Problematic aspects of Weinberg's nonlinear theory

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ABSTRACT. We point our a number of fundamental problematic aspects of a nonlinear generalization of quantum mechanics recently proposed by S. Weinberg, such as : the lack of a consistent, left and right unit of the underling operator algebra ; the lack of a consistent measurement theory ; the lack of Planck's quantum of energy ; the lack of quantization ; the lack of consistent Casimir's invariants ; the lack of well defined characteristics of particles ; the lack of well defined exponentiations into finite space-time symmetries ; the inequivalence of the Heisenberg-type with the Schrödinger's-type representation ; the absence of any essential remnant of nonlinearity at the abstract, realization-free level, and others caused by Okubo's no-go theorems on the inconsistencies of operator formulations with nonassociative envelopes. We finally indicate the existence of a generalization of quantum mechanics, known as "hadronic mechanics" and based on the so-called axiom-preserving isotopies of quantum mechanics, which : 1) avoids all problematic aspects of Weinberg's nonlinear theory because it is characterized by an associative envelope, although realiezd in its most general possible unity-preserving, isotopic form ; 2) includes all possible Weinberg's equations nonlinear in Ψ and Ψ^*, plus all possible additional equations which are nonlinear in $r, p, \dot{p}, \partial\Psi, \partial\Psi^*, \ldots$ as well as nonlocal-integral in all these quantities ; and 3) is a covering of quantum mechanics because it admits the latter as a particular case. A primary function of this paper is therefore that of stressing that the established knowledge on operator theories is currently restricted to those with an associative enveloping algebras, and that care should be exercised for all nonassociative extensions of the envelope.

1. Introduction.

As is well known (see, e.g., ref.s [1, 2]), the mathematical structure of quantum mechanics can be essentially reduced to that of the universal enveloping associative algebra $\mathbf{A}$ of operators $A, B, \ldots$ with trivial associative product AB and unit $I = Diag(1, 1, \ldots, 1)$

$$\mathbf{A} \quad \begin{cases} AB = Ass, \\ IA = AI = A, \forall A \epsilon \mathbf{A}, \end{cases} \tag{1.1 a et b}$$

on a Hilbert space $\mathcal{H}$ with conventional inner product

$$\mathcal{H} : \int \psi^* \psi dv, \tag{1.2}$$

over the field $\mathbf{C}$ of complex numbers. In fact, the brackets of the first fundamental representation of the theory, Heisenberg's representation, are characterized by the Lie algebras $\mathbf{A}^-$ attached to $\mathbf{A}$ according to the familiar equation

$$i\dot{A} = [A, H]_{\mathbf{A}} = AH - HA, \quad \hbar = 1 \tag{1.3}$$

Similarly, the modular-associative structure of the second fundamental representation of the theory, Schrödinger's representation, is centrally dependent on the conventional associative character of the original enveloping algebra, as expressed by the celebrated equation.

$$i\frac{\partial}{\partial t}\psi = H\psi \tag{1.4}$$

where the modular-associative character is expressed by the fact that the action $A\psi$ of an element $A \epsilon A$ on an element $\psi \epsilon \mathcal{H}$ is associative.

In a recent article [3], S. Weinberg has proposed a nonlinear generalization of quantum mechanics which is centered in the generalizatiion of the conventional associative envelope $\mathbf{A}$ into the form

$$\mathbf{U} : a \times b = \frac{\partial a}{\partial \psi_k} \quad \frac{\partial b}{\partial \psi_k^*} \tag{1.5}$$

where the elements $a, b, \ldots \epsilon \mathbf{U}$ are trilinear or higher-linear functions of the states ψ, ψ^*. The nonlinear of the theory evidently results from the non-linearity of the elements $a, b, \ldots$ in the states ψ, ψ^*. The generalization of Heisenberg's law [1.3] proposed by Weinberg is characterized by the brackets of the algebra $\mathbf{U}-$ attached to $\mathbf{U}$ to the forms [3]

$$i\dot{a} = [a, h]_{\mathbf{U}} = \frac{\partial a}{\partial \psi_k} \quad \frac{\partial h}{\partial \psi_k^*} - \frac{\partial h}{\partial \psi_k} \quad \frac{\partial a}{\partial \psi_k^*} \tag{1.6}$$

where the Hamiltonian is given by

$$h = \frac{1}{2m} \int \psi_k^*(\vec{x}) \Delta \psi_k(\vec{x}) d\vec{x} + h' \tag{1.7}$$

and the expression

$$h' = \int H[\psi(\vec{x}), \psi^*(\vec{x})] d\vec{x} \tag{1.8}$$

holds under certain restrictions for the functional H [3].

The generalization of Schrödinger's equation [1.4] proposed by Weinberg is given by [3]

$$i\frac{\partial}{\partial t}\psi_k = -\frac{1}{2m}\Delta\psi_k + \frac{\partial H}{\partial \psi_k^*} \tag{1.9}$$

The remaining parts of paper [3] are devoted to a generalization of various aspects of quantum mechanics, such as eigenvalues, expectation values, and others topics. An illustration of Eq. [1.9] is provided by the equation

$$i\frac{\partial}{\partial t}\psi = [-\frac{1}{2m}\Delta + V(\vec{x})]\psi + V_{RR}(\psi^*\psi)\psi \tag{1.10}$$

which was originally suggested by Fermi [4].

Weinberg's proposal [3] was rapidly considered by several authors [5-12]. From an experimental viewpoint, tests on hyperfine transitions of Be [8] and Ne [9] as well as on hydrogen maser transitions [10] have established that possible nonlinear corrections to conventional, linear, atomic, transitions are very small, with an upper limit of the order of 10^27. Previous experiments looking for possible nonlinear corrections to conventional quantum mechanics, based on neutron interferometry, have also set very stringent limits on the contributions from such nonlinearity [13, 16]. We can therefore conclude that current experiments do not support nonlinear contributions for the conditions considered.

In this paper we shall point out certain fundamental problematic aspects of Weinberg's nonlinear theory which appear to originate from the selection of the particular form [1.5] of the enveloping operator algebra. In the concluding remarks we shall briefly touch the problem of a nonlinear generalization of quantum mechanics capable of resolving these problematic aspects.

2. Fundamental problematic aspects of Weinberg's nonlinear theory.

It is our duty to report that, to the best of our knowledge, Weinberg's nonlinear generalization of quantum mechanics [3] via structures [1.5], [1.6] and [1.9] is afflicted by fundamental problematic aspects, some of which are reviewed below.

Problematic aspect 1: Weinberg's nonassociative generalization U of the conventional associative enveloping algebra of quantum mechanics does generally not admit a consistent unit, except for the trivial case in which U is one-dimensional with the sole element given by the unit itself.

Let U be an (abstract) algebra with elements $a, b, c, \ldots$ and product ab over a field F (hereinafter assumed of characteristics zero). The *unit* of U, when it exists, is the element 1 of the center of U which verifies the left and right identities

$$1a = a1 \equiv a \tag{2.1}$$

for *all* elements $a \epsilon U$. The conventional associative envelope of quantum mechancis, Eq. [1.1], does admit a consistent, left and right unit, which is usually given by the trivial unit matrix $I = diag.(1, 1, \ldots, 1)$.

For the case of Weinberg's envelope U, Eq. [1.5], conditions [2.1] require the existence of an element $1(\psi, \psi^*) \epsilon U$ such that

$$\frac{\partial^1}{\partial\psi_k}\frac{\partial a}{\partial\psi_k^*} = \frac{\partial a}{\partial\psi_k}\frac{\partial^1}{\partial\psi_k^*} \equiv a \tag{2.2}$$

for all elements $a \epsilon U$. These conditions are evidently not possible for algebra [1.5] in the necessary generality. The lack of existence of the unit is, in reality, a direct consequence of the nature of the algebra U which was not identified in Weinberg's original paper [3], nor in any of the subsequent investigations [5-12]. It can be readily seen that Weinberg's

product ab as per Eq. [1.b) characterizes a *general nonassociative Lie-admissible algebra* [1.18] in the terminology of ref. [18]. In fact, the algebra U is *nonassociative*, trivially, because it violates the associativity law

$$(a \times b) \times c = \frac{\partial}{\partial\psi_k}(\frac{\partial a}{\partial\psi_k} \quad \frac{\partial b}{\partial\psi_k^*})\frac{\partial c}{\partial\psi_k^*} \neq a \times (b \times c) = \frac{\partial a}{\partial\psi_k} \quad \frac{\partial}{\partial\psi_k^*}(\frac{\partial b}{\partial\psi_k}\frac{\partial c}{\partial\psi_k^*}) \tag{2.3}$$

Second, the algebra U is *Lie-admissible* in the sense originally identified by Albert [17], that is, the attached algebra U^- is Lie. In fact, the attached antisymmetric product

$$[a, b]_U = \frac{\partial a}{\partial\psi_k} \quad \frac{\partial b}{\partial\psi_k^*} - \frac{\partial b}{\partial\psi_k} \quad \frac{\partial a}{\partial\psi_k^*} \tag{2.4}$$

verifies the Lie algebra axioms

$$[a, b]_U + [b, a]_U = 0 \tag{2.5a}$$

$$[[a, b]_U, c]_U + [[b, c]_U, a]_U + [[c, a]_U, b]_U = 0 \tag{2.5b}$$

as the reader is encouraged to verify.

Finally, the algebra U is a nonassociative Lie-admissible algebra of the *general* type, in the sense of ref. [18] of not verifying the *flexibility law*

$$(a \times b) \times a = a \times (b \times a), \forall A \quad \epsilon \mathbf{U} \tag{2.6}$$

The point is that, as well known in the theory of abstract algebras, general nonassociative Lie-admissible algebras do not generally admit the unit [18]. An intriguing exception is the case when the algebra U is one-dimensional. In fact, one can assume the element $I = \psi_k^*\psi_k$, in which case the identities $Ixa = axI \equiv$ a hold for all elements a given by the scalar products of $I, a = kI, k\epsilon I$ (but excluding the functions $a(I)$ of I, as required by the last identity). An inspection then easily reveals that the algebra U is composed by the element I alone, because its scalar extensions belong to the center [19]. It is evident that a one-dimensional algebra U is insufficient for a quantitative description of physical reality. Thus, to have a nontrivial theory, one must have the identities $Ixa = axI \equiv a$ for all possibler elements $a \neq kI$, which is impossible for general nonassociative Lie-admissible algebras. For completeness we should recall here that the nonassociative character of the algebra with product axb was fully identified by Weinberg [3]. What we are referring here is the important Lie-admissible character of the nonassociative algebra which was not identified in the quoted paper.

Problematic aspect 2: Weinberg's nonlinear theory does not admit a consistent measurement theory.

As well known in the axiomatic theory of quantum mechanics, the admission of a consistent measurement theory by ordinary quantum mechanics is due not only to the existence of consistent expectation values, but also to the possibility of their "measurement" when referred to a consistent unit of the theory. In Weinberg's nonlinear theory, expectation values can indeed be formally defined [3], but their "scale" cannot be uniquely and unequivocally introduced, thus preventing a consistent notion of measurement. More specifically, the notion of *dilations*, e.g.

$$A \to NA, \quad N\epsilon\mathcal{H}, \quad A\epsilon\mathbf{A} \tag{2.7}$$

can be fully defined in ordinary quantum mechanics because the enveloping algebra always admit a consistent unit, but the same notion is absent in Weinberg's nonlinear theory. This results in rather serious problems of consistency from an experimental viewpoint (see the concluding remarks).

Problematic aspect 3: Weinberg's nonlinear theory does not admit Planck's quantum of energy.

In fact, as well known, *Planck's constant $\hbar = 1$ is the algebraic unit of quantum mechanics*, that is, the (left and right) unit of the enveloping associative algebra A. The lack of a unit in Weinberg's non-associative generalization U of A evidently implies the impossibility of defining in a consistent way Planck's fundamental notion of quantum of action. It also prevents any meaningful attempt at its possible generalization for physical conditions more complex than those of the atomic structure (see next section). Note that Planck's constant $\hbar = 1$ fixes the scale [.7] of the algebraic unit of quantum mechanics. As a result, Problematic Aspects 1, 2 and 3 are deeply interrelated.

Problematic aspect 4: Weinberg's nonlinear theory does not admit a consistent quantization.

As also well known, quantum mechanics can be reached as an operator image of classical Hamiltonian mechanics via various quantization techniques. All these techniques, however, are centrally dependent on

the existence of the unit in the operator formalism. For instance, the socalled *naive quantization* can be expressed via the mapping of the classical action functional a into the unit $\hbar I$ of quantum mechanics times $(-ilog\psi)$

$$a \to \hbar I(-ilog\psi) \tag{2.8}$$

under which the conventional Hamilton-Jacobi equations

$$-\frac{\partial a}{\partial t} = H \tag{2.9a}$$

$$\frac{\partial a}{\partial \vec{x}} = \vec{P} \tag{2.9b}$$

are mapped into Schrödinger's equations

$$i\frac{1}{\psi}\frac{\partial}{\partial t}\psi = H \tag{2.10a}$$

$$-i\frac{1}{\psi}\vec{\nabla}\psi = \vec{P} \tag{2.10b}$$

Similar quantization procedures leading to Weinberg's nonlinear theory are evidently impossible because of the lack of center of U. This implies that the classical counterpart of Weinberg's theory cannot be consistently identified.

Problematic aspect 5: Weinberg's nonlinear theory does not admit well defined Casimir invariants.

As well known in the mathematical theory of Lie's algebras (see, e.g., ref [19], the Casimir invariants are characterized by the center of the universal enveloping associative algebra and, as such, they are not elements of the Lie algebra. Weinberg's nonlinear theory does indeed admit a well defined Lie algebra, as characterized by brackets [2.4]. Nevertheless, the Casimir invariants of the theory cannot be consistently defined because of the lack of the center of the underlying nonassociative Lie-admissible envelope $\mathbf{U}$. The physical implications of the latter problematic aspect are also rather serious. As an example, the indefinabilities implies that fundamental algebras such as the Galilei algebra can indeed be consistently defined, as done in paper [3], Sect. 2. Nevertheless, the crucial Casimir invariants of the Galilei algebra cannot be generally defined in a consistent way.

Consider, for instance, the central case of the magnitude of the angular momentum $\vec{j}$. In conventional quantum mechanics it is given by the "square" $\vec{j}^2$ which is characterized, of course, by the product of the underlying associative envelope A, *i.e.*$\vec{j}^{def=} j_k j_k$, where "$j_k j_k$" is the associative product. In Weinberg's nonlinear theory, the associative product $j_k j_k$ must necessarily be replaced, for consistency, with the nonassociative product $j_k x j_k$ of U, resulting in the form

$$\vec{j}^{2def=} \frac{\partial j_r}{\partial \psi_k} \frac{\partial j_r}{\partial \psi_k^*} \tag{2.11}$$

The above quantity can indeed be consistently defined. Nevertheless, its invariant character cannot be established. Even when, in some particular case, quantity [2.11] is indeed invariant, its eigenvalues cannot be consistently defined owing to the lack of the center of the nonassociative envelope. The above results can also be independently obtained via a property indentified in paper [3], to the effect that elements of U do not generally commute with their own powers. This evidently implies the inability to define all the infinite elements of the center which, in turn, results in the general lack of consistent Casimir invariants. The physical implications of Problematic Aspects 5 are equally deep. In fact, it implies the inability to define a *particle* (whether a physical particle such as the proton or a quark) as currently done, via the Casimir invariants of the Galilei (or Poincaré) algebra.

For completeness we mention that Weinberg did indicate in his paper [3] that *observables such as linear or angular momentum* should be represented by bilinear functions which evidently commute with their powers. The issue here addressed is the restrictions on the algebra imposed by a consistent center. In fact, to have a center, we need first a consistent unit which, as mentioned after Problematic Aspect 1, restricts all elements of the algebra U to the scalar multiples $a = kI, k\epsilon F, I = \psi_k^* \psi_k$. The issue here addressed is therefore whether a consistent algebra made up of the quantity $I = \psi_k^* \psi_k$ and of its scalar multiples kI, and with Casimirs provides by powers kI^n is indeed sufficient for a quantitative representation of physical reality.

Problematic aspect 6: The particles of Weinberg's nonlinear theory do not possess well defined intrinsic characteristics such as spin.

Here, we would like to stress that the problematic aspects in the definition of a particle are not of mathematical nature, but refer, specifically, to the inability to define consistent intrinsic characteristics, such as spin, trivially, which follow from the lack of existence of consistent invariants such as [2.11].

It is evident that we can always talk about spin $\frac{1}{2}$. The technical issue here addressed is whether such a familiar notion can indeed be consistently formulated for Weinberg's algebra, namely, whether the SU [2] symmetry *algebra* can be consistently defined for product $axb - bxa$, whether it admits a consistent *enveloping* algebra with producty axb and with a consistent unit (i.e., a consistent Poincaré-Birkhoff-Witt theorem), and whether it admits a consistent exponentiation to the corresponding SU [2] symmetry *group.* It is evident from the preceding analysis that there are problematic aspect in *each* of these basic aspects, thus requiring care before claiming that the traditional spin $\frac{1}{2}$ can indeed be consistently formulated for Weinberg's theory [3].

The above comments refer to the case of closed-isolated systems. If we have a particle under *external* forces, the problems for the spin are compounded. In fact, in this case the product appearing in the time evolution is itself a nonassociative Lie-admissible product, as discussed in detail in ref.s [18, 20, 23]. But, a central condition of physical consistency requires that the brackets of the time evolution must be those characterizing the spin algebra. This evidently implies a further departure from the conventional *Lie* algebra SU [2], to a *Lie-admissible* SU [2] spin algebra, that is, an algebra whose product is non-Lie and Lie-admissible, for which the value $\frac{1}{2}$ is known to be generally lost.

Problematic aspect 7: Weinberg's nonlinear theory does admit consistent space-time symmetries in their finite form.

As well known, the existence of quantum mechanical, space-time (and other) symmetries in their finite, group theoretical form, is due to the existence of a consistent infinite-dimensional basis in the underlying envelope. For a given Lie algebra of n-dimension and generators $X_j, j = 1, 2, \ldots n$, such a basis can be written

$$I, \quad X_i, \quad \underset{i \leq j}{X_i X_j}, \quad \underset{i \leq j \leq k}{X_i X_j X_k}, \ldots \tag{2.12}$$

as ensured by the celebrated Poincaré-Birkhoff-Witt theorem [19]. In fact, the exponentiation into a finite (unitary) symmetry group is precisely a power series expansion in $\mathbf{A}$ (here considered for the simple one-dimensional case without loss of generality)

$$G(w) : e^{iXw} = e_A^{iXw} = I + \frac{iXw}{1!} + \frac{(iXw)^2}{2!} + \ldots \tag{2.13}$$

In particular, its convergence into the finite form $exp(iXw)$ is precisely due to the existence of a consistent basis in A. It is known to experts in Lie-admissible algebras that the Poincaré-Birkhoff-Witt theorem can indeed be extended to *flexible* nonassociative Lie-admissible algebras (see ref. [18]) and the specific study [20], but its extension to the case of *general* Lie-admissible algebras does not exist, because of numerous technical problems, beginning from the lack of existence of the first term in the infinite basis [2.12] (the unit), and then passing to the lack of a consistent ordering of the basis due to the intrinsic nonassociativity of the product, i.e.,

$$(X_i \times X_j) \times X_k \neq X_i \times (X_j \times X_k) \quad (i \leq j \leq k) \tag{2.14}$$

As a result, the process of esponentiation can indeed be defined up to flexible Lie-admissible algebras, but not, in general, for the Lie-admissible algebras of the general class selected by Weinberg. The physical implications of the latter problematic aspect are also far reaching. For instance, it implies that fundamental symmetries, such as the Galilei symmetry, the Lorentz symmetry and the SU [3] symmetry, cannot be defined in a clearly consistent way.

Problematic aspect 8 : In Weinberg's nonlinear theory, the Heisenberg-type representation [1.6] and the Schrödinger-type one [1.9] are generally inequivalent.

As recalled in the introductory section, the conventional modular-associative structure of the Schrödinger's action $H\psi, H\epsilon A, \psi\epsilon\mathcal{H}$, is directly due to the conventional associative structure of the underlying envelope A. This, in turn, is at the foundation of the proof of the equivalence of the conventional Heisenberg's and Schrödinger's representations. In different terms, the algebraic structure of action $H\psi$ of Schrödinger's representation and of the product AB underlying Heisenberg's representation are the same. In particular, the modular action $H\psi$ is associative in the conventional sense, i.e.

$$ABC\psi = (AB)(C\psi) = A(BC)\psi = (ABC)\psi \tag{2.15}$$

The above equivalence of mathematical structures between the Heisenberg-type and the Schrödinger-type representations is lost in Weinberg's nonlinear theory. In fact, to achieve such an equivalence, instead of Weinberg's action

$$Ho\psi_k = \frac{\partial H}{\partial \psi_k^*} \tag{2.16}$$

the Schrödinger's type representation should have the action

$$Hx\psi = \frac{\partial H}{\partial \psi_k} \frac{\partial \psi}{\partial \psi_k^*} \tag{2.17}$$

Again, to clarify this important point, the equivalence between the Heisenberg-type and Schrödinger-type representations requires, first, the same algebraic structure in both representations. For the case of ordinary quantum mechanics, the Heisenberg representation is based on the conventional associative product AB which is evidently equivalent to the Schrödinger's conventional associative product $H\psi$. In Weinberg's case, the Heisenberg-type representation is now characterized by the nonassociative Lie-admissible product $AxB = (\partial A/\partial\psi_k)(\partial B/\psi_k^*)$. As a result, the corresponding product in the Schrödinger-type representation should be exactly the same, that is, we should have $HxA = (\partial H/\partial\psi_k)(\partial\psi/\partial\psi^*)$. But this is not the case for Weinberg's Schrödinger-type equation [1.6], thus implying the inequivalence of the two representations.

Even assuming that the above problematic aspect can be resolved, the rigorous proof of the equivalence of the two representations requires the existence of a unitary equivalence mapping, that is, of a unitary transformation that uniquely transforms the Heisenberg-type into the Schrödinger-type representation, and viceversa, exactly as it occurs in the conventional case. The proof of the inequivalence of Weinberg's Heisenberg-type and Schrödinger-type representations is then completed by the lack of a consistent generalization of the conventional unitary transformations.

Problematic aspect 9 : The nonlinearity of Weinberg's equations is not essential, in the sense that it can be made to disappear at the abstract, realization-free level.

This is the most abstract of the problematic aspects of Weinberg's theory, inasmuch as it requires an in depth knowledge of the abstract transformation theory, algebras and groups. Nevertheless, it is the most penetrating from a physical viewpoint inasmuch as it pre-empties the need for experiments on possible atomic nonlinearity.

The latter problematic aspect can be better understood following the analysis of the next section. At this point, let us recall the result reached by one of us (R.M.S.) via the use of the so-called *Lie-isotopic* *theory* [22], according to which a given nonlinear transformation on a manifold M with local coordinates x

$$x \to x' = x'(x, w) \tag{2.18}$$

can always be expressed in an identical isolinear form of the type

$$x \to x' = A(w)^* x \stackrel{\text{def}}{=} A(w)T(x;w)x \tag{2.19}$$

by embedding all the nonlinear terms in the so-called isotopic element $T(x;w)$. The point is that nonlinear transformations [2.19] coincide with conventional, linear, transformations

$$x \to x' = A(w)x \tag{2.20}$$

at the abstract, realization-free level.

To put it differently, and as stressed in ref. [22], nonlinearity is not a structure-characterizing feature for operator theories. The lack of an essential remnant of nonlinearity which survives at the abstract level is then sufficient to void the need for conventional, atomic experiments, particularly when keeping in mind the preceding problematic aspects, such as the lack of a consistent measurement theory.

It should be stressed that Problematic aspect 9 is specifically referred to "Weinberg's equations" and not to "Weinberg's nonlinear theory", where the former terms refer to "the actual equations" in their explicit form after working out the partial derivatives of Eq. [1.9], while with the latter terms refer to "the theory itself", thus including its nonassociative Lie-admissible operator algebra. In fact, the isotopic techniques can eliminate at the abstract level the nonlinearity of the actual equations, but certainly not the nonassociativity of Weinberg's envelope. As a result, all preceding problematic aspects persist because they cannot be eliminated via isotopies, as the reader is encouraged to verify. [The Authors are grateful to the Referee for suggesting this important clarification].

By no means the above problematic aspects exhaust all the problems afflicting Weinberg's nonlinear theory [3] (one can find additional problematic aspects en ref.S [7, 8, 15, 16]).

However, the major problematic aspect of Weinberg's theory is that it violate *Okubo's no-go theorems* [21], which essentially state that *a generalization of quantum mechanics based on the transition from the conventional associative envelope to a nonassociative form is structurally inconsistent.* In fact, the inequivalence of Heisenberg-type and Schrödinger-type representations (problematic aspect 8) is a consequence of Okubo's theorems. This is the central point of the analysis of this section because it shows that, even assuming that the individual problematic aspects outlined above can be somewhat resolved, one still remains with the main problematic aspect that *a theory based on a nonassociative, enveloping operator algebra is structurally inconsistent.*

It is hoped the reader has noted our emphasis on the terms "problematic aspects" rather than "inconsistencies", because the above issues deserve additional inspections by other colleagues, and suggest caution prior to claiming final conclusions whether in favor or against Weinberg's nonlinear theory.

3. The isotopies of quantum mechanics.

While studies on the consistency or inconsistency of Weinberg's theory [3] will take their predictable time, the remaining issue is whether there exists a generalization of quantum mechanics which includes all possible Weinberg's nonlinear equations, while bypassing all problematic aspects of Weinberg's theory considered in Sect. 2. It may be of some interest to the interested reader to known that such a generalized mechanics does indeed exist, and it is provided by the so-called *hadronic generalization of quantum mechanics*, or *hadronic mechanics* for short, as originally proposed by one of us [23] and then developed by a number of authors (see ref.s [24 - 27] and quoted papers).

The mechanics is based on the preservation of the associative character of the conventional enveloping algebra, although expressed in its most general possible form. Hadronic mechanics therefore by-passes Okubo's no-go theorems [21] by its very conception. The associativity of the envelope ensures the existence of a bona-fide, left and right unit, as we shall see. The resolution of all problematic aspects of Weinberg's nonlinear theory [3] is then consequential.

Moreover, hadronic mechanics is characterized by the *isotopies* of quantum mechanics, which essentially are given by the most general possible nonlinear (in all variables and their derivatives), nonlocal (integral) and nonpotential (nonhamiltonian) generalization of quantum mechanics capable of preserving its original axioms unchanged at the abstract, realization free-level. This implies that hadronic mechanics, not only contains the totality of Weinberg's nonlinear equations, but also a considerably broder class of nonlinear equations (such as those nonlinear in the *derivatives* of the wavefunctions, as we shall see shortly).

It should be indicated that hadronic mechanics was specifically conceived for the *interior* dynamics of strong interactions, in the hope of achieving a compatibility between the established local, differential and potential (center-of-mass) behaviour of hadrons (say, in a particle accelerator), with the historical open legacy of their ultimate nonlocality. Hadronic mechanics is therefore intrinsically nonlocal, nondifferential (integral) and nonhamiltonian. As such, it can only be applied under the existence of the latter forces.

The physical arenas of intended applicability of Weinberg's and hadronic mechanics are, therefore, profoundly different. The former mechanics has been used to attempt the identification of possible nonlinearity under electromagnetic interactions (e.g., in atomic structure). The latter mechanics assumes the current quantum mechanical description of the electromagnetic interactions as being exact "ab initio", and searches for possible nonlinearity (or, more generally, nonlocality) in the interior dynamics of strong interactions, e.g., for a hadron in the core of a star, or, much equivalently, for a quark with extended wavepacket of the size of all masive particle ($1F$) which coincides with the size of all hadrons. This implies a hadronic structure characterized by motion of extended wavepackets *within* the volume occupied by all remaining constituents, with consequent expected nonlinear, as well as nonlocal and nonhamiltonian effects.

The experimental profiles of the two mechanics are consequentially fifferent. In fact, a conceivable nonlinearity is studied by hadronic mechanics as an approximation of the expected nonlocality of the strong interactions. The experimental results on the lack of appreciable nonlinearity [13 - 16] are therefore strictly inapplicable for hadronic mechanics, as well as yet inconclusive, in the sense that they are insufficient for a final claim on the lack of appreciable nonlinearity in particle physics.

Recall that all current measures are reached via the use of external *electromagnetic* interactions. By recalling that the dynamics of the center-of-mass of a composite system with strong internal forces is conventional, the possible future experimental detection of the nonlinearity predicted by hadronic mechanics will require a duplication, this time at the level of strong interactions, of the current experimental measures, i.e.,

a new technology capable of providing measures under external *strong* interactions.

To begin, the central assumption of hadronic mechanics is that of preserving axioms [1.1] of the conventional associative enveloping algebra, i.e., of *preserving the crucial associativity of the enveloping operator algebra* and merely assuming its most general possible form admitting of a unit. Among the various possible forms, the realization of axioms [1.1] suggested at the foundation of hadronic mechanics is given by [18, 23]

$$\hat{\mathbf{A}} \quad \begin{cases} A * B = ATB, & T \text{fixed} \\ \hat{I} * A = A * \hat{I} = A & \text{for all} \quad A \epsilon \hat{\mathbf{A}}, \quad \hat{I} = T^{-1} \end{cases} \tag{3.1}$$

where the new product $A * B$ (called *isoassociative product*) is still associative, as one can easily verify, and the quantity $\hat{I}$ (called *isounit*) is still the correct right and left unit of the theory.

For mathematical consistency, isoassociative envelope [3.1] must be defined on the following generalization of the conventional notion of complex field $\mathbf{C}$

$$\hat{\mathbf{C}} = \{\hat{c}I\hat{c} = c\hat{i}, \quad c\epsilon\mathbf{C}, \quad \hat{i}\epsilon\hat{\mathbf{A}}\} \tag{3.2}$$

where the sum is the conventional one and the product is isotopic. Thus $\hat{c} * \hat{d} = \hat{c}d = (cd)\hat{i}$. The numbers of the theory then remain the conventional ones because $\hat{c} * \psi \equiv c\psi$. The Hilbert space was also subjected to a lifting with generalized composition [27]

$$\hat{\kappa} :< \hat{I} >=< IGI > \hat{I}\epsilon\hat{\mathbf{C}}, \quad G > 0 \tag{3.3}$$

which is evidently still inner. The space κ is then called an *isohilbert space.*

Howing to the isolinear structure of the theory, all conventional concepts on a Hilbert space (Hermiticity, unitarity, etc.) admit a consistent operator formulation [26, 27]. In particular, for $T = G > 0$, conventional Hermiticity and isohermiticity coincide. Thus *observables of quantum mechanics remain observable of hadronic mechanics* under isotopies [3.1] and [3.4] with the same (positive-definite) isotopic element T.

The Heisenberg-type equation was identified in the original proposal [23] as being characterized by the antisymmetric brackets attached to $\hat{\mathbf{A}}$

$$i\dot{A} = [a, H]_{\hat{\mathbf{A}}} = A * H - H * A = ATH - HTA \tag{3.4}$$

and it is called the *isoheisenberg's equation.* Note its nonhamiltonian character, in the sense that, besides the conventional Hamiltonian H including all contemporary models, the generalized equation admits an *additional operator T which multiplies the Hamiltonian from the right and from the left*, and which is representative precisely of the nonhamiltonian forces.

By central condition, the isotopic element T becomes the identity I for all particles moving in vacuum at mutual distances larger than their wavepackets (e.g., in the atomic structure). Eq.s [3.4] therefore recover the conventional Heisenberg's equations identically when all nonhamiltonian components of the strong interactions are null. In this sense, hadronic mechanics can be considered as a *covering* of quantum mechanics (i.e., a generalized theory based on a mathematical structure and intended for physical conditions broder than those of quantum mechanics, yet including the latter as a particular case). Notice that conventional quantum mechanics cannot be admitted as a particular case by Weinberg's nonlinear theory.

The corresponding Schrödinger-type equation was identified in ref.s [25, 26]

$$i\frac{\partial}{\partial t}\psi = H * \psi = HT\psi \tag{3.5}$$

and it is called the *isoschrödinger's equation.* Notice, again, the appearance of nonhamiltonian terms, as well as the admittance of the conventional Schrödinger's equation as a particular case. For numerous additional properties, see ref.s [25, 26]. It is evident that hadronic mechanics avoids the problematic aspects of Weinberg's nonlinear theory. In fact, the theory is centered on the preservation of the *associative* character of the enveloping operator algebra and in the existence of a generalized, but consistent (right and left) *unit*, the isounit $\hat{I}$, with consequently well defined *center.*

To illustrate this crucial point, note that one could select another form of isoassociative product, e.g., of the type [23]

$$A * B = WAWBW, \quad W^2 = W \neq 0 \tag{3.6}$$

which is evidently still associative. Nevertheless, the use of the above product in the construction of hadronic mechanics was *discouraged* because it does not admit a consistent (right and left) unit.

The preservation of the unit avoids Problematic Aspects [1.6]. The associative character of the envelope ensures the existence of a consistent isotopic generalization of the Poincaré-Birkhoff-Witt Theorem [18] and the isoenvelope $\hat{\mathbf{A}}$ admits a consistent infinite-dimensional basis, which resolves Problematic Aspect [7] (see the isotopic lifting of the Lorentz group of ref. [24]). The equivalence between the isoheisenberg equation [3.4] and the isoschrödinger's one [3.5] is predictable from the preservation of the associative character of the isomodular action $H * \psi$, and it was explicitly proved in ref. [26]. This avoids Problematic Aspect 8. Finally, Problematic Aspect 9 *cannot* be eliminated by hadronic mechanics (or any other generalized theory for that matter), because of the universal character of the isolinear theory [22]. In fact, hadronic mechanics is formally linear at the abstract, realization-free level, yet intrinsically nonlinear. We are now sufficiently equipped to prove the following.

Proposition 1: *All Weinberg's nonlinear equations [1.9] in their explicit form (but not Weinberg's nonlinear theory) are a particular case of isoschrödinger's equation [3.5]*

Proof - Except for the positive-definiteness, $T > 0$, the functional dependence of the isotopic operator remains completely unrestricted in hadronic mechanics. As such, T can have *the most general possible nonlinear and well as nonlocal dependence on all possible local variables and quantities.*

$$T = T(t, r, p, \psi, \psi^*, \partial\psi, \partial\psi^*, \ldots), \quad \hat{I} = (t, r, p, \psi, \psi^*, \partial\psi, \partial\psi^*, \ldots) \tag{3.7}$$

Eq. [3.5] can then be written

$$i\frac{\partial}{\partial t}\psi = H * \psi = HT\psi = H(t, r, p)T(t, r, p, \psi, \psi^*, \partial\psi, \partial\psi^*, \ldots)\psi \tag{3.8}$$

This results in the most general possible nonlinear equations not only in the quantities ψ, ψ^*, and all their possible derivatives, but also in the variables t, r, p, as well as in the most general possible nonlocality in the same variables. The admittance of Weinberg's equations [1.9] as a particular case then follows. Q.E.D.

Note the use in Proposition [1] of the differentiation between "Weinberg's nonlinear equations" and "Weinberg's nonlinear theory" made in Sect. [2] after Problematic aspect 9. In fact, hadronic mechanics can include the explicit form of Weinberg's "equations", but evidently not