of Jordan's form (2.2).

Santilli is therefore the originator on both mathematical and physical grounds of theories today known as *Lie-admissible formulations*, and referred to a step-by-step generalization of Lie's theory, with realizations in classical, operator and statistical mechanics. This priority is now acknowledged in mathematical circles (see, e.g., the historical charts of ref. [10], p. 13, or the mathematical monographs of ref.s [34]). In fact, following Albert [6] and prior to paper [8], only two short mathematical notes in Lie-admissibility had appeared (see [8] and bibliography [9]), also without any specific realization.

On mathematical grounds, Lie-admissible algebras had been studied as *nonassociative* algebras, an approach still continuing in the mathematical literature [9]. On the contrary, Santilli constructed a generalization of enveloping *associative* algebras characterizing Lie-admissible algebras, groups, representation theory, etc., which subsequently resulted to be crucial for the axiomatization of q-deformations presented in below.

On physical grounds, Santilli studied already in 1968 [11] the *classical limit* of the  $(\lambda, \mu)$ -mutations (2.1), by proving that they are a particular case of Hamilton's equations with external terms. This established that the mutations  $AB - BA \rightarrow \lambda AB - \mu BA$  imply the transition from closed-conservative to open-nonconservative systems, because of the loss of total antisymmetry of the product.

These initial classical studies were then complemented in 1978 [12] with the identification that *the brackets of Hamilton's equation with external terms, when properly written, characterize a general Lie-admissible algebra*. In fact, we can write for N particles in "phase space" with unified coordinates  $a = (a^\mu) = (r_a^k, p_{ak})$ ,  $\mu = 1, 2, \dots, 6N$ ,  $k = 1, 2, 3$ ,  $a = 1, 2, \dots, N$ ,

$$(a^\mu) = \begin{pmatrix} r_a^k \\ p_{ak} \end{pmatrix} = \left\{ \begin{array}{l} \partial H / \partial p_{ak} \\ -\partial H / \partial r_a^k + F_k \end{array} \right\} = ( \omega^{\mu\alpha} \gamma^>_{\alpha}{}^\nu(t, a, \dot{a}, \dots) \frac{\partial H}{\partial a^\nu} ), \quad (2.3)$$

where  $\omega^{\mu\alpha}$  is the conventional canonical Lie tensor,  $\gamma^>_{\alpha}{}^\nu = \delta_{\alpha}{}^\nu + \omega_{\alpha\rho} s^{\rho\nu}$ ,  $s = \text{diag. } (0, F/(\partial H/\partial p))$  and the meaning of the symbol ">" will be identified later on. The corresponding brackets among functions in "phase space"

$$(A, B) = \frac{\partial A}{\partial a^\mu} \omega^{\mu\alpha} \gamma^>_{\alpha}{}^\nu(t, a, \dot{a}, \dots) \frac{\partial B}{\partial a^\nu} \quad (2.4)$$

are then Lie-admissible because the attached brackets are twice the conventional Poisson brackets,  $(A, B) - (B, A) = 2(A, B)$ .

The need for the reformulation emerges from the fact that the brackets of the original Hamilton's equations with external terms violate the right scalar and distributive law and, as such, they do not characterize any algebra (see (see Vol. II of ref. [13] for details). Intriguingly, the classical brackets (2.4) are *not* Jordan-admissible, as one can verify. Only their operator counterparts (see Eq. (2.8) below) are Jordan-admissible.

These classical studies were systematically continued in monographs [13,14] via: the classical version of the Lie-admissible formulations with exponentiated group structure called *classical Lie-admissible group*

$$a' = ( e^{\omega^{\mu\alpha} \gamma^>_{\alpha}{}^\nu} a )_a, \quad (2.5)$$

admitting a non-Lie, Lie-admissible structure in the neighborhood of the identity; the Lie-admissible generalization of Lie's first, second and third theorems; the identification of the *exterior-admissible calculus*, as a generalization of the conventional exterior calculus; the introduction of the main lines of the *symplectic-admissible geometry* as the classical geometry underlying brackets (2.4); the derivation of Hamilton's equations with external term from the variational principle (despite their *variational nonselfadjointness* -NSA- [12])

$$\delta \hat{A}^> = \delta \int_{-\infty}^{+\infty} (p > dr - H > dt) = 0, \quad (2.6)$$

where  $\Phi_1^> = p > dr := p T_0^> dr$  is the *exterior-admissible one-form* characterized by a nonsymmetric matrix  $T_0^>$ ; the Hamilton-Jacobi equations for principle (2.6), etc.

To understand the significance of these studies it is sufficient to note that they imply a generalization of Noether's theorem in which the Lie-admissible symmetry characterizes *time-rate-of-variations of physical quantities*. The conventional Noether's theorem is then an evident particular case when the time-rate-of-variation is null.

On operator grounds, Santilli was the first to introduced back in 1978 [4]: the *general Lie-admissible and Jordan admissible algebras* with brackets

$$(A, B) = APB - BQA, \quad (2.7)$$

where P and Q are operators; the well known *Lie-admissible equations* in the infinitesimal form [4], p. 746 ( $\hbar=1$ ),

$$i \hat{A} = (A, H) = A P H - H Q A, \quad (2.8)$$

with corresponding finite form [4], p. 783,

$$A(t) = e^{iHQ t} A(0) e^{-i t P H}; \quad (2.9)$$

the *fundamental Lie-admissible commutation rules* [4], p. 746,

$$(a^\mu, a^\nu) = a^\mu P a^\nu - a^\nu Q a^\mu = i \omega^{\mu\alpha} \gamma_\alpha^\nu; \quad (2.10)$$

the first formulation of Lie-admissible operator algebras on bimodular Hilbert spaces; and other advances.

Subsequently, Fronteau, Tellez-Arenas and Santilli [15] were the first to identify in 1979 the Lie-admissible structure of the most general possible equations in statistical mechanics, those with an arbitrary collisions term C,

$$i \rho = (\rho, H) = \rho P H - H Q \rho = \rho H - H \rho + C. \quad (2.11)$$

The need for the Lie-admissible reformulation stems from the fact that the brackets  $\rho \times \eta = \rho H - H \rho + C$  violate the scalar and distributive laws and, therefore, do not characterize any algebra of any kind. This implies that familiar notions such as "a proton with spin  $\uparrow$ " which are well defined for brackets  $[\rho, H] = \rho H - H \rho$  have no mathematical or physical sense for brackets  $\rho \times H = \rho H - H \rho + C$ .

The generalized Schrödinger's counterpart of Lie-admissible equations (2.8) was identified by Myung and Santilli [16] in 1982 and, independently, Mignani [17] according to the expressions

$$i \frac{\partial}{\partial t} |\hat{\psi}\rangle = H Q |\hat{\psi}\rangle, \quad -i \langle \hat{\psi} | \frac{\partial}{\partial t} = \langle \hat{\psi} | P H. \quad (2.12)$$

The identification of the correct form of the linear momentum operator required considerable additional studies at the *classical* level [13,14], which eventually permitted Santilli [18] to reach the axiomatically correct form

$$P_k Q |\hat{\psi}\rangle = -i (Q^{-1})_k^j \nabla_j |\hat{\psi}\rangle, \quad \langle \hat{\psi} | P_k = i \langle \hat{\psi} | \nabla_k (P^{-1})_k^j. \quad (2.13)$$

achieved via the prior identification of the Hamilton-Jacobi equations for principle (2.6). The above classical and operator formulations were then interconnected by a unique map called *isoquantization*, first identified by Animalu and Santilli (see ref.s [2]). The simplest possible case, called *naive isoquantization*, maps the Hamilton-Jacobi equations for principle (2.6) into Eq.s (2.12) via the rule

$$\hat{A} \hat{\psi} \Rightarrow -i \hat{\psi} \hat{A} \text{Log } \psi, \quad (2.14)$$

where  $\hat{\psi} = Q^{-1}$  for the envelope acting to the right, with corresponding conjugate quantities for the envelope acting to the left.

Note for subsequent needs the primary role of the universal enveloping associative algebras in the above Lie-admissible formulations, exactly as it is the case for the conventional Lie formulations [5].

Additional biographical data worth an indication are the following. The first deformation of the  $(\lambda, \mu)$ -mutation of SU(2) spin was presented by Santilli at the Clausthal Conference on *Differential Geometric Methods in Physics* of 1980 [19]. The first generalizations of the rotational and Lorentz symmetries for operators  $P = Q$  was reached in [20,21]. The first identification of the underlying generalizations of symplectic, affine and Riemannian geometries was done in [22]; the first Q-operator generalization of gauge theories was reached by Gasperini [23] in 1983; the first studies of the Lie-admissible generalization of creation and annihilation operators were conducted by Jannussis *et al.* [24] beginning from 1981; Mignani [25] initiated the construction of a Lie-admissible scattering theory, subsequently completed by Santilli [2,18] via the use of special P-Q-functions; Okubo [26] identified certain "no go" theorems for operator formulations with nonassociative envelopes; Kalnay and Santilli [27] discovered the operator form of Nambu's mechanics for triplets with an essential Lie-admissible structure; Animalu [28] was the first to apply the methods to electron pairing in superconductivity; Kadelsvili [29] initiated the systematic study of special functions, distributions and transforms compatible with Lie-admissible structures; additional studies were conducted by Nishioka [30], Aringazin [31], Lopez [32], and others.

A comprehensive presentation of all these operator studies is now available in the three volumes on HM [2] (see also ref. [33] for a recent review), which is based on the main classification of HM into:

**Lie-admissible formulations**, applicable when the energy is not conserved, i.e., from Eq.s (2.8),  $i H = (H, H) = H(P - Q)H \neq 0$ ; and the simpler

**Lie-isotopic formulations**, applicable when the energy is conserved, which occur when in Eq.s (2.8)  $P = Q$ ,  $i \hat{A} = [A, \hat{H}] = A Q H - H Q A$ , in which case the algebra is still Lie, although of a more general type.

Equivalently, the two branches can be identified via their underlying methods, which were called in ref. [12]:

**Isotopies**, when the original axioms are preserved, as it is the case for the Lie-isotopic branch of HM; and

**Genotopies**, which apply when the original axioms are replaced by covering axioms, as it is the case for the Lie-admissible branch.

As marginal comments, we should note that the scripture  $A * B = q A B$  is

correct but only when  $q$  is in the center of the algebra. In fact, the "product"  $A*B = qAB$  for  $q$  a fixed operator violates the left distributive and scalar laws and, as such, it does not characterize any algebra of any type. This is the reason why Santilli's writes the deformation in the form  $A*B = AqB$  which now verifies the left and right scalar and distributive laws for arbitrary operator realizations of  $q$ . Similarly, the correct form of writing deformation (1.2) for arbitrary  $q$  is  $rp - pqr$  because the form  $rp - qpr$  for  $q$  a fixed operator does not characterize any algebra of any kind [2,8,12].

### 3: SANTILLI'S AXIOMATIZATION OF $q$ -DEFORMATIONS

We now present a dual axiomatization of  $q$ -deformations worked out by Santilli [2,12,18] which avoids the problematic aspects of Sect. 1. The first is of Lie-isotopic type, and the second is of the more general Lie-admissible type. The former is sufficient for  $q$ -deformations of Type I and III, while those of Type II demand the full Lie-admissible treatment.

The emerging axiomatization is naturally applicable for operator  $Q$  with an arbitrary, nonlinear, nonlocal and noncanonical dependence  $Q = Q(t, r, \dot{r}, \psi, \partial\psi, \partial\partial\psi, \dots)$ . Within this context, QM emerges as describing the *exterior particle problem*, that is, motion of point-like particles in vacuum, while HM applies for the *interior particle problems*, that is, extended-deformable particles moving within hyperdense physical media, thus resulting in the most general known equations of motions with an arbitrary nonlinearity and nonlocality (in  $x, \dot{x}, \partial\psi, \dots$ ). Also, the operator  $Q$  is restricted, by construction, to recover the identity when motion returns to be in vacuum. In this way, HM is a *covering* of QM.

Before entering in the field, the reader should be aware of its dimension. HM is first divided into the the Lie-isotopic and Lie-admissible branches, and then each of them is classified into *Kadetsvill five classes*: I (when the isotopic elements are sufficiently smooth, bounded, nowhere singular, Hermitean and positive-definite), II (when the isotopic elements are the same as in I but negative-definite), III (the union of I and II), IV (when the isotopic elements are degenerate), and V (when the isotopic elements are arbitrary, i.e., discrete structures, distributions, lattices, etc.) [29]. For the Lie-isotopic cases these characterizations refer to the operator  $Q$ , while for the Lie-admissible case they refer to the maximal Hermitean part of  $P$  and  $Q$ . In this note we can only consider for brevity HM formulations of Class I (see ref.s [2] for the other classes).

We shall now first study the isotopic axiomatization, which can be summarized via the following basic points.

1) Recall that Lie algebras  $L$  with product  $[A, B] = AB - BA$  over  $F$  are the antisymmetric algebras  $[\xi(L)]$  attached to the universal enveloping algebra  $\xi(L)$  with conventional associative product  $AB$  [5]. The first point is to focus the attention of the deformations of  $\xi$ , and construct the brackets of the time evolution only thereafter.

2) Consider the  $q$ -deformations  $\xi_q$  of  $\xi$  characterized by

$$\xi : AB \Rightarrow \xi_q : qAB, \quad (3.1)$$

Santilli's fundamental point is that *any deformation of the conventional product  $AB$  necessarily requires a corresponding generalization of the basic (multiplicative) unit*. In fact, it is "anathema" in number theory to change the product and keep the old unit, or viceversa, because units and products are deeply inter-related. Recall that the basic (left and right) unit of  $\xi$  is the trivial unit matrix  $I$ ,  $IA = AI = A, \forall A \in \xi$ .

The fundamental assumption is the interpretation of deformations (3.1) as a redefinition of the basic associative product in term of the  $Q$ -operator (of Class I) called *isotopic element* [4,12]

$$\xi_Q : A * B := A Q B, \quad Q = \text{fixed} \quad (3.2)$$

We then have the consequential generalization of the unit  $I$  into the form  $\lambda = Q^{-1}$  called *isounit*, which is such to be the correct left and right unit of the  $Q$ -theory

$$\lambda = Q^{-1}, \quad \lambda * A = A * \lambda = A, \quad \forall A \in \xi_Q. \quad (3.3)$$

Santilli identified other isotopies of associative algebras, such as the form  $AB \Rightarrow A*B = WAWBW$  with  $W$  idempotent,  $W^2 = W$ , which preserves associativity. The latter isotopies were however rejected for the construction of physical theories because they do not admit a unit. This is further illustration of the emphasis throughout Santilli's studied on the preservation of the basic unit.

3) The generalization of the multiplication and related unit requires, for mathematical consistency, a generalization of the notion of "numbers". Recall that a field  $F(\alpha, +, *)$  is a set of elements  $\alpha, \beta, \gamma$ , equipped with two operations and related units, the (associative and commutative) sum  $+$  with *additive unit*  $0, \alpha + 0 = 0 + \alpha = \alpha$ , and the (associative but not necessarily commutative) multiplication  $\times, \alpha \times \beta = \alpha\beta$ , with *multiplicative unit*  $1, 1 \times \alpha = \alpha \times 1 = \alpha$ , which is closed under sum, multiplication and their combinations (left and right distributive laws). At the 1980 Clausthal Conference on *Differential Geometric Methods in Physics*, Santilli [22] introduced the isotopies

$$F(\alpha, +, \times) \Rightarrow F_Q(\hat{\alpha}, +, *) , \quad \times \Rightarrow * := \times Q \times, \quad \alpha \Rightarrow \hat{\alpha} := \alpha \lambda, \quad 1 \Rightarrow \hat{1} := Q^{-1}, \quad (3.4)$$

characterizing *isofields*. In particular, for  $Q = q \in F$  the lifting  $\alpha \Rightarrow \hat{\alpha} = \alpha \lambda$  is unnecessary because the set  $F_q(\alpha, +, *)$  is a field (see Propositions 1.2.3.1 and 1.2.3.2 of ref. [2]). However, the generalization of numbers  $\alpha \Rightarrow \hat{\alpha} = \alpha \lambda$  is needed whenever  $Q$  is not an element of the original field  $F$ , as a necessary condition for isotopies, i.e.,

for  $\mathbb{F}$  to preserve all the original axioms of  $\mathbb{F}$  [22]. It is evident that this third step requires a suitable isotopic generalization of all operations on numbers, e.g.,  $\hat{a}^n = \hat{a} * \hat{a} * \dots * \hat{a} = \alpha^n$  (n times),  $\hat{a} \hat{b} = a \hat{b}$ ,  $\hat{1} = 1$ ;  $\hat{a}^{-1} = a^{-1} \hat{1}$  (see [22]).

4) Recall that conventional carrier spaces are defined over conventional fields. The generalization of multiplication, unit and fields evidently requires, also for mathematical consistency, a compatible generalization of conventional carrier spaces, introduced for the first time by Santilli [21] in 1983. Let  $S(x, g, \mathbb{R})$  be a metric or pseudo-metric space with local coordinates  $x$  and (Hermitian, nowhere singular) metric  $g$  over the reals  $\mathbb{R}$ . The isotopies necessary under  $Q$ -deformations are

$$S(x, g, \mathbb{R}) : x^2 = x^t g x \in \mathbb{R} \Rightarrow S(x, \hat{g}, \hat{\mathbb{R}}) : x^2 = (x^t \hat{g} x) \hat{1} \in \hat{\mathbb{R}}, \hat{g} = Qg, \hat{1} = Q^{-1}. \quad (3.5)$$

Isospaces  $S(x, \hat{g}, \hat{\mathbb{R}})$  characterize fundamentally novel geometries called *isoeuclidean*, *isominkowskian* and *isoriemannian*, with intriguing mathematical and physical implications, such as the isotopic generalization of conventional angles, the geometric unification of spheres, ellipsoids and hyperboloids, etc. [2, 14, 20-22].

Note that the original geometries are local-differential while Santilli's isogeometries are nonlocal-integral, as well as nonlinear in the velocities and the derivatives of the wavefunction, as needed for interior dynamical problem. This is due to the arbitrary functional dependence of the isometric  $\hat{g}(t, x, \dot{x}, \ddot{x}, \psi, \partial\psi, \dots)$ .

A most intriguing property of the isogeometries is that they deform any given structure. However, this deformation is seen only in the projection to the original space because in isospace the original structure is preserved in its entirety. Thus, isotopies deform straight lines, circles and cones into geometric structures called *isostraight lines*, *isocircles* and *isocones*, which are perfect straight lines, circles and cones, respectively, in isospaces, but are deformed when projected in our space.

This remarkable occurrence is due to the joint lifting of metric  $g \rightarrow \hat{g} = Qg$  and of the unit in the amount which is the inverse of the deformation of the metric,  $1 \Rightarrow \hat{1} = Q^{-1}$  and is at the foundation of the resolution of problematic aspect  $E$  (loss of Einsteinian axioms for conventional deformations). In fact, the preservation of the perfect light cone under deformations evidently permits the preservation of the basic axioms of the special relativity (see ref.s [2] for brevity).

As an example, the perfect sphere in Euclidean space  $E(r, \delta, \mathbb{R})$  represented by the metric  $g = \delta = \text{diag.} (1, 1, 1)$  can be deformed into the ellipsoids  $\hat{g} = \hat{\delta} = Q\delta = \text{diag.} (b_1^2, b_2^2, b_3^2)$ ,  $b_k \neq 0$ . However, in isospace the original sphere remains perfectly spherical because of the joint lifting of the unit  $1 \Rightarrow \hat{1} = \text{diag.} (b_1^{-2}, b_2^{-2}, b_3^{-2})$ , that is, for each semiaxis we have the lifting  $1 \Rightarrow b_k^{-2}$  which is compensated by the opposite lifting of the relative unit  $1 \Rightarrow b_k^{-2}$ . One of the intriguing

consequences is that the conventional rotational symmetry  $O(3)$  is evidently lost for the ellipsoidal deformations of the sphere, but it is reconstructed as an exact symmetry under Santilli's joint lifting  $\delta \Rightarrow \hat{\delta} = Q\delta$  and  $1 \Rightarrow \hat{1} = Q^{-1}$ .

These novel geometric properties have predictable fundamental implications, such as the reconstruction of the exact *light cone* in vacuum for electromagnetic waves propagating within physical media with a locally varying speed [2].

The reader can begin to see the horizon of novel applications which is available to  $q$ -deformations, but only when lifted into an axiomatic  $Q$ -operator/matrix form. In fact, for  $q$ -number there is no meaningful deformation of Minkowski or Riemann, trivially, because in this latter case  $x^2 = (x^t q g x) q^{-1} = (x^t g x) q q^{-1} = x^t g x \equiv x^2$ .

4) The liftings of multiplication, unit, fields and carrier spaces require a compatible lifting of the transformation theory into the so-called *isotransformations*

$$x' = U x \Rightarrow x' = \hat{U} * x = H Q x, \quad Q = \text{fixed}. \quad (3.6)$$

first introduced in ref. [12] of 1978. Note that the preservation of the old transformation  $x' = Ux$  under isotopies implies the loss of linearity, transitivity, superposition principle, etc.

Note also that isotransforms are nonlinear, nonlocal and noncanonical only when projected in the original space, while they verify the axioms of linearity, locality and canonicity at the isotopic level. For this reason they are called *isolinear*, *isolocal* and *isocanonical*.

This is another property of isotopic methods permitting further applications of  $Q$ -deformations via the turning of given nonlinear-integral theories into *identical* isolinear and isolocal forms, thus being manifestly more manageable.

5) The generalization of the multiplication, unit, field, carrier spaces and transformation theory then requires a step-by-step generalization of the entire Lie theory into a form originally submitted as *Lie-isotopic theory* [12] and today known as the *Lie-Santilli theory* (see papers [23-33] and monographs [34]). We are here referring to the isotopies of all structural parts of Lie's theory, such as enveloping algebras, Lie algebras, Lie groups, representation theory, symmetries and first integrals, etc.

The fundamental isotopies, those of enveloping associative algebras, were the central topic of the original proposal [12]. Most important is the first achievement of the isotopies of the Poincaré-Birkhoff-Witt theorem on the infinite-dimensional basis of  $\xi$ , which provides the new basis of  $\xi_Q$  and the correct exponentiation under isotopies, called *isoexponentiation*

$$e_{\xi_Q}^{X \cdot \hat{w}} = 1 + (\hat{w} * X) / 1! + (\hat{w} * X) * (\hat{w} * X) / 2! + \dots = (e^{X \cdot \hat{w}}) 1. \quad (3.7)$$

Particularly important is the *uniqueness* of the above isoexponentiation (up to isoequivalence transformations studied below), which should be compared to the various types of q-exponentiation in the literature [1].

Yet another horizon of applications for q-deformations emerge from isoexponentiation (3.7), such as the isotopic lifting of Dirac's  $\delta(x)$  to spread its singularity at  $x = 0$  over a finite region of space, thus removing the singularities afflicting conventional theories from the beginning [2].

6) The above isotopies imply corresponding lifting of Lie algebras into the *Lie-Santilli algebras* [12,34]

$$[X_i, X_j] = X_i X_j - X_j X_i = C_{ij}^k X_k \Rightarrow [X_i, \hat{X}_j] = X_i Q X_j - X_j Q X_i = \hat{C}_{ij}^k X_k, \quad (3.8)$$

where the  $\hat{C}$ 's are called *structure isofunctions*, and depend on all needed local variables and their derivatives. Note the preservation of the Lie axioms by the isotopic product  $AQB - BQA$  (and *not* by  $QAB - QBA$ ).

The existence of a unique infinite-dimensional basis for the isoexponentiations then permits the identification of the (connected) *Lie-Santilli groups*

$$x' = \hat{U}(\hat{w}) * x = (e_{\xi_Q}^{X \cdot \hat{w}}) * x = (e^{X \cdot Q w}) x, \quad (3.9a)$$

$$\hat{U}(0) = \hat{U}(\hat{w}) * (\hat{U} - \hat{w}) = 1 = Q^{-1}, \quad \hat{U}(\hat{w}) * \hat{U}(\hat{w}') = \hat{U}(\hat{w}) * \hat{U}(\hat{w}') = \hat{U}(\hat{w} + \hat{w}'), \quad (3.9b)$$

$$(e_{\xi_Q}^{X_1}) * (e_{\xi_Q}^{X_2}) = e_{\xi_Q}^{X_3}, \quad X_3 = X_1 + X_2 + [X_1, \hat{X}_2] + \dots \quad (3.9c)$$

In turn, the above liftings imply the isotopies of the representation theory, symmetries and first integrals, etc. Note the nontriviality of the isotopies, as transparently exhibited by the appearance of an *unrestricted, nonlinear, integro-differential operator Q in the exponent of the group structure (3.9a)*. In fact, the isotopic image of the conventional linear-local-canonical rotations, Lorentz and Poincaré transformations are given by highly nonlinear-nonlocal-noncanonical generalizations.

The remarkable property of Santilli's isotopies is that, despite these differences, the isotopic groups are isomorphic to the original groups for all positive-definite Q,  $\hat{O}_Q(3) \sim O(3)$ ,  $\hat{O}_Q(3,1) \sim O(3,1)$ ,  $\hat{P}_Q(3,1) \sim P(3,1)$ ,  $\hat{S}\hat{U}_Q(2) \sim SU(2)$ ,  $\hat{S}\hat{O}_Q(3) \sim SU(3)$ , etc.

In fact, the isotopies are introduced as methods for the reconstruction of exact space-time and internal symmetries when *believed* to be broken. One should expect this property from the preservation of the geodesics of the original symmetry in isospace mentioned earlier. In fact, the *isorotational group*  $\hat{O}_Q(3)$  was introduced [20] to show that the *rotational symmetry remains exact* for all

the *ellipsoidal deformations* of the sphere  $\delta = \text{diag. } (b_1^2, b_2^2, b_3^2)$ ,  $b_k \neq 0$ . Similarly, the Lorentz and Poincaré symmetries remain *exact* for all signature preserving *nonlinear-nonlocal-noncanonical deformations* of the Minkowski metric  $\hat{\eta} = Q\eta$  [21], etc.

Intriguingly, when the Q-element depends only on the local coordinates,  $Q = Q(x)$ , the *isopoincaré symmetry*  $\hat{P}_Q(3.1)$  provides the *universal invariance of all possible conventional Riemannian metrics*  $g(x) = Q(x)\eta$ .

Further physical applications of the Q-deformations then emerge in conventional gravitation, such as the characterization of the gravitational horizon as the zeros of Q, and the gravitational singularities as the zeros of the isounit  $\hat{1}$  (see [2] for detail).

7) The preceding isotopies further imply a step-by-step generalization of functional analysis into a new discipline called *functional isoanalysis* [29], in which all conventional operations (say, log, derivative, integral, etc.), distributions (Dirac's delta, etc.), transforms (Fourier, Laplace and other transforms), special polynomials (Legendre polynomials, spherical harmonics, etc.), weak and strong continuity, etc. are generalized into a *unique* form compatible with the basic isounit  $\hat{1} = Q^{-1}$  which is applicable at all times. See ref.s [2] for a comprehensive presentation with applications.

8) The above chain of interconnected isotopies can indeed be formulated on a *conventional* Hilbert space  $\mathcal{H}$ , as done in the original proposal [4]. However, this implies the general loss of Hermiticity because isohermiticity is now defined by

$$H \hat{1} = Q H \hat{1} Q^{-1}. \quad (3.10)$$

**Lemma 3.1:** *An operator  $H \in \xi_Q$  which is originally Hermitean under q-number-deformations at time  $t = 0$ , over a conventional Hilbert space  $\mathcal{H}$ , becomes generally nonhermitean over the same space  $\mathcal{H}$  under nonunitary time evolutions leading to a Q-operator-deformation, unless Q and H commute.*

For this reason, Myung and Santilli [18] introduced in 1982 the *isohilbert space*  $\mathcal{H}_Q$  characterized by the lifting

$$\mathcal{H}: \langle \psi | \phi \rangle = \int d^3r \psi^\dagger(r) \phi(r) \in \mathbb{C} \Rightarrow \mathcal{H}_Q: \langle \psi | \phi \rangle = \int d^3r \psi^\dagger(r) Q(r, \dots) \phi(r) \in \mathbb{C} \quad (3.11)$$

in which case isohermiticity coincides with Hermiticity. *This is a first manifestly fundamental property of Santilli's axiomatization of Q-deformations because it permits the preservation of observability under arbitrary time evolutions* (the issue whether the observable is conserved or not is a separate one treated below). Note that for Q positive-definite the composition is still inner and  $\mathcal{H}_Q$  is still Hilbert. Note also that for Q independent of the integration variables (or constant),  $\mathcal{H}_Q \equiv \mathcal{H}$  because in this case