ALGEBRAS, GROUPS, AND GEOMETRIES 9, 319-342 (1992)

ELEMENTS OF FOURIER–SANTILLI ISOTRANSFORMS

J. V. Kadeisvili*
International Center of Physics
Institute of Nuclear Physics
Alma-Ata 480082, Kazakhstan

Received December 5, 1992; revised February 25, 1993

Abstract

In this second paper we study the axiom-preserving, nonlinear, nonlocal and noncanonical isotopies of the Fourier transforms, here called *Fourier-Santilli isotransforms*, which were introduced by R. M. Santilli in a little known physical publication of 1989, as part of his isotopies of contemporary mathematical structures recently outlined in this Journal. The mathematical relevance of the isotransforms is pointed out by showing that they are based on the new branch of functional analysis, called *functional isoanalysis*, as presented in the preceding paper, including its classification into ten mathematically and physically distinct classes. The Fourier-Santilli isotransforms are studied in only four of these classes for brevity. The physical relevance is shown by proving that, when applied to a Gauss distribution, the isotransforms imply a generalization of Heisenberg's uncertainties for particles in vacuum $\Delta x\ \Delta k \approx 1$, into the isotopic form proposed by Santilli for particles within physical media $\Delta x\ \Delta k \approx \hat{1}$, where the isounit $\hat{1}$ geometrizes the inhomogenuity and anisotropy of physical media. We finally show that the functional isoanalysis of the preceding paper and the Fourier-Santilli isotransforms of this paper confirm the isotopic completion of quantum mechanics essentially along the historical argument by Einstein, Podolsky, Rosen and others.

*) Address for 1992–1993: The Institute for Basic Research, P. O. Box 1577, Palm Harbor, FL 34682 USA

Introduction. The theoretical physicist R. M. Santilli, when at the Department of Mathematics of Harvard University, discovered [1,2] certain isotopies of contemporary mathematical structures, including units, fields, vector spaces, transformation theory, algebras, groups, geometries, etc., recently outlined for this Journal in memoirs [3,4].

In the preceding paper [5] (hereinafter tacitly assumed as an integral part of this paper and referred to as I), the author pointed out that these isotopic structures imply a genuinely new branch of functional analysis, submitted under the name of *functional isoanalysis*, and classified into ten mathematically and physically significant classes.

In this paper we continue the study of functional isoanalysis, with particular reference to the isotopies of the Dirac delta function, the Fourier series and the Fourier transforms, which were introduced in the little known physical memoir [6] and were not reviewed in ref.s [3,4].

As we shall see, the same isotopic techniques are applicable for the generalization of other special functions (such as gamma, beta, Legendre and other functions), as well as of other transforms (such as Laplace, Hankel, Mellin and other transforms), which are not studied in this paper for brevity. This indicates that the covering functional isoanalysis includes isotopic generalizations not only of its main structural foundations, as outlined in Sect. I.3, but also of all conventional special functions and transforms, as emerging from the studies of this paper.

Our primary interest is to identify the structure of the isotopic delta functions, isotopic Fourier series and isotopic Fourier transforms, and point out their novel mathematical and physical significance. Full technical languages and treatments in the necessary details are deferred to interested mathematicians.

The primary result of this paper is that the application of the isotopic Fourier transforms to Gauss distribution provides an independent confirmation of the generalization of Heisenberg's uncertainties for particles of the exterior problem $\Delta x\ \Delta k \approx 1$, into the isotopic form identified by Santilli [7] for particles of the interior problem $\Delta x\ \Delta k \approx \hat{1}$, where the isounit $\hat{1}$ geometrizes the inhomogenuity and anisotropy of the physical medium considered.

Moreover, the isouncertainties approach asymptotically the classical determinism at the limit of gravitational collapse into a singularity [10]. Thus, functional isoanalysis at large, and isotopic uncertainties in particular, confirm the existence of the isotopic completion of quantum mechanics [6-11] much along the historical argument of Einstein, Podolsky, Rosen [12] and others.

2: Dirac-Santilli isodelta functions. As well known (see, e.g., ref. [13] and quoted bibliography), the conventional *Dirac delta function* is not a function, but a distribution representing a rather delicate limit procedure in a conventional functional space, such as the Hilbert space $\mathcal{H}$, with a mathematically well defined meaning only when it appears under an integral.

When the singularity is at the point x = 0, the δ-function can be defined in terms of a well behaved function f(x) on a one-dimensional space $S(x,\Re)$ over the reals $\Re$ by [loc. cit.]

$$\int_{-\infty}^{+\infty} f(x)\, \delta(x)\, dx = f(0), \qquad \int_{-\infty}^{+\infty} \delta(x)\, dx = 1 \tag{2.1}$$

This essentially means that δ(x) = 0 everywhere except at x = 0 where it is singular. Nevertheless, what is mathematically and physically significant is the behaviour near that point, which permits explicit realization, such as the familiar integral form

$$\delta(x) = (1/2\pi) \int_{-\infty}^{+\infty} e^{ixy}\, dy . \tag{2.2}$$

If the singularity is at a point x ≠ 0, then we can write [loc. cit.]

$$f(x) = \int_{-\infty}^{+\infty} f(x')\, \delta(x' - x)\, dx' . \tag{2.3}$$

Finally, the δ-function verifies the basic properties

$$\delta(x) = \delta(-x) \quad , \quad \delta(x - x') = \int_{-\infty}^{+\infty} dz\, \delta(x - z)\, \delta(z - x') . \tag{2.4}$$

The delta function is evidently inapplicable when dealing with functional isospaces, such as the isohilbert spaces $\hat{\mathcal{H}}$ (Sect. I.3). In particular, exponentials of the type appearing in the integrand of Eq. (2.2) are no longer defined in isospaces, and must be replaced by the isoexponentials of type (I.2.29).

These occurrences, known to Santilli since the late 70's, rendered mandatory the studied of the isotopies of the delta function. The origin of these isotopies can therefore be traced back to the isotopies of the Poincaré-Birkhoff-Witt theorem of ref. [1] of 1978 (see Theorem 2.1, p. 295 of paper I) which play a fundamental role in their very definition; they

were studied in a number of contributions quoted in paper I; they received a first formal treatment in ref. [14] of 1982; they were subjected to a systematic study in ref. [6] of 1989; and then used for a number of physical applications in the recent ref.s [8,9].

In particular, six mathematically and physically distinguishable isotopies of the Dirac delta function were identified in ref. [6], and are now called *Dirac-Santilli isodelta functions* (see, e.g., ref. [11]). Their outline is recommendable as an application of functional isoanalysis, and as a pre-requisite for the isotopies of the Fourier transforms studied in the subsequent sections.

Consider a one-dimensional isospace of Class IA (Sect. I.2C), denoted $\hat{S}_{IA}(x,\hat{\Re})$ with (conventional) real coordinates x over the isofield of real numbers $\hat{\Re}(n,+,*)$ with conventional elements n and sum +, but isotopic multiplication $n_1 * n_2 := n_1 T n_2$, where T is the isotopic element and $\hat{1} = T^{-1}$ is the multiplicative isounit of Class I.

Let f(x) be an ordinary function defined on $\hat{S}_{IA}(x,\hat{\Re})$ which verifies the conditions of strong isocontinuity of Sect. I.3 in all possible subintervals of $[-\infty,+\infty]$. Recall that the isotopic element T of Class I is a strongly isocontinuous, bounded, real valued, and positive-definite function of the coordinate x as well as its derivatives with respect to an independent variable of arbitrary order and any other needed quantity, $T = T(x, \dot{x}, \ddot{x}, ...)$.

Then, the *Dirac-Santilli isodelta function of the first kind*, denoted δ_1, can be defined in terms of the expression

$$\int_{-\infty}^{+\infty} f(x) * \delta_1(x)\, dx = f(0), \tag{2.5}$$

from which we obtain for f = 1

$$\int_{-\infty}^{+\infty} T(x, \dot{x}, \ddot{x}, ...)\, \delta_1(x)\, dx = 1 . \tag{2.6}$$

The isotopic image of (2.3) is then given by

$$f(Tx) = \int_{-\infty}^{+\infty} f(Tx') * \delta_1(x' - x)\, dx' . \tag{2.7}$$

namely, it is not possible any longer to map the dependence on x to the dependence at x′, but rather the dependence on Tx to Tx′. This confirms the very peculiar nonlocality of the topology underlying Santilli's isotopies discussed in paper I.

In fact, the isotopic element T can have an integral dependence on the interval $x \in [a, b]$ centered at x. In this case the singularity of the Dirac δ at x can be spread over the interval [a, b] by the Dirac–Santilli δ_1-function for a suitable selection of T.

In several cases of physical interest, T can be assumed as having an explicit dependence only on the variables $\dot{x}, \ddot{x}, \ldots$, $T = T(\dot{x}, \ddot{x}, \ldots)$, with consequential identity $T dx \equiv d(Tx)$. In this case, the projection of the δ_1-function into the original functional space $S(x,\mathfrak{R})$ implies the equivalence

$$\delta_1(x) \approx \delta(Tx). \qquad (2.8)$$

It is easy to see that, under the above assumption of T being independent from x, the δ_1-function admits the integral representation [6]

$$\delta_1(x) = (1/2\pi) \int_{-\infty}^{+\infty} T\, e_\xi^{ixy}\, dy = (1/2\pi) \int_{-\infty}^{+\infty} e^{ixTy}\, dy \qquad (2.9)$$

(where we have used the fundamental Theorem I.2.1 on isoexponentiation), and verifies the properties

$$\delta_1(x) = \delta_1(-x) \quad , \quad \delta_1(x - x') = \int_{-\infty}^{+\infty} dz\, \delta_1(x - z) * \delta_1(z - x') . \qquad (2.10)$$

For the case of an isospace of Class IB, $\hat{S}_{IB}(x,\hat{\mathfrak{R}})$, with isofunctions $\hat{f}(x) = f(x)\,\hat{1}$, a different isotopic expression emerged in ref. [6], here called *Dirac-Santilli isodelta function of the second kind*, and denoted δ_2, which is characterized by the property

$$\int_{-\infty}^{+\infty} \hat{f}(x) * \delta_2(x)\, dx = \int_{-\infty}^{+\infty} f(x)\, \delta_2(x)\, dx = \hat{f}(0) = f(0)\,\hat{1}, \qquad (2.12)$$

In this case the δ_2-function must necessarily be an isofunction, i.e., admitting a structure of the type $\delta_2(x) = \tilde{\delta}_2(x)\, \hat{1}(x, \dot{x}, \ddot{x}, \ldots)$. Then, for $\hat{1} = \hat{1}$, we have

$$\int_{-\infty}^{+\infty} \delta_2(x)\, dx = \int_{-\infty}^{+\infty} \tilde{\delta}_2(x)\, \hat{1}(x, \dot{x}, \ldots)\, dx = \hat{1} \qquad (2.13)$$

and the isotopic image of (2.3) is given by

$$\hat{f}(x) = \int_{-\infty}^{+\infty} \hat{f}(x') * \delta_2(x' - x)\, dx' . \qquad (2.14)$$

One can see that the projection of the δ_2-function in the original functional space $S(x,\mathfrak{R})$ implies the equivalence (again for isounits independent of the integration variable)

$$\delta_2(x) \approx \delta(x)\, \hat{1}(\dot{x}, \ddot{x}, \ldots) . \qquad (2.15)$$

It is easy to see that, under the same assumptions, the δ_2-function admits the integral representation [6]

$$\delta_2(x) = 1/2\pi \int_{-\infty}^{+\infty} T\, e_\xi^{ixy}\, dy = \hat{1}/2\pi \int_{-\infty}^{+\infty} e_\xi^{ixTz}\, d(Tz) \qquad (2.16)$$

and verifies the properties

$$\delta_2(x) = \delta_2(-x) \quad , \quad \delta_2(x - x') = \int_{-\infty}^{+\infty} dz\, \delta_2(x - z) * \delta_2(z - x') . \qquad (2.17)$$

It is an intriguing exercise for the reader interested in learning the isotopic techniques to prove that *the first and second kind isodelta functions can be interconnected by the reciprocity transformation* $T \to \hat{1}$ (Sect. I.2c).

To present the *Dirac-Santilli function of the third kind*, let us recall that the separation on a generic, n-dimensional isospace $\hat{S}(x,\hat{g},\hat{\mathfrak{R}})$, $\hat{g} = Tg$, $\hat{\mathfrak{R}} \approx \mathfrak{R}\,\hat{1}$, $\hat{1} = T^{-1}$ (see Sect. I.2..C for details), can be formally written as that of a fictitious conventional space in the same dimension $S(\tilde{x},g,\mathfrak{R})$, according to the simple rule [6,9]

$$x^{\hat{2}} = x^t \hat{g}\, x \equiv \tilde{x}^t \tilde{x} = \tilde{x}^{\,2}, \quad \tilde{x} = T^{\frac{1}{2}} x . \qquad (2.18)$$

This implies that a number of problems in isospaces can be worked out in this fictitious conventional space in the $\tilde{x}$-variables, and the results then re-expressed in the x-variables.

The Dirac-Santilli δ_3-function emerged precisely from reduction of this type. It can be defined via the conditions [6]

$$\int_{-\infty}^{+\infty} f(\tilde{x})\, \delta_3(\tilde{x})\, d\tilde{x} = \int_{-\infty}^{+\infty} f(T^{\frac{1}{2}} x)\, \delta_3(T^{\frac{1}{2}} x)\, d(T^{\frac{1}{2}} x) = f(0), \quad T^{\frac{1}{2}} = T^{\frac{1}{2}}(x, \dot{x}, ...) \tag{2.19}$$

from which we obtain for f = 1

$$\int_{-\infty}^{+\infty} \delta_3(T^{\frac{1}{2}} x)\, d(T^{\frac{1}{2}} x) = 1 \ . \tag{2.20}$$

with realization in terms of the conventional δ-function

$$\delta_3(x) \approx \delta(\tilde{x}) = \delta(T^{\frac{1}{2}} x) \ . \tag{2.21}$$

It should be stressed that, while the isodelta functions of the first and second kind are bona-fide isotopies of the conventional expression, this is not the case for δ_3 which is merely a pragmatic tool for simplifying calculations, rather than a rigorous structure.

The above expressions have been presented for the case of one-dimensional coordinates x. The extension to three-dimensions is trivial, and given by isotopic products of the type

$$\delta_1(\mathbf{r}) = \delta_1(x) * \delta_1(y) * \delta_1(z) \ . \tag{2.22}$$

Consider now the isodual image of Class IIA, $\hat{S}_{IIA}{}^d(x^d, \hat{\mathfrak{R}}^d)$ of isospace $\hat{S}_{IA}(x, \hat{\mathfrak{R}})$ which is defined over the isodual isoreals $\hat{\mathfrak{R}}^d$, with isotopic element $T^d = -T$ and isounit $\hat{1}^d = -\hat{1}$. Santilli [6] also studied the isodelta functions on isodual isospaces, by reaching the following

PROPOSITION 2.1: The isodual isodelta functions of the first and second kind are isoselfdual in their structure, and only change their overall sign under isoduality.

In fact, by recalling that $x^d = -x$, $y^d = -y$, $i^d = -i$, we have the *isodual isodelta function of the first kind*

$$\delta^d_1(x^d) = (1/2\pi) \int_{-\infty}^{+\infty} T^d\, e_{\xi^d}{}^{i^d x^d y^d}\, dy^d = -(1/2\pi) \int_{-\infty}^{+\infty} T\, e_{\xi}{}^{ixy}\, dy$$

, (2.23)

with a similar expression for the second kind.

However, δ_3 has no isoselfdual structure, evidently because under isoduality $T \rightarrow T^{d\frac{1}{2}} = i\, T^{\frac{1}{2}}$, thus altering the structure of the original function. This confirms that δ_3 has a mere pragmatic value for practical calculations without an isotopic structure.

The properties of the Dirac-Santilli delta functions for all the remaining Classes III, IV and V is unexplored at this writing. Additional generalizations of the delta functions are expected from the extension of the isofield to include the isotopies of the addition [15].

Comment 2.1: While the Dirac delta function is unique, there exist infinitely possible Dirac-Santilli isodelta functions for each of the above six kinds, evidently because of the infinitely possible isounits or isotopic elements. The mathematician has certainly noted the intriguing character of the general case of isodelta functions (2.5) and (2.12) for T = T(x,...), which are hoped to receive an attention in the literature much needed for physical advances.

Comment 2.2: As well known, the locality of quantum mechanics is ultimately expressible via the Dirac delta function. The nonlocality of the isotopies of quantum mechanics [8] is then expressed by the Dirac-Santilli isodelta functions. In turn, such nonlocality is necessary for a quantitative treatment of the extended character of hadrons with consequential nonlocal components in the strong interactions due to mutual overlapping of the wavepackets and charge distributions of the particles.

Comment 2.3: While the Dirac delta is a *bona fide* distribution, the Dirac-Santilli isodelta functions are not necessarily so because the original singularity at x can be spread over an interval of which x is the center. Nevertheless, in specific cases, such as when T = cost., the isotopic δ-functions are distributions similar to δ(x).

Comment 2.4: Owing to the property of spreading out the δ-singularity over a finite region of space, the isodelta functions have important physical applications. In fact, they permit the isotopic completion of quantum mechanics, that is, its reformulation in terms of functional isoanalysis which is much along the historical argument by Einstein, Rosen and Podolsky [10] (see ref. [10,11] for brevity).

Comment 2.5: The topology of the isodelta functions is unknown at