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Volume 1

ORIGIN AND AXIOMATIZATION OF *Q*-DEFORMATIONS

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1. Problematic aspects of *q*-deformations.

The so-called *q*-deformations (see, e.g., [1]) include a great variety of deformations of conventional quantum mechanical (QM) formulations, such as: deformations of fundamental commutation rules

$$rp - qpr = if(q), \quad rp - pr = ig(q); \tag{1.1}$$

deformations of creation and annihilation operators; deformations of Lie algebras; Hopf algebras; mixtures of conventional Lie-Hamiltonian time evolutions and *q*-commutators; and others.

All the above deformations are clearly correct on *mathematical* grounds as currently treated, that is, under the conditions that: I) *q* is a *number* (e.g., an element of the base field of reals *R* or complex numbers *C*); II) the quantities $f(q)$, $g(q)$, etc. are ordinary *functions* on *R* or *C*; and III) there is no time evolution (see, e.g., the mathematical rigor of the *q*-special functions of [2]).

However, a number of rather fundamental problematic aspects emerge in *physical* applications whenever one attempts to do dynamics, thus including time evolution. Their origin was first identified by Santilli [3] back in 1978 (see also Jannussis [4], and others) who showed that departures (1.1) from QM structures generally imply a *nonunitary time evolution*, with consequential loss of form-invariance, loss of the original Hermiticity of the Hamiltonian (see later on Lemma 3.1), loss of probabilities, loss of applicability of the q-number-special functions, etc.

In fact, under a nonunitary time evolution $UU^\dagger \neq I$, an arbitrary *q*-"commutator" becomes [3]

$$U(AB - qBA)U^\dagger = A'PB' - B'QA', \ A' = UAU^\dagger, \ B' = UBUt, \ P' = (UU^\dagger)^{-1}, \ Q = qP, \tag{1.2}$$

thus showing the inevitability of the transition from *q*-numbers to *Q*-operators (as well as to a more general $P - Q$ deformations identified below).

Moreover, in general, *q*-deformations are theories without the unit which implies a number of additional problematic aspects, such as the loss of the uniqueness of the exponentiation, or the lack of applicability to measurements. Let $F(n, +, \times)$ be a field of real or complex numbers *n* with conventional sum + and multiplication $n \times m = nm$. As well known, a necessary condition for *F* to be a field is the existence of the element $1 \in F$ called *multiplicative unit*, such that $1n \equiv n1 \equiv n$, $\forall n \in F$. Let ξ be an enveloping associative albebra with elements *A*, *B*, ... and conventional associative product *AB* over *F*, that is, with the same associative product of *F*, $AB = A \times B$. Let $I \in \xi$ be the unit of ξ, i.e., $IA = AI \equiv A$, $\forall A \in \xi$.

Most *q*-deformations are deformations of the basic associative product $\times$, i.e,

$$A \times B = AB \Rightarrow A * B = qAB \tag{1.3}$$

without jointly redefining the unit as done in Santilli's axiomatization (Sec. 3). As a consequence, deformations (1.3) imply the evident loss of the unit $I \in \xi$. In turn, this implies the inapplicability of the Poincaré–Birkhoff–Witt theorem on the infinite-dimensional basis of ξ, with consequential loss of a basic property of Lie algebras, the uniqueness of the exponentiation from algebras to groups.

In addition, in specific cases one may find the need of keeping the identification of the associative product of ξ with that of *F*. In this case, deformations (1.3) evidently imply the additional loss of the unit of *F*, that is, the reduction of the *field* $F(n, +, \times)$ to a *commutative-ring without the unit* $F^*(n, +, *)$. Under these circumstances, the underlying Hilbert space of

the theory is defined over F^* and not over *F*. This results in *the general inapplicability of the measurement theory to q-deformations, with consequential inapplicability to experiments, because of the necessary condition for any measure to have a well defined unit.* As we shall see, Santilli's axiomatization is conceived precisely to prevent these problematic aspects.

But perhaps the most serious uneasiness is created by the fact that *q-deformation imply necessary deviations from the axiomatic structure of the Poincaré symmetry and of the special relativity, with a number of consequential open problems, e.g., for causality.* On the contrary, Santilli's axiomatization is conceived to preserve these basic axioms and only realize them in more general way.

To avoid misrepresentations, we would like to stress that we are referring here to "problematic aspects" which are essentially "physical" in nature, and not to "mathematical inconsistencies". Also, these problematic aspects are not referred to "all" *q*-deformations, but only to those for which the time evolution can be proved to be nonunitary or which lack the unit.

2. Bibliographical notes.

When studying the axiomatic structure of quantum mechanics, the first and most fundamental task is the identification of the algebra characterized by the commutator $[A, B] = AB - BA$, the Lie algebra [5]. Similarly, when studying *q*-deformations, the identification of the algebra characterized by the "commutator" $[A, B]_q = AB - qBA$ is an evident pre-requisite for the achievement of a consistent axiomatization.

The algebra characterized by the product $[A, B]_q$ was first introduced by the American mathematician Albert [6] back in 1948, via the following two notions.

Lie-admissibility: a (generally nonassociative) algebra *U* with elements *a*, *b*, *c*,... and (abstract) product ab over a field *F* is said to be *Lie-admissible* when the attached algebra U^-, which is the same vector space as *U* but equipped with the product $[A, b]_U = ab - ba$, is a Lie algebra; and

Jordan-admissibility: the algebra *U* is said to be *Jordan-admissible* if the attached algebra U^+, which is the vector space *U* equipped with the product $\{a, b\}_U = ab + ba$, is a (commutative) Jordan algebra [7].

It is easy to see that the algebra *U* characterized by the *q*-product $[A, B]_q$ is a *nonassociative, Lie-admissible and Jordan-admissible algebra*, because $[A, B]_U = (1 - q)(AB - BA)$ and $\{A, B\}_U = (1 - q)(AB + BA)$. We can therefore say that in the transition from the quantum to the *q*-"commutator" we do preserve a well defined content of Lie algebra and, in addition, we gain a new content of Jordan algebra which is notoriously absent in Lie algebras. The emergence of the still open Jordan legacy alone should be sufficient reason for its quotation in the specialized literature on *q*-deformations.

The above characterization of *q*-deformations was first introduced in the mathematical and physical literature by the physicist Santilli [8] in 1967 as part of his Ph.D. in physics at the University of Turin, Italy. In fact, in [8], p.573, one can see the product

$$(a, b) = \lambda ab - \mu ba = \rho[a, b] + \sigma\{a, b\}, \quad \lambda, \ \mu, \ \rho, \ \sigma \in F, \tag{2.1}$$

which was introduced as the (λ, μ)-*mutation* [1] of a generic algebra *U* with product *ab*, as well as a particular class of algebras called *flexible Lie-admissible algebras*. The *q*-deformation $[A, B]_q = AB - qBA$ is an evident particular case of Santilli's mutation occurring for $\lambda = 1$, $\mu = q$ and *ab* associative.

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[1]Santilli [8] introduced the term "mutations" rather than "deformations" to stress the fact that structures (2.1) are different than the deformations of Lie algebras as understood in mathematics, e.g., because the former are non-Lie, while the latter remain Lie. As we shall see shortly, Santilli also introduced the name of "Lie-admissible group" [12] to characterize generalized group structures with non-Lie, Lie-admissible brackets in the neighborhood of the identity. By comparison, the name subsequently introduced of "quantum groups" is doubly misleading, because *q*-deformations imply a generalization of both, the notion of group and the quantum of energy.

Albert presented in [6] an abstract (and relatively short) treatment of Lie-admissibly, with more emphasis on the Jordan-admissibly because of its greater interest in the mathematics of the time. In fact, the sole explicit realization of the product in Albert's paper is given by the known realization of *noncommutative Jordan algebras* [6]-[9].

$$(a,b) = \lambda ab - (1 - \lambda)ba, \tag{2.2}$$

for ab associative. The point is that q-deformations are a particular case of Santilli's mutation (2.1) and not of Jordan's form (2.2).

Santilli is therefore the originator on both mathematical and physical grounds of theories today known as *Santilli's Lie-admissible formulations*, constituting of a step-by-step generalization of Lie's theory, with realization on classical, operator and statistical mechanics, a priority now acknowledge in mathematical circles (see, e.g., the historical charts of [10], p. 13). In fact, following Albert [6], only two short mathematical notes in Lie-admissibility had appeared in 1967 (see [8] and bibliography [9]), also without any specific realization.

On mathematical grounds, Lie-admissible algebras had been studied as *nonassociative* algebras, an approach still continuing in the mathematical literature [9]. On the contrary, Santilli constructed a generalization of enveloping *associative* algebras characterizing Lie-admissible algebras, groups, representation theory, etc., which subsequently resulted to be crucial for the axiomatization of q-deformations presented on Sect. 3 below.

On physical grounds, Santilli studied already in 1968 [11] the *classical limit* of the (λ, μ)-mutations (2.1), by proving that they are a particular case of Hamilton's equations with external terms. This established that the mutations $AB - BA \to \lambda AB - \mu BA$ imply the transition from closed-conservative to open-nonconservative systems, because of the loss of total antisymmetry of the product.

These initial classical studies were then complemented in 1978 [12] with the identification that *the brackets of Hamilton's equation with external terms, when properly written, characterize a general Lie-admissible algebra*.

These classical studies were systematically continued in monographs [13, 14] via: the classical version of the Lie-admissible formulations with exponentiated group structure, called *classical Lie-admissible group*, admitting a non-Lie, Lie-admissible structure in the neighborhood of the identity; the Lie-admissible generalization of Lie's first, second and third theorems; the identification of the *exterior-admissible calculus*, as a generalization of the conventional exterior calculus; the introduction of the main lines of the *symplectic-admissible geometry* as the classical geometry underlying the Lie-admissible brackets the derivation of Hamilton's equations with external term from variational principle on the *exterior-admissible one-form* (despite their variational nonselfadjointness-NSA- [12]).

On operator grounds, Santilli was the first to introduced back in 1978 [3]: the *general Lie-admissible and Jordan admissible algebras* with product [2]

$$(A, B) = APB - BQA; \tag{2.3}$$

the now well known *Lie-admissible generalization of Heisenberg's equations*, which can be written in the infinitesimal form [3], p. 746 ($\hbar = 1$),

$$i\dot{A} = (A, H) = APH - HQA, \tag{2.4}$$

and finite form [3], p.783, [3]

$$A(t) = e^{iHQt}A(0)e^{-itPH}; \tag{2.5}$$

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[2]One should note that the scripture $AB - QBA$, with Q-operator, violates the left distributive and scalar laws and, therefore, characterizes no algebra of any kind [3].

[3]Note for future use that *nonassociative* algebras emerge only in the product of the time evolution (2.4), while the envelope remains strictly *associative*, as shown by each term APH, HQA, $\exp(iHQt)$ and $\exp(-itPH)$.

the *fundamental Lie-admissible commutation rules* [3], p.746,

$$(a^\mu, a^\nu) = a^\mu P a^\nu - a^\nu Q a^\mu = i\omega^{\mu\alpha}\hat{I}_\alpha^\nu, \quad a = (r, p), \tag{2.6}$$

where $\omega^{\mu\alpha}$ is the (antisymmetric) canonical Lie tensor and $\hat{I}$ is a general, non-Hermitean operator for $P = Q^+$; the first formulation of Lie-admissible operator algebras on bimodular Hilbert spaces; and other advances.

Subsequently, Fronteau, Tellez-Arenas and Santilli [15] were the first to identify in 1979 the Lie-admissible structure of the most general possible equations in statistical mechanics, those with arbitrary collisions terms

$$i\dot{\rho} = (\rho, H) = \rho PH - HQ\rho \equiv [\rho, H] + C \tag{2.7}$$

The generalized Schrödinger's counterpart of Lie-admissible equations (2.4) was identified by Myung and Santilli [16] and, idependently, Mignani [17] in 1982 according to the expressions

$$i\frac{\partial}{\partial t}|\hat{\psi}> = HQ|\hat{\psi}>, \quad -i<\hat{\psi}|\frac{\partial}{\partial t} = <\hat{\psi}|PH. \tag{2.8}$$

The identification of the correct form of the linear momentum operator required considerable additional studies at the *classical* level [13, 14], which eventually permitted Santilli [18] to reach the expressions

$$p_k Q|\hat{\psi}> = -i(Q^{-1})_k^i \nabla_i|\hat{\psi}>, \quad <\hat{\psi}|Pp_k = i<\hat{\psi}|\nabla_i(P^{-1})_k^i, \tag{2.9}$$

achieved via the identification of the Hamilton-Jacobi equations for the classical Lie-admissible variational and its map into the operator form called *isoquantization*.

Additional biographical data worth an indication are the following. The first study of the (λ, μ)-mutation of $SU(2)$ spin was presented by Santilli at the Clausthal *Conference on Differential Geometric Methods in Physics* of 1980 [19]. The first generalizations of the rotational and Lorentz symmetries for operators $P = Q$ was reached in [20, 21]. The first identification of the Lie-admissible generalizations of symplectic, affine and Riemannian geometries was done in [13, 22]; the first Q-operator deformation of gauge theories was reached by Gasperini [23] in 1983; the first studies on the Lie-admissible generalization of creation and annihilation operators were conducted by Jannussis et al. [24] beginning from 1981; Mignani [25] initiated the construction of a Lie-admissible scattering theory, subsequently completed by Santilli [18] via the use of special $P - Q$-functions; Okubo [26] identified certain "no go"-theorems for operator formulations with nonassociative envelopes; Kalnay and Santilli [27] discovered the operator form of Nambu's mechanics for triplets with an essential Lie-admissible $P - Q$-structure; Animalu [28] was the first to apply the methods to electron pairing in superconductivity; Kadeisvili [29] initiated the systematic study of special functions, distributions and transforms compatible for Q-operator deformations; additional studies were conducted by Nishioka [30] Aringazin [31]. Lopez [32], and others.

A comprehensive presentation of all these operator studies is now forthcoming in the three volumes [33] under the name of *hadronic mechanics* (HM), which is based on the main classification into:

Lie-admissible formulations, applicable when the energy is not conserved, i.e., from Eq.s. (2.4), $i\dot{H} = (H, H) = H(P - Q)H \neq 0$, and the simpler

Lie-isotopic formulations, applicable when the energy is conserved, which occur when in Eq.s (2.4) $P = Q$, $i\dot{A} = [A, H] = AQH - HQA$, in which case the algebra is still Lie, although of a more general type. The latter theories were called *isotopic* [12], in the sense of being axiom-preserving, while the former theories were called *genotopic* [loc. cit.] in the sense of inducing covering axioms.

In conclusions, the studies here considered have essentially established: 1) the Lie-admissible/Jordan-admissible character of the deformation/mutation $AB - qBA$; 2) the existence of a step-by-step generalization of Lie's theory of Lie-admissible character; 3) the existence of Lie-admissible formulations in classical, quantum and statistical mechanics; 4) the existence of a unique map interconnecting classical and operator formulations; and 5) the existence of an axiomatization of the *operator* deformations of *associative* algebras which is at the foundation of both the Lie-isotopic and Lie-admissible theories.

3. Santilli's axiomatization of q-deformations.

An axiomatization of q-deformations which avoids the problematic aspects mentioned in Sect. 1, will now be presented via the so-called *isotopies of enveloping associative algebra* [12, 14, 33]. The formulations can then be used for the axiomatization, first, of Lie-isotopic theories, and then of the more general Lie-admissible theories.

The emerging axiomatization is naturally applicable for operator Q with an arbitrary, non-linear, nonlocal and noncanonical dependence $Q = Q(t, r, \dot{r}, \ddot{r}, \hat{\psi}, \partial\hat{\psi}, \partial\partial\hat{\psi}, \ldots)$, which is classified into *Kadeisvili five topologically different classes* [29]. In this note we can only consider for brevity Q-operators of Class I (smooth, bounded, nowhere singular, Hermitean and positive-definite). The axiomatization can be summarized via the following basic points.

Consider a universal enveloping associative algebra ξ of a Lie-algebra $L \approx \xi^-$ of Sect. 1. Santilli's fundamental point is that *any deformation-mutation of the associative product AB necessary requires a corresponding generalization of the basic (multiplicative) unit*. In fact, it is "anathema" in number theory to change the product and keep the old unit, or viceversa, because units and products are deeply inter-related.

Santilli therefore introduced the general isotopic deformation of the associative product AB of ξ in terms of an *operator* Q called *isotopic element* [12] [4]

$$\hat{\xi}_Q : A * B := AQB, \quad Q = \text{fixed} \tag{3.1}$$

and, jointly, redefined the basic unit $I \in \xi$ into the form $\hat{I} = Q^{-1}$ called *isounit*, which is the correct left and right unit of the Q-theory

$$\hat{I} = Q^{-1}, \qquad \hat{I} * A = A * \hat{I} \equiv A, \ \forall A \in \hat{\xi}_Q. \tag{3.2}$$

The generalization of the basic multiplication and related unit then requires, for mathematical consistency, a generalization of the notion of "numbers". In fact at the 1980 Clausthal *Conference on Differential Geometric Methods in Physics*, Santilli [19], [20]–[22] introduced the isotopies

$$F(a, +, \times) \Rightarrow \hat{F}_Q(\hat{a}, +, *), \quad \times \Rightarrow * := \times Q \times, \quad a \Rightarrow \hat{a} := a\hat{I}, \quad 1 \Rightarrow \hat{I} := Q^{-1}, \tag{3.3}$$

characterizing *isofields*, i.e., $\hat{F}_Q$ verifies all axioms of F. Note that for $Q = q \in F$ the lifting $a \Rightarrow \hat{a}$ is un-necessary because the set $\hat{F}_Q(a, \hat{+}, *)$ in this case is a field. However, the generalization of numbers $a \Rightarrow \hat{a}$ is needed whenever Q is not an element of the original field F. It is evident that this third step requires a suitable isotopic generalization of all operations on numbers, e.g.: $\hat{a}^{\hat{n}} = \hat{a} * \hat{a} * \ldots * \hat{a}$ (n times); $\hat{a}\hat{/}\hat{b}$, $\hat{/} = \hat{I}/$; $\hat{a}^{\frac{1}{2}} = a^{\frac{1}{2}}\hat{I}$ (see [22]).

Recall that conventional carrier spaces are defined over conventional fields. The generalization of multiplication, unit and fields evidently requires, also for mathematical consistency, a

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[4] Santilli identified three isotopies of associative algebras: I) the q-isotopy with q scalar, $AB \Rightarrow A * B = qAB$; II) the Q-isotopy with Q operator (not necessarily Hermitean), $AB \Rightarrow A * B = AQB$; and III) the isotopy $AB \Rightarrow A * B = WAWBW$ with W idempotent, $W^2 = W$. All of them preserve the original associativity as a necessary condition for an isotopies, $A * (B * C) = (A * B) * C$. Of these, Santilli selected the second because the third does not admit a left and right unit, thus implying the problematic aspects for physical applications of Sect. 1.

compatible generalization of conventional carrier spaces, introduced for the first time by Santilli [21] in 1983. Let $S(x, g, R)$ be a metric or pseudo-metric space with local coordinates x, (Hermitean and nowhere singular) metric g over the reals R. The isotopies necessary under Q-deformations are

$$S(x, g, R) : x^2 = x^t g x \in R \Rightarrow \hat{S}(x, \hat{g}, \hat{R}) : x^2 = (x^t \hat{g} x)\hat{I} \in \hat{R}, \ \hat{g} = Qg, \ \hat{I} = Q^{-1}. \tag{3.4}$$

Isospaces $\hat{S}(x, \hat{g}, \hat{R})$ characterize fundamentally novel geometries called *isoeuclidean*, *isominkowskian* and *isoriemannian* with intriguing mathematical and physical implications, such as a certain isotopic generalization of conventional angles, a geometric unification of spheres, ellipsoids and hyperboloids, etc. [14], [20]–[22], [33].

The generalization of multiplication, unit, fields and carrier spaces then requires a compatible lifting of the transformation theory, from the conventional linear expression, into the isotopic form

$$x' = Ux \Rightarrow x' = \hat{U} * x = HQx, \quad Q = \text{fixed}. \tag{3.5}$$

first introduced in ref. [12] of 1978. Note that the preservation of the old transformations $x' = Ux$ under isotopies implies the loss of linearity, transitivity, superposition principle, etc.

The generalization of the multiplication, unit, field, carrier spaces and transformation theory then requires a step-by-step generalization of the theory of enveloping associative algebras which was the central topic of study of memoir [12]. Most important is the first achievement of the Poincaré–Birkhoff–Witt theorem on the infinite-dimensional basis of ξ, which provides the new basis of $\hat{\xi}$ and the correct exponentiation under isotopies, called *isoexponentiation* [12]

$$e_{\hat{\xi}_Q}^{iXw} = \hat{I} + (iwX)/1! + (iwX) * (iwX)/2! + \ldots = \{e^{iXQw}\}\hat{I}, \ w \in F, \ X \in \xi_Q \tag{3.6}$$

Particularly important is the *uniqueness* of the above isoexponentiation (up to isoequivalence transformations studied below), which should be compared to the various types of q-exponentiation in the literature [1].

The preceding isotopies then imply a step-by-step generalization of Lie's theory into a form submitted in memoir [12] and today called *Lie-Santilli theory* [23]–[32], [34], which includes the isotopies of Lie algebras, Lie groups, representation theory, the notion of symmetry, etc. (see [14, 33]) for details).

The preceding isotopies further imply a step-by-step generalization of functional analysis into a new discipline called *functional isoanalysis* [29], in which all conventional operations (say, log, derivative, integral, etc.), distributions (Dirac's delta, etc.), transforms (Fourier, Laplace and other transforms), special polynomials (Legendre polynomials, spherical harmonics, etc.) weak and strong continuity, etc. are generalized into a form compatible with the basic isounit $\hat{I} = Q^{-1}$. See ref.s [33] for a comprehensive presentation with applications.

The above chain of interconnected isotopies can indeed be formulated on a *conventional* Hilbert space $\mathcal{H}$, as done in the original proposal [3]. However, this implies the general loss of Hermiticity because isohermiticity is now defined by we therefore have the following $H^{\hat{\dagger}} = QH^\dagger Q^{-1}$.

Lemma 3.1: *An operator $H \in \xi_q$ which is originally Hermitean under q-number-deformations at time $t = 0$, over a conventional Hilbert space $\mathcal{H}$, becomes nonhermitean over the same space $\mathcal{H}$ under nonunitary time evolutions leading to a Q-operator-deformation, unless Q and H commute.*

For this reason, Myung and Santilli [18] introduced in 1982 the *isohilbert space* $\hat{\mathcal{H}}_Q$ characterized by the lifting

$$\mathcal{H} : <\psi|\phi> = \int d^3r \psi^\dagger \phi(r) \in C \Rightarrow \hat{\mathcal{H}}_Q : <\hat{\psi}|\hat{\phi}> := \hat{I} \int d^3r \hat{\psi}^\dagger(r) Q(r, \ldots)\hat{\phi}(r) \in \hat{C} \tag{3.7}$$

in which case isohermiticity coincides with Hermiticity. *This is a first fundamental property of Santilli's axiomatization of Q-deformations because it permits the preservation of observability under arbitrary time evolutions* (the issue of conserved *vs* nonconserved quantities is a separate one under isotopies treated below). Note that for Q of Class I (positive-definite) the composition is still inner and $\hat{\mathcal{H}}_Q$ is still Hilbert. Note also that for Q independent of the integration variables (or constant), $\hat{\mathcal{H}}_Q \equiv \mathcal{H}$ because in this case

$$< \hat{\psi}|\hat{\phi} > = < \hat{\psi}|Q|\hat{\phi} > \hat{I} \equiv < \hat{\psi}|\hat{\phi} > . \qquad (3.8)$$

In this sense, Myung-Santilli isohilbert spaces are "hidden" in the conventional formulation of q-deformations. Interested mathematicians are encouraged to extend the results to formal aspects, such as selfadjointness.

9) The preceding isotopies then imply, also for mathematical consistency, compatible generalizations of all operations on $\mathcal{H}$ into a form called isolinear operations on $\hat{\mathcal{H}}_Q$ [33]. We here limit ourselves to indicate the *isounitarity laws*

$$\hat{U} * \hat{U}^\dagger = \hat{U}^\dagger * \hat{U} = \hat{I}, \text{ or } \hat{U}^{-\hat{I}} = U^\dagger; \qquad (3.9)$$

the isounitary transformations $A' = \hat{U} * A * \hat{U}^\dagger$, with realization in term of an isohermitean operator X,

$$\hat{U} = e_{\hat{\xi}_Q}^{iXw}; \qquad (3.10)$$

the notions of determinant and trace of a matrix A

$$\text{Dêt } A = [\text{Det}(AQ)]\hat{I} \in \hat{F}, \quad \text{Trî } A = (\text{Tr } A)\hat{I} \in \hat{F}; \qquad (3.11)$$

the isotopies of eigenvalue equations [5]

$$H * |\hat{\psi} > = \hat{E} * |\hat{\psi} >, \quad \hat{E} \in \hat{F}_Q, \qquad (3.12)$$

and similarly for all other operations.

Then next step requires the selection of the specific dynamical brackets constructed via the isotopies of envelopes. We now assume that the Q-operator is independent from the coordinates r to avoid gravitational profiles within physical conditions in which they are generally ignored.

Axiomatization of Lie-isotopic q-deformations [18, 33]

Fundamental assumption: *Integro-differential generalization $\hat{I} = Q^{-1}$ of Planck's unit $\hbar = 1$ [3, 12], with reconstruction of the entire QM formalism to admit $\hat{I}$ as the correct left and right unit, as per the preceding mathematical notions and the following physical axioms:*

Axiom I: *The states are elements of a isohilbert space $\hat{\mathcal{H}}_Q$ interpreted as (left or right) isomodule with isoschödinger equations and isonormalization*

$$i\frac{\partial}{\partial t}|\hat{\psi} > = H * |\hat{\psi} > := HQ|\hat{\psi} > = \hat{E} * |\hat{\psi}, \quad < \hat{\psi}|\hat{\psi} > = \hat{I} = Q^{-1}. \qquad (3.13)$$

Axiom II: *Measurable quantities are represented by isocommuting isohermitean operators on $\hat{\mathcal{H}}_Q$ whose eigenvalues are conventional real numbers, e.g.,*

$$H^\dagger \equiv H^\dagger, \quad H * |\hat{\psi} > = \hat{E} * |\hat{\psi} > \equiv E|\hat{\psi} >, \quad \hat{E} \in \hat{R}, E \in R, . \qquad (3.14)$$

[5]This latter lifting is intriguing indeed for Q-deformations because they are turned into an explicit realization of the theory of "hidden variables", with significant epistemological implications. In fact, we can say that hadronic mechanics in general, and the Q-deformations in particular, are a "completion" of quantum mechanics essentially along the historical argument of Einstein, Podolsky, Rosen and others [33].

Axiom III: *The fundamental dynamical operators, the coordinates r^k and momenta p_k, are characterized by isoeigenvalue equations and isocommutation rules*

$$p_k * |\hat{\psi} > = -i\hat{I}_k^i \nabla_i|\hat{\psi} >, \quad r_{op}^k * |\hat{\psi} > = \hat{r}_{scal.}^k * |\hat{\psi} > \equiv r^k|\hat{\psi} >, \quad \hat{r} \in \hat{R}, r \in R, \qquad (3.15a)$$

$$[a^\mu, \hat{\ } a^\nu] := a^\mu Q a^\nu - a^\nu Q a^\mu = i\omega^{\mu\alpha}\hat{I}_\alpha^\nu, \quad a = (p,r), \quad (\hat{I}_\alpha^\nu) = \text{diag.}(Q^{-1}, Q^{-1}). \qquad (3.15b)$$

Axiom IV: *The time evolution of states is characterized by isounitary transformations with the (isohermitean) Hamiltonian as generator*

$$|\hat{\psi}(t) > = \hat{U}(t,t_0) * |\hat{\psi}(t_0) > = \{e_{\hat{\xi}_Q}^{iH(t_0-t)}\} * |\hat{\psi}(t_0) > \equiv e^{iHQ(t_0-t)}|\hat{\psi}(t_0) > . \qquad (3.16)$$

while the time evolution of operators is characterized by an equivalent, one-dimensional, Lie-Santilli group of isounitary transformations with the same Hamiltonian as generators, expressible in the finite form

$$A(t) = \hat{U} * A(t_0) * \hat{U}^\dagger = \{e_{\hat{\xi}_Q}^{iH(t_0-t)}\} * A(t_0) * \{e_{\hat{\xi}_Q}^{iH(t_0-t)}H\} \qquad (3.17)$$

with infinitesimal version provided by the Heisenberg-Santilli isotopic equations

$$i\frac{\hat{d}A}{\hat{d}t} = [A, \hat{\ } H] = AQH - HQA \qquad (3.18)$$

where $\hat{d}/\hat{d}t$ is the isoderivative. [6]

Axiom V: *The values expected in measurements of observables are given by the isoexpectation values*

$$\hat{<}A\hat{>} = \frac{< \hat{\psi}| * A * |\hat{\psi} >}{< \psi| * |\hat{\psi} >} = \frac{< \hat{\psi}|QAQ|\hat{\psi} >}{< \psi|Q|\hat{\psi} >}, \qquad (3.19)$$

The following comments are in order. The first fundamental result is that *Santilli's axiomatization of Q-operator-deformations of Class I coincides with conventional quantum mechanics at the abstract level.* In fact, at the abstract level, all distinction cease to exist between $\hat{F}_Q$ and F, $\hat{E}(r,\hat{\delta},\hat{R})$ and $E(r,\delta,R)$, $\hat{\xi}_Q$ and ξ, $\hat{\mathcal{H}}_Q$ and $\mathcal{H}$, etc. A subtle implication is that criticisms on the above axiomatization may eventually result to be criticisms on the axiomatic structure of quantum mechanics itself.

The second fundamental result is that *Santilli's axiomatization is form-invariant under its own transformation theory, the isounitary transformations.* This can be seen from the fact that the Lie-Santilli isocommutator is invariant under isounitary transformations. $\hat{U} * [A, \hat{\ } B] * \hat{U}^\dagger = [A', \hat{\ } B']$, or the invariance of eigenvalues and isoexpectation values under isounitary transformation, etc. This form-invariance should be compared with the general lack of invariance of q-deformations under time evolution, Eq. (1.2).

The third fundamental result is that *Santilli's isotopies achieve a true axiomatization of the quantity Q^{-1} assumed as the isounit of the theory.* In fact, $\hat{I} = Q^{-1}$ verifies the following properties: 1) $\hat{I}$ is isoidempotent of arbitrary (finite) degree, $\hat{I}^{\hat{n}} = \hat{I} * \hat{I} * \ldots * \hat{I} \equiv \hat{I}$; 2) The isoquotient of $\hat{I}$ by itself is $\hat{I}$, $\hat{I}/\hat{I} \equiv \hat{I}$; 3) The isosquare root of $\hat{I}$ is $\hat{I}$, $\hat{I}^{\frac{1}{2}} \equiv \hat{I}$; 4) $\hat{I}$ isocommutes with all possible operators, $[A, \hat{\ } \hat{I}] = A - A \equiv 0$; 5) $\hat{I}$ is left invariant by isounitary transformations, $\hat{U} * \hat{I} * \hat{U}^\dagger \equiv U * \hat{U}^\dagger = \hat{I}$; 6) $\hat{I}$ is conserved in time, $i\,d\hat{I}/\hat{d}t \equiv$

[6]For isounits independent of local variables (but dependent on the velocities and other quantities) $d/dt = \hat{I}_t d/dt$, where $\hat{I}_t$ is a new isounit of time independent of that of space $\hat{I} = Q^{-1}$. The appearance of the new isounit $\hat{I}_t$ has crucial importance for the overall consistency of the theory, e.g., for the compatibility of nonrelativistic and relativistic "isoplane-waves" $\hat{\psi}(r,t) = N \exp\{kQr - EQ_t t\}$, $\hat{I}_t = Q_t^{-1}$, for the axiomatic formulation of discrete-time theories, and others aspects.